Chapter 5
Dimensions of Spaces

For an open cover $\mathcal{U}$ of a space X, $\operatorname{ord}\mathcal{U} = \sup\{\operatorname{card}\mathcal{U}[x] \mid x \in X\}$ is called the **order** of $\mathcal{U}$. Note that $\operatorname{ord}\mathcal{U} = \dim N(\mathcal{U}) + 1$, where $N(\mathcal{U})$ is the nerve of $\mathcal{U}$. The (**covering**) **dimension** of X is defined as follows: $\dim X \leq n$ if each *finite* open cover of X has a *finite* open refinement $\mathcal{U}$ with $\operatorname{ord}\mathcal{U} \leq n + 1$. and then, $\dim X = n$ if $\dim X \leq n$ and $\dim X \not\leq n$. By $\dim X = -1$, we mean that $X = \emptyset$. We say that X is ***n*-dimensional** if $\dim X = n$ and that X is **finite-dimensional** (**f.d.**) ($\dim X < \infty$) if $\dim X \leq n$ for some $n \in \omega$. Otherwise, X is said to be **infinite-dimensional** (**i.d.**) ($\dim X = \infty$). The dimension is a topological invariant (i.e., $\dim X = \dim Y$ if $X \approx Y$).

This chapter is devoted to lectures on Dimension Theory. Fundamental theorems are proved and some examples of infinite-dimensional spaces are given. In this context, we discuss the Brouwer Fixed Point Theorem and the characterization of the Cantor set. We also construct finite-dimensional universal spaces such as the Nöbeling spaces and the Menger compacta.

We will use the results in Chaps. 2 and 4. In particular, we will need the combinatorial techniques treated in Chap. 4. Also, the concept of the nerves of open covers is very important in Dimension Theory.

5.1 The Brouwer Fixed Point Theorem

It is said that a space X has the **fixed point property** if any map $f : X \to X$ has a fixed point, i.e., $f(x) = x$ for some $x \in X$. In this section, we prove the following Brouwer Fixed Point Theorem:

Theorem 5.1.1 (BROUWER FIXED POINT THEOREM). *For every $n \in \mathbb{N}$, the n-cube $\mathbf{I}^n$ has the fixed point property.*

To prove this theorem, we need two lemmas. Let K be a simplicial complex and K' a simplicial subdivision of K. A simplicial map $h : K' \to K$ is called a

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Sperner map if for each $v \in K'^{(0)}$, $h(v)$ is a vertex of the carrier $c_K(v)^{(0)}$ of v in K, equivalently $v \in O_K(h(v))$. In other words, h is a simplicial approximation of $\mathrm{id}_{|K|}$. Indeed, for each $x \in |K'| = |K|$, $c_{K'}(x) \subset c_K(x)$. Since $c_K(v) \leq c_K(x)$ for every $v \in c_{K'}(x)^{(0)}$, it follows that $h(c_{K'}(x)^{(0)}) \subset c_K(x)^{(0)}$, hence $h(x) \in h(c_{K'}(x)) \leq c_K(x)$.

Lemma 5.1.2 (Sperner). *Let τ be an n-simplex, and K' a subdivision of $F(\tau)$, where $F(\tau)$ is the natural triangulation of τ. If $h : K' \to F(\tau)$ is a Sperner map, then the number of n-simplexes $\tau' \in K'$ such that $h(\tau') = \tau$ is odd; hence, there exists such an n-simplex $\tau' \in K'$.*

Proof. We prove the lemma by induction with respect to n. The case $n = 0$ is obvious. Assume the lemma has been established for any $(n-1)$-simplex. Let σ be an $(n-1)$-face of τ. Then, $h(\sigma) \subset \sigma$. The natural triangulation $F(\sigma)$ of σ is a subcomplex of $F(\tau)$. Let L' be the subdivision of $F(\sigma)$ induced by K'. As is easily observed, $h|\sigma : L' \to F(\sigma)$ is also a Sperner map. Let a be the number of $(n-1)$-simplexes $\sigma' \in L'$ such that $h(\sigma') = \sigma$. Then, a is odd by the inductive assumption. Let $\mathcal{S}$ be the set of all $(n-1)$-simplexes $\sigma' \in K'$ such that $h(\sigma') = \sigma$. For each n-simplex $\tau' \in K'$, let $b(\tau')$ denote the number of faces σ' of τ' that belong to $\mathcal{S}$, i.e., $h(\sigma') = \sigma$. Then, it follows that

$$b(\tau') = \begin{cases} 2 & \text{if } h(\tau') = \sigma; \\ 1 & \text{if } h(\tau') = \tau; \\ 0 & \text{otherwise.} \end{cases}$$

Let c be the number of n-simplexes $\tau' \in K'$ such that $h(\tau') = \tau$. Then,

$$\sum_{\tau' \in K' \setminus K'^{(n-1)}} b(\tau') - c \quad \text{is even.}$$

On the other hand, a is equal to the number of $(n-1)$-simplexes σ' of $\mathcal{S}$ such that $\sigma' \subset \sigma$. For each $\sigma' \in \mathcal{S}$, σ' is a common face of exactly two n-simplexes of K' if and only if $\sigma' \not\subset \sigma$. Hence,

$$\sum_{\tau' \in K' \setminus K'^{(n-1)}} b(\tau') - a \quad \text{is even.}$$

Therefore, $a - c$ is also even. Recall that a is odd. Thus, c is also odd. □

Lemma 5.1.3. *Let $\tau = \langle v_1, \ldots, v_{n+1} \rangle$ be an n-simplex and $F_1, \ldots, F_{n+1}$ be closed sets in τ. If $\langle v_{i(1)}, \ldots, v_{i(m)} \rangle \subset F_{i(1)} \cup \cdots \cup F_{i(m)}$ for each $1 \leq i(1) < \cdots < i(m) \leq n+1$, then $F_1 \cap \cdots \cap F_{n+1} \neq \emptyset$.*

Proof. Assume that $F_1 \cap \cdots \cap F_{n+1} = \emptyset$. Then,

$$\mathcal{U} = \{\tau \setminus F_1, \ldots, \tau \setminus F_{n+1}\} \in \operatorname{cov}(\tau).$$

Let K' be a subdivision of $F(\tau)$ that refines $\mathcal{U}$. For each $v \in K'^{(0)}$, choose a vertex v_i of the carrier of v in $F(\tau)$ so that $v \in F_i$, and let $h(v) = v_i$. Then, we have a Sperner map $h : K' \to F(\tau)$. By Lemma 5.1.2, there is a simplex $\tau' \in K'$ such that $h(\tau') = \tau$. Write $\tau' = \langle v'_1, \ldots, v'_{n+1}\rangle$ so that $h(v'_i) = v_i$. By the definition of h, $v'_i \in F_i$ for each $i = 1, \ldots, n+1$. Thus, τ' is not contained in any $\tau \setminus F_i$, which is a contradiction. □

Proof of Theorem 5.1.1. It suffices to show that any map $f : \Delta^n \to \Delta^n$ has a fixed point, where $\Delta^n \subset \mathbb{R}^{n+1}$ is the standard n-simplex. For each $i = 1, \ldots, n+1$, let

$$F_i = \{x \in \Delta^n \mid \operatorname{pr}_i(f(x)) \le \operatorname{pr}_i(x)\},$$

where $\operatorname{pr}_i : \mathbb{R}^{n+1} \to \mathbb{R}$ is the projection onto the i-th factor. Then, F_i is closed in Δ^n. Moreover, each face $\sigma = \langle \mathbf{e}_{i(1)}, \ldots, \mathbf{e}_{i(m)}\rangle \le \Delta^n$ is contained in $F_{i(1)} \cup \cdots \cup F_{i(m)}$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ is the canonical orthonormal basis for $\mathbb{R}^{n+1}$. In fact, if $x \in \sigma$ then

$$\sum_{j=1}^{m} \operatorname{pr}_{i(j)}(f(x)) \le 1 = \sum_{j=1}^{m} \operatorname{pr}_{i(j)}(x),$$

which implies that $\operatorname{pr}_{i(j)}(f(x)) \le \operatorname{pr}_{i(j)}(x)$ for some $j = 1, \ldots, m$. By Lemma 5.1.3, we have a point $a \in F_1 \cap \cdots \cap F_{n+1}$. Since $0 \le \operatorname{pr}_i(f(a)) \le \operatorname{pr}_i(a)$ for each $i = 1, \ldots, n+1$ and

$$\sum_{i=1}^{n+1} \operatorname{pr}_i(f(a)) = 1 = \sum_{i=1}^{n+1} \operatorname{pr}_i(a),$$

it follows that $\operatorname{pr}_i(f(a)) = \operatorname{pr}_i(a)$ for each $i = 1, \ldots, n+1$, which means that $f(a) = a$. □

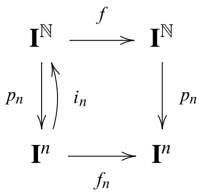
The following is the infinite-dimensional version of Theorem 5.1.1:

Corollary 5.1.4. *The Hilbert cube $\mathbf{I}^{\mathbb{N}}$ has the fixed point property.*

Proof. For each $n \in \mathbb{N}$, let $p_n : \mathbf{I}^{\mathbb{N}} \to \mathbf{I}^n$ be the projection onto the first n factors and $i_n : \mathbf{I}^n \to \mathbf{I}^{\mathbb{N}}$ the natural injection defined by

$$i_n(x) = (x(1), \ldots, x(n), 0, 0, \ldots).$$

For each map $f : \mathbf{I}^{\mathbb{N}} \to \mathbf{I}^{\mathbb{N}}$, consider the map $f_n = p_n f i_n : \mathbf{I}^n \to \mathbf{I}^n$.



By the Brouwer Fixed Point Theorem 5.1.1, f_n has a fixed point. We define

$$K_n = \{x \in \mathbf{I}^{\mathbb{N}} \mid p_n f(x) = p_n(x)\},$$

which is closed in $\mathbf{I}^{\mathbb{N}}$ and $K_n \supset K_{n+1}$ for each $n \in \mathbb{N}$. Moreover, $K_n \neq \emptyset$. Indeed, if $y \in \mathbf{I}^n$ is a fixed point of f_n, then $p_n f(i_n(y)) = f_n(y) = y = p_n(i_n(y))$, i.e., $i_n(y) \in K_n$. By compactness, we have $a \in \bigcap_{n\in\mathbb{N}} K_n$. Since $p_n f(a) = p_n(a)$ for every $n \in \mathbb{N}$, we have $f(a) = a$. □

As another corollary of the Brouwer Fixed Point Theorem 5.1.1, we have the following:

Corollary 5.1.5 (No Retraction Theorem). *There does not exist any map* $r : \mathbf{B}^n \to \mathbf{S}^{n-1}$ *with* $r|\mathbf{S}^{n-1} = \mathrm{id}$.[1]

Proof. Suppose that there is a map $r : \mathbf{B}^n \to \mathbf{S}^{n-1}$ with $r|\mathbf{S}^{n-1} = \mathrm{id}$. We define a map $f : \mathbf{B}^n \to \mathbf{B}^n$ by $f(x) = -r(x)$. Then, f has no fixed points, which contradicts the Brouwer Fixed Point Theorem 5.1.1. □

Remark 1. It should be noted that the Brouwer Fixed Point Theorem 5.1.1 can be derived from the No Retraction Theorem 5.1.5. Indeed, if there is a map $f : \mathbf{B}^n \to \mathbf{B}^n$ without fixed points, then we have a map $r : \mathbf{B}^n \to \mathbf{S}^{n-1}$ such that $x \in \langle f(x), r(x)\rangle$ for each $x \in \mathbf{B}^n$, which implies that $r|\mathbf{S}^{n-1} = \mathrm{id}$. In fact, such a map r can be defined as follows:

$$r(x) = (1 + \alpha(x))x - \alpha(x) f(x),$$

where $\alpha(x) \geq 0$ can be obtained by solving the equation

$$\alpha(x)^2 \|x - f(x)\|^2 + 2\alpha(x)\langle x - f(x), x\rangle + \|x\|^2 - 1 = 0,$$

where $\langle y, z\rangle = \sum_{i=1}^n y(i)z(i)$ is the inner product (Fig. 5.1). Therefore, the No Retraction Theorem 5.1.5 implies that $\mathbf{I}^n \approx \mathbf{B}^n$ has the fixed point property. Thus, the Brouwer Fixed Point Theorem 5.1.1 and the No Retraction Theorem 5.1.5 are equivalent.

[1] Such a map r is called a **retraction**, which will be discussed in Chap. 6.

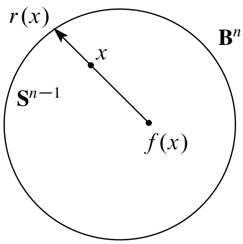


Fig. 5.1 The construction of r

Note. In Algebraic Topology, the homotopy groups or the homology groups are used to prove the No Retraction Theorem 5.1.5, and then the Brouwer Fixed Point Theorem 5.1.1 is proved as the above Remark 1.

Using the Tietze Extension Theorem 2.2.2, we have the following extension theorem:

Theorem 5.1.6. *Let A be a closed set in a normal space X and $n \in \mathbb{N}$.*

(1) *Every map $f : A \to \mathbf{B}^n$ extends over X.*
(2) *Every map $f : A \to \mathbf{S}^{n-1}$ extends over a neighborhood of A in X.*

Proof. By the coordinate-wise application of the Tietze Extension Theorem 2.2.2, each map $f : A \to \mathbf{I}^n$ can be extended over X, which implies (1) because $\mathbf{B}^n \approx \mathbf{I}^n$.

To prove (2), let $f : A \to \mathbf{S}^{n-1}$ be a map. By (1), f extends to a map $\tilde{f} : X \to \mathbf{B}^n$. Then, $W = \tilde{f}^{-1}(\mathbf{B}^n \setminus \{0\})$ is an open neighborhood of A in X. Let $r : \mathbf{B}^n \setminus \{0\} \to \mathbf{S}^{n-1}$ be the radial projection, i.e., $r(x) = x/\|x\|$. Then, $r\tilde{f}|W : W \to \mathbf{S}^{n-1}$ is an extension of f. □

Using the No Retraction Theorem 5.1.5 and Theorem 5.1.6, we can obtain the following characterization of boundary points of a closed set X in Euclidean space $\mathbb{R}^n$:

Theorem 5.1.7. *Let X be a closed subset of Euclidean space $\mathbb{R}^n$. For a point $x \in X$, $x \in \operatorname{bd} X$ if and only if each neighborhood U of x in X contains a neighborhood V of x in X such that every continuous map $f : X \setminus V \to \mathbf{S}^{n-1}$ extends to a continuous map $\tilde{f} : X \to \mathbf{S}^{n-1}$.*

Proof. To show the "only if" part, for each neighborhood U of x in X, choose $\varepsilon > 0$ so that $\overline{\mathrm{B}}(x, \varepsilon) \cap X \subset U$. Then, $V = \mathrm{B}(x, \varepsilon) \cap X$ is the desired neighborhood of x in X. Indeed, every map $f : X \setminus V \to \mathbf{S}^{n-1}$ can be extended to a map $g : X \to \mathbf{B}^n$ by Theorem 5.1.6. Choose $0 < \delta < \varepsilon$ so that $g(X \setminus \mathrm{B}(x, \delta)) \subset \mathbf{B}^n \setminus \{0\}$. Let $r : \mathbf{B}^n \setminus \{0\} \to \mathbf{S}^{n-1}$ be the canonical radial retraction (i.e., $r(y) = \|y\|^{-1}y$). Because $x \in \operatorname{bd} X$, we have $z \in \mathrm{B}(x, \frac{1}{2}(\varepsilon - \delta)) \setminus X$. Let $\lambda = \frac{1}{2}(\varepsilon + \delta) > 0$. Observe that $\mathrm{B}(x, \delta) \subset \mathrm{B}(z, \lambda) \subset \mathrm{B}(x, \varepsilon)$. We define a map $h : X \to X \setminus \mathrm{B}(z, \lambda) \subset X \setminus \mathrm{B}(x, \delta)$ by $h|X \setminus \mathrm{B}(z, \lambda) = \mathrm{id}$ and

$$h(y) = z + \frac{\lambda}{\|y - z\|}(y - z) \text{ for } y \in X \cap \mathrm{B}(z, \lambda).$$

Then, $rgh : X \to \mathbf{S}^{n-1}$ is a continuous extension of f.

To prove the "if" part, assume that $x \in \operatorname{int} X$. Then, $\overline{\mathrm{B}}(x, \delta) \subset X$ for some $\delta > 0$. By the condition, $\mathrm{B}(x, \delta)$ contains a neighborhood V of x such that every map $f : X \setminus V \to \mathbf{S}^{n-1}$ extends to a map $\tilde{f} : X \to \mathbf{S}^{n-1}$. It is easy to construct a retraction

$$r : \mathbb{R}^n \setminus \{x\} \to \operatorname{bd} \mathrm{B}(x, \delta) \approx \mathbf{S}^{n-1}.$$

Then, $r|X \setminus V$ extends to a retraction $\tilde{r} : X \to \operatorname{bd} \mathrm{B}(x, \delta)$. Since $\overline{\mathrm{B}}(x, \delta) \subset X$, $\operatorname{bd} \mathrm{B}(x, \delta)$ is a retract of $\overline{\mathrm{B}}(x, \delta)$, which contradicts the No Retraction Theorem 5.1.5 because $(\overline{\mathrm{B}}(x, \delta), \operatorname{bd} \mathrm{B}(x, \delta)) \approx (\mathbf{B}^n, \mathbf{S}^{n-1})$. Thus, we have $x \in \operatorname{bd} X$. □

As a corollary of Theorem 5.1.7, we have the so-called INVARIANCE OF DOMAIN:

Corollary 5.1.8 (INVARIANCE OF DOMAIN). *For each* $X, Y \subset \mathbb{R}^n$, $X \approx Y$ *implies* $\operatorname{int} X \approx \operatorname{int} Y$.

Proof. Let $h : X \to Y$ be a homeomorphism. For each $x \in \operatorname{bd} X$ and each neighborhood U of $h(x)$ in Y, $h^{-1}(U)$ is a neighborhood of x in X that contains a neighborhood V of x such that every map $f : X \setminus V \to \mathbf{S}^{n-1}$ extends to a map $\tilde{f} : X \to \mathbf{S}^{n-1}$. Then, $h(V)$ is a neighborhood of $h(x)$ in Y such that $h(V) \subset U$, and every continuous map $g : Y \setminus h(V) \to \mathbf{S}^{n-1}$ extends to a continuous map $\tilde{g} : Y \to \mathbf{S}^{n-1}$. Indeed, $gh : X \setminus V \to \mathbf{S}^{n-1}$ extends to a continuous map $\tilde{f} : X \to \mathbf{S}^{n-1}$. Then, $\tilde{f}h^{-1} : Y \to \mathbf{S}^{n-1}$ is a continuous extension of g. □

5.2 Characterizations of Dimension

Recall that we define $\dim X \le n$ if each *finite* open cover of X has a *finite* open refinement $\mathcal{U}$ with $\operatorname{ord} \mathcal{U} \le n + 1$. The following lemma shows that the refinement $\mathcal{U}$ in this definition need not be finite.

Lemma 5.2.1. *Let* $\mathcal{U}$ *be an open cover of a space* X *and* $\mathcal{V}$ *an open refinement of* $\mathcal{U}$. *Then,* $\mathcal{U}$ *has an open refinement* $\mathcal{W} = \{W_U \mid U \in \mathcal{U}\}$ *such that* $W_U \subset U$ *for each* $U \in \mathcal{U}$ *and* $\operatorname{card} \mathcal{W}[x] \le \operatorname{card} \mathcal{V}[x]$ *for each* $x \in X$, *which implies that* $\operatorname{ord} \mathcal{W} \le \operatorname{ord} \mathcal{V}$ *and if* $\mathcal{U}$ *is (locally) finite (or* σ*-discrete) then so is* $\mathcal{W}$.

Proof. Let $\varphi : \mathcal{V} \to \mathcal{U}$ be a function such that $V \subset \varphi(V)$ for each $V \in \mathcal{V}$. For each $U \in \mathcal{U}$, define

$$W_U = \bigcup \varphi^{-1}(U) = \bigcup \{V \in \mathcal{V} \mid \varphi(V) = U\}.$$

Then, $\mathcal{W} = \{W_U \mid U \in \mathcal{U}\}$ is the desired refinement of $\mathcal{U}$. □

The following is a particular case of the Open Cover Shrinking Lemma 2.7.1, which is easily proved directly.

Lemma 5.2.2. *Each finite open cover* $\{U_1, \ldots, U_n\}$ *of a normal space* X *has an open refinement* $\{V_1, \ldots, V_n\}$ *such that* $\operatorname{cl} V_i \subset U_i$ *for each* $i = 1, \ldots, n$.

Proof. Using the normality of X, V_i can be inductively chosen so that

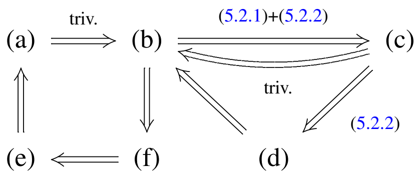
$$\operatorname{cl} V_i \subset U_i \quad \text{and} \quad V_1 \cup \cdots \cup V_i \cup U_{i+1} \cup \cdots \cup U_n = X. \quad \square$$

We now prove the following characterizations of dimension:

Theorem 5.2.3. *For* $n \in \omega$ *and a normal space* X, *the following are equivalent:*

(a) $\dim X \leq n$;
(b) *Every open cover* $\{U_1, \ldots, U_{n+2}\}$ *of* X *has an open refinement* $\mathcal{V}$ *with* $\operatorname{ord} \mathcal{V} \leq n + 1$;
(c) *For each open cover* $\{U_1, \ldots, U_{n+2}\}$ *of* X, *there exists an open cover* $\{V_1, \ldots, V_{n+2}\}$ *of* X *such that* $V_1 \cap \cdots \cap V_{n+2} = \emptyset$ *and* $\operatorname{cl} V_i \subset U_i$ *for each* $i = 1, \ldots, n + 2$;
(d) *For every open cover* $\{U_1, \ldots, U_{n+2}\}$ *of* X, *there exists a closed cover* $\{A_1, \ldots, A_{n+2}\}$ *of* X *such that* $A_1 \cap \cdots \cap A_{n+2} = \emptyset$ *and* $A_i \subset U_i$ *for each* $i = 1, \ldots, n+2$;
(e) *For every* $k \geq n$, *each map* $f : A \to \mathbf{S}^k$ *of any closed set* A *in* X *extends over* X;
(f) *Each map* $f : A \to \mathbf{S}^n$ *of any closed set* A *in* X *extends over* X.

Proof. Consider the following diagram of implications:



The implications (a) ⇒ (b) and (c) ⇒ (b) are obvious. By Lemmas 5.2.1 and 5.2.2, we have (b) ⇒ (c), hence (b) ⇔ (c). The implication (c) ⇒ (d) follows from Lemma 5.2.2 (or, (d) can be obtained by twice using (c)). Lastly, we prove the implications (d) ⇒ (b) ⇒ (f) ⇒ (e) ⇒ (a).

(d) ⇒ (b): In condition (d), note that

$$\{X \setminus A_1, \ldots, X \setminus A_{n+2}\} \in \operatorname{cov}(X).$$

By Lemma 5.2.2, we have a closed cover $\{B_1, \ldots, B_{n+2}\}$ of X such that $B_i \subset X \setminus A_i$ for each $i = 1, \ldots, n + 2$. Observe

$$(X \setminus B_1) \cap \cdots \cap (X \setminus B_{n+2}) = X \setminus (B_1 \cup \cdots \cup B_{n+2}) = \emptyset.$$

For each $i = 1, \dots, n+2$, let $V_i = U_i \setminus B_i \subset U_i$. Since $A_i \subset U_i \cap (X \setminus B_i) = V_i$, we have $\mathcal{V} = \{V_1, \dots, V_{n+2}\} \in \operatorname{cov}(X)$. Moreover, $V_1 \cap \cdots \cap V_{n+2} = \emptyset$, which means $\operatorname{ord}\mathcal{V} \le n+1$.

(b) ⇒ (f): Let Δ^{n+1} be the standard $(n+1)$-simplex and $K = F(\partial\Delta^{n+1})$ (i.e., the simplicial complex consisting of all proper faces of Δ^{n+1}). Then, $|K| = \partial\Delta^{n+1} \approx \mathbf{S}^n$. To extend a given map $f : A \to \mathbf{S}^n$ over X, we consider $\mathbf{S}^n = |K|$. By Theorem 5.1.6(2), $f : A \to |K|$ is extended to a map $\tilde{f} : \operatorname{cl} W \to |K|$, where W is an open neighborhood of A in X. Note that $\operatorname{card} K^{(0)} = n+2$. By (b), X has a finite open cover $\mathcal{V}$ such that $\operatorname{ord}\mathcal{V} \le n+1$ and

$$\mathcal{V} \prec \left\{ \tilde{f}^{-1}(O_K(v)) \cup (X \setminus \operatorname{cl} W) \;\middle|\; v \in K^{(0)} \right\}.$$

We have a function $\varphi : \mathcal{V} \to K^{(0)}$ such that

$$V \subset \tilde{f}^{-1}(O_K(\varphi(V))) \cup (X \setminus \operatorname{cl} W) \quad \text{for each } V \in \mathcal{V},$$

which defines a simplicial map $\varphi : N(\mathcal{V}) \to K$ because every $n+1$ many vertices span a simplex of K and each simplex of $N(\mathcal{V})$ has at most $n+1$ many vertices. Since $\mathcal{V}$ is finite, there is a canonical map $g : X \to |N(\mathcal{V})|$ for $N(\mathcal{V})$ by Theorem 4.9.4. For each $x \in W$, $\tilde{f}(x)$ and $\varphi g(x)$ are contained in the same simplex of K. In fact, let $\tau \in K$ be the carrier of $\tilde{f}(x)$, i.e., $\tilde{f}(x) \in \operatorname{rint}\tau$. Then, for each $V \in \mathcal{V}[x]$,

$$x \in V \cap W \subset \tilde{f}^{-1}(O_K(\varphi(V))),$$

hence $\tilde{f}(x) \in O_K(\varphi(V))$. Thus, we have $\tilde{f}(x) \in \bigcap_{V \in \mathcal{V}[x]} O_K(\varphi(V))$, which implies that $\varphi(V) \in \tau^{(0)}$ for each $V \in \mathcal{V}[x]$, i.e., $\langle \varphi(\mathcal{V}[x]) \rangle \le \tau$. On the other hand, $g(x) \in \langle \mathcal{V}[x] \rangle$, which implies

$$\varphi g(x) \in \varphi(\langle \mathcal{V}[x] \rangle) = \langle \varphi(\mathcal{V}[x]) \rangle \le \tau,$$

so $\varphi g(x), \tilde{f}(x) \in \tau$. Thus, we can define a map $h : X \times \{0\} \cup W \times \mathbf{I} \to |K|$ as follows:

$$h(x,0) = \varphi g(x) \quad \text{for } x \in X \quad \text{and}$$

$$h(x,t) = (1-t)\varphi g(x) + t\tilde{f}(x) \quad \text{for } (x,t) \in W \times \mathbf{I}.$$

Let $k : X \to \mathbf{I}$ be an Urysohn map with $X \setminus W \subset k^{-1}(0)$ and $A \subset k^{-1}(1)$. Then, an extension $f^* : X \to |K|$ of f can be defined by $f^*(x) = h(x,0)$ $(= \varphi g(x))$ for $x \in X \setminus W$ and $f^*(x) = h(x, k(x))$ for $x \in W$.

(f) ⇒ (e): By induction on $k \ge n$, we show that each map $f : A \to \mathbf{S}^{k+1}$ of any closed set A in X extends over X. Let

$$\mathbf{S}^{k+1}_+ = \mathbf{S}^{k+1} \cap (\mathbb{R}^{k+1} \times \mathbb{R}_+) \;\text{ and }\; \mathbf{S}^{k+1}_- = -\mathbf{S}^{k+1}_+,$$

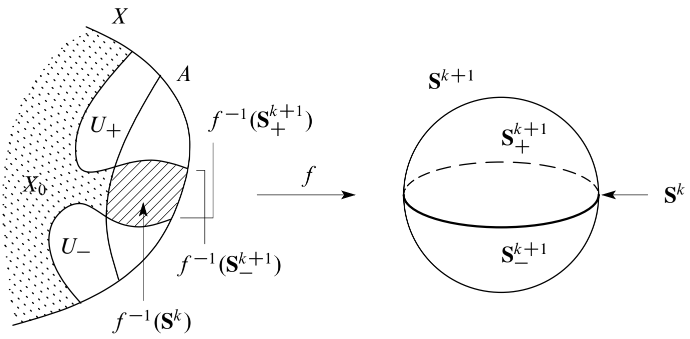


Fig. 5.2 Extending a map $f : A \to \mathbf{S}^{k+1}$

where we identify $\mathbf{S}^k = \mathbf{S}^k \times \{0\} = \mathbf{S}_+^{k+1} \cap \mathbf{S}_-^{k+1} \subset \mathbf{S}^{k+1}$. We have disjoint open sets U_+ and U_- in X such that

$$U_+ \cap A = A \setminus f^{-1}(\mathbf{S}_-^{k+1}) \text{ and } U_- \cap A = A \setminus f^{-1}(\mathbf{S}_+^{k+1}).$$

In fact, by Theorem 5.1.6(2), f extends to a map $f' : U \to \mathbf{S}^{k+1}$ of an open neighborhood of A in X. Then, $U_\pm = f'^{-1}(\mathbf{S}^{k+1} \setminus \mathbf{S}_\mp^{k+1})$ are the desired open sets.

Now, let $X_0 = X \setminus (U_+ \cup U_-)$ and $A_0 = A \cap X_0 = f^{-1}(\mathbf{S}^k)$ (Fig. 5.2). Since $f|A_0 : A_0 \to \mathbf{S}^k$ extends over X by the inductive assumption, $f|A_0$ extends to a map $f_0 : X_0 \to \mathbf{S}^k$. Let

$$X_+ = X_0 \cup U_+ = X \setminus U_- \text{ and } X_- = X_0 \cup U_- = X \setminus U_+,$$

which are closed in X, and hence they are normal. Note that X_0 is closed in both X_+ and X_-. Since $\mathbf{S}_+^{k+1} \approx \mathbf{S}_-^{k+1} \approx \mathbf{B}^{k+1}$, f_0 extends to maps $f_+ : X_+ \to \mathbf{S}_+^{k+1}$ and $f_- : X_- \to \mathbf{S}_-^{k+1}$ by Theorem 5.1.6(1). Then, the desired extension $\tilde{f} : X \to \mathbf{S}^{k+1}$ of f can be defined by $\tilde{f}|X_+ = f_+$ and $\tilde{f}|X_- = f_-$.

(e) $\Rightarrow$ (a): For each finite open cover $\mathcal{U}$ of X, let $K = N(\mathcal{U})$ be the nerve of $\mathcal{U}$ with $f : X \to |K|$ a canonical map (cf. Theorem 4.9.4). If $f(X) \subset |K^{(n)}|$, $f^{-1}(\mathcal{O}_{K^{(n)}}) \in \operatorname{cov}(X)$ is a finite open refinement of $\mathcal{U}$ and

$$\operatorname{ord} f^{-1}(\mathcal{O}_{K^{(n)}}) \leq \operatorname{ord} \mathcal{O}_{K^{(n)}} = \dim K^{(n)} + 1 \leq n + 1.$$

Otherwise, choose $m > n$ so that $f(X) \subset |K^{(m)}|$ but $f(X) \not\subset |K^{(m-1)}|$. Let $\tau_1, \ldots, \tau_k$ be the m-simplexes of K. Since $\partial\tau_i \approx \mathbf{S}^{m-1}$ and $m - 1 \geq n$, we have maps $f_i : X \to \partial\tau_i$ such that $f_i|f^{-1}(\partial\tau_i) = f|f^{-1}(\partial\tau_i)$ by (e). Let $f' : X \to |K|$ be the map defined by

$$f'|f^{-1}(|K^{(m-1)}|) = f|f^{-1}(|K^{(m-1)}|) \text{ and}$$
$$f'|f^{-1}(\tau_i) = f_i|f^{-1}(\tau_i) \text{ for each } i = 1, \ldots, k.$$

Then, $f'(X) \subset |K^{(m-1)}|$. Since $c_K(f'(x)) \leq c_K(f(x)) \leq \langle \mathcal{U}[x] \rangle$ for each $x \in X$, f' is still a canonical map. By the downward induction on $m \geq n$, we can obtain a canonical map $f : X \to |K|$ such that $f(X) \subset |K^{(n)}|$. This completes the proof. □

Remark 2. In the above proof of (e) ⇒ (a), instead of a finite open cover $\mathcal{U}$ of X, let us take a local finite open cover $\mathcal{U}$ whose nerve $K = N(\mathcal{U})$ is *locally finite-dimensional* (l.f.d.). It can be shown that $\mathcal{U}$ has a locally finite open refinement $\mathcal{V}$ with ord $\mathcal{V} \leq n + 1$ (i.e., $\dim N(\mathcal{V}) \leq n$).

Indeed, since K is the nerve of a locally finite open cover, by Theorem 4.9.4, we have a canonical map $f : X \to |K|$ such that each $x \in X$ has a neighborhood V_x in X with $f(V_x) \subset |K_x|$ for some finite subcomplex K_x of K. Note that K might be infinite-dimensional.

Now, consider the following subcomplexes of K:

$$K_i = K \setminus \{\tau \in K \mid \dim \tau > n,\ \tau \text{ is principal in } K_{i-1}\}$$
$$= K^{(n)} \cup \{\tau \in K_{i-1} \mid \tau \text{ is not principal in } K_{i-1}\},\ i \in \mathbb{N},$$

where $K_0 = K$. Then, $K^{(n)} = \bigcap_{i \in \mathbb{N}} K_i$ because K is l.f.d. We will inductively construct canonical maps $f_i : X \to |K|$, $i \in \mathbb{N}$, such that

$$f_i|f_{i-1}^{-1}(|K_i|) = f_{i-1}|f_{i-1}^{-1}(|K_i|),\ f_i(X) \subset |K_i| \text{ and}$$
$$f_i(V_x) \subset |K_x| \text{ for each } x \in X,$$

where $f_0 = f$. Suppose f_{i-1} have been constructed. For each $\tau \in K_{i-1} \setminus K_i$, since $\dim \tau > n$, we can apply (e) to obtain an extension $f_\tau : X \to \partial\tau$ of $f_{i-1}|f_{i-1}^{-1}(\partial\tau)$. We can define $f_i : X \to |K|$ as follows:

$$f_i|f_{i-1}^{-1}(|K_i|) = f_{i-1}|f_{i-1}^{-1}(|K_i|) \text{ and}$$
$$f_i|f_{i-1}^{-1}(\tau) = f_\tau|f_{i-1}^{-1}(\tau) \text{ for each } \tau \in K_{i-1} \setminus K_i.$$

Then, $f_i(X) \subset |K_i|$. Since $f_i(f_{i-1}^{-1}(\tau)) \subset \partial\tau \subset \tau$ for each $\tau \in K_{i-1} \setminus K_i$, it follows that $f_i(V_x) \subset |K_x|$ for each $x \in X$, so f_i is continuous because each K_x is finite. Moreover, $c_K(f_i(x)) \leq c_K(f_{i-1}(x))$ for each $x \in X$, hence f_i is also a canonical map.

For each $x \in X$, since K_x is finite, $K_x^{(n)} = K_x \cap K_{i(x)}$ for some $i(x) \in \mathbb{N}$. For every $i \geq i(x)$, because $K_x \cap K_i = K_x \cap K_{i(x)}$, we have $f_i|V_x = f_{i(x)}|V_x$. Therefore, we can define a map $\tilde{f} : X \to |K^{(n)}|$ by $\tilde{f}|V_x = f_{i(x)}|V_x$ for each $x \in X$. Then, $\mathcal{V} = \tilde{f}^{-1}(\mathcal{O}_{K^{(n)}}) \in \operatorname{cov}(X)$ is an open refinement of $\mathcal{U}$ with ord $\leq n + 1$. By applying Lemma 5.2.1, we can obtain the desired refinement $\mathcal{V}$ of $\mathcal{U}$.

When X is paracompact, since every open cover of X has a locally finite (and σ-discrete) open refinement with the l.f.d. nerve by Theorem 4.9.9, if $\dim X \leq n$, then an arbitrary open cover of X has a (locally finite σ-discrete) open refinement $\mathcal{V}$ with $\operatorname{ord} \mathcal{V} \leq n + 1$ by the above remark. Since the converse obviously holds, we have the following characterization:

Theorem 5.2.4. *For $n \in \omega$ and a paracompact space X, $\dim X \leq n$ if and only if an arbitrary open cover of X has a (locally finite σ-discrete) open refinement $\mathcal{V}$ with $\operatorname{ord} \mathcal{V} \leq n + 1$.* □

Instead of Theorem 4.9.9, we can use Theorem 4.9.10 to obtain the following corollary:

Corollary 5.2.5. *Let X be regular Lindelöf and $n \in \omega$. Then, $\dim X \leq n$ if and only if an arbitrary open cover of X has a (star-finite and countable) open refinement $\mathcal{V}$ with $\operatorname{ord} \mathcal{V} \leq n + 1$.* □

In the proof of Theorem 4.10.10, we can apply Theorem 5.2.4 (Corollary 5.2.5) to obtain $\mathcal{U}_i$ with $\operatorname{ord} \mathcal{U}_i \leq n + 1$, namely $\dim K_i \leq n$. By Remark 16 at the end of Sect. 4.10, we have the following version of Theorem 4.10.10 (Corollaries 4.10.11 and 4.10.12).

Corollary 5.2.6. *Every completely metrizable space X with $\dim X \leq n < \infty$ is homeomorphic to the inverse limit of an inverse sequence $(|K_i|_{\mathrm{m}}, f_i)_{i \in \mathbb{N}}$ of metric polyhedra and PL maps such that $\dim K_i \leq n$, $\operatorname{card} K_i \leq \aleph_0 w(X)$, and $f_i : K_{i+1} \to \operatorname{Sd} K_i$ is simplicial. Moreover, if X is compact metrizable (resp. separable and completely metrizable), then each $|K_i|_{\mathrm{m}} = |K_i|$ is compact (resp. locally compact). If X is separable and locally compact metrizable, each $|K_i|_{\mathrm{m}} = |K_i|$ is locally compact and each f_i is proper.* □

Now, we can prove the following theorem:

Theorem 5.2.7. *For each $n \in \mathbb{N}$, $\dim \mathbf{B}^n = n$.*

Proof. For any $\mathcal{U} \in \operatorname{cov}(\Delta^n)$, Δ^n has a triangulation K such that $\mathcal{O}_K \prec \mathcal{S}_K \prec \mathcal{U}$ (Corollary 4.7.7). Since $\operatorname{ord} \mathcal{O}_K = \dim K + 1 = n + 1$ and $|K| = \Delta^n \approx \mathbf{B}^n$, it follows that $\dim \mathbf{B}^n \leq n$. If $\dim \mathbf{B}^n \leq n - 1$, then we apply Theorem 5.2.3 to obtain a map $r : \mathbf{B}^n \to \mathbf{S}^{n-1}$ such that $r|\mathbf{S}^{n-1} = \mathrm{id}$, which contradicts the No Retraction Theorem 5.1.5. Consequently, we have $\dim \mathbf{B}^n = n$. □

Proposition 5.2.8. *For a normal space X, if there exists a map $f : X \to \mathbf{S}^n$ that is not null-homotopic, then $\dim X \geq n$.*

Proof. Define $\mathbf{S}^n_+$ and $\mathbf{S}^n_-$ as in the proof of Theorem 5.2.3 (f) ⇒ (e) and identify $\mathbf{S}^{n-1} = \mathbf{S}^n_+ \cap \mathbf{S}^n_- \subset \mathbf{S}^n$. If $\dim X \leq n - 1$ then $f|f^{-1}(\mathbf{S}^{n-1})$ extends to a map $f' : X \to \mathbf{S}^{n-1}$ by Theorem 5.2.3. We can define a map $g : X \to \mathbf{S}^n$ as follows:

$$g|f^{-1}(\mathbf{S}^n_+) = f'|f^{-1}(\mathbf{S}^n_+) \text{ and } g|f^{-1}(\mathbf{S}^n_-) = f|f^{-1}(\mathbf{S}^n_-).$$

Then, $g \simeq f$ rel. $f^{-1}(\mathbf{S}^n_-)$. Indeed, we have a homeomorphism $\varphi : \mathbf{S}^n_+ \to \mathbf{B}^n$ with $\varphi|\mathbf{S}^{n-1} = \mathrm{id}$. Then, $\varphi f|f^{-1}(\mathbf{S}^n_+) \simeq \varphi f'|f^{-1}(\mathbf{S}^n_+)$ rel. $f^{-1}(\mathbf{S}^{n-1})$ in $\mathbf{B}^n$, which is realized by the straight-line homotopy. Hence, $f|f^{-1}(\mathbf{S}^n_+) \simeq f'|f^{-1}(\mathbf{S}^n_+)$ rel. $f^{-1}(\mathbf{S}^{n-1})$ in $\mathbf{S}^n_+$, which implies $g \simeq f$ rel. $f^{-1}(\mathbf{S}^n_-)$. Since $g(X) \subset \mathbf{S}^n_- \approx \mathbf{B}^n$, it follows that $f \simeq g \simeq 0$. This is a contradiction. □

Remark 3. The converse of Proposition 5.2.8 does not hold. In fact, if X is an n-dimensional contractible space then every map $f : X \to \mathbf{S}^n$ is null-homotopic.

