Chapter 2
Metrization and Paracompact Spaces

In this chapter, we are mainly concerned with metrization and paracompact spaces. We also derive some properties of the products of compact spaces and perfect maps. Several metrization theorems are proved, and we characterize completely metrizable spaces. We will study several different characteristics of paracompact spaces that indicate, in many situations, the advantages of paracompactness. In particular, there exists a useful theorem showing that, if a paracompact space has a certain property *locally*, then it has the same property *globally*. Furthermore, paracompact spaces have partitions of unity, which is also a very useful property.

2.1 Products of Compact Spaces and Perfect Maps

In this section, we present some theorems regarding the products of compact spaces and compactifications. In addition, we introduce perfect maps. First, we present a proof of the TYCHONOFF THEOREM.

Theorem 2.1.1 (TYCHONOFF). *The product space $\prod_{\lambda\in\Lambda} X_\lambda$ of compact spaces X_λ, $\lambda \in \Lambda$, is compact.*

Proof. Let $X = \prod_{\lambda\in\Lambda} X_\lambda$. We may assume that $\Lambda = (\Lambda, \leq)$ is a well-ordered set. For each $\mu \in \Lambda$, let $p_\mu : X \to \prod_{\lambda\leq\mu} X_\lambda$ and $q_\mu : X \to \prod_{\lambda<\mu} X_\lambda$ be the projections.

Let $\mathcal{A}$ be a collection of subsets of X with the finite intersection property (f.i.p.). Using transfinite induction, we can find $x_\lambda \in X_\lambda$ such that $\mathcal{A}|p_\lambda^{-1}(U)$ has the f.i.p. for every neighborhood U of $(x_\nu)_{\nu\leq\lambda}$ in $\prod_{\nu\leq\lambda} X_\nu$. Indeed, suppose that $x_\lambda \in X_\lambda$, $\lambda < \mu$, have been found, but there exists no $x_\mu \in X_\mu$ with the above property, i.e., any $y \in X_\mu$ has an open neighborhood V_y with an open neighborhood U_y of $(x_\lambda)_{\lambda<\mu}$ in $\prod_{\lambda<\mu} X_\lambda$ such that $\mathcal{A}|q_\mu^{-1}(U_y) \cap \mathrm{pr}_\mu^{-1}(V_y)$ does not have the f.i.p. Because X_μ is compact, we have $y_1, \ldots, y_n \in X_\mu$ such that $X_\mu = \bigcup_{i=1}^n V_{y_i}$. Since $\bigcap_{i=1}^n U_{y_i}$ is a neighborhood of $(x_\lambda)_{\lambda<\mu}$ in

K. Sakai, *Geometric Aspects of General Topology*, Springer Monographs in Mathematics, DOI 10.1007/978-4-431-54397-8_2, © Springer Japan 2013

$\prod_{\lambda<\mu} X_\lambda$, we have $\nu_1, \dots, \nu_m < \mu$ and neighborhoods W_i of x_{ν_i} in X_{ν_i} such that $\bigcap_{i=1}^m \mathrm{pr}_{\nu_i}^{-1}(W_i) \subset q_\mu^{-1}(\bigcap_{i=1}^n U_{y_i})$. Let $\nu = \max\{\nu_1, \dots, \nu_m\} < \mu$. Then, we can write $\bigcap_{i=1}^m \mathrm{pr}_{\nu_i}^{-1}(W_i) = p_\nu^{-1}(W)$ for some neighborhood W of $(x_\lambda)_{\lambda\le\nu}$ in $\prod_{\lambda\le\nu} X_\lambda$. Because $p_\nu^{-1}(W) \subset \bigcap_{i=1}^n q_\mu^{-1}(U_{y_i})$, no $\mathcal{A}|p_\nu^{-1}(W) \cap \mathrm{pr}_\mu^{-1}(V_{y_i})$ have the f.i.p. Since $X = \bigcup_{i=1}^n \mathrm{pr}_\mu^{-1}(V_{y_i})$, it follows that $\mathcal{A}|p_\nu^{-1}(W)$ does not have the f.i.p., which contradicts the inductive assumption.

Now, we have obtained the point $x = (x_\lambda)_{\lambda\in\Lambda} \in X$. For each neighborhood U of x in X, we have $\lambda_1, \dots, \lambda_n \in \Lambda$ and neighborhoods U_i of x_{λ_i} in X_{λ_i} such that $\bigcap_{i=1}^n \mathrm{pr}_{\lambda_i}^{-1}(U_i) \subset U$. Let $\lambda_0 = \max\{\lambda_1, \dots, \lambda_n\} \in \Lambda$. Then, we can write $\bigcap_{i=1}^n \mathrm{pr}_{\lambda_i}^{-1}(U_i) = p_{\lambda_0}^{-1}(U_0)$ for some neighborhood U_0 of $(x_\nu)_{\nu\le\lambda_0}$ in $\prod_{\nu\le\lambda_0} X_\nu$. Since $p_{\lambda_0}^{-1}(U_0) \subset U$, $\mathcal{A}|U$ has the f.i.p. Consequently, every neighborhood U of x in X meets every member of $\mathcal{A}$. This means that $x \in \bigcap_{A\in\mathcal{A}} \mathrm{cl}\, A$, and so $\bigcap_{A\in\mathcal{A}} \mathrm{cl}\, A \neq \emptyset$. □

Note. There are various proofs of the Tychonoff Theorem. In one familiar proof, Zorn's Lemma is applied instead of the transfinite induction. Let $\mathcal{A}$ be a collection of subsets of X with the f.i.p. and Φ be all of collections $\mathcal{A}'$ of subsets of X such that $\mathcal{A}'$ has the f.i.p. and $\mathcal{A} \subset \mathcal{A}'$. Applying Zorn's Lemma to the ordered set $\Phi = (\Phi, \subset)$, we can obtain a maximal element $\mathcal{A}^* \in \Phi$. Because of the maximality, $\mathcal{A}^*$ has the following properties:

(1) The intersection of any finite members of $\mathcal{A}^*$ belongs to $\mathcal{A}^*$;
(2) If $B \subset X$ meets every member of $\mathcal{A}^*$, then $B \in \mathcal{A}^*$.

