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Optimal Control Problems Governed by a Second Order Ordinary Differential Equation with *m*-Point Boundary Condition

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Abstract. Using a new Green type function we present a study of optimal control problem where the dynamic is governed by a second order ordinary differential equation (SODE) with m-point boundary condition.

Key words: Differential game, Green function, m-Point boundary, Optimal control, Pettis, Strategy, Sweeping process, Viscosity

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1. Introduction

The pioneering works concerning control systems governed by second order ordinary differential equations (SODE) with three point boundary condition are developed in [2, 16]. In this paper we present some new applications of the Green function introduced in [11] to the study of viscosity problem in Optimal Control Theory where the dynamic is governed by (SODE) with m-point boundary condition. The paper is organized as follows. In Sect. 2 we recall and summarize the properties of a new Green function (Lemma 2.1) with application to a second order differential equation with m-point boundary condition in a separable Banach space E of the form

$$(SODE)\begin{cases} \ddot{u}_{\tau,x,f}(t) + \gamma \dot{u}_{\tau,x,f}(t) = f(t),\ t \in [\tau, 1] \\ u_{\tau,x,f}(\tau) = x,\ u_{\tau,x,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f}(\eta_i). \end{cases}$$

Here γ is positive, $f \in L^1_E([0,1])$, m is an integer number > 3, $0 \leq \tau < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\alpha_i \in \mathbf{R}$ $(i = 1, 2, \ldots, m-2)$ satisfying the condition

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma(1-\tau)) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - \tau)) \neq 0 \tag{1.1.1}$$

and $u_{\tau,x,f}$ is the trajectory $W^{2,1}_E([\tau, 1])$-solution to (SODE) associated with $f \in L^1_E([0,1])$ starting at the point $x \in E$ at time $\tau \in [0, 1[$. By Lemma 2.1, $u_{\tau,x,f}$ and $\dot{u}_{\tau,x,f}$ are represented, respectively, by

$$\begin{cases} u_{\tau,x,f}(t) = e_{\tau,x}(t) + \int_0^1 G_\tau(t,s) f(s)ds,\ \forall t \in [\tau, 1] \\ \dot{u}_{\tau,x,f}(t) = \dot{e}_{\tau,x}(t) + \int_0^1 \frac{\partial G_\tau}{\partial t}(t,s) f(s)ds,\ \forall t \in [\tau, 1] \end{cases}$$

where G_τ is the Green function defined in Lemma 2.1 with

$$\begin{cases} e_{\tau,x}(t) = x + A_\tau(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma(t-\tau)))x,\ \forall t \in [\tau, 1] \\ \dot{e}_{\tau,x}(t) = \gamma A_\tau \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma(t-\tau))x,\ \forall t \in [\tau, 1] \\ A_\tau = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma(1-\tau)) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - \tau))\right)^{-1}. \end{cases}$$

We stress that both existence and uniqueness and the integral representation formulas of solution and its derivative for (SODE) via the new Green function are of importance of this work. Indeed this allows to treat several new applications to optimal control problems and also some viscosity solutions for the value function governed by (SODE) with m-point boundary condition. In Sect. 3, we treat an optimal control problem governed by (SODE) in a separable Banach space

$$(SODE)_\Gamma \begin{cases} \ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t), \ f \in S^1_\Gamma \\ u_f(0) = x, \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i) \end{cases}$$

where Γ is a measurable and integrably bounded convex compact valued mapping and S^1_Γ is the set of all integrable selections of Γ. We show the compactness of the solution set and the existence of optimal control for the problem

$$\begin{cases} \ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t), \ f \in S^1_\Gamma \\ u_f(0) = x, \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i), \end{cases}$$

$$\inf_{f \in S^1_\Gamma} \int_0^1 J(t, u_f(t), \dot{u}_f(t), \ddot{u}_f(t))dt.$$

These results lead naturally to the problem of viscosity for the value function associated with this class of (SODE) which is presented in Sect. 4. In Sect. 5 we deal with a class of (SODE) with Pettis integrable second member. Existence and compactness of the solution set are also provided. Open problems concerning differential game governed by (SODE) and (ODE) with strategies are given in Sect. 6. We finish the paper by providing an application to the dynamic programming principle (DPP) and viscosity property for the value function associated with a sweeping process related to a model in Mathematical Economics [25].

2. Existence and Uniqueness

Let E be a separable Banach space. We denote by E^* the topological dual of E; $\overline{B}_E$ is the closed unit ball of E; $\mathcal{L}([0, 1])$ is the σ algebra of Lebesgue measurable sets on [0, 1]; $\lambda = dt$ is the Lebesgue measure on [0, 1]; $\mathcal{B}(E)$ is the σ algebra of Borel subsets of E. By $L^1_E([0, 1])$, we denote the space of all Lebesgue–Bochner integrable E-valued functions defined on [0, 1]. Let

$C_E([0,1])$ be the Banach space of all continuous functions $u : [0,1] \to E$ endowed with the sup-norm and let $C^1_E([0,1])$ be the Banach space of all functions $u \in C_E([0,1])$ with continuous derivative, endowed with the norm

$$\max\left\{\max_{t\in[0,1]} \|u(t)\|\,,\ \max_{t\in[0,1]} \|\dot{u}(t)\|\right\}.$$

We also denote $W^{2,1}_E([0,1])$ the space of all continuous functions in $C_E([0,1])$ such that their first derivatives are continuous and their second weak derivatives belong to $L^1_E([0,1])$.

We recall and summarize a new Green type function given in [11] that is a key ingredient in the statement of the problems under consideration.

Lemma 2.1. *Let* $0 \le \tau < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\gamma > 0$, $m > 3$ *be an integer number, and* $\alpha_i \in \mathbf{R}$ $(i = 1, \ldots, m-2)$ *satisfying the condition*

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp\left(-\gamma(1-\tau)\right) - \sum_{i=1}^{m-2} \alpha_i \exp\left(-\gamma(\eta_i - \tau)\right) \neq 0. \quad (1.1.1)$$

Let E *be a separable Banach space and let* $G_\tau : [\tau,1] \times [\tau,1] \to \mathbf{R}$ *be the function defined by*

$$G_\tau(t,s) = \begin{cases} \dfrac{1}{\gamma}\left(1 - \exp(-\gamma(t-s))\right), & \tau \le s \le t \le 1 \\ 0, & \tau \le t < s \le 1 \end{cases}$$
$$+\frac{A_\tau}{\gamma}\left(1 - \exp(-\gamma(t-\tau))\right)\phi_\tau(s), \quad (2.1)$$

where

$$\phi_\tau(s) = \begin{cases} 1 - \exp(-\gamma(1-s)) - \displaystyle\sum_{i=1}^{m-2} \alpha_i \left(1 - \exp(-\gamma(\eta_i - s))\right), & \tau \le s < \eta_1 \\ 1 - \exp(-\gamma(1-s)) - \displaystyle\sum_{i=2}^{m-2} \alpha_i \left(1 - \exp(-\gamma(\eta_i - s))\right), & \eta_1 \le s \le \eta_2 \\ \cdots\cdots & \\ 1 - \exp(-\gamma(1-s)), & \eta_{m-2} \le s \le 1, \end{cases}$$
$$(2.2)$$

and

$$A_\tau = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma(1-\tau)) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - \tau)) \right)^{-1}. \tag{2.3}$$

Then the following assertions hold

(i) *For every fixed* $s \in [\tau, 1]$, *the function* $G_\tau(., s)$ *is right derivable on* $[\tau, 1[$ *and left derivable on* $]\tau, 1]$. *Its derivative is given by*

$$\left(\frac{\partial G_\tau}{\partial t}\right)_+ (t,s) = \begin{cases} \exp(-\gamma(t-s)), & \tau \le s \le t < 1 \\ 0, & \tau \le t < s < 1 \end{cases} + A_\tau \exp(-\gamma(t-\tau))\phi_\tau(s), \tag{2.4}$$

$$\left(\frac{\partial G_\tau}{\partial t}\right)_- (t,s) = \begin{cases} \exp(-\gamma(t-s)), & \tau \le s < t \le 1 \\ 0, & \tau < t \le s \le 1 \end{cases} + A_\tau \exp(-\gamma(t-\tau))\phi_\tau(s). \tag{2.5}$$

(ii) $G_\tau(\cdot,\cdot)$ *and* $\frac{\partial G_\tau}{\partial t}(\cdot,\cdot)$ *satisfies*

$$|G_\tau(t,s)| \le M_{G_\tau} \text{ and } \left|\frac{\partial G_\tau}{\partial t}(t,s)\right| \le M_{G_\tau}, \quad \forall (t,s) \in [\tau,1] \times [\tau,1],$$

where

$$M_{G_\tau} = \max\{\gamma^{-1}, 1\}\left[1 + |A_\tau| \left(1 + \sum_{i=1}^{m-2} |\alpha_i|\right)\right].$$

(iii) *If* $u \in W_E^{2,1}([\tau, 1])$ *with* $u(\tau) = x$ *and* $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$, *then*

$$u(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s)(\ddot{u}(s) + \gamma \dot{u}(s))ds, \quad \forall t \in [\tau, 1],$$

where

$$e_{\tau,x}(t) = x + A_\tau(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma(t-\tau)))x.$$

(iv) *Let* $f \in L_E^1([\tau, 1])$ *and let* $u_f : [\tau, 1] \to E$ *be the function defined by*

$$u_f(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s) f(s)ds, \quad \forall t \in [\tau, 1].$$

Then we have

$$u_f(\tau) = x, \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i).$$

Further the function u_f *is derivable on* $[\tau, 1]$ *and its derivative* $\dot{u}_f$ *is defined by*

$$\dot{u}_f(t) = \lim_{h \to 0} \frac{u_f(t+h) - u_f(t)}{h} = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s) f(s) ds,$$

with

$$\dot{e}_{\tau,x}(t) = \gamma A_\tau (1 - \sum_{i=1}^{m-2} \alpha_i) \exp(-\gamma(t-\tau))x.$$

(v) If $f \in L_E^1([\tau, 1])$*, the function* $\dot{u}_f$ *is scalarly derivable, and its weak derivative* $\ddot{u}_f$ *satisfies*

$$\ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t) \quad a.e. \quad t \in [\tau, 1].$$

Proof. (i) Let $s \in [\tau, 1]$ and $t \in [\tau, 1]$. We consider two following cases.

Case 1 $t \neq s$. For every small $h > 0$ with $h < \min\{|t-s|, 1-t\}$, we have

$$\frac{G_\tau(t+h,s) - G_\tau(t,s)}{h} = \begin{cases} (\gamma h)^{-1} \exp(-\gamma(t-s))(1 - \exp(-\gamma h)), \\ \quad \tau \le s < t < 1 \\ 0, \\ \quad \tau \le t < s \le 1 \end{cases}$$
$$+ A_\tau \exp(-\gamma(t-\tau))$$
$$\times (1 - \exp(-\gamma h))(\gamma h)^{-1} \phi_\tau(s).$$

Hence $G_\tau(\cdot, s)$ is right derivable at $t \in [\tau, 1[\setminus \{s\}$ and

$$\left(\frac{\partial G_\tau}{\partial t}\right)_+ (t,s) = \begin{cases} \exp(-\gamma(t-s)), & \tau \le s < t < 1 \\ 0, & \tau \le t < s \le 1 \end{cases}$$
$$+ A_\tau \exp(-\gamma(t-\tau)) \phi_\tau(s).$$

Similarly, it is not difficult to check that $G_\tau(\cdot, s)$ is left derivable at $t \in]\tau, 1] \setminus \{s\}$ and

$$\left(\frac{\partial G_\tau}{\partial t}\right)_- (t,s) = \begin{cases} \exp(-\gamma(t-s)), & \tau \le s < t \le 1 \\ 0, & \tau < t < s \le 1 \end{cases}$$
$$+ A_\tau \exp(-\gamma(t-\tau)) \phi_\tau(s).$$

Case 2 $t = s$. Given $0 < h < 1 - s$. We have

$$\frac{G_\tau(t+h,s) - G_\tau(t,s)}{h} = (\gamma h)^{-1}(1 - \exp(-\gamma h)) + A_\tau \exp(-\gamma(t-\tau))(1 - \exp(-\gamma h)) \times (\gamma h)^{-1} \phi_\tau(s),$$

hence

$$\left(\frac{\partial G_\tau}{\partial t}\right)_+ (s,s) = 1 + A_\tau \exp(-\gamma(s-\tau))\phi_\tau(s).$$

Now given $0 < h < s - \tau$. We have

$$\frac{G_\tau(t-h,s) - G_\tau(t,s)}{h} = A_\tau \exp(-\gamma(t-\tau)) \times (1 - \exp(-\gamma h))(\gamma h)^{-1} \phi_\tau(s),$$

hence

$$\left(\frac{\partial G_\tau}{\partial t}\right)_+ (s,s) = A_\tau \exp(-\gamma(s-\tau))\phi_\tau(s).$$

(ii) It is easy to see that $|\phi_\tau(s)| \leq 1 + \sum_{i=1}^{m-2} |\alpha_i|$ for all $s \in [0,1]$. So, from the definition of G_τ we deduce that for all $s, t \in [\tau, 1]$

$$|G_\tau(t,s)| \leq \frac{1}{\gamma}\left[1 + |A_\tau|\left(1 + \sum_{i=1}^{m-2} |\alpha_i|\right)\right] \leq M_{G_\tau}.$$

Similarly we deduce that for all $s, t \in [\tau, 1]$

$$\left|\frac{\partial G_\tau}{\partial t}(t,s)\right| \leq 1 + |A_\tau|\,|\phi_\tau(s)| \leq 1 + |A_\tau|\left(1 + \sum_{i=1}^{m-2} |\alpha_i|\right) \leq M_{G_\tau}.$$

(iii) Let $x^* \in E^*$. By definition of G_τ, for all $t \in [\tau, 1]$, we have

$$\left\langle x^*, \int_\tau^1 G_\tau(t,s)\ddot{u}(s)ds \right\rangle = \int_\tau^1 \left\langle x^*, G_\tau(t,s)\ddot{u}(s)\right\rangle ds$$
$$= \frac{1}{\gamma}\int_\tau^t (1 - \exp(-\gamma(t-s)))\left\langle x^*, \ddot{u}(s)\right\rangle ds + \frac{A_\tau}{\gamma}(1 - \exp(-\gamma(t-\tau)))\int_\tau^1 \left\langle x^*, \phi_\tau(s)\ddot{u}(s)\right\rangle ds.$$

On the other hand

$$\int_\tau^t (1-\exp(-\gamma(t-s)))\,\langle x^*, \ddot{u}(s)\rangle ds$$
$$= (\exp(-\gamma(t-\tau))-1)\,\langle x^*, \dot{u}(\tau)\rangle + \gamma\int_\tau^t \exp(-\gamma(t-s))\langle x^*, \dot{u}(s)\rangle ds$$

and $\int_\tau^1 \langle x^*, \phi_\tau(s)\ddot{u}(s)\rangle ds = I_1 + I_2$ where

$$I_1 = \sum_{i=1}^{m-1}\int_{\eta_i-1}^{\eta_i} (1-\exp(-\gamma(1-s)))\,\langle x^*, \ddot{u}(s)\rangle ds$$
$$= (\exp(-\gamma(1-\tau))-1)\,\langle x^*, \dot{u}(\tau)\rangle + \gamma\int_\tau^1 \exp(-\gamma(1-s))\langle x^*, \dot{u}(s)\rangle ds$$
$$I_2 = -\sum_{i=1}^{m-2}\sum_{j=i}^{m-2}\alpha_j\int_{\eta_i-1}^{\eta_i}(1-\exp(-\gamma(\eta_j-s)))\langle x^*, \dot{u}(s)\rangle ds$$
$$= -\sum_{i=1}^{m-2}\alpha_i\,(\exp(-\gamma(\eta_i-\tau))-1)\,\langle x^*, \dot{u}(\tau)\rangle$$
$$-\gamma\sum_{i=1}^{m-2}\sum_{j=i}^{m-2}\int_{\eta_{i-1}}^{\eta_i}\exp(-\gamma(\eta_i-s))\langle x^*, \dot{u}(s)\rangle ds$$

with $\eta_0 = \tau$, $\eta_{m-1} = 1$.
Hence

$$\left\langle x^*, \int_\tau^1 G_\tau(t,s)(\ddot{u}(s)+\gamma\dot{u}(s))ds\right\rangle$$
$$= \frac{1}{\gamma}(\exp(-\gamma(t-\tau))-1)\langle x^*, \dot{u}(\tau)\rangle$$
$$+ \frac{A_\tau}{\gamma}(1-\exp(-\gamma(t-\tau)))\langle x^*, \dot{u}(\tau)\rangle$$
$$\times\left[\exp(-\gamma(1-\tau))-1-\sum_{i=1}^{m-2}\alpha_i\,(\exp(-\gamma(\eta_i-\tau))-1)\right]$$
$$+ \int_\tau^t \langle x^*, \dot{u}(s)\rangle ds + A_\tau\,(1-\exp(-\gamma t))\sum_{i=1}^{m-2}\left(1-\sum_{j=i}^{m-2}\alpha_j\right)$$
$$\times\int_{\eta_{i-1}}^{\eta_i}\langle x^*, \dot{u}(s)\rangle ds.$$

This implies that

$$\langle x^*, \int_0^1 G_\tau(t,s)(\ddot{u}(s)+\gamma\dot{u}(s))ds\rangle = \langle x^*, u(t)-e_{\tau,x}(t)\rangle, \quad \forall t \in [\tau, 1].$$

Since this equality holds for every $x^* \in E^*$, we get

$$u(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s)(\ddot{u}(s) + \gamma\dot{u}(s))ds, \quad \forall t \in [\tau, 1].$$

(iv) Let $f \in L^1_E([0,1])$ and $u_f(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s)f(s)ds$, $\forall t \in [0,1]$. Then, by definition of G_τ in (i), we have $u_f(\tau) = x$ and

$$\begin{aligned}
u_f(1) &= e_{\tau,x}(1) + \frac{1}{\gamma}\int_\tau^1 (1-\exp(-\gamma(1-s)))f(s)ds \\
&\quad + \frac{A_\tau}{\gamma}(1-\exp(-\gamma(1-\tau)))\int_\tau^1 \phi_\tau(s)f(s)ds \\
&= e_{\tau,x}(1) + \frac{1}{\gamma}\int_\tau^1 \left[1-\exp(-\gamma(1-s)) - \phi_\tau(s)\right]f(s)ds \\
&\quad + \frac{1}{\gamma}\left[A_\tau(1-\exp(-\gamma(1-\tau)))+1\right]\int_\tau^1 \phi_\tau(s)f(s)ds \\
&= e_{\tau,x}(1) + \frac{1}{\gamma}\sum_{i=1}^{m-2}\sum_{j=i}^{m-2}\alpha_j \int_{\eta_i-1}^{\eta_i}(1-\exp(-\gamma(\eta_i - s)))f(s)ds \\
&\quad + \frac{A_\tau}{\gamma}\sum_{i=1}^{m-2}\alpha_i(1-\exp(-\gamma(\eta_i-\tau)))\int_\tau^1 \phi_\tau(s)f(s)ds \\
&= e_{\tau,x}(1) + \frac{1}{\gamma}\sum_{i=1}^{m-2}\alpha_i \int_0^{\eta_i}(1-\exp(-\gamma(\eta_i - s)))f(s)ds \\
&\quad + \frac{A_\tau}{\gamma}\sum_{i=1}^{m-2}\alpha_i(1-\exp(-\gamma(\eta_i-\tau)))\int_\tau^1 \phi_\tau(s)f(s)ds.
\end{aligned}$$