Using simplicial complexes, we can characterize the dimension of paracompact spaces as follows:

Theorem 5.2.9. *Let X be paracompact and $n \in \omega$. Then,* $\dim X \le n$ *if and only if, for every simplicial complex K, each map $f : X \to |K|$ (or $f : X \to |K|_{\mathrm{m}}$) is contiguous to a map $g : X \to |K^{(n)}|$ (or $g : X \to |K^{(n)}|_{\mathrm{m}}$). In this case, each $g(x)$ is contained in the carrier $c_K(f(x)) \in K$ of $f(x)$.*

Proof. First, we will show the "if" part. Each (finite) open cover $\mathcal{U}$ of X has an open star-refinement $\mathcal{V}$. Let $K = N(\mathcal{V})$ be the nerve of $\mathcal{V}$. A canonical map $f : X \to |K|$ is contiguous to a map $g : X \to |K^{(n)}|$. Then, $g^{-1}(\mathcal{O}_{K^{(n)}}) \in \mathrm{cov}(X)$ with

$$\mathrm{ord}\, g^{-1}(\mathcal{O}_{K^{(n)}}) \le \mathrm{ord}\, \mathcal{O}_{K^{(n)}} = \dim K^{(n)} + 1 \le n + 1.$$

Let $V \in \mathcal{V} = K^{(0)}$ and $x \in g^{-1}(O_{K^{(n)}}(V))$. We have $\sigma \in K$ such that $f(x), g(x) \in \sigma$. Then, $c_K(f(x)) \le \sigma$ and $V \in \sigma^{(0)}$. Since f is canonical, we have $c_K(f(x))^{(0)} \subset \mathcal{V}[x]$ (Proposition 4.9.1). It follows that $V \cap V' \ne \emptyset$ and $x \in V'$ for any $V' \in c_K(f(x))^{(0)}$, which implies $x \in \mathrm{st}(V, \mathcal{V})$. Thus, $g^{-1}(O_{K^{(n)}}(V)) \subset \mathrm{st}(V, \mathcal{V})$, which means $g^{-1}(\mathcal{O}_{K^{(n)}}) \prec \mathcal{U}$. Therefore, $\dim X \le n$.

To prove the "only if" part, let $f : X \to |K|$ be a map. Because $\dim X \le n$, X has an open cover $\mathcal{U} \prec f^{-1}(\mathcal{O}_K)$ with $\mathrm{ord}\,\mathcal{U} \le n + 1$ by Theorem 5.2.4. Let $L = N(\mathcal{U})$ be the nerve of $\mathcal{U}$ with $\varphi : X \to |L|$ a canonical map. Then, we have a function $\psi : L^{(0)} = \mathcal{U} \to K^{(0)}$ such that $U \subset f^{-1}(O_K(\psi(U)))$, i.e., $f(U) \subset O_K(\psi(U))$. By Proposition 4.4.5, $\psi : L^{(0)} \to K^{(0)}$ induces the simplicial map $\psi : L \to K$. Since $\dim L \le n$, it follows that $\psi\varphi(X) \subset \psi(|L|) \subset |K^{(n)}|$. Thus, we have a map $g = \psi\varphi : X \to |K^{(n)}|$.

We will show that $g(x) \in c_K(f(x))$ for every $x \in X$. For each $x \in X$, $\varphi(x) \in \langle \mathcal{U}[x] \rangle \in L$ because φ is canonical. Then, $g(x) = \psi\varphi(x) \in \psi(\langle \mathcal{U}[x] \rangle) \in K$. For each $U \in \mathcal{U}[x]$, $f(x) \in f(U) \subset O_K(\psi(U))$, which means $\psi(U) \in c_K(f(x))^{(0)}$. Hence, $\psi(\langle \mathcal{U}[x] \rangle) \le c_K(f(x))$. Thus, $g(x) \in c_K(f(x))$ for every $x \in X$. □

Remark 4. In the above proof of the "only if" part, when K is *locally finite-dimensional*, we can apply the same argument used in Remark 2 to obtain a map $g : X \to |K^{(n)}|$ contiguous to f.

As a corollary of Theorems 5.2.7 and 5.2.9, we have the following:

Corollary 5.2.10. *For any simplicial complex K,*

$$\dim K = \dim |K| = \dim |K|_{\mathrm{m}}.$$

Proof. An n-simplex $\tau \in K$ is closed in both $|K|$ and $|K|_{\mathrm{m}}$, and $\dim \tau = n$ by Theorem 5.2.7. By the definition of dimension, $\dim |K| \geq \dim \tau$ and $\dim |K|_{\mathrm{m}} \geq \dim \tau$. On the other hand, combining Theorem 5.2.9 with the Simplicial Approximation Theorem 4.7.14, we arrive at $\dim |K| \leq \dim K$ and $\dim |K|_{\mathrm{m}} \leq \dim K$. □

Since the n-dimensional Euclidean space $\mathbb{R}^n$ has an n-dimensional triangulation, we have the following corollary:

Corollary 5.2.11. *For each $n \in \mathbb{N}$, $\dim \mathbb{R}^n = n$.* □

Let A and B be disjoint closed sets in a space X. A closed set C in X is called a **partition** between A and B in X if there exist disjoint open sets U and V in X such that $A \subset U$, $B \subset V$, and $X \setminus C = U \cup V$. A family $(A_\gamma, B_\gamma)_{\gamma \in \Gamma}$ of pairs of disjoint closed sets in X is **inessential** in X if there are partitions L_γ between A_γ and B_γ with $\bigcap_{\gamma \in \Gamma} L_\gamma = \emptyset$. Note that if one of A_γ or B_γ is empty then (A_γ, B_γ) is inessential. If $(A_\gamma, B_\gamma)_{\gamma \in \Gamma}$ is not inessential in X (i.e., $\bigcap_{\gamma \in \Gamma} L_\gamma \neq \emptyset$ for any partitions L_γ between A_γ and B_γ), it is said to be **essential** in X.

A map $f : X \to \mathbf{I}^n$ is said to be **essential** if every map $g : X \to \mathbf{I}^n$ with $g|f^{-1}(\partial \mathbf{I}^n) = f|f^{-1}(\partial \mathbf{I}^n)$ is surjective, where it should be noted that g is also essential. It is said that f is **inessential** if it is not essential, i.e., there is a map $g : X \to \mathbf{I}^n$ such that $g|f^{-1}(\partial \mathbf{I}^n) = f|f^{-1}(\partial \mathbf{I}^n)$ and $g(X) \neq \mathbf{I}^n$. Then, for an inessential map $f : X \to \mathbf{I}^n$, there is a map $g : X \to \partial \mathbf{I}^n$ such that $g|f^{-1}(\partial \mathbf{I}^n) = f|f^{-1}(\partial \mathbf{I}^n)$.

Lemma 5.2.12. *For two maps $f, g : X \to \mathbf{B}^n$, if $f(x) \neq g(x)$ for any $x \in X$, then there is a map $h : X \to \mathbf{S}^{n-1}$ such that $h|f^{-1}(\mathbf{S}^{n-1}) = f|f^{-1}(\mathbf{S}^{n-1})$.*

Proof. In the same way as for the map r in the remark for the No Retraction Theorem 5.1.5, we can obtain a map $h : X \to \mathbf{S}^{n-1}$ such that $f(x) \in \langle h(x), g(x) \rangle$ for each $x \in X$, which implies $h|f^{-1}(\mathbf{S}^{n-1}) = f|f^{-1}(\mathbf{S}^{n-1})$. □

For a map $f : X \to \mathbf{B}^n$ with $f(X) \neq \mathbf{B}^n$, by taking g as a constant map, the following is a special case of Lemma 5.2.12.

Lemma 5.2.13. *If a map $f : X \to \mathbf{B}^n$ is not surjective, then there is a map $h : X \to \mathbf{S}^{n-1}$ such that $h|f^{-1}(\mathbf{S}^{n-1}) = f|f^{-1}(\mathbf{S}^{n-1})$.* □

Proposition 5.2.14. *Let X be a normal space and $h : X \times \mathbf{I} \to \mathbf{I}^n$ be a homotopy such that h_0 is essential and $h(f^{-1}(\partial \mathbf{I}^n) \times \mathbf{I}) \subset \partial \mathbf{I}^n$. Then, h_1 is also essential, hence it is surjective.*

Proof. Let $h_0 = f$ and assume that h_1 is inessential. By Lemma 5.2.13, there is a map $g : X \to \partial \mathbf{I}^n$ such that $g|h_1^{-1}(\partial \mathbf{I}^n) = h_1|h_1^{-1}(\partial \mathbf{I}^n)$. Then, $f^{-1}(\partial \mathbf{I}^n) \subset h_1^{-1}(\partial \mathbf{I}^n)$ and $h_1 \simeq g$ rel. $h_1^{-1}(\partial \mathbf{I}^n)$ by the straight-line homotopy:

$$(1-t)h_1(x) + tg(x) \quad \text{for each } (x,t) \in X \times \mathbf{I}.$$

Connecting this to h, we obtain a homotopy $\varphi : X \times \mathbf{I} \to \mathbf{I}^n$ such that

$$\varphi(f^{-1}(\partial \mathbf{I}^n) \times I) \subset \partial \mathbf{I}^n, \ \varphi_0 = f \ \text{ and } \ \varphi_1 = g.$$

Then, $A = \mathrm{pr}_X(\varphi^{-1}([\frac{1}{3}, \frac{2}{3}]^n))$ is a closed set in X. Observe

$$\varphi^{-1}([\tfrac{1}{3}, \tfrac{2}{3}]^n) \cap (f^{-1}(\partial \mathbf{I}^n) \times \mathbf{I}) = \emptyset,$$

which implies $A \cap f^{-1}(\partial \mathbf{I}^n) = \emptyset$. Taking an Urysohn map $k : X \to \mathbf{I}$ with $k(f^{-1}(\partial \mathbf{I}^n)) = 0$ and $k(A) = 1$, we define a map $g' : X \to \mathbf{I}^n$ as follows:

$$g'(x) = \varphi(x, k(x)) \ \text{ for each } x \in X.$$

Then, $g'|f^{-1}(\partial \mathbf{I}^n) = f|f^{-1}(\partial \mathbf{I}^n)$ but $g'(X) \neq \mathbf{I}^n$. In fact, $g'(A) = g(A) \subset \partial \mathbf{I}^n$ and

$$g'(X \setminus A) \subset \varphi((X \setminus A) \times \mathbf{I}) \subset \varphi\big((X \times \mathbf{I}) \setminus \varphi^{-1}([\tfrac{1}{3}, \tfrac{2}{3}]^n)\big) \subset \mathbf{I}^n \setminus [\tfrac{1}{3}, \tfrac{2}{3}]^n.$$

This is a contradiction because $h_0 = f$ is essential. □

Essential maps can be characterized as follows:

Theorem 5.2.15. *Let X be a normal space. For a map $f : X \to \mathbf{I}^n$, the following are equivalent:*

(a) *f is essential;*
(b) *For each map $g : X \to \mathbf{I}^n$, there is some $x \in X$ such that $f(x) = g(x)$;*
(c) *$(f^{-1}(\mathrm{pr}_i^{-1}(0)), f^{-1}(\mathrm{pr}_i^{-1}(1)))_{i=1}^n$ is essential in X.*

Proof. The implication (a) ⇒ (b) follows from Lemma 5.2.12.

(b) ⇒ (c): Assume that $(f^{-1}(\mathrm{pr}_i^{-1}(0)), f^{-1}(\mathrm{pr}_i^{-1}(1)))_{i=1}^n$ is inessential, that is, there are partitions L_i between $f^{-1}(\mathrm{pr}_i^{-1}(0))$ and $f^{-1}(\mathrm{pr}_i^{-1}(1))$ such that $\bigcap_{i=1}^n L_i = \emptyset$. Then, we have disjoint open sets U_i and V_i in X such that

$$X \setminus L_i = U_i \cup V_i, \ f^{-1}(\mathrm{pr}_i^{-1}(0)) \subset U_i \ \text{ and } \ f^{-1}(\mathrm{pr}_i^{-1}(1)) \subset V_i.$$

Applying Lemma 5.2.2 to the open cover $\{X \setminus L_i \mid i = 1, \dots, n\}$ of X, we have a closed cover $\{F_i \mid i = 1, \dots, n\}$ of X such that $F_i \subset X \setminus L_i = U_i \cup V_i$, where we may assume that

$$f^{-1}(\mathrm{pr}_i^{-1}(0)) \cup f^{-1}(\mathrm{pr}_i^{-1}(1)) \subset F_i.$$

Each $U_i \cap F_i = F_i \setminus V_i$ and $V_i \cap F_i = F_i \setminus U_i$ are disjoint closed sets in X. Using Urysohn maps for $U_i \cap F_i$ and $V_i \cap F_i$, we can define a map $g : X \to \mathbf{I}^n$ such that $\mathrm{pr}_i g(U_i \cap F_i) = 1$ and $\mathrm{pr}_i g(V_i \cap F_i) = 0$. Observe

$$\bigcup_{i=1}^{n}(U_i \cap F_i) \cup (V_i \cap F_i) = \bigcup_{i=1}^{n}(U_i \cup V_i) \cap F_i = \bigcup_{i=1}^{n} F_i = X,$$

$$(\mathrm{pr}_i f)^{-1}(1) \subset V_i \subset X \setminus U_i \ \text{ and } \ (\mathrm{pr}_i f)^{-1}(0) \subset U_i \subset X \setminus V_i.$$

It follows that $g(x) \neq f(x)$ for any $x \in X$.

(c) ⇒ (a): Suppose that f is inessential. Then, there is a map $h : X \to \partial\mathbf{I}^n$ with $h|f^{-1}(\partial\mathbf{I}^n) = f|f^{-1}(\partial\mathbf{I}^n)$ by Lemma 5.2.13. Note that

$$f^{-1}(\mathrm{pr}_i^{-1}(0)) \subset h^{-1}(\mathrm{pr}_i^{-1}(0)) \ \text{ and } \ f^{-1}(\mathrm{pr}_i^{-1}(1)) \subset h^{-1}(\mathrm{pr}_i^{-1}(1)).$$

Each $h^{-1}(\mathrm{pr}_i^{-1}(\frac{1}{2}))$ is a partition between $f^{-1}(\mathrm{pr}_i^{-1}(0))$ and $f^{-1}(\mathrm{pr}_i^{-1}(1))$, and then

$$\bigcap_{i=1}^{n} h^{-1}(\mathrm{pr}_i^{-1}(\tfrac{1}{2})) = h^{-1}(\tfrac{1}{2}, \ldots, \tfrac{1}{2}) = \emptyset.$$

Thus, $(f^{-1}(\mathrm{pr}_i^{-1}(0)), f^{-1}(\mathrm{pr}_i^{-1}(1)))_{i=1}^{n}$ is inessential. □

The Brouwer Fixed Point Theorem 5.1.1 means that the identity map of $\mathbf{I}^n$ satisfies condition (b) in Theorem 5.2.15, hence we have the following corollary:

Corollary 5.2.16. *The family* $(\mathrm{pr}_i^{-1}(0), \mathrm{pr}_i^{-1}(1))_{i=1}^{n}$ *is essential in* $\mathbf{I}^n$. □

Remark 5. Due to Theorem 5.2.15, this Corollary 5.2.16 is equivalent to the Brouwer Fixed Point Theorem 5.1.1.

Using essential families and essential maps, we can also characterize dimension as follows:

Theorem 5.2.17 (Eilenberg–Otto; Alexandroff). *Let X be a normal space and $n \in \mathbb{N}$. Then, the following are equivalent:*

(a) $\dim X \geq n$;
(b) *X has an essential map $f : X \to \mathbf{I}^n$;*
(c) *X has an essential family of n pairs of disjoint closed sets.*

Proof. The implication (b) ⇒ (c) follows from Theorem 5.2.15. For an essential map $f : X \to \mathbf{I}^n$, $f|f^{-1}(\partial\mathbf{I}^n) : f^{-1}(\partial\mathbf{I}^n) \to \partial\mathbf{I}^n$ cannot extend to any map from X to $\partial\mathbf{I}^n$, which means $\dim X \geq n$ by Theorem 5.2.3. Thus, we have also (b) ⇒ (a). The implications (a) ⇒ (b) and (c) ⇒ (b) remain to be proved.

(a) ⇒ (b): By Theorem 5.2.3, there exists a map $f' : A \to \partial\mathbf{I}^n$ of a closed set A in X that cannot extend over X. Nevertheless, f' can be extended to a map $f : X \to \mathbf{I}^n$ by Theorem 5.1.6(1). If there is a map $g : X \to \mathbf{I}^n$ such that $g(x) \neq f(x)$ for any $x \in X$, then we have a map $h : X \to \partial\mathbf{I}^n$ such that $h|f^{-1}(\partial\mathbf{I}^n) = f^{-1}|f^{-1}(\partial\mathbf{I}^n)$ by Lemma 5.2.12. This is a contradiction because h is an extension of f'. Therefore, for each map $g : X \to \mathbf{I}^n$, there is some $x \in X$ such that $f(x) = g(x)$. By Theorem 5.2.15, f is essential.

(c) ⇒ (b): Let $(A_i, B_i)_{i=1}^n$ be an essential family of n pairs of disjoint closed sets in X. Using Urysohn maps for A_i and B_i, we can define a map $f : X \to \mathbf{I}^n$ so that $\mathrm{pr}_i f(A_i) = 0$ and $\mathrm{pr}_i f(B_i) = 1$ for each $i = 1, \dots, n$. Since $A_i \subset f^{-1}(\mathrm{pr}_i^{-1}(0))$ and $B_i \subset f^{-1}(\mathrm{pr}_i^{-1}(1))$, it follows that $(f^{-1}(\mathrm{pr}_i^{-1}(0)), f^{-1}(\mathrm{pr}_i^{-1}(1)))_{i=1}^n$ is essential, which means that f is essential according to Theorem 5.2.15. □

Conditions (b) and (c) are called the ALEXANDROFF CHARACTERIZATION the EILENBERG–OTTO CHARACTERIZATION of dimension. Using Theorem 5.2.17, we can easily show the following corollary:

Corollary 5.2.18. *Every non-degenerate* 0*-dimensional normal space is disconnected. Equivalently, every non-degenerate connected normal space is positive dimensional.* □

5.3 Dimension of Metrizable Spaces

In this section, we will give characterizations of dimension for metrizable spaces. For metric spaces, the following characterization can be established:

Theorem 5.3.1. *Let $X = (X, d)$ be a metric space. Then,* $\dim X \leq n$ *if and only if X has a sequence $\mathcal{U}_1 \succ \mathcal{U}_2 \succ \cdots$ of (locally finite σ-discrete) open covers such that* $\operatorname{ord} \mathcal{U}_i \leq n + 1$ *and* $\lim_{i\to\infty} \operatorname{mesh} \mathcal{U}_i = 0$.

Proof. When $\dim X \leq n$, using the "only if" of Theorem 5.2.4, we can inductively construct locally finite σ-discrete open covers $\mathcal{U}_1 \succ \mathcal{U}_2 \succ \cdots$ of X such that $\operatorname{ord} \mathcal{U}_i \leq n + 1$ and $\lim_{i\to\infty} \operatorname{mesh} \mathcal{U}_i = 0$. Thus, the "only if" part holds.

To show the "if" part, let $\mathcal{W}$ be a finite open cover of X. We have a function $\varphi_i : \mathcal{U}_{i+1} \to \mathcal{U}_i$ such that $U \subset \varphi_i(U)$ for each $U \in \mathcal{U}_{i+1}$. For each $j > i$, let $\varphi_{i,j} = \varphi_i \circ \cdots \circ \varphi_{j-1} : \mathcal{U}_j \to \mathcal{U}_i$ and $\varphi_{i,i} = \mathrm{id}_{\mathcal{U}_i}$.

For each $i \in \mathbb{N}$, let

$$X_i = \bigcup \{ U \in \mathcal{U}_i \mid \mathrm{st}(U, \mathcal{U}_i) \text{ is contained in some } W \in \mathcal{W} \}.$$

Then, $X_1 \subset X_2 \subset \cdots$ and $X = \bigcup_{i\in\mathbb{N}} X_i$ because $\lim_{i\to\infty} \operatorname{mesh} \mathcal{U}_i = 0$. Moreover, let $\mathcal{U}_i' = \mathcal{U}_i[X_i]$ and $\mathcal{U}_i'' = \mathcal{U}_i' \setminus \mathcal{U}_i[X_{i-1}]$, where $X_0 = \emptyset$.

For each $i \in \mathbb{N}$ and $U \in \mathcal{U}_i'$, we define

$$k_i(U) = \min \{ k \leq i \mid \varphi_{k,i}(U) \cap X_k \neq \emptyset \}.$$

Observe that $\varphi_{k_i(U),i}(U) \in \mathcal{U}_{k_i(U)}''$ and $k_{k_i(U)}(\varphi_{k_i(U),i}(U)) = k_i(U)$. As is easily seen, $\mathcal{U}_i'' \cap \mathcal{U}_j'' = \emptyset$ if $i \neq j$. For each $U \in \bigcup_{i\in\mathbb{N}} \mathcal{U}_i''$, there is a unique $j(U) \in \mathbb{N}$ such that $U \in \mathcal{U}_{j(U)}''$. Then, we can define

$$U^* = \bigcup \{ U' \cap X_i \mid U' \in \mathcal{U}_i',\ i \geq j(U) = k_i(U'),\ \varphi_{j(U),i}(U') = U \} \subset U.$$

Note that if $k_{j(U)}(U) < j(U)$ then $U^* = \emptyset$.

Each $x \in X$ is contained in some X_i, hence $x \in U' \cap X_i$ for some $U' \in \mathcal{U}'_i$. Let $U = \varphi_{k_i(U'),i}(U') \in \mathcal{U}''_{k_i(U')}$. Then, $k_i(U') = j(U)$ and $x \in U' \cap X_i \subset U^*$. Thus, we have

$$\mathcal{V} = \{U^* \mid U \in \bigcup_{i\in\mathbb{N}} \mathcal{U}''_i\} \in \operatorname{cov}(X).$$

Each $U \in \mathcal{U}''_i$ meets X_i, hence it meets some $U' \in \mathcal{U}_i$ such that $\operatorname{st}(U', \mathcal{U}_i)$ is contained in some $W \in \mathcal{W}$. Then, $U^* \subset U \subset \operatorname{st}(U', \mathcal{U}_i) \subset W$. Therefore, $\mathcal{V} \prec \mathcal{W}$.

For each $x \in X$, choose $k \in \mathbb{N}$ so that $x \in X_k \setminus X_{k-1}$. For each $U^* \in \mathcal{V}[x]$, we can find $U' \in \mathcal{U}'_i$ such that $i \geq j(U) = k_i(U')$, $\varphi_{j(U),i}(U') = U$, and $x \in U' \cap X_i$. Then, $k \leq i$ because $x \in X_i$ and $x \notin X_{k-1}$. Thus, we have $\varphi_{k,i}(U') \in \mathcal{U}_k[x]$. On the other hand, $j(U) \leq k$ because $U \cap X_k \neq \emptyset$ and $U \cap X_{j(U)-1} = \emptyset$. Then, $\varphi_{j(U),k}\varphi_{k,i}(U') = \varphi_{j(U),i}(U') = U$. This means that $\mathcal{V}[x] \ni U^* \mapsto \varphi_{k,i}(U') \in \mathcal{U}_k[x]$ is a well-defined injection. Therefore,

$$\operatorname{card} \mathcal{V}[x] \leq \operatorname{card} \mathcal{U}_k[x] \leq \operatorname{ord} \mathcal{U}_k \leq n + 1.$$

The proof is complete. □

Applying Theorem 5.3.1, we can show that the inverse limit preserves the dimension.

Theorem 5.3.2. *Let $X = \varprojlim(X_i, f_i)$ be the inverse limit of an inverse sequence $(X_i, f_i)_{i\in\mathbb{N}}$ of metrizable spaces. If $\dim X_i \leq n$ for infinitely many $i \in \mathbb{N}$ then $\dim X \leq n$.*

Proof. By Corollary 4.10.4, we may assume that $\dim X_i \leq n$ for every $i \in \mathbb{N}$. Recall that X is the following subspace of the product space $\prod_{i\in\mathbb{N}} X_i$:

$$X = \{x \in \prod_{i\in\mathbb{N}} X_i \mid x(i) = f_i(x(i+1)) \text{ for every } i \in \mathbb{N}\}.$$

We define $d \in \operatorname{Metr}(X)$ as follows:

$$d(x, y) = \sup_{i\in\mathbb{N}} \min\{d_i(x(i), y(i)), 2^{-i}\},$$

where $d_i \in \operatorname{Metr}(X_i)$. For each $i \in \mathbb{N}$, we can inductively choose $\mathcal{V}_i \in \operatorname{cov}(X_i)$ so that $\operatorname{ord} \mathcal{V}_i \leq n + 1$,

$$\mathcal{V}_i \prec (f_j \ldots f_{i-1})^{-1}(\mathcal{V}_j) \text{ and } \operatorname{mesh} f_j \ldots f_{i-1}(\mathcal{V}_i) < 2^{-i} \text{ for } j < i.$$

Let $\mathcal{U}_i = p_i^{-1}(\mathcal{V}_i) \in \operatorname{cov}(X)$, where $p_i = \operatorname{pr}_i|X : X \to X_i$ is the inverse limit projection. Then, $\mathcal{U}_1 \succ \mathcal{U}_2 \succ \cdots$, $\operatorname{ord} \mathcal{U}_i \leq n + 1$, and $\operatorname{mesh} \mathcal{U}_i < 2^{-i}$. Therefore, $\dim X \leq n$ by Theorem 5.3.1. □

The following is obvious by definition:

- If Y is a *closed* set in X then $\dim Y \leq \dim X$.

There exists a 0-dimensional compact space X that contains a subspace Y with $\dim Y > 0$. Such a space will be constructed in Sect. 5.5 (cf. Theorem 5.5.3). However, when X is metrizable, we have the following theorem as a corollary of Theorem 5.3.1.

Theorem 5.3.3 (SUBSET THEOREM). *For every subset Y of a metrizable space X,* $\dim Y \leq \dim X$. □

We can apply Theorem 5.3.1 to prove the following completion theorem:

Theorem 5.3.4. *Every n-dimensional metrizable space X can be embedded in an n-dimensional completely metrizable space as a dense set.*

Proof. We can regard X as a dense subset of a complete metric space $Y = (Y, d)$ (Corollary 2.3.10). Applying Theorem 5.3.1, we can obtain a sequence $\mathcal{U}_1 \succ \mathcal{U}_2 \succ \cdots \in \operatorname{cov}(X)$ such that $\operatorname{ord} \mathcal{U}_i \leq n+1$ and $\operatorname{mesh}_d \mathcal{U}_i \to 0$ as $i \to \infty$. For each $i \in \mathbb{N}$, there is a collection $\tilde{\mathcal{U}}_i$ of open sets in Y such that $\tilde{\mathcal{U}}_i|X = \mathcal{U}_i$. Then, $\operatorname{ord} \tilde{\mathcal{U}}_i = \operatorname{ord} \mathcal{U}_i \leq n+1$ and $\operatorname{mesh}_d \tilde{\mathcal{U}}_i = \operatorname{mesh}_d \mathcal{U}_i$. For each $i \in \mathbb{N}$, $G_i = \bigcup \tilde{\mathcal{U}}_i$ is an open set in Y. Thus, we have a G_δ-set $\tilde{X} = \bigcap_{i\in\mathbb{N}} G_i$ in Y and X is dense in $\tilde{X}$. According to Theorem 2.5.3(2), $\tilde{X}$ is completely metrizable. Moreover, $\dim \tilde{X} \leq n$ by Theorem 5.3.1 and $\dim \tilde{X} \geq n$ by Theorem 5.3.3. Consequently, we have $\dim \tilde{X} = n$. □

A subset of a space X is called a **clopen set** in X if it is both closed and open in X. A **clopen basis** for X is an open basis consisting of clopen sets. For metrizable spaces, we characterize the 0-dimensionality as follows:

Theorem 5.3.5. *For a metrizable space X ($\neq \emptyset$),* $\dim X = 0$ *if and only if X has a σ-locally finite clopen basis.*

Proof. First, assume that $\dim X = 0$ and let $d \in \operatorname{Metr}(X)$. By Theorem 5.3.1, X has a sequence of locally finite open covers $\mathcal{B}_1 \succ \mathcal{B}_2 \succ \cdots$ such that $\operatorname{ord} \mathcal{B}_i = 1$ and $\lim_{i\to\infty} \operatorname{mesh} \mathcal{B}_i = 0$. Note that each $B \in \mathcal{B}_i$ is clopen in X because $B = X \setminus \bigcup\{B' \in \mathcal{B}_i \mid B' \neq B\}$. It is easy to see that $\mathcal{B} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i$ is a σ-locally finite clopen basis for X.

To show the "if" part, let $\mathcal{B} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i$ be a σ-locally finite clopen basis for X, where each $\mathcal{B}_i$ is locally finite. Let $\{U_1, U_2\} \in \operatorname{cov}(X)$. For each $i \in \mathbb{N}$, let

$$V_{2i-1} = \bigcup \{B \in \mathcal{B}_i \mid B \subset U_1\} \text{ and } V_{2i} = \bigcup \{B \in \mathcal{B}_i \mid B \subset U_2\}.$$

Because V_i is clopen, we have an open set $W_i = V_i \setminus \bigcup_{j<i} V_j$ in X. Then, $\mathcal{W} = \{W_i \mid i \in \mathbb{N}\}$ is an open refinement of $\{U_1, U_2\}$. Indeed, $W_{2i-1} \subset U_1$, $W_{2i} \subset U_2$, and

$$\bigcup_{i\in\mathbb{N}} W_i = \bigcup_{i\in\mathbb{N}} V_i = \bigcup_{i\in\mathbb{N}} V_{2i-1} \cup \bigcup_{i\in\mathbb{N}} V_{2i} = U_1 \cup U_2 = X.$$

Since $\operatorname{ord} \mathcal{W} \leq 1$ by definition, we have $\dim X \leq 0$ by Theorem 5.2.3. □

Using the above characterization, we can easily show that $\dim \mathbb{Q} = \dim(\mathbb{R} \setminus \mathbb{Q}) = 0$ and $\dim \mu^0 = 0$, where μ^0 is the Cantor (ternary) set. The following theorem can also be easily proved by applying this characterization.

Theorem 5.3.6. *The countable product of* 0*-dimensional metrizable spaces is* 0*-dimensional.* □

The following simple lemma is very useful in Dimension Theory.

Lemma 5.3.7 (Partition Extension). *Let A, B be closed and U, V be open sets in a metrizable space X such that $A \subset U$ and $B \subset V$ and $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$. For any subspace Y of X, if Y has a partition S between $Y \cap \operatorname{cl} U$ and $Y \cap \operatorname{cl} V$, then X has a partition L between A and B with $Y \cap L \subset S$.*

Proof. Let U' and V' be disjoint open sets in Y such that $Y \setminus S = U' \cup V'$, $Y \cap \operatorname{cl} U \subset U'$, and $Y \cap \operatorname{cl} V \subset V'$. From $U \cap V' = \emptyset$, it follows that $A \cap \operatorname{cl} V' = \emptyset$. Then,

$$(A \cup U') \cap \operatorname{cl}(B \cup V') = (A \cup U') \cap (B \cup \operatorname{cl} V') = \emptyset.$$

Similarly, we have $(B \cup V') \cap \operatorname{cl}(A \cup U') = \emptyset$. Let $d \in \operatorname{Metr}(X)$ and define

$$U'' = \{x \in X \mid d(x, A \cup U') < d(x, B \cup V')\} \quad \text{and}$$
$$V'' = \{x \in X \mid d(x, B \cup V') < d(x, A \cup U')\}.$$

Then, U'' and V'' are disjoint open sets in X, $A \cup U' \subset U''$, and $B \cup V' \subset V''$. Hence, $L = X \setminus (U'' \cup V'')$ is the desired partition. □

Note that it does not suffice to assume that S is a partition between $A \cap Y$ and $B \cap Y$ in Y. In fact, $A = [-1, 0] \times \{0\}$ and $B = [0, 1] \times \{1\}$ are disjoint closed sets in $X = \mathbb{R}^2$. Let

$$Y = \mathbb{R}^2 \setminus (\mathbb{Q} \times \mathbf{2}) \subset X,$$

where $\mathbf{2} = \{0, 1\}$ is the discrete space of two points. Then, $S = \{0\} \times \mathbb{R}$ is a partition between $A \cap Y$ and $B \cap Y$ in Y but X has no partition L between A and B such that $Y \cap L \subset S$.

Using partitions, we can characterize the dimension for metrizable spaces as in the following theorem:

Theorem 5.3.8. *Let X be metrizable and $n \in \omega$. Then, $\dim X \leq n$ if and only if, for any pair of disjoint closed sets A and B in X, there is a partition L in X between A and B with $\dim L \leq n - 1$.*

Proof. To prove the "if" part, let $(A_i, B_i)_{i=1}^{n+1}$ be a family of pairs of disjoint closed sets in X. Let L_{n+1} be a partition between A_{n+1} and B_{n+1} with $\dim L_{n+1} \leq n - 1$. For each $i = 1, \ldots, n$, let U_i and V_i be open sets in X such that $A_i \subset U_i$, $B_i \subset V_i$ and $\operatorname{cl} U_i \cap \operatorname{cl} V_i = \emptyset$. By Theorem 5.2.17, L_{n+1} has partitions S_i between $L_{n+1} \cap \operatorname{cl} U_i$ and $L_{n+1} \cap \operatorname{cl} V_i$ such that $\bigcap_{i=1}^n S_i = \emptyset$. By the Partition Extension Lemma 5.3.7, X has partitions L_i between A_i and B_i such that $L_i \cap L_{n+1} \subset S_i$.

Then, $\bigcap_{i=1}^{n+1} L_i \subset \bigcap_{i=1}^{n} S_i = \emptyset$. Therefore, $(A_i, B_i)_{i=1}^{n+1}$ is inessential. Thus, we have $\dim X \leq n$ by Theorem 5.2.17.

To show the "only if" part, let $d \in \mathrm{Metr}(X)$ such that $\mathrm{dist}(A, B) > 1$. (Such a metric can be obtained by a metric for X and an Urysohn map for A and B.) Using Theorem 5.2.4 (cf. Theorem 5.3.1), we can construct a sequence $(\mathcal{U}_i)_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\mathrm{ord}\,\mathcal{U}_i \leq n+1$, $\mathrm{mesh}\,\mathcal{U}_i < i^{-1}$, and $\mathcal{U}_{i+1}^{\mathrm{cl}} \prec \mathcal{U}_i$. Let A_0 and B_0 be open neighborhoods of A and B in X, respectively, such that $\mathrm{dist}(A_0, B_0) > 1$. We inductively define sets A_i and B_i ($i \in \mathbb{N}$) as follows:

$$A_i = X \setminus \bigcup \left\{ \mathrm{cl}\, U \;\middle|\; U \in \mathcal{U}_i[B_{i-1}] \right\} \quad \text{and}$$

$$B_i = X \setminus \bigcup \left\{ \mathrm{cl}\, U \;\middle|\; U \in \mathcal{U}_i \setminus \mathcal{U}_i[B_{i-1}] \right\}.$$

Then, $A_i \cap B_i = \emptyset$. Because of the local finiteness of $\mathcal{U}_i^{\mathrm{cl}}$, A_i and B_i are open in X. Since $B_{i-1} \cap U = \emptyset$ if and only if $B_{i-1} \cap \mathrm{cl}\, U = \emptyset$ for each $U \in \mathcal{U}_i$, it follows that $B_{i-1} \subset B_i$ for each $i \in \mathbb{N}$. Then, $\mathcal{U}_i[B_{i-1}] \subset \mathcal{U}_i[B_i]$. We also have $\mathcal{U}_i[B_i] \subset \mathcal{U}_i[B_{i-1}]$. Indeed, each $U \in \mathcal{U}_i[B_i]$ contains some point of B_i, where that point does not belong to any member of $\mathcal{U}_i \setminus \mathcal{U}_i[B_{i-1}]$. This means $U \in \mathcal{U}_i[B_{i-1}]$. Therefore, $\mathcal{U}_i[B_i] = \mathcal{U}_i[B_{i-1}]$ for each $i \in \mathbb{N}$.

We will show that $\mathrm{cl}\, A_{i-1} \subset A_i$ for each $i \in \mathbb{N}$. This follows from the fact that $\mathrm{cl}\, A_{i-1} \cap \mathrm{cl}\, U = \emptyset$ for each $U \in \mathcal{U}_i[B_{i-1}]$. This fact can be shown as follows: The case $i = 1$ follows from $\mathrm{mesh}\,\mathcal{U}_1 < 1$ and $\mathrm{dist}(A_0, B_0) > 1$. When $i > 1$, for each $U \in \mathcal{U}_i[B_{i-1}]$, $\mathrm{cl}\, U$ is contained in some $V \in \mathcal{U}_{i-1}$. Since $V \in \mathcal{U}_{i-1}[B_{i-1}] = \mathcal{U}_{i-1}[B_{i-2}]$, it follows that $A_{i-1} \cap V = \emptyset$, and hence $\mathrm{cl}\, A_{i-1} \cap V = \emptyset$, which implies $\mathrm{cl}\, A_{i-1} \cap \mathrm{cl}\, U = \emptyset$.

For each $i \in \mathbb{N}$, let $L_i = X \setminus (A_i \cup B_i)$ and let $L = \bigcap_{i\in\mathbb{N}} L_i$. Then, L is a partition between A and B. Indeed, $X \setminus L = \left(\bigcup_{i\in\mathbb{N}} A_i\right) \cup \left(\bigcup_{i\in\mathbb{N}} B_i\right)$, $A \subset \bigcup_{i\in\mathbb{N}} A_i$, $B \subset \bigcup_{i\in\mathbb{N}} B_i$, and

$$\left(\bigcup_{i\in\mathbb{N}} A_i\right) \cap \left(\bigcup_{i\in\mathbb{N}} B_i\right) = \bigcup_{i,j\in\mathbb{N}} (A_i \cap B_j) = \bigcup_{i,j\in\mathbb{N}} (A_{\max\{i,j\}} \cap B_{\max\{i,j\}})$$
$$= \bigcup_{i\in\mathbb{N}} (A_i \cap B_i) = \emptyset.$$

For each $i \in \mathbb{N}$, we have

$$\mathcal{W}_i = \left\{ U \cap L \;\middle|\; U \in \mathcal{U}_i[B_{i-1}] \right\} \in \mathrm{cov}(L).$$

Indeed, each $x \in L$ is not contained in A_{i+1}, so $x \in \mathrm{cl}\, V$ for some $V \in \mathcal{U}_{i+1}[B_i]$. Choose $U \in \mathcal{U}_i$ so that $\mathrm{cl}\, V \subset U$. Then, $U \in \mathcal{U}_i[B_i] = \mathcal{U}_i[B_{i-1}]$, hence $x \in U \cap L \in \mathcal{W}_i$. Therefore, $\mathcal{W}_i \in \mathrm{cov}(L)$. Note that $\mathrm{mesh}\,\mathcal{W}_i \leq \mathrm{mesh}\,\mathcal{U}_i < i^{-1}$. Moreover, $\mathcal{W}_{i+1} \prec \mathcal{W}_i$ because each $V \in \mathcal{U}_{i+1}[B_i]$ is contained in some $U \in \mathcal{U}_i$, where $U \in \mathcal{U}_i[B_i] = \mathcal{U}_i[B_{i-1}]$.

We will show that $\operatorname{ord} \mathcal{W}_i \leq n$. Suppose that there are $n + 1$ many distinct $U_1, \ldots, U_{n+1} \in \mathcal{U}_i[B_{i-1}]$ that contain a common point $x \in L$. Since $x \notin B_i$, $x \in \operatorname{cl} U_{n+2}$ for some $U_{n+2} \in \mathcal{U}_i \setminus \mathcal{U}_i[B_{i-1}]$. Since $\bigcap_{j=1}^{n+1} U_j$ is a neighborhood of x, it follows that $\bigcap_{j=1}^{n+2} U_j \neq \emptyset$, which is contrary to $\operatorname{ord} \mathcal{U}_i \leq n + 1$. Therefore, we have $\dim L \leq n - 1$ by Theorem 5.3.1 □

Remark 6. It should be noted that the Partition Extension Lemma 5.3.7 and the "if" part of Theorem 5.3.8 are valid for completely normal (= hereditarily normal) spaces (cf. Sect. 2.2).

5.4 Fundamental Theorems on Dimension

In this section, we prove several fundamental theorems on dimension. We begin with two types of sum theorem.

Theorem 5.4.1 (Countable Sum Theorem). *Let $X = \bigcup_{i \in \mathbb{N}} F_i$ be normal and $n \in \omega$, where each F_i is closed in X. If* $\dim F_i \leq n$ *for every $i \in \mathbb{N}$, then* $\dim X \leq n$.