For each $\lambda \in \Lambda$, $\mathrm{pr}_\lambda(\mathcal{A}^*)$ has the f.i.p. Since X_λ is compact, we have $x_\lambda \in \bigcap_{A\in\mathcal{A}^*} \mathrm{cl}\, \mathrm{pr}_\lambda(A)$. It follows from (2) that $\mathrm{pr}_\lambda^{-1}(V) \in \mathcal{A}^*$ for every neighborhood V of x_λ in X_λ. Now, it is easy to see that

$$x = (x_\lambda)_{\lambda\in\Lambda} \in \bigcap_{A\in\mathcal{A}^*} \mathrm{cl}\, A \subset \bigcap_{A\in\mathcal{A}} \mathrm{cl}\, A.$$

Next, we prove WALLACE'S THEOREM:

Theorem 2.1.2 (WALLACE). *Let $A = \prod_{\lambda\in\Lambda} A_\lambda \subset X = \prod_{\lambda\in\Lambda} X_\lambda$, where each A_λ is compact. Then, for each open set W in X with $A \subset W$, there exists a finite subset $\Lambda_0 \subset \Lambda$ and open sets V_λ in X_λ, $\lambda \in \Lambda_0$, such that $A \subset \bigcap_{\lambda\in\Lambda_0} \mathrm{pr}_\lambda^{-1}(V_\lambda) \subset W$.*

Proof. When Λ is finite, we may take $\Lambda_0 = \Lambda$. Then, $\bigcap_{\lambda\in\Lambda_0} \mathrm{pr}_\lambda^{-1}(V_\lambda)$ coincides with $\prod_{\lambda\in\Lambda} V_\lambda$. This case can be proved by induction on $\mathrm{card}\, \Lambda$, which is reduced to the case $\mathrm{card}\, \Lambda = 2$. Proving the case $\mathrm{card}\, \Lambda = 2$ is an excellent exercise.[1]

We will show that the general case is derived from the finite case. For each $x \in A$, we have a finite subset $\Lambda(x) \subset \Lambda$ and an open set $U(x)$ in $\prod_{\lambda\in\Lambda(x)} X_\lambda$ such that $x \in \mathrm{pr}_{\Lambda(x)}^{-1}(U(x)) \subset W$. Because of the compactness of A, there exist finite $x_1, \dots, x_n \in A$ such that $A \subset \bigcup_{i=1}^n \mathrm{pr}_{\Lambda(x_i)}^{-1}(U(x_i))$. Thus, we have a finite subset

[1] Use the same strategy used in the proof of normality of a compact Hausdorff space.

$\Lambda_0 = \bigcup_{i=1}^n \Lambda(x_i) \subset \Lambda$. For each $i = 1, \ldots, n$, let $p_i : \prod_{\lambda \in \Lambda_0} X_\lambda \to \prod_{\lambda \in \Lambda(x_i)} X_\lambda$ be the projection. Then, $W_0 = \bigcup_{i=1}^n p_i^{-1}(U(x_i))$ is an open set in $\prod_{\lambda \in \Lambda_0} X_\lambda$.

Note that $\bigcup_{i=1}^n \mathrm{pr}_{\Lambda(x_i)}^{-1}(U(x_i)) = \mathrm{pr}_{\Lambda_0}^{-1}(W_0)$. From the finite case, we obtain open sets V_λ, $\lambda \in \Lambda_0$, such that $\prod_{\lambda \in \Lambda_0} A_\lambda \subset \prod_{\lambda \in \Lambda_0} V_\lambda \subset W_0$. Hence,

$$A \subset \bigcap_{\lambda \in \Lambda_0} \mathrm{pr}_\lambda^{-1}(V_\lambda) \subset \mathrm{pr}_{\Lambda_0}^{-1}(W_0) \subset W. \qquad \square$$

For any space X, we define the evaluation map $e_X : X \to \mathbf{I}^{\mathrm{C}(X,\mathbf{I})}$ by $e_X(x) = (f(x))_{f \in \mathrm{C}(X,\mathbf{I})}$ for each $x \in X$. The continuity of e_X follows from the fact that $\mathrm{pr}_f \circ e_X = f$ is continuous for each $f \in \mathrm{C}(X, \mathbf{I})$, where $\mathrm{pr}_f : \mathbf{I}^{\mathrm{C}(X,\mathbf{I})} \to \mathbf{I}$ is the projection (i.e., $\mathrm{pr}_f(\xi) = \xi(f)$).