From the definition of $e_{\tau,x}(t)$ and A_τ, we deduce that

$$\begin{aligned}
e_{\tau,x}(1) &= x + A_\tau\left(1 - \sum_{i=1}^{m-2}\alpha_i\right)(1-\exp(-\gamma(1-\tau)))\,x \\
&= A_\tau\left[A_\tau^{-1} + 1 - \exp(-\gamma(1-\tau)) + \sum_{i=1}^{m-2}\alpha_i\left(\exp(-\gamma(1-\tau))-1\right)\right]x \\
&= A_\tau\left[\sum_{i=1}^{m-2}\alpha_i\exp(-\gamma(1-\tau)) - \sum_{i=1}^{m-2}\alpha_i\exp(-\gamma(\eta_i-\tau))\right]x
\end{aligned}$$

and

$$e_{\tau,x}(\eta_i) = x + A_\tau \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) (1 - \exp(-\gamma(\eta_i - \tau)))\, x$$

$$= A_\tau \left[A_\tau^{-1} + 1 - \exp(-\gamma(\eta_i - \tau)) - \sum_{j=1}^{m-2} \alpha_j + \exp(-\gamma(\eta_i - \tau)) \sum_{j=1}^{m-2} \alpha_j\right] x$$

$$= A_\tau \left[\exp(-\gamma(1 - \tau)) - \exp(-\gamma(\eta_i - \tau)) + \exp(-\gamma(\eta_i - \tau)) \sum_{j=1}^{m-2} \alpha_j \right.$$

$$\left. + \sum_{j=1}^{m-2} \alpha_j \exp(-\gamma(\eta_j - \tau))\right] x.$$

Hence we deduce that

$$\sum_{i=1}^{m-2} \alpha_i e_{\tau,x}(\eta_i)$$

$$= A_\tau \left[\sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(1 - \tau)) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - \tau)) \right.$$

$$+ \left(\sum_{j=1}^{m-2} \alpha_j\right) \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - \tau)) - \left(\sum_{i=1}^{m-2} \alpha_i\right)$$

$$\left. \times \sum_{j=1}^{m-2} \alpha_j \exp(-\gamma(\eta_j - \tau))\right] x = e_{\tau,x}(1).$$

So, by combining the above relations, we get

$$u_f(1) = \sum_{i=1}^{m-2} \alpha_i e_{\tau,x}(\eta_i) + \frac{1}{\gamma} \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (1 - \exp(-\gamma(\eta_i - s))) f(s) ds$$

$$+ \frac{A_\tau}{\gamma} \sum_{i=1}^{m-2} \alpha_i (1 - \exp(-\gamma(\eta_i - \tau))) \int_\tau^1 \phi_\tau(s) f(s) ds$$

$$= \sum_{i=1}^{m-2} \alpha_i \left[e_{\tau,x}(\eta_i) + \frac{1}{\gamma} \int_0^{\eta_i} (1 - \exp(-\gamma(\eta_i - s))) f(s) ds\right.$$

$$+ \frac{A_\tau}{\gamma}(1 - \exp(-\gamma(\eta_i - \tau))) \int_\tau^1 \phi_\tau(s) f(s) ds \Bigg]$$

$$= \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i).$$

On the other hand, by the same arguments as in [2] we can conclude that u_f is derivable and its derivative $\dot{u}_f$ is defined by

$$\dot{u}_f(t) = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G}{\partial t}(t,s) f(s) ds, \quad \forall t \in [0,1].$$

(v) Indeed, let $t \in [0,1]$. Using the expression of $\frac{\partial G}{\partial t}$ in (i) we have

$$\dot{u}_f(t) = \dot{e}_{\tau,x}(t) + \int_\tau^t \exp(-\gamma(t-s)) f(s) ds$$
$$+ A_\tau \exp(-\gamma(t-\tau)) \int_\tau^1 \phi_\tau(s) f(s) ds.$$

Whence

$$\langle x^*, \ddot{u}_f(t)\rangle = \frac{d}{dt}\langle x^*, \dot{u}_f(t)\rangle$$
$$= \langle x^*, \ddot{e}_{\tau,x}(t)\rangle + \frac{d}{dt}\int_\tau^t \exp(-\gamma(t-s))\langle x^*, f(s)\rangle ds$$
$$-A_\tau \gamma \exp(-\gamma(t-\tau)) \int_\tau^1 \langle x^*, \phi_\tau(s) f(s)\rangle ds$$
$$= \langle x^*, \ddot{e}_{\tau,x}(t)\rangle + \langle x^*, f(t)\rangle - \gamma \int_\tau^t \exp(-\gamma(t-s))\langle x^*, f(s)\rangle ds$$
$$-A_\tau \gamma \exp(-\gamma(t-\tau)) \int_\tau^1 \langle x^*, \phi_\tau(s) f(s)\rangle ds.$$

We also note that $\ddot{e}_{\tau,x}(t) = -\gamma \dot{e}_{\tau,x}(t)$. Therefore

$$\langle x^*, \ddot{u}_f(t)\rangle = \langle x^*, f(t)\rangle - \langle x^*, \gamma \dot{u}_f(t)\rangle.$$

This implies that $\dot{u}_f$ is scalarly derivable and

$$\ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t) \quad a.e. \quad t \in [0,1].$$

□

The following result is a direct application of Lemma 2.1.

Lemma 2.2. *With the notations of Lemma 2.1, assume* $0 \le \tau < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\gamma > 0$, $m > 3$ *be an integer number, and* $\alpha_i \in \mathbf{R}$ $(i = 1, \ldots, m-2)$ *and (1.1.1). Let* $f \in C_E([\tau, 1])$ *(resp.* $f \in L^1_E([\tau, 1])$*). Then the m-point boundary problem*

$$\begin{cases} \ddot{u}_{\tau,x,f}(t) + \gamma \dot{u}_{\tau,x,f}(t) = f(t), \ t \in [\tau, 1] \\ u_{\tau,x,f}(\tau) = x, u_{\tau,x,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f}(\eta_i) \end{cases}$$

has a unique $C^2_E([\tau, 1])$*-solution (resp.* $W^{2,1}_E([\tau, 1])$*-solution) which is given by the integral representation formulas*

$$\begin{cases} u_{\tau,x,f}(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t, s) f(s) ds, \ t \in [\tau, 1] \\ \dot{u}_{\tau,x,f}(t) = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t, s) f(s) ds, \ t \in [\tau, 1] \end{cases}$$

where

$$\begin{cases} e_{\tau,x}(t) = x + A_\tau (1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma(t - \tau)))x, \\ \dot{e}_{\tau,x}(t) = \gamma A_\tau \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma(t - \tau))x, \\ A_\tau = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma(1 - \tau)) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - \tau))\right)^{-1}. \end{cases}$$

Remark. It is clear that the Green function G_τ depends on τ. When $\tau = 0$, (1.1.1) is reduced to

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i)) \neq 0 \tag{1.1.2}$$

where m is an integer number > 3, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\alpha_i \in \mathbf{R}$ $(i = 1, 2, \ldots, m-2)$. Then the m-point boundary problem

$$\begin{cases} \ddot{u}_{x,f}(t) + \gamma \dot{u}_{x,f}(t) = f(t), \ t \in [0, 1] \\ u_{x,f}(0) = x, u_{x,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{x,f}(\eta_i) \end{cases}$$

has a unique $C^2_E([0,1])$-solution (resp. $W^{2,1}_E([0,1])$-solution), $u_{x,f}$, with integral representation formulas

$$\begin{cases} u_{x,f}(t) = e_x(t) + \int_0^1 G_0(t,s)f(s)ds, \ t \in [0,1] \\ \dot{u}_{x,f}(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G_0}{\partial t}(t,s)f(s)ds, \ t \in [0,1] \end{cases}$$

where

$$\begin{cases} e_x(t) = x + A_0(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x, \\ \dot{e}_x(t) = \gamma A_0 \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t)x, \\ A_0 = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1}. \end{cases}$$

This remark and its notation will be used in the next section.

3. Existence of Optimal Controls

Let us recall the following denseness result based on Lyapunov theorem. See e.g. [12, 28].

Proposition 3.1. *Let E be a separable Banach space. Let $\Gamma : [0,T] \to cwk(E)$ be a convex weakly compact valued measurable and integrably bounded mapping. Let $ext(\Gamma) : t \mapsto ext(\Gamma(t))$ where $ext(\Gamma(t))$ is the set of extreme points of $\Gamma(t)(t \in [0,T])$. Then the set S^1_Γ of all integrable selections of Γ is convex and $\sigma(L^1_E, L^\infty_{E^*})$-compact and the set of all integrable selections $S^1_{ext(\Gamma)}$ of $ext(\Gamma)$ is dense in S^1_Γ with respect to this topology.*

Proof. See e.g. [12, 28]. □

In this section we will assume that the hypotheses and notations of Lemma 2.1 hold with $\tau = 0$.

Theorem 3.1. *With the hypotheses and notations of Proposition 3.1, let E be a separable Banach space and let $\Gamma : [0,T] \to ck(E)$ be a convex compact valued measurable and integrably bounded mapping. Let us following (SODE)*

$$(SODE)_\Gamma \begin{cases} \ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t), \ f \in S^1_\Gamma \\ u_f(0) = x, \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i) \end{cases}$$

$$(SODE)_{ext(\Gamma)} \begin{cases} \ddot{u}_g(t) + \gamma \dot{u}_g(t) = g(t), \ g \in S^1_{ext(\Gamma)} \\ u_g(0) = x, \quad u_g(1) = \sum_{i=1}^{m-2} \alpha_i u_g(\eta_i). \end{cases}$$

Then the set $\{u_f : f \in S^1_\Gamma\}$ *of* $W^{2,1}_E([0,1])$*-solutions to* $(SODE)_\Gamma$ *is compact in* $C^1_E([0,1])$ *and the set* $\{u_g : g \in S^1_{ext(\Gamma)}\}$ *of* $W^{2,1}_E([0,1])$*-solutions to* $(SODE)_{ext(\Gamma)}$ *is dense in the compact set* $\{u_f : f \in S^1_\Gamma\}$ *of* $W^{2,1}_E([0,1])$*-solutions to* $(SODE)_\Gamma$.

Proof. Step 1. Compactness of the solution set $\{u_f : f \in S^1_\Gamma\}$ in $C^1_E([0,1])$.

Let (u_{f_n}) be a sequence of $W^{2,1}_E([0,1])$-solutions to $(SODE)_\Gamma$. As S^1_Γ is $\sigma(L^1_E, L^\infty_{E^*})$-compact, by Eberlein–Smulian theorem, we may assume that (f_n) $\sigma(L^1_E, L^\infty_{E^*})$-converges to $f_\infty \in S^1_\Gamma$. From the properties of the Green function G_0 in Lemma 2.1 (by taking $\tau = 0$) we have, for each $n \in \mathbf{N}$,

$$u_{f_n}(t) = e_x(t) + \int_0^1 G_0(t,s) f_n(s)ds, \ \ t \in [0,1], \tag{3.1.1}$$

$$\dot{u}_{f_n}(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G_0}{\partial t}(t,s) f_n(s)ds, \ \ t \in [0,1], \tag{3.1.2}$$

$$\ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) = f_n(t) \in \Gamma(t), a.e. \ t \in [0,1] \tag{3.1.3}$$

with

$$\begin{cases} e_x(t) = x + A_0(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x, \ t \in [0,1] \\ \dot{e}_x(t) = \gamma A_0 \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t)x, \ \ t \in [0,1] \\ A_0 = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1}. \end{cases}$$

On the other hand, from definition of the Green function G_0 in Lemma 2.1(iv) and (3.1.1), it is not difficult to show that $\{u_{f_n} : n \in \mathbf{N}\}$ is equicontinuous in $C_E([0,1])$. Indeed, let $t, t' \in [0,1]$, from (3.1.1) and (iv), we have the estimate

$$\begin{aligned}
&||u_{f_n}(t) - u_{f_n}(t')|| \\
&\leq ||e_x(t) - e_x(t')|| + \int_0^1 |G_0(t,s) - G_0(t',s)|\, ||\ddot{u}_{f_n}(s) + \gamma \dot{u}_{f_n}(s)||ds \\
&\leq ||e_x(t) - e_x(t')|| + \int_0^1 |G_0(t,s) - G_0(t',s)|\, |\Gamma(s)|ds.
\end{aligned}$$

Further, for each $t \in [0,1]$ $\{u_{f_n}(t) : n \in \mathbf{N}\}$ is relatively compact because it is included in the norm compact set $e_x(t) + \int_0^1 G_0(t,s)\Gamma(s)ds$ (see e.g. [12, 14]). So by Ascoli's theorem, $\{u_{f_n} : n \in \mathbf{N}\}$ is relatively compact in $C_E([0,1])$. Similarly using the properties of $\frac{\partial G_0}{\partial t}$ in Lemma 2.1 and (3.1.2) we deduce that $\left\{\dot{u}_{f_n} : n \in \mathbf{N}\right\}$ is equicontinuous in $C_E([0,1])$. In addition, the set $\left\{\dot{u}_{f_n}(t) : n \in \mathbf{N}\right\}$ is included in the compact set $\dot{e}_x(t) + \int_0^1 \frac{\partial G_0}{\partial t}(t,s)\,\Gamma(s)\,ds$. So $\left\{\dot{u}_{f_n} : n \in \mathbf{N}\right\}$ is relatively compact in $C_E([0,1])$ by Ascoli's theorem. From the above facts, we deduce that there exists a subsequence of $\left(u_{f_n}\right)_{n\in\mathbf{N}}$ still denoted by $\left(u_{f_n}\right)_{n\in\mathbf{N}}$ which converges uniformly to $u^\infty \in C_E([0,1])$ with $u^\infty(0) = x$, $u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i)$. Similarly, we may assume that $\left(\dot{u}_{f_n}\right)$ converges uniformly to $v^\infty \in C_E([0,1])$. Furthermore, by the above facts, it is easy to see that $\left(\ddot{u}_{f_n}\right)$ $\sigma(L^1_E, L^\infty_{E^*})$-converges to $w^\infty \in L^1_E([0,1])$. For every $t \in [0,1]$, using the representation formula (3.1.1), we have

$$\begin{aligned}
u^\infty(t) = \lim_{n\to\infty} u_{f_n}(t) &= e_x(t) + \lim_{n\to\infty} \int_0^1 G_0(t,s)(\ddot{u}_{f_n}(s) + \gamma \dot{u}_{f_n}(s))ds \\
&= e_x(t) + \lim_{n\to\infty} \int_0^1 G_0(t,s)\ddot{u}_{f_n}(s)ds + \gamma \lim_{n\to\infty} \int_0^1 G_0(t,s)\dot{u}_{f_n}(s)ds \\
&= e_x(t) + \int_0^1 G_0(t,s)w^\infty(s)ds + \gamma \int_0^1 G_0(t,s)v^\infty(s)ds \\
&= e_x(t) + \int_0^1 G_0(t,s)(w^\infty(s) + \gamma v^\infty(s))ds. \qquad (3.1.4)
\end{aligned}$$

From (3.1.4) and Lemma 2.1(iv), we deduce that u^∞ is derivable and its derivative $\dot{u}^\infty$ is given by

$$\dot{u}^\infty(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G_0}{\partial t}(t,s)(w^\infty(s) + \gamma v^\infty(s))ds, \forall t \in [0,1]. \quad (3.1.5)$$

Now using the integral representation formula (3.1.2) we have, for every $t \in [0, 1]$,

$$
\begin{aligned}
v^\infty(t) &= \lim_{n\to\infty} \dot u_{f_n}(t) = \dot e_x(t) + \lim_{n\to\infty} \int_0^1 \frac{\partial G_0}{\partial t}(t,s)(\ddot u_{f_n}(s) + \gamma \dot u_{f_n}(s))ds \\
&= \dot e_x(t) + \lim_{n\to\infty} \int_0^1 \frac{\partial G_0}{\partial t}(t,s)\ddot u_{f_n}(s)ds \\
&\quad + \gamma \lim_{n\to\infty} \int_0^1 \frac{\partial G_0}{\partial t}(t,s)\dot u_{f_n}(s)ds \\
&= \dot e_x(t) + \int_0^1 \frac{\partial G_0}{\partial t}(t,s) w^\infty(s)ds + \gamma \int_0^1 \frac{\partial G_0}{\partial t}(t,s) v_\infty(s)ds \\
&= \dot e_x(t) + \int_0^1 \frac{\partial G_0}{\partial t}(t,s)(w^\infty(s) + \gamma v^\infty(s))ds
\end{aligned} \tag{3.1.6}
$$

so that by (3.1.5) and (3.1.6) we get $v^\infty = \dot u^\infty$. Now invoking Lemma 2.1(v) and using (3.1.4) we get

$$
\ddot u^\infty(t) + \gamma \dot u^\infty(t) = w^\infty(t) + \gamma v^\infty(t) = w^\infty(t) + \gamma \dot u^\infty(t) \quad a.e. \quad t \in [0, 1].
$$

Thus we get $\ddot u^\infty(t) = w^\infty(t)\, a.e.\, t \in [0, 1]$ so that by (3.1.4)

$$
\begin{cases}
u^\infty(t) = e_x(t) + \int_0^1 G_0(t,s)(\ddot u^\infty(s) + \gamma \dot u^\infty(s))ds, \quad t \in [0,1] \\
u^\infty(0) = x, \quad u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i).
\end{cases} \tag{3.1.7}
$$

Step 2. Main fact: u^∞ coincides with the $W_E^{2,1}([0,1])$-solution u_{f_∞} associated with $f_\infty \in S_\Gamma^1$ to

$$
\begin{cases}
\ddot u_{f_\infty}(t) + \gamma \dot u_{f_\infty}(t) = f_\infty(t), \\
u_{f_\infty}(0) = x, \quad u_{f_\infty}(1) = \sum_{i=1}^{m-2} \alpha_i u_{f_\infty}(\eta_i).
\end{cases} \tag{3.1.8}
$$

Remember that

$$
\begin{cases}
\ddot u_{f_n}(t) + \gamma \dot u_{f_n}(t) = f_n(t), \\
u_{f_n}(0) = x, \quad u_{f_n}(1) = \sum_{i=1}^{m-2} \alpha_i u_{f_n}(\eta_i)
\end{cases}
$$

and by the above fact, $(\ddot{u}_{f_n} + \gamma \dot{u}_{f_n})$ converges weakly in $L^1_E([0, 1])$ to $\ddot{u}^\infty + \gamma \dot{u}^\infty$. Let $v \in L^\infty_{E^*}([0, 1])$. Multiply scalarly the equation

$$\ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) = f_n(t)$$

by $v(t)$ and integrating on [0, 1] yields

$$\int_0^1 \langle v(t), \ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) \rangle dt = \int_0^1 \langle v(t), f_n(t) \rangle dt. \tag{3.1.9}$$