Proof. It suffices to show the case $n < \infty$. Let $\{U_1, \ldots, U_{n+2}\} \in \operatorname{cov}(X)$. By induction on $i \in \mathbb{N}$, we can define $\mathcal{U}_i = \{U_{i,1}, \ldots, U_{i,n+2}\} \in \operatorname{cov}(X)$ so that

$$\operatorname{cl} U_{i,j} \subset U_{i-1,j} \quad \text{and} \quad U_{i,1} \cap \cdots \cap U_{i,n+2} \cap F_i = \emptyset,$$

where $U_{0,j} = U_j$. Indeed, assume that $\mathcal{U}_{i-1}$ has been defined, where $F_0 = \emptyset$. By Theorem 5.2.3, we have $\{V_{i,1}, \ldots, V_{i,n+2}\} \in \operatorname{cov}(F_i)$ such that

$$V_{i,j} \subset U_{i-1,j} \quad \text{and} \quad V_{i,1} \cap \cdots \cap V_{i,n+2} = \emptyset.$$

Let $W_{i,j} = V_{i,j} \cup (U_{i-1,j} \setminus F_i)$. Then, $\{W_{i,1}, \ldots, W_{i,n+2}\} \in \operatorname{cov}(X)$. By normality, we can find $\mathcal{U}_i = \{U_{i,1}, \ldots, U_{i,n+2}\} \in \operatorname{cov}(X)$ such that $\operatorname{cl} U_{i,j} \subset W_{i,j}$. Observe that $\mathcal{U}_i$ is as desired.

For each $j = 1, \ldots, n+2$, let $A_j = \bigcap_{i \in \mathbb{N}} U_{i,j}$. Observe that $A_j = \bigcap_{i \in \mathbb{N}} \operatorname{cl} U_{i,j}$ is closed in X. Since

$$A_1 \cap \cdots \cap A_{n+2} \cap F_i \subset U_{i,1} \cap \cdots \cap U_{i,n+2} \cap F_i = \emptyset,$$

we have $A_1 \cap \cdots \cap A_{n+2} = \emptyset$. For each $x \in X$, $\{i \in \mathbb{N} \mid x \in U_{i,j}\}$ is infinite for some j. Then, $x \in \bigcap_{i \in \mathbb{N}} U_{i,j} = A_j$. Hence, $X = A_1 \cup \cdots \cup A_{n+2}$. According to Theorem 5.2.3, we have $\dim X \leq n$. □

Theorem 5.4.2 (Locally Finite Sum Theorem). *Let X be normal and $n \in \omega$. If X has a locally finite closed cover $\{F_\lambda \mid \lambda \in \Lambda\}$ such that* $\dim F_\lambda \leq n$ *for each $\lambda \in \Lambda$, then* $\dim X \leq n$.

Proof. We may assume that $n < \infty$, $\Lambda = (\Lambda, \leq)$ is a well-ordered set, and $F_{\min \Lambda} = \emptyset$. Let $\{U_1, \dots, U_{n+2}\} \in \operatorname{cov}(X)$. Using transfinite induction, we will define $\mathcal{U}_\lambda = \{U_{\lambda,1}, \dots, U_{\lambda,n+2}\} \in \operatorname{cov}(X)$ so that

$$U_{\min \Lambda, j} = U_j, \; U_{\lambda,1} \cap \dots \cap U_{\lambda,n+2} \cap F_\lambda = \emptyset, \text{ and}$$
$$\mu < \lambda \Rightarrow U_{\lambda,j} \subset U_{\mu,j}, \; U_{\mu,j} \setminus U_{\lambda,j} \subset \bigcup_{\mu \leq \nu \leq \lambda} F_\nu.$$

Suppose that $\mathcal{U}_\mu$ has been obtained for $\mu < \lambda$. Let $U'_{\lambda,j} = \bigcap_{\mu<\lambda} U_{\mu,j}$. Then, $\{U'_{\lambda,1}, \dots, U'_{\lambda,n+2}\} \in \operatorname{cov}(X)$. Indeed, if there exists $\mu_0 = \max\{\mu \in \Lambda \mid \mu < \lambda\}$, then $U'_{\lambda,j} = U_{\mu_0,j}$ for each $j = 1, \dots, n+2$. Otherwise, for each $x \in X$, choose an open neighborhood U of x in X that meets only finitely many F_μ. Then, there exists $\mu_1 < \lambda$ such that $U \cap F_\mu = \emptyset$ for $\mu_1 \leq \mu < \lambda$. If $x \in U_{\mu_1,j} \in \mathcal{U}_{\mu_1}$, then $U \cap U_{\mu_1,j} \subset U_{\mu,j}$ for $\mu_1 \leq \mu < \lambda$ because

$$(U \cap U_{\mu_1,j}) \setminus U_{\mu,j} \subset \bigcup_{\mu_1 \leq \nu \leq \mu} (U \cap F_\nu) = \emptyset.$$

It follows that

$$x \in U \cap U_{\mu_1,j} \subset \bigcap_{\mu_1 \leq \mu < \lambda} U_{\mu,j} = \bigcap_{\mu<\lambda} U_{\mu,j} = U'_{\lambda,j}.$$

We apply Theorem 5.2.3 to obtain $\{V_{\lambda,1}, \dots, V_{\lambda,n+2}\} \in \operatorname{cov}(F_\lambda)$ such that $V_{\lambda,j} \subset U'_{\lambda,j}$ and $V_{\lambda,1} \cap \dots \cap V_{\lambda,n+2} = \emptyset$. Now, let $U_{\lambda,j} = V_{\lambda,j} \cup (U'_{\lambda,j} \setminus F_\lambda)$. Then, $\{U_{\lambda,1}, \dots, U_{\lambda,n+2}\} \in \operatorname{cov}(X)$ is the desired open cover. In fact, if $\mu < \lambda$ then

$$U_{\mu,j} \setminus U_{\lambda,j} \subset F_\lambda \cup \big((U_{\mu,j} \setminus F_\lambda) \setminus (U'_{\lambda,j} \setminus F_\lambda)\big) \subset F_\lambda \cup \left(U_{\mu,j} \setminus \bigcap_{\nu<\lambda} U_{\nu,j}\right)$$
$$\subset F_\lambda \cup \bigcup_{\mu<\nu<\lambda} (U_{\mu,j} \setminus U_{\nu,j}) = \bigcup_{\mu \leq \nu \leq \lambda} F_\nu.$$

The proofs of the other properties are easy.

For each $j = 1, \dots, n+2$, let $U^*_j = \bigcap_{\lambda \in \Lambda} U_{\lambda,j}$. Then, similar to the above, $\{U^*_1, \dots, U^*_{n+2}\} \in \operatorname{cov}(X)$. Clearly, $U^*_j \subset U_j$ and $U^*_1 \cap \dots \cap U^*_{n+2} = \emptyset$. Therefore, $\dim X \leq n$ by Theorem 5.2.3. □

The following corollary is a combination of Theorems 5.4.1 and 5.4.2:

Corollary 5.4.3. *Let X be a normal space and $n \in \omega$. If X has a σ-locally finite closed cover $\{F_\lambda \mid \lambda \in \Lambda\}$ such that* $\dim F_\lambda \leq n$ *for each $\lambda \in \Lambda$, then* $\dim X \leq n$. □

The next corollary follows from Theorem 5.4.2:

Corollary 5.4.4. *Let X be a paracompact space and $n \in \omega$. If each point of X has a closed neighborhood with* $\dim \leq n$*, then* $\dim X \leq n$.

Proof. Since X is paracompact and each point of X has a closed neighborhood with $\dim \leq n$, X has a locally finite open cover $\mathcal{U}$ such that $\dim \operatorname{cl} U \leq n$ for each $U \in \mathcal{U}$. Then, $\mathcal{U}^{\mathrm{cl}}$ is also locally finite in X, hence $\dim X \leq n$ by Theorem 5.4.2. □

Remark 7. Corollary 5.4.4 can also be proved by applying Michael's Theorem on local properties of closed sets (Corollary 2.6.6). In this case, the proof is reduced to showing that if X is the union of two closed sets X_1 and X_2 with $\dim X_i \leq n$, $i = 1, 2$, then $\dim X \leq n$.

In the remainder of this section, we consider only metrizable spaces. The real line $\mathbb{R}$ is 1-dimensional and we can decompose $\mathbb{R}$ into two 0-dimensional subsets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. This can be generalized as follows:

Theorem 5.4.5 (Decomposition Theorem). *Let X be metrizable and $n \in \omega$. Then,* $\dim X \leq n$ *if and only if X is covered by $n+1$ many subsets $X_1, \ldots, X_{n+1}$ with* $\dim X_i \leq 0$.

Proof. To prove the "if" part, let $\mathcal{U}$ be a finite open cover of X. Since $\dim X_i \leq 0$, X_i has a finite open cover $\mathcal{V}_i$ such that $\mathcal{V}_i \prec \mathcal{U}$ and $\operatorname{ord} \mathcal{V}_i \leq 1$. For each $V \in \mathcal{V}_i$, choose an open set $W(V)$ in X so that $W(V) \cap X_i = V$ and $W(V)$ is contained in some member of $\mathcal{U}$. Note that $\operatorname{cl} W(V) \cap X_i = V$ because V is also closed in X_i. For each $V \in \mathcal{V}_i$, let

$$\tilde{V} = W(V) \setminus \bigcup \{ \operatorname{cl} W(V') \mid V \neq V' \in \mathcal{V}_i \}.$$

Then, $\tilde{\mathcal{V}}_i = \{\tilde{V} \mid V \in \mathcal{V}_i\}$ is a collection of disjoint open sets in X that covers X_i and refines $\mathcal{U}$. Observe that $\tilde{\mathcal{V}} = \bigcup_{i=1}^{n+1} \tilde{\mathcal{V}}_i$ is an open refinement of $\mathcal{U}$ with $\operatorname{ord} \mathcal{V} \leq n+1$. Therefore, $\dim X \leq n$.

The "only if" part can be easily obtained by induction once the following proposition has been proved. □

Proposition 5.4.6. *Let X be metrizable and* $\dim X \leq n < \infty$*. Then, $X = Y \cup Z$ for some $Y, Z \subset X$ with* $\dim Y \leq n - 1$ *and* $\dim Z \leq 0$.

Proof. Assume that X is a metric space. For each $i \in \mathbb{N}$, let $\mathcal{U}_i$ be a locally finite open cover of X with $\operatorname{mesh} \mathcal{U}_i < i^{-1}$. By paracompactness (Lemma 2.6.2) or normality (Lemma 2.7.1), X has a closed cover $\{F_U \mid U \in \mathcal{U}_i\}$ such that $F_U \subset U$ for all $U \in \mathcal{U}_i$. For each $U \in \mathcal{U}_i$, we apply Theorem 5.3.8 to obtain an open set B_U such that

$$F_U \subset B_U \subset \operatorname{cl} B_U \subset U \quad \text{and} \quad \dim \operatorname{bd} B_U \leq n - 1.$$

It is easy to see that $\mathcal{B} = \{B_U \mid U \in \mathcal{U}_i, i \in \mathbb{N}\}$ is a σ-locally finite open basis for X. Let

$$Y = \bigcup \{ \operatorname{bd} B \mid B \in \mathcal{B} \} \quad \text{and} \quad Z = X \setminus Y.$$

According to Corollary 5.4.3, $\dim Y \le n-1$. Since $\{B \cap Z \mid B \in \mathcal{B}\}$ is a σ-locally finite clopen basis for Z, we have $\dim Z \le 0$ by Theorem 5.3.5. □

In the above proof of Proposition 5.4.6, the following two facts have been proved:

(1) Each metrizable space X with $\dim X \le n$ has a σ-locally finite open basis $\mathcal{B}$ such that $\dim \operatorname{bd} B \le n-1$ for every $B \in \mathcal{B}$.
(2) If a metrizable space X has such a basis $\mathcal{B}$ then $X = Y \cup Z$ for some $Y, Z \subset X$ with $\dim Y \le n-1$ and $\dim Z \le 0$.

In (2), Y is covered by n many subsets with $\dim \le 0$ by the Decomposition Theorem 5.4.5. Hence, X is covered by $n+1$ many subsets with $\dim \le 0$. By the Decomposition Theorem 5.4.5 again, we have $\dim X \le n$. Thus, (1) implies $\dim X \le n$. Consequently, we have the following characterization of dimension, which is a generalization of Theorem 5.3.5:

Theorem 5.4.7. *Let X be metrizable and $n \in \omega$. Then,* $\dim X \le n$ *if and only if X has a σ-locally finite open basis $\mathcal{B}$ such that* $\dim \operatorname{bd} B \le n-1$ *for each $B \in \mathcal{B}$.* □

The following theorem is obtained as a corollary of the Decomposition Theorem 5.4.5:

Theorem 5.4.8 (Addition Theorem). *For any two subspaces X and Y of a metrizable space,*

$$\dim X \cup Y \le \dim X + \dim Y + 1. \qquad \square$$

Regarding the dimension of product spaces, we have the following theorem:

Theorem 5.4.9 (Product Theorem). *For any metrizable spaces X and Y,*

$$\dim X \times Y \le \dim X + \dim Y.$$

Proof. If $\dim X = \infty$ or $\dim Y = \infty$, the theorem is obvious.

When $\dim X, \dim Y < \infty$, we prove the theorem by induction on $\dim X + \dim Y$. The case $\dim X = \dim Y = 0$ is a consequence of Theorem 5.3.6. Assume the theorem is true for any two spaces X and Y with $\dim X + \dim Y < k$. Now, let $\dim X = m$, $\dim Y = n$, and $m + n = k$. According to Theorem 5.4.7, X and Y have σ-locally finite open bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ such that $\dim \operatorname{bd} B \le m-1$ for each $B \in \mathcal{B}_X$ and $\dim \operatorname{bd} B \le n-1$ for each $B \in \mathcal{B}_Y$. Then,

$$\mathcal{B} = \left\{ B_1 \times B_2 \;\middle|\; B_1 \in \mathcal{B}_X \text{ and } B_2 \in \mathcal{B}_Y \right\}$$

is a σ-locally finite open basis for $X \times Y$. For each $B_1 \in \mathcal{B}_X$ and $B_2 \in \mathcal{B}_Y$,

$$\operatorname{bd}(B_1 \times B_2) = (\operatorname{bd} B_1 \times \operatorname{cl} B_2) \cup (\operatorname{cl} B_1 \times \operatorname{bd} B_2).$$

Hence, $\dim \operatorname{bd}(B_1 \times B_2) \leq m+n-1$ by the inductive assumption and Theorems 5.4.1 or 5.4.2. Then, we have $\dim X \times Y \leq m+n$ by Theorem 5.4.7. □

Remark 8. In Theorem 5.4.9, the equality $\dim X \times Y = \dim X + \dim Y$ does not hold in general. In fact, there exists a separable metrizable space X such that $\dim X^2 \neq 2 \dim X$. Such a space will be constructed in Theorem 5.12.1. However, if one of X or Y is a locally compact polyhedron or a metric polyhedron (cf. Sect. 4.5), the equality does hold. This will be proved in Theorem 7.9.7.

5.5 Inductive Dimensions

In this section, we introduce two types of dimension defined by induction. First, the **large inductive dimension** $\operatorname{Ind} X$ of X can be defined as follows: $\operatorname{Ind} \emptyset = -1$ and $\operatorname{Ind} X \leq n$ if each closed set $A \subset X$ has an arbitrarily small open neighborhood V with $\operatorname{Ind} \operatorname{bd} V \leq n-1$. Then, we define $\operatorname{Ind} X = n$ if $\operatorname{Ind} X \leq n$ and $\operatorname{Ind} X \not\leq n-1$. We write $\operatorname{Ind} X < \infty$ if $\operatorname{Ind} X \leq n$ for some $n \in \mathbb{N}$, and otherwise $\operatorname{Ind} X = \infty$. Observe the following:

- If Y is a closed set in X then $\operatorname{Ind} Y \leq \operatorname{Ind} X$.

For an open set G and a closed set F in X,

$$\operatorname{bd} \operatorname{cl} G = \operatorname{cl} G \setminus \operatorname{int} \operatorname{cl} G \subset \operatorname{cl} G \setminus G = \operatorname{bd} G \quad \text{and}$$
$$\operatorname{bd} \operatorname{int} F = \operatorname{cl} \operatorname{int} F \setminus \operatorname{int} F \subset F \setminus \operatorname{int} F = \operatorname{bd} F.$$

Then, $\operatorname{Ind} X \leq n$ if and only if each closed set A in X has an arbitrarily small *closed* neighborhood V with $\operatorname{Ind} \operatorname{bd} V \leq n-1$.

As is easily observed, $\operatorname{Ind} X \leq n$ if and only if, for any two disjoint closed sets A and B in X, there is a partition L between A and B with $\operatorname{Ind} L \leq n-1$. Note that $\operatorname{Ind} \emptyset = \dim \emptyset = -1$. The next theorem follows, by induction, from Theorem 5.3.8.

Theorem 5.5.1. *For every metrizable space X, $\dim X = \operatorname{Ind} X$.* □

Next, the **small inductive dimension** $\operatorname{ind} X$ of X is defined as follows[2]: $\operatorname{ind} \emptyset = -1$ and $\operatorname{ind} X \leq n$ if each point $x \in X$ has an arbitrarily small open neighborhood V with $\operatorname{ind} \operatorname{bd} V \leq n-1$; and then $\operatorname{ind} X = n$ if $\operatorname{ind} X \leq n$ and $\operatorname{ind} X \not\leq n-1$. We write $\operatorname{ind} X < \infty$ if $\operatorname{ind} X \leq n$ for some $n \in \mathbb{N}$, and otherwise $\operatorname{ind} X = \infty$. Now,

- $\operatorname{ind} Y \leq \operatorname{ind} X$ for an *arbitrary* subset $Y \subset X$.

Then, $\operatorname{ind} X \leq n$ if and only if each point x of X has an arbitrarily small *closed* neighborhood V with $\operatorname{ind} \operatorname{bd} V \leq n-1$.

[2]In this chapter, spaces are assumed normal, but the small inductive dimension also makes sense for regular spaces.

By definition, $\operatorname{ind} X \leq \operatorname{Ind} X$ and $\operatorname{ind} \emptyset = \operatorname{Ind} \emptyset = \dim \emptyset = -1$. As is easily shown, $\operatorname{ind} X \leq n$ if and only if X has an open basis $\mathcal{B}$ such that $\operatorname{ind} \operatorname{bd} B \leq n-1$ for every $B \in \mathcal{B}$. Comparing this with Theorem 5.4.7, one might expect that the equality $\operatorname{ind} X = \operatorname{Ind} X = \dim X$ holds for an arbitrary metrizable space X. However, there exists a completely metrizable space X such that $\operatorname{ind} X \neq \operatorname{Ind} X$. Before constructing such a space, we first prove the following Coincidence Theorem:

Theorem 5.5.2 (COINCIDENCE THEOREM). *For every separable metrizable space X, the equality* $\dim X = \operatorname{Ind} X = \operatorname{ind} X$ *holds.*

Proof. Because $\operatorname{ind} X \leq \operatorname{Ind} X$ and $\operatorname{Ind} X = \dim X$, it is enough to show that $\dim X \leq \operatorname{ind} X$ when $\operatorname{ind} X < \infty$. We will prove this by induction on $\operatorname{ind} X$. Assume that $\dim X \leq \operatorname{ind} X$ for every separable metrizable space X with $\operatorname{ind} X < n$. Now, let $\operatorname{ind} X = n$. Then, X has an open basis $\mathcal{B}$ such that $\operatorname{ind} \operatorname{bd} B \leq n - 1$ for every $B \in \mathcal{B}$. Since X is separable metrizable, X has a countable open basis $\{V_i \mid i \in \mathbb{N}\}$. We define

$$P = \big\{(i, j) \in \mathbb{N}^2 \bigm| V_i \subset B \subset V_j \ \text{ for some } \ B \in \mathcal{B}\big\}.$$

For each $p = (i, j) \in P$, choose $B_p \in \mathcal{B}$ so that $V_i \subset B_p \subset V_j$. Then, $\{B_p \mid p \in P\}$ is a countable open basis for X such that $\dim \operatorname{bd} B_p \leq \operatorname{ind} \operatorname{bd} B_p \leq n - 1$ for each $p \in P$. By Theorem 5.4.7, we have $\dim X \leq n$. □

In the non-separable case, we have the following theorem:

Theorem 5.5.3. *There exists a completely metrizable space Z such that* $\operatorname{ind} Z = 0$ *but* $\operatorname{Ind} Z = \dim Z = 1$. *Furthermore, Z has a 0-dimensional compactification.*

Example and Proof. Let $\Omega = [0, \omega_1)$ be the space of all countable ordinals with the order topology. Note that the space $\overline{\Omega} = [0, \omega_1]$ is compact and 0-dimensional. In fact, for each open cover $\mathcal{U}$ of $\overline{\Omega}$, we can inductively choose $\omega_1 = \alpha_0 > \alpha_1 > \alpha_2 > \cdots$ so that each $(\alpha_i, \alpha_{i-1}]$ is contained in some member of $\mathcal{U}$. Since Ω is well-ordered, some α_n is equal to 0. Thus, $\mathcal{U}$ has a finite open refinement $\{0\} \cup \{(\alpha_i, \alpha_{i-1}] \mid i = 1, \ldots, n\}$, which is pair-wise disjoint.

Our space is constructed as a subspace of the product $\Omega^{\mathbb{N}}$. Let L be the subset of Ω consisting of infinite limit ordinals and $S = \Omega \setminus L$. For each $k \in \mathbb{N}$, let $S_k = \{\alpha + k \mid \alpha \in L\}$. We define

$$Z = \big\{z \in \Omega^{\mathbb{N}} \bigm| z(k) \in L \Rightarrow z(k+1) = z(k) + k, z(k+j) \in S_k \ \text{ for } j > 1\big\}.$$

By definition, we have $\operatorname{ind} \Omega^{\mathbb{N}} = 0$, so $\operatorname{ind} Z = 0$. On the other hand, we can write $Z = S^{\mathbb{N}} \cup \bigcup_{k \in \mathbb{N}} Z_k$, where

$$Z_k = \big\{z \in Z \bigm| z(k) \in L\big\} \subset S^{k-1} \times L \times S_k^{\mathbb{N}}.$$

Since S is a discrete space, it follows from Theorem 5.3.6 that $\dim S^{\mathbb{N}} = 0$. As is easily seen, each Z_k is homeomorphic to $S^{k-1} \times S_k^{\mathbb{N}}$ via the following correspondence:

$$Z_k \ni z \mapsto (z(1), \dots, z(k-1), z(k+1), z(k+2), \dots) \in S^{k-1} \times S_k^{\mathbb{N}},$$

where $z(k+1) = z(k) + k$. Then, it follows that $\dim Z_k = 0$ for each $k \in \mathbb{N}$.

(*Neighborhood bases*) For each $\alpha \in L$, choose $\xi_1(\alpha) < \xi_2(\alpha) < \cdots < \alpha$ so that $\sup_{i \in \mathbb{N}} \xi_i(\alpha) = \alpha$. Each $z \in Z$ has the neighborhood basis $\{U_n(z) \mid n \in \mathbb{N}\}$ defined as follows:

$$U_n(z) = \begin{cases} \{x \in Z \mid x(i) = z(i) \ \text{ for } i \leq n\} & \text{if } z \in S^{\mathbb{N}}, \\ \{x \in Z \mid x(i) = z(i) \ \text{ for } k \neq i \leq k+n, \\ \qquad \text{and } \ \xi_n(z(k)) < x(k) \leq z(k)\} & \text{if } z \in Z_k. \end{cases}$$

Note that each $U_n(z)$ is clopen in Z, but $\{U_n(z) \mid z \in S^{\mathbb{N}}\}$ is not locally finite at the point $(\omega, \omega+1, \omega+1, \dots)$ in Z (cf. Theorem 5.3.5). The following statements can be easily proved:

(1) If $z, z' \in S^{\mathbb{N}}$ or $z, z' \in Z_k$, then $U_n(z) \cap U_n(z') \neq \emptyset \Rightarrow U_n(z) = U_n(z')$.
(2) If $z \in Z_k$, $z' \in Z_{k'}$, and $k < k' < n + k$, then $U_n(z) \cap U_{n'}(z') = \emptyset$ for every $n' \in \mathbb{N}$.
(3) If $z \in S^{\mathbb{N}}$, $z' \in Z_k$, and $n < k$, then $U_n(z) \cap U_{n'}(z') \neq \emptyset \Rightarrow U_{n'}(z') \subset U_n(z)$.

Furthermore, we have the following:

(4) For any $z \in S^{\mathbb{N}}$ and $n \in \mathbb{N}$, there exists some $m > n$ such that $U_m(z) \cap U_m(z') = \emptyset$ for every $z' \in \bigcup_{k \leq n} Z_k$.

In fact, if $z(n+1) \notin S_k$ for any $k \leq n$, then $U_{n+1}(z) \cap U_{n+1}(z') = \emptyset$ for every $z' \in \bigcup_{k \leq n} Z_k$. If $z(n+1) \in S_k$ for some $k \leq n$, then $U_m(z) \cap U_m(z') = \emptyset$ for every $m > n$ and $z' \in \bigcup_{k \neq j \leq n} Z_j$. On the other hand, because $z(k+1) \in S$, we can write $z(k+1) = \alpha + r$, where $\alpha \in L \cup \{0\}$ and $r \in \mathbb{N}$. If $\alpha = 0$ or $r \neq k$, then $U_m(z) \cap U_m(z') = \emptyset$ for every $m > k$ and $z' \in Z_k$. When $\alpha \in L$ and $r = k$, choose $m > k$ so that $z(k) \notin (\xi_m(\alpha), \alpha]$. Then, it follows that $U_m(z) \cap U_m(z') = \emptyset$ for every $z' \in Z_k$.

Note that each Z_k is closed in Z by (2) and (4). Then, as mentioned before, we have $\dim Z \leq 1$.

(*Metrizability*) To prove the metrizability, by the Frink Metrization Theorem 2.4.1 it suffices to show that, for each $z \in Z$ and $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ so that $U_m(z) \cap U_m(z') \neq \emptyset$ implies $U_m(z') \subset U_n(z)$.

When $z \in Z_k$ for some $k \in \mathbb{N}$, if $z' \in \bigcup_{k' < k} Z_{k'}$ or $z' \in \bigcup_{k < k' < n+2k} Z_{k'}$ then $U_{n+k}(z) \cap U_{n+k}(z') = \emptyset$ by (2). Assume $U_{n+k}(z) \cap U_{n+k}(z') \neq \emptyset$. If $z' \in S^{\mathbb{N}} \cup \bigcup_{k' > n+k} Z_{k'}$, then $U_{n+k}(z') \subset U_n(z)$ by definition. If $z' \in Z_k$, then $U_{n+k}(z') = U_{n+k}(z) \subset U_n(z)$ by (1). Thus, we have

$$U_{n+k}(z) \cap U_{n+k}(z') \neq \emptyset \Rightarrow U_{n+k}(z') \subset U_n(z).$$

For $z \in S^{\mathbb{N}}$, we can choose $m > n$ by (4) such that $U_m(z) \cap U_m(z') = \emptyset$ for every $z' \in \bigcup_{k \leq n} Z_k$. Assume $U_m(z) \cap U_m(z') \neq \emptyset$. Then, $z' \in S^{\mathbb{N}}$ or $z' \in Z_k$ for some $k > n$. If $z' \in S^{\mathbb{N}}$, then $U_m(z') = U_m(z) \subset U_n(z)$ by (1). If $z' \in Z_k$ for some $k > n$, then $U_m(z') \subset U_n(z)$ by (3). Thus, we have

$$U_m(z) \cap U_m(z') \neq \emptyset \Rightarrow U_m(z') \subset U_n(z).$$

(*Complete metrizability*) Because of Theorem 2.5.5, to show the complete metrizability of Z, it is enough to prove that Z is a G_δ-set in the compact space $\overline{\Omega}^{\mathbb{N}}$. Extend each $U_n(z)$ to a neighborhood of z in $\overline{\Omega}^{\mathbb{N}}$ as follows:

$$\tilde{U}_n(z) = \begin{cases} \left\{x \in \overline{\Omega}^{\mathbb{N}} \;\middle|\; x(i) = z(i) \ \text{ for } i \leq n\right\} & \text{for } z \in S^{\mathbb{N}}, \\ \left\{x \in \overline{\Omega}^{\mathbb{N}} \;\middle|\; x(i) = z(i) \ \text{ for } k \neq i \leq k+n, \right. & \\ \qquad\qquad \left. \text{and } \ \xi_n(z(k)) < x(k) \leq z(k)\right\} & \text{for } z \in Z_k. \end{cases}$$

Then, each $W_n = \bigcup_{z \in Z} \tilde{U}_n(z)$ is an open neighborhood of Z in $\overline{\Omega}^{\mathbb{N}}$ and $Z = \bigcap_{n \in \mathbb{N}} W_n$. Indeed, if $x \in \bigcap_{n \in \mathbb{N}} W_n \setminus S^{\mathbb{N}}$, then $x(k) \in L$ for some $k \in \mathbb{N}$. For $n > k$, choose $z_n \in Z$ so that $x \in \tilde{U}_n(z_n)$. Since $x(k) \in L$ and $k < n$, it follows that $z_n \notin S^{\mathbb{N}} \cup \bigcup_{k' \neq k} Z_{k'}$, i.e., $z_n \in Z_k$. Then, $x(k+i) = z_n(k+i) \in S_k$ for each $0 < i \leq n$ and $\xi_n(z_n(k)) < x(k) \leq z_n(k)$. Since $x(k+1) = z_n(k+1) = z_n(k)+k$, every $z_n(k)$ is identical, say $z(k)$. Since $z(k) = \sup \xi_n(z(k))$, we have $x(k) = z(k)$. Taking $n \in \mathbb{N}$ arbitrarily large, we can see that $x(i) \in S_k$ for any $i > k$. Hence, $x \in Z_k \subset Z$.

(1-*dimensionality*) It has been shown that Z is metrizable and each Z_k is closed in Z. Then, applying the Countable Sum Theorem (5.4.1) and the Addition Theorem (5.4.8), we have $\dim Z \leq 1$.

To see that $\dim Z > 0$, assume $\dim Z = 0$. Let $\mathcal{W} = \{W_\alpha \mid \alpha \in \Omega\} \in \operatorname{cov}(Z)$, where $W_\alpha = \{z \in Z \mid 0 \leq z(2) \leq \alpha\}$. By the assumption, $\mathcal{W}$ has an open refinement $\mathcal{V}$ with $\operatorname{ord} \mathcal{V} \leq 1$. Then, $\mathcal{V}$ is discrete in Z. Here, we call $s \in S^n$ **regular** if there exist $f : \bigoplus_{i \in \mathbb{N}} S^i \to S$ and $V \in \mathcal{V}$ such that $R(s; f) \subset V$, where

$$R(s; f) = \left\{x \in S^{\mathbb{N}} \;\middle|\; x(i) = s(i) \ \text{ for } i \leq n \text{ and } x(n+i) \geq f(x(n), \dots, x(n+i-1)) \ \text{ for } i \in \mathbb{N}\right\}.$$

Otherwise, s is said to be **irregular**.

First, we verify the following fact:

(5) Every $s \in S$ is irregular.

For each $f : \bigoplus_{i \in \mathbb{N}} S^i \to S$ and $\alpha \in \Omega$, define $s_f^\alpha \in S^{\mathbb{N}}$ as follows: $s_f^\alpha(1) = s$, $s_f^\alpha(2) = \max\{s, f(s), \alpha + 1\}$, and $s_f^\alpha(i+1) = f(s_f^\alpha(1), \dots, s_f^\alpha(i))$ for $i \geq 2$. Then, $s_f^\alpha \in R(s; f) \setminus W_\alpha$. Hence, $R(s; f)$ is not contained in any $V \in \mathcal{V}$.

Next, we show the following fact:

(6) If $s \in S^n$ is irregular, then $(s, t) \in S^{n+1}$ is irregular for some $t \in S$.

Suppose that (s, t) is regular for every $t \in S$, that is, there are $f_t : \bigoplus_{i \in \mathbb{N}} S^i \to S$ and $V_t \in \mathcal{V}$ such that $R(s, t; f_t) \subset V_t$. When there exist $a \in S$ and $V \in \mathcal{V}$ such that $V_t = V$ for $t \geq \max\{a, f_a(a)\}$, we define $f : \bigoplus_{i \in \mathbb{N}} S^i \to S$ by

$$f(t) = \max\{a, f_a(a)\} \ \text{ for } t \in S \text{ and}$$
$$f(t_1, \dots, t_i) = f_{t_2}(t_2, \dots, t_i) \ \text{ for } (t_1, \dots, t_i) \in S^i, i \geq 2.$$

For $x \in R(s; f)$, let $t = x(n + 1)$. Then, $x \in R(s, t; f_t)$ because

$$\begin{aligned} x(n + 1 + i) &\geq f(x(n), \dots, x(n + i)) \\ &= f_t(x(n + 1), \dots, x(n + 1 + (i - 1))) \ \text{ for } i \in \mathbb{N}. \end{aligned}$$

Moreover, $t = x(n + 1) \geq f(x(n)) = \max\{a, f_a(a)\} \geq a$. Hence, $R(s; f) \subset \bigcup_{t \geq a} R(s, t; f_t) \subset V$, which contradicts the irregularity of s. Therefore, we can obtain an increasing sequence $a_1 < a_2 < \cdots$ in S such that $V_{a_i} \neq V_{a_{i+1}}$ and $a_{i+1} \geq f_{a_i}(a_i)$. Let $\alpha = \sup_{i \in \mathbb{N}} a_i \in L$ and $b_0 = \alpha + n + 1$. For each $j \in \mathbb{N}$, we can inductively choose $b_j \in S_{n+1}$ so that

$$b_j \geq \sup_{i \in \mathbb{N}} f_{a_i}(a_i, b_0, \dots, b_{j-1}).$$

Then, we have

$$z = (s(1), \dots, s(n), \alpha, b_0, b_1, b_2, \dots) \in Z_{n+1} \ \text{ and}$$
$$z_i = (s(1), \dots, s(n), a_i, b_0, b_1, b_2, \dots) \in R(s, a_i; f_{a_i}) \subset V_{a_i},$$

where $\lim_{i \to \infty} z_i = z$. This contradicts the discreteness of $\mathcal{V}$ because $V_{a_i} \neq V_{a_{i+1}}$.

By (5) and (6), we obtain $s \in S^{\mathbb{N}}$ such that each $(s(1), \dots, s(n)) \in S^n$ is irregular. Then, s is contained in some $V \in \mathcal{V}$, from which $U_n(s) \subset V$ for some $n \in \mathbb{N}$, which implies that $(s(1), \dots, s(n))$ is regular. This is a contradiction.

(0-*dimensional compactification*) Finally, we will show that $\mathrm{cl}_{\overline{\Omega}^{\mathbb{N}}} Z$ is a 0-dimensional compactification of Z. It suffices to show that $\dim \overline{\Omega}^{\mathbb{N}} = 0$. Because $\overline{\Omega}^{\mathbb{N}}$ is compact, each open cover $\mathcal{U}$ of $\overline{\Omega}^{\mathbb{N}}$ has a finite refinement

$$\left\{ p_{m_i}^{-1}\left(\prod_{j=1}^{m_i} [\alpha_{i,j}, \beta_{i,j}] \right) \,\middle|\, i = 1, \dots, n \right\},$$

where $p_k : \overline{\Omega}^{\mathbb{N}} \to \overline{\Omega}^k$ is the projection onto the first k factors. We write

$$\{\alpha_{i,j}, \beta_{i,j} \mid i = 1, \dots, n; j = 1, \dots, m_i\} = \{\gamma_k \mid k = 1, \dots, \ell\},$$

where $\gamma_k < \gamma_{k+1}$ for each $k = 1, \dots, \ell - 1$. Note that $\gamma_1 = 0$ and $\gamma_\ell = \omega_1$. Then, $\mathcal{U}$ has the following pair-wise disjoint open refinement:

$$\left\{p_m^{-1}\left(\prod_{j=1}^{m}(\gamma_{k_j-1}, \gamma_{k_j}]\right) \;\middle|\; k_j = 1, \dots, \ell\right\},$$

where $m = \max\{m_1, \dots, m_n\}$ and $(\gamma_0, \gamma_1] = \{0\}$. Therefore, $\dim \overline{\Omega}^{\mathbb{N}} = 0$. This completes the proof. □

Remark 9. According to Theorem 5.5.3, there exists a 0-dimensional compact space that contains a 1-dimensional subspace. Thus, in the Subset Theorem 5.3.3, metrizability cannot be replaced by compactness.

Remark 10. The inequality $\dim X \leq \operatorname{Ind} X$ holds for any completely normal (= hereditarily normal) space X because the "if" part of Theorem 5.3.8 is valid for such a space, as was pointed out in Remark 6 (at the end of Sect. 5.3).

5.6 Infinite Dimensions

In this section, several types of infinite dimensions are defined and discussed. According to Theorem 5.2.17, $\dim X = \infty$ if and only if X has an essential family of n pairs of disjoint closed sets for any $n \in \mathbb{N}$. A space X is said to be **strongly infinite-dimensional** (**s.i.d.**) if X has an infinite essential family of pairs of disjoint closed sets. Obviously, if X is s.i.d. then $\dim X = \infty$. It is said that X is **weakly infinite-dimensional** (**w.i.d.**) if $\dim X = \infty$ and X is not s.i.d.,[3] that is, for every family $(A_i, B_i)_{i \in \mathbb{N}}$ of pairs of disjoint closed sets in X, there are partitions L_i between A_i and B_i such that $\bigcap_{i \in \mathbb{N}} L_i = \emptyset$.

Theorem 5.6.1. *The Hilbert cube* $\mathbf{I}^{\mathbb{N}}$ *is strongly infinite-dimensional.*

Proof. For each $i \in \mathbb{N}$, let

$$A_i = \left\{x \in \mathbf{I}^{\mathbb{N}} \;\middle|\; x(i) = 0\right\} \text{ and } B_i = \left\{x \in \mathbf{I}^{\mathbb{N}} \;\middle|\; x(i) = 1\right\}.$$

Then, $(A_i, B_i)_{i \in \mathbb{N}}$ is essential in $\mathbf{I}^{\mathbb{N}}$. Indeed, for each $i \in \mathbb{N}$, let L_i be a partition between A_i and B_i. For each $n \in \mathbb{N}$, let $j_n : \mathbf{I}^n \to \mathbf{I}^{\mathbb{N}}$ be the natural injection defined by

$$j_n(x) = (x(1), \dots, x(n), 0, 0, \dots).$$

Then, for each $i \leq n$, $j_n^{-1}(L_i)$ is a partition between

$$j_n^{-1}(A_i) = \left\{x \in \mathbf{I}^n \;\middle|\; x(i) = 0\right\} \text{ and } j_n^{-1}(B_i) = \left\{x \in \mathbf{I}^n \;\middle|\; x(i) = 1\right\}.$$

[3] In many articles, the infinite dimensionality is not assumed, i.e., w.i.d. = not s.i.d., so f.d. implies w.i.d. However, here we assume the infinite dimensionality because we discuss the difference among infinite-dimensional spaces.

Since $(j_n^{-1}(A_i), j_n^{-1}(B_i))_{i=1}^n$ is essential in $\mathbf{I}^n$ (Corollary 5.2.16), we have $\bigcap_{i=1}^n j_n^{-1}(L_i) \neq \emptyset$, hence $\bigcap_{i=1}^n L_i \neq \emptyset$. Since $\mathbf{I}^{\mathbb{N}}$ is compact, it follows that $\bigcap_{i\in\mathbb{N}} L_i \neq \emptyset$. □

By definition, a space is strongly infinite-dimensional if it contains an s.i.d. closed subspace. Then, it follows from Theorem 5.6.1 that every space containing a copy of $\mathbf{I}^{\mathbb{N}}$ is strongly infinite-dimensional. For example, ℓ_1, ℓ_2, and $\mathbb{R}^{\mathbb{N}}$ are s.i.d.[4] Moreover, rint $\boldsymbol{Q} = \bigcup_{n\in\mathbb{N}}[-1+2^{-n}, 1-2^{-n}]^{\mathbb{N}}$ and $\mathbf{I}^{\mathbb{N}} \setminus (0,1)^{\mathbb{N}}$ are also s.i.d.[5]

It is said that X is **countable-dimensional** (**c.d.**) if X is a countable union of f.d. normal subspaces, where it should be noted that subspaces of normal spaces need not be normal (cf. Sect. 2.10). A metrizable space is countable-dimensional if and only if it is a countable union of 0-dimensional subspaces, because an f.d. metrizable space is a finite union of 0-dimensional subspaces by the Decomposition Theorem 5.4.5.