Proposition 2.1.3. *For every Tychonoff space X, the map $e_X : X \to \mathbf{I}^{\mathrm{C}(X,\mathbf{I})}$ is an embedding.*

Proof. Let U be an open set in X and $x \in U$. Since X is a Tychonoff space, we have some $f \in \mathrm{C}(X, \mathbf{I})$ such that $f(x) = 0$ and $f(X \setminus U) \subset \{1\}$. Then, $V = \mathrm{pr}_f^{-1}([0, 1))$ is an open set in $\mathbf{I}^{\mathrm{C}(X,\mathbf{I})}$. Since $\mathrm{pr}_f(e_X(x)) = f(x) = 0$, it follows that $e_X(x) \in V$. Since $\mathrm{pr}_f \circ e_X(X \setminus U) = f(X \setminus U) \subset \{1\}$, we have $e_X(X \setminus U) \cap V = \emptyset$. Therefore, $e_X(x) \in V \cap e_X(X) \subset e_X(U)$. This implies that $e_X : X \to e_X(X)$ is an open map.

For $x \neq y \in X$, applying the above argument to $U = X \setminus \{y\}$, we can see that $e_X(x)(f) = 0 \neq 1 = e_X(y)(f)$. Thus, e_X is an embedding. $\square$

From Tychonoff's Theorem, it follows that the product space $\mathbf{I}^{\mathrm{C}(X,\mathbf{I})}$ is compact. Then, identifying X with $e_X(X)$, we define a compactification βX of X as follows:

$$\beta X = \mathrm{cl}_{\mathbf{I}^{\mathrm{C}(X,\mathbf{I})}} e_X(X),$$

which is called the **Stone–Čech compactification**.

Now, let $f : X \to Y$ be a map between Tychonoff spaces. The map $f_* : \mathbf{I}^{\mathrm{C}(X,\mathbf{I})} \to \mathbf{I}^{\mathrm{C}(Y,\mathbf{I})}$ is defined as $f_*(\xi) = (\xi(kf))_{k \in \mathrm{C}(Y,\mathbf{I})}$ for each $\xi \in \mathbf{I}^{\mathrm{C}(X,\mathbf{I})}$, where the continuity of f_* follows from the continuity of $\mathrm{pr}_k \circ f_* = \mathrm{pr}_{kf}$, $k \in \mathrm{C}(Y, \mathbf{I})$. Then, we have $f_* \circ e_X = e_Y \circ f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_X"'] & Y \arrow[d, "e_Y"] \\ \mathbf{I}^{\mathrm{C}(X,\mathbf{I})} \arrow[r, "f_*"'] & \mathbf{I}^{\mathrm{C}(Y,\mathbf{I})} \end{tikzcd}$$

Indeed, for each $x \in X$ and $k \in \mathrm{C}(Y, \mathbf{I})$,

$$f_*(e_X(x))(k) = e_X(x)(kf) = k(f(x)) = e_Y(f(x))(k).$$

Since f_* is continuous, it follows that $f_*(\beta X) \subset \beta Y$. Thus, f extends to the map $\beta f = f_*|\beta X : \beta X \to \beta Y$.

Further, let $g : Y \to Z$ be another map, where Z is Tychonoff. Then, for each $\xi \in \mathbf{I}^{\mathrm{C}(X,\mathbf{I})}$ and $k \in \mathrm{C}(Z, \mathbf{I})$,

$$g_*(f_*(\xi))(k) = f_*(\xi)(kg) = \xi(kgf) = (gf)_*(\xi)(k),$$

that is, $g_* f_* = (gf)_*$. Therefore, $\beta(gf) = \beta g \beta f$.

The Stone–Čech compactification βX can be characterized as follows:

Theorem 2.1.4 (Stone; Čech). *Let X be a Tychonoff space. For any compactification γX of X, there exists the (unique) map $f : \beta X \to \gamma X$ such that $f|X = \mathrm{id}_X$. If a compactification $\beta' X$ has the same property as above, then there exists a homeomorphism $h : \beta X \to \beta' X$ such that $h|X = \mathrm{id}_X$.*

Proof. Note that $\beta(\gamma X) = \gamma X$ because γX is compact. Let $i : X \hookrightarrow \gamma X$ be the inclusion and let $f = \beta i : \beta X \to \beta(\gamma X) = \gamma X$. Then, $f|X = \mathrm{id}_X$ and f is unique because X is dense in βX.

If a compactification $\beta' X$ of X has the same property, then we have two maps $h : \beta X \to \beta' X$ and $h' : \beta' X \to \beta X$ such that $h|X = h'|X = \mathrm{id}_X$. It follows that $h'h = \mathrm{id}_{\beta X}$ and $hh' = \mathrm{id}_{\beta' X}$, which means that h is a homeomorphism. □

A **perfect map** $f : X \to Y$ is a closed map such that $f^{-1}(y)$ is compact for each $y \in Y$. A map $f : X \to Y$ is said to be **proper** if $f^{-1}(K)$ is compact for every compact set $K \subset Y$.

Proposition 2.1.5. *Every perfect map $f : X \to Y$ is proper. If Y is locally compact, then every proper map $f : X \to Y$ is perfect.*

Proof. To prove the first assertion, let $K \subset Y$ be compact and $\mathcal{U}$ an open cover of $f^{-1}(K)$ in X. For each $y \in K$, choose a finite subcollection $\mathcal{U}_y \subset \mathcal{U}$ so that $f^{-1}(y) \subset \bigcup \mathcal{U}_y$. Since f is closed, each $V_y = Y \setminus f(X \setminus \bigcup \mathcal{U}_y)$ is an open neighborhood of y in Y, where $f^{-1}(V_y) \subset \bigcup \mathcal{U}_y$. We can choose $y_1, \ldots, y_n \in K$ so that $K \subset \bigcup_{i=1}^n V_{y_i}$. Thus, we have a finite subcollection $\mathcal{U}_0 = \bigcup_{i=1}^n \mathcal{U}_{y_i} \subset \mathcal{U}$ such that $f^{-1}(K) \subset \bigcup \mathcal{U}_0$. Hence, $f^{-1}(K)$ is compact.