It is clear that

$$\lim_{n\to\infty} \int_0^1 \langle v(t), \ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) \rangle dt = \int_0^1 \langle v(t), \ddot{u}^\infty(t) + \gamma \dot{u}^\infty(t) \rangle dt$$
$$= \lim_{n\to\infty} \int_0^1 \langle v(t), f_n(t) \rangle dt = \int_0^1 \langle v(t), f_\infty(t) \rangle dt$$

so that

$$\ddot{u}^\infty + \gamma \dot{u}^\infty = f_\infty. \tag{3.1.10}$$

Using (3.1.7), (3.1.8), and (3.1.10) and uniqueness of solutions we get $u^\infty = u_{f_\infty}$. This proves the first part of the theorem, while the second part follows from Proposition 3.1 and the integral representation formulas. □

Now comes a direct application to the existence of optimal controls for the problem

$$\begin{cases} \ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t), \ f \in S^1_\Gamma \\ u_f(0) = x, \quad u_f(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u_f(\eta_i), \end{cases} \tag{*}$$

$$\inf_{f \in S^1_\Gamma} \int_0^1 J(t, u_f(t), \dot{u}_f(t), \ddot{u}_f(t)) dt. \tag{**}$$

Theorem 3.2. *Under the hypotheses and notations of Theorem 3.1, problem (*)–(**) admits an optimal control.*

Proof. Let us set $m := \inf_{f \in S^1_\Gamma} \int_0^1 J(t, u_f(t), \dot{u}_f(t), \ddot{u}_f(t)) dt$. Let us consider a minimizing sequence $(u_{f_n}, \dot{u}_{f_n}, \ddot{u}_{f_n})$, that is

$$\lim_{n\to\infty} \int_0^1 J(t, u_{f_n}(t), \dot{u}_{f_n}(t), \ddot{u}_{f_n}(t)) dt = m.$$

Since (f_n) is relatively weakly compact in $L^1_E([0, 1])$, we may assume that (f_n) converges weakly in $L^1_E([0, 1])$ to $\overline{f}$. Applying the arguments in the proof of Theorem 3.1 shows that (u_{f_n}) converges uniformly to $(u_{\overline{f}})$, $(\dot{u}_{f_n})$ converges uniformly to $\dot{u}_{\overline{f}}$ and $(\ddot{u}_{f_n})$ $\sigma(L^1_E, L^\infty_{E^*})$ -converges to $\ddot{u}_{\overline{f}}$ with

$$\ddot{u}_{\overline{f}}(t) + \gamma \dot{u}_{\overline{f}}(t) = \overline{f}(t),$$

$$u_{\overline{f}}(0) = x, \quad u_{\overline{f}}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\overline{f}}(\eta_i).$$

Now apply the lower semicontinuity for integral functionals ([14], Theorem 8.1.6) yields

$$\liminf_{n\to\infty} \int_0^1 J(t, u_{f_n}(t), \dot{u}_{f_n}(t), \ddot{u}_{f_n}(t))dt \geq \int_0^1 J(t, u_{\overline{f}}(t), \dot{u}_{\overline{f}}(t), \ddot{u}_{\overline{f}}(t))dt \geq m.$$

Hence we conclude that

$$m = \inf_{f\in S^1_\Gamma} \int_0^1 J(t, u_f(t), \dot{u}_f(t), \ddot{u}_f(t))dt = \int_0^1 J(t, u_{\overline{f}}(t), \dot{u}_{\overline{f}}(t), \ddot{u}_{\overline{f}}(t))dt.$$

□

Now along the paper we will assume that the hypotheses and notations of Lemma 2.1 hold.

4. Viscosity Property of the Value Function

The results given in Sect. 3 lead naturally to the problem of viscosity for the value function associated with a second order differential inclusion. Similar results dealing with ordinary differential equation (ODE) and evolution inclusion with control measures are available in [2, 7, 14, 16]. In this section we treat a new problem of value function in the context of second order ordinary differential equations (SODE) with m-point boundary condition. Assume that E is a separable Banach space, Z is a convex compact subset of E and S^1_Z is the set of all Lebesgue measurable mappings $f : [0, 1] \to Z$ (alias measurable selections of the constant mapping Z). For each $f \in S^1_Z$, let us denote by $u_{\tau,x,f}$ the trajectory solution associated with the control $f \in S^1_Z$ starting from x at time $\tau \in [0, \eta_1[$ to

$$(SODE) \begin{cases} \ddot{u}_{\tau,x,f}(t) + \gamma \dot{u}_{\tau,x,f}(t) = f(t),\ t \in [\tau, 1] \\ u_{\tau,x,f}(\tau) = x, u_{\tau,x,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f}(\eta_i) \end{cases}$$

with the integral representation formulas

$$\begin{cases} u_{\tau,x,f}(t) = e_{\tau,x}(t) + \int_{\tau}^{1} G_{\tau}(t,s) f(s) ds, \ t \in [\tau, 1] \\ \dot{u}_{\tau,x,f}(t) = \dot{e}_{\tau,x}(t) + \int_{\tau}^{1} \frac{\partial G_{\tau}}{\partial t}(t,s) f(s) ds, \ t \in [\tau, 1] \end{cases} \tag{4.1}$$

and

$$\begin{cases} e_{\tau,x}(t) = x + A_{\tau}(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma(t-\tau)))x, \ t \in [\tau, 1] \\ \dot{e}_{\tau,x}(t) = \gamma A_{\tau} \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma(t-\tau))x, \ t \in [\tau, 1] \\ A_{\tau} = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma(1-\tau)) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - \tau))\right)^{-1} \end{cases} \tag{4.2}$$

where the coefficient A_τ and the Green function G_τ are given in Lemma 2.1.

By the above considerations and Lemma 2.1(ii), it is easy to check that $\dot{u}_{\tau,x,f}$ are uniformly majorized by a continuous function $c_\tau : [\tau, 1] \to \mathbf{R}^+$, namely

$$\begin{aligned} ||\dot{u}_{\tau,x,f}(t)|| &\leq ||e_{\tau,x}(t)|| + \int_{\tau}^{1} |\frac{\partial G_{\tau}}{\partial t}(t,s)| \, ||f(s)|| ds \\ &\leq ||e_{\tau,x}(t)|| + \int_{\tau}^{1} |\frac{\partial G_{\tau}}{\partial t}(t,s)||Z| ds = c_{\tau}(t), \ \forall t \in [\tau, 1]. \end{aligned} \tag{4.3}$$

It is worth mentioning that integral representation formulas (4.1) and (4.2) will be useful in the study of the value function we present below. Let us mention a useful lemma that is borrowed from ([16], Lemma 6.3) and ([7], Lemma 3.1).

Lemma 4.1. *Assume that (1.1.1) is satisfied. Let* $(t_0, x_0) \in [0, \eta_1[\times E$ *and let* Z *be a convex compact subset in* E. *Let* $\Lambda : [0, T] \times E \times Z \to \mathbf{R}$ *be an upper semicontinuous function such that the restriction of* Λ *to* $[0, T] \times B \times Z$ *is bounded on any bounded subset* B *of* E. *If*

$$max_{z \in Z} \Lambda(t_0, x_0, z) < -\eta < 0$$

for some $\eta > 0$, *then there exists* $\sigma > 0$ *such that*

$$\sup_{f \in S_Z^1} \left\{ \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,f}(t), f(t)) dt \right\} < -\frac{\sigma \eta}{2}$$

where $u_{t_0,x_0,f}$ is the trajectory solution associated with the control $f \in S^1_Z$ starting from x_0 at time t_0 to

$$(SODE)\begin{cases} \ddot{u}_{t_0,x_0,f}(t) + \gamma \dot{u}_{t_0,x_0,f}(t) = f(t),\ t \in [t_0, 1] \\ u_{t_0,x_0,f}(t_0) = x_0,\ u_{t_0,x_0,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{t_0,x_0,f}(\eta_i). \end{cases}$$

Proof. By hypothesis, one has $\max_{z\in Z} \Lambda(t_0, x_0, z) < -\eta < 0$. As Λ is upper semi continuous, so is the function

$$(t, x) \mapsto \max_{z\in Z} \Lambda(t, x, z).$$

Hence there is $\varepsilon > 0$ such that

$$\max_{z\in Z} \Lambda(t, x, z) < -\frac{\eta}{2}$$

for $0 \le t - t_0 \le \varepsilon$ and $||x - x_0|| \le \varepsilon$. As $\dot{u}_{t_0,x_0,f}$ is uniformly bounded for all $f \in S^1_Z$ and for all $t \in [t_0, 1]$ by using the estimate (4.3) we can take $\sigma > 0$ such that $||u_{t_0,x_0,f}(t) - u_{t_0,x_0,f}(t_0)|| \le \varepsilon$ for all $t \in [t_0, t_0 + \sigma]$ and for all $f \in S^1_Z$. Then by integrating

$$\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,f}(t), f(t))dt \le \int_{t_0}^{t_0+\sigma} [\max_{z\in Z} \Lambda(t, u_{t_0,x_0,f}(t), z)]dt < -\frac{\sigma\eta}{2}$$

for all $f \in S^1_Z$ and the result follows. □

For simplicity we deal first with a dynamic programming principle (DPP) for a value function V_J related to a bounded continuous function $J : [0, 1] \times E \times Z \to \mathbf{R}$ associated with

$$(SODE)\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = f(t),\ f \in S^1_Z,\ t \in [\tau, 1] \\ u(\tau) = x,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{cases}$$

The following result is of importance in the statement of viscosity.

Theorem 4.1 (of Dynamic Programming Principle). *Let (1.1.1) holds. Let $x \in E$, $0 \le \tau < \eta_1 < .. < \eta_{m-2} < 1$ and $\sigma > 0$ such that $\tau + \sigma < \eta_1$. Assume that $J : [0, 1] \times E \times E \to \mathbf{R}$ is bounded continuous such that $J(t, x, .)$ is convex on E for every $(t, x) \in [0, 1] \times E$. Let us consider the value function*

$$V_J(\tau, x) = \sup_{f\in S^1_Z} \{\int_\tau^1 J(t, u_{\tau,x,f}(t), f(t))dt\}, \quad (\tau, x) \in [0, \eta_1[\times E$$

where $u_{\tau,x,f}$ is the trajectory solution on $[\tau, 1]$ associated the control $f \in S^1_Z$ starting from x at time τ to

$$(SODE)\begin{cases} \ddot{u}_{\tau,x,f}(t) + \gamma \dot{u}_{\tau,x,f}(t) = f(t),\ t \in [\tau, 1] \\ u_{\tau,x,f}(\tau) = x,\ u_{\tau,x,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f}(\eta_i). \end{cases} \quad (4.4)$$

Then the following hold

$$V_J(\tau, x) = \sup_{f \in S^1_Z} \left\{ \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt + V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)) \right\}$$

with

$$V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)) = \sup_{g \in S^1_Z} \left\{ \int_{\tau+\sigma}^1 J(t, v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}(t), g(t))dt \right\}$$

where $v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}$ denotes the trajectory solution on $[\tau+\sigma, 1]$ associated with the control $g \in S^1_Z$ starting from $u_{\tau,x,f}(\tau+\sigma)$ at time $\tau+\sigma$ to[1]

$$(SODE)\begin{cases} \ddot{v}_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}(t) + \gamma \dot{v}_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}(t) = g(t), \\ \quad t \in [\tau+\sigma, 1] \\ v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}(\tau+\sigma) = u_{\tau,x,f}(\tau+\sigma), \\ v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}(1) = \sum_{i=1}^{m-2} \alpha_i v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}(\eta_i). \end{cases} \quad (4.5)$$

Proof. Let

$$W_J(\tau, x) := \sup_{f \in S^1_Z} \left\{ \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt + V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)) \right\}.$$

For any $f \in S^1_Z$, we have

$$\int_\tau^1 J(t, u_{\tau,x,f}(t), f(t))dt$$
$$= \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt + \int_{\tau+\sigma}^1 J(t, u_{\tau,x,f}(t), f(t))dt.$$

[1] It is necessary to write completely the expression of the trajectory $v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}$ that depends on $(f, g) \in S^1_Z \times S^1_Z$ in order to get the lower semi-continuous dependence with respect to $f \in S^1_Z$ of $V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma))$.

By the definition of $V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma))$ we have

$$V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)) \geq \int_{\tau+\sigma}^1 J(t, u_{\tau,x,f}(t), f(t))dt.$$

It follows that

$$\int_\tau^1 J(t, u_{\tau,x,f}(t), f(t))dt$$
$$\leq \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt + V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)).$$

By taking the supremum on S_Z^1 in this inequality we get

$$V_J(\tau, x) \leq \sup_{f\in S_Z^1} \{\int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt + V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma))\}$$
$$= W_J(\tau, x).$$

Let us prove the converse inequality.

Main Fact: $f \to V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma))$ is lower semicontinuous on S_Z^1 (endowed with the $\sigma(L_E^1, L_{E^*}^\infty)$-topology).

Let us focus on the expression of $V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma))$

$$V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)) = \sup_{g\in S_Z^1} \left\{\int_{\tau+\sigma}^1 J(t, v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}(t), g(t))dt\right\}$$

where $v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}$ denotes the trajectory solution on $[\tau+\sigma, 1]$ associated with the control $g \in S_Z^1$ starting from $u_{\tau,x,f}(\tau+\sigma)$ at time $\tau+\sigma$ to (SODE) (4.5). By the integral representation formulas (4.1) (4.2) given above we have

$$v_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma), g}(t) = e_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)}(t) + \int_{\tau+\sigma}^1 G_{\tau+\sigma}(t, s)g(s)ds$$

with

$$e_{\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)}(t)$$
$$= u_{\tau,x,f}(\tau+\sigma) + A_{\tau+\sigma}(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma(t-(\tau+\sigma)))u_{\tau,x,f}(\tau+\sigma).$$

It is already seen in the proof of Step 1 of Theorem 3.1 that $f \mapsto u_{\tau,x,f}$ from S_Z^1 into $C_E([\tau, 1])$ is continuous when S_Z^1 is endowed with the $\sigma(L_E^1, L_{E^*}^\infty)$

topology and $C_E([\tau, 1])$ is endowed with the norm of uniform convergence, namely, when f_n $\sigma(L^1_E, L^\infty_{E^*})$-converges to $f \in S^1_Z$, then u_{τ,x,f_n} converges uniformly to $u_{\tau,x,f}$, this entails that

$$e_{\tau+\sigma,u_{\tau,x,f_n}}(\tau+\sigma)(t) \to e_{\tau+\sigma,u_{\tau,x,f}}(\tau+\sigma)(t)$$

for every $t \in [\tau, 1]$. Further, when g_n $\sigma(L^1_E, L^\infty_{E^*})$-converges to $g \in S^1_Z$, by compactness of Z, and the boundedness property of $G_{\tau+\sigma}(t, s)$ in Lemma 2.1, it is not difficult to check that

$$\int_{\tau+\sigma}^1 G_{\tau+\sigma}(t,s)g_n(s)ds \to \int_{\tau+\sigma}^1 G_{\tau+\sigma}(t,s)g(s)ds$$

for every $t \in [\tau, 1]$. Therefore

$$v_{\tau+\sigma,u_{\tau,x,f_n}(\tau+\sigma),g_n}(t) \to v_{\tau+\sigma,u_{\tau,x,f}(\tau+\sigma),g}(t)$$

for every $t \in [\tau, 1]$. Hence in view of ([14], Theorem 8.1.6) we deduce that

$$(f, g) \mapsto \int_{\tau+\sigma}^1 J(t, v_{\tau+\sigma,u_{\tau,x,f}(\tau+\sigma),g}(t), g(t))dt$$

is lower semicontinuous on $S^1_Z \times S^1_Z$ using the above fact and the convexity assumption on the integrand $J(t, x, .)$. Consequently $f \to V_J(\tau + \sigma, u_{\tau,x,f}(\tau+\sigma))$ is lower semicontinuous on S^1_Z. Hence the mapping

$$f \mapsto \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt + V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma))$$

is lower semicontinuous on S^1_Z. Since S^1_Z is weakly compact in $L^1_E([0, 1])$, there is $f^1 \in S^1_Z$ such that

$$\begin{aligned} W_J(\tau, x) &= \sup_{f \in S^1_Z} \left\{ \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt + V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma)) \right\} \\ &= \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f^1}(t), f^1(t))dt + V_J(\tau+\sigma, u_{\tau,x,f^1}(\tau+\sigma)). \end{aligned}$$

Similarly there is $g^2 \in S^1_Z$ such that

$$\begin{aligned} V_J(\tau+\sigma, u_{\tau,x,f^1}(\tau+\sigma)) &= \sup_{g \in S^1_Z} \left\{ \int_{\tau+\sigma}^1 J(t, v_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g}(t), g(t))dt \right\} \\ &= \int_{\tau+\sigma}^1 J(t, v_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(t), g^2(t))dt \end{aligned}$$

where $v_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(t)$ denotes the trajectory solution on $[\tau+\sigma,1]$ associated with the control $g^2 \in S^1_Z$ starting from $u_{\tau,x,f^1}(\tau+\sigma)$ at time $\tau+\sigma$ to

$$(SODE)\begin{cases} \ddot{v}_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(t) + \gamma\dot{v}_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(t) = g^2(t), \\ \quad t \in [\tau+\sigma,1] \\ v_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(\tau+\sigma) = u_{\tau,x,f^1}(\tau+\sigma), \\ v_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(1) = \sum_{i=1}^{m-2} \alpha_i v_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(\eta_i). \end{cases}$$

Let us set

$$\overline{f} := 1_{[\tau,\tau+\sigma]} f^1 + 1_{[\tau+\sigma,1]} f^2.$$

Then $\overline{f} \in S^1_Z$ (because S^1_Z is decomposable). Let $w_{\tau,x,\overline{f}}$ be the trajectory solution on $[\tau,1]$ associated with $\overline{f} \in S^1_Z$, that is

$$\ddot{w}_{\tau,x,\overline{f}}(t) + \gamma\ddot{w}_{\tau,x,\overline{f}}(t) = \overline{f}(t),\ t \in [\tau,1],$$

$$w_{\tau,x,\overline{f}}(\tau) = x,\ w_{\tau,x,\overline{f}}(1) = \sum_{i=1}^{m-2} \alpha_i w_{\tau,x,\overline{f}}(\eta_i).$$

By uniqueness of solution we have

$$w_{\tau,x,\overline{f}}(t) = u_{\tau,x,f^1}(t),\ \forall t \in [\tau,\tau+\sigma],$$
$$w_{\tau,x,\overline{f}}(t) = v_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(t),\ \forall t \in [\tau+\sigma,1].$$

Coming back to the expression of V_J and W_J we have

$$\begin{aligned} &W_J(\tau,x) \\ &= \int_\tau^{\tau+\sigma} J(t,u_{\tau,x,f^1}(t), f^1(t))dt + \int_{\tau+\sigma}^1 J(t, v_{\tau+\sigma,u_{\tau,x,f^1}(\tau+\sigma),g^2}(t), g^2(t))dt \\ &= \int_\tau^1 J(t, w_{\tau,x,\overline{f}}(t), \overline{f}(t))dt \\ &\le \sup_{f\in S^1_Z} \{\int_\tau^1 J(t,u_{\tau,x,f}(t), f(t))dt\} = V_J(\tau,x). \end{aligned}$$

□

Here are our results on viscosity of solutions for the value function.