Theorem 5.6.2. *A countable-dimensional metrizable space X with* $\dim X = \infty$ *is weakly infinite-dimensional. In other words, any strongly infinite-dimensional metrizable space is not countable-dimensional.*

Proof. Let $(A_i, B_i)_{i\in\mathbb{N}}$ be a family of pairs of disjoint closed sets in X. We can write $X = \bigcup_{i\in\mathbb{N}} X_i$, where $\dim X_i = 0$. From Theorem 5.2.17 and the Partition Extension Lemma 5.3.7, it follows that for each $i \in \mathbb{N}$, X has a partition L_i between A_i and B_i such that $L_i \cap X_i = \emptyset$. Then, we have

$$\bigcap_{i\in\mathbb{N}} L_i = \Big(\bigcap_{i\in\mathbb{N}} L_i\Big) \cap \Big(\bigcup_{i\in\mathbb{N}} X_i\Big) = \bigcup_{i\in\mathbb{N}}\bigg(\Big(\bigcap_{j\in\mathbb{N}} L_j\Big) \cap X_i\bigg)$$
$$\subset \bigcup_{i\in\mathbb{N}}(L_i \cap X_i) = \emptyset.$$

Therefore, X is w.i.d. □

According to Theorem 5.6.2, the space $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$ and its one-point compactification are c.d., hence they are w.i.d. The following space is also c.d. (so w.i.d.):

$$\mathbf{I}_f^{\mathbb{N}} = \big\{x \in \mathbf{I}^{\mathbb{N}} \bigm| x(i) = 0 \text{ except for finitely many } i\big\}.$$

There exists a w.i.d. compactum that is not c.d. As is easily seen, any subspace of a c.d. metrizable space is also c.d. However, a subspace of a w.i.d. metrizable space need not be w.i.d. Such a compactum will be constructed in Theorem 5.13.1.

[4]It is known that $\ell_1 \approx \ell_2 \approx \mathbb{R}^{\mathbb{N}}$, where the latter homeomorphy was proved by R.D. Anderson.

[5]Since rint $\boldsymbol{Q}$ and $\mathbf{I}^{\mathbb{N}} \setminus (0,1)^{\mathbb{N}}$ are not completely metrizable, they are not homeomorphic to $\mathbb{R}^{\mathbb{N}}$, but it is known that rint $\boldsymbol{Q} \approx \mathbf{I}^{\mathbb{N}} \setminus (0,1)^{\mathbb{N}}$.

Now, we introduce a strong version of countable dimensionality. We say that X is **strongly countable-dimensional** (**s.c.d.**) if X is a countable union of f.d. *closed* subspaces. The space $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$, its one-point compactification, and the space $\mathbf{I}^{\mathbb{N}}_f$ are s.c.d. Every s.c.d. space is c.d. but the converse does not hold. Let ν_ω be the subspace of the Hilbert cube $\mathbf{I}^{\mathbb{N}}$ defined as follows:

$$\nu_\omega = \{x \in \mathbf{I}^{\mathbb{N}} \mid x(i) \in \mathbf{I} \setminus \mathbb{Q} \text{ except for finitely many } i\}.$$

Theorem 5.6.3. *The space ν_ω is countable-dimensional but not strongly countable-dimensional.*

Proof. Since ν_ω is the countable union of subspaces

$$\{x \in \mathbf{I}^{\mathbb{N}} \mid x(i) \in \mathbf{I} \setminus \mathbb{Q} \text{ for } i \geq n\} \approx \mathbf{I}^n \times (\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}},$$

it follows that ν_ω is c.d. Moreover, $\dim \nu_\omega = \infty$ because $\mathbf{I}^n \times \{0\} \subset \nu_\omega$ for any $n \in \mathbb{N}$.

Assume that ν_ω is s.c.d., that is, $\nu_\omega = \bigcup_{n\in\mathbb{N}} F_n$, where each F_n is f.d. and closed in ν_ω. Consider the subspace $(\mathbf{I}\setminus\mathbb{Q})^{\mathbb{N}} \subset \nu_\omega$. Since $(\mathbf{I}\setminus\mathbb{Q})^{\mathbb{N}}$ is completely metrizable, at least one $F_n \cap (\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}}$ has the non-empty interior in $(\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}}$ by the Baire Category Theorem 2.5.1. Then, we have a non-empty open set U in ν_ω such that $U \cap (\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}} \subset F_n \cap (\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}}$. Since U contains a copy of every n-cube $\mathbf{I}^n$, it follows that $\dim U = \infty$, hence $U \setminus F_n \neq \emptyset$ because $\dim F_n < \infty$. Since $(\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}}$ is dense in ν_ω, we have

$$\big(U \cap (\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}}\big) \setminus \big(F_n \cap (\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}}\big) = (U \setminus F_n) \cap (\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}} \neq \emptyset,$$

which is a contradiction. Therefore, ν_ω is not s.c.d. □

A collection $\mathcal{A}$ of subsets of X is **locally countable** if each $x \in X$ has a neighborhood U that meets only countably many members of $\mathcal{A}$, i.e., $\operatorname{card} \mathcal{A}[U] \leq \aleph_0$.

Basic Properties of (Strong) Countable-Dimension 5.6.4.

(1) If X is a countable union of countable-dimensional subspaces, then X is countable-dimensional.
(2) If X is a countable union of strongly countable-dimensional closed subspaces, then X is strongly countable-dimensional.
(3) Every *closed* subspace of a (strongly) countable-dimensional space is (strongly) countable-dimensional. For a metrizable space, this is valid for a non-closed subspace, that is, every subspace of a (strongly) countable-dimensional *metrizable* space is (strongly) countable-dimensional.

The proofs of the above three items are trivial by definition.

(4) A paracompact space X is (strongly) countable-dimensional if each point $x \in X$ has a (strongly) countable-dimensional neighborhood.

Sketch of Proof. Let $\mathcal{P}$ be the property of closed sets in X being c.d. (or s.c.d.). Apply Michael's Theorem on local properties (Corollary 2.6.6). To show (F-3), use the Locally Finite Sum Theorem 5.4.2.

(5) If a paracompact space X has a locally countable union of countable-dimensional subspaces then X is countable-dimensional.
(6) If a paracompact space X has a locally countable union of strongly countable-dimensional closed subspaces then X is strongly countable-dimensional.

Sketch of Proof of (5) *(and* (6)*).* Let $\mathcal{A}$ be a locally countable (closed) cover of X such that each $A \in \mathcal{A}$ is c.d. (s.c.d.). Each $x \in X$ has an open neighborhood V_x in X such that $\mathcal{A}[V_x]$ is countable. Then, $\mathrm{st}(V_x, \mathcal{A}) = \bigcup \mathcal{A}[V_x]$ is a c.d. (s.c.d.) neighborhood of x in X.

From Theorem 5.3.8, it follows that any finite-dimensional metrizable space X contains an n-dimensional closed set for every $n \leq \dim X$. However, this is not true for an infinite-dimensional space. Namely, there exists an infinite-dimensional compactum such that every subset with $\dim \not\leq 0$ is infinite-dimensional. Such a space is called a **hereditarily infinite-dimensional** (**h.i.d.**) space. We will construct an h.i.d. compactum in Theorem 5.13.4.

Next, we introduce infinite-dimensional versions of inductive dimensions. By transfinite induction on ordinals $\alpha \geq \omega$, the **large transfinite inductive dimension** $\mathrm{trInd}\, X$ and the **small transfinite inductive dimension** $\mathrm{trind}\, X$ are defined as follows: $\mathrm{trInd}\, X < \omega$ means that $\mathrm{Ind}\, X < \infty$ and $\mathrm{trInd}\, X \leq \alpha$ if each closed set $A \subset X$ has an arbitrarily small open neighborhood V with $\mathrm{trInd}\,\mathrm{bd}\, V < \alpha$. Similarly, $\mathrm{trind}\, X < \omega$ means that $\mathrm{ind}\, X < \infty$ and $\mathrm{trind}\, X \leq \alpha$ if each $x \in X$ has an arbitrarily small open neighborhood V with $\mathrm{trind}\,\mathrm{bd}\, V < \alpha$. Then, we define $\mathrm{trInd}\, X = \alpha$ (resp. $\mathrm{trind}\, X = \alpha$) if $\mathrm{trInd}\, X \leq \alpha$ (resp. $\mathrm{trind}\, X \leq \alpha$) and $\mathrm{trInd}\, X \not\leq \beta$ (resp. $\mathrm{trind}\, X \not\leq \beta$) for any $\beta < \alpha$. It should be noted that $\mathrm{trInd}\, X < \omega$ (resp. $\mathrm{trind}\, X < \omega$) implies $\mathrm{trInd}\, X = \mathrm{Ind}\, X < \infty$ (resp. $\mathrm{trind}\, X = \mathrm{ind}\, X < \infty$). Using transfinite induction, we can show that if $\mathrm{trInd}\, X = \alpha$ (resp. $\mathrm{trind}\, X = \alpha$) and $\beta < \alpha$, then X contains a closed set A with $\mathrm{trInd}\, A = \beta$ (resp. $\mathrm{trind}\, A = \beta$).

Lemma 5.6.5. *If* $\mathrm{trInd}\, X = \alpha$ *(resp.* $\mathrm{trind}\, X = \alpha$*) and* $\beta < \alpha$*, then* X *has a closed set* Y *such that* $\mathrm{trInd}\, Y = \beta$ *(resp.* $\mathrm{trind}\, Y = \beta$*).*

Proof. Because of the similarity, we prove the lemma only for trInd. Assume that the lemma holds for any ordinal $< \alpha$. Since $\mathrm{trInd}\, X \not\leq \beta$, X has disjoint closed sets A and B such that $\mathrm{trInd}\, L \not< \beta$ for any partition L between A and B. On the other hand, since $\mathrm{trInd}\, X \leq \alpha$, there is a partition L between A and B such that $\mathrm{trInd}\, L < \alpha$. If $\beta = \mathrm{trInd}\, L$, then L is the desired Y. When $\beta < \mathrm{trInd}\, L$, by the inductive assumption, L has a closed set Y with $\mathrm{trInd}\, Y = \beta$. □

It is said that a space X has **large** (or **small**) **transfinite inductive dimension** (abbrev. trInd (or trind)) if $\mathrm{trInd}\, X \leq \alpha$ (or $\mathrm{trind}\, X \leq \alpha$) for some ordinal α.

Proposition 5.6.6. *For a space* X*, the following statements hold:*

(1) *If* X *has* trInd*, then* X *has* trind *and* $\mathrm{trind}\, X \leq \mathrm{trInd}\, X$.

(2) *If X has* trind*, then every subspace A of X also has* trind*, where* $\operatorname{trind} A \leq \operatorname{trind} X$.
(3) *If X has* trInd*, then every* closed *subspace A of X has* trInd*, where* $\operatorname{trInd} A \leq \operatorname{trInd} X$.
(4) *If X has no* trInd*, then X has a closed set A with an open neighborhood U such that the boundary of each neighborhood of A contained in U has no* trInd.
(5) *If X has no* trind*, then X has a point $x \in X$ with an open neighborhood U such that the boundary of each neighborhood of x contained in U has no* trind.

Proof. Statements (1)–(3) are easily proved by the definitions.

(4): Let $\mathcal{P}$ be the collection of pairs (A, U) of closed sets A in X and open sets U in X with $A \subset U$. Suppose that for each $(A, U) \in \mathcal{P}$, A has a neighborhood $V_{(A,U)}$ in X such that $\operatorname{cl} V_{(A,U)} \subset U$ and $\operatorname{bd} V_{(A,U)}$ has trInd. Take an ordinal α so that $\alpha > \operatorname{trInd} \operatorname{bd} V_{(A,U)}$ for every $(A, U) \in \mathcal{P}$. Then, $\operatorname{Ind} X \leq \alpha$, so X has trInd.

(5): In the proof of (4), replace the closed sets A in X with points $x \in X$. □

We now prove that the converse of Proposition 5.6.6(1) does not hold.

Theorem 5.6.7. *The strongly countable-dimensional space* $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$ *has no* trInd *but* $\operatorname{trind} \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n = \omega$.

Proof. Each point of $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$ is contained in some $\mathbf{I}^n$, hence $\operatorname{trind} \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n \leq \omega$. Because $\operatorname{ind} \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n = \infty$, we have $\operatorname{trind} \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n = \omega$.

On the other hand, assume that $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$ has trInd, i.e., $\operatorname{trInd} \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n = \alpha$ for some ordinal α. Then, $\alpha \geq \omega$ because $\dim \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n = \infty$. By Lemma 5.6.5, $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$ contains a closed set X with $\operatorname{trInd} X = \omega$. For each $n \in \mathbb{N}$, let $X_n = X \cap \mathbf{I}^n$. Then, each X_n is finite-dimensional, but $\sup_{n\in\mathbb{N}} \dim X_n = \infty$ because $X = \bigoplus_{n\in\mathbb{N}} X_n$. By Theorem 5.3.8, we have disjoint closed sets A_n and B_n in X_n such that $\dim L \geq \dim X_n - 1$ for any partition L between A_n and B_n in X_n. Then, $A = \bigoplus_{n\in\mathbb{N}} A_n$ and $B = \bigoplus_{n\in\mathbb{N}} B_n$ are disjoint closed sets in X. Since $\operatorname{trInd} X = \omega$, we have a partition L in X between A and B such that $\operatorname{trInd} L < \omega$, i.e., $\dim L < \infty$. Choose $n \in \mathbb{N}$ so that $\dim X_n > \dim L + 1$. Then, $X_n \cap L$ is a partition in X_n between A_n and B_n and $\dim X_n \cap L \leq \dim L < \dim X_n - 1$. This is a contradiction. Therefore, $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$ has no trInd. □

The above Theorem 5.6.7 also shows that the converse of the following theorem does not hold.

Theorem 5.6.8. *A metrizable space is countable-dimensional if it has* trInd.

Proof. This can be proved by transfinite induction. Assume that all metrizable spaces with $\operatorname{trInd} < \alpha$ are c.d. and let X be a metrizable space with $\operatorname{trInd} X = \alpha$. By the analogy of Proposition 5.4.6, we can construct a σ-locally finite basis $\mathcal{B}$ for X such that $\operatorname{trInd} \operatorname{bd} B < \alpha$ for each $B \in \mathcal{B}$. Let

$$Y = \bigcup \{ \operatorname{bd} B \mid B \in \mathcal{B} \} \quad \text{and} \quad Z = X \setminus Y.$$

Then, $\dim Z \leq 0$ by Theorem 5.3.5. On the other hand, by the assumption, $\operatorname{bd} B$ is c.d. for all $B \in \mathcal{B}$. Then, Y is also c.d. by 5.6.4(5) and (1). Therefore, X is c.d. □

The following theorem can be proved in a similar manner (cf. the proof of Theorem 5.5.2).

Theorem 5.6.9. *A separable metrizable space is countable-dimensional if it has* trind. □

Remark 11. In Theorem 5.6.9, it is unknown whether the separability is necessary or not, that is, the existence of a metrizable space that has trind but is not c.d. is unknown.

As we saw in Theorem 5.6.7, the converse of Theorem 5.6.8 is not true in general, but it is true for compacta. Namely, the following theorem holds:

Theorem 5.6.10. *A compactum has* trInd *if and only if it is countable-dimensional.*

Proof. It is enough to prove the "if" part. Let X be compact and $X = \bigcup_{n\in\mathbb{N}} A_n$, where $\dim A_n \leq 0$ for each $n \in \mathbb{N}$. Suppose that X has no trInd. Then, by Proposition 5.6.6(4), X has a closed set A with an open neighborhood U such that the boundary of each neighborhood of A contained in U has no trInd. Since $\dim A_1 \leq 0$, we can use the Partition Extension Lemma 5.3.7 to find a closed neighborhood V_1 of A contained in U such that $\operatorname{bd} V_1 \cap A_1 = \emptyset$. Then, $X_1 = \operatorname{bd} V_1$ has no trInd and $X_1 \cap A_1 = \emptyset$. By the same argument, we have a closed set $X_2 \subset X_1$ that misses A_2 and has no trInd. Thus, by induction, we can obtain closed sets $X_1 \supset X_2 \supset \cdots$ such that each X_n has no trInd and $X_n \cap A_n = \emptyset$. Then,

$$\bigcap_{n\in\mathbb{N}} X_n = \bigcap_{n\in\mathbb{N}} X_n \cap \bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{n\in\mathbb{N}} (X_n \cap A_n) = \emptyset,$$

which contradicts the compactness of X. □

Although $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$ has no trInd (Theorem 5.6.7), the one-point compactification of $\bigoplus_{n\in\mathbb{N}} \mathbf{I}^n$ has trInd by Theorem 5.6.10. Thus, even if a space X has trInd, it does not imply that a subspace A of X has trInd, that is, Theorem 5.6.6(3) does not hold without the closedness of A.

Theorem 5.6.11. *A completely metrizable space has* trind *if it is countable-dimensional.*

Proof. Let $X = (X, d)$ be a complete metric space and $X = \bigcup_{n\in\mathbb{N}} A_n$, where $\dim A_n \leq 0$ for each $n \in \mathbb{N}$. Suppose that X has no trind. Then, by Proposition 5.6.6(5), X has a point a with an open neighborhood U such that the boundary of each neighborhood of a contained in U has no trind, where we may assume that $\operatorname{diam} U < 2^{-1}$. In the same way as for Theorem 5.6.10, we can inductively obtain non-empty closed sets $X_1 \supset X_2 \supset \cdots$ such that $X_n \cap A_n = \emptyset$ and $\operatorname{diam} X_n < 2^{-n}$ for each $n \in \mathbb{N}$. Then,

$$\bigcap_{n\in\mathbb{N}} X_n = \bigcap_{n\in\mathbb{N}} X_n \cap \bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{n\in\mathbb{N}} (X_n \cap A_n) = \emptyset.$$

However, the completeness of X implies $\bigcap_{n\in\mathbb{N}} X_n \neq \emptyset$. This is a contradiction. □

Combining Theorems 5.6.9 and 5.6.11, we have the following corollary:

Corollary 5.6.12. *A* separable *completely metrizable space has* trind *if and only if it is countable-dimensional.* □

The next theorem shows that the "if" part of Theorem 5.6.11 does not hold without the completeness.

Theorem 5.6.13. *The strongly countable-dimensional space $\mathbf{I}_f^{\mathbb{N}}$ has no* trind*.*

To prove this theorem, we need the following two lemmas:

Lemma 5.6.14. *Let X be a subspace of a metrizable space M. Then, every open set U in M contains an open set U' in M such that $X \cap U' = X \cap U$ and $X \cap \mathrm{cl}_M U' = \mathrm{cl}_X(X \cap U')$, hence $X \cap \mathrm{bd}_M U' = \mathrm{bd}_X(X \cap U')$.*

Proof. Take $d \in \mathrm{Metr}\, M$ and define

$$U' = \{x \in U \mid d(x, X \cap U) < d(x, X \setminus U)\}.$$

Then, $X \cap U' = X \cap U$. Evidently, $\mathrm{cl}_X(X \cap U') \subset X \cap \mathrm{cl}_M U'$. Assume that $\mathrm{cl}_X(X \cap U') = \mathrm{cl}_X(X \cap U) \neq X \cap \mathrm{cl}_M U'$, that is, we have $x \in X \cap \mathrm{cl}_M U' \setminus \mathrm{cl}_X(X \cap U)$. For each $\varepsilon > 0$, we have $y \in U'$ so that $d(x, y) < \frac{1}{2}\min\{\varepsilon, d(x, X \cap U)\}$. Since $d(y, X \cap U) < d(y, X \setminus U)$, it follows that

$$\begin{aligned} d(x, X \setminus U) &\geq d(y, X \setminus U) - d(x, y) \\ &> d(y, X \cap U) - \tfrac{1}{2}d(x, X \cap U) = \tfrac{1}{2}d(x, X \cap U) > 0. \end{aligned}$$

On the other hand, $x \notin X \cap U$, i.e., $x \in X \setminus U$, which is a contradiction. □

Lemma 5.6.15. *Let M be a separable metrizable space and $X \subset M$ with* $\mathrm{trind}\, X \leq \alpha$*. Then, X is contained in some G_δ-set X^* in M with* $\mathrm{trind}\, X^* \leq \alpha$*.*

Proof. Assuming that the lemma is true for any ordinal $< \alpha$, we will show the lemma for α. For each $i \in \mathbb{N}$, applying Lemma 5.6.14, we can find a countable open collection $\mathcal{U}_i$ in M such that $X \subset X_i = \bigcup \mathcal{U}_i$, $\mathrm{mesh}\, \mathcal{U}_i < 1/i$, and $\mathrm{trind}\, X \cap \mathrm{bd}_M U < \alpha$ for each $U \in \mathcal{U}_i$, where $X \cap \mathrm{bd}_M U = \mathrm{bd}_X(X \cap U)$ for each $U \in \mathcal{U}_i$. By the inductive assumption, for each $U \in \mathcal{U}_i$, there is a G_δ-set G_U in M such that $X \cap \mathrm{bd}_M U \subset G_U$ and $\mathrm{trind}\, G_U < \alpha$. Then,

$$\begin{aligned} X^* &= \bigcap_{i\in\mathbb{N}} X_i \cap \bigcap_{i\in\mathbb{N}} \bigcap_{U\in\mathcal{U}_i} (G_U \cup (M \setminus \mathrm{bd}_M U)) \\ &= \bigcap_{i\in\mathbb{N}} X_i \setminus \bigcup_{i\in\mathbb{N}} \bigcup_{U\in\mathcal{U}_i} (\mathrm{bd}_M U \setminus G_U) \end{aligned}$$

is a G_δ-set in M and $X \subset X^*$. For any $i \in \mathbb{N}$, every $x \in X^*$ is contained in some $U \in \mathcal{U}_i$. Then, $\operatorname{diam} X^* \cap U < 1/i$ and

$$\begin{aligned}\operatorname{bd}_{X^*}(X^* \cap U) &= \operatorname{cl}_{X^*}(X^* \cap U) \setminus U \\ &\subset (X^* \cap \operatorname{cl}_M U) \setminus U = X^* \cap \operatorname{bd}_M U \subset G_U,\end{aligned}$$

which implies $\operatorname{trind} X^* \cap \operatorname{bd}_M U < \alpha$. Thus, each point $x \in X^*$ has an arbitrarily small neighborhood V with $\operatorname{trind} \operatorname{bd}_{X^*} V < \alpha$. Hence, $\operatorname{trind} X^* \leq \alpha$. □

Proof of Theorem 5.6.13. Assume that $\mathbf{I}^{\mathbb{N}}_f$ has trind. According to Lemma 5.6.15, $\mathbf{I}^{\mathbb{N}}_f$ is contained in some G_δ-set G in $\mathbf{I}^{\mathbb{N}}$ that also has trind. Then, G is c.d. by Theorem 5.6.9. We show that G contains a copy of $\mathbf{I}^{\mathbb{N}}$, hence G is s.i.d., which contradicts Theorem 5.6.2. Thus, we obtain the desired result.

Let $G = \bigcap_{k \in \mathbb{N}} U_k$, where U_k is open in $\mathbf{I}^{\mathbb{N}}$. Note that $0 = (0, 0, \dots) \in \mathbf{I}^{\mathbb{N}}_f \subset G \subset U_1$. Choose $n_1 \in \mathbb{N}$ and $a_1, \dots, a_{n_1} \in (0, 1)$ so that

$$\{x \in \mathbf{I}^{\mathbb{N}} \mid x(i) \leq a_i \text{ for } i = 1, \dots, n_1\} \subset U_1.$$

Note that $\prod_{i=1}^{n_1}[0, a_i] \times \{0\} \subset \mathbf{I}^{\mathbb{N}}_f \subset U_2$. According to the Wallace Theorem 2.1.2, we can choose $n_2 \in \mathbb{N}$ and $a_{n_1+1}, \dots, a_{n_2} \in (0, 1)$ so that $n_2 > n_1$ and

$$\{x \in \mathbf{I}^{\mathbb{N}} \mid x(i) \leq a_i \text{ for } i = 1, \dots, n_2\} \subset U_2.$$

By induction, we can obtain an increasing sequence n_i of natural numbers and a sequence $a_i \in (0, 1)$ such that

$$\{x \in \mathbf{I}^{\mathbb{N}} \mid x(i) \leq a_i \text{ for } i = 1, \dots, n_k\} \subset U_k \quad \text{for each } k \in \mathbb{N}.$$

Then, $G = \bigcap_{k \in \mathbb{N}} U_k$ contains $\prod_{i \in \mathbb{N}}[0, a_i] \approx \mathbf{I}^{\mathbb{N}}$. □

Remark 12. There exists a slightly stronger version of the weak infinite dimension. We say that X is **weakly infinite-dimensional in the sense of Smirnov (S-w.i.d.)** if $\dim X = \infty$, and for every family $(A_i, B_i)_{i \in \mathbb{N}}$ of pairs of disjoint closed sets in X, there are partitions L_i between A_i and B_i such that $\bigcap_{i=1}^n L_i = \emptyset$ for some $n \in \mathbb{N}$. To distinguish w.i.d. from S-w.i.d. the term "**weakly infinite-dimensional in the sense of Alexandroff (A-w.i.d.)**" is used. Obviously, every S-w.i.d. space is (A-)w.i.d. For compact spaces, the converse is also true, that is, the two notions of weak infinite dimension are equivalent. It was shown in [32] that the Stone–Čech compactification of a normal space X is w.i.d. if and only if X is S-w.i.d.[6]

[6]Refer to Engelking's book "Theory of Dimensions, Finite and Infinite," Problem 6.1.E.

5.7 Compactification Theorems

Note that every separable metrizable space X has a metrizable compactification. Indeed, embedding X into the Hilbert cube $\mathbf{I}^{\mathbb{N}}$ (Corollary 2.3.8), the closure of X in $\mathbf{I}^{\mathbb{N}}$ is a metrizable compactification of X. On the other hand, every n-dimensional metrizable space can be embedded in an n-dimensional completely metrizable space as a dense set (Theorem 5.3.4). In this section, we show that every n-dimensional separable metrizable space has an n-dimensional metrizable compactification and that every c.d. (resp. s.c.d.) separable completely metrizable space has a c.d. (resp. s.c.d.) metrizable compactification.

Note. Here is an alternative proof of Corollary 2.3.8. Let $X = (X, d)$ be a separable metric space with $\{a_i \mid i \in \mathbb{N}\}$ a countable dense set. For each $i \in \mathbb{N}$, we define a map $f_i : X \to \mathbf{I}$ by $f_i(x) = \min\{1, d(x, a_i)\}$ for each $x \in X$. Then, the map $f : X \to \mathbf{I}^{\mathbb{N}}$ defined by $f(x) = (f_i(x))_{i\in\mathbb{N}}$ is an embedding. Indeed, for $x \neq y \in X$, choose a_i so that $d(x, a_i) < \min\{1, \frac{1}{2}d(x, y)\}$. Then, $f_i(x) < f_i(y)$ because

$$f_i(x) = d(x, a_i) < \tfrac{1}{2}d(x, y) < d(x, y) - d(x, a_i) \leq d(y, a_i).$$

Thus, f is injective. If f is not an embedding, then there are $x, x_n \in X$, $n \in \mathbb{N}$, and $0 < \delta < 1$ such that $\lim_{n\to\infty} f(x_n) = f(x)$ but $d(x_n, x) \geq \delta$ for all $n \in \mathbb{N}$. Choose a_i so that $d(x, a_i) < \frac{1}{3}\delta$. Then, we have $f_i(x) < 1$. For sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} f_i(x_n) - f_i(x) &= d(x_n, a_i) - d(x, a_i) \\ &\geq d(x_n, x) - 2d(x, a_i) > \delta - \tfrac{2}{3}\delta = \tfrac{1}{3}\delta, \end{aligned}$$

which contradicts $\lim_{n\to\infty} f_i(x_n) = f_i(x)$. Therefore, f is an embedding.

Recall that a metric space $X = (X, d)$ or a metric d is said to be **totally bounded** provided that, for any $\varepsilon > 0$, there is a finite set $A \subset X$ such that $d(x, A) < \varepsilon$ for every $x \in X$, i.e., $X = \bigcup_{a\in A} \mathrm{B}_d(a, \varepsilon)$. It is now easy to show that X is totally bounded if and only if, for any $\varepsilon > 0$, X has a finite open cover $\mathcal{U}$ with mesh $\mathcal{U} < \varepsilon$. Then, every compact metric space is totally bounded. As is easily seen, any subspace of a totally bounded metric space X is also totally bounded with respect to the metric inherited from X.

Theorem 5.7.1. *A metrizable space is separable if and only if it has an admissible totally bounded metric.*

Proof. If a metrizable space X is separable, then X can be embedded in the Hilbert cube $\mathbf{I}^{\mathbb{N}}$. Restricting a metric for $\mathbf{I}^{\mathbb{N}}$, we can obtain an admissible totally bounded metric on X.

Conversely, if X has an admissible totally bounded metric d, then X has finite subsets A_i, $i \in \mathbb{N}$, so that $d(x, A_i) < 2^{-i}$ for every $x \in X$. Then, $A = \bigcup_{i\in\mathbb{N}} A_i$ is a countable dense subset of X. Hence, X is separable. □

Theorem 5.7.2 (COMPACTIFICATION THEOREM). *Every n-dimensional separable metrizable space has an n-dimensional metrizable compactification.*

Proof. Let X be a separable metrizable space with $\dim X = n$. By Theorem 5.7.1, X has an admissible totally bounded metric d. For each $i \in \mathbb{N}$, X has a finite open cover $\mathcal{U}_i = \{U_{i,j} \mid j = 1, \dots, m_i\}$ such that $\operatorname{ord}\mathcal{U}_i \le n+1$, $\operatorname{mesh}_d \mathcal{U}_i < 2^{-i}$, and $\operatorname{mesh} f_{i',j'}(\mathcal{U}_i) < 2^{-i}$ for $i' < i$ and $j' \le m_{i'}$, where $f_{i,j} : X \to \mathbf{I}$ is the map defined by

$$f_{i,j}(x) = \frac{d(x, X \setminus U_{i,j})}{\sum_{k=1}^{m_i} d(x, X \setminus U_{i,k})}.$$

For each $i \in \mathbb{N}$, we define a map $f_i : X \to \mathbf{I}^{m_i}$ by

$$f_i(x) = (f_{i,1}(x), \dots, f_{i,m_i}(x)).$$

Then, the map $f : X \to \prod_{i\in\mathbb{N}} \mathbf{I}^{m_i}$ defined by $f(x) = (f_i(x))_{i\in\mathbb{N}}$ is an embedding. Indeed, $\bigcup_{i\in\mathbb{N}} \mathcal{U}_i = \{U_{i,j} \mid i \in \mathbb{N},\ j \le m_i\}$ is an open basis for X. Since $x \in U_{i,j}$ if and only if $f_{i,j}(x) > 0$, it follows that f is injective, and

$$f(U_{i,j}) = f(X) \cap \left\{z \in \textstyle\prod_{i\in\mathbb{N}} \mathbf{I}^{m_i} \;\middle|\; z(i)(j) > 0\right\}.$$

The closure $\tilde{X}$ of $f(X)$ in $\prod_{i\in\mathbb{N}} \mathbf{I}^{m_i}$ is a metrizable compactification of X. Let ρ be the admissible metric for $\prod_{i\in\mathbb{N}} \mathbf{I}^{m_i}$ defined by $\rho(z, z') = \sup_{i\in\mathbb{N}} 2^{-i} \rho_i(z(i), z'(i))$, where ρ_i is the metric for $\mathbf{I}^{m_i}$ defined by

$$\rho_i(x, y) = \max\left\{|x(j) - y(j)| \;\middle|\; j = 1, \dots, m_i\right\} \text{ for } x, y \in \mathbf{I}^{m_i}.$$

For each $i \in \mathbb{N}$ and $j \le m_i$, let $W_{i,j} = \{z \in \tilde{X} \mid z(i)(j) > 0\}$. Then, $W_{i,j} \cap f(X) = f(U_{i,j})$ is dense in $W_{i,j}$. For $i' < i$

$$\operatorname{diam}_{\rho_{i'}} f_{i'}(U_{i,j}) = \max\left\{\operatorname{diam} f_{i',j'}(U_{i,j}) \;\middle|\; j' \le m_{i'}\right\} < 2^{-i}.$$

Thus, it follows that $\operatorname{diam}_\rho W_{i,j} = \operatorname{diam}_\rho f(U_{i,j}) \le 2^{-i}$. For each $z \in \tilde{X}$, we have $x_n \in X$, $n \in \mathbb{N}$, such that $f(x_n) \to z$ $(n \to \infty)$. Note that $\sum_{j=1}^{m_i} f_{i,j}(x_n) = 1$. For each $i \in \mathbb{N}$, we can find $j \le m_i$ such that $f_{i,j}(x_n) \ge 1/m_i$ for infinitely many $n \in \mathbb{N}$. Because $f_{i,j}(x_n) \to z(i)(j)$ $(n \to \infty)$, we have $z(i)(j) \ge 1/m_i$, i.e., $z \in W_{i,j}$. Therefore, $\mathcal{W}_i = \{W_{i,j} \mid j = 1, \dots, m_i\} \in \operatorname{cov}(\tilde{X})$ with $\operatorname{mesh}_\rho \mathcal{W}_i \le 2^{-i}$. Since $W_{i,j} \cap f(X) = f(U_{i,j})$ and $f(X)$ is dense in $\tilde{X}$, it follows that $\operatorname{ord}\mathcal{W}_i = \operatorname{ord} f(\mathcal{U}_i) = \operatorname{ord}\mathcal{U}_i \le n+1$. Since $\tilde{X}$ is compact, we can find $i_1 < i_2 < \cdots$ in $\mathbb{N}$ so that $\mathcal{W}_{i_1} \succ \mathcal{W}_{i_2} \succ \cdots$. Then, $\dim \tilde{X} \le n$ by Theorem 5.3.1. On the other hand, $\dim X \le \dim \tilde{X}$ by the Subset Theorem 5.3.3. Thus, we have $\dim \tilde{X} = n$. □

In the above proof, suppose now that X is a closed subset of a separable metrizable space Y and d is an admissible totally bounded metric for Y. Then, Y has open covers $\mathcal{V}_i = \{V_{i,j} \mid j = 1, \dots, m_i\}$ such that $\operatorname{ord}\mathcal{V}_i[X] \le n+1$,

$\operatorname{mesh}_d \mathcal{V}_i < 2^{-i}$, and $\operatorname{mesh} g_{i',j'}(\mathcal{V}_i) < 2^{-i}$ for $i' < i$ and $j' \leq m_{i'}$, where $g_{i,j} : Y \to \mathbf{I}$ is the map defined by

$$g_{i,j}(y) = \frac{d(y, Y \setminus V_{i,j})}{\sum_{k=1}^{m_i} d(y, Y \setminus V_{i,k})}.$$

As for f in the above proof, using maps $g_{i,j}$, we can define an embedding $g : Y \to \prod_{i \in \mathbb{N}} \mathbf{I}^{m_i}$. The closure $\tilde{Y}$ of $g(Y)$ in $\prod_{i \in \mathbb{N}} \mathbf{I}^{m_i}$ is a metrizable compactification of Y such that $\dim \operatorname{cl}_{\tilde{Y}} X = \dim X$. Furthermore, we can strengthen this as follows:

Theorem 5.7.3. *Let X be a separable metrizable space and $X_1, X_2, \ldots$ be closed sets in X. Then, there exists a metrizable compactification $\tilde{X}$ of X such that* $\dim \operatorname{cl}_{\tilde{X}} X_i = \dim X_i$.

Sketch of Proof. Assume that $\dim X_i = n_i < \infty$. Let d be an admissible totally bounded metric for X. Construct open covers $\mathcal{U}_{i,j} = \{U_{i,j,k} \mid k = 1, \ldots, m(i,j)\}$ of X so that $\operatorname{ord} \mathcal{U}_{i,j}[X_i] \leq n_i + 1$, $\operatorname{mesh}_d \mathcal{U}_{i,j} < 2^{-i-j}$, and $\operatorname{mesh} f_{i',j',k'}(\mathcal{U}_{i,j}) < 2^{-i-j}$ for $i' + j' < i + j$ and $k' \leq m(i',j')$, where $f_{i,j,k} : X \to \mathbf{I}$ is the map defined by

$$f_{i,j,k}(x) = \frac{d(x, X \setminus U_{i,j,k})}{\sum_{l=1}^{m(i,j)} d(x, X \setminus U_{i,j,l})}.$$

As above, we can now use these maps $f_{i,j,k}$ to define an embedding

$$f : X \to \prod_{n \in \mathbb{N}} \prod_{i+j=n+1} \mathbf{I}^{m(i,j)}.$$

The desired compactification of X is obtained as the closure $\tilde{X}$ of $f(X)$ in the compactum $\prod_{n \in \mathbb{N}} \prod_{i+j=n+1} \mathbf{I}^{m(i,j)}$.

Next, we show the following theorem:

Theorem 5.7.4. *Every separable completely metrizable space X has a metrizable compactification γX such that the remainder $\gamma X \setminus X$ is a countable union of finite-dimensional compact sets, hence it is strongly countable-dimensional.*

Proof. We may assume that X is a subset of a compact metric space $Z = (Z, d)$ with $\operatorname{diam} Z \leq 1$. Since X is completely metrizable, we can write $X = \bigcap_{i \in \mathbb{N}} G_i$, where $G_1 \supset G_2 \supset \cdots$ are open in Z. Since each G_i is totally bounded, G_i has a finite open cover $\mathcal{U}_i$ with $\operatorname{mesh} \mathcal{U}_i < 2^{-i}$. We can write $\bigcup_{i \in \mathbb{N}} \mathcal{U}_i = \{U_n \mid n \in \mathbb{N}\}$. Let $f : Z \to \mathbf{I}^{\mathbb{N}}$ be a map defined by

$$f(z)(n) = d(z, X \setminus U_n), \ n \in \mathbb{N}.$$

Then, $f|X$ is an embedding. In fact, if $x \neq y \in X$, there exists some U_n such that $x \in U_n$ but $y \notin U_n$. Then, $f(x)(n) \neq 0 = f(y)(n)$, which implies that $f(x) \neq f(y)$. For each $x \in X$ and each neighborhood U of x in X, choose $n \in \mathbb{N}$ so that $x \in U_n \cap X \subset U$. Since

$$f(U_n \cap X) = f(X) \cap \{x \in \mathbf{I}^{\mathbb{N}} \mid x(n) > 0\},$$

$f(U)$ is a neighborhood of $f(x)$ in $f(X)$.

Let γX be the closure of $f(X)$ in $\mathbf{I}^{\mathbb{N}}$. Identifying X with $f(X)$, γX is a compactification of X. Note that $\gamma X \subset f(Z)$. If $f(z)(n) > 0$ for infinitely many $n \in \mathbb{N}$, then z is contained in infinitely many G_i, which implies that $z \in \bigcap_{i\in\mathbb{N}} G_i = X$. Thus, we have $f(Z) \setminus f(X) \subset \mathbf{I}^{\mathbb{N}}_f$, hence $\gamma X \setminus X \subset \mathbf{I}^{\mathbb{N}}_f$. Since X is completely metrizable, X is G_δ in γX, hence $\gamma X \setminus X$ is F_σ in γX. Consequently, $\gamma X \setminus X$ is a countable union of f.d. compact sets. □

Now, we prove a compactification theorem for (strongly) countable-dimensional spaces:

Theorem 5.7.5. *Every (strongly) countable-dimensional separable completely metrizable space has a (strongly) countable-dimensional metrizable compactification.*

Proof. The c.d. case is a direct consequence of Theorem 5.7.4. To prove the s.c.d. case, let X be a separable completely metrizable space with $X = \bigcup_{i\in\mathbb{N}} X_i$, where each X_i is closed in X, $\dim X_i < \infty$, and $X_1 \subset X_2 \subset \cdots$. By Theorem 5.7.3, X has a metrizable compactification Y such that $\dim \mathrm{cl}_Y X_i = \dim X_i$. By the complete metrizability of X, we can write $X = \bigcap_{i\in\mathbb{N}} U_i$, where each U_i is open in Y and $Y = U_1 \supset U_2 \supset \cdots$. Let $Z = \bigcup_{i\in\mathbb{N}} U_i \cap \mathrm{cl}_Y X_i$. Then, $X = \bigcup_{i\in\mathbb{N}} X_i \subset Z$. Since each $U_i \cap \mathrm{cl}_Y X_i$ is an F_σ-set in Y, Z is a countable union of f.d. compact sets.

We show that $Y \setminus Z = \bigcup_{i\in\mathbb{N}}((Y \setminus \mathrm{cl}_Y X_i) \setminus U_{i+1})$, which is an F_σ-set in Y. For each $y \in Y \setminus Z$, let $i_0 = \max\{i \in \mathbb{N} \mid y \in U_i\}$. Then, $y \in U_{i_0} \setminus U_{i_0+1}$, which implies $y \notin \mathrm{cl}_Y X_{i_0}$ because $y \notin Z$. Hence, $y \in (Y \setminus \mathrm{cl}_Y X_{i_0}) \setminus U_{i_0+1}$. On the other hand, for each $z \in Z$, we have i_1 such that $z \in U_{i_1} \cap \mathrm{cl}_Y X_{i_1}$. For $i \geq i_1$, $z \notin (Y \setminus \mathrm{cl}_Y X_i) \setminus U_{i+1}$ because $z \in \mathrm{cl}_Y X_{i_1} \subset \mathrm{cl}_Y X_i$. For $i < i_1$, $z \notin (Y \setminus \mathrm{cl}_Y X_i) \setminus U_{i+1}$ because $z \in U_{i_1} \subset U_{i+1}$. Thus, Z is a G_δ-set in a compactum Y, hence it is completely metrizable.