To show the second assertion, it suffices to prove that a proper map f is closed. Let $A \subset X$ be closed and $y \in \mathrm{cl}\, f(A)$. Since Y is locally compact, y has a compact neighborhood N in Y. Note that $N \cap f(A) \neq \emptyset$, which implies $f^{-1}(N) \cap A \neq \emptyset$. Since f is proper, $f^{-1}(N)$ is compact, and hence $f^{-1}(N) \cap A$ is also compact. Thus, $f(f^{-1}(N) \cap A)$ is compact, so it is closed in Y. If $y \notin f(f^{-1}(N) \cap A)$, y has a compact neighborhood $M \subset N$ with $M \cap f(f^{-1}(N) \cap A) = \emptyset$. Then, observe that

$$f(f^{-1}(M) \cap A) \subset M \cap f(f^{-1}(N) \cap A) = \emptyset,$$

which means that $f^{-1}(M) \cap A = \emptyset$. However, using the same argument as for $f^{-1}(N) \cap A \neq \emptyset$, we can see that $f^{-1}(M) \cap A \neq \emptyset$, which is a contradiction. Thus, $y \in f(f^{-1}(N) \cap A) \subset f(A)$. Therefore, $f(A)$ is closed in Y. □

It follows from the first assertion of Proposition 2.1.5 that the composition of any two perfect maps is also perfect. In the second assertion, the local compactness of Y is not necessary if X and Y are metrizable, which allows the following proposition:

Proposition 2.1.6. *For a map $f : X \to Y$ between metrizable spaces, the following are equivalent:*

(a) *$f : X \to Y$ is perfect;*
(b) *$f : X \to Y$ is proper;*
(c) *Any sequence $(x_n)_{n\in\mathbb{N}}$ in X has a convergent subsequence if $(f(x_n))_{n\in\mathbb{N}}$ is convergent in Y.*

Proof. The implication (a) $\Rightarrow$ (b) has been shown in Proposition 2.1.5.

(b) $\Rightarrow$ (c): Let $y = \lim_{n\to\infty} f(x_n) \in Y$ and $K = \{f(x_n) \mid n \in \mathbb{N}\} \cup \{y\}$. Since K is compact, (b) implies the compactness of $f^{-1}(K)$, whose sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

(c) $\Rightarrow$ (a): For each $y \in Y$, every sequence $(x_n)_{n\in\mathbb{N}}$ in $f^{-1}(y)$ has a convergent subsequence due to (c), which means that $f^{-1}(y)$ is compact because $f^{-1}(y)$ is metrizable.

To see that f is a closed map, let $A \subset X$ be a closed set and $y \in \mathrm{cl}_Y f(A)$. Then, we have a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $y = \lim_{n\to\infty} f(x_n)$. Due to (c), $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_i})_{i\in\mathbb{N}}$, and since A is closed in X, we have $\lim_{i\to\infty} x_{n_i} = x \in A$. Then, $y = f(x) \in f(A)$, and therefore $f(A)$ is closed in Y. This completes the proof. □

Lemma 2.1.7. *Let D be a dense subset of X such that $D \neq X$. Any perfect map $f : D \to Y$ cannot extend over X.*

Proof. Assume that f extends to a map $\tilde{f} : X \to Y$. Let $x_0 \in X \setminus D$, $y_0 = \tilde{f}(x_0)$, $\widetilde{D} = D \cup \{x_0\}$, and $g = \tilde{f}|\widetilde{D} : \widetilde{D} \to Y$. Since $f^{-1}(y_0)$ is compact and $x_0 \notin f^{-1}(y_0)$, $\widetilde{D}$ has disjoint open sets U and V such that $x_0 \in U$ and $f^{-1}(y_0) \subset V$. Since f is a closed map, $f(D \setminus V)$ is closed in Y, hence $g^{-1}(f(D \setminus V))$ is closed in $\widetilde{D}$. Because $g^{-1}(y) = f^{-1}(y)$ for any $y \in Y \setminus \{y_0\}$, we have

$$D \setminus V \subset g^{-1}(f(D \setminus V)) = f^{-1}(f(D \setminus V)) \subset D.$$

On the other hand, $x_0 \notin \mathrm{cl}_{\widetilde{D}} V$. Therefore, $D = \mathrm{cl}_{\widetilde{D}} V \cup g^{-1}(f(D \setminus V))$ is closed in $\widetilde{D}$, which contradicts the fact that D is dense in $\widetilde{D}$. □

Theorem 2.1.8. *For a map $f : X \to Y$ between Tychonoff spaces, the following are equivalent:*

(a) *f is perfect;*
(b) *For any compactification γY of Y, f extends to a map $\tilde{f} : \beta X \to \gamma Y$ so that $\tilde{f}(\beta X \setminus X) \subset \gamma Y \setminus Y$;*
(c) *$\beta f(\beta X \setminus X) \subset \beta Y \setminus Y$.*

Proof. The implication (b) $\Rightarrow$ (c) is obvious.