Theorem 4.2 (of Viscosity Subsolution). *Assume that E is a separable Hilbert space. Assume (1.1.1) and $J : [0,1] \times E \times E \to \mathbf{R}$ is bounded* *continuous such that $J(t, x, .)$ is convex on E for every $(t, x) \in [0, 1] \times E$. Let us consider the value function*

$$V_J(\tau, x) = \sup_{f \in S_Z^1} \left\{ \int_\tau^1 J(t, u_{\tau,x,f}(t), f(t))dt \right\}, \ (\tau, x) \in [0, \eta_1[\times E$$

where $u_{\tau,x,f}$ is the trajectory solution on $[\tau, 1]$ associated the control $f \in S_Z^1$ starting from $x \in E$ at time τ to

$$(SODE) \begin{cases} \ddot{u}_{\tau,x,f}(t) + \gamma \dot{u}_{\tau,x,f}(t) = f(t), \ t \in [\tau, 1] \\ u_{\tau,x,f}(\tau) = x, u_{\tau,x,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f}(\eta_i). \end{cases}$$

Then V_J satisfies a viscosity property: For any $\varphi \in C^1([0, 1] \times E)$ such that V_J reaches a local maximum at $(t_0, x_0) \in [0, \eta_1[\times E$, then

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \max_{z \in Z}\{J(t_0, x_0, z)\} + \delta^*(\nabla\varphi(t_0, x_0), \dot{e}_{t_0,x_0}(t_0) + \int_{t_0}^1 \frac{\partial G_{t_0}}{\partial t}(t_0, s)Zds) \geq 0.$$

Proof. Assume by contradiction that there exist a $\varphi \in C^1([0, 1] \times E)$ such that V_J reaches a local maximum at $(t_0, x_0) \in [0, \eta_1[\times E$ for which

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \max_{z \in Z}\{J(t_0, x_0, z)\} + \delta^*(\nabla\varphi(t_0, x_0), \dot{e}_{t_0,x_0}(t_0) + \int_{t_0}^1 \frac{\partial G_{t_0}}{\partial t}(t_0, s)Zds) \\ \leq -\eta < 0$$

for some $\eta > 0$. Applying Lemma 4.1, by taking

$$\Lambda(t, x, z) = J(t, x, z) + \delta^*(\nabla\varphi(t, x), \dot{e}_{t_0,x_0}(t) + \int_{t_0}^1 \frac{\partial G_{t_0}}{\partial t}(t, s)Zds) + \frac{\partial \varphi}{\partial t}(t, x)$$

yields $\sigma > 0$ such that

$$\sup_{f \in S_Z^1} \left\{ \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f}(t), f(t))dt \right. \\ + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,f}(t)), \dot{e}_{t_0,x_0}(t) + \int_{t_0}^1 \frac{\partial G_{t_0}}{\partial t}(t, s)Zds)dt \\ \left. + \int_{t_0}^{t_0+\sigma} \frac{\partial \varphi}{\partial t}(t, u_{t_0,x_0,f}(t))dt \right\} \\ < -\frac{\sigma\eta}{2} \tag{4.2.1}$$

where $u_{t_0,x_0,f}$ is the trajectory solution associated with the control $f \in S^1_Z$ starting from x_0 at time t_0 to

$$(SODE)\begin{cases} \ddot{u}_{t_0,x_0,f}(t) + \gamma \dot{u}_{t_0,x_0,f}(t) = f(t),\ t \in [t_0, 1] \\ u_{t_0,x_0,f}(t_0) = x_0,\ u_{t_0,x_0,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{t_0,x_0,f}(\eta_i). \end{cases}$$

Applying the dynamic programming principle (Theorem 4.1) gives

$$V_J(t_0, x_0) = \sup_{f \in S^1_Z} \left\{ \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f}(t), f(t))dt + V_J(t_0+\sigma, u_{t_0,x_0,f}(t_0+\sigma)) \right\}. \tag{4.2.2}$$

Since $V_J - \varphi$ has a local maximum at (t_0, x_0), for small enough σ

$$V_J(t_0, x_0) - \varphi(t_0, x_0) \geq V_J(t_0+\sigma, u_{t_0,x_0,f}(t_0+\sigma)) \\ -\varphi(t_0+\sigma, u_{t_0,x_0,f}(t_0+\sigma)) \tag{4.2.3}$$

for all $f \in S^1_Z$. By (4.2.2), for each $n \in \mathbf{N}$, there is $f^n \in S^1_Z$ such that

$$V_J(t_0, x_0) \leq \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))dt + V_J(t_0 \\ +\sigma, u_{t_0,x_0,f^n}(t_0+\sigma)) + \frac{1}{n}. \tag{4.2.4}$$

From (4.2.3) and (4.2.4) we deduce that

$$V_J(t_0+\sigma, u_{t_0,x_0,f^n}(t_0+\sigma)) - \varphi(t_0+\sigma, u_{t_0,x_0,f^n}(t_0+\sigma)) \\ \leq \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))dt + \frac{1}{n} \\ -\varphi(t_0, x_0) + V_J(t_0+\sigma, u_{t_0,x_0,f^n}(t_0+\sigma)).$$

Therefore we have

$$0 \leq \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))dt \\ +\varphi(t_0+\sigma, u_{t_0,x_0,f^n}(t_0+\sigma)) - \varphi(t_0, x_0) + \frac{1}{n}. \tag{4.2.5}$$

As $\varphi \in C^1([0, 1] \times E)$

$$\varphi(t_0+\sigma, u_{t_0,x_0,f^n}(t_0+\sigma)) - \varphi(t_0, x_0) \\ = \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,f^n}(t)), \dot{u}_{t_0,x_0,f^n}(t)\rangle dt + \int_{t_0}^{t_0+\sigma} \frac{\partial \varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))dt. \tag{4.2.6}$$

Applying the integral representation formulas (4.1) and (4.2) gives

$$\begin{cases} u_{t_0,x_0,f^n}(t) = e_{t_0,x_0}(t) + \int_{t_0}^{1} G_{t_0}(t,s) f^n(s)ds, \ t \in [t_0,1] \\ \dot{u}_{t_0,x_0,f^n}(t) = \dot{e}_{t_0,x_0}(t) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t,s) f^n(s)ds, \ t \in [t_0,1] \end{cases}$$

with

$$\begin{cases} e_{t_0,x_0}(t) = x_0 + A_{t_0}(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma(t - t_0)))x_0, \ \forall t \in [t_0,1] \\ \dot{e}_{t_0,x_0}(t) = \gamma A_{t_0} \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma(t - t_0))x_0, \ \forall t \in [t_0,1] \\ A_{t_0} = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma(1 - t_0)) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - t_0))\right)^{-1} \end{cases}$$

where the coefficient A_{t_0} and the Green function G_{t_0} are defined in Lemma 2.1. Then from (4.2.6) we get the estimation

$$\begin{aligned} &\varphi(t_0 + \sigma, u_{t_0,x_0,f^n}(t_0 + \sigma)) - \varphi(t_0, x_0) \\ &\quad = \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,f^n}(t)), \dot{e}_{t_0,x_0}(t) + \int_{0}^{1} \frac{\partial G_{t_0}}{\partial t}(t,s) f^n(s)ds \rangle dt \\ &\qquad + \int_{t_0}^{t_0+\sigma} \frac{\partial \varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))dt. \end{aligned} \tag{4.2.7}$$

Since $f^n(s) \in Z$ for all $s \in [t_0, 1]$, it follows that

$$\frac{\partial G_{t_0}}{\partial t}(t,s) f^n(s) \in \frac{\partial G_{t_0}}{\partial t}(t,s) Z$$

for all $t, s \in [t_0, 1]$. From (4.2.7) and this inclusion we get

$$\begin{aligned} &\varphi(t_0 + \sigma, u_{t_0,x_0,f^n}(t_0 + \sigma)) - \varphi(t_0, x_0) \\ &\quad \leq \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,f^n}(t)), \dot{e}_{t_0,x_0}(t) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t,s) Z ds) dt \\ &\qquad + \int_{t_0}^{t_0+\sigma} \frac{\partial \varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))dt. \end{aligned} \tag{4.2.8}$$

Put the estimation (4.2.8) in (4.2.5) we get

$$
\begin{aligned}
0 \le & \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))dt \\
& + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,f^n}(t)), \dot{e}_{t_0,x_0}(t) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t,s)Zds)dt \\
& + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))dt + \frac{1}{n}. \qquad (4.2.9)
\end{aligned}
$$

By combining (4.2.1) and (4.2.9) we get the estimation

$$
\begin{aligned}
0 \le & \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))dt \\
& + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,f^n}(t)), \dot{e}_{t_0,x_0}(t) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t,s)Zds)dt \\
& + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))dt + \frac{1}{n} < -\frac{\sigma\eta}{2} + \frac{1}{n}. \qquad (4.2.10)
\end{aligned}
$$

Therefore we have that $0 < \frac{\sigma\eta}{2} < \frac{1}{n}$ for every $n \in \mathbf{N}$. Passing to the limit when n goes to ∞ in the preceding inequality yields a contradiction. □

5. Optimal Control Problem in Pettis Integration

We provide in this section some results in optimal control problems governed by an (SODE) with m-point boundary condition where the controls are Pettis-integrable. Here E is a separable Banach space. We recall and summarize some needed results on the Pettis integrability. Let $f : [0, 1] \to E$ be a scalarly integrable function, that is, for every $x^* \in E^*$, the scalar function $t \mapsto \langle x^*, f(t)\rangle$ is Lebesgue-integrable on $[0, 1]$. A scalarly integrable function $f : [0, 1] \to E$ is Pettis-integrable if, for every Lebesgue-measurable set A in $[0, 1]$, the weak integral $\int_A f(t)dt$ defined by $\langle x^*, \int_A f(t)dt\rangle = \int_A \langle x^*, f(t)\rangle\, dt$ for all $x^* \in E^*$ belongs to E. We denote by $P^1_E([0, 1], dt)$ the space of all Pettis-integrable functions $f : [0, 1] \to E$ endowed with the Pettis norm $||f||_{Pe} = \sup_{x^* \in \overline{B}_{E^*}} \int_0^1 |\langle x^*, f(t)\rangle| dt$. A mapping $f : [0, 1] \to E$ is Pettis-integrable iff the set $\{\langle x^*, f\rangle : ||x^*|| \le 1\}$ is uniformly integrable in the space $L^1_{\mathbf{R}}([0, 1], dt)$. More generally a convex compact valued mapping $\Gamma : [0, 1] \rightrightarrows E$ is scalarly integrable, if, for every $x^* \in E^*$, the scalar function $t \mapsto \delta^*(x^*, \Gamma(t))$ is Lebesgue-integrable on $[0, 1]$, Γ is Pettis-integrable if the set $\{\delta^*(x^*, \Gamma(.)) : ||x^*|| \le 1\}$ is uniformly integrable in the space $L^1_{\mathbf{R}}([0, 1], dt)$. In view of [[6], Theorem 4.2; or [14], Cor. 6.3.3] the set S_{Γ}^{Pe} of all Pettis-integrable selections of a convex compact valued Pettis-integrable mapping $\Gamma : [0, 1] \rightrightarrows E$ is sequentially $\sigma(P_E^1, L^\infty \otimes E^*)$-compact. We refer to [19], for related results on the integration of Pettis-integrable multifunctions.

We provide some useful lemmas.

Lemma 5.1. *Let* $G : [0, 1] \times [0, 1] \to \mathbf{R}$ *be a mapping with the following properties*

(i) for each $t \in [0, 1]$*,* $G(t, .)$ *is Lebesgue-measurable on* $[0, 1]$*,*
(ii) for each $s \in [0, 1]$*,* $G(., s)$ *is continuous on* $[0, 1]$*,*
(iii) there is a constant $M > 0$ *such that* $|G(t, s)| \leq M$ *for all* $(t, s) \in [0, 1] \times [0, 1]$.

Let $f : [0, 1] \to E$ *be a Pettis-integrable mapping. Then the mapping*

$$u_f : t \mapsto \int_0^1 G(t, s) f(s) ds$$

is continuous from $[0, 1]$ *into* E*, that is,* $u_f \in C_E([0, 1])$.

Proof. Let (t_n) be a sequence in $[0, 1]$ such that $t_n \to t \in [0, 1]$. Then we have the estimation

$$\sup_{x^* \in \overline{B}_{E^*}} |\langle x^*, \int_0^1 G(t_n, s) f(s) ds - \int_0^1 G(t, s) f(s) ds \rangle|$$
$$\leq \sup_{x^* \in \overline{B}_{E^*}} \int_0^1 |G(t_n, s) - G(t, s)| |\langle x^*, f(s) \rangle| ds.$$

As the sequence $(|G(t_n, .) - G(t, .)|)$ is bounded in $L^\infty_{\mathbf{R}}([0, 1])$ and pointwise converges to 0, it converges to 0 uniformly on uniformly integrable subsets of $L^1_{\mathbf{R}}([0, 1])$ in view of a lemma due to Grothendieck's [24], in others terms it converges to 0 with respect to the Mackey topology $\tau(L^\infty, L^1)$, see also [5] for a more general result concerning the Mackey topology for bounded sequences in L^∞_{E*}. Since the set $\{|\langle x^*, f(s)\rangle| : ||x^*|| \leq 1\}$ is uniformly integrable in $L^1_{\mathbf{R}}([0, 1])$, the second term in the above estimation goes to 0 when $t_n \to t$ showing that u_f is continuous on $[0, 1]$ with respect to the norm topology of E. □

The following is a generalization of Lemma 5.1.

Lemma 5.2. *Let* $G : [0,1] \times [0,1] \to \mathbf{R}$ *be a mapping with the following properties*

(i) for each $t \in [0,1]$, $G(t,.)$ *is Lebesgue-measurable on* $[0,1]$,
(ii) for each $s \in [0,1]$, $G(.,s)$ *is continuous on* $[0,1]$,
(iii) there is a constant $M > 0$ *such that* $|G(t,s)| \leq M$ *for all* $(t,s) \in [0,1] \times [0,1]$.

Let $\Gamma : [0,1] \to E$ *be a convex compact valued measurable and Pettis-integrable mapping. Then the set*

$$\{u_f : u_f(t) = \int_0^1 G(t,s)f(s)ds : t \in [0,1],\ f \in S_\Gamma^{Pe}\}$$

is equicontinuous in $C_E([0,1])$.

Proof. By Lemma 5.1 it is clear that

$$\{u_f : u_f(t) = \int_0^1 G(t,s)f(s)ds : t \in [0,1],\ f \in S_\Gamma^{Pe}\} \subset C_E([0,1]).$$

Let us check the equicontinuity property. Indeed, let $t, t_k \in [\tau, 1]$ such that $t_k \to t$, we have the estimation

$$||u_f(t) - u_f(t_k)|| \leq \sup_{x^* \in \overline{B}_{E^*}} \int_0^1 |G(t_k, s) - G(t,s)||\delta^*(x^*, \Gamma(s))|ds.$$

As the sequence $(|G(t_k,.) - G(t,.)|)$ is bounded in $L_{\mathbf{R}}^\infty([0,1])$ and the set $\{|\delta^*(x^*, \Gamma(.))| : ||x^*|| \leq 1\}$ is uniformly integrable in $L_{\mathbf{R}}^1([0,1])$, by invoking again Grothendieck lemma [24] as in the proof of Lemma 5.1, the second term goes to 0 when $t_k \to t$ showing that $\{u_f : f \in S_\Gamma^{Pe}\}$ is equicontinuous in $C_E([0,1])$. □

The following lemma is crucial in the statement of the (SODE) with Pettis-integrable second member and m-point boundary condition. Here we suppose that the hypotheses and notations of Lemma 2.1 hold.

Lemma 5.3. *Let* $x \in E$, *let* G_τ *be the Green function,* $e_{\tau,x}$ *and* $\dot{e}_{\tau,x}$ *in Lemma 2.1*

$$\begin{cases} e_{\tau,x}(t) = x + A_\tau(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma(t-\tau)))x,\ \forall t \in [\tau, 1] \\ \dot{e}_{\tau,x}(t) = \gamma A_\tau \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma(t-\tau))x,\ \forall t \in [\tau, 1] \\ A_\tau = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma(1-\tau)) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i - \tau))\right)^{-1} \end{cases}$$

and let f be a Pettis-integrable function. Let us consider the mapping

$$u_{\tau,x,f}(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s)f(s)ds, \quad \tau \in [0,\eta_1[, \quad t \in [0,1].$$

Then the following assertions hold

(1) $u_{\tau,x,f}$ is continuous i.e. $u_{\tau,x,f} \in C_E([0,1])$,
(2) $u_{\tau,x,f}(\tau) = x, \quad u_{\tau,x,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f}(\eta_i)$,
(3) The function $u_{\tau,x,f}$ is scalarly derivable, that is, for every $x^* \in E^*$, the scalar function $\langle x^*, u_{\tau,x,f}\rangle$ is derivable and its weak derivative $\dot{u}_{\tau,x,f}$ satisfies

$$\dot{u}_{\tau,x,f}(t) = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)f(s)ds, \quad \tau \in [0,\eta_1[, \quad t \in [\tau,1].$$

(4) The function $\dot{u}_{\tau,x,f}$ is continuous and scalarly derivable, that is, for every $x^* \in E^*$, the scalar function $\langle x^*, \dot{u}_{\tau,x,f}\rangle$ is derivable and its weak derivative $\ddot{u}_{\tau,x,f}$ satisfies

$$\ddot{u}_{\tau,x,f}(t) + \gamma u_{\tau,x,f}(t) = f(t) \quad a.e. \quad t \in [\tau,1].$$

Proof. (1) Since $e_{\tau,x} \in C_E([0,1])$ and G_τ is a Carathéodory and bounded function, $u_{\tau,x,f}$ is continuous on $[\tau,1]$ with respect to the norm topology of E in view of Lemma 5.1.
(2) follows from Lemma 2.1(iv).
(3)–(4) Similarly, using the property of $\frac{\partial G_\tau}{\partial t}$ in Lemma 2.1 we infer that $t \mapsto \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)f(s)ds$ is continuous on $[\tau,1]$ with respect to the norm topology of E in view of Lemma 5.1 and so is the mapping $t \mapsto \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)f(s)ds$. Now (3)–(4) follow from the computation used in (iv)–(v) in Lemma 2.1. □

By $W^{2,1}_{P,E}([\tau,1])$ we denote the space of all continuous functions in $C_E([\tau,1])$ such that their first weak derivatives are continuous and their second weak derivatives are Pettis-integrable on $[\tau,1]$. By Lemma 5.3, given a Pettis-integrable function $f : [\tau,1] \to E$ (shortly $f \in P^1_E([\tau,1])$ the (SODE)

$$\begin{cases} \ddot{u}_{\tau,x,f}(t) + \gamma \dot{u}_{\tau,x,f}(t) = f(t), t \in [\tau,1], \ \tau \in [0,\eta_1[\\ u_{\tau,x,f}(\tau) = x, \quad u_{\tau,x,f}(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f}(\eta_i) \end{cases}$$

admits a unique $W^{2,1}_{P,E}([\tau, 1])$-solution with integral representation formulas

$$u_{\tau,x,f}(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s) f(s) ds, \quad \tau \in [0, \eta_1[, \quad t \in [\tau, 1],$$

$$\dot{u}_{\tau,x,f}(t) = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s) f(s) ds, \quad \tau \in [0, \eta_1[, \quad t \in [\tau, 1].$$

The following result provides the compactness of solutions for a class of (SODE) with m ($m > 3$) point boundary condition and Pettis-integrable controls.