Now, applying Theorem 5.7.4, we have a metrizable compactification $\tilde{Z}$ of Z such that $\tilde{Z} \setminus Z$ is a countable union of f.d. compact sets. Then, $\tilde{Z}$ is a compactification of X and it is a countable union of f.d. compact sets, hence it is s.c.d. □

5.8 Embedding Theorem

Recall that every separable metrizable space X can be embedded into the Hilbert cube $\mathbf{I}^{\mathbb{N}}$ (Corollary 2.3.8). As a finite-dimensional version of this result, we prove the following theorem:

Theorem 5.8.1 (Embedding Theorem). *Every separable metrizable space with* $\dim \leq n$ *can be embedded in* $\mathbf{I}^{2n+1}$*, and can hence be embedded in the Euclidean space* $\mathbb{R}^{2n+1}$.

Remark 13. In Theorem 5.8.1, the cube $\mathbf{I}^{2n+1}$ cannot be replaced by a smaller dimensional cube. In fact, there exist n-dimensional compact polyhedra that cannot be embedded into $\mathbf{I}^{2n}$. See Fig. 5.3.

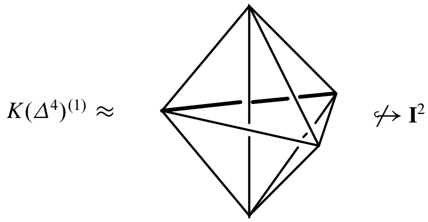


Fig. 5.3 A 1-dimensional polyhedron that cannot be embedded in $\mathbf{I}^2$

To prove Theorem 5.8.1, we introduce a new notion. Now, let $X = (X, d)$ be a *compact* metric space. Given $\varepsilon > 0$, a map $f : X \to Y$ is called an **ε-map** if diam $f^{-1}(y) < \varepsilon$ for each $y \in Y$. Then, a map $f : X \to Y$ is an embedding if and only if $f : X \to Y$ is an ε-map for every $\varepsilon > 0$.

Lemma 5.8.2. *Let $f : X \to Y$ be an ε-map from a compact metric space $X = (X, d)$ to a metric space $Y = (Y, \rho)$. Then, there is some $\delta > 0$ such that any map $g : X \to Y$ with $\rho(f, g) < \delta$ is an ε-map.*

Proof. Since f is a closed map, each $y \in Y$ has an open neighborhood V_y in Y such that diam $f^{-1}(V_y) < \varepsilon$. Since X is compact, we can choose $\delta > 0$ so that each $\mathrm{B}_\rho(f(x), 2\delta)$ is contained in some V_y, hence diam $f^{-1}(\mathrm{B}_\rho(f(x), 2\delta)) < \varepsilon$. Let $g : X \to Y$ be a map with $\rho(f, g) < \delta$. For $y \in Y$ and $x, x' \in g^{-1}(y)$,

$$\rho(f(x), f(x')) \leq \rho(f(x), g(x)) + \rho(f(x'), g(x')) < 2\delta,$$

which implies that $g^{-1}(y) \subset f^{-1}(\mathrm{B}_\rho(f(x), 2\delta))$. Therefore, diam $g^{-1}(y) < \varepsilon$, that is, g is an ε-map. □

For spaces X and Y, let $\mathrm{Emb}(X, Y)$ denote the subspace of $\mathrm{C}(X, Y)$ consisting of all closed embeddings.

Theorem 5.8.3. *Let $X = (X, d)$ be a compact metric space and $Y = (Y, \rho)$ a complete metric space. Assume that for each $\varepsilon > 0$ and $\delta > 0$, every map $f : X \to Y$ is δ-close to an ε-map. Then, every map $f : X \to Y$ can be approximated by an embedding, that is,* $\mathrm{Emb}(X, Y)$ *is dense in the space* $\mathrm{C}(X, Y)$ *with the sup-metric.*

Proof. For each $n \in \mathbb{N}$, let G_n be the set of all 2^{-n}-maps from X to Y. Then, G_n is open and dense in the space $\mathrm{C}(X, Y)$ by Lemma 5.8.2 and the assumption. By the Baire Category Theorem 2.5.1, $\mathrm{Emb}(X, Y) = \bigcap_{n \in \mathbb{N}} G_n$ is also dense in $\mathrm{C}(X, Y)$, hence so is the set of embeddings of X into Y. □

The following is called the GENERAL POSITION LEMMA:

Lemma 5.8.4 (GENERAL POSITION). *Let $\{U_i \mid i \in \mathbb{N}\}$ be a countable open collection in $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$ with* card $A \leq \aleph_0$ *such that each $n + 1$ many points of*

A are affinely independent. Then, there exists $B = \{v_i \mid i \in \mathbb{N}\}$ such that $v_i \in U_i \setminus A$ for each $i \in \mathbb{N}$ and each $n+1$ many points of $A \cup B$ are affinely independent.

Proof. Assume that $v_1 \in U_1, \ldots, v_k \in U_k$ have been chosen so that each $n+1$ many points of $A \cup \{v_1, \ldots, v_k\}$ are affinely independent. Using the Baire Category Theorem 2.5.1 and the fact that every $(n-1)$-dimensional flat (= hyperplane) in $\mathbb{R}^n$ is nowhere dense in $\mathbb{R}^n$, we can find a point

$$v_{k+1} \in U_{k+1} \setminus \bigcup \big\{ \mathrm{fl}\{x_1, \ldots, x_k\} \bigm| x_i \in A \cup \{v_1, \ldots, v_k\} \big\}.$$

Then, each $n+1$ many points of $A \cup \{v_1, \ldots, v_{k+1}\}$ are affinely independent. By induction, we can obtain the desired set $B = \{v_i \mid i \in \mathbb{N}\} \subset \mathbb{R}^n$. □

Because every separable metrizable space has a metrizable compactification with the same dimension by the Compactification Theorem 5.7.2, the Embedding Theorem 5.8.1 can be obtained as a corollary of the next theorem:

Theorem 5.8.5 (Embedding Approximation). *Let X be a compact metric space with $\dim X \leq n$. Then, every map $f : X \to \mathbf{I}^{2n+1}$ can be approximated by embeddings, that is, for each $\varepsilon > 0$, there is an embedding $h : X \to \mathbf{I}^{2n+1}$ that is ε-close to f. In particular, every compact metrizable space with $\dim \leq n$ can be embedded in $\mathbf{I}^{2n+1}$.*

Proof. Because of Theorem 5.8.3, it is enough to show that for each $\varepsilon > 0$ and $\delta > 0$, every map $f : X \to \mathbf{I}^{2n+1}$ is δ-close to an ε-map. We have a finite open cover $\mathcal{U}$ of X such that $\operatorname{ord} \mathcal{U} \leq n+1$, $\operatorname{mesh} \mathcal{U} < \varepsilon$, and $\operatorname{mesh} f(\mathcal{U}) < \delta/2$. Let $K = N(\mathcal{U})$ be the nerve of $\mathcal{U}$. A canonical map $\varphi : X \to |K|$ is an ε-map because $\varphi^{-1}(\mathcal{O}_K) \prec \mathcal{U}$. By the General Position Lemma 5.8.4, we have points $v_U \in \mathbf{I}^{2n+1}$, $U \in \mathcal{U}$, such that $d(v_U, f(U)) < \delta/2$ and every $2n+2$ many points $v_{U_1}, \ldots, v_{U_{2n+2}}$ are affinely independent. We can define a map $g : |K| \to \mathbf{I}^{2n+1}$ as follows: $g(U) = v_U$ for each $U \in \mathcal{U} = K^{(0)}$ and g is linear on each simplex of K_1. Then, g is injective. Hence, $h = g\varphi : X \to \mathbf{I}^{2n+1}$ is an ε-map. For each $x \in X$, let $\mathcal{U}[x] = \{U_1, \ldots, U_k\}$. Then,

$$\|v_{U_i} - f(x)\| \leq d(v_{U_i}, f(U_i)) + \operatorname{diam} f(U_i) < \delta.$$

Since $\mathrm{B}(f(x), \delta)$ is convex, it follows that

$$g\varphi(x) \in g(\langle U_1, \ldots, U_k \rangle) = \langle v_{U_1}, \ldots, v_{U_k} \rangle \subset \mathrm{B}(f(x), \delta).$$

Therefore, $h = g\varphi$ is δ-close to f. □

We generalize a non-compact version of the Embedding Approximation Theorem 5.8.5. Given $\mathcal{U} \in \operatorname{cov}(X)$, a map $f : X \to Y$ is called a ***$\mathcal{U}$*-map** if $f^{-1}(\mathcal{V}) \prec \mathcal{U}$ for some $\mathcal{V} \in \operatorname{cov}(Y)$. By $\mathrm{C}_{\mathcal{U}}(X, Y)$, we denote the subspace of $\mathrm{C}(X, Y)$ consisting of all $\mathcal{U}$-maps. In the case that X is a compact metric space, let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{U} \in \operatorname{cov}(X)$. Then, every $\mathcal{U}$-map is an ε-map.

Conversely, if $\mathcal{U} = \{\mathrm{B}(x, \varepsilon) \mid x \in X\}$, then every ε-map $f : X \to Y$ is a $\mathcal{U}$-map. Indeed, f is closed because of the compactness of X. For each $y \in f(X)$, take $x_y \in f^{-1}(y)$. Since $f^{-1}(y) \subset \mathrm{B}(x_y, \varepsilon)$, y has an open neighborhood V_y in Y such that $f^{-1}(V_y) \subset \mathrm{B}(x_y, \varepsilon)$. Then,

$$\mathcal{V} = \{V_y \mid y \in f(X)\} \cup \{Y \setminus f(X)\} \in \operatorname{cov}(Y) \text{ and } f^{-1}(\mathcal{V}) \prec \mathcal{U}.$$

Recall that if Y is completely metrizable then the space $\mathrm{C}(X, Y)$ with the limitation topology is a Baire space (Theorem 2.9.4). The limitation topology is the topology in which $\{\mathcal{V}(f) \mid \mathcal{V} \in \operatorname{cov}(Y)\}$ is a neighborhood basis of each $f \in \mathrm{C}(X, Y)$,[7] where

$$\mathcal{V}(f) = \left\{g \in \mathrm{C}(X, Y) \;\middle|\; g \text{ is } \mathcal{V}\text{-close to } f\right\}.$$

In the following two lemmas, let Y be an arbitrary paracompact space.

Lemma 5.8.6. *For each $\mathcal{U} \in \operatorname{cov}(X)$, $\mathrm{C}_{\mathcal{U}}(X, Y)$ is open in the space $\mathrm{C}(X, Y)$ with the limitation topology.*

Proof. For each $f \in \mathrm{C}_{\mathcal{U}}(X, Y)$, $f^{-1}(\mathcal{V}) \prec \mathcal{U}$ for some $\mathcal{V} \in \operatorname{cov}(Y)$. Let $\mathcal{W} \in \operatorname{cov}(Y)$ such that $\operatorname{st}\mathcal{W} \prec \mathcal{V}$. For each $g \in \mathcal{W}(f)$, $f(g^{-1}(\mathcal{W})) \prec \operatorname{st}\mathcal{W} \prec \mathcal{V}$, so $g^{-1}(\mathcal{W}) \prec f^{-1}(\mathcal{V}) \prec \mathcal{U}$, which implies $g \in \mathrm{C}_{\mathcal{U}}(X, Y)$. □

Lemma 5.8.7. *For each complete metric space $X = (X, d)$, $\mathrm{Emb}(X, Y) = \bigcap_{n \in \mathbb{N}} \mathrm{C}_{\mathcal{U}_n}(X, Y)$, where $\mathcal{U}_n \in \operatorname{cov}(X)$ with $\operatorname{mesh}\mathcal{U}_n < 2^{-n}$. Thus, when X is a completely metrizable space, $\mathrm{Emb}(X, Y)$ is a G_δ-set in the space $\mathrm{C}(X, Y)$ with the limitation topology.*

Proof. Obviously, $\mathrm{Emb}(X, Y) \subset \bigcap_{n \in \mathbb{N}} \mathrm{C}_{\mathcal{U}_n}(X, Y)$. Every $f \in \bigcap_{n \in \mathbb{N}} \mathrm{C}_{\mathcal{U}_n}(X, Y)$ is injective. For $x_n \in X$, $n \in \mathbb{N}$, if $(f(x_n))_{n \in \mathbb{N}}$ is convergent, then $(x_n)_{n \in \mathbb{N}}$ is Cauchy, so it is convergent. This means that f is closed, hence $f \in \mathrm{Emb}(X, Y)$. Thus, we have $\mathrm{Emb}(X, Y) = \bigcap_{n \in \mathbb{N}} \mathrm{C}_{\mathcal{U}_n}(X, Y)$. □

When Y is completely metrizable, the space $\mathrm{C}(X, Y)$ with the limitation topology is a Baire space by Theorem 2.9.4. Then, by Lemmas 5.8.6 and 5.8.7, Theorem 5.8.3 can be generalized as follows:

Theorem 5.8.8. *Let X and Y be completely metrizable spaces. Suppose that, for each $\mathcal{U} \in \operatorname{cov}(X)$, $\mathrm{C}_{\mathcal{U}}(X, Y)$ is dense in the space $\mathrm{C}(X, Y)$ with the limitation topology. Then, $\mathrm{Emb}(X, Y)$ is also dense in $\mathrm{C}(X, Y)$. In other words, if every map $f : X \to Y$ is approximated by $\mathcal{U}$-maps for each $\mathcal{U} \in \operatorname{cov}(X)$, then every map $f : X \to Y$ is approximated by closed embeddings.* □

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[7]When Y is paracompact, $\{\mathcal{V}(f) \mid \mathcal{V} \in \operatorname{cov}(Y)\}$ is a neighborhood basis of each $f \in \mathrm{C}(X, Y)$ and the topology is Hausdorff.

We consider the case that X and Y are locally compact metrizable. Let $\mathrm{C}^P(X, Y)$ be the subspace of the space $\mathrm{C}(X, Y)$ with the limitation topology consisting of all proper maps.[8] Then, the space $\mathrm{C}^P(X, Y)$ is a Baire space by Theorem 2.9.8. It should be noted that $\mathrm{Emb}(X, Y) \subset \mathrm{C}^P(X, Y)$. Moreover, if X is non-compact, then any constant map of X to Y is not proper, which implies that $\mathrm{Emb}(X, Y)$ is not dense in the space $\mathrm{C}(X, Y)$ with the limitation topology because $\mathrm{C}^P(X, Y)$ is clopen in $\mathrm{C}(X, Y)$ due to Corollary 2.9.7. For an open cover $\mathcal{U} \in \mathrm{cov}(X)$ consisting of open sets with the compact closures, we have $\mathrm{C}_{\mathcal{U}}(X, Y) \subset \mathrm{C}^P(X, Y)$.

Indeed, for each $f \in \mathrm{C}_{\mathcal{U}}(X, Y)$, let $\mathcal{V}$ be a locally finite open cover of Y such that $f^{-1}(\mathcal{V}) \prec \mathcal{U}$. Each compact set A in Y meets only finitely many $V_1, \ldots, V_n \in \mathcal{V}$, where each $f^{-1}(V_i)$ is contained in some $U_i \in \mathcal{U}$. Then, $f^{-1}(A) \subset \bigcup_{i=1}^n \mathrm{cl}\, U_i$. Since $\bigcup_{i=1}^n \mathrm{cl}\, U_i$ is compact, $f^{-1}(A)$ is also compact.

The following theorem is the locally compact version of Theorem 5.8.8:

Theorem 5.8.9. *Let X and Y be locally compact metrizable spaces. If $\mathrm{C}_{\mathcal{U}}(X, Y)$ is dense in the space $\mathrm{C}^P(X, Y)$ with the limitation topology for each open cover $\mathcal{U}$ of X consisting of open sets with the compact closures, then* $\mathrm{Emb}(X, Y)$ *is also dense in* $\mathrm{C}^P(X, Y)$. □

Now, we show the following locally compact version of the Embedding Approximation Theorem 5.8.5:

Theorem 5.8.10 (Embedding Approximation). *Let X be a locally compact separable metrizable space with* $\dim X \leq n$. *Then,* $\mathrm{Emb}(X, \mathbb{R}^{2n+1})$ *is dense in the space* $\mathrm{C}^P(X, \mathbb{R}^{2n+1})$ *with the limitation topology, that is, for each open cover $\mathcal{U}$ of $\mathbb{R}^{2n+1}$, every proper map $f : X \to \mathbb{R}^{2n+1}$ is $\mathcal{U}$-close to a closed embedding $h : X \to \mathbb{R}^{2n+1}$.*

Proof. Because of Theorem 5.8.9, it suffices to show that $\mathrm{C}_{\mathcal{U}}(X, Y)$ is dense in $\mathrm{C}^P(X, \mathbb{R}^{2n+1})$ for each $\mathcal{U} \in \mathrm{cov}(X)$, that is, for any $\mathcal{V} \in \mathrm{cov}(\mathbb{R}^{2n+1})$, every proper map $f : X \to \mathbb{R}^{2n+1}$ is $\mathcal{V}$-close to some $\mathcal{U}$-map $h : X \to \mathbb{R}^{2n+1}$.

We can find $\mathcal{W} \in \mathrm{cov}(\mathbb{R}^{2n+1})$ such that $\mathcal{W}$ is star-finite ($\mathrm{ord}\, \mathcal{W} \leq 2n + 2$), $\mathrm{cl}\, W$ is compact for each $W \in \mathcal{W}$, and $\{\langle \mathrm{st}(x, \mathcal{W}) \rangle \mid x \in X\} \prec \mathcal{V}$. By replacing a refinement with $\mathcal{U}$, we may assume that $\mathcal{U} \prec f^{-1}(\mathcal{W})$ (i.e., $f(\mathcal{U}) \prec \mathcal{W}$), $\mathcal{U}$ is countable, and $\mathrm{ord}\, \mathcal{U} \leq n + 1$ (cf. Corollary 5.2.5). Write $\mathcal{U} = \{U_i \mid i \in \mathbb{N}\}$ and choose $W_i \in \mathcal{W}$, $i \in \mathbb{N}$, so that $f(U_i) \subset W_i$. Let $K = N(\mathcal{U})$ be the nerve of $\mathcal{U}$ with $\varphi : X \to |K|$ a canonical map. Then, $\dim K \leq n$ and φ is a $\mathcal{U}$-map because $\varphi^{-1}(O_K(U)) \subset U$ for each $U \in \mathcal{U} = K^{(0)}$.

By the General Position Lemma 5.8.4, we have points $v_i \in \mathbb{R}^{2n+1}$, $i \in \mathbb{N}$, such that $v_i \in W_i$ and every $2n + 2$ many points $v_{i_1}, \ldots, v_{i_{2n+2}}$ are affinely independent. Then, we have a PL-map $g : |K| \to \mathbb{R}^{2n+1}$ such that $g(U_i) = v_i \in W_i$ for each $U_i \in K^{(0)} = \mathcal{U}$ and $g|\sigma$ is affine on each simplex $\sigma \in K$. For each pair of simplexes $\sigma, \tau \in K$, $g(\sigma^{(0)} \cup \tau^{(0)})$ is affinely independent, which implies that $g|\sigma \cup \tau$ is an embedding. Hence, g is injective.

[8] In this case, a proper map coincides with a perfect map (Proposition 2.1.5).

To prove that g is a closed embedding, let A be a closed set in $|K|$. Each $y \in \operatorname{cl} g(A)$ is contained in some $W \in \mathcal{W}$. By the star-finiteness of $\mathcal{W}$, $\mathcal{W}[W]$ is finite, hence $g(K^{(0)}) \cap W$ is finite. Since K is star-finite, $W \cap g(\sigma) \neq \emptyset$ for only finitely many simplexes $\sigma \in K$. Let

$$\{\sigma \in K \mid W \cap g(\sigma) \neq \emptyset\} = \{\sigma_1, \ldots, \sigma_m\}.$$

Since g is injective, it follows that $W \cap g(A) = \bigcup_{i=1}^m W \cap g(A \cap \sigma_i)$, which is closed in W, and hence $y \in g(A)$. Therefore, $g(|K|)$ is closed in $\mathbb{R}^{2n+1}$.

It remains to be shown that the $\mathcal{U}$-map $g\varphi : X \to Y$ is $\mathcal{V}$-close to f. For each $x \in X$, take the carrier $\sigma \in K$ of $\varphi(x)$ and let $\sigma^{(0)} = \{U_{i_1}, \ldots, U_{i_k}\}$. Then,

$$g(\varphi(x)) \in g(\sigma) = \langle g(\sigma^{(0)}) \rangle = \langle v_{i_1}, \ldots, v_{i_k} \rangle.$$

On the other hand, since $x \in U_{i_1} \cap \cdots \cap U_{i_k}$, we have $f(x) \in W_{i_1} \cap \cdots \cap W_{i_k}$. Then, it follows that $v_{i_1}, \ldots, v_{i_k} \in \operatorname{st}(f(x), \mathcal{W})$. Recall that $\langle \operatorname{st}(f(x), \mathcal{W}) \rangle$ is contained in some $V \in \mathcal{V}$. Then, we have $g(\varphi(x)), f(x) \in V$. Thus, $g\varphi$ is $\mathcal{V}$-close to f. □

Remark 14. In the Embedding Approximation Theorem 5.8.10, a map $f : X \to \mathbb{R}^{2n+1}$ cannot be approximated by closed embeddings if f is not proper. Indeed, $A = f^{-1}(a)$ is not compact for some $a \in \mathbb{R}^{2n+1}$. If $h : X \to \mathbb{R}^{2n+1}$ is a closed embedding then $h(A)$ is closed in $\mathbb{R}^{2n+1}$. Because $h(A)$ is non-compact, it is unbounded, hence $\sup_{x \in A} \|h(x) - f(x)\| = \infty$.

We now show the following proposition:

Proposition 5.8.11. *Let X be a paracompact space and $n \in \omega$. Suppose that for each $\mathcal{U} \in \operatorname{cov}(X)$, there exist a paracompact space Y with $\dim Y \leq n$ and a $\mathcal{U}$-map $f : X \to Y$. Then, $\dim X \leq n$.*

Proof. For each $\mathcal{U} \in \operatorname{cov}(X)$, we have a $\mathcal{U}$-map $f : X \to Y$ such that $\dim Y \leq n$. Then, by Theorem 5.2.4, we have $\mathcal{V} \in \operatorname{cov}(Y)$ such that $f^{-1}(\mathcal{V}) \prec \mathcal{U}$ and $\operatorname{ord} \mathcal{V} \leq n + 1$. Note that $\operatorname{ord} f^{-1}(\mathcal{V}) \leq n + 1$. Therefore, $\dim X \leq n$ by Theorem 5.2.4. □

When X is a metric space, using Theorem 5.3.1 instead of Theorem 5.2.4, we have the following:

Proposition 5.8.12. *Let X be a metric space and $n \in \omega$. Suppose that for each $\varepsilon > 0$, there exist a paracompact space Y with $\dim Y \leq n$ and a closed ε-map $f : X \to Y$. Then, $\dim X \leq n$.* □

5.9 Universal Spaces

Given a class $\mathcal{C}$ of spaces, a space $Y \in \mathcal{C}$ is called a **universal space** for $\mathcal{C}$ if every space $X \in \mathcal{C}$ can be embedded into Y. The Hilbert cube $\mathbf{I}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{N}}$ are universal spaces for separable metrizable spaces (Corollary 2.3.8) and the countable power

$J(\Gamma)^{\mathbb{N}}$ of the hedgehog is the universal space for metrizable spaces with weight $\leq \operatorname{card}\Gamma$ (Corollary 2.3.7).[9]

In this section, we show the existence of universal spaces for metrizable spaces with $\dim \leq n$, and for countable-dimensional and strongly countable-dimensional metrizable spaces.

First, we will show that the space $\mathbf{I}^{\mathbb{N}}_f$ is also a universal space for strongly countable-dimensional separable metrizable spaces.

Lemma 5.9.1. *Let X be a separable metrizable space and $X_0 \subset X_1$ be closed sets in X with $\dim X_1 \leq n$. Then, there exists a map $f : X \to \mathbf{I}^{2n+2}$ such that $X_0 = f^{-1}(0)$ and $f|X_1 \setminus X_0$ is an embedding.*

Proof. Applying the Tietze Extension Theorem 2.2.2 coordinate-wise, we can extend an embedding of X_1 into $\mathbf{I}^{2n+1}$ obtained by Theorem 5.8.1 to a map $h : X \to \mathbf{I}^{2n+1}$. Let $g : X \to \mathbf{I}$ be a map with $g^{-1}(0) = X_0$. We define a map $f : X \to \mathbf{I}^{2n+2} = \mathbf{I}^{2n+1} \times \mathbf{I}$ by $f(x) = (g(x)h(x), g(x))$. Then, $f^{-1}(0) = X_0$. It is easy to prove that $f|X_1 \setminus X_0$ is injective. To see that $f|X_1 \setminus X_0$ is an embedding, let $x, x_i \in X_1 \setminus X_0$, $i \in \mathbb{N}$, and assume that $f(x) = \lim_{i\to\infty} f(x_i)$. Since $g(x_i) \to g(x)$ and $g(x_i), g(x) > 0$, we have $g(x_i)^{-1} \to g(x)^{-1}$, which implies that $h(x_i) \to h(x)$ in $\mathbf{I}^{2n+1}$, hence $x_i \to x$ in X. Therefore, $f|X_1 \setminus X_0$ is an embedding. □

Theorem 5.9.2. *The space $\mathbf{I}^{\mathbb{N}}_f$ is a universal space for strongly countable-dimensional separable metrizable spaces.*

Proof. Let X be an s.c.d. separable metric space. We can write $X = \bigcup_{k\in\mathbb{N}} X_k$, where $X_1 \subsetneqq X_2 \subsetneqq \cdots$ are closed in X and $\dim X_k = n_k < \infty$. By Theorem 5.7.3, X has a metrizable compactification Y such that $\dim \operatorname{cl}_Y X_k = \dim X_k = n_k$. By Lemma 5.9.1, we have maps $f_k : Y \to \mathbf{I}^{2n_k+2}$ ($k \in \mathbb{N}$) such that $f_k^{-1}(0) = \operatorname{cl}_Y X_{k-1}$ and $f_k|\operatorname{cl}_Y X_k \setminus \operatorname{cl}_Y X_{k-1}$ is an embedding for each $k \in \mathbb{N}$, where $X_0 = \emptyset$. We define a map $f : Y \to \prod_{k\in\mathbb{N}} \mathbf{I}^{2n_k+2} = \mathbf{I}^{\mathbb{N}}$ by $f(x) = (f_k(x))_{k\in\mathbb{N}}$. Then, $f|\bigcup_{k\in\mathbb{N}} \operatorname{cl}_Y X_k$ is injective. By definition, $f(\bigcup_{k\in\mathbb{N}} \operatorname{cl}_Y X_k) \subset \mathbf{I}^{\mathbb{N}}_f$. For $y \in Y$, if $f(y) \in \mathbf{I}^{\mathbb{N}}_f$ then $f_{k+1}(y) = 0$ for some $k \in \mathbb{N}$, which means that $y \in \operatorname{cl}_Y X_k$. Then, it follows that

$$f(A) \cap \mathbf{I}^{\mathbb{N}}_f = f\left(A \cap \bigcup_{k\in\mathbb{N}} \operatorname{cl}_Y X_k\right) \text{ for each } A \subset Y.$$

Since f is a closed map, the restriction $f|\bigcup_{k\in\mathbb{N}} \operatorname{cl}_Y X_k : \bigcup_{k\in\mathbb{N}} \operatorname{cl}_Y X_k \to \mathbf{I}^{\mathbb{N}}_f$ is also a closed map. Therefore, $f|\bigcup_{k\in\mathbb{N}} \operatorname{cl}_Y X_k$ is an embedding, hence so is $f|X$. This completes the proof. □

For each $n \in \omega$, let

$$\nu_n = \left\{x \in \mathbf{I}^{\mathbb{N}} \;\middle|\; x(i) \in \mathbf{I} \setminus \mathbb{Q} \text{ except for } n \text{ many } i\right\}.$$[10]

[9]Usually, the phrase "the class of" is omitted.

[10]Recall that ν^0 denotes the space $\mathbb{R} \setminus \mathbb{Q}$. Then, $\nu_0 \subsetneqq \nu^0$ but $\nu_0 \approx ((-1,1) \setminus \mathbb{Q})^{\mathbb{N}} \approx \nu^0$.

Then, $\nu_0 = (\mathbf{I} \setminus \mathbb{Q})^{\mathbb{N}} \subset \nu_1 \subset \nu_2 \subset \cdots \subset \nu_\omega = \bigcup_{n\in\omega} \nu_n$. Recall that ν_ω is c.d. but not s.c.d. (Theorem 5.6.3). We will show that ν_n is a universal space for separable metrizable spaces with $\dim \le n$ and that ν_ω is the universal space for c.d. separable metrizable spaces. To avoid restricting ourselves to separable spaces, we construct non-separable analogues to ν_n and ν_ω.

Let Γ be an infinite set. Recall that the hedgehog $J(\Gamma)$ is the closed subspace of $\ell_1(\Gamma)$ defined as

$$\begin{aligned} J(\Gamma) &= \bigl\{x \in \ell_1(\Gamma) \bigm| x(\gamma) \in \mathbf{I} \text{ for all } \gamma \in \Gamma \text{ and} \\ &\qquad\qquad x(\gamma) \neq 0 \text{ for at most one } \gamma \in \Gamma\bigr\} \\ &= \bigcup_{\gamma\in\Gamma} \langle 0, \mathbf{e}_\gamma \rangle = \bigcup_{\gamma\in\Gamma} \mathbf{I}\mathbf{e}_\gamma \subset \ell_1(\Gamma). \end{aligned}$$

Then, $\dim J(\Gamma) = 1$. Let

$$P(\Gamma) = \bigl\{x \in J(\Gamma) \bigm| x(\gamma) \in (\mathbf{I} \setminus \mathbb{Q}) \cup \{0\}\bigr\} = \{0\} \cup \bigcup_{\gamma\in\Gamma} (\mathbf{I} \setminus \mathbb{Q})\mathbf{e}_\gamma.$$

Observe $P(\Gamma) = \{0\} \cup \bigcup_{i\in\mathbb{N}} P_i$, where $P_i = P(\Gamma) \setminus \mathrm{B}(0, 1/i)$. Each P_i is the discrete union of 0-dimensional closed sets in $P(\Gamma)$ that are homeomorphic to $\mathbf{I}\setminus\mathbb{Q}$, hence $\dim P_i = 0$ by the Locally Finite Sum Theorem 5.4.2. Then, $\dim P(\Gamma) = 0$ by the Countable Sum Theorem 5.4.1. Now, we define

$$\nu_\omega(\Gamma) = \bigl\{z \in J(\Gamma)^{\mathbb{N}} \bigm| z(i) \in P(\Gamma) \text{ except for finitely many } i\bigr\}.$$

Observe that $\nu_\omega(\Gamma)$ is the countable union of subspaces that are homeomorphic to $J(\Gamma)^n \times P(\Gamma)^{\mathbb{N}}$. Since $\dim J(\Gamma)^n \times P(\Gamma)^{\mathbb{N}} \le n$ (Product Theorem 5.4.9 and Theorem 5.3.6) and $J(\Gamma)^n$ contains a copy of $\mathbf{I}^n$, we have $\dim J(\Gamma)^n \times P(\Gamma)^{\mathbb{N}} = n$. Therefore, it follows that $\nu_\omega(\Gamma)$ is c.d. For each $n \in \omega$, we define

$$\nu_n(\Gamma) = \bigl\{z \in J(\Gamma)^{\mathbb{N}} \bigm| z(i) \in P(\Gamma) \text{ except for } n \text{ many } i\bigr\}.$$

Then, $\nu_0(\Gamma) = P(\Gamma)^{\mathbb{N}} \subset \nu_1(\Gamma) \subset \nu_2(\Gamma) \subset \cdots \subset \nu_\omega(\Gamma) = \bigcup_{n\in\omega} \nu_n(\Gamma)$.

Theorem 5.9.3. *For each $n \in \omega$,* $\dim \nu_n(\Gamma) = \dim \nu_n = n$.

Proof. We only give a proof of $\dim \nu_n(\Gamma) = n$ because $\dim \nu_n = n$ is similar and simpler.

We already proved that $\dim \nu_0(\Gamma) = \dim P(\Gamma)^{\mathbb{N}} = 0$. Assuming that $\dim \nu_{n-1}(\Gamma) = n - 1$ and $n > 0$, we now prove that $\dim \nu_n(\Gamma) = n$. We can write

$$\nu_n(\Gamma) = \nu_0(\Gamma) \cup \bigcup_{i\in\mathbb{N}} \bigcup_{q\in((0,1]\cap\mathbb{Q})} \bigcup_{\gamma\in\Gamma} \nu_n(i, q, \gamma),$$

where $\nu_n(i, q, \gamma)$ is a closed set in $\nu_n(\Gamma)$ defined as follows:

$$\nu_n(i, q, \gamma) = \{z \in \nu_n(\Gamma) \mid z(i)(\gamma) = q\}.$$

Since $\{\nu_n(i, q, \gamma) \mid \gamma \in \Gamma\}$ is discrete in $\nu_n(\Gamma)$ and $\nu_n(i, q, \gamma) \approx \nu_{n-1}(\Gamma)$, $\bigcup_{\gamma \in \Gamma} \nu_n(i, q, \gamma)$ is an $(n-1)$-dimensional closed set in $\nu_n(\Gamma)$ by the Locally Finite Sum Theorem 5.4.2. Then, $\dim \nu_n(\Gamma) \leq n$ by the Countable Sum Theorem 5.4.1 and the Addition Theorem 5.4.8. Since $\nu_n(\Gamma)$ contains an n-dimensional subspace $J(\Gamma)^n \times P(\Gamma)^{\mathbb{N}}$, we have $\dim \nu_n(\Gamma) \geq n$, hence $\dim \nu_n(\Gamma) = n$. The result follows by induction. □

We will show that the space $\nu_n(\Gamma)$ is a universal space for metrizable spaces with $\dim \leq n$ and weight $\leq \operatorname{card} \Gamma$, and that the space $\nu_\omega(\Gamma)$ is a universal space for c.d. metrizable spaces with weight $\leq \operatorname{card} \Gamma$.

Lemma 5.9.4. *Let X be a metrizable space and $X_0, X_1, \dots \subset X$ with* $\dim X_n \leq 0$. *Suppose that $L_0 = \emptyset$, $L_1, \dots, L_{m-1}$ are closed sets in X satisfying the following condition:*

$(*)$ *No $x \in X_n$ are contained in $n+1$ many sets L_i.*

Then, for each pair (A, B) of disjoint closed sets in X, there exists a partition L_m in X between A and B that does not violate the condition $(*)$.

Proof. Let $C_0 = X_0$. For $n < m$, define

$$C_n = \bigcup \{X_n \cap \bigcap_{j=1}^n L_{i_j} \mid 0 \leq i_1 < i_2 < \dots < i_n < m\}.$$

Then, $C_i \cap C_j = \emptyset$ for $i \neq j$ by $(*)$. Let $D = \bigcup_{i=0}^{m-1} C_i$. For each $n < m-1$, $\bigcup_{i=n+1}^{m-1} C_i$ is contained in the closed set

$$F = \bigcup \{\bigcap_{j=1}^{n+1} L_{i_j} \mid 0 \leq i_1 < i_2 < \dots < i_{n+1} < m\}.$$

Note that $F \cap \bigcup_{i=0}^n X_i = \emptyset$ by $(*)$. For this reason, $\bigcup_{i=0}^n C_i = D \setminus F$ is open in D. Therefore, each $C_n = \bigcup_{i=0}^n C_i \setminus \bigcup_{i=0}^{n-1} C_i$ is an F_σ-set in D. It follows from the Subset Theorem 5.3.3 and the Countable Sum Theorem 5.4.1 that $\dim D \leq 0$.

Using Theorem 5.2.17 and the Partition Extension Lemma 5.3.7, we obtain a partition L_m between A and B such that $L_m \cap D = \emptyset$. Condition $(*)$ is trivial for $n \geq m$. For $n < m$, if $x \in X_n$ is contained in n many sets L_i $(i < m)$, then $x \in C_n \subset D$, which implies $x \notin L_m$. Therefore, condition $(*)$ is satisfied. □

Lemma 5.9.5. *Let X be a metrizable space and $X_0, X_1, \dots \subset X$ with* $\dim X_n \leq 0$ *and let $a < b \in \mathbb{R}$. Then, for any sequence $(A_i, B_i)_{i \in \mathbb{N}}$ of pairs of disjoint closed sets in X, there exist maps $f_i : X \to [a, b]$, $i \in \mathbb{N}$, such that $A_i = f_i^{-1}(a)$, $B_i = f_i^{-1}(b)$, and*

$$\operatorname{card}\{i \in \mathbb{N} \mid f_i(x) \in (a, b) \cap \mathbb{Q}\} \leq n \quad \text{for } x \in X_n.$$

Proof. Let $\{q_j \mid j \in \mathbb{N}\} = (a,b) \cap \mathbb{Q}$, where $q_i \neq q_j$ if $i \neq j$. For each $j \in \mathbb{N}$, let

$$\delta_j = \min\{q_i - a, b - q_i, |q_i - q_{i'}| \mid i, i' \leq j,\ i \neq i'\},$$

and define $a_j = q_j - 2^{-j-1}\delta_j$ and $b_j = q_j + 2^{-j-1}\delta_j$. For each $i \in \mathbb{N}$, let $f_{i,0} : X \to [a,b]$ be a map with $A_i = f_{i,0}^{-1}(a)$ and $B_i = f_{i,0}^{-1}(b)$. We construct maps $f_{i,j} : X \to [a,b]$, $i, j \in \mathbb{N}$, so as to satisfy the following conditions:

(1) $A_i = f_{i,j}^{-1}(a)$ and $B_i = f_{i,j}^{-1}(b)$;
(2) $f_{i,j}(x) \neq f_{i,j-1}(x) \Rightarrow f_{i,j-1}(x), f_{i,j}(x) \in (a_j, b_j)$
(i.e., $f_{i,j}|f_{i,j-1}^{-1}([a,a_j] \cup [b_j,b]) = f_{i,j-1}|f_{i,j-1}^{-1}([a,a_j] \cup [b_j,b])$);
(3) No $x \in X_n$ are contained in $n+1$ many $f_{i,j}^{-1}(q_j)$.

For each $(i,j) \in \mathbb{N}^2$, let $k(i,j) = \frac{1}{2}(i+j-2)(i+j-1) + j \in \mathbb{N}$. Then, (i,j) is the $k(i,j)$-th element of $\mathbb{N}^2$ in the ordering

$$(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), \ldots.$$

By induction on $k(i,j)$, we construct maps $f_{i,j}$ satisfying conditions (1), (2), and (3) above. Assume that $f_{i',j'}$ have been defined for $k(i',j') < m$. We will define $f_{i,j}$ for $k(i,j) = m$. Applying Lemma 5.9.4 to $L_0 = \emptyset$, $L_{k(i',j')} = f_{i',j'}^{-1}(q_{j'})$, $k(i',j') < m$, $A = f_{i,j-1}^{-1}([a,a_j])$, and $B = f_{i,j-1}^{-1}([b_j,b])$, we obtain a partition L_m in X between A and B such that

(∗) No $x \in X_n$ are contained in $n+1$ many sets L_i.

Then, we can easily obtain a map $f_{i,j} : X \to [a,b]$ such that $L_m = f_{i,j}^{-1}(q_j)$ and $f_{i,j}|A \cup B = f_{i,j-1}|A \cup B$, for which conditions (1), (2), and (3) are satisfied.

Since $b_j - a_j = 2^{-j}\delta_j$, it follows from (2) that $|f_{i,j}(x) - f_{i,j-1}(x)| < 2^{-j}\delta_j$ for each $x \in X$. Then, $(f_{i,j})_{j\in\mathbb{N}}$ uniformly converges to a map $f_i : X \to [a,b]$ and $|f_{i,j}(x) - f_i(x)| \leq 2^{-j}\delta_j$. For each $x \in A_i$, $f_i(x) = \lim_{j\to\infty} f_{i,j}(x) = a$ by (1). For each $x \in X \setminus A_i$, we have $k = \min\{j \in \mathbb{N} \mid f_{i,0}(x) > a_j\}$ because $f_{i,0}(x) > a = \inf_{j\in\mathbb{N}} a_j$. Then, $f_{i,0}(x) = f_{i,1}(x) = \cdots = f_{i,k-1}(x) > a_k$ and $f_{i,k}(x) > a_k = q_k - 2^{-k-1}\delta_k$, hence

$$f_i(x) \geq f_{i,k}(x) - 2^{-k}\delta_k > q_k - \delta_k \geq q_k - (q_k - a) = a.$$

Therefore, $A_i = f_i^{-1}(a)$. Similarly, we have $B_i = f_i^{-1}(b)$.