(a) $\Rightarrow$ (b): Applying Theorem 2.1.4, we can obtain a map $g : \beta Y \to \gamma Y$ with $g|Y = \mathrm{id}$. Then, $\tilde{f} = g(\beta f)$ is an extension of f. Moreover, we can apply Lemma 2.1.7 to see that $\tilde{f}(\beta X \setminus X) \subset \gamma Y \setminus Y$.

(c) $\Rightarrow$ (a): For each $y \in Y$, $f^{-1}(y) = (\beta f)^{-1}(y)$ is compact. For each closed set A in X,

$$(\beta f)(\mathrm{cl}_{\beta X} A) \cap Y = f(\mathrm{cl}_{\beta X} A \cap X) = f(A),$$

which implies that $f(A)$ is closed in Y. Therefore, f is perfect. □

Remark 1. In Theorem 2.1.4, the map $f : \beta X \to \gamma X$ with $f|X = \mathrm{id}_X$ satisfies the condition $f(\beta X \setminus X) \subset \gamma X \setminus X$ that follows from Theorem 2.1.8.

Using Tychonoff's Theorem 2.1.1 and Wallace's Theorem 2.1.2, we can prove the following:

Theorem 2.1.9. *For each $\lambda \in \Lambda$, let $f_\lambda : X_\lambda \to Y_\lambda$ be a perfect map. Then, the map $f = \prod_{\lambda\in\Lambda} f_\lambda : X = \prod_{\lambda\in\Lambda} X_\lambda \to Y = \prod_{\lambda\in\Lambda} Y_\lambda$ is also perfect.*

Proof. Owing to Tychonoff's Theorem 2.1.1, $f^{-1}(y) = \prod_{\lambda\in\Lambda} f_\lambda^{-1}(y(\lambda))$ is compact for each $y \in Y$. To show that f is a closed map, let A be a closed set in X and $y \in Y \setminus f(A)$. Since $f^{-1}(y) \subset X \setminus A$, we can apply Wallace's Theorem 2.1.2 to obtain $\lambda_1, \dots, \lambda_n \in \Lambda$ and open sets U_i in X_{λ_i}, $i = 1, \dots, n$, such that

$$f^{-1}(y) = \prod_{\lambda\in\Lambda} f_\lambda^{-1}(y(\lambda)) \subset \bigcap_{i=1}^{n} \mathrm{pr}_{\lambda_i}^{-1}(U_i) \subset X \setminus A.$$

Since f_{λ_i} is a closed map, $V_i = Y_{\lambda_i} \setminus f_{\lambda_i}(X_{\lambda_i} \setminus U_i)$ is an open neighborhood of $y(\lambda_i)$ in Y_{λ_i} and $f_{\lambda_i}^{-1}(V_i) \subset U_i$. Then, $V = \bigcap_{i=1}^{n} \mathrm{pr}_{\lambda_i}^{-1}(V_i)$ is a neighborhood of y in Y and $f^{-1}(V) \subset X \setminus A$, i.e., $V \cap f(A) = \emptyset$. Therefore, f is a closed map. □

2.2 The Tietze Extension Theorem and Normalities

In this section, we prove the Tietze Extension Theorem and present a few concepts that strengthen normality. For $A, B \subset X$, it is said that A and B are **separated** in X if $A \cap \mathrm{cl}\, B = \emptyset$ and $B \cap \mathrm{cl}\, A = \emptyset$.

Lemma 2.2.1. *Let A and B be separated F_σ sets in a normal space X. Then, X has disjoint open sets U and V such that $A \subset U$ and $B \subset V$.*

Proof. Let $A = \bigcup_{n\in\mathbb{N}} A_n$ and $B = \bigcup_{n\in\mathbb{N}} B_n$, where $A_1 \subset A_2 \subset \cdots$ and $B_1 \subset B_2 \subset \cdots$ are closed in X. Set $U_0 = V_0 = \emptyset$. Using normality, we can inductively choose open sets $U_n, V_n \subset X$, $n \in \mathbb{N}$, so that

$$A_n \cup \mathrm{cl}\, U_{n-1} \subset U_n \subset \mathrm{cl}\, U_n \subset X \setminus (\mathrm{cl}\, B \cup \mathrm{cl}\, V_{n-1}) \quad \text{and}$$

$$B_n \cup \mathrm{cl}\, V_{n-1} \subset V_n \subset \mathrm{cl}\, V_n \subset X \setminus (\mathrm{cl}\, A \cup \mathrm{cl}\, U_n).$$

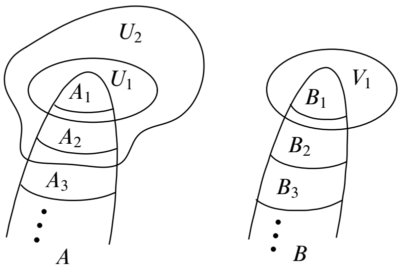

Fig. 2.1 Construction of U_n and V_n

Then, $U = \bigcup_{n\in\mathbb{N}} U_n$ and $V = \bigcup_{n\in\mathbb{N}} V_n$ are disjoint open sets in X such that $A \subset U$ and $B \subset V$ — Fig. 2.1. □

We can now prove the following extension theorem:

Theorem 2.2.2 (Tietze Extension Theorem). *Let A be a closed set in a normal space X. Then, every map $f : A \to \mathbf{I}$ extends over X.*

Proof. We first construct the open sets $W(q)$ in X, $q \in \mathbf{I} \cap \mathbb{Q}$, so that

(1) $q < q' \Rightarrow \operatorname{cl} W(q) \subset W(q')$,
(2) $A \cap W(q) = f^{-1}([0, q))$.