Theorem 5.1. *Let E be a separable Banach space and let $\Gamma : [0, 1] \to ck(E)$ be a convex compact valued measurable and Pettis-integrable mapping. Let us consider the following*

$$(SODE)_\Gamma \begin{cases} \ddot{u}_{\tau,x,f}(t) + \gamma \dot{u}_{\tau,x,f}(t) = f(t), t \in [\tau, 1], \ \tau \in [0, \eta_1[, \quad f \in S^{Pe}_\Gamma \\ u_{\tau,x,f}(\tau) = x, \quad u_{\tau,x,f}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f}(\eta_i). \end{cases}$$

Then the set $\{u_{\tau,x,f} : f \in S^{Pe}_\Gamma\}$ of $W^{2,1}_{P,E}([\tau, 1])$-solutions to $(SODE)_\Gamma$ is compact in $C_E([\tau, 1])$.

Proof. Let (u_{τ,x,f_n}) be a sequence of $W^{2,1}_{P,E}([\tau, 1])$-solutions to $(SODE)_\Gamma$. As S^{Pe}_Γ is sequentially $\sigma(P^1_E, L^\infty \otimes E^*)$-compact, by extracting a subsequence we may assume that (f_n) converges with respect to the $\sigma(P^1_E, L^\infty \otimes E^*)$ topology to $f_\infty \in S^{Pe}_\Gamma$. Using Lemma 5.3, we have, for each $n \in \mathbf{N}$,

$$u_{\tau,x,f_n}(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s) f_n(s) ds, \ \ t \in [\tau, 1] \tag{5.1.1}$$

$$\dot{u}_{\tau,x,f_n}(t) = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s) f_n(s) ds, \ \ t \in [\tau, 1] \tag{5.1.2}$$

$$\ddot{u}_{\tau,x,f_n}(t) + \gamma \dot{u}_{\tau,x,f_n}(t) = f_n(t) \in \Gamma(t), a.e. \ t \in [\tau, 1]. \tag{5.1.3}$$

From the property the Green function G_τ in Lemma 2.1, (5.1.1) and Lemma 5.2, we infer that $\{u_{\tau,x,f_n} : n \in \mathbf{N}\}$ is equicontinuous in $C_E([0, 1])$. Further, for each $t \in [\tau, 1]$, $\{u_{\tau,x,f_n}(t) : n \in \mathbf{N}\}$ is relatively compact because it is included in the norm compact set $e_{\tau,x}(t) + \int_0^1 G_\tau(t,s)\Gamma(s) ds$ (see e.g. [12, 14]). So by Ascoli's theorem, $\{u_{\tau,x,f_n} : n \in \mathbf{N}\}$ is relatively compact in $C_E([\tau, 1])$. Similarly using the properties of $\frac{\partial G_\tau}{\partial t}$ in Lemma 2.1, (5.1.2) and Lemma 5.2, we deduce that $\left\{\dot{u}_{\tau,x,f_n} : n \in \mathbf{N}\right\}$ is equicontinuous in $C_E([\tau, 1])$. In addition, the set $\left\{\dot{u}_{\tau,x,f_n}(t) : n \in \mathbf{N}\right\}$ is included in the compact set $\dot{e}_{\tau,x}(t) + \int_0^1 \frac{\partial G_\tau}{\partial t}(t,s)\,\Gamma(s)\,ds$. So $\left\{\dot{u}_{\tau,x,f_n} : n \in \mathbf{N}\right\}$ is relatively compact in $C_E([\tau, 1])$ using the Ascoli's theorem. From the above facts, we deduce that there exists a subsequence of $\left(u_{\tau,x,f_n}\right)_{n\in\mathbf{N}}$ still denoted by $\left(u_{\tau,x,f_n}\right)_{n\in\mathbf{N}}$ which converges uniformly to $u^\infty \in C_E([\tau, 1])$ with $u^\infty(0) = x$ and $u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i)$. Similarly, we may assume that $\left(\dot{u}_{\tau,x,f_n}\right)$ converges uniformly to $v^\infty \in C_E([\tau, 1])$. Furthermore, by the above facts, it is easy to see that $\left(\ddot{u}_{\tau,x,f_n}\right)$ converges $\sigma(P_E^1, L^\infty \otimes E^*)$ to a Pettis integrable function $w^\infty \in P_E^1([\tau, 1])$. For every $t \in [\tau, 1]$, using the representation formula (5.1.1), we have

$$
\begin{aligned}
&u^\infty(t)\\
&= \lim_{n\to\infty} u_{\tau,x,f_n}(t) = e_{\tau,x}(t) + \lim_{n\to\infty} \int_\tau^1 G_\tau(t,s)(\ddot{u}_{\tau,x,f_n}(s) + \gamma \dot{u}_{\tau,x,f_n}(s))ds\\
&= e_{\tau,x}(t) + \lim_{n\to\infty} \int_0^1 G_\tau(t,s)\ddot{u}_{\tau,x,f_n}(s)ds + \gamma \lim_{n\to\infty} \int_\tau^1 G_\tau(t,s)\dot{u}_{\tau,x,f_n}(s)ds\\
&= e_{\tau,x}(t) + \int_0^1 G_\tau(t,s)w^\infty(s)ds + \gamma \int_0^1 G_0(t,s)v^\infty(s)ds\\
&= e_{\tau,x}(t) + \int_0^1 G_\tau(t,s)(w^\infty(s) + \gamma v^\infty(s))ds. \qquad (5.1.4)
\end{aligned}
$$

From (5.1.4) and Lemma 5.3, we deduce that u^∞ is scalarly derivable and its weak derivative $\dot{u}^\infty$ is given by

$$
\dot{u}^\infty(t) = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)(w^\infty(s) + \gamma v^\infty(s))ds, \forall t \in [\tau, 1]. \quad (5.1.5)
$$

Now using the integral representation formula (5.1.2) we have, for every $t \in [\tau, 1]$,

$$
\begin{aligned}
v^\infty(t) &= \lim_{n\to\infty} \dot{u}_{\tau,x,f_n}(t)\\
&= \dot{e}_{\tau,x}(t) + \lim_{n\to\infty} \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)(\ddot{u}_{\tau,x,f_n}(s) + \gamma \dot{u}_{\tau,x,f_n}(s))ds\\
&= \dot{e}_{\tau,x}(t) + \lim_{n\to\infty} \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)\ddot{u}_{f_n}(s)ds\\
&\quad + \gamma \lim_{n\to\infty} \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)\dot{u}_{f_n}(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)w^\infty(s)ds + \gamma \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)v^\infty(s)ds \\
&= \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s)(w^\infty(s) + \gamma v^\infty(s))ds
\end{aligned}
\tag{5.1.6}
$$

so that by (5.1.5) and (5.1.6) we get $v^\infty = \dot{u}^\infty$. Now using (5.1.4) and invoking Lemma 5.3(4) we get

$$
\ddot{u}^\infty(t) + \gamma \dot{u}^\infty(t) = w^\infty(t) + \gamma v^\infty(t) = w^\infty(t) + \gamma \dot{u}^\infty(t) \quad a.e. \quad t \in [\tau, 1].
$$

Thus we get $\ddot{u}^\infty(t) = w^\infty(t)\, a.e.\, t \in [\tau, 1]$ so that

$$
\begin{cases}
u^\infty(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s)(\ddot{u}^\infty(s) + \gamma \dot{u}^\infty(s))ds, \quad t \in [\tau, 1] \\
u^\infty(\tau) = x, \quad u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i).
\end{cases}
\tag{5.1.7}
$$

Step 2. Main fact: u^∞ coincides with the $W^{2,1}_{P,E}([\tau, 1])$-solution u_{f_∞} associated with $f_\infty \in S^{Pe}_\Gamma$ to

$$
\begin{cases}
\ddot{u}_{f_\infty}(t) + \gamma \dot{u}_{f_\infty}(t) = f_\infty(t), \; t \in \tau, 1] \\
u_{f_\infty}(\tau) = x, \quad u_{f_\infty}(1) = \sum_{i=1}^{m-2} \alpha_i u_{f_\infty}(\eta_i).
\end{cases}
\tag{5.1.8}
$$

Remember that

$$
\begin{cases}
\ddot{u}_{\tau,x,f_n}(t) + \gamma \dot{u}_{f_n}(t) = f_n(t), \\
u_{\tau,x,f_n}(\tau) = x, \quad u_{\tau,x,f_n}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f_n}(\eta_i)
\end{cases}
$$

and by the above fact, $(\ddot{u}_{\tau,x,f_n} + \gamma \dot{u}_{\tau,x,f_n})$ $\sigma(P^1_E, L^\infty \otimes E^*)$-converges in $P^1_E([\tau, 1])$ to $\ddot{u}^\infty + \gamma \dot{u}^\infty$. Let $v = h \otimes x^* \in L^\infty([\tau, 1]) \otimes E^*$. Multiply scalarly the equation

$$
\ddot{u}_{\tau,x,f_n}(t) + \gamma \dot{u}_{\tau,x,f_n}(t) = f_n(t)
$$

by $v(t)$ and integrating on $[\tau, 1]$ yields

$$
\int_\tau^1 \langle h(t) \otimes x^*, \ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) \rangle dt = \int_\tau^1 \langle h(t) \otimes x^*, f_n(t) \rangle dt. \tag{5.1.9}
$$

It is clear that

$$\lim_{n\to\infty}\int_\tau^1 \langle h(t)\otimes x^*, \ddot{u}_{f_n}(t)+\gamma\dot{u}_{f_n}(t)\rangle dt = \int_\tau^1 \langle h(t)\otimes x^*, \ddot{u}^\infty(t)+\gamma\dot{u}^\infty(t)\rangle dt$$

$$= \lim_{n\to\infty}\int_\tau^1 \langle h(t)\otimes x^*, f_n(t)\rangle dt = \int_\tau^1 \langle h(t)\otimes x^*, f_\infty(t)\rangle dt$$

so that by invoking the separability of E

$$\ddot{u}^\infty(t) + \gamma\dot{u}^\infty(t) = f_\infty(t) \quad a.e. \quad t \in [\tau, 1]. \tag{5.1.10}$$

Using (5.1.7), (5.1.8), and (5.1.10) and uniqueness of solution we obtain $u^\infty = u_{f_\infty}$. □

Remark. In the context of Theory of Control, we have stated in the proof of Theorem 5.1, the dependence of the trajectory solution with respect to the Pettis controls. Namely, with the notations of Theorem 5.1, if u_{τ,x,f_n} is the $W^{2,1}_{P,E}([\tau, 1])$-solution of

$$\begin{cases} \ddot{u}_{\tau,x,f_n}(t) + \gamma\dot{u}_{f_n}(t) = f_n(t), & t \in [\tau, 1] \\ u_{\tau,x,f_n}(\tau) = x, \quad u_{\tau,x,f_n}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f_n}(\eta_i) \end{cases}$$

and if (f_n) $\sigma(P^1_E, L^\infty \otimes E^*)$-converges to $f_\infty \in S^{Pe}_\Gamma$, then (u_{τ,x,f_n}) converges uniformly to u_{τ,x,f_∞}, $(\dot{u}_{\tau,x,f_n})$ converges uniformly to $\dot{u}_{\tau,x,f_\infty}$ and $(\ddot{u}_{\tau,x,f_n})$ $\sigma(P^1_E, L^\infty \otimes E^*)$-converges to $\ddot{u}_{\tau,x,f_\infty}$ where u_{τ,x,f_∞} is the $W^{2,1}_{P,E}([\tau, 1])$-solution of

$$\begin{cases} \ddot{u}_{\tau,x,f_\infty}(t) + \gamma\dot{u}_{\tau,x,f_\infty}(t) = f_\infty(t), & t \in [\tau, 1] \\ u_{\tau,x,f_\infty}(\tau) = x, \quad u_{\tau,x,f_\infty}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,f_\infty}(\eta_i). \end{cases}$$

The above remark is of importance since it allows to prove further results. Here is an application to the existence of $W^{2,1}_{P,E}([\tau, 1])$-solution of a (SODE) with m-point boundary condition.

Theorem 5.2. *Let $F : [0, 1] \times (E \times E) \to E$ be a Carathéodory mapping satisfying*

$$F(t, x, y) \in \Gamma(t)$$

for all $(t, x, y) \in [0, 1] \times E \times E$ *where* $\Gamma : [0, 1] \rightrightarrows E$ *is a convex compact valued Pettis-integrable mapping. Then the (SODE)*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = F(t, u(t), \dot{u}(t)), \quad t \in [\tau, 1] \\ u(\tau) = x, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{cases}$$

has a $W^{2,1}_{P,E}([\tau, 1])$*-solution.*

Proof. Let us set

$$\mathcal{X} := \{u_{\tau,x,f} : [\tau, 1] \to E : u_{\tau,x,f}(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t, s) f(s) ds, \; t \in [\tau, 1], \; f \in S^{Pe}_\Gamma\}.$$

Then Theorem 5.1 shows that $\mathcal{X}$ is compact and convex in $C_E([\tau, 1])$. For each $u \in \mathcal{X}$, let us set

$$\Phi(u) := \{w \in \mathcal{X} : \ddot{w}(t) + \gamma \dot{w}(t) = F(t, u(t), \dot{u}(t)), \; t \in [\tau, 1]\}.$$

In view of Lemma 5.3, $\Phi(u)$ is non empty. Let us prove that the mapping $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous. Let $(u_n, v_n) \in$ Graph Φ such that $u_n \to u$ and $v_n \to v$ in $\mathcal{X}$. We need to check that $v = \Phi(u)$. Taking account of the particular structure of $\mathcal{X}$ and the remark of Theorem 5.1, we have that $\dot{u}_n \to \dot{u}$ uniformly and $\ddot{u}_n$ $\sigma(P^1_E, L^\infty \otimes E^*)$-converges to $\ddot{u}$ and that $\dot{v}_n \to \dot{v}$ uniformly and $\ddot{v}_n$ $\sigma(P^1_E, L^\infty \otimes E^*)$-converges to $\ddot{v}$. Multiply scalarly the equality

$$\ddot{v}_n(t) + \gamma \dot{v}_n(t) = F(t, u_n(t), \dot{u}_n(t)), \quad t \in [\tau, 1]$$

by $h(t) \otimes x^*$ where $h \in L^\infty_{\mathbf{R}^+}([\tau, 1])$ and $x^* \in \overline{B}_{E^*}$ and integrating on $[\tau, 1]$ gives

$$\int_\tau^1 \langle h(t) \otimes x^*, \ddot{v}_n(t) + \gamma \dot{v}_n(t) \rangle dt = \int_\tau^1 \langle h(t) \otimes x^*, F(t, u_n(t), \dot{u}_n(t)) \rangle dt. \tag{5.2.1}$$

By passing to the limit when $n \to \infty$ in (5.2.1) we get

$$\begin{aligned} \lim_{n\to\infty} \int_\tau^1 \langle h(t) \otimes x^*, \ddot{v}_n(t) + \gamma \dot{v}_n(t) \rangle dt &= \int_\tau^1 \langle h(t) \otimes x^*, \ddot{v}(t) + \gamma \dot{v}(t) \rangle dt \\ = \lim_{n\to\infty} \int_\tau^1 \langle h(t) \otimes x^*, F(t, u_n(t), \dot{u}_n(t)) \rangle dt &= \int_\tau^1 \langle h(t) \otimes x^*, F(t, u(t), \dot{u}(t)) \rangle dt \end{aligned} \tag{5.2.2}$$

by Lebesgue dominated convergence theorem, because

$$|\langle h(t) \otimes x^*, F(t,x,y)\rangle| \le h(t)|\delta^*(x^*, \Gamma(t))|$$

for all $(t,x,y) \in [0,1] \times E \times E$. By (5.2.2) we deduce that

$$\int_\tau^1 \langle h(t) \otimes x^*, \ddot{v}(t) + \gamma \dot{v}(t)\rangle dt = \int_\tau^1 \langle h(t) \otimes x^*, F(t, u(t), \dot{u}(t))\rangle dt.$$

Whence we get

$$\langle x^*, \ddot{v}(t) + \gamma \dot{v}(t)\rangle = \langle x^*, F(t, u(t), \dot{u}(t))\rangle, \quad a.e. \tag{5.2.4}$$

for every $x^* \in \overline{B}_{E^*}$. By taking a dense sequence $(e_k^*)_{k\in\mathbf{N}}$ in $\overline{B}_{E^*}$ for the Mackey topology we get

$$\langle e_k^*, \ddot{v}(t) + \gamma \dot{v}(t)\rangle = \langle e_k^*, F(t, u(t), \dot{u}(t))\rangle, \quad a.e. \tag{5.2.5}$$

for all $k \in \mathbf{N}$. Finally we get

$$\ddot{v}(t) + \gamma \dot{v}(t) = F(t, u(t), \dot{u}(t)), \quad a.e.$$

proving that Graph Φ is compact. By applying the Kakutani–Ky Fan fixed point theorem to Φ, we find $u \in \mathcal{X}$ such that $u = \Phi(u)$ which is a $W^{2,1}_{P,E}([\tau,1])$-solution of the (SODE) under consideration. □

The compactness in $C_E([\tau,1])$ of

$$\mathcal{X} := \{u_{\tau,x,f} : [\tau,1] \to E : u_{\tau,x,f}(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t,s) f(s)ds,\ t \in [\tau,1],\ f \in S_\Gamma^{Pe}\}$$

$$\mathcal{Y} := \{\dot{u}_{\tau,x,f} : [\tau,1] \to E : \dot{u}_{\tau,x,f}(t) = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t,s) f(s)ds,\ t \in [\tau,1],\ f \in S_\Gamma^{Pe}\}$$

are of importance and rely on some delicate arguments in the pioneering work of [1, 2] involving the Pettis uniformly integrable condition, Grothendieck lemma characterizing the Mackey topology for bounded sets in $L^\infty_{\mathbf{R}}$ [24] and other compactness results. Second order differential inclusions with three point boundary condition in case where the second member is a Pettis-integrable convex compact valued multifunction is initiated in [2]. At this point a second order differential inclusion with upper semicontinuous convex compact valued multifunction and three point boundary condition of the form

$$\begin{cases} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)) \ a.e \ t \in [0, 1], \\ u(0) = 0; \ \ u(\theta) = u(1). \end{cases}$$

is available in [2, 27]. Taking account into the above facts, one may state the

validity of Theorem 5.2 when F is a convex compact valued upper semicontinuous mapping. Since we don't focus on differential inclusion in the paper, we only mention a closure type lemma which may have an independent interest and solves this problem.