For each $x \in X_n$, let

$$M = \{i \in \mathbb{N} \mid f_{i,j}(x) = q_j \text{ for some } j \in \mathbb{N}\}.$$

Then, M has at most n many elements by (3). For $i \in \mathbb{N} \setminus M$ and $j \in \mathbb{N}$, let $K = \{k > j \mid f_{i,k}(x) \neq f_{i,j}(x)\}$. If $K = \emptyset$, then $f_i(x) = f_{i,j}(x) \neq q_j$ because $i \notin M$. Otherwise, let $k = \min K > j \geq 1$. Since $f_{i,k-1}(x) = f_{i,j}(x) \neq f_{i,k}(x)$, we have $a_k < f_{i,k}(x) < b_k$ by (2). Then, $|f_{i,k}(x) - q_j| \geq \delta_k - 2^{-k-1}\delta_k$. On the other hand, $|f_i(x) - f_{i,k}(x)| < 2^{-k+1}\delta_k$. Therefore,

$$\begin{aligned} |f_i(x) - q_j| &\geq |f_{i,k}(x) - q_j| - |f_i(x) - f_{i,k}(x)| \\ &> \delta_k - 2^{-k-1}\delta_k - 2^{-k+1}\delta_k > \tfrac{1}{4}\delta_k > 0. \end{aligned}$$

Thus, $\mathrm{card}\{i \mid f_i(x) \in (a,b) \cap \mathbb{Q}\} \leq n$ for $x \in X_n$. □

Proposition 5.9.6. *Let X be a metrizable space and Γ be an infinite set with $w(X) \leq \mathrm{card}\,\Gamma$. For each sequence $X_0, X_1, \dots \subset X$ with $\dim X_n \leq 0$, there exists an embedding $h : X \to J(\Gamma)^{\mathbb{N}}$ such that $h(X_n) \subset \nu_n(\Gamma)$.*

Proof. By Corollary 2.3.2, X has an open basis $\mathcal{B} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i$, where each $\mathcal{B}_i$ is discrete in X. Then, as is easily observed, $\mathrm{card}\,\mathcal{B}_i \leq w(X) \leq \mathrm{card}\,\Gamma$, hence we have $\Gamma_i \subset \Gamma$, $i \in \mathbb{N}$, such that $\mathrm{card}\,\mathcal{B}_i = \mathrm{card}\,\Gamma_i$ and $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$. For each $i \in \mathbb{N}$, we write $\mathcal{B}_i = \{B_\gamma \mid \gamma \in \Gamma_i\}$, where $B_\gamma \neq B_{\gamma'}$ if $\gamma \neq \gamma'$. Let $A_i = X \setminus \bigcup_{\gamma\in\Gamma_i} B_\gamma$. We apply Lemma 5.9.5 to obtain maps $f_i : X \to [0,1]$, $i \in \mathbb{N}$, such that $A_i = f_i^{-1}(0)$ and $\mathrm{card}\{i \in \mathbb{N} \mid f_i(x) \in (0,1) \cap \mathbb{Q}\} \leq n$ for $x \in X_n$. We define $h_i : X \to J(\Gamma)$ by

$$h_i(x)(\gamma) = \begin{cases} f_i(x) & \text{if } x \in B_\gamma, \gamma \in \Gamma_i, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $h_i(x) = f_i(x)\mathbf{e}_\gamma$ for $x \in B_\gamma$, $\gamma \in \Gamma_i$, and $h_i(x) = 0$ for $x \in A_i$. The desired embedding $h : X \to J(\Gamma)^{\mathbb{N}}$ can be defined by $h(x) = (h_i(x))_{i\in\mathbb{N}}$. Indeed, if $x \neq y \in X$, then $x \in B_\gamma$ and $y \notin B_\gamma$ for some $\gamma \in \Gamma_i$. Then, $h_i(x)(\gamma) = f_i(x) > 0 = h_i(y)(\gamma)$. Thus, h is injective. For each $\gamma \in \Gamma_i$, $U_\gamma = \{z \in J(\Gamma)^{\mathbb{N}} \mid z(i)(\gamma) > 0\}$ is open in $J(\Gamma)^{\mathbb{N}}$. Observe that $h(B_\gamma) = U_\gamma \cap h(X)$. Therefore, h is an embedding of X into $J(\Gamma)^{\mathbb{N}}$. For $x \in X_n$,

$$\mathrm{card}\{i \in \mathbb{N} \mid h_i(x) \notin P(\Gamma)\} = \mathrm{card}\{i \in \mathbb{N} \mid f_i(x) \in \mathbb{Q} \setminus \{0\}\} \leq n.$$

Then, it follows that $h(X_n) \subset \nu_n(\Gamma)$. □

Theorem 5.9.7. *Let Γ be an infinite set. The space $\nu_n(\Gamma)$ is a universal space for metrizable spaces X with $w(X) \leq \mathrm{card}\,\Gamma$ and $\dim X \leq n$, and the space $\nu_\omega(\Gamma)$ is a universal space for countable-dimensional metrizable spaces X with $w(X) \leq \mathrm{card}\,\Gamma$.*

Proof. We can write $X = \bigcup_{i\in\omega} X_i$, where $\dim X_i \leq 0$ and $X_i = \emptyset$ for $i > n$ if $\dim X = n$. The theorem follows from Proposition 5.9.6. □

Let X be a separable metrizable space with $\dim X \leq n$. In the proof of Proposition 5.9.6, we can take a $\mathcal{B}_i$ with only one element. Then, replacing $\mathbf{I}$ with $[a,1]$ where $a \in \mathbf{I}\setminus\mathbb{Q}$, the map $h : X \to [a,1]^{\mathbb{N}} \subset \mathbf{I}^{\mathbb{N}}$ defined by $h(x) = (f_i(x))_{i\in\mathbb{N}}$ is an embedding such that $h(X_n) \subset \nu_n$. Similar to Theorem 5.9.7, we have the following separable version:

Theorem 5.9.8. *The space* ν_n *is a universal space for separable metrizable spaces with* $\dim \leq n$ *and the space* ν_ω *is a universal space for countable-dimensional separable metrizable spaces.* □

Next, recall that $\mathbf{I}_f^{\mathbb{N}}$ is s.c.d. We now define

$$K_\omega = \bigcup_{n\in\mathbb{N}} \nu_n \times \big((0,1]^n \times \{0\}\big) \subset \nu_\omega \times \mathbf{I}_f^{\mathbb{N}}.$$ [11]

For an infinite set Γ, we define

$$K_\omega(\Gamma) = \bigcup_{n\in\mathbb{N}} \nu_n(\Gamma) \times \big((0,1]^n \times \{0\}\big) \subset \nu_\omega(\Gamma) \times \mathbf{I}_f^{\mathbb{N}} \subset J(\Gamma)^{\mathbb{N}} \times \mathbf{I}^{\mathbb{N}}.$$

Then, K_ω is separable and $w(K_\omega(\Gamma)) = \operatorname{card}\Gamma$. Moreover, $K_\omega(\Gamma)$ and K_ω are s.c.d. Indeed, for $(x,y) \in K_\omega(\Gamma)$,

$$(x,y) \in \bigcup_{i=1}^{n} \nu_i(\Gamma) \times \big((0,1]^i \times \{0\}\big) \Leftrightarrow y(n+1) = 0.$$

Hence, $\bigcup_{i=1}^n \nu_i(\Gamma) \times ((0,1]^i \times \{0\})$ is a closed set in $K_\omega(\Gamma)$, which is finite-dimensional by the Product Theorem 5.4.9 and the Addition Theorem 5.4.8.

Theorem 5.9.9. *Let* Γ *be an infinite set. The space* $K_\omega(\Gamma)$ *is a universal space for strongly countable-dimensional metrizable spaces* X *with* $w(X) \leq \operatorname{card}\Gamma$.

Proof. We can write $X = \bigcup_{i\in\mathbb{N}} F_i$, where each F_i is closed in X, $\dim F_i \leq i-1$, and $F_i \subset F_{i+1}$ for each $i \in \mathbb{N}$. By the Decomposition Theorem 5.4.5, we have a sequence $X_1, X_2, \dots \subset X$ such that $\dim X_n \leq 0$ and

$$F_1 = X_1,\ F_2 \setminus F_1 = X_2 \cup X_3,\ F_3 \setminus F_2 = X_4 \cup X_5 \cup X_6,\ \dots,$$

i.e., $F_i \setminus F_{i-1} = \bigcup_{n=k(i-1)+1}^{k(i)} X_n$, where $F_0 = \emptyset$ and $k(i) = \frac{1}{2}i(i+1)$. We apply Proposition 5.9.6 to obtain an embedding $h : X \to J(\Gamma)^{\mathbb{N}}$ such that $h(X_n) \subset \nu_n(\Gamma)$ for each $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $f_i : X \to \mathbf{I}$ be a map with $f_i^{-1}(0) = F_{i-1}$, and define a map $f : X \to \mathbf{I}^{\mathbb{N}}$ as follows:

$$f(x) = (f_1(x), f_2(x), f_2(x), f_3(x), f_3(x), f_3(x), \dots),$$

where each $f_i(x)$ appears i times, i.e., $\operatorname{pr}_n f = f_i$ for $k(i-1)+1 \leq n \leq k(i)$. Now, we can define the embedding $g : X \to J(\Gamma)^{\mathbb{N}} \times \mathbf{I}^{\mathbb{N}}$ by $g(x) = (h(x), f(x))$. For each $x \in X$, choose $i \in \mathbb{N}$ and $k(i-1)+1 \leq n \leq k(i)$ so that $x \in X_n \subset F_i \setminus F_{i-1}$. Then, $h(x) \in h(X_n) \subset \nu_n(\Gamma) \subset \nu_{k(i)}(\Gamma)$. Since $x \in F_i \setminus F_{i-1}$, it follows that

[11] This is different from the usual notation. In the literature for Dimension Theory, this space is represented by $K_\omega(\aleph_0)$ and K_ω stands for $\mathbf{I}_f^{\mathbb{N}}$.

$f_j(x) > 0$ for $j \leq i$ and $f_j(x) = 0$ for $j \geq i + 1$, i.e., $\mathrm{pr}_j f(x) > 0$ for $j \leq k(i)$ and $\mathrm{pr}_j f(x) = 0$ for $j \geq k(i) + 1$. Therefore, $f(x) \in (0, 1]^{k(i)} \times \{0\} \subset \mathbf{I}^{\mathbb{N}}$. Thus, we have

$$g(x) = (h(x), f(x)) \in \nu_{k(i)}(\Gamma) \times \big((0, 1]^{k(i)} \times \{0\}\big) \subset K_\omega(\Gamma).$$

Consequently, X can be embedded into $K_\omega(\Gamma)$. □

Similarly, we can obtain the following separable version:

Theorem 5.9.10. *The space K_ω is a universal space for strongly countable-dimensional separable metrizable spaces.* □

5.10 Nöbeling Spaces and Menger Compacta

In this section, we shall construct two universal spaces for separable metrizable spaces with $\dim \leq n$, which are named the n-dimensional Nöbeling space and the n-dimensional Menger compactum.

In the previous section, we defined the universal space ν_n. In the definition of ν_n, we replace $\mathbf{I}^{\mathbb{N}}$ with $\mathbb{R}^{2n+1}$ to define the ***n*-dimensional Nöbeling space** ν^n, that is,

$$\begin{aligned}\nu^n &= \big\{x \in \mathbb{R}^{2n+1} \bigm| x(i) \in \mathbb{R} \setminus \mathbb{Q} \text{ except for } n \text{ many } i\big\}\\ &= \big\{x \in \mathbb{R}^{2n+1} \bigm| x(i) \in \mathbb{Q} \text{ at most } n \text{ many } i\big\},\end{aligned}$$

which is the n-dimensional version of the space of irrationals $\nu^0 = \mathbb{R} \setminus \mathbb{Q}$. Similar to Theorem 5.9.3, we can see $\dim \nu^n = n$. Observe

$$\mathbb{R}^{2n+1} \setminus \nu^n = \big\{x \in \mathbb{R}^{2n+1} \bigm| x(i) \in \mathbb{Q} \text{ at least } n + 1 \text{ many } i\big\},$$

which is a countable union of n-dimensional flats that are closed in $\mathbb{R}^{2n+1}$. Then, ν^n is a G_δ-set in $\mathbb{R}^{2n+1}$, hence it is completely metrizable. Thus, we have the following proposition:

Proposition 5.10.1. *The space ν^n is a separable completely metrizable space with* $\dim \nu^n = n$. □

Moreover, ν^n has the additional property:

Proposition 5.10.2. *Each point of ν^n has an arbitrarily small neighborhood that is homeomorphic to ν^n. In fact, $\nu^n \cap \prod_{i=1}^{2n+1}(a_i, b_i) \approx \nu^n$ for each $a_i < b_i \in \mathbb{Q}$, $i = 1, \ldots, 2n + 1$.*

Proof. Let $\varphi : \mathbb{R} \to (-1, 1)$ be the homeomorphism defined by

$$\varphi(t) = \frac{t}{1 + |t|} \quad \left(\varphi_i^{-1}(s) = \frac{s}{1 - |s|}\right).$$

We define a homeomorphism $h : \mathbb{R}^{2n+1} \to \prod_{i=1}^{2n+1}(a_i, b_i)$ as follows:

$$h(x) = (h_1(x(1)), \ldots, h_{2n+1}(x(2n+1))),$$

where $h_i : \mathbb{R} \to (a_i, b_i)$ is the homeomorphism defined by

$$h_i(t) = \frac{b_i - a_i}{2}(\varphi(t) + 1) + a_i.$$

Since $h_i(\mathbb{Q}) = \mathbb{Q} \cap (a_i, b_i)$, we have $h(\nu^n) = \nu^n \cap \prod_{i=1}^{2n+1}(a_i, b_i)$. □

We will show the universality of ν^n.

Theorem 5.10.3. *The n-dimensional Nöbeling space ν^n is a universal space for separable metrizable spaces with* $\dim \leq n$.

According to the Compactification Theorem 5.7.2, every n-dimensional separable metrizable space X has an n-dimensional metrizable compactification. Theorem 5.10.3 comes from the following proposition:

Proposition 5.10.4. *For each locally compact separable metrizable space X with* $\dim X \leq n$ *and $\mathcal{U} \in \operatorname{cov}(\mathbb{R}^{2n+1})$, every proper map $f : X \to \mathbb{R}^{2n+1}$ is $\mathcal{U}$-close to a closed embedding $g : X \to \nu^n$. If X is compact, then $\mathbb{R}^{2n+1}$ can be replaced by* $\mathbf{I}^{2n+1}$.

This can be shown by a modification of the proof of the Embedding Approximation Theorem 5.8.10 (or 5.8.5). To this end, we need the following generalization of Theorem 5.8.9:

Lemma 5.10.5. *Let X and Y be locally compact metrizable spaces and $Y_0 = \bigcap_{n\in\mathbb{N}} G_n \subset Y$, where each G_n is open in Y (hence Y_0 is a G_δ-set in Y). Suppose that for each $n \in \mathbb{N}$ and each open cover $\mathcal{U}$ of X consisting of open sets with the compact closures,* $\mathrm{C}_{\mathcal{U}}(X, G_n)$ *is dense in the space* $\mathrm{C}^P(X, Y)$ *with the limitation topology. Then,* $\operatorname{Emb}(X, Y_0)$ *is dense in* $\mathrm{C}^P(X, Y)$.

Proof. Observe that

$$\begin{aligned}\operatorname{Emb}(X, Y_0) &= \operatorname{Emb}(X, Y) \cap \mathrm{C}(X, Y_0)\\ &= \bigcap_{n\in\mathbb{N}} \mathrm{C}_{\mathcal{U}_n}(X, Y) \cap \bigcap_{n\in\mathbb{N}} \mathrm{C}(X, G_n) = \bigcap_{n\in\mathbb{N}} \mathrm{C}_{\mathcal{U}_n}(X, G_n),\end{aligned}$$

where $\mathcal{U}_n \in \operatorname{cov}(X)$ consists of open sets with the compact closures and mesh $\mathcal{U}_n < 2^{-n}$. By the assumption, each $\mathrm{C}_{\mathcal{U}}(X, G_n)$ is open and dense in $\mathrm{C}^P(X, Y)$. Since $\mathrm{C}^P(X, Y)$ is a Baire space by Theorem 2.9.8, we have the desired result. □

Proof of Proposition 5.10.4. According to the definition of ν^n, we can write

$$\nu^n = \mathbb{R}^{2n+1} \setminus \bigcup_{i\in\mathbb{N}} H_i = \bigcap_{i\in\mathbb{N}} (\mathbb{R}^{2n+1} \setminus H_i),$$

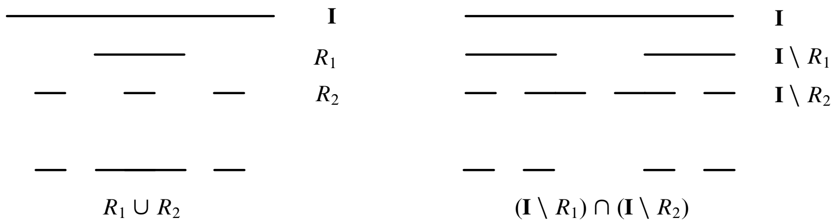

Fig. 5.4 R_k and $\mathbf{I} \setminus R_k$

where each H_i is an n-dimensional flat. Because of Lemma 5.10.5, it suffices to show that, for each n-dimensional flat H in $\mathbb{R}^{2n+1}$ and each $\mathcal{U} \in \operatorname{cov}(X)$ consisting of open sets with the compact closure, $\mathrm{C}_{\mathcal{U}}(X, \mathbb{R}^{2n+1} \setminus H)$ is dense in the space $\mathrm{C}^P(X, \mathbb{R}^{2n+1})$ with the limitation topology.

In the proof of 5.8.10, we can choose $v_i \in \mathbb{R}^{2n+1}$, $i \in \mathbb{N}$, to satisfy the additional condition that the flat hull of every $n+1$ many points $v_{i_1}, \ldots, v_{i_{n+1}}$ misses H (i.e., $\mathrm{fl}\{v_{i_1}, \ldots, v_{i_{n+1}}\} \cap H = \emptyset$). Thus, we can obtain the PL embedding $g : |K| \to \mathbb{R}^{2n+1}$ such that $g(|K|) \cap H = \emptyset$. The map $g\varphi : X \to \mathbb{R}^{2n+1} \setminus H$ is a $\mathcal{U}$-map that is $\mathcal{V}$-close to f.

If X is compact, we can replace $\mathbb{R}^{2n+1}$ by $\mathbf{I}^{2n+1}$ to obtain the additional statement. □

Remark 15. It is known that if X is a separable completely metrizable space with $\dim X \leq n$, then every map $f : X \to \nu^n$ can be approximated by closed embeddings $h : X \to \nu^n$. Refer to Remark 14.

Before defining the n-dimensional Menger compactum, let us recall the construction of the Cantor (ternary) set μ^0. We can geometrically describe $\mu^0 \subset \mathbf{I}$ as follows: For each $k \in \mathbb{N}$, let

$$R_k = \bigcup_{m=0}^{3^{k-1}-1} (m/3^{k-1} + 1/3^k, m/3^{k-1} + 2/3^k) \subset \mathbf{I}.$$

Then, $\mu^0 = \bigcap_{k \in \mathbb{N}} (\mathbf{I} \setminus R_k) = \mathbf{I} \setminus \bigcup_{k \in \mathbb{N}} R_k$ (Fig. 5.4). Observe that

$$\bigcap_{i=1}^{k} (\mathbf{I} \setminus R_i) = [0, 3^{-k}] + V_k^0, \quad \text{where} \quad V_k^0 = \left\{ \sum_{i=1}^{k} \frac{2x(i)}{3^i} \;\middle|\; x \in \mathbf{2}^k \right\}.$$

Moreover, $\{3^{-k}\mu^0 + v \mid v \in V_k^0\}$ is an open cover of μ^0 with ord $= 1$, where

$$\begin{aligned} \mu^0 \approx 3^{-k}\mu^0 + v &= \mu^0 \cap ([0, 3^{-k}] + v) \\ &= \mu^0 \cap ((-3^{-k-1}, 3^{-k} + 3^{-k-1}) + v). \end{aligned}$$

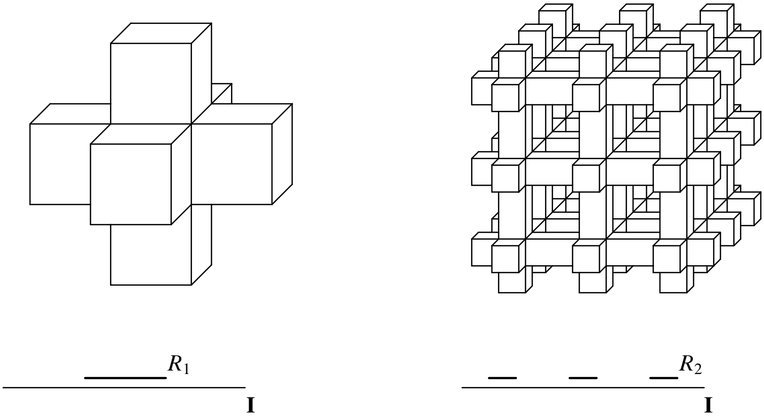

Fig. 5.5 $\mathbf{I}^3 \setminus M_1^3$ and $\mathbf{I}^3 \setminus M_2^3$

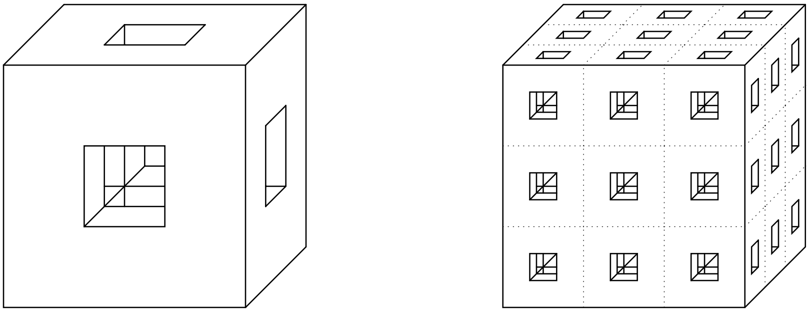

Fig. 5.6 M_1^3 and M_2^3

As the n-dimensional version of μ^0, the ***n*-dimensional Menger compactum** μ^n is defined as follows: For each $k \in \mathbb{N}$, let

$$M_k^{2n+1} = \{x \in \mathbf{I}^{2n+1} \mid x(i) \in \mathbf{I} \setminus R_k \text{ except for } n \text{ many } i\}$$
$$= \{x \in \mathbf{I}^{2n+1} \mid x(i) \in R_k \text{ at most } n \text{ many } i\},$$

where it should be noted that

$$\mathbf{I} \setminus M_k^{2n+1} = \{x \in \mathbf{I}^{2n+1} \mid x(i) \in R_k \text{ at least } n+1 \text{ many } i\}.$$

Now, we define $\mu^n = \bigcap_{k \in \mathbb{N}} M_k^{2n+1}$. Since each M_k^{2n+1} is compact, μ^n is also compact. See Figs. 5.5–5.7.

Proposition 5.10.6. *For each $n \in \mathbb{N}$, $\dim \mu^n = n$.*

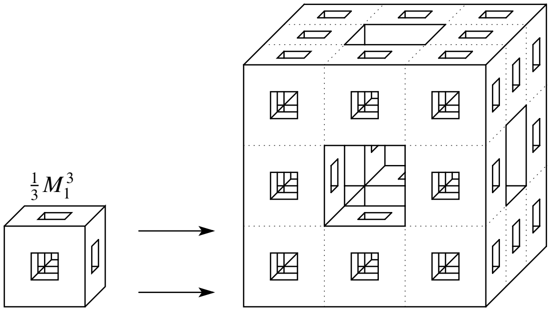

Fig. 5.7 $M_1^3 \cap M_2^3$

Proof. Since μ^n contains every n-face of $\mathbf{I}^{2n+1}$, it follows that $\dim \mu^n \geq n$. We can apply Proposition 5.8.12 to see $\dim \mu^n \leq n$. We use the metric $d \in \mathrm{Metr}(\mathbf{I}^{2n+1})$ defined by

$$d(x, y) = \max\{|x(i) - y(i)| \mid i = 1, \ldots, 2n + 1\}.$$

For each $\varepsilon > 0$, choose $k \in \mathbb{N}$ so large that $2/3^k < \varepsilon$. Let K be the cell complex consisting of all faces of $(2n + 1)$-cubes

$$\prod_{i=1}^{2n+1} \left[\frac{m_i - 1}{3^{k-1}}, \frac{m_i}{3^{k-1}}\right] \subset \mathbf{I}^{2n+1},\ m_i = 1, \ldots, 3^k.$$

Since $\mu^n \subset M_k^{2n+1}$, it suffices to construct an ε-map of M_k^{2n+1} to $|K^{(n)}|$, where $K^{(n)}$ is the n-skeleton of K.

For each $C \in K$ with $\dim C > n$, let $r_C : C \setminus \{\hat{C}\} \to \partial C$ be the radial retraction, where $\hat{C}$ is the barycenter of C and ∂C is the radial boundary of C. Observe that $M_k^{2n+1} \cap C \subset C \setminus \{\hat{C}\}$ and $r_C(M_k^{2n+1} \cap C) \subset M_k^{2n+1} \cap \partial C$. For each $m \geq n$, we can define a retraction

$$r_m : M_k^{2n+1} \cap |K^{(m+1)}| \to M_k^{2n+1} \cap |K^{(m)}|$$

by $r_m|C = r_C$ for each $(m + 1)$-cell $C \in K$. Since $|K^{(n)}| \subset M_k^{2n+1}$, we have a retraction

$$r = r_n \cdots r_{2n} : M_k^{2n+1} \to |K^{(n)}|.$$

By construction, $r^{-1}(x) \subset \mathrm{st}(x, K)$ for each $x \in |K^{(n)}|$. Since $\mathrm{mesh}\, K = 1/3^{k-1} < \varepsilon/2$, it follows that r is an ε-map. □

For each $k \in \mathbb{N}$, $\mu^n \approx 3^{-k}\mu^n \subset [0, 3^{-k}]$. Let

$$V_k^n = \left\{v \in 3^{-k}\mathbb{Z}^{2n+1} \;\middle|\; [0, 3^{-k}]^{2n+1} + v \subset M_k^{2n+1}\right\}.$$

Then, $M_k^{2n+1} = \bigcup_{v \in V_k^n}([0, 3^{-k}]^{2n+1} + v)$ and $\mu^n = 3^{-k}\mu^n + V_k^n$. Thus, we have the following proposition:

Proposition 5.10.7. *Every neighborhood of each point of μ^n contains a copy of μ^n.* □

We will show the universality of μ^n.

Theorem 5.10.8. *The n-dimensional Menger compactum μ^n is a universal space for separable metrizable spaces with* $\dim \leq n$.

Proof. By Theorem 5.10.3, it suffices to prove that every compact set X in $\mathbf{I}^{2n+1} \cap \nu^n$ can be embedded in μ^n.

First, note that

$$X \subset \{x \in \mathbf{I}^{2n+1} \mid x(i) \neq 1/2 \text{ except for } n \text{ many } i\}.$$

Then, we have a rational $q_1 > 0$ such that

$$X \subset \{x \in \mathbf{I}^{2n+1} \mid x(i) \in \mathbf{I} \setminus R_1^X \text{ except for } n \text{ many } i\},$$

where $R_1^X = (1/2 - q_1, 1/2 + q_1) \subset (0, 1)$. Let $A_1^X = \{1/2 - q_1, 1/2 + q_1\}$ be the set of end-points of R_1^X and let $g_1 : \mathbf{I} \to \mathbf{I}$ be the PL homeomorphism defined by $g_1(0) = 0$, $g_1(1) = 1$, and $g_1(1/2 \pm q_1) = 1/2 \pm 1/6$, i.e., $g_1(A_1^X) = \{1/3, 2/3\}$. Observe that $|g_1(s) - s| < 3^{-1}$ for every $s \in \mathbf{I}$.

Let B_1^X be the set of mid-points of components of $\mathbf{I} \setminus A_1^X$, i.e., $B_1^X = \{1/2, 1/2^2 - q_1/2, 3/2^2 + q_1/2\} \subset \mathbb{Q}$. Note that

$$X \subset \{x \in \mathbf{I}^{2n+1} \mid x(i) \in \mathbf{I} \setminus B_1^X \text{ except for } n \text{ many } i\}.$$

Then, we have a rational $q_2 > 0$ such that $2q_2$ is smaller than the diameter of each component of $\mathbf{I} \setminus A_1^X$, and

$$X \subset \{x \in \mathbf{I}^{2n+1} \mid x(i) \in \mathbf{I} \setminus R_2^X \text{ except for } n \text{ many } i\},$$

where $R_2^X = \bigcup_{b \in B_1^X}(b - q_2, b + q_2)$. Let A_2^X be the set of end-points of components of R_2^X and let $g_2 : \mathbf{I} \to \mathbf{I}$ be the PL homeomorphism defined by $g_2(0) = 0$, $g_2(1) = 1$, $g_2(A_1^X \cup A_2^X) = \{m/3^2 \mid m = 1, \ldots, 3^2 - 1\}$. Then, $g_2|A_1^X = g_1|A_1^X$ and $|g_2(s) - g_1(s)| < 3^{-2}$ for every $s \in \mathbf{I}$.

Let B_2^X be the set of mid-points of components of $\mathbf{I} \setminus (A_1^X \cup A_2^X)$. Then, $B_2^X \subset \mathbb{Q}$. Since

$$X \subset \{x \in \mathbf{I}^{2n+1} \mid x(i) \in \mathbf{I} \setminus B_2^X \text{ except for } n \text{ many } i\},$$

we have a rational $q_3 > 0$ such that $2q_3$ is smaller than the diameter of each component of $\mathbf{I} \setminus (A_1^X \cup A_2^X)$, and

$$X \subset \{x \in \mathbf{I}^{2n+1} \mid x(i) \in \mathbf{I} \setminus R_3^X \text{ except for } n \text{ many } i\},$$

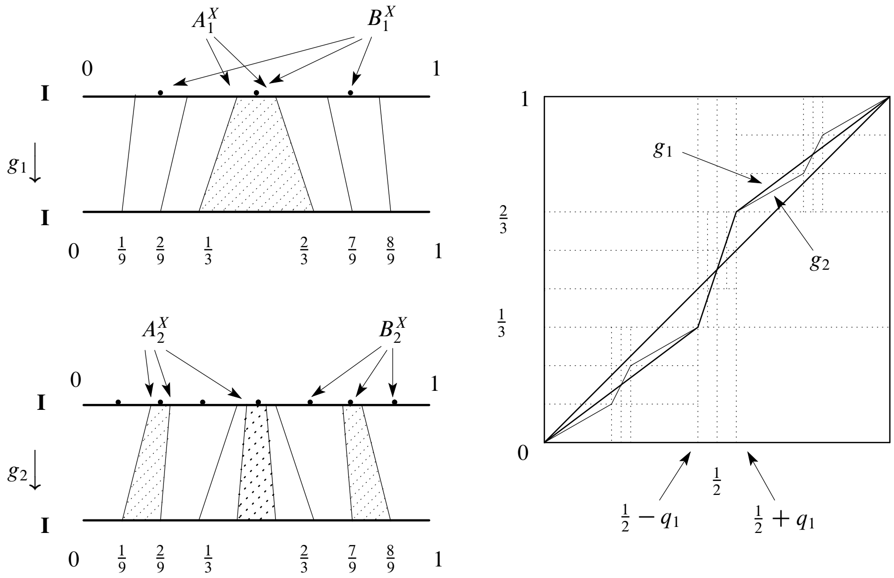

Fig. 5.8 Homeomorphisms $g_1, g_2, \ldots$

where $R_3^X = \bigcup_{b \in B_2^X} (b - q_2, b + q_2)$. Let A_3^X be the set of end-points of components of R_3^X and let $g_3 : \mathbf{I} \to \mathbf{I}$ be the PL homeomorphism defined by $g_3(0) = 0$, $g_3(1) = 1$, $g_3(A_1^X \cup A_2^X \cup A_3^X) = \{m/3^3 \mid m = 1, \ldots, 3^3 - 1\}$. Then, $g_3|A_1^X \cup A_2^X = g_2|A_1^X \cup A_2^X$ and $|g_3(s) - g_2(s)| < 3^{-3}$ for every $s \in \mathbf{I}$ — (Fig. 5.8).

By induction, we obtain $R_k^X, A_k^X \subset \mathbf{I}$ ($k \in \mathbb{N}$) such that R_k^X is the union of 3^{k-1} many disjoint open intervals, A_k^X is the set of all end-points of components of R_k^X, each component of R_k^X is contained in some component of $\mathbf{I} \setminus A_{k-1}^X$, and

$$X \subset \left\{ x \in \mathbf{I}^{2n+1} \;\middle|\; x(i) \in \mathbf{I} \setminus R_k^X \text{ except for } n \text{ many } i \right\}.$$

Hence, X is contained in

$$\mu_X^n = \bigcap_{k \in \mathbb{N}} \left\{ x \in \mathbf{I}^{2n+1} \;\middle|\; x(i) \in \mathbf{I} \setminus R_k^X \text{ except for } n \text{ many } i \right\}.$$

At the same time, we have the PL homeomorphisms $g_k : \mathbf{I} \to \mathbf{I}$, $k \in \mathbb{N}$, such that

$$g_k(0) = 0,\; g_k(1) = 1,\; g_k\big(\textstyle\bigcup_{i=1}^k A_i^X\big) = \{m/3^k \mid m = 1, \ldots, 3^k - 1\},$$

$$g_k \Big| \bigcup_{i=1}^{k-1} A_i^X = g_{k-1} \Big| \bigcup_{i=1}^{k-1} A_i^X \text{ and } \big|g_k(s) - g_{k-1}(s)\big| < 3^{-k} \text{ for every } s \in \mathbf{I}.$$

Then, $(g_k)_{k \in \mathbb{N}}$ uniformly converges to a map $g : \mathbf{I} \to \mathbf{I}$. Since $A = \bigcup_{i=1}^\infty A_i^X$ is dense in $\mathbf{I}$ and g maps A onto $\{m/3^k \mid k \in \mathbb{N},\ m = 1, \ldots, 3^k - 1\}$ in the same order, it follows that g is bijective, hence g is a homeomorphism. Let $h : \mathbf{I}^{2n+1} \to \mathbf{I}^{2n+1}$

be the homeomorphism defined by $h(x) = (g(x(1)), \dots, g(x(2n+1)))$. As is easily observed, $h(\mu^n_X) = \mu^n$, hence $h(X) \subset \mu^n$. □

We also have the following theorem:

Theorem 5.10.9. *Let X be a compactum with* $\dim X \leq n$. *Then, every map $f : X \to \mu^n$ can be approximated by embeddings into μ^n.*

Proof. By Proposition 5.10.4, f can be approximated by embeddings f' into $M^{2n+1}_k \cap \nu^n$ for an arbitrarily large $k \in \mathbb{N}$. Replacing X by $f'(X)$ in the proof of Theorem 5.10.8, we can take $R^X_i = R_i$ and $g_i = \mathrm{id}$ for $i \leq k$. Therefore, f can be approximated by embeddings like hf'. □

5.11 Total Disconnectedness and the Cantor Set

A space X is said to be **totally disconnected** provided that, for any two distinct points $x \neq y \in X$, there is a clopen set H in X such that $x \in H$ but $y \notin H$ (i.e., the empty set is a partition between any two distinct points). Equivalently, for each $x \in X$ the intersection of all clopen sets containing x is the singleton $\{x\}$. According to Theorem 5.3.8, the 0-dimensionality implies the total disconnectedness. We say that X is **hereditarily disconnected** if every non-degenerate subset of X is disconnected (i.e., every component of X is a singleton). Clearly, the total disconnectedness implies the hereditary disconnectedness. Therefore, we have the following fact:

Fact. *Every* 0*-dimensional space is totally disconnected, and every totally disconnected space is hereditarily disconnected.*

The converse assertions are true for compact spaces. To see this, we prove the following lemma:

Lemma 5.11.1. *Let X be compact, $x \in X$, and C be the intersection of all clopen sets in X containing x.*

(1) *For each open neighborhood U of C in X, there is a clopen set H in X such that $C \subset H \subset U$.*
(2) *C is the component of X containing x.*

Proof. (1): Let $\mathcal{H}$ be all the clopen sets in X containing x. Since $X \setminus U$ is compact and $\{X \setminus H \mid H \in \mathcal{H}\}$ is its open cover in X, there are $H_1, \dots, H_n \in \mathcal{H}$ such that

$$X \setminus U \subset \bigcup_{i=1}^{n} (X \setminus H_i) = X \setminus \bigcap_{i=1}^{n} H_i .$$

Thus, we have $H = \bigcap_{i=1}^{n} H_i \in \mathcal{H}$ and $C \subset H \subset U$.

(2): Since C clearly contains the component of X containing x, it suffices to show that C is connected. Now assume that $C = A \cup B$, where A and B are disjoint closed sets in C and $x \in A$. From the normality, it follows that there are disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$. By (1), we have a clopen set H in X such that $C \subset H \subset U \cup V$. Since $H \cap U$ is open in X and $H \setminus V$ is closed in X, $H \cap U = H \setminus V$ is clopen in X. Then, $C \subset H \cap U \subset U$, which implies that $B \subset C \cap V = \emptyset$. Thus, C is connected. □

Theorem 5.11.2. *For every non-empty compact space X, the following are equivalent:*

(a) $\dim X = 0$;
(b) *X is totally disconnected;*
(c) *X is hereditarily disconnected.*

Proof. The implications (a) ⇒ (b) ⇒ (c) follow from the above Fact. Here, we will prove the converse implications.

(c) ⇒ (b): For each $x \in X$, the intersection of all clopen sets in X containing x is a component of X by Lemma 5.11.1(2). It is, in fact, the singleton $\{x\}$, which means that X is totally disconnected.

(b) ⇒ (a): Let $\mathcal{U}$ be a finite open cover of X. Each $x \in X$ belongs to some $U \in \mathcal{U}$. Because of the total disconnectedness of X, the singleton $\{x\}$ is the intersection of all clopen sets in X containing x. By Lemma 5.11.1(1), we have a clopen set H_x in X such that $x \in H_x \subset U$. From the compactness, it follows that $X = \bigcup_{i=1}^n H_{x_i}$ for some $x_1, \ldots, x_n \in X$. Let

$$V_1 = H_{x_1}, V_2 = H_{x_2} \setminus H_{x_1}, \ldots, V_n = H_{x_n} \setminus (H_{x_1} \cup \cdots \cup H_{x_{n-1}}).$$

Then, $\mathcal{V} = \{V_1, \ldots, V_n\}$ is an open refinement of $\mathcal{U}$ and $\operatorname{ord} \mathcal{V} = 1$. Hence, we have $\dim X = 0$. □

The implications (c) ⇒ (b) ⇒ (a) in Theorem 5.11.2 do not hold in general. In the next section, we will show the existence of nonzero-dimensional totally disconnected spaces, i.e., counter-examples for (b) ⇒ (a). Here, we give a counter-example for (c) ⇒ (b) via the following theorem:

Theorem 5.11.3. *There exists a separable metrizable space that is hereditarily disconnected but not totally disconnected.*

Example and Proof. Take a countable dense set D in the Cantor set μ^0 and define

$$X = D \times \mathbb{Q} \cup (\mu^0 \setminus D) \times (\mathbb{R} \setminus \mathbb{Q}) \subset \mu^0 \times \mathbb{R}.$$

Let $p : X \to \mu^0$ be the restriction of the projection of $\mu^0 \times \mathbb{R}$ onto μ^0.

First, we show that X is hereditarily disconnected. Let $A \subset X$ be a non-degenerate subset. When $\operatorname{card} p(A) > 1$, since μ^0 is hereditarily disconnected, $p(A)$ is disconnected, which implies that A is disconnected. When $\operatorname{card} p(A) = 1$, $A \subset p(A) \times \mathbb{Q} \approx \mathbb{Q}$ or $A \subset p(A) \times (\mathbb{R} \setminus \mathbb{Q}) \approx \mathbb{R} \setminus \mathbb{Q}$. Since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are hereditarily disconnected, A is disconnected.

Next, we prove that X is not totally disconnected. Assume that X is totally disconnected and let $x_0 \in D \subset \mu^0$. Because $(x_0, 0), (x_0, 1) \in X$, we have closed sets F_0 and F_1 in $\mu^0 \times \mathbb{R}$ such that $X \subset F_0 \cup F_1$, $F_0 \cap F_1 \cap X = \emptyset$, and $(x_0, i) \in F_i$. Then, $F_0 \cap X$ and $F_1 \cap X$ are clopen in X. Choose an open neighborhood U_0 of x_0 in μ^0 so that

$$(U_0 \cap D) \times \{i\} = U_0 \times \{i\} \cap X \subset F_i \quad \text{for } i = 0, 1.$$

Since F_i is closed in $\mu^0 \times \mathbb{R}$ and D is dense in μ^0, it follows that $U_0 \times \{i\} \subset F_i$. For each $r \in \mathbb{Q}$, let

$$C_r = \{x \in U_0 \mid (x, r) \in F_0 \cap F_1\}.$$

Then, each C_r is closed and nowhere dense in U_0. Indeed, for each $x \in U_0 \setminus C_r$, because $(x, r) \notin F_0 \cap F_1$ and $F_0 \cap F_1$ is closed in $\mu^0 \times \mathbb{R}$, x has a neighborhood U in U_0 such that $U \times \{r\} \cap F_0 \cap F_1 = \emptyset$. Then, $(y, r) \notin F_0 \cap F_1$ for all $y \in U$, i.e., $U \cap C_r = \emptyset$, so C_r is closed in U_0. Since $F_0 \cap F_1 \cap X = \emptyset$ and $(x, r) \in X$ for $x \in D$, we have $C_r \subset U_0 \setminus D$, which implies that C_r is nowhere dense in U_0.