To this end, let $\{q_n \mid n \in \mathbb{N}\} = \mathbf{I} \cap \mathbb{Q}$, where $q_1 = 0$, $q_2 = 1$ and $q_i \neq q_j$ if $i \neq j$. We define $W(q_1) = W(0) = \emptyset$ and $W(q_2) = W(1) = X \setminus f^{-1}(1)$. Assume that $W(q_1), W(q_2), \cdots, W(q_n)$ have been defined so as to satisfy (1) and (2). Let

$$q_l = \min \{q_i \bigm| q_i > q_{n+1}, i = 1, \cdots, n\} \quad \text{and}$$
$$q_m = \max \{q_i \bigm| q_i < q_{n+1}, i = 1, \cdots, n\}.$$

Note that $f^{-1}([0, q_{n+1}))$ and $f^{-1}((q_{n+1}, 1])$ are separated F_σ sets in X. Using Lemma 2.2.1, we can find an open set U in X such that $f^{-1}([0, q_{n+1})) \subset U$ and $f^{-1}((q_{n+1}, 1]) \cap \operatorname{cl} U = \emptyset$. Then, $V = U \setminus f^{-1}(q_{n+1})$ is open in X and $A \cap V = f^{-1}([0, q_{n+1}))$. Again, using normality, we can obtain an open set G in X such that

$$\operatorname{cl} W(q_m) \cup f^{-1}([0, q_{n+1}]) \subset G \subset \operatorname{cl} G \subset W(q_l).$$

Then, $A \cap (V \cap G) = f^{-1}([0, q_{n+1}))$ and $\operatorname{cl}(V \cap G) \subset W(q_l)$. Yet again, using normality, we can take an open set H in X such that

$$\operatorname{cl} W(q_m) \subset H \subset \operatorname{cl} H \subset G \setminus f^{-1}([q_{n+1}, 1]) \; (\subset W(q_l)).$$

Then, $W(q_{n+1}) = (V \cap G) \cup H$ is the desired open set in X (Fig. 2.2).

Now, we define $\tilde{f} : X \to \mathbf{I}$ as follows:

$$\tilde{f}(x) = \begin{cases} 1 & \text{if } x \notin W(1), \\ \inf \{q \in \mathbf{I} \cap \mathbb{Q} \bigm| x \in W(q)\} & \text{if } x \in W(1). \end{cases}$$

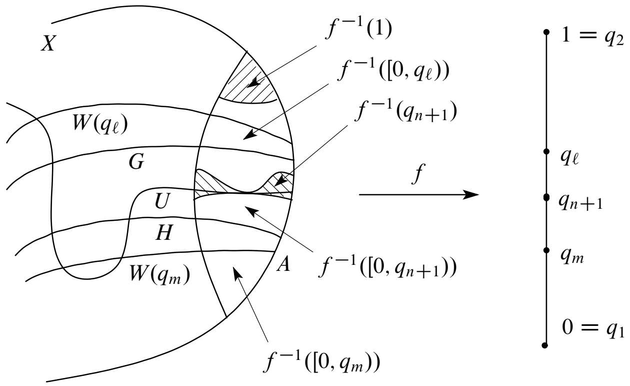

Fig. 2.2 $W(q_{n+1}) = ((U \setminus f^{-1}(q_{n+1})) \cap G) \cup H$

Then, $\tilde{f}|A = f$ because, for each $x \in A \cap W(1) = A \setminus f^{-1}(1)$,

$$\tilde{f}(x) = \inf\{q \in \mathbf{I} \cap \mathbb{Q} \mid x \in f^{-1}([0, q))\} = f(x).$$

To see the continuity of $\tilde{f}$, let $0 < a \leq 1$ and $0 \leq b < 1$. Since $\tilde{f}(x) < a$ if and only if $x \in W(q)$ for some $q < a$, it follows that $\tilde{f}^{-1}([0, a)) = \bigcup_{q<a} W(q)$ is open in X. Moreover, from (1), it follows that $\tilde{f}(x) > b$ if and only if $x \notin \operatorname{cl} W(q)$ for some $q > b$. Then, $\tilde{f}^{-1}((b, 1]) = X \setminus \bigcap_{q>b} \operatorname{cl} W(q)$ is also open in X. Therefore, $\tilde{f}$ is continuous. □

As a corollary, we have Urysohn's Lemma:

Corollary 2.2.3 (URYSOHN'S LEMMA). *For each disjoint pair of closed sets A and B in a normal space X, there exists a map $f : X \to \mathbf{I}$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.* □

Such a map f as in the above is called a **Urysohn map**.

Note. In the standard proof of the Tietze Extension Theorem 2.2.2, the desired extension is obtained as the uniform limit of a sequence of approximate extensions that are sums of Urysohn maps. On the other hand, Urysohn's Lemma is directly proved as follows:

Using the normality property yields the open sets $W(q)$ in X corresponding to all $q \in \mathbf{I} \cap \mathbb{Q}$ satisfying condition (1) in our proof of the Tietze Extension Theorem and

$$A \subset W(0) \subset \operatorname{cl} W(0) \subset W(1) = X \setminus B.$$

A Urysohn map $f : X \to \mathbf{I}$ can be defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \notin W(1), \\ \inf\{q \in \mathbf{I} \cap \mathbb{Q} \mid x \in W(q)\} & \text{if } x \in W(1). \end{cases}$$

In general, a subspace of a normal space is not normal (cf. Sect. 2.10). However, we have the following proposition:

Proposition 2.2.4. *Every F_σ set in a normal space is also normal.*

Proof. Let Y be an F_σ set in a normal space X. Every pair of disjoint closed sets in Y are F_σ sets in X that are separated in X. Then, the normality of Y follows from Lemma 2.2.1. □

A space X is **hereditarily normal** if every subspace of X is normal. Evidently, every metrizable space is hereditarily normal. It is said that X is **completely normal** provided that, for each pair of separated subsets $A, B \subset X$, there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$. These concepts meet in the following theorem:

Theorem 2.2.5. *For a space X, the following are equivalent:*

(a) *X is hereditarily normal;*
(b) *Every open set in X is normal;*
(c) *X is completely normal.*

Proof. The implication (a) ⇒ (b) is obvious.