Theorem 5.3. *Let* $F : [0, 1] \times E \times E \to E$ *be a convex compact valued upper semicontinuous mapping satisfying*

$$F(t, x, y) \subset \Gamma(t)$$

for all $(t, x, y) \in [0, 1] \times E \times E$ *where* $\Gamma : [0, 1] \rightrightarrows E$ *is a convex compact valued Pettis-integrable mapping. Let* $(u_n, v_n) \in \mathcal{X} \times \mathcal{X}$ *such that* $u_n \to u$ *and* $v_n \to v$ *in* $\mathcal{X}$ *and that*

$$\ddot{v}_n(t) + \gamma v_n(t) \in F(t, u_n(t), \dot{u}_n(t))$$

for all $n \in \mathbf{N}$ *and for all* $t \in [\tau, 1]$. *Then we have* $\ddot{v}(t) + \gamma v(t) \in F(t, u(t), \dot{u}(t))$ *a.e.*

Proof. Let $h \otimes x^*$ where $h \in L^\infty_{\mathbf{R}^+}([\tau, 1])$ and $x^* \in \overline{B}_{E^*}$. From

$$\ddot{v}_n(t) + \gamma v_n(t) \in F(t, u_n(t), \dot{u}_n(t))$$

we have

$$\langle h(t) \otimes x^*, \ddot{v}_n(t) + \gamma v_n(t)\rangle \leq \delta^*(h(t) \otimes x^*, F(t, u_n(t), \dot{u}_n(t))).$$

Integrating on $[\tau, 1]$ this inequality yields

$$\int_\tau^1 \langle h(t) \otimes x^*, \ddot{v}_n(t) + \gamma v_n(t)\rangle dt \leq \int_\tau^1 \delta^*(h(t) \otimes x^*, \times F(t, u_n(t), \dot{u}_n(t)))dt. \quad (5.3.1)$$

Repeating the arguments of the proof of Theorem 5.2, we have that $\dot{u}_n \to \dot{u}$ uniformly and $\ddot{u}_n$ $\sigma(P^1_E, L^\infty \otimes E^*)$-converges to $\ddot{u}$ and that $\dot{v}_n \to \dot{v}$ uniformly and $\ddot{v}_n$ $\sigma(P^1_E, L^\infty \otimes E^*)$-converges to $\ddot{v}$. Then by passing to the limit when $n \to \infty$ in (5.3.1) we get

$$\begin{aligned}
\lim_{n\to\infty} \int_\tau^1 & \langle h(t)\otimes x^*, \ddot{v}_n(t) + \gamma \dot{v}_n(t)\rangle dt \\
&= \int_\tau^1 \langle h(t)\otimes x^*, \ddot{v}(t) + \gamma \dot{v}(t)\rangle dt \\
&\leq \limsup_{n\to\infty} \int_\tau^1 h(t)\delta^*(x^*, F(t, u_n(t), \dot{u}_n(t)))dt \\
&\leq \int_\tau^1 h(t) \limsup_{n\to\infty} \delta^*(x^*, F(t, u_n(t), \dot{u}_n(t)))dt \\
&\leq \int_\tau^1 h(t)\delta^*(x^*, F(t, u(t), \dot{u}(t)))dt \qquad (5.3.2)
\end{aligned}$$

because

$$|\delta^*(h(t)\otimes x^*, F(t,x,y))| \leq |\delta^*(h(t)\otimes x^*, \Gamma(t))| = h(t)|\delta^*(x^*, \Gamma(t))|$$

for all $(t,x,y) \in [0,1]\times E\times E$ and the mapping F is upper semicontinuous. By (5.3.2) we deduce that

$$\int_\tau^1 h(t)\langle x^*, \ddot{v}(t) + \gamma\dot{v}(t)\rangle dt \leq \int_\tau^1 h(t)\delta^*(x^*, F(t,u(t),\dot{u}(t)))dt.$$

Whence we get

$$\langle x^*, \ddot{v}(t) + \gamma\dot{v}(t)\rangle \leq \delta^*(x^*, F(t,u(t),\dot{u}(t))) \quad a.e.$$

for every $x^* \in \overline{B}_{E^*}$. By taking a dense sequence $(e^*_k)_{k\in\mathbf{N}}$ in $\overline{B}_{E^*}$ for the Mackey topology we get

$$\langle e^*_k, \ddot{v}(t) + \gamma\dot{v}(t)\rangle \leq \delta^*(e^*_k, F(t,u(t),\dot{u}(t))) \quad a.e.$$

for all $k \in \mathbf{N}$ so that

$$\ddot{v}(t)+\gamma\dot{v}(t) \in F(t,u(t),\dot{u}(t)) \quad a.e. \qquad \square$$

6. Open Problems: Differential Game Governed by (SODE), (ODE) and Sweeping Process with Strategies

To finish the paper we discuss some viscosity problems in a differential game governed by a class of (ODE) with strategy in the line of Elliot [20], Elliot–Kalton [21] and Evans–Souganides [22]. For simplicity we assume that E is a separable Hilbert space. Let us consider two compact subsets Y and Z in E. Set

$$\mathcal{Y}(\tau) = \{y : [\tau, 1] \to Y |\, y \text{ measurable}\}$$
$$\mathcal{Z}(\tau) = \{z : [\tau, 1] \to Z |\, z \text{ measurable}\}$$

Denote by $\Gamma(\tau)$ the set of all strategies $\alpha : \mathcal{Z}(\tau) \to \mathcal{Y}(\tau)$ and $\Delta(\tau)$ the set of all strategies $\beta : \mathcal{Y}(\tau) \to \mathcal{Z}(\tau)$. Let us given a Carathéodory integrable mapping $F : [0, 1] \times (Y \times Z) \to E$ such that $F(t, y, z) \subset K(t)$ for all $(t, y, z) \in [0, 1] \times Y \times Z$ where $K : [0, 1] \Rightarrow E$ is a convex compact valued integrably bounded mapping, a bounded continuous integrand $J : [0, 1] \times E \times Y \times Z \to \mathbf{R}$ and let us define the upper–lower value function

$$U_J(\tau, x) = \sup_{\alpha \in \Gamma(\tau)} \inf_{z \in \mathcal{Z}(\tau)} \{\int_\tau^1 J(t, u_{\tau,x,\alpha(z),z}(t), \alpha(z)(t), z(t))dt\}, \tau \in [0, \eta_1]$$

$$V_J(\tau, x) = \inf_{\beta \in \Delta(\tau)} \sup_{y \in \mathcal{Y}(\tau)} \{\int_\tau^1 J(t, u_{\tau,x,y,\beta(y)}(t), y(t), \beta(y)(t))dt\}, \tau \in [0, \eta_1]$$

where $u_{\tau,x,\alpha(z),z}$ is the trajectory $W_E^{2,1}([\tau, 1])$-solution of second order differential game

$$\begin{cases} \ddot{u}_{\tau,x,\alpha(z),z}(t) + \gamma \dot{u}_{\tau,x,\alpha(z),z}(t) = F(t, \alpha(z)(t), z(t)), t \in [\tau, 1],\ \tau \in [0, \eta_1] \\ u_{\tau,x,\alpha(z),z}(\tau) = x, \\ u_{\tau,x,\alpha(z),z}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,\alpha(z),z}(\eta_i), \end{cases} \tag{6.1.1}$$

with the integral representation formulas

$$u_{\tau,x,\alpha(z),z}(t)(t) = e_{\tau,x}(t) + \int_\tau^1 G_\tau(t, s)F(s, \alpha(z)(s), z(s))ds,\ \ t \in [\tau, 1]$$

$$\dot{u}_{\tau,x,\alpha(z),z}(t) = \dot{e}_{\tau,x}(t) + \int_\tau^1 \frac{\partial G_\tau}{\partial t}(t, s)F(s, \alpha(z)(s), z(s))ds,\ \ t \in [\tau, 1]$$

and similarly $u_{\tau,x,y,\beta(y)}$ is the trajectory $W_E^{2,1}([\tau, 1])$-solution of second order differential game

$$\begin{cases} \ddot{u}_{\tau,x,y,\beta(y)}(t) + \gamma \ddot{u}_{\tau,x,y,\beta(y)}(t) = F(t, y(t), \beta(y)(t)), t \in [\tau, 1], \tau \in [0, \eta_1] \\ u_{\tau,x,y,\beta(y)}(\tau) = x, \\ u_{\tau,x,y,\beta(y)}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,y,\beta(y)}(\eta_i). \end{cases} \tag{6.1.2}$$

We aim to generalize the viscosity problem in Theorem 4.2 to the case of strategies in the following

Proposition 6.1. *Let* $J : [0, 1] \times E \times Y \times Z \to \mathbf{R}$ *be a bounded continuous integrand,* $\tau, \sigma \in [0, 1]$ *such that* $\tau \in [0, \eta_1[$ *and* $\tau + \sigma < 1$ *and let us consider the upper value function*

$$U_J(\tau, x) = \sup_{\alpha \in \Gamma(\tau)} \inf_{z \in Z(\tau)} \left\{ \int_\tau^1 J(t, u_{\tau,x,\alpha(z),z}(t), \alpha(z)(t), z(t))dt \right\},$$
$$\tau \in [0, \eta_1], x \in E.$$

where $u_{\tau,x,\alpha(z),z}$ *is the trajectory* $W_E^{2,1}([\tau, 1])$*-solution of second order differential game*

$$\begin{cases} \ddot{u}_{\tau,x,\alpha(z),z}(t) + \gamma \dot{u}_{\tau,x,\alpha(z),z}(t) = F(t, \alpha(z)(t), z(t)), t \in [\tau, 1], \ \tau \in [0, \eta_1] \\ u_{\tau,x,\alpha(z),z}(\tau) = x, \\ u_{\tau,x,\alpha(z),z}(1) = \sum_{i=1}^{m-2} \alpha_i u_{\tau,x,\alpha(z),z}(\eta_i). \end{cases} \tag{6.1.1}$$

Then U_J *satisfies a sub-viscosity property: For any* $\varphi \in C^1([0, 1] \times E)$ *such that* $U_J - \varphi$ *reaches a local maximum at* $(t_0, x_0) \in [0, \eta_1[\times E$*, then*

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \min_{z \in Z} \max_{y \in Y} \{J(t_0, x_0, y, z)\} + \delta^*(\nabla \varphi(t_0, x_0), \dot{e}_{t_0,x_0}(t_0)$$
$$+ \int_{t_0}^1 \frac{\partial G_{t_0}}{\partial t}(t_0, s)K(s)ds) \geq 0$$

provides that U_J *satisfies the DPP*

$$U_J(\tau, x) = \sup_{\alpha \in \Gamma(\tau)} \inf_{z \in Z(\tau)} \left\{ \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,\alpha(z),z}(s), \alpha(z)(s), z(s))ds \right.$$
$$\left. + U_J(\tau + \sigma), u_{\tau,x,\alpha(z),z}(\tau + \sigma) \right\}.$$

Proof. Assume there is a $\varphi \in C^1([0, 1] \times E)$ such that $U_J - \varphi$ reaches a local maximum at $(t_0, x_0) \in [0, \eta_1[\times E$ for which

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \min_{z \in Z} \max_{y \in Y} \{J(t_0, x_0, y, z)\} + \delta^*(\nabla \varphi(t_0, x_0), \dot{e}_{t_0,x_0}(t_0)$$
$$+ \int_{t_0}^1 \frac{\partial G_{t_0}}{\partial t}(t_0, s)K(s)ds) < 0.$$

Hence there exists some $\eta > 0$ such that

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \min_{z \in Z} \max_{y \in Y}\{J(t_0, x_0, y, z)\} + \delta^*(\nabla\varphi(t_0, x_0), \dot{e}_{t_0,x_0}(t_0) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t_0, s)K(s)ds) \leq -\eta < 0.$$

Set

$$\Lambda(t, x, y, z) = \frac{\partial \varphi}{\partial t}(t, x) + J(t, x, y, z) + \delta^*(\nabla\varphi(t, x), \dot{e}_{t_0,x_0}(t) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t, s)K(s)ds).$$

Then we have

$$\min_{z \in Z} \max_{y \in Y} \Lambda(t_0, x_0, y, z) \leq -\eta < 0.$$

Hence there exists some $\overline{z} \in Z$ such that

$$\max_{y \in Y} \Lambda(t_0, x_0, y, \overline{z}) \leq -\eta < 0.$$

Since the mapping

$$(t, x) \mapsto \max_{y \in Y} \Lambda(t_0, x_0, y, \overline{z})$$

is continuous there is $\varepsilon > 0$ such that

$$\max_{y \in Y} \Lambda(t, x, y, \overline{z}) < -\frac{\eta}{2}$$

for $0 \leq t - t_0 \leq \varepsilon$ and $||x - x_0|| \leq \varepsilon$. As $\dot{u}_{t_0,x_0,\alpha(z),z}$ is estimated by

$$||\dot{u}_{t_0,x_0,\alpha(z),z}(t)|| \leq ||\dot{e}_{t_0,x_0}(t)|| + \int_{t_0}^{1} |\frac{\partial G_{t_0}}{\partial t}(t, s)||K(s)|ds = c(t)$$

with $c \in C_{\mathbf{R}}([t_0, 1])$ for all $z \in Z(t_0)$ and for all $\alpha \in \Gamma(t_0)$ in view of the above integral representation formula, so we can choose $\sigma > 0$ such that $||u_{t_0,x_0,\alpha(z),z}(t) - u_{t_0,x_0,\alpha(z),z}(t_0)|| \leq \int_{t_0}^{t_0+\sigma} c(t)dt \leq \varepsilon$ for all $t \in [t_0, t_0 + \sigma]$ and for all $z \in Z(t_0)$ and for all $\alpha \in \Delta(t_0)$. Then the constant control $\overline{z}(t) = \overline{z}, \forall t \in [t_0, 1]$ belongs to $Z(t_0)$ and $\alpha(\overline{z})$ belongs to $Y(t_0)$ for all $\alpha \in \Gamma(t_0)$ so that by integrating we have

$$\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,\alpha(\overline{z}),\overline{z}}(t), \alpha(\overline{z})(t), \overline{z}(t))dt < -\frac{\sigma\eta}{2}$$

for all $\alpha \in \Gamma(t_0)$. Thus

$$\sup_{\alpha\in\Gamma(t_0)} \left\{ \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,\alpha(\overline{z}),\overline{z}}(t), \alpha(\overline{z})(t), \overline{z}(t))dt \right. \tag{6.1.3}$$

$$+\delta^*(\nabla\varphi(t, u_{t_0,x_0,\alpha(\overline{z}),\overline{z}}(t)), \dot{e}_{t_0,x_0}(t) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t, s)K(s)ds)$$

$$\left. + \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha(\overline{z}),\overline{z}}(t) \right\} < -\frac{\sigma\eta}{2}.$$

As U_J satisfies the DPP property, we have

$$U_J(t_0, x_0) \leq \sup_{\alpha\in\Gamma(t_0)} \left\{ \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,\alpha(\overline{z}),\overline{z}}(t), \alpha(\overline{z})(t), \overline{z}(t))dt \right.$$

$$\left. + U_J(t_0 + \sigma, u_{t_0,x_0,\alpha(\overline{z}),\overline{z}}(t_0 + \sigma)) \right\}.$$

Hence, for every $n \in \mathbf{N}$, there exists $\alpha^n \in \Gamma(t_0)$ such that

$$U_J(t_0, x_0) \leq \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t), \alpha^n(\overline{z})(t), \overline{z}(t))dt$$

$$+ U_J(t_0 + \sigma, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t_0 + \sigma)) + \frac{1}{n}. \tag{6.1.4}$$

But $U_J - \varphi$ has a local maximum at (t_0, x_0), for small enough σ

$$U_J(t_0, x_0) - \varphi(t_0, x_0) \geq U_J(t_0 + \sigma, u_{t_0,x_0,\alpha(z),z}(t_0 + \sigma))$$

$$-\varphi(t_0 + \sigma, u_{t_0,x_0,\alpha(z),z}(t_0 + \sigma)) \tag{6.1.5}$$

for any trajectory solution $u_{t_0,x_0,\alpha(z),z}$ associated with control $(\alpha(z), z)$ ($\alpha \in \Gamma(t_0)$, $z \in Z$). From (6.1.4) and (6.1.5) we deduce

$$U_J(t_0 + \sigma, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t_0 + \sigma)) - \varphi(t_0 + \sigma, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t_0 + \sigma))$$

$$\leq \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t), \alpha^n(\overline{z})(t), \overline{z}(t))dt$$

$$+U_J(t_0 + \sigma, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t_0 + \sigma) + \frac{1}{n} - \varphi(t_0, x_0).$$

Thus we have

$$0 \le \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t), \alpha^n(\overline{z})(t), \overline{z}(t))dt + \varphi(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t_0+\sigma)) - \varphi(t_0, x_0) + \frac{1}{n}. \quad (6.1.6)$$

But

$$\varphi(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t_0+\sigma)) - \varphi(t_0, x_0) = \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t)), \dot{u}_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t)\rangle dt + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t))dt \quad (6.1.7)$$

and

$$\dot{u}_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t) = \dot{e}_{t_0,x_0}(t) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t,s)F(s, \alpha^n(\overline{z})(s), \overline{z}(s))ds$$

because $u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}$ is the $W^{2,1}([\tau, 1])$-solution to (SODE)

$$\ddot{u}_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t) + \gamma\dot{u}_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t) = F(t, \alpha^n(\overline{z})(t), \overline{z}(t)),$$

$$u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t_0) = x_0,$$

$$u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(1) = \sum_{i=1}^{m-2} \alpha_i u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(\eta_i).$$

From (6.1.6) and (6.1.7) we deduce

$$0 \le \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t), \alpha^n(\overline{z})(t), \overline{z}(t))dt + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,\alpha^n(\overline{z})(t),\overline{z}}(t)), \dot{e}_{t_0,x_0}(t) + \int_{t_0}^{1} \frac{\partial G_{t_0}}{\partial t}(t,s)K(s)ds)dt + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha^n(\overline{z}),\overline{z}}(t)) + \frac{1}{n}. \quad (6.1.8)$$

Using (6.1.3) and (6.1.8) it follows that $0 < \frac{\sigma\eta}{2} < \frac{1}{n}$ for every $n \in \mathbf{N}$. Passing to the limit when n goes to ∞ in the preceding inequality yields a contradiction. □