We will show that $U_0 \setminus D = \bigcup_{r \in \mathbb{Q}} C_r$. Then,

$$U_0 = \bigcup_{r \in \mathbb{Q}} C_r \cup \bigcup_{x \in D \cap U_0} \{x\},$$

which is contrary to the Baire Category Theorem 2.5.1. Thus, it would follow that X is not totally disconnected. For each $x \in U_0 \setminus D$,

$$\{x\} \times \mathbb{R} = \mathrm{cl}_{\mu^0 \times \mathbb{R}} \{x\} \times (\mathbb{R} \setminus \mathbb{Q}) \subset \mathrm{cl}_{\mu^0 \times \mathbb{R}} X \subset F_0 \cup F_1.$$

If $x \notin \bigcup_{r \in \mathbb{Q}} C_r$, then $F_0 \cap F_1 \cap \{x\} \times \mathbb{Q} = \emptyset$ because $x \notin C_r$ for all $r \in \mathbb{Q}$. Therefore,

$$F_0 \cap F_1 \cap \{x\} \times \mathbb{R} = F_0 \cap F_1 \cap \{x\} \times (\mathbb{R} \setminus \mathbb{Q}) \subset F_0 \cap F_1 \cap X = \emptyset.$$

Because $(x, i) \in F_i \cap \{x\} \times \mathbb{R}$, this contradicts the connectedness of $\mathbb{R}$. Therefore, $x \in \bigcup_{r \in \mathbb{Q}} C_r$ and the proof is complete. □

In the remainder of this section, we give a characterization of the Cantor set μ^0 and show that every compactum is a continuous image of μ^0. Recall that $\mu^0 \approx \mathbf{2}^{\mathbb{N}}$, where $\mathbf{2} = \{0, 1\}$ is the discrete space of two points. In the following, μ^0 can be replaced by $\mathbf{2}^{\mathbb{N}}$ (cf. Sect. 1.1).

Theorem 5.11.4 (Characterization of the Cantor Set). *A space X is homeomorphic to the Cantor set μ^0 if and only if X is a totally disconnected compactum with no isolated points.*

Proof. It suffices to show the "if" part. Since $\mu^0 \approx \mathbf{2}^{\mathbb{N}}$, we will construct a homeomorphism $h : \mathbf{2}^{\mathbb{N}} \to X$. Let $d \in \mathrm{Metr}(X)$ with $\mathrm{diam}\, X < 1$. First, note that

(∗) Each non-empty open set in X can be written as the disjoint union of an arbitrary finite number of non-empty open sets.

In fact, because X has no isolated points, each non-empty open set U in X is non-degenerate and $\dim U = 0$ by Theorem 5.11.2 and the Subset Theorem 5.3.3. We apply Theorem 5.2.3 iteratively to obtain the fact $(*)$.

Using the fact $(*)$, we will construct a sequence $1 = n_0 < n_1 < \cdots$ in $\mathbb{N}$ and

$$\mathcal{E}_n = \{E(x) \mid x \in \mathbf{2}^n\} \in \operatorname{cov}(X), \quad n \in \mathbb{N},$$

so that

(1) Each $E(x) \in \mathcal{E}_n$ is non-empty, so non-degenerate;
(2) $\operatorname{mesh} \mathcal{E}_{n_i} < 2^{-i}$;
(3) $E(x) \cap E(y) = \emptyset$ if $x \neq y \in \mathbf{2}^n$; and
(4) $E(x) = E(x(1), \ldots, x(n), 0) \cup E(x(1), \ldots, x(n), 1)$ for all $x \in \mathbf{2}^n$.

By $(*)$, we have $\mathcal{E}_1 = \{E(0), E(1)\} \in \operatorname{cov}(X)$ such that $E(0)$ and $E(1)$ are non-empty, $E(0) \cap E(1) = \emptyset$, and $\operatorname{mesh} \mathcal{E}_1 \leq \operatorname{diam} X < 1 = 2^0$. Assume that $1 = n_0 < \cdots < n_{i-1}$ and $\mathcal{E}_1, \ldots, \mathcal{E}_{n_{i-1}}$ have been defined. For each $x \in \mathbf{2}^{n_{i-1}}$, $E(x) \in \mathcal{E}_{n_{i-1}}$ is a compactum as a clopen set in X. Since $\dim E(x) = 0$, $E(x)$ has a finite open cover $\mathcal{U}_x$ with $\operatorname{ord} \mathcal{U}_x = 1$ and $\operatorname{mesh} \mathcal{U}_x < 2^{-i}$ (Theorem 5.3.1). Choose $m \in \mathbb{N}$ so that $\operatorname{card} \mathcal{U}_x \leq 2^m$ for each $x \in \mathbf{2}^{n_{i-1}}$. Using the fact $(*)$, as a refinement of $\mathcal{U}_x$, we can obtain

$$\mathcal{E}_x = \{E(x, y) \mid y \in \mathbf{2}^m\} \in \operatorname{cov}(E(x)),$$

where $E(x, y) \neq \emptyset$ for every $y \in \mathbf{2}^m$. Then, $\operatorname{mesh} \mathcal{E}_x < 2^{-i}$. We define $n_i = m + n_{i-1} > n_{i-1}$ and

$$\mathcal{E}_{n_i} = \bigcup_{x \in \mathbf{2}^{n_{i-1}}} \mathcal{E}_x = \{E(x, y) \mid (x, y) \in \mathbf{2}^{n_{i-1}} \times \mathbf{2}^m = \mathbf{2}^{n_i}\}.$$

Thus, we have $\mathcal{E}_{n_i} \in \operatorname{cov}(X)$ with $\operatorname{mesh} \mathcal{E}_{n_i} < 2^{-i}$. By the downward induction using formula (4), we can define $\mathcal{E}_{n_i - 1}, \ldots, \mathcal{E}_{n_{i-1}+1} \in \operatorname{cov}(X)$. Therefore, we obtain $\mathcal{E}_1, \ldots, \mathcal{E}_{n_i} \in \operatorname{cov}(X)$.

For each $x \in \mathbf{2}^{\mathbb{N}}$, $\bigcap_{n \in \mathbb{N}} E(x(1), \ldots, x(n)) \neq \emptyset$ because of the compactness of X. Since

$$\lim_{n \to \infty} \operatorname{diam} E(x(1), \ldots, x(n)) = 0,$$

we can define $h : \mathbf{2}^{\mathbb{N}} \to X$ by

$$\{h(x)\} = \bigcap_{n \in \mathbb{N}} E(x(1), \ldots, x(n)).$$

To show that h is a homeomorphism, it suffices to prove that h is a continuous bijection because $\mathbf{2}^{\mathbb{N}}$ is compact. For each $\varepsilon > 0$, choose $i \in \mathbb{N}$ so that $2^{-i} < \varepsilon$. Then, $\operatorname{mesh} \mathcal{E}_{n_i} < \varepsilon$ by (2). For each $x, y \in \mathbf{2}^{\mathbb{N}}$, $x(1) = y(1), \ldots, x(n_i) = y(n_i)$ imply $h(x), h(y) \in E(x(1), \ldots, x(n_i)) \in \mathcal{E}_{n_i}$, so $d(h(x), h(y)) < \varepsilon$. Hence, h is continuous. It easily follows from (3) that h is injective. By (4), for each $y \in X$, we can inductively choose $x(n) \in \mathbf{2}$, $n \in \mathbb{N}$, so that $y \in E(x(1), \ldots, x(n))$. Then, we have $x \in \mathbf{2}^{\mathbb{N}}$ such that $y \in \bigcap_{n \in \mathbb{N}} E(x(1), \ldots, x(n))$, i.e., $y = h(x)$. Hence, h is surjective. This completes the proof. □

The Cantor set is very important because of the following theorem:

Theorem 5.11.5. *Every compactum X is a continuous image of the Cantor set, that is, there exists a continuous surjection $f : \mu^0 \to X$.*

The proof consists of a combination of the following two propositions.

Proposition 5.11.6. *Every separable metrizable space X is a continuous image of a subspace of the Cantor set.*

Proof. We have a natural continuous surjection $\varphi : \mu^0 \to \mathbf{I}$ defined by $\varphi(\sum_{i \in \mathbb{N}} 2x_i/3^i) = \sum_{i \in \mathbb{N}} x_i/2^i$, where $x_i \in \mathbf{2} = \{0, 1\}$. Since $(\mu^0)^{\mathbb{N}} \approx \mu^0$, the Hilbert cube $\mathbf{I}^{\mathbb{N}}$ is a continuous image of the Cantor set. Therefore, the result follows from the fact that every separable metrizable space can be embedded in $\mathbf{I}^{\mathbb{N}}$ (Corollary 2.3.8). □

Proposition 5.11.7. *Any non-empty closed set A in μ^0 is a retract of μ^0, that is, there is a map $r : \mu^0 \to A$ with $r|A = \mathrm{id}$.*

Proof. Since $\mu^0 \approx \mathbf{2}^{\mathbb{N}}$, we may replace μ^0 by $\mathbf{2}^{\mathbb{N}}$. For each $x \in \mathbf{2}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we inductively define $x^A(n) \in \mathbf{2}$ as follows:

$$x^A(n) = \begin{cases} x(n) & \text{if } (x^A(1), \ldots, x^A(n-1), x(n)) \in p_n(A), \\ 1 - x(n) & \text{otherwise,} \end{cases}$$

where $p_n : \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^n$ is the projection onto the first n factors. Since $A \neq \emptyset$, $(x^A(1), \ldots, x^A(n)) \in p_n(A)$ for each $n \in \mathbb{N}$. Since A is closed in $\mathbf{2}^{\mathbb{N}}$, it follows that $x^A = (x^A(n))_{n \in \mathbb{N}} \in A$. It is obvious that $x^A = x$ for $x \in A$. We can define a retraction $r : \mathbf{2}^{\mathbb{N}} \to A$ by $r(x) = x^A$. For each $x, y \in \mathbf{2}^{\mathbb{N}}$,

$$p_n(x) = p_n(y) \Rightarrow p_n(r(x)) = p_n(x^A) = p_n(y^A) = p_n(r(y)),$$

hence r is continuous. □

5.12 Totally Disconnected Spaces with dim ≠ 0

In this section, we will construct totally disconnected separable metrizable spaces X with $\dim X \neq 0$. The first example called the **Erdös space** is constructed in the proof of the following theorem. This space is also an example of spaces X such that $\dim X^2 \neq 2 \dim X$.

Theorem 5.12.1. *There exists a 1-dimensional totally disconnected separable metrizable space X that is homeomorphic to $X^2 = X \times X$.*

Example and Proof. The desired space X is a subspace of the Hilbert space ℓ_2 defined as follows:

$$X = \{x \in \ell_2 \mid x(i) \in \mathbb{Q} \text{ for all } i \in \mathbb{N}\}.$$

The space $\ell_2 \times \ell_2$ has the norm $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$. Then, the map $h : \ell_2 \times \ell_2 \to \ell_2$ defined by $h(x, y)(2i - 1) = x(i)$ and $h(x, y)(2i) = y(i)$ is an isometry, hence it is a homeomorphism. Since $h(X \times X) = X$, we have $X \times X \approx X$.

To prove the total disconnectedness of X, let $x \neq y \in X$. Then, $x(i_0) \neq y(i_0)$ for some $i_0 \in \mathbb{N}$. Without loss of generality, we may assume that $x(i_0) < y(i_0)$. Choose $t \in \mathbb{R} \setminus \mathbb{Q}$ so that $x(i_0) < t < y(i_0)$. Then, $H = \{z \in X \mid z(i_0) < t\}$ is clopen in X and $x \in H$ but $y \notin H$. Hence, X is totally disconnected.

Note that $\dim X = \operatorname{ind} X$ by the Coincidence Theorem 5.5.2. Next, we show that $\operatorname{ind} X > 0$ and $\operatorname{ind} X \leq 1$. If so, we would have $\dim X = \operatorname{ind} X = 1$.

To show that $\operatorname{ind} X > 0$, it suffices to prove that $\operatorname{bd} U \neq \emptyset$ for every open neighborhood U of 0 contained in $\mathrm{B}(0, 1) = \{x \in X \mid \|x\| < 1\}$. We can inductively choose $a_1, a_2, \dots \in \mathbb{Q}$ so that

$$x_n = (a_1, \dots, a_n, 0, 0, \dots) \in U \quad \text{and} \quad d(x_n, X \setminus U) < 1/n.$$

In fact, when $a_1, \dots, a_n$ have been chosen, let

$$k_0 = \min \{k \in \mathbb{N} \mid (a_1, \dots, a_n, k/(n + 2), 0, 0, \dots) \notin U\}.$$

Then, $(k_0 - 1)/(n + 2) \in \mathbb{Q}$ is the desired a_{n+1}. Since $\sum_{i=1}^{n} a_i^2 < 1$ for each n, it follows that $\sum_{i=1}^{\infty} a_i^2 \leq 1 < \infty$, hence $x_0 = (a_i)_{i \in \mathbb{N}} \in X$. Since $x_n \to x_0$ $(n \to \infty)$, it follows that $x_0 \in \operatorname{cl} U$. On the other hand, since $d(x_n, X \setminus U) < 1/n$, we have $x_0 \in \operatorname{cl}(X \setminus U)$. Therefore, $x_0 \in \operatorname{bd} U$.

To show that $\operatorname{ind} X \leq 1$, it suffices to prove that each $F_n = \{x \in X \mid \|x\| = 1/n\}$ is 0-dimensional. Note that $F_n \subset \mathbb{Q}^{\mathbb{N}}$ as sets. Furthermore, the topology on F_n coincides with the product inherited from the product space $\mathbb{Q}^{\mathbb{N}}$ (Proposition 1.2.4). Since $\dim \mathbb{Q}^{\mathbb{N}} = 0$, we have $\dim F_n = 0$ by the Subset Theorem 5.3.3. The proof is complete. □

To construct totally disconnected metrizable spaces X of arbitrarily large dimensions, we need the following lemmas:

Lemma 5.12.2. *Let $(A_\gamma, B_\gamma)_{\gamma \in \Gamma}$ be an essential family of pairs of disjoint closed sets in a compact space X and $\gamma_0 \in \Gamma$. For each $\gamma \in \Gamma \setminus \{\gamma_0\}$, let L_γ be a partition between A_γ and B_γ in X and $L = \bigcap_{\gamma \in \Gamma \setminus \{\gamma_0\}} L_\gamma$. Then, L has a component that meets both A_{γ_0} and B_{γ_0}.*

Proof. Assume that L has no components that meet both A_{γ_0} and B_{γ_0}. Let D be the union of all components of L that meet A_{γ_0}, where we allow the case $D = \emptyset$ or $D = L$. For each $x \in L \setminus D$, the component C_x of L containing x misses A_{γ_0}. By Lemma 5.11.1(1), we have a clopen set E_x in L such that $C_x \subset E_x \subset L \setminus A_{\gamma_0}$. For each $y \in E_x$, the component C_y of L with $y \in C_y$ is contained in E_x, hence $C_y \cap A_{\gamma_0} = \emptyset$. Then, it follows that $E_x \subset L \setminus D$. Therefore, $L \setminus D$ is open in L, that is, D is closed in L, so it is compact.

For each $x \in D$, the component of L containing x misses B_{γ_0} by the assumption. As above, we have a clopen set E_x in L such that $x \in E_x \subset L \setminus B_{\gamma_0}$. Since D is compact, $D \subset \bigcup_{i=1}^n E_{x_i}$ for some $x_1, \dots, x_n \in D$. Then, $E = \bigcup_{i=1}^n E_{x_i}$ is clopen in L and $A_{\gamma_0} \cap L \subset D \subset E \subset L \setminus B_{\gamma_0}$.

By the normality of X, we have disjoint open sets U and V in X such that $A_{\gamma_0} \cup E \subset U$ and $B_{\gamma_0} \cup (L \setminus E) \subset V$. Then, $L_{\gamma_0} = X \setminus (U \cup V)$ is a partition between A_{γ_0} and B_{γ_0} in X and $\bigcap_{\gamma \in \Gamma} L_\gamma = L \cap L_{\gamma_0} = \emptyset$. This is contrary to the essentiality of $(A_\gamma, B_\gamma)_{\gamma \in \Gamma}$. Therefore, L has a component that meets both A_{γ_0} and B_{γ_0}. □

Lemma 5.12.3. *Let X be a compactum and $f : X \to Y$ be a continuous surjection. Then, X has a G_δ-subset S that meets each fiber of f at precisely one point, that is,*

$$\operatorname{card}(f^{-1}(y) \cap S) = 1 \quad \textit{for each } \ y \in Y.$$

Proof. We may assume that $X \subset \mathbf{I}^{\mathbb{N}}$. For each $y \in Y$, since $f^{-1}(y)$ is non-empty and compact, we can define $g(y) \in X$ as follows:

$$g(y)(1) = \min \operatorname{pr}_1(f^{-1}(y)) \quad \text{and}$$

$$g(y)(n) = \min \operatorname{pr}_n\big(f^{-1}(y) \cap \textstyle\bigcap_{i=1}^{n-1} \operatorname{pr}_i^{-1}(g(y)(i))\big) \ \text{ for } n > 1.$$

Then, $\emptyset \neq f^{-1}(y) \cap \bigcap_{i=1}^n \operatorname{pr}_i^{-1}(g(y)(i)) \subset \operatorname{pr}_n^{-1}(g(y)(n))$. By the compactness of $f^{-1}(y)$, we have

$$\emptyset \neq f^{-1}(y) \cap \bigcap_{i \in \mathbb{N}} \operatorname{pr}_i^{-1}(g(y)(i)) \subset \bigcap_{n \in \mathbb{N}} \operatorname{pr}_n^{-1}(g(y)(n)) = \{g(y)\},$$

which means $g(y) \in f^{-1}(y)$. Thus, the set $S = \{g(y) \mid y \in Y\}$ meets each fiber of f at precisely one point.

For each $n, m \in \mathbb{N}$, let

$$F_{n,m} = \Big\{x \in X \ \Big|\ \exists z \in X \ \text{ such that } \ z(i) = x(i) \ \text{ for } i < n, \\ z(n) \leq x(n) - \tfrac{1}{m} \ \text{ and } \ f(z) = f(x)\Big\}.$$

Since X is a compactum, it is easy to see that $F_{n,m}$ is closed in X, hence $U_{n,m} = X \setminus F_{n,m}$ is open in X. We show that $S = \bigcap_{n,m \in \mathbb{N}} U_{n,m}$, which is a G_δ-set in X. For each $y \in Y$, if $z \in X$, $z(i) = g(y)(i)$ for all $i < n$ and $z(n) \leq g(y)(n) - \frac{1}{m}$, then $f(z) \neq y = f(g(y))$; otherwise $g(y)(n) \leq z(n)$ $(< g(y)(n))$ by the definition of $g(y)$. Thus, $g(y) \in U_{n,m}$ for all $n, m \in \mathbb{N}$, i.e., $S \subset \bigcap_{n,m \in \mathbb{N}} U_{n,m}$. Conversely, for each $x \in \bigcap_{n,m \in \mathbb{N}} U_{n,m}$, let $y = f(x)$ (i.e., $x \in f^{-1}(y)$). Then, $x = g(y) \in S$. Otherwise, let $n = \min\{i \in \mathbb{N} \mid x(i) \neq g(y)(i)\}$. Since $g(y)(n) < x(n)$ by the definition of $g(y)$, it follows that $g(y)(n) \leq x(n) - \frac{1}{m}$ for some $m \in \mathbb{N}$, i.e., $x \in F_{n,m} = X \setminus U_{n,m}$, which is a contradiction. □

For a metric space $X = (X, d)$, let $\mathrm{Comp}(X)$ be the space of all non-empty compact sets in X that admits the **Hausdorff metric** d_H defined as follows:

$$\begin{aligned} d_H(A, B) &= \inf\big\{r > 0 \bigm| A \subset \mathrm{N}_d(B, r),\ B \subset \mathrm{N}_d(A, r)\big\} \\ &= \max\big\{\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A)\big\}. \end{aligned}$$

According to the following proposition, the topology of $\mathrm{Comp}(X)$ induced by the Hausdorff metric d_H coincides with the Vietoris topology defined in Sect. 3.8.

Proposition 5.12.4. *For a metric space $Y = (Y, d)$, the Vietoris topology on* $\mathrm{Comp}(Y)$ *is induced by the Hausdorff metric d_H. Consequently, the space* $\mathrm{Comp}(X)$ *with the Vietoris topology is metrizable if Y is metrizable.*

Proof. For each $A \in \mathrm{Comp}(Y)$ and $r > 0$, we can choose $a_1, \ldots, a_n \in A$ so that $A \subset \bigcup_{i=1}^n \mathrm{B}(a_i, r/2)$. Then,

$$A \in \left(\bigcup_{i=1}^n \mathrm{B}(a_i, r/2)\right)^+ \cap \bigcap_{i=1}^n \mathrm{B}(a_i, r/2)^- \cap \mathrm{Comp}(Y) \subset \mathrm{B}_{d_H}(A, r),$$

which means that $\mathrm{B}_{d_H}(A, r)$ is a neighborhood of A in the Vietoris topology.[12]

Let $A \in \mathrm{Comp}(Y)$. For each open set U in Y with $A \in U^-$, taking $a \in A \cap U$, we have $\mathrm{B}_{d_H}(A, d(a, Y \setminus U)) \subset U^-$. On the other hand, for each open set U in Y with $A \in U^+$, we have $\mathrm{B}_{d_H}(A, \delta) \subset U^+$, where $\delta = \mathrm{dist}(A, Y \setminus U) > 0$. Thus, $\{\mathrm{B}_{d_H}(A, r) \mid r > 0\}$ is a neighborhood basis at $A \in \mathrm{Comp}(Y)$. □

Note. When $Y = (Y, d)$ is a bounded metric space, the Hausdorff metric d_H is defined on the set $\mathrm{Cld}(Y)$ consisting of all non-empty closed sets in Y, which induces a topology different from the Vietoris topology if Y is non-compact. If Y is unbounded, then $d_H(A, B) = \infty$ for some $A, B \in \mathrm{Cld}(Y)$. But, even in this case, d_H induces the topology on $\mathrm{Cld}(Y)$. We should note that this topology is dependent on the metric d. For example, $\mathrm{Cld}(\mathbb{R})$ is non-separable with respect to the Hausdorff metric induced by the usual metric. In fact, it has no countable open basis because $\mathfrak{P}_0(\mathbb{N})$ is an uncountable discrete set of $\mathrm{Cld}(\mathbb{R})$. On the other hand, $\mathbb{R}$ is homeomorphic to the unit open interval $(0, 1)$ and $\mathrm{Cld}((0, 1))$ is separable with respect to the Hausdorff metric induced by the usual metric because $\mathrm{Fin}((0, 1))$ is dense in $\mathrm{Cld}((0, 1))$.

As observed in Sect. 3.8, the space $\mathrm{Cld}(Y)$ with the Vietoris topology is Hausdorff if and only if Y is regular. Here, it is remarked that $\mathrm{Cld}(Y)$ is metrizable if and only if Y is compact and metrizable. Indeed, if Y is compact metrizable then $\mathrm{Cld}(Y) = \mathrm{Comp}(Y)$ is metrizable by Proposition 5.12.4. Conversely, if Y is non-compact then Y contains a countable discrete set. Then, $\mathfrak{P}_0(\mathbb{N}) = \mathrm{Cld}(\mathbb{N})$ can be embedded into $\mathrm{Cld}(Y)$ as a subspace, which implies that $\mathfrak{P}_0(\mathbb{N})$ is metrizable. Note that $\mathfrak{P}_0(\mathbb{N})$ is separable because $\mathrm{Fin}(\mathbb{N})$ is dense in $\mathfrak{P}_0(\mathbb{N})$. Thus, $\mathfrak{P}_0(\mathbb{N})$ is second countable. Let $\mathcal{B}$ be a countable open base for $\mathfrak{P}_0(\mathbb{N})$. For each $A \in \mathfrak{P}_0(\mathbb{N})$, choose $B_A \in \mathcal{B}$ so that $A \in B_A \subset A^+$. When $A \neq A' \in \mathfrak{P}_0(\mathbb{N})$, we may assume $A \setminus A' \neq \emptyset$. Then, $A \notin B_{A'}$. Hence, we have $B_A \neq B_{A'}$. Consequently, $\mathrm{card}\, \mathcal{B} \geq \mathrm{card}\, \mathfrak{P}_0(\mathbb{N}) = 2^{\aleph_0}$, which is a contradiction.

[12]Recall $U^- = \{A \subset Y \mid A \cap U \neq \emptyset\}$ and $U^+ = \{A \subset Y \mid A \subset U\}$.

Theorem 5.12.5. *Let $X = (X, d)$ be a metric space.*

(1) *If X is totally bounded then so is* $\mathrm{Comp}(X)$ *with respect to* d_H.
(2) *If X is complete then so is* $\mathrm{Comp}(X)$ *with respect to* d_H.
(3) *If X is compact then so is* $\mathrm{Comp}(X)$.

Proof. (1): For each $\varepsilon > 0$, we have $F \in \mathrm{Fin}(X)$ such that $d(x, F) < \varepsilon$ for every $x \in X$. Then, $\mathrm{Fin}(F)$ is a finite subset of $\mathrm{Comp}(X)$. For each $A \in \mathrm{Comp}(X)$, let $F_A = \{z \in F \mid d(z, A) < \varepsilon\}$. For each $x \in A$, we have $z \in F$ such that $d(x, z) < \varepsilon$, which implies that $z \in F_A$. Then, $F_A \neq \emptyset$ (i.e., $F_A \in \mathrm{Fin}(F)$) and $d_H(A, F_A) < \varepsilon$. Hence, $\mathrm{Comp}(X)$ is totally bounded.

(2): Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{Comp}(X)$. If $(A_n)_{n\in\mathbb{N}}$ has a convergent subsequence, then $(A_n)_{n\in\mathbb{N}}$ itself is convergent. Hence, it can be assumed that $d_H(A_n, A_i) < 2^{-n-1}$ for each $n < i$. Then, we prove that $(A_n)_{n\in\mathbb{N}}$ converges to

$$A_0 = \bigcap_{n\in\mathbb{N}} \mathrm{cl}\,\mathrm{N}(A_n, 2^{-n}) \in \mathrm{Comp}(X).$$

To this end, since A_0 is closed in X and $A_0 \subset \mathrm{N}(A_n, 2^{-n+1})$ for each $n \in \mathbb{N}$, it suffices to show that A_0 is totally bounded and $A_n \subset \mathrm{N}(A_0, 2^{-n})$ for each $n \in \mathbb{N}$.

First, we show that $A_n \subset \mathrm{N}(A_0, 2^{-n})$. For each $x \in A_n$, inductively choose $x_i \in A_i$, $i > n$, so that $d(x_i, x_{i-1}) < 2^{-i}$, where $x = x_n$. Since $(x_i)_{i\geq n}$ is a Cauchy sequence in X, it converges to some $x_0 \in X$. For each $i \geq n$,

$$d(x_i, x_0) \leq \sum_{j=i}^{\infty} d(x_j, x_{j+1}) < \sum_{j=i}^{\infty} 2^{-j-1} = 2^{-i},$$

hence $d(x_0, A_i) < 2^{-i}$ and $d(x_0, x) < 2^{-n}$. Moreover, for each $i < n$,

$$d(x_0, A_i) \leq d(x_0, x) + d(x, A_i) < 2^{-n} + 2^{-i-1} \leq 2^{-i}.$$

Therefore, $x_0 \in \bigcap_{i\in\mathbb{N}} \mathrm{N}(A_i, 2^{-i}) \subset A_0$, so $A_0 \neq \emptyset$ and $x \in \mathrm{N}(A_0, 2^{-n})$.

To see the total boundedness of A_0, let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ so that $2^{-n+1} < \varepsilon/3$, and take a finite $\varepsilon/3$-dense subset $\{u_1, \ldots, u_k\}$ of A_n.[13] For each $i = 1, \ldots, k$, choose $v_i \in A_0$ so that $d(u_i, v_i) < 2^{-n}$. Then, $\{v_1, \ldots, v_k\}$ is an ε-dense subset of A_0. Indeed, for each $x \in A_0$, we have $y \in A_n$ such that $d(x, y) < 2^{-n+1}$. Then, $d(y, u_i) < \varepsilon/2$ for some $i = 1, \ldots, k$. Hence,

$$d(x, v_i) \leq d(x, y) + d(y, u_i) + d(u_i, v_i) < 2^{-n+1} + \varepsilon/3 + 2^{-n} < \varepsilon.$$

(3): This is a combination of (1) and (2). □

Theorem 5.12.6. *For each $n \in \mathbb{N}$, there exists an n-dimensional totally disconnected separable completely metrizable space. In addition, there exists a strongly infinite-dimensional totally disconnected separable completely metrizable space.*

[13] In a metric space $X = (X, d)$, $A \subset X$ is said to be **ε-dense** if $d(x, A) < \varepsilon$ for each $x \in X$.

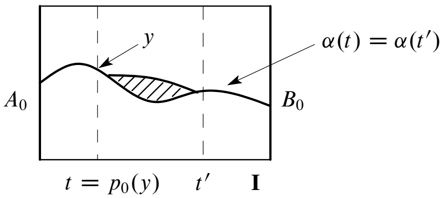

Fig. 5.9 $\alpha(t), t \in \mu^0$

Proof (Example and Proof). To construct the examples simultaneously, let $X = \mathbf{I} \times \mathbf{I}^\Gamma$ and $d \in \mathrm{Metr}(X)$ where $\Gamma = \{1, \ldots, n\}$ in the n-dimensional case and $\Gamma = \mathbb{N}$ in the infinite-dimensional case. Let $p_0 : X \to \mathbf{I}$ be the projection onto the first factor. Put $A_0 = p_0^{-1}(0)$ and $B_0 = p_0^{-1}(1)$ and define

$$\mathcal{E} = \{E \in \mathrm{Comp}(X) \mid E \text{ is connected, } E \cap A_0 \neq \emptyset,\ E \cap B_0 \neq \emptyset\}.$$

Then, $\mathcal{E}$ is closed in $\mathrm{Comp}(X)$. Indeed, let $D \in \mathrm{Comp}(X) \setminus \mathcal{E}$. When D is not connected, it can be written as the disjoint union of two non-empty closed subsets D_1 and D_2. Let $\varepsilon = \frac{1}{2}\mathrm{dist}_d(D_1, D_2) > 0$. Then, every $E \in \mathrm{B}_{d_H}(D, \varepsilon)$ is not connected because E is contained in $\mathrm{N}_d(D_1, \varepsilon) \cup \mathrm{N}_d(D_2, \varepsilon)$ and meets both $\mathrm{N}_d(D_1, \varepsilon)$ and $\mathrm{N}_d(D_2, \varepsilon)$. Hence, $\mathrm{B}_{d_H}(D, \varepsilon) \cap \mathcal{E} = \emptyset$. If $D \cap A_0 = \emptyset$, then $\mathrm{N}_d(D, \delta) \cap A_0 = \emptyset$, where $\delta = \mathrm{dist}_d(A_0, D) > 0$. Every $E \in \mathrm{B}_{d_H}(D, \delta)$ also misses A_0, which implies $\mathrm{B}_{d_H}(D, \delta) \cap \mathcal{E} = \emptyset$. The case $D \cap B_0 = \emptyset$ is identical.

Since $\mathrm{Comp}(X)$ is compact by Theorem 5.12.5(3), $\mathcal{E}$ is also compact. Then, we have a map $\alpha : \mu^0 \to \mathcal{E}$ of the Cantor set μ^0 onto $\mathcal{E}$ by Theorem 5.11.5. We define

$$Y = \{y \in p_0^{-1}(\mu^0) \mid y \in \alpha p_0(y)\} \subset X.$$

Obviously, $p_0(Y) \subset \mu^0$. For each $t \in \mu^0$, since $\alpha(t)$ is a continuum that meets both A_0 and B_0, it follows that $p_0\alpha(t) = \mathbf{I}$, so $t = p_0(y)$ for some $y \in \alpha(t)$, where $y \in Y$ (Fig. 5.9). Thus, we have $p_0(Y) = \mu^0$. Moreover, Y is closed in X, so is compact. Indeed, let $(y_i)_{i\in\mathbb{N}}$ be a sequence in Y converging to $y \in X$. Since $p_0(y_i) \in \mu^0$ for every $i \in \mathbb{N}$ and $(p_0(y_i))_{i\in\mathbb{N}}$ converges to $p_0(y)$, we have $p_0(y) \in \mu^0$. Since $y_i \in \alpha p_0(y_i)$ for every $i \in \mathbb{N}$ and $(\alpha p_0(y_i))_{i\in\mathbb{N}}$ converges to $\alpha p_0(y)$ in $\mathcal{E}$, it easily follows that $y \in \alpha p_0(y)$, hence $y \in Y$.

By Lemma 5.12.3, Y has a G_δ-subset S such that

$$\mathrm{card}(p_0^{-1}(t) \cap S) = 1 \ \text{ for each } t \in \mu^0.$$

Since Y is compact, S is completely metrizable. Since $p_0|S : S \to \mu^0$ is a continuous bijection and μ^0 is totally disconnected, it follows that S is also totally disconnected. Moreover, $S \cap E \neq \emptyset$ for every $E \in \mathcal{E}$. Indeed, because $\mathcal{E} = \alpha p_0(S)$, we can find $y \in S \subset Y$ such that $E = \alpha p_0(y)$, where $y \in \alpha p_0(y) = E$.

Now, for each $i \in \Gamma$, let $p_i : X \to \mathbf{I}$ be the projection onto the i-th coordinates of the second factor $\mathbf{I}^\Gamma$. Since $p_i^{-1}(0), p_i^{-1}(1) \in \mathcal{E}$, it follows that $A_i = S \cap p_i^{-1}(0) \neq \emptyset$ and $B_i = S \cap p_i^{-1}(1) \neq \emptyset$. Then, $(A_i, B_i)_{i\in\Gamma}$ is essential in S. In fact, by the

Partition Extension Lemma 5.3.7, for each partition L_i between A_i and B_i in S, we have a partition $\tilde{L}_i$ between $p_i^{-1}(0)$ and $p_i^{-1}(1)$ in X such that $\tilde{L}_i \cap S \subset L_i$. According to Lemma 5.12.2, the intersection of the partions $\tilde{L}_i$ has a component $E \in \mathcal{E}$. Then, $\bigcap_{i \in \Gamma} L_i \supset E \cap S \neq \emptyset$. Therefore, S is s.i.d. when $\Gamma = \mathbb{N}$. In the case that $\Gamma = \{1, \ldots, n\}$, $\dim S \geq n$ by Theorem 5.2.17. Since $S \subset \mu^0 \times \mathbf{I}^\Gamma$, $\dim S \leq n$ by the Subset Theorem 5.3.3 and the Product Theorem 5.4.9, hence $\dim S = n$. □

5.13 Examples of Infinite-Dimensional Spaces

In this section, we construct two infinite-dimensional compacta. One is weakly infinite-dimensional but not countable-dimensional. The other is hereditarily infinite-dimensional. First, we present the following theorem:

Theorem 5.13.1. *There exists a weakly infinite-dimensional compact metrizable space that contains a strongly infinite-dimensional subspace, and hence it is not countable-dimensional.*

Example and Proof. Let S be an s.i.d. totally disconnected separable completely metrizable space (Theorem 5.12.6) and let $X = \gamma S$ be a compactification of S with the c.d. remainder (Theorem 5.7.4). Then, we show that X is the required example.

First, X contains the s.i.d. subset S, so X is not c.d. (Theorem 5.6.2). To see that X is w.i.d., let $(A_i, B_i)_{i \in \omega}$ be a family of pairs of disjoint closed sets in X. Since $X \setminus S$ is c.d., $X \setminus S = \bigcup_{i \in \mathbb{N}} X_i$, where $\dim X_i = 0$ for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, by Theorem 5.3.8 and the Partition Extension Lemma 5.3.7, X has a partition L_i between A_i and B_i such that $L_i \cap X_i = \emptyset$. Then,

$$L = \bigcap_{i \in \mathbb{N}} L_i \subset \bigcap_{i \in \mathbb{N}} X \setminus X_i = X \setminus \bigcup_{i \in \mathbb{N}} X_i = S.$$

If $L \neq \emptyset$, then L is compact and totally disconnected, which implies $\dim L = 0$ by Theorem 5.11.2. Again by Theorem 5.3.8 and the Partition Extension Lemma 5.3.7, X has a partition L_0 between A_0 and B_0 such that $L_0 \cap L = \emptyset$, which means $\bigcap_{i \in \omega} L_i = \emptyset$. □

For a compact space X and a metric space $Y = (Y, d)$, let $\mathrm{C}(X, Y)$ be the space of all maps from X to Y admitting the topology induced by the sup-metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$, which is identical to the compact-open topology because X is compact (cf. 1.1.3(6)). Then, from 1.1.3(5), we have the following lemma:

Lemma 5.13.2. *Let X be a compactum and $Y = (Y, d)$ be a separable metric space. The space $\mathrm{C}(X, Y)$ is separable.*

Note. This lemma can be proved directly as follows:

Sketch of Direct Proof. Let $\{U_i \mid i \in \mathbb{N}\}$ and $\{V_j \mid j \in \mathbb{N}\}$ be open bases for X and Y, respectively. For each $i, j \in \mathbb{N}$, let

$$W_{i,j} = \{f \in \mathrm{C}(X, Y) \mid f(\mathrm{cl}\, U_i) \subset V_j\}.$$

It is easy to prove that each $W_{i,j}$ is open in $\mathrm{C}(X, Y)$.

To see that $\{W_{i,j} \mid i, j \in \mathbb{N}\}$ is an open subbase for $\mathrm{C}(X, Y)$, let $f \in \mathrm{C}(X, Y)$ and $\varepsilon > 0$. For each $x \in X$, choose $i(x), j(x) \in \mathbb{N}$ so that $x \in U_{i(x)}$, $\operatorname{diam} V_{j(x)} < \varepsilon$ and $f(\mathrm{cl}\, U_{i(x)}) \subset V_{j(x)}$, i.e., $f \in W_{i(x),j(x)}$. Because of the compactness of X, we have $x_1, \dots, x_n \in X$ such that $X = U_{i(x_1)} \cup \cdots U_{i(x_n)}$. Then, observe $f \in \bigcap_{k=1}^n W_{i(x_k),j(x_k)} \subset \mathrm{B}_d(f, \varepsilon)$.

To construct a hereditarily infinite-dimensional space, we need the following key lemma:

Lemma 5.13.3. *Let $C \subset \mathbf{I}$ be homeomorphic to the Cantor set, $n \in \mathbb{N}$, and $\Gamma \subset \mathbb{N} \setminus \{n\}$ such that Γ and $\mathbb{N} \setminus \Gamma$ are infinite. Then, there exists a collection $\{S_i \mid i \in \Gamma\}$ of partitions S_i between $A_i = \mathrm{pr}_i^{-1}(0)$ and $B_i = \mathrm{pr}_i^{-1}(1)$ in $\mathbf{I}^{\mathbb{N}}$ such that every subset $X \subset \bigcap_{i\in\Gamma} S_i$ is strongly infinite-dimensional if $C \subset \mathrm{pr}_n(X)$, where $\mathrm{pr}_i : \mathbf{I}^{\mathbb{N}} \to \mathbf{I}$ is the projection of $\mathbf{I}^{\mathbb{N}}$ onto the i-th factor.*

Proof. Without loss of generality, we may assume that $n = 1$ and $\Gamma = \{2i \mid i \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, let $C_i = \mathrm{pr}_i^{-1}([0, \frac{1}{4}])$ and $D_i = \mathrm{pr}_i^{-1}([\frac{3}{4}, 1])$. We define

$$\Omega = \{f \in \mathrm{C}(\mathbf{I}^{\mathbb{N}}, \mathbf{I}^{\mathbb{N}}) \mid \forall i \in \mathbb{N},\ f^{-1}(A_i) = C_{2i},\ f^{-1}(B_i) = D_{2i}\}.$$

Since Ω is separable by Lemma 5.13.2, there exist $T \subset C$ and a continuous surjection $\psi : T \to \Omega$ by Proposition 5.11.6. Let $E = \mathrm{pr}_1^{-1}(T) \subset \mathbf{I}^{\mathbb{N}}$ and define a map $\varphi : E \to \mathbf{I}^{\mathbb{N}}$ by $\varphi(x) = (\psi\mathrm{pr}_1(x))(x)$. For each $i \in \mathbb{N}$,

$$\begin{aligned}\varphi^{-1}(A_i) &= \{x \in E \mid \varphi(x) = (\psi\mathrm{pr}_1(x))(x) \in A_i\} \\ &= \{x \in E \mid x \in (\psi\mathrm{pr}_1(x))^{-1}(A_i) = C_{2i}\} = E \cap C_{2i}\end{aligned}$$

and similarly $\varphi^{-1}(B_i) = E \cap D_{2i}$. Since $\mathrm{pr}_i^{-1}(\frac{1}{2})$ is a partition between A_i and B_i in $\mathbf{I}^{\mathbb{N}}$, $\varphi^{-1}(\mathrm{pr}_i^{-1}(\frac{1}{2}))$ is a partition between $C_{2i} \cap E$ and $D_{2i} \cap E$ in E. By the Partition Extension Lemma 5.3.7, we have a partition S_{2i} between A_{2i} and B_{2i} in $\mathbf{I}^{\mathbb{N}}$ such that $S_{2i} \cap E \subset \varphi^{-1}(\mathrm{pr}_i^{-1}(\frac{1}{2}))$. It should be noted that $(A_{2i} \cap \mathrm{pr}_1^{-1}(x), B_{2i} \cap \mathrm{pr}_1^{-1}(x))_{i\in\mathbb{N}}$ is essential in $\mathrm{pr}_1^{-1}(x)$ for every $x \in C$. Then, $\mathrm{pr}_1^{-1}(x) \cap \bigcap_{i\in\mathbb{N}} S_{2i} \neq \emptyset$ for every $x \in C$, hence $C \subset \mathrm{pr}_1(\bigcap_{i\in\mathbb{N}} S_{2i})$.