(c) ⇒ (a): For an arbitrary subspace $Y \subset X$, each pair of disjoint closed sets A and B in Y are separated in X. Then, (a) follows from (c).

(b) ⇒ (c): Let $A, B \subset X$ be separated, i.e., $A \cap \operatorname{cl} B = \emptyset$ and $B \cap \operatorname{cl} A = \emptyset$. Then, $W = X \setminus (\operatorname{cl} A \cap \operatorname{cl} B)$ is open in X and $A, B \subset W$. Moreover,

$$\operatorname{cl}_W A \cap \operatorname{cl}_W B = W \cap \operatorname{cl} A \cap \operatorname{cl} B = \emptyset.$$

From the normality of W, we have disjoint open sets U and V in W such that $A \subset U$ and $B \subset V$. Then, U and V are open in X, and hence we have (c). □

A normal space X is **perfectly normal** if every closed set in X is G_δ in X (equivalently, every open set in X is F_σ in X). Clearly, every metrizable space is perfectly normal. A closed set $A \subset X$ is called a **zero set** in X if $A = f^{-1}(0)$ for some map $f : X \to \mathbb{R}$, where $\mathbb{R}$ can be replaced by $\mathbf{I}$. The complement of a zero set in X is called a **cozero set**.

Theorem 2.2.6. *For a space X, the following conditions are equivalent:*

(a) *X is perfectly normal;*
(b) *Every closed set in X is a zero set (equivalently, every open set in X is a cozero set);*
(c) *For every pair of disjoint closed sets A and B in X, there exists a map $f : X \to \mathbf{I}$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

Proof. The implication (c) ⇒ (a) is trivial.

(a) ⇒ (b): Let A be a closed set in X. Then, we can write $A = \bigcap_{n\in\mathbb{N}} G_n$, where each G_n is open in X. Using Urysohn's Lemma, we take maps $f_n : X \to \mathbf{I}$, $n \in \mathbb{N}$, such that $f_n(A) \subset \{0\}$ and $f_n(X \setminus G_n) \subset \{1\}$. We can define a map $f : X \to \mathbf{I}$ as $f(x) = \sum_{n\in\mathbb{N}} 2^{-n} f_n(x)$. Then, it is easy to see that $A = f^{-1}(0)$.

(b) $\Rightarrow$ (c): Let A and B be disjoint closed sets in X. Condition (b) provides two maps $g, h : X \to \mathbf{I}$ such that $g^{-1}(0) = A$ and $h^{-1}(0) = B$. Then, the desired map $f : X \to \mathbf{I}$ can be defined as follows:

$$f(x) = \frac{g(x)}{g(x) + h(x)}. \qquad \square$$

Theorem 2.2.7. *Every perfectly normal space is hereditarily normal (= completely normal).*

Proof. Let X be perfectly normal. Then, each open set in X is an F_σ set, which is normal as a consequence of Proposition 2.2.4. Hence, it follows from Theorem 2.2.5 that X is hereditarily normal. $\square$

Remark 2. Let $A_0, A_1, \ldots, A_n$ be pairwise disjoint closed sets in a normal space X. We can apply the Tietze Extension Theorem 2.2.2 to obtain a map $f : X \to \mathbf{I}$ such that $A_i \subset f^{-1}(i/n)$ (i.e., $f(A_i) \subset \{i/n\}$) for each $i = 0, 1, \ldots, n$. When X is perfectly normal and $n > 2$, the condition $A_i \subset f^{-1}(i/n)$ cannot be replaced by $A_i = f^{-1}(i/n)$. For example, let $X = \mathbf{S}^1$ be the unit circle (the unit 1-sphere of $\mathbb{R}^2$), $A_0 = \{\mathbf{e}_1\}$, $A_1 = \{\mathbf{e}_2\}$, and $A_2 = \{-\mathbf{e}_1\}$, where $\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1) \in \mathbb{R}^2$. Since $X \setminus A_1$ is (path-)connected, there does not exist a map $f : X \to \mathbf{I}$ such that $A_0 = f^{-1}(0)$, $A_1 = f^{-1}(1/2)$ and $A_2 = f^{-1}(1)$.