The viscosity property for the lower–upper value function is an open problem in the present context. Proposition 6.1 is a step forward in the problem under consideration. Compare with earlier result in the literature dealing with viscosity problem governed by (ODE) in $\mathbf{R}^n$ involving differential games and strategies, e.g. [4, 20, 22], evolution inclusions, e.g. [7, 8, 13–17] involving Young control measures, and Relaxation and Bolza problems governed by (SODE), e.g. [3, 9–11]. In order to illustrate the comparison, let us come back to a differential game governed by ordinary differential equation (ODE). Let $\mathcal{M}^1_+(Y)$ and $\mathcal{M}^1_+(Z)$ be the set of all probability Radon measures on compact metric space Y and Z, respectively, endowed with the narrow topology so that $\mathcal{M}^1_+(Y)$ and $\mathcal{M}^1_+(Z)$ are compact metrizable. Consider the space of Young measures (alias relaxed controls)

$$\mathcal{Y}(\tau) = \{y : [\tau, 1] \to \mathcal{M}^1_+(Y) \mid y \ \text{ measurable}\}$$

$$\mathcal{Z}(\tau) = \{z : [\tau, 1] \to \mathcal{M}^1_+(Z) \mid z \ \text{ measurable}\}$$

and as above denote by $\Gamma(\tau)$ the set of all strategies $\alpha : \mathcal{Z}(\tau) \to \mathcal{Y}(\tau)$ and $\Delta(\tau)$ the set of all strategies $\beta : \mathcal{Y}(\tau) \to \mathcal{Z}(\tau)$. Let $J : [0, 1] \times (E \times Y \times Z) \to \mathbf{R}$ be a bounded Carathéodory integrand and let $F : [0, 1] \times (E \times Y \times Z) \to E$ be a Carathéodory mapping satisfying $F(t, x, y, z) \in K(t)$ for all $(t, x, y, z) \in [0, 1]\times E\times Y\times Z$ where $K : [0, 1] \rightrightarrows E$ is a convex compact valued integrably bounded mapping and a Lipschitz type condition $||F(t, x_1, y, z)-F(t, x_2, y, z)|| \le \lambda||x_1-x_2||$ for all (t, x_1, y, z), (t, x_2, y, z) in $[0, 1] \times E \times Y \times Z$. Then one may consider the lower value function

$$V_J(\tau, x) = \inf_{\beta\in\Delta(\tau)} \sup_{\mu\in\mathcal{Y}(\tau)} \left\{\int_\tau^1 \left[\int_Z \left[\int_Y J(t, u_{\tau,x,\mu,\beta(\mu)}(t), y, z)\mu_t(dy)\right] \times\beta(\mu)_t(dz)\right] dt\right\}$$

where $u_{\tau,x,\mu,\beta(\mu)}$ is the absolutely continuous solution to (ODE)

$$\dot{u}_{\tau,x,\mu,\beta(\mu)}(t) = \int_Z \left[\int_Y F(t, u_{\tau,x,\mu,\beta(\mu)}(t), y, z)\mu_t(dy)\right] \beta(\mu)_t(dz), t \in [\tau, 1]$$

$$u_{\tau,x,\mu,\beta(\mu)}(\tau) = x$$

and the upper value function

$$U_J(\tau, x) = \sup_{\alpha\in\Gamma(\tau)} \inf_{\nu\in\mathcal{Z}(\tau)} \left\{\int_\tau^1 \left[\int_Z \left[\int_Y J(t, u_{\tau,x,\alpha(\nu),\nu}(t), y, z)\alpha(\nu)_t(dy)\right] \nu_t(dz)\right]\right\} dt,$$
$$\tau \in [0, 1], x \in E$$

where $u_{\tau,x,\alpha(\nu),\nu}$ is the absolutely continuous solution to (ODE)

$$\dot{u}_{\tau,x,\alpha(\nu),\nu}(t) = \int_Z \left[\int_Y F(t, u_{\tau,x,\alpha(\nu)(t),\nu}(t), y, z)\alpha(\nu)_t(dy)\right]\nu_t(dz), t \in [\tau, 1]$$

$$u_{\tau,x,\alpha(\nu),\nu}(\tau) = x$$

and state the viscosity properties for these functions. In the sequel, we will make some additional assumptions on J and F, namely, J and F are continuous and the family $(J(.,.,y,z))_{(y,z)\in Y\times Z}$ is equicontinuous and the family $(F(.,.,y,z))_{(y,z)\in Y\times Z}$ is equicontinuous.

Proposition 6.2. *Let $J : [0,1]\times E\times Y\times Z \to \mathbf{R}$ be a bounded continuous integrand, and let us consider the upper value function*

$$U_J(\tau, x) = \sup_{\alpha\in\Gamma(\tau)} \inf_{\nu\in\mathcal{Z}(\tau)} \left\{\int_\tau^1 \left[\int_Z \left[\int_Y J(t, u_{\tau,x,\alpha(\nu),\nu}(t), y, z)\alpha(\nu)_t(dy)\right]\nu_t(dz)\right]\right\} dt,$$
$$\tau \in [0,1], x \in E.$$

Let us consider the Hamiltonian

$$H^+(t,x,\rho) = \min_{\nu\in\mathcal{M}^1_+(Z)} \max_{\mu\in\mathcal{M}^1_+(Y)} \left\{\langle\rho, \int_Z \left[\int_Y F(t,x,y,z)d\mu(y)\right]d\nu(z) + \int_Z \left[\int_Y J(t,x,y,z)d\mu(y)\right]d\nu(z)\right\}.$$

Then U_J is a viscosity solution to the HJB equation $\frac{\partial U}{\partial t} + H^+(t,x,\nabla U)) = 0$, that is, for any $\varphi \in C^1([0,1]\times E)$ for which $U_J - \varphi$ reaches a local maximum at $(t_0,x_0)\in[0,1]\times E$ we have

$$\frac{\partial\varphi}{\partial t}(t_0,x_0) + H^+(t_0,x_0,\nabla\varphi(t_0,x_0)) \geq 0$$

and for any $\varphi \in C^1([0,1]\times E)$ for which $U_J - \varphi$ reaches a local minimum at $(t_0,x_0)\in[0,1]\times E$, we have

$$\frac{\partial\varphi}{\partial t}(t_0,x_0) + H^+(t_0,x_0,\nabla\varphi(t_0,x_0)) \leq 0$$

provided that U_J satisfies the DPP

$$U_J(\tau,x) = \sup_{\alpha\in\Gamma(\tau)} \inf_{\nu\in\mathcal{Z}(\tau)} \left\{\int_\tau^{\tau+\sigma} \left[\int_Z \left[\int_Y J(t, u_{\tau,x,\alpha(\nu),\nu}(s), y, z) \times\alpha(\nu)_s(dy)\right]\nu_s(dz)\right]ds + U_J(\tau+\sigma), u_{\tau,x,\alpha(\nu),\nu}(\tau+\sigma)\right\}.$$

Proof. See Proposition 6.1 and ([14], Theorem 8.3.12). We will sketch the proof. Assume there is a $\varphi \in C^1([0,1] \times E)$ such that $U_J - \varphi$ reaches a local maximum at $(t_0, x_0) \in [0,1] \times E$ for which

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \min_{\nu \in \mathcal{M}^1_+(Z)} \max_{\mu \in \mathcal{M}^1_+(Y)} \left\{ \int_Z \left[\int_Y J(t_0, x_0, y, z) d\mu(y) \right] d\nu(z) \right. \\ \left. + \langle \nabla \varphi(t_0, x_0), \int_Z \left[\int_Y F(t_0, x_0, y, z) d\mu(y) \right] d\nu(z) \rangle \right\} < 0.$$

Hence there exists some $\eta > 0$ such that

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \min_{\nu \in \mathcal{M}^1_+(Z)} \max_{\mu \in \mathcal{M}^1_+(Y)} \left\{ \int_Z \left[\int_Y J(t_0, x_0, y, z) d\mu(y) \right] d\nu(z) \right. \\ \left. + \langle \nabla \varphi(t_0, x_0), \int_Z \left[\int_Y F(t_0, x_0, y, z) d\mu(y) \right] d\nu(z) \rangle \right\} \le -\eta < 0.$$

Set

$$\Lambda(t, x, \mu, \nu) = \frac{\partial \varphi}{\partial t}(t, x) + \int_Z \left[\int_Y J(t, x, y, z) d\mu(y) \right] d\nu(z) \\ + \langle \nabla \varphi(t, x), \int_Z \left[\int_Y F(t, x, y, z) d\mu(y) \right] d\nu(z) \rangle.$$

Then we have

$$\min_{\nu \in \mathcal{M}^1_+(Z)} \max_{\mu \in \mathcal{M}^1_+(Y)} \Lambda(t_0, x_0, \mu, \nu) \le -\eta < 0.$$

Hence there exists some $\overline{\nu} \in \mathcal{M}^1_+(Z)$ such that

$$\max_{\mu \in \mathcal{M}^1_+(Y)} \Lambda(t_0, x_0, \mu, \overline{\nu}) \le -\eta < 0.$$

Since the mapping

$$(t, x) \mapsto \max_{\mu \in \mathcal{M}^1_+(Y)} \Lambda(t_0, x_0, \mu, \overline{\nu})$$

is continuous there is $\varepsilon > 0$ such that

$$\max_{\mu \in \mathcal{M}^1_+(Y)} \Lambda(t, x, \mu, \overline{\nu}) < -\frac{\eta}{2}$$

for $0 \le t - t_0 \le \varepsilon$ and $||x - x_0|| \le \varepsilon$. As $\dot{u}_{t_0, x_0, \alpha(\nu), \nu}$ is estimated by $||\dot{u}_{t_0, x_0, \alpha(\nu), \nu}(t)|| \le |K(t)|$ with $|K| \in L^1_{\mathbf{R}}([t_0, 1])$ for all $\nu \in \mathcal{Z}(t_0)$ and for all $\alpha \in \Gamma(t_0)$ $\Gamma(t_0)$ so we can choose $\sigma > 0$ such that $||u_{t_0, x_0, \alpha(\nu), \nu}(t) - u_{t_0, x_0, \alpha(\nu), \nu}(t_0)|| \le \int_{t_0}^{t_0 + \sigma} |K(t)| dt \le \varepsilon$ for all $t \in [t_0, t_0 + \sigma]$ and for all

$\nu \in \mathcal{Z}(t_0)$ and for all $\alpha \in \Gamma(t_0)$. Then the constant control $\overline{\nu}_t = \overline{\nu}, \forall t \in [t_0, 1]$ belongs to $\mathcal{Z}(t_0)$ and $\alpha(\overline{\nu})$ belongs to $\mathcal{Y}(t_0)$ for all $\alpha \in \Gamma(t_0)$ so that by integrating we have

$$\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,\alpha(\overline{\nu}),\overline{\nu}}(t), \alpha(\overline{\nu})_t, \overline{\nu}_t)dt < -\frac{\sigma\eta}{2}$$

for all $\alpha \in \Gamma(t_0)$. Thus

$$\begin{aligned}
&\sup_{\alpha\in\Gamma(t_0)} \Bigg\{\int_{t_0}^{t_0+\sigma} [\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha(\overline{\nu})_t(dy)\right] \overline{\nu}_t(dz)]dt \\
&+\langle \nabla\varphi(t, u_{t_0,x_0,\alpha(\overline{\nu}),\overline{\nu}}(t)), \int_Z \left[\int_Y F(t, u_{t_0,x_0,\alpha(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha(\overline{\nu})_t(dy)\right] \overline{\nu}_t(dz)\rangle \\
&+ \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha(\overline{\nu}),\overline{\nu}}(t))\Bigg\} < -\frac{\sigma\eta}{2}. \qquad (6.2.3)
\end{aligned}$$

As U_J satisfies the DPP, we have

$$\begin{aligned}
U_J(t_0, x_0) \le &\sup_{\alpha\in\Gamma(t_0)} \Bigg\{\int_{t_0}^{t_0+\sigma} \left[\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha(\overline{\nu})_t(dy)\right] \overline{\nu}_t(dz)\right] dt \\
&+ U_J(t_0+\sigma, u_{t_0,x_0,\alpha(\overline{\nu}),\overline{\nu}}(t_0+\sigma))\Bigg\}.
\end{aligned}$$

Hence, for every $n \in \mathbf{N}$, there exists $\alpha^n \in \Gamma(t_0)$ such that

$$\begin{aligned}
U_J(t_0, x_0) \le &\int_{t_0}^{t_0+\sigma} \left[\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha^n(\overline{\nu})_t(dy)\right], \overline{\nu}_t(dz)\right] dt \\
&+U_J(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t_0+\sigma)) + \frac{1}{n}. \qquad (6.2.4)
\end{aligned}$$

But $U_J - \varphi$ has a local maximum at (t_0, x_0), for small enough σ

$$\begin{aligned}
U_J(t_0, x_0) - \varphi(t_0, x_0) \ge &U_J(t_0+\sigma, u_{t_0,x_0,\alpha(\nu),\nu}(t_0+\sigma)) \\
&-\varphi(t_0+\sigma, u_{t_0,x_0,\alpha(\nu),\nu}(t_0+\sigma)) \qquad (6.2.5)
\end{aligned}$$

for any trajectory solution $u_{t_0,x_0,\alpha(\nu),\nu}$ associated with control $(\alpha(\nu), \nu)$ $(\alpha \in \Gamma(t_0), \nu \in \mathcal{Z}(t_0))$. From (6.2.4) and (6.2.5) we deduce

$$\begin{aligned}
&U_J(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t_0+\sigma)) - \varphi(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t_0+\sigma)) \\
&\le \int_{t_0}^{t_0+\sigma} \left[\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha^n(\overline{\nu})_t(dy)\right] \overline{\nu}_t(dz)\right] dt \\
&+ U_J(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t_0+\sigma) + \frac{1}{n} - \varphi(t_0, x_0).
\end{aligned}$$

Thus we have

$$0 \le \int_{t_0}^{t_0+\sigma} \left[\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha^n(\overline{\nu})_t(dy) \right], \overline{\nu}_t(dz) \right] dt + \varphi(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t_0+\sigma)) - \varphi(t_0, x_0) + \frac{1}{n} \tag{6.2.6}$$

But

$$\begin{aligned} &\varphi(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t_0+\sigma)) - \varphi(t_0, x_0) \\ &= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t)), \dot{u}_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t)\rangle dt \\ &\quad + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t))dt \end{aligned} \tag{6.2.7}$$

and

$$\dot{u}_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t) = \int_Z \left[\int_Y F(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha^n(\overline{\nu})_t(dy) \right] \overline{\nu}_t(dz)$$

so that by combining with (6.2.7)

$$\begin{aligned} &\varphi(t_0+\sigma, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t_0+\sigma)) - \varphi(t_0, x_0) \\ &= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t)), \int_Z \left[\int_Y F(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha^n(\overline{\nu})_t(dy) \right] \overline{\nu}_t(dz)\rangle dt \\ &+ \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t)). \end{aligned} \tag{6.2.8}$$

From (6.2.6) and (6.2.8) we deduce

$$\begin{aligned} 0 &\le \int_{t_0}^{t_0+\sigma} \left[\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha^n(\overline{\nu})_t(dy) \right] \overline{\nu}_t(dz) \right] dt \\ &\int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t)), \int_Z \left[\int_Y F(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t), y, z)\alpha^n(\overline{\nu})_t(dy) \right] \overline{\nu}_t(dz)\rangle dt \\ &+ \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha^n(\overline{\nu}),\overline{\nu}}(t)) + \frac{1}{n}. \end{aligned} \tag{6.2.9}$$

Using (6.2.3) and (6.2.9) it follows that $0 < \frac{\sigma\eta}{2} < \frac{1}{n}$ for every $n \in \mathbf{N}$. Passing to the limit when n goes to ∞ in the preceding inequality yields a contradiction.

Next assume that $U_J - \varphi$ has a local minimum at $(t_0, x_0) \in [0, 1] \times E$. We must prove that

$$\frac{\partial\varphi}{\partial t}(t_0,x_0)+\min_{\nu\in\mathcal{M}^1_+(Z)}\max_{\nu\in\mathcal{M}^1_+(Z)}\left\{\int_Z\left[\int_Y J(t_0,x_0,y,z)d\mu(y)\right]d\nu(z)\right.$$
$$\left.+\langle\nabla\varphi(t_0,x_0),\int_Z\left[\int_Y F(t_0,x_0,y,z)d\mu(y)\right]d\nu(z)\rangle\right\}\leq 0$$

and so will assume the contrary that

$$\frac{\partial\varphi}{\partial t}(t_0,x_0)+\min_{\nu\in\mathcal{M}^1_+(Z)}\max_{\mu\in\mathcal{M}^1_+(Y)}\left\{\int_Z\left[\int_Y J(t_0,x_0,y,z)d\mu(y)\right]d\nu(z)\right.$$
$$\left.+\langle\nabla\varphi(t_0,x_0),\int_Z\left[\int_Y F(t_0,x_0,y,z)d\mu(y)\right]d\nu(z)\rangle\right\}>\eta>0.$$

Arguing as in ([14], Lemma 8.3.11(b)) asserts that there exists for all sufficiently small $\sigma>0$ some $\alpha\in\Gamma(t_0)$ such that

$$\int_{t_0}^{t_0+\sigma}\left[\int_Z\left[\int_Y J(t,u_{t_0,x_0,\alpha(\nu),\nu}(t),y,z)\alpha(\nu)_t(dy)\right]\nu_t(dz)\right]dt$$
$$+\int_{t_0}^{t_0+\sigma}\langle\nabla\varphi(t,u_{t_0,x_0,\alpha(\nu),\nu}(t)),\int_Z\left[\int_Y F(t,u_{t_0,x_0,\alpha(\nu)\nu}(t),y,z)\alpha(\nu)_t(dy)\right]$$
$$\times\nu_t(dz)\rangle dt$$
$$+\int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{t_0,x_0,\alpha(\nu),\nu}(t))\geq\frac{\sigma\eta}{2}\tag{6.2.10}$$

for all $\nu\in\mathcal{Z}(t_0)$. According to the DPP property we have

$$U_J(t_0,x_0)$$
$$\geq\inf_{\nu\in\mathcal{Z}(t_0)}\left\{\int_{t_0}^{t_0+\sigma}\left[\int_Z\left[\int_Y J(t,u_{t_0,x_0,\alpha(\nu),\nu}(t),y,z)\alpha(\nu)_t(dy)\right]\nu_t(dz)\right]dt\right.$$
$$\left.+U_J(t_0+\sigma,u_{t_0,x_0,\alpha(\nu),\nu}(t_0+\sigma))\right\}.$$

Hence, for every $n\in\mathbf{N}$, there exists $\nu^n\in\mathcal{Z}(t_0)$ such that

$$U_J(t_0,x_0)\geq\int_{t_0}^{t_0+\sigma}\left[\int_Z\left[\int_Y J(t,u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t),y,z)\alpha(\nu^n)_t(dy)\right]\nu^n_t(dz)\right]dt$$
$$+U_J(t_0+\sigma,u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t_0+\sigma))-\frac{1}{n}.\tag{6.2.11}$$

But $U_J - \varphi$ has a local minimum at (t_0, x_0), for small enough σ

$$U_J(t_0, x_0) - \varphi(t_0, x_0) \le U_J(t_0 + \sigma, u_{t_0,x_0,\alpha(\nu),\nu}(t_0 + \sigma)) - \varphi(t_0 + \sigma, u_{t_0,x_0,\alpha(\nu),\nu}(t_0 + \sigma)) \quad (6.2.12)$$

for any trajectory solution $u_{t_0,x_0,\alpha(\nu),\nu}$ associated with control $(\alpha(\nu), \nu)$ ($\alpha \in \Gamma(t_0)$, $\nu \in \mathcal{Z}(t_0)$). From (6.2.11) and (6.2.12) we deduce

$$\begin{aligned} &U_J(t_0 + \sigma, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t_0 + \sigma)) - \varphi(t_0 + \sigma, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t_0 + \sigma)) \\ &\ge \int_{t_0}^{t_0+\sigma} \left[\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t), y, z)\alpha(\nu^n)_t(dy)\right] \nu_t^n(dz)\right] dt \\ &+U_J(t_0 + \sigma, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t_0 + \sigma) - \frac{1}{n} - \varphi(t_0, x_0). \end{aligned}$$

Thus we have

$$\begin{aligned} 0 \ge &\int_{t_0}^{t_0+\sigma} \left[\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t), y, z)\alpha(\nu^n)_t(dy)\right] \nu_t^n(dz)\right] dt \\ &+\varphi(t_0 + \sigma, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t_0 + \sigma)) - \varphi(t_0, x_0) - \frac{1}{n}. \end{aligned} \quad (6.2.13)$$

But

$$\begin{aligned} &\varphi(t_0 + \sigma, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t_0 + \sigma)) - \varphi(t_0, x_0) \\ &= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t)), \dot{u}_{t_0,x_0,\alpha(\nu^n),\nu^n}(t)\rangle dt \\ &\quad + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t))dt \end{aligned} \quad (6.2.14)$$

and

$$\dot{u}_{t_0,x_0,\alpha(\nu^n),\nu^n}(t) = \int_Z \left[\int_Y F(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t), y, z)\alpha(\nu^n)_t(dy)\right] \nu_t^n(dz)$$

so that from (6.2.14)

$$\begin{aligned} &\varphi(t_0 + \sigma, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t_0 + \sigma)) - \varphi(t_0, x_0) \\ &= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t)), \\ &\quad \int_Z \left[\int_Y F(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t), y, z)\alpha(\nu^n)_t(dy)\right] \nu_t^n(dz)\rangle dt \\ &\quad + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t)). \end{aligned} \quad (6.2.15)$$

From (6.2.13) and (6.2.15) we deduce

$$
\begin{aligned}
0 \geq & \int_{t_0}^{t_0+\sigma} \left[\int_Z \left[\int_Y J(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t), y, z)\alpha(\nu^n)_t(dy) \right] \nu_t^n(dz) \right] dt \\
& + \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t)), \\
& \int_Z \left[\int_Y F(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t), y, z)\alpha(\nu^n)_t(dy) \right] \nu_t^n(dz)\rangle dt \\
& + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,\alpha(\nu^n),\nu^n}(t)) - \frac{1}{n}. \qquad (6.2.16)
\end{aligned}
$$

Using (6.2.10) and (6.2.16) it follows that $\frac{1}{n} \geq \frac{\sigma\eta}{2} > 0$ for every $n \in \mathbf{N}$. Passing to the limit when n goes to ∞ in the preceding inequality yields a contradiction. □

Taking account into the sweeping process introduced by J.J. Moreau [26] and its modelisation in Mathematical Economics [25], we finish the paper with an application to the DPP and viscosity property for the value function associated with a sweeping process. Compare with Theorem 3.5 in [17] dealing with sweeping process involving Young measure control and Theorem 4.2 dealing with (SODE). Here E is a separable Hilbert space.