Take $X \subset \bigcap_{i\in\mathbb{N}} S_{2i}$ such that $C \subset \mathrm{pr}_1(X)$. We will show that X is s.i.d., that is, X has an infinite essential family of pairs of disjoint closed sets. For each $i \in \mathbb{N}$, let $C'_i = \mathrm{pr}_i^{-1}([0, \frac{1}{3}]) \cap X$ and $D'_i = \mathrm{pr}_i^{-1}([\frac{2}{3}, 1]) \cap X$. To see that $(C'_{2i}, D'_{2i})_{i\in\mathbb{N}}$ is essential, let L_i be a partition between C'_{2i} and D'_{2i} in X. By the Partition Extension Lemma 5.3.7, we have a partition H_i between C_{2i} and D_{2i} in $\mathbf{I}^{\mathbb{N}}$ such that $H_i \cap X \subset L_i$. There is a map $f_i : \mathbf{I}^{\mathbb{N}} \to \mathbf{I}$ such that $f_i^{-1}(0) = C_{2i}$, $f_i^{-1}(1) = D_{2i}$, and

$f_i^{-1}(\frac{1}{2}) = H_i$.[14] Indeed, let U_i and V_i be disjoint open sets in $\mathbf{I}^{\mathbb{N}}$ such that $C_{2i} \subset U_i$, $D_{2i} \subset V_i$, and $X \setminus H_i = U_i \cup V_i$. We can take maps $g_i : X \setminus V_i \to \mathbf{I}$ and $h_i : X \setminus U_i \to \mathbf{I}$ such that $g_i^{-1}(0) = C_{2i}$, $g_i^{-1}(1) = H_i$, $h_i^{-1}(0) = H_i$, and $h_i^{-1}(1) = D_{2i}$ (cf. Theorem 2.2.6). The desired f_i can be defined by

$$f_i(x) = \begin{cases} \frac{1}{2}g_i(x) & \text{if } x \in X \setminus V_i, \\ \frac{1}{2} + \frac{1}{2}h_i(x) & \text{if } x \in X \setminus U_i. \end{cases}$$

Now, we define a map $f : \mathbf{I}^{\mathbb{N}} \to \mathbf{I}^{\mathbb{N}}$ by $f(x) = (f_i(x))_{i\in\mathbb{N}}$. For each $i \in \mathbb{N}$, $f^{-1}(A_i) = f^{-1}(\mathrm{pr}_i^{-1}(0)) = f_i^{-1}(0) = C_{2i}$ and similarly $f^{-1}(B_i) = D_{2i}$, which implies that $f \in \Omega = \psi(T)$, hence $f = \psi(t)$ for some $t \in T$. Since $T \subset C \subset \mathrm{pr}_1(X)$, we have $x \in X$ such that $t = \mathrm{pr}_1(x)$. Then, $\varphi(x) = (\psi\mathrm{pr}_1(x))(x) = f(x)$. On the other hand, since $x \in \mathrm{pr}_1^{-1}(T) = E$, we have

$$x \in X \cap E \subset \bigcap_{i\in\mathbb{N}} S_{2i} \cap E \subset \bigcap_{i\in\mathbb{N}} \varphi^{-1}(\mathrm{pr}_i^{-1}(\tfrac{1}{2})) = \varphi^{-1}(\tfrac{1}{2}, \tfrac{1}{2}, \dots).$$

Then, $f(x) = \varphi(x) = (\frac{1}{2}, \frac{1}{2}, \dots)$, i.e., $f_i(x) = \frac{1}{2}$ for each $i \in \mathbb{N}$, hence $x \in \bigcap_{i\in\mathbb{N}} H_i \cap X \subset \bigcap_{i\in\mathbb{N}} L_i$. Therefore, $(C'_{2i}, D'_{2i})_{i\in\mathbb{N}}$ is essential. □

Theorem 5.13.4. *There exists a hereditarily infinite-dimensional compact metrizable space.*

Example and Proof. Let $\{C_n \mid n \in \mathbb{N}\}$ be a collection of Cantor sets in $\mathbf{I}$ such that every non-degenerate subinterval of $\mathbf{I}$ contains some C_n. Let $\Gamma_{i,n}$ $(i, n \in \mathbb{N})$ be disjoint infinite subsets of $\mathbb{N} \setminus \{1\}$ such that $i \notin \Gamma_{i,n}$. For each $i, n \in \mathbb{N}$, by Lemma 5.13.3, we have a compact set $S_{i,n} \subset \mathbf{I}^{\mathbb{N}}$ that is the intersection of partitions between $A_j = \mathrm{pr}_j^{-1}(0)$ and $B_j = \mathrm{pr}_j^{-1}(1)$ $(j \in \Gamma_{i,n})$ and has the property that $X \subset S_{i,n}$ is s.i.d. if $C_n \subset \mathrm{pr}_i(X)$.

We will show that $S = \bigcap_{i,n\in\mathbb{N}} S_{i,n}$ is h.i.d. Since S is the intersection of partitions between A_j and B_j $(j \in \bigcup_{i,n\in\mathbb{N}} \Gamma_{i,n})$ and $(A_j, B_j)_{j\in\mathbb{N}}$ is essential, S meets every partition between A_1 and B_1, which implies that $\dim S \neq -1, 0$. Now, let $\emptyset \neq X \subset S$. In the case that $\dim \mathrm{pr}_i(X) = 0$ for every $i \in \mathbb{N}$, since $\dim \prod_{i\in\mathbb{N}} \mathrm{pr}_i(X) = 0$ by Theorem 5.3.6 and $X \subset \prod_{i\in\mathbb{N}} \mathrm{pr}_i(X)$, we have $\dim X = 0$ by the Subset Theorem 5.3.3. When $\dim \mathrm{pr}_i(X) \neq 0$ for some $i \in \mathbb{N}$, $\mathrm{pr}_i(X)$ contains a non-degenerate subinterval of $\mathbf{I}$, hence it contains some C_n. Then, it follows that X is s.i.d. □

[14] Refer to the last Remark of Sect. 2.2.

5.14 Appendix: The Hahn–Mazurkiewicz Theorem

The content of this section is not part of Dimension Theory but is related to the content of Sect. 5.11. According to Theorem 5.11.5, every compact metrizable space is the continuous image of the Cantor (ternary) set μ^0. In this section, we will prove the following characterization of the continuous image of the interval $\mathbf{I}$:

Theorem 5.14.1 (HAHN–MAZURKIEWICZ). *A space X is the continuous image of the interval $\mathbf{I}$ if and only if X is a locally connected continuum.*[15]

Here, X is **locally connected** if each point $x \in X$ has a neighborhood basis consisting of connected neighborhoods. Because of Theorem 5.14.1, a locally connected continuum is called a **Peano continuum** in honor of the first mathematician who showed that the square $\mathbf{I}^2$ is the continuous image of the interval $\mathbf{I}$.

The continuous image of a continuum is also a continuum, where the metrizability follows from 2.4.5(1). Since every closed map is a quotient map, the "only if" part of Theorem 5.14.1 comes from the following proposition:

Proposition 5.14.2. *Let $f : X \to Y$ be a quotient map. If X is locally connected, then so is Y. Namely, the quotient space of a locally connected space is also locally connected.*

Proof. Let $y \in Y$. For each open neighborhood U of y in Y, let C be the connected component of U with $y \in C$. Since X is locally connected, each $x \in f^{-1}(C)$ has a connected neighborhood $V_x \subset f^{-1}(U)$. Note that $f(V_x)$ is connected, $f(V_x) \subset U$, and $f(V_x) \cap C \neq \emptyset$. Since C is a connected component of U, it follows that $f(V_x) \subset C$, hence $V_x \subset f^{-1}(C)$. Therefore, $f^{-1}(C)$ is open in X, which means that C is open in Y. Thus, C is a connected neighborhood of y in Y with $C \subset U$. □

To prove the "if" part of Theorem 5.14.1, we introduce a simple chain in a metric space $X = (X, d)$. A finite sequence $(U_1, \ldots, U_n)$ of *connected open* sets[16] in X is called a **chain** (an **ε-chain**) if

$$U_i \cap U_{i+1} \neq \emptyset \ \text{ for each } i = 1, \ldots, n-1$$

(and $\operatorname{diam} U_i < \varepsilon$ for every $i = 1, \ldots, n$), where n is called the **length** of this chain. A chain is said to be **simple** provided that

$$\operatorname{cl} U_i \cap \operatorname{cl} U_j = \emptyset \ \text{ if } |i - j| > 1.^{17}$$

[15] Recall that a continuum is a compact connected metrizable space.

[16] In general, each link U_i is not assumed to be connected and open.

[17] This condition is stronger than usual, and is adopted to simplify our argument. Usually, it is said that $(U_1, \ldots, U_n)$ is a simple chain if $U_i \cap U_j \neq \emptyset \Leftrightarrow |i - j| \leq 1$. However, in our definition, $U_i \cap U_j \neq \emptyset \Leftrightarrow \operatorname{cl} U_i \cap \operatorname{cl} U_j \neq \emptyset \Leftrightarrow |i - j| \leq 1$.

It is said that two distinct points $a, b \in X$ are connected by a simple (ε-)chain $(U_1, \ldots, U_n)$ if $a \in U_1 \setminus \operatorname{cl} U_2$ and $b \in U_n \setminus \operatorname{cl} U_{n-1}$ (when $n = 1$, this means $a, b \in U_1$), where $(U_1, \ldots, U_n)$ is called a simple (ε-)chain from a to b. Given open sets U and V in X with $\operatorname{dist}(\operatorname{cl} U, \operatorname{cl} V) > 0$, it is said that U and V are connected by a simple (ε-)chain $(U_1, \ldots, U_n)$ if

$$U \cap U_1 \neq \emptyset,\ \operatorname{cl} U \cap \operatorname{cl}(U_2 \cup \cdots \cup U_n) = \emptyset,$$
$$V \cap U_n \neq \emptyset,\ \text{and}\ \ \operatorname{cl} V \cap \operatorname{cl}(U_1 \cup \cdots \cup U_{n-1}) = \emptyset,$$

where $(U_1, \ldots, U_n)$ is called a simple (ε-)chain from U to V. When U is connected (and $\operatorname{diam} U < \varepsilon$), $(U, U_1, \ldots, U_n, V)$ is a simple (ε-)chain.

Lemma 5.14.3. *Let $X = (X, d)$ be a connected, locally connected metric space, and $a \neq b \in X$. Then, the following hold:*

(1) *Each pair of distinct points are connected by a simple ε-chain for any $\varepsilon > 0$.*
(2) *Each pair of open sets U and V in X with $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$ are connected by a simple ε-chain for any $\varepsilon > 0$.*
(3) *Each pair of open sets U and V in X with $\operatorname{dist}(U, V) > 0$ are connected by a simple chain of length n for any $n \in \mathbb{N}$.*

Proof. (1): Let W be the subset of X consisting of all points $x \in X$ satisfying the following condition:

- a and x are connected by a simple ε-chain.

Then, W is open in X by the definition. Using the local connectedness of X, we can easily show that $a \in W$ and $X \setminus W$ is open in X. Since X is connected, it follows that $W = X$. Then, we have $b \in W$. This gives (1).

(2): Take points $a \in U$ to $b \in V$ and apply (1) to them, we have a simple ε-chain $(W_1, \ldots, W_m)$ from a to b. Let

$$k_0 = \max\{i \mid \operatorname{cl} W_i \cap \operatorname{cl} U \neq \emptyset\} > 1\ \ \text{and}$$
$$k_1 = \min\{i \geq k_0 \mid \operatorname{cl} W_i \cap \operatorname{cl} V \neq \emptyset\} \geq k_0.$$

If $W_{k_0} \cap U \neq \emptyset$, then $(W_{k_0}, \ldots, W_{k_1})$ is a simple ε-chain from U to V. When $W_{k_0} \cap U = \emptyset$ or $W_{k_1} \cap V = \emptyset$ (except for the case that $k_0 = k_1$ and $W_{k_0} \cap U = W_{k_0} \cap V = \emptyset$), we take a connected open neighborhood U' of some $x \in \operatorname{cl} W_{k_0} \cap \operatorname{cl} U$ with $\operatorname{diam} U' < \varepsilon - \operatorname{diam} W_{k_0}$ or a connected open neighborhood V' of some $y \in \operatorname{cl} W_{k_1} \cap \operatorname{cl} V$ with $\operatorname{diam} V' < \varepsilon - \operatorname{diam} W_{k_1}$ (in the except case, $\operatorname{diam} U'$, $\operatorname{diam} V' < \frac{1}{2}(\varepsilon - \operatorname{diam} W_{k_0})$). Then, replacing W_{k_0} by $U' \cup W_{k_0}$ or W_{k_1} by $V' \cup W_{k_1}$ (in the except case, replacing $W_{k_0} = W_{k_1}$ by $U' \cup V' \cup W_{k_0}$), we can obtain a simple ε-chain from U to V.

(3): For each $n \in \mathbb{N}$, let $\varepsilon = n^{-1}\operatorname{dist}(U,V) > 0$. By (2), we have a simple ε-chain $(W_1,\dots,W_k)$ from U to V. Then, $n < k$ because

$$\operatorname{dist}(U,V) \le \operatorname{diam} W_1 + \cdots + \operatorname{diam} W_k < k\varepsilon = n^{-1}k\operatorname{dist}(U,V).$$

Hence, U and V are connected by a simple chain $(W_1,\dots,\bigcup_{i=n}^{k} W_i)$ of length n. □

Recall that X is **path-connected** if every pair of points $x, y \in X$ can be connected by a path, i.e., there is a path $f : \mathbf{I} \to X$ with $f(0) = x$ and $f(1) = y$. It is said that X is **arcwise connected** if every two distinct points $x, y \in X$ can be connected by an arc, i.e., there is an arc $f : \mathbf{I} \to X$ with $f(0) = x$ and $f(1) = y$.[18] A space X is **locally path-connected** (or **locally arcwise connected**) if each neighborhood U of each point $x \in X$ contains a neighborhood V of x such that every two (distinct) points $y, y' \in V$ can be connected by a path (or an arc) in U. According to the following lemma, the local path-connectedness and the local arcwise connectedness can be defined in the same manner as the local connectedness.

Lemma 5.14.4. *For a locally path-connected (or locally arcwise connected) space X, the following hold:*

(1) *Every component of X is open and path-connected (or arcwise connected).*
(2) *Each point of a locally path-connected (or locally arcwise connected) space X has a neighborhood basis consisting of path-connected (or arcwise connected) open neighborhoods.*

Proof. (1): For each $x \in X$, let W be a subset of X consisting of all points connected with x by a path (or an arc) in X (and x itself). Then, it is easy to see that W is a connected clopen set in X, and hence it is a component of X.

(2): Every open neighborhood U of each $x \in X$ is also locally path-connected (or locally arcwise connected). It follows from (1) that the component of U containing x is a path-connected (or arcwise connected) open neighborhood of x. □

Obviously, every arcwise connected (resp. locally arcwise connected) space is path-connected (resp. locally path-connected), and every path-connected (resp. locally path-connected) space is connected (resp. locally connected). However, according to the following theorem, for connected locally compact metrizable spaces, the local connectedness implies the local arcwise connectedness.

Theorem 5.14.5. *Every connected, locally connected, locally compact metrizable space X is arcwise connected and locally arcwise connected.*

Proof. Because of the local compactness of X and 2.7.7(1), it can be assumed that $X = (X,d)$ is a metric space such that $\overline{\mathrm{B}}(x,1)$ is compact for each $x \in X$, so $X = (X,d)$ is complete. Let $a, b \in X$ be two distinct points. By induction on

[18] Recall that an arc is an injective path, i.e., an embedding of $\mathbf{I}$.

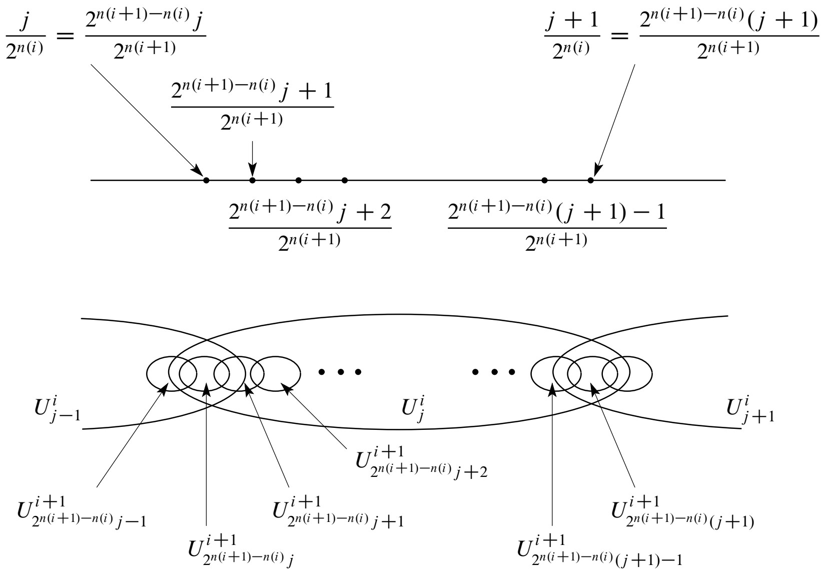

Fig. 5.10 Illustration of condition (2)

$i \in \mathbb{N}$, we will construct a simple 2^{-i}-chain $(U^i_0, U^i_1, \ldots, U^i_{2^{n(i)}-1})$ from a to b so that

(1) $n(1) < n(2) < \cdots$; and
(2) $U^{i+1}_k \subset U^i_j$ for $2^{n(i+1)-n(i)}j \le k < 2^{n(i+1)-n(i)}(j+1)$ (Fig. 5.10).

Since X is locally connected, a and b have connected open neighborhoods U and V, respectively, such that $\operatorname{diam} U$, $\operatorname{diam} V < 2^{-1}$, and $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$. Using Lemma 5.14.3(2), we can obtain $n(1) \ge 2$ and a simple 2^{-1}-chain $(U^1_1, \ldots, U^1_{2^{n(1)}-2})$ in X from U to V. Let $U^1_0 = U$ and $U^1_{2^{n(1)}-1} = V$. Thus, we have a simple 2^{-1}-chain $(U^1_0, \ldots, U^1_{2^{n(1)}-1})$ from a to b.

Next, suppose that a simple 2^{-i}-chain $(U^i_0, U^i_1, \ldots, U^i_{2^{n(i)}-1})$ from a to b has been obtained. Let U and V be connected open neighborhoods of a and b in X, respectively, such that $\operatorname{cl} U \subset U^i_0$ and $\operatorname{cl} V \subset U^i_{2^{n(i)}-1}$. Since each U^i_j is connected and locally connected, we can apply inductively Lemma 5.14.3(2) to obatin a simple $2^{-(i+1)}$-chain $(V^j_0, \ldots, V^j_{k(j)})$ in U^i_j from $U^i_j \cap V^{j-1}_{k(j-1)}$ to $U^i_j \cap U^i_{j+1}$, where $V^{-1}_{k(-1)} = U$ and $U^i_{2^{n(i)}} = V$. Choose $n(i+1) > n(i)$ so that

$$2^{n(i+1)-n(i)} > \max\{k(j) \mid j = 0, 1, \ldots, 2^{n(i)} - 1\}.$$

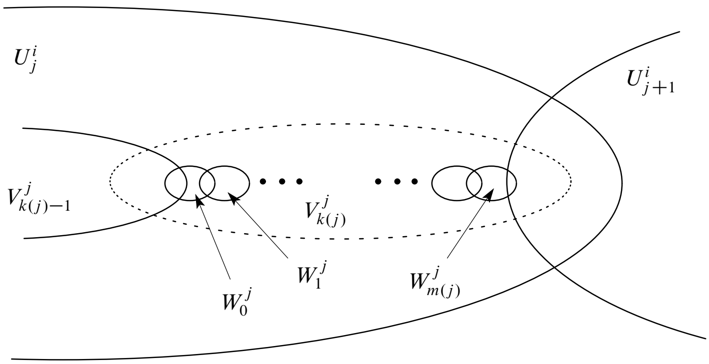

Fig. 5.11 A simple chain $(W_0^j, W_1^j, \ldots, W_{m(j)}^j)$ in $V_{k(j)}^j$

For each $j = 0, 1, \ldots, 2^{n(i)} - 1$, let $m(j) = 2^{n(i+1)-n(i)} - k(j) - 1$ (i.e., $k(j) + m(j) = 2^{n(i+1)-n(i)} - 1$). By Lemma 5.14.3(3), we have a simple chain $(W_0^j, \ldots, W_{m(j)}^j)$ in $V_{k(j)}^j$ from $V_{k(j)}^j \cap V_{k(j)-1}^j$ and $V_{k(j)}^j \cap V_1^{j+1}$ (Fig. 5.11). Now, we define

$$U_{2^{n(i+1)-n(i)}j}^{i+1} = V_0^j, \ldots, U_{2^{n(i+1)-n(i)}j+k(j)-1}^{i+1} = V_{k(j)-1}^j,$$
$$U_{2^{n(i+1)-n(i)}j+k(j)}^{i+1} = W_0^j, \cdots, U_{2^{n(i+1)-n(i)}j+2^{n(i+1)-n(i)}-1}^{i+1} = W_{m(j)}^j,$$

which are contained in U_j^i. Let $U_0^{i+1} = U$ and $U_{2^{n(i+1)}-1}^{i+1} = V$. Then, $(U_0^{i+1}, U_1^{i+1}, \ldots, U_{2^{n(i+1)}-1}^{i+1})$ is the desired simple $2^{-(i+1)}$-chain.

For each $x \in \mathbf{2}^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$, observe $0 \leq \sum_{j=1}^{n(i)} 2^{n(i)-j} x(j) \leq 2^{n(i)} - 1$ and

$$\sum_{j=1}^{n(i)} 2^{n(i)-j} x(j) = 2^{n(i)-n(i-1)} \sum_{j=1}^{n(i-1)} 2^{n(i-1)-j} x(j) + \sum_{j=n(i-1)+1}^{n(i)} 2^{n(i)-j} x(j),$$

where $0 \leq \sum_{j=n(i-1)+1}^{n(i)} 2^{n(i)-j} x(j) < 2^{n(i)-n(i-1)}$. Then, it follows from (4) that

$$U_{\sum_{j=1}^{n(i)} 2^{n(i)-j} x(j)}^i \subset U_{\sum_{j=1}^{n(i-1)} 2^{n(i-1)-j} x(j)}^{i-1}.$$

By (3) and the completeness of X, the following is a singleton:

$$\bigcap_{i \in \mathbb{N}} \operatorname{cl} U_{\sum_{j=1}^{n(i)} 2^{n(i)-j} x(j)}^i \neq \emptyset.$$

Then, we have a map $f : \mathbf{2}^{\mathbb{N}} \to X$ such that

$$\{f(x)\} = \bigcap_{i \in \mathbb{N}} \operatorname{cl} U^i_{\sum_{j=1}^{n(i)} 2^{n(i)-j} x(j)},$$

where $f(0) = a$ and $f(1) = b$. For $x, y \in \mathbf{2}^{\mathbb{N}}$, if $x(j) = y(j)$ for $j < 2^{n(i)}$, then

$$f(x), f(y) \in U^i_{\sum_{j=1}^{n(i)} 2^{n(i)-j} x(j)} = U^i_{\sum_{j=1}^{n(i)} 2^{n(i)-j} y(j)},$$

hence $d(f(x), f(y)) < 2^{-i}$ by (2), which implies that f is continuous.

Let $\varphi : \mathbf{2}^{\mathbb{N}} \to \mathbf{I}$ be the quotient map defined by $\varphi(x) = \sum_{i=1}^{\infty} 2^{-i} x(i)$. For each $x, y \in \mathbf{2}^{\mathbb{N}}$, we will show that $\varphi(x) = \varphi(y)$ if and only if $f(x) = f(y)$, hence f induces the embedding $h : \mathbf{I} \to X$ with $h(0) = a$ and $h(1) = b$.

First, suppose that $\varphi(x) = \varphi(y)$, i.e., $\sum_{i=1}^{\infty} 2^{-i} x(i) = \sum_{i=1}^{\infty} 2^{-i} y(i)$. When $x \neq y$, let $k = \min\{i \in \mathbb{N} \mid x(i) \neq y(i)\}$, where we may assume that $x(k) = 1$ and $y(k) = 0$. Then,

$$\begin{aligned}\sum_{i=1}^{\infty} 2^{-i} x(i) \geq \sum_{i=1}^{k} 2^{-i} x(i) &= \sum_{i=1}^{k-1} 2^{-i} x(i) + 2^{-k} \\ &= \sum_{i=1}^{k} 2^{-i} y(i) + \sum_{j=k+1}^{\infty} 2^{-j} \geq \sum_{i=1}^{k} 2^{-i} y(i),\end{aligned}$$

which implies that $x(i) = 0$ and $y(i) = 1$ for every $i > k$. Thus, we have

$$\sum_{j=1}^{k-1} 2^{k-1-j} x(j) = \sum_{j=1}^{k-1} 2^{k-1-j} y(j) \text{ and}$$

$$\sum_{j=1}^{m} 2^{m-j} x(j) = \sum_{j=1}^{m} 2^{m-j} y(j) + 1 \text{ for every } m \geq k.$$

Then, it follows that

$$U^i_{\sum_{j=1}^{n(i)} 2^{n(i)-j} x(j)} \cap U^i_{\sum_{j=1}^{n(i)} 2^{n(i)-j} y(j)} \neq \emptyset \text{ for every } i \in \mathbb{N},$$

which implies that $d(f(x), f(y)) = 0$ by (3), hence $f(x) = f(y)$.

Conversely, suppose that $f(x) = f(y)$. For every $i \in \mathbb{N}$,

$$U^i_{\sum_{j=1}^{n(i)-1} 2^{n(i)-j} x(j)} \cap U^i_{\sum_{j=1}^{n(i)-1} 2^{n(i)-j} y(j)} \neq \emptyset,$$

which means $\left| \sum_{j=1}^{n(i)} 2^{n(i)-j} x(j) - \sum_{j=1}^{n(i)} 2^{n(i)-j} y(j) \right| \leq 1$. Therefore,

$$|\varphi(x) - \varphi(y)| = \left| \sum_{j=1}^{\infty} 2^{-j} x(j) - \sum_{j=1}^{\infty} 2^{-j} y(j) \right|$$

$$= \lim_{i\to\infty} \left| \sum_{j=1}^{n(i)-1} 2^{-j} x(j) - \sum_{j=1}^{n(i)-1} 2^{-j} y(j) \right|$$

$$= \lim_{i\to\infty} 2^{-n(i)} \left| \sum_{j=1}^{n(i)-1} 2^{n(i)-j} x(j) - \sum_{j=1}^{n(i)-1} 2^{n(i)-j} y(j) \right|$$

$$\leq \lim_{i\to\infty} 2^{-n(i)} = 0,$$

that is, $\varphi(x) = \varphi(y)$. Thus, we have proved that X is arcwise connected.

Finally, note that every neighborhood of each point $x \in X$ contains a connected open neighborhood U in X. Since U is also completely metrizable, it follows that U is also arcwise connected. This means that X is locally arcwise connected. □

By the "only if" part of Theorems 5.14.1 and 5.14.5, we have the following corollary:

Corollary 5.14.6. *Let X be an arbitrary space. Then, each pair of distinct points $x \neq y \in X$ are connected by a path in X if and only if they are connected by an arc in X. In this case, the image of the arc is contained in the image of the path.*

Proof. The "if" part is obvious. To see the "only if" part, let $f : \mathbf{I} \to X$ be a path with $f(0) = x$ and $f(1) = y$. Since the image $f(\mathbf{I})$ is a locally connected continuum (i.e., a Peano continuum) by the "only if" part of Theorem 5.14.1, we have an arc from x to y in $f(\mathbf{I})$ ($\subset X$) by Theorem 5.14.5. □

Thus, we know that there is no difference between the (local) path-connectedness and the (local) arcwise connectedness of an arbitrary space. This allows us to sate the following:

Corollary 5.14.7. *An arbitrary space X is path-connected if and only if X is arcwise connected. Moreover, X is locally path-connected if and only if X is locally arcwise connected.* □

A metric space $X = (X, d)$ is said to be **uniformly locally path-connected** provided that, for every $\varepsilon > 0$, there is $\delta > 0$ such that each pair of points $x, y \in X$ with $d(x, y) < \delta$ can be connected by a path with diam $< \varepsilon$.

Proposition 5.14.8. *A compact metric space X is uniformly locally path-connected if it is locally path-connected.*

Proof. For each $\varepsilon > 0$, we apply Lemma 5.14.4(2) to obtain $\mathcal{U} \in \operatorname{cov}(X)$ consisting of path-connected open sets with mesh $\mathcal{U} < \varepsilon$. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Then, each pair of points $x, y \in X$ with $d(x, y) < \delta$ can be connected by a path with diam $< \varepsilon$. □

We are now ready to prove the "if" part of the Hahn–Mazurkiewicz Theorem 5.14.1.

Proof of the "if" part of Theorem 5.14.1. We may assume that $X = (X, d)$ is a compact connected metric space. Let μ^0 be the Cantor (ternary) set in $\mathbf{I}$. By Theorem 5.11.5, there exists a continuous surjection $f : \mu^0 \to X$. By Theorem 5.14.5, X is path-connected and locally path-connected (arcwise connected and locally arcwise connected). According to Proposition 5.14.8, we have $\delta_1 > \delta_2 > \cdots > 0$ such that every two distinct points within δ_n can be connected by a path with diam $< 1/n$, where we may assume that $\delta_n \leq 1/n$.

Because of the construction of μ^0, the complement $\mathbf{I} \setminus \mu^0$ has only finitely many components $C_i = (a_i, b_i)$, $i = 1, \ldots, m$, such that $d(f(a_i), f(b_i)) \geq \delta_1$. Indeed, there is some $k \in \mathbb{N}$ such that

$$a, b \in \mu^0,\ |a - b| < 3^{-k} \Rightarrow d(f(a), f(b)) < \delta_1,$$

$$\big(\text{i.e., } d(f(a), f(b)) \geq \delta_1 \Rightarrow |a - b| \geq 3^{-k}\big),$$

which implies that $m \leq \sum_{i=1}^{k} 2^{i-1}$. For each $i = 1, \ldots, m$, let $f_i : \operatorname{cl} C_i = [a_i, b_i] \to X$ be a path with $f_i(a_i) = f(a_i)$ and $f_i(b_i) = f(b_i)$. Then, we can extend f to the map

$$f' : M = \mu^0 \cup \bigcup_{i=1}^{m} \operatorname{cl} C_i \to X$$

that is defined by $f'|\operatorname{cl} C_i = f_i$ for each $i = 1, \ldots, m$.

For each component $C = (a, b)$ of $\mathbf{I} \setminus M$ (which is a component of $\mathbf{I} \setminus \mu^0$), $f(a) = f(b)$ or $0 < d(f(a), f(b)) < \delta_1$. In the former case, let $f_C : \operatorname{cl} C = [a, b] \to X$ be the constant path with $f_C([a, b]) = \{f(a)\}$ $(= \{f(b)\})$. In the latter case, choose $n \in \mathbb{N}$ so that $\delta_{n+1} \leq d(f(a), f(b)) < \delta_n$ and take a path $f_C : \operatorname{cl} C = [a, b] \to X$ such that $f_C(a) = f(a)$, $f_C(b) = f(b)$, and diam $f_C([a, b]) < 1/n$. Then, f' can be extended to the map $f^* : \mathbf{I} \to X$ by $f^*|\operatorname{cl} C = f_C$ for every component C of $\mathbf{I} \setminus M$.

It remains to verify the continuity of f^*. Since each component C of $\mathbf{I} \setminus M$ is an open interval, the continuity of f^* at a point of $\mathbf{I} \setminus M$ follows from the continuity of f_C. The continuity of f^* at a point of $\operatorname{int} M$ comes from the continuity of f'. We will show the continuity of f^* at a point $x \in \operatorname{bd} M$ $(= \mu^0)$. For each $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $1/n < \varepsilon/2$. Since f' is continuous at x, we have a neighborhood U of x in $\mathbf{I}$ such that $f'(U \cap M) \subset \mathrm{B}(f'(x), \delta_n/2)$ $(\subset \mathrm{B}(f^*(x), \varepsilon/2)$ because $\delta_n \leq 1/n < \varepsilon/2)$. In the case that $x \notin \operatorname{bd} C$ for any component $C = (a, b)$ of $\mathbf{I} \setminus M$ with $d(f(a), f(b)) \geq \delta_n$, U can be chosen so that $U \cap \operatorname{cl} C = \emptyset$ for any component $C = (a, b)$ of $\mathbf{I} \setminus M$ with $d(f(a), f(b)) \geq \delta_n$. In the case $x \in \operatorname{bd} C_0$ for some component $C_0 = (a_0, b_0)$ of $\mathbf{I} \setminus M$ with $d(f(a_0), f(b_0)) \geq \delta_n$ (such a component C_0 is unique if it exists), U can be chosen so that $f_{C_0}(U \cap C_0) \subset \mathrm{B}(f'(x), \varepsilon/2)$. Now, let $C = (a, b)$ be a component of $\mathbf{I} \setminus M$ with $\operatorname{cl} C \cap U \neq \emptyset$. Then, $a \in U \cap M$ or $b \in U \cap M$, and so $d(f'(a), f'(x)) < \varepsilon/2$ or $d(f'(b), f'(x)) < \varepsilon/2$, respectively. If $f'(a) = f'(b)$, then $f^*(C) = f_C(C) = \{f(a)\} \subset \mathrm{B}(f'(x), \varepsilon/2)$. If $0 < d(f(a), f(b)) < \delta_n$, then diam $f_C([a, b]) < 1/n < \varepsilon/2$, which implies that $f^*(C) = f_C([a, b]) \subset$

$B(f'(x), \varepsilon)$. When $d(f(a), f(b)) \geq \delta_n$, it follows that $x \in \operatorname{bd} C$, which means that $C = C_0$. Then, $f^*(U \cap C) = f_{C_0}(U \cap C_0) \subset B(f'(x), \varepsilon/2)$. Consequently, we have $f^*(U) \subset B(f^*(x), \varepsilon)$. This completes the proof. □

Notes for Chap. 5

Below, we list only three among textbooks on Dimension Theory:

- R. Engelking, *Theory of Dimensions, Finite and Infinite*, Sigma Ser. in Pure Math. **10** (Heldermann Verlag, Lembo, 1995)
- W. Hurewicz and H. Wallman, *Dimension Theory* (Princeton University Press, Princeton, 1941)
- K. Nagami, *Dimension Theory* (Academic Press, Inc., New York, 1970)

For a more comprehensive study of Dimension Theory, we refer to Engelking's book, which also contains excellent historical notes. Nagami's book is quite readable and contains an appendix titled "Cohomological Dimension Theory" by Kodama. The classical book by Hurewicz and Wallman is still a worthwhile read. Nothing fundamental has yet changed in the framework of Dimension Theory since its publication. In this book, Hurewicz and Wallman discuss the Hausdorff dimension, which is useful in the field of Fractal Geometry. However, we do not discuss this here. In the following textbook of van Mill, Chap. 5 is devoted to Dimension Theory, and was used to prepare the last two sections of this chapter.

- J. van Mill, *Infinite-Dimensional Topology, Prerequisites and Introduction*, North-Holland Math. Library **43** (Elsevier Sci. Publ. B.V., Amsterdam, 1989)

The definition of dim, which is due to Čech [11], is based on a property of covers of $\mathbf{I}^n$ discovered by Lebesgue [28]. The Brouwer Fixed Point Theorem 5.1.1 was established in [8]. The proof using Sperner's Lemma 5.1.2 in [53] is due to Knaster et al. [26].

The equivalence between (a), (b), and (d) in Theorem 5.2.3 was established by Hemmingsen [20] and the equivalence between (a) and (d) was proved independently by Alexandroff [2] and Dowker [12]. The equivalence between (a) and (f) was first established for compact metrizable spaces by Hurewicz [23] and for normal spaces by Alexandroff [2], Hemmingsen [20], and Dowker [12], independently.

The compact case of Corollary 5.2.6 was established by Freudenthal [17], and was generalized to compact Hausdorff spaces by Mardešić [33].

In [22], a map $f : X \to \mathbf{I}^n$ is called a **universal map** if it satisfies condition (b) in Theorem 5.2.15. The equivalence between (b) and (c) in Theorem 5.2.15 is due to Holszyński [22]. The equivalence between (a) and (b) in Theorem 5.2.17 was established by Alexandroff [1]. The equivalence between (a) and (c) in Theorem 5.2.17 was first established by Eilenberg and Otto [14] in the separable metrizable case and extended to normal spaces by Hemmingsen [20].

Theorem 5.3.1 was established by Vopěnka [55] and Theorem 5.3.2 was proved by Nagami [40]. The Subset Theorem was proved by Dowker [13]. The Countable Sum Theorem (5.4.1) was established by Čech [11] and the Locally Finite Sum Theorem (5.4.2) was proved independently by Morita [Mo] and Katětov [24]. The Addition Theorem (5.4.8) was proved by Smirnov [52]. The Decomposition and Product Theorems (5.4.5, 5.4.9) were proved independently by Katětov [24] and Morita [39].

An inductive definition of dimension was outlined by Poincaré [44]. The first precise definition of a dimension function was introduced by Brouwer [9]. His function coincides with Ind in the class of locally connected compact metrizable spaces. The definition of Ind was formulated by Čech [10]. On the other hand, the definition of ind was formulated by Urysohn [54] and Menger [37]. The first example in Theorem 5.5.3 was constructed by Roy [47,48] but the example presented here was constructed by Kulesza [27] and the proof of dim > 0 was simplified by Levin [31].

The weak infinite dimension was first introduced by Alexandroff in [3]. In Remark 12, we mentioned the weak infinite dimension in the sense of Smirnov, which was first studied in [32] and [51].

Theorem 5.7.4 is due to Lelek [30] and the simple proof presented here is taken from Engelking and Pol [15].

In [42], Nöbeling introduced the spaces ν^n and showed their universality. The spaces μ^n were introduced by Menger [38], who showed that the universality μ^1 is a universal space for compacta with dim ≤ 1. Theorem 5.10.8 is due to Bothe [6]. In [29], Lefschetz constructed a universal space for compacta with dim $\leq n$. In [5], Bestvina gave the topological characterization of μ^n. Using Bestvina's characterization, we can see that Lefschetz' universal space is homeomorphic to μ^n; the result for $n = 1$ had been obtained by Anderson [4]. Recently, in [41], Nagórko established the topological characterization of ν^n.

The total disconnectedness and the hereditary disconnectedness were respectively introduced by Sierpiński [50] and Hausdorff [19]. The example of Theorem 5.11.3 is due to Knaster and Kuratowski [25] (their example is the one in the Remark).

The example of Theorem 5.12.1 was described by Erdös [16]. Lemma 5.12.3 is due to Bourbaki [7, Chap. 9] and the proof presented here is due to van Mill (Chap. 5 in his book listed above). The first completely metrizable nonzero-dimensional totally disconnected space was constructed by Sierpiński [50] (his example is 1-dimensional). Theorem 5.12.6 was established by Mazurkiewicz [36] but the example and proof presented here is due to Rubin et al. [49] with some help from [45].

The example of Theorem 5.13.1 is presented by Pol [45]. Theorem 5.13.4 is due to Walsh [56] but the example given here is due to Pol [46]. The earlier example of a compact metrizable space, whose compact subsets are all either 0-dimensional or infinite-dimensional, was constructed by Henderson [21].

In 1890, Peano [43] showed that the square $\mathbf{I}^2$ is the continuous image of $\mathbf{I}$. The Hahn–Mazurkiewicz Theorem 5.14.1 was independently proved by Hahn [18] for planar sets and by Mazurkiewicz [34] for subspaces of Euclidean space. In [35], Mazurkiewicz gave a systematic exposition.

For more details, consult the historical and bibliographical notes at the end of each section of Engelking's book.

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