Proposition 6.3. *Let $C : [0, T] \to ck(E)$ be a convex compact valued L-Lipschitzean mapping:*

$$
|d(x, C(t)) - d(y, C(\tau))| \leq L|t - \tau| + ||x - y||, \forall x, y \in E \times E, \forall t, \tau \in [0, T] \times [0, T].
$$

Let Z be a convex compact subset in E and $\mathcal{S}_Z^1$ is the set of all integrable mappings $f : [0, T] \to Z$. Assume that $J : [0, T] \times E \times E \to \mathbf{R}$ is bounded and continuous such that $J(t, x, .)$ is convex for every $(t, x) \in [0, T] \times E$. Let us consider the value function

$$
V_J(\tau, x) = \sup_{f \in \mathcal{S}_Z^1} \left\{ \int_\tau^T J(t, u_{\tau,x,f}(t), f(t))\, dt \right\}, \ (\tau, x) \in [0, T] \times E
$$

where $u_{\tau,x,f}$ is the trajectory solution on $[\tau, T]$ associated the control $f \in \mathcal{S}_Z^1$ starting from $x \in E$ at time τ to the sweeping process $(\mathcal{PSW})(C; f; x)$

$$
\begin{cases} -\dot{u}_{\tau,x,f}(t) - f(t) \in N_{C(t)}(u_{\tau,x,f}(t)),\ t \in [\tau, T] \\ u_{\tau,x,f}(\tau) = x \end{cases}
$$

and the Hamiltonian

$$H(t,x,\rho) = \sup_{z\in Z}\{-\langle \rho, z\rangle + J(t,x,z)\} + \delta^*(\rho, -M\partial[d_{C(t)}](x))$$

where $M := L + 2|Z|$, $(t,x,\rho) \in [0,T]\times E\times E$ *and* $\partial[d_{C(t)]}](x)$ *denotes the subdifferential of the distance functions* $x \mapsto d_{C(t)}x$. *Then* V_J *has the DPP property*

$$V_J(\tau,x) = \sup_{f\in\mathcal{S}^1_Z}\left\{\int_\tau^{\tau+\sigma} J(t,u_{\tau,x,f}(t), f(t))dt + V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma))\right\}$$

and is a viscosity subsolution to the HJB equation

$$\frac{\partial U}{\partial t}(t,x) + H(t,x,\nabla U(t,x)) = 0$$

that is, for any $\varphi \in C^1([0,T])\times E)$ *for which* $V_J - \varphi$ *reaches a local maximum at* $(t_0,x_0)\in[0,T]\times E$, *we have*

$$H(t_0,x_0,\nabla\varphi(t_0,x_0)) + \frac{\partial\varphi}{\partial t}(t_0,x_0) \geq 0.$$

Proof. We prove first that V_J has the DPP property by applying the continuous property of the solution with respect to the state and the control (see Lemma 6.1 below) and lower semicontinuity of the integral functional ([14], Theorem 8.1.6). We omit the proof of Lemma 6.1 because it is an adaptation of the proof of Lemma 4.1 in [17].

Lemma 6.1. *Let* u_{τ,x^n,f^n} *be the trajectory solution on* $[\tau,T]$ *associated the control* $f^n \in \mathcal{S}^1_Z$ *starting from* $x^n \in E$ *at time* τ *to the sweeping process* $(\mathcal{PSW})(C; f^n; x)$

$$\begin{cases} -\dot{u}_{\tau,x^n,f^n}(t) - f^n(t) \in N_{C(t)}(u_{\tau,x^n,f^n}(t)) \\ u_{\tau,x^n,f^n}(\tau) = x^n \in C(\tau) \end{cases}$$

(a) *If* (x^n) *converges to* x^∞ *and* f^n *converges* $\sigma(L^1_E, L^\infty_E)$ *to* f^∞, *then* u_{τ,x^n,f^n} *converges uniformly to* $u_{\tau,x^\infty,f^\infty}$, *which is the Lipschitz solution of the sweeping process* $(\mathcal{PSW})(C; f^\infty; x^\infty)$

$$\begin{cases} -\dot{u}_{\tau,x^\infty,f^\infty}(t) - f^\infty(t) \in N_{C(t)}(u_{\tau,x^\infty,f^\infty}(t)) \\ u_{\tau,x^\infty,f^\infty}(\tau) = x^\infty \in C(\tau) \end{cases}$$

(b) Let $J : [0, 1] \times (E \times E) \to]-\infty, +\infty]$ *be a normal integrand such that* $J(t, x, .)$ *is convex on* E *for all* $(t, x) \in [0, T] \times E$ *and that*

$$J(t, u_{\tau,x^n,f^n}(t), f^n(t)) \geq \beta_n(t)$$

for all $n \in \mathbf{N}$ *and for all* $t \in [0, T]$ *for some uniformly integrable sequence* $(\beta_n)_{n\in\mathbf{N}}$ *in* $L^1_{\mathbf{R}}([0, T])$*, then we have*

$$\liminf_{n\to\infty} \int_\tau^T J(t, u_{\tau,x^n,f^n}(t), f^n(t))\, dt \geq \int_\tau^T J(t, u_{\tau,x^\infty,f^\infty}(t), f^\infty(t))\, dt.$$

Let us focus on the expression of $V_J(\tau + \sigma, u_{\tau,x,f}(\tau + \sigma))$

$$V_J(\tau + \sigma, u_{\tau,x,f}(\tau + \sigma)) = \sup_{g\in S^1_Z} \left\{ \int_{\tau+\sigma}^T J(t, v_{\tau+\sigma,u_{\tau,x,f}(\tau+\sigma),g}(t), g(t))dt \right\}$$

where $v_{\tau+\sigma,u_{\tau,x,f}(\tau+\sigma),g}$ denotes the trajectory solution on $[\tau + \sigma, T]$ associated with the control $g \in S^1_Z$ starting from $u_{\tau,x,f}(\tau + \sigma)$ at time $\tau + \sigma$.

Main Fact: $f \to V_J(\tau + \sigma, u_{\tau,x,f}(\tau + \sigma))$ is lower semicontinuous on S^1_Z (endowed with the $\sigma(L^1_E, L^\infty_E)$-topology). Let $(f_n, g_n) \in \mathcal{S}^1_Z \times \mathcal{S}^1_Z$ such that $f_n \to f \in \mathcal{S}^1_Z$ and $g_n \to g \in \mathcal{S}^1_Z$. By Lemma 6.1, u_{τ,x,f_n} converges uniformly to $u_{\tau,x,f}$ and $v_{\tau+\sigma,u_{\tau,x,f_n}(\tau+\sigma),g_n}$ converges uniformly to $v_{\tau+\sigma,u_{\tau,x,f}(\tau+\sigma),g}$ so that by invoking the lower semicontinuity of integral functional ([14], Theorem 8.1.6) we get

$$\liminf_{n\to\infty} \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f_n}(t), f_n(t))dt \geq \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt$$

$$\liminf_{n\to\infty} \int_{\tau+\sigma}^T J(t, v_{\tau+\sigma,u_{\tau,x,f_n}(\tau+\sigma),g_n}(t), g_n(t))dt$$
$$\geq \int_{\tau+\sigma}^T J(t, v_{\tau+\sigma,u_{\tau,x,f}(\tau+\sigma),g}(t), g(t))dt$$

proving that the mapping $f \mapsto \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt$ is lower semicontinuous on $\mathcal{S}^1_Z$ and the mapping $(f, g) \mapsto \int_{\tau+\sigma}^T J(t, v_{\tau+\sigma,u_{\tau,x,f}(\tau+\sigma),g}$ $(t), g(t))dt$ is lower semicontinuous on $\mathcal{S}^1_Z \times \mathcal{S}^1_Z$. It follows that the mapping $f \mapsto V_J(\tau + \sigma, u_{\tau,x,f}(\tau + \sigma))$ is lower semicontinuous on $\mathcal{S}^1_Z$ and so is the mapping $f \mapsto \int_\tau^{\tau+\sigma} J(t, u_{\tau,x,f}(t), f(t))dt + V_J(\tau+\sigma, u_{\tau,x,f}(\tau+\sigma))$. Now the DPP property for V_J follows the same line of the proof of Theorem 4.1. This fact allows to obtain the required viscosity property. Let us recall the following

Lemma 6.2. *Let* $(t_0, x_0) \in [0, T] \times E$ *and let* Z *be a convex compact subset in* E. *Let* $\Lambda : [0, T] \times E \times Z \to \mathbf{R}$ *be an upper semicontinuous function such that the restriction of* Λ *to* $[0, T] \times B \times Z$ *is bounded on any bounded subset* B *of* E. *If*

$$max_{z \in Z} \Lambda(t_0, x_0, z) < -\eta < 0$$

for some $\eta > 0$, *then there exists* $\sigma > 0$ *such that*

$$\sup_{f \in S_Z^1} \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{t_0,x_0,f}(t), f(t))dt < -\frac{\sigma\eta}{2}$$

where $u_{t_0,x_0,f}$ *is the trajectory solution associated with the control* $f \in S_Z^1$ *starting from* x_0 *at time* t_0 *to*

$$\begin{cases} -\dot{u}_{t_0,x_0,f}(t) - f(t) \in N_{C(t)}(u_{t_0,x_0,f}(t)), \ t \in [t_0, T] \\ u_{t_0,x_0,f}(t_0) = x_0. \end{cases}$$

Assume by contradiction that there exists a $\varphi \in C^1([0, T] \times E)$ and a point $(t_0, x_0) \in [0, T] \times E$ for which

$$\frac{\partial\varphi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla\varphi(t_0, x_0)) \leq -\eta < 0 \quad \text{for} \quad \eta > 0.$$

Applying Lemma 6.2 by taking

$$\Lambda(t, x, z) = J(t, x, z) - \langle \nabla\varphi(t, x), z \rangle + \delta^*(\nabla\varphi(t, x), \\ - M\,\partial[d_{C(t)}](x)) + \frac{\partial\varphi}{\partial t}(t, x)$$

provides some $\sigma > 0$ such that

$$\sup_{f \in \mathcal{S}_Z^1} \left\{ \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f}(t), f(t))\,dt - \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,f}(t)), f(t) \rangle\,dt \right. \\ + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,f}(t)), -M\,\partial[d_{C(t)}](u_{t_0,x_0,f}(t)))\,dt \\ \left. + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,f}(t))\,dt \right\} < -\frac{\sigma\eta}{2} \tag{6.3.1}$$

where $u_{t_0,x_0,f}$ is the trajectory solution associated with the control $f \in \mathcal{S}_Z^1$ starting from x_0 at time t_0 to the sweeping process $(\mathcal{PSW})(C; f; x)$

$$\begin{cases} -\dot{u}_{t_0,x_0,f}(t) - f(t) \in N_{C(t)}(u_{t_0,x_0,f}(t)), \ t \in [t_0, T] \\ u_{t_0,x_0,f}(t_0) = x_0. \end{cases}$$

Applying the dynamic programming principle gives

$$V_J(t_0, x_0) = \sup_{f \in \mathcal{S}^1_Z} \left\{ \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f}(t), f(t))dt + V_J(t_0 + \sigma, u_{t_0,x_0,f}(\tau + \sigma)) \right\}. \tag{6.3.2}$$

Since $V_J - \varphi$ has a local maximum at (t_0, x_0), for small enough σ

$$V_J(t_0, x_0) - \varphi(t_0, x_0) \geq V_J(t_0 + \sigma, u_{t_0,x_0,f}(t_0 + \sigma)) - \varphi(t_0 + \sigma, u_{t_0,x_0,f}(t_0 + \sigma)) \tag{6.3.3}$$

for all $f \in \mathcal{S}^1_Z$. For each $n \in \mathbf{N}$, there exists $f^n \in \mathcal{S}^1_Z$ such that

$$V_J(t_0, x_0) \leq \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))\,dt + V_J(t_0+\sigma, u_{t_0,x_0,f^n}(t_0+\sigma)) + \frac{1}{n}. \tag{6.3.4}$$

From (6.3.3) and (6.3.4) we deduce that

$$\begin{aligned} &V_J(t_0 + \sigma, u_{t_0,x_0,f^n}(t_0 + \sigma)) - \varphi(t_0 + \sigma, u_{t_0,x_0,f^n}(t_0 + \sigma)) \\ &\leq \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))dt + \frac{1}{n} \\ &\quad -\varphi(t_0, x_0) + V_J(t_0 + \sigma, u_{t_0,x_0,f^n}(t_0 + \sigma)). \end{aligned}$$

Therefore we have

$$0 \leq \int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))\,dt + \varphi(t_0+\sigma, u_{t_0,x_0,f^n}(t_0+\sigma) - \varphi(t_0, x_0) + \frac{1}{n}. \tag{6.3.5}$$

As $\varphi \in C^1([0, T] \times E)$ we have

$$\begin{aligned} &\varphi(t_0 + \sigma, u_{t_0,x_0,f^n}(t_0 + \sigma)) - \varphi(t_0, x_0) \\ &= \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,f^n}(t)), \dot{u}_{t_0,x_0,f^n}(t) \rangle\,dt + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))\,dt. \end{aligned} \tag{6.3.6}$$

Since u_{t_0,x_0,f^n} is the trajectory solution starting from x_0 at time t_0 to the sweeping process $(\mathcal{PSW})(C; f^n; x)$

$$\begin{cases} -\dot{u}_{t_0,x_0,f^n}(t) - f^n(t) \in N_{C(t)}(u_{t_0,x_0,f^n}(t)),\ t \in [t_0, T] \\ u_{t_0,x_0,f^n}(t_0) = x_0 \end{cases}$$

by the classical property of normal convex cone and the estimation $||\dot{u}_{t_0,x_0,f^n}(t) - f^n(t)|| \leq L + 2|Z| = M$ we get

$$-\dot{u}_{t_0,x_0,f^n}(t) - f^n(t) \in M\,\partial[d_{C(t)}](u_{t_0,x_0,f^n}(t))$$

so that (6.3.6) yields

$$\begin{aligned}
&\varphi(t_0+\sigma, u_{t_0,x_0,f^n}(t_0+\sigma)) - \varphi(t_0,x_0)\\
&\quad = \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,f^n}(t)), \dot{u}_{t_0,x_0,f^n}(t)\rangle\, dt\\
&\qquad + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))\, dt\\
&\quad \leq -\int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,f^n}(t)), f^n(t)\rangle\, dt\\
&\qquad + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,f^n}(t)), -M\,\partial[d_{C(t)}](u_{t_0,x_0,f^n}(t)))\, dt\\
&\qquad + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))\, dt. \qquad (6.3.7)
\end{aligned}$$

Putting the estimate (6.3.7) in (6.3.5) we get

$$\begin{aligned}
0 \leq &\int_{t_0}^{t_0+\sigma} J(t, u_{t_0,x_0,f^n}(t), f^n(t))dt - \int_{t_0}^{t_0+\sigma} \langle \nabla\varphi(t, u_{t_0,x_0,f^n}(t)), f^n(t)\rangle\, dt\\
&+ \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{t_0,x_0,f^n}(t)), -M\partial[d_{C(t)}](u_{t_0,x_0,f^n}(t)))\, dt\\
&+ \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{t_0,x_0,f^n}(t))\, dt + \frac{1}{n} \qquad (6.3.8)
\end{aligned}$$

so that (6.3.1) and (6.3.8) give $0 < \frac{\sigma\eta}{2} < \frac{1}{n}$ for all $n \in \mathbf{N}$. Passing to the limit when n goes ∞ in this inequality gives a contradiction. □

Viscosity problem governed by sweeping process with strategies and Young measures

$$\begin{aligned}
&\dot{u}_{\tau,x,\alpha(\nu),\nu}(t) \in \int_Z \left[\int_Y F(t, u_{\tau,x,\alpha(\nu),\nu}(t), y, z)\alpha(\nu)_t(dy)\right] \nu_t(dz)\\
&- N_{C(t)}(u_{\tau,x,\alpha(\nu),\nu}(t)), t \in [\tau, 1],\\
&u_{\tau,x,\alpha(\nu),\nu}(\tau) = x \in C(\tau),\\
&U_J(\tau,x) = \sup_{\alpha\in\Gamma(\tau)} \inf_{\nu\in\mathcal{Z}(\tau)} \left\{\int_\tau^1 \left[\int_Z \left[\int_Y J(t, u_{\tau,x,\alpha(\nu),\nu}(t), y, z)\alpha(\nu)_t(dy)\right] \nu_t(dz)\right] dt\right\}
\end{aligned}$$

where the integrand J, the upper value function U_J, the data Y, Z and F are defined as in Proposition 6.2, is an open problem. Further related results dealing with continuous and bounded variation (BVC) solution in sweeping process governed by non empty interior closed convex valued continuous mappings are available in [17, 18].

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