2.3 Stone's Theorem and Metrization

In this section, we prove Stone's Theorem and characterize the metrizability using open bases. Let $\mathcal{A}$ be a collection of subsets of a space X and $B \subset X$. Recall that

$$\mathcal{A}[B] = \{A \in \mathcal{A} \mid A \cap B \neq \emptyset\}.$$

When $B = \{x\}$, we write $\mathcal{A}[\{x\}] = \mathcal{A}[x]$. It is said that $\mathcal{A}$ is **locally finite** (resp. **discrete**) in X if each $x \in X$ has a neighborhood U that meets only finite members (resp. at most one member) of $\mathcal{A}$, i.e., $\operatorname{card} \mathcal{A}[U] < \aleph_0$ (resp. $\operatorname{card} \mathcal{A}[U] \leq 1$). When $w(X) \geq \aleph_0$, if $\mathcal{A}$ is locally finite in X, then $\operatorname{card} \mathcal{A} \leq w(X)$. For the sake of convenience, we introduce the notation $\mathcal{A}^{\mathrm{cl}} = \{\operatorname{cl} A \mid A \in \mathcal{A}\}$. The following is easily proved and will be used frequently:

Fact. *If $\mathcal{A}$ is locally finite (or discrete) in X, then so is $\mathcal{A}^{\mathrm{cl}}$ and also* $\operatorname{cl} \bigcup \mathcal{A} = \bigcup \mathcal{A}^{\mathrm{cl}}$ $(= \bigcup_{A\in\mathcal{A}} \operatorname{cl} A)$.

A collection of subsets of X is said to be **σ-locally finite** (resp. **σ-discrete**) in X if it can be represented as a countable union of locally finite (resp. discrete) collections.

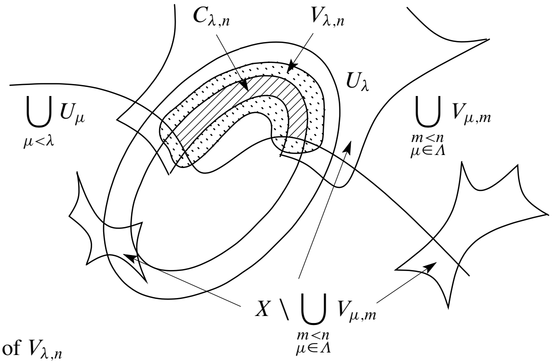

Fig. 2.3 Definition of $V_{\lambda,n}$

Theorem 2.3.1 (A.H. Stone). *Every open cover of a metrizable space has a locally finite and σ-discrete open refinement.*

Proof. Let $X = (X, d)$ be a metric space and $\mathcal{U} \in \operatorname{cov}(X)$. We may index all members of $\mathcal{U}$ by a well-ordered set $\Lambda = (\Lambda, \leq)$, that is, $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$. By induction on $n \in \mathbb{N}$, we define open collections $\mathcal{V}_n = \{V_{\lambda,n} \mid \lambda \in \Lambda\}$ as follows:

$$V_{\lambda,n} = \mathrm{N}(C_{\lambda,n}, 2^{-n}) = \left\{x \in X \;\middle|\; d(x, C_{\lambda,n}) < 2^{-n}\right\},$$

where

$$C_{\lambda,n} = \left\{x \in X \;\middle|\; d(x, X \setminus U_\lambda) > 2^{-n}3\right\} \setminus \left(\bigcup_{\mu<\lambda} U_\mu \cup \bigcup_{\substack{m<n\\ \mu\in\Lambda}} V_{\mu,m} \right).$$

For each $x \in X$, let $\lambda(x) = \min\{\lambda \in \Lambda \mid x \in U_\lambda\}$ and choose $n \in \mathbb{N}$ so that $2^{-n}3 < d(x, X \setminus U_{\lambda(x)})$. Then, $x \in C_{\lambda(x),n} \subset V_{\lambda(x),n}$ or $x \in V_{\mu,m}$ for some $\mu \in \Lambda$ and $m < n$. Hence, we have $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n \in \operatorname{cov}(X)$. Since each $V_{\lambda,n}$ is contained in U_λ, it follows that $\mathcal{V} \prec \mathcal{U}$. See Fig. 2.3.

The discreteness of each $\mathcal{V}_n$ follows from the claim:

Claim (1). If $\lambda \neq \mu$ then $\operatorname{dist}_d(V_{\lambda,n}, V_{\mu,n}) \geq 2^{-n}$.

To prove this claim, we may assume $\mu < \lambda$. For each $x \in V_{\lambda,n}$ and $y \in V_{\mu,n}$, choose $x' \in C_{\lambda,n}$ and $y' \in C_{\mu,n}$ so that $d(x, x') < 2^{-n}$ and $d(y, y') < 2^{-n}$, respectively. Then, $x' \notin U_\mu$ and $d(y', X \setminus U_\mu) > 2^{-n}3$, hence $d(x', y') > 2^{-n}3$. Therefore,

$$d(x, y) \geq d(x', y') - d(x, x') - d(y, y') > 2^{-n}.$$

The local finiteness of $\mathcal{V}$ follows from the discreteness of each $\mathcal{V}_n$ and the claim:

Claim (2). If $\mathrm{B}(x, 2^{-k}) \subset V_{\mu,m}$, then $\mathrm{B}(x, 2^{-k-1}) \cap V_{\lambda,n} = \emptyset$ for all $\lambda \in \Lambda$ and $n > \max\{k, m\}$.

For each $y \in V_{\lambda,n}$, choose $y' \in C_{\lambda,n}$ so that $d(y, y') < 2^{-n}$. Since $y' \notin V_{\mu,m}$, it follows that $d(x, y') \geq 2^{-k}$. Hence,

$$d(x, y) \geq d(x, y') - d(y, y') > 2^{-k} - 2^{-n} \geq 2^{-k-1}.$$

The proof is complete. □

Applying Theorem 2.3.1 to the open covers $\mathcal{B}_n = \{\mathrm{B}(x, 2^{-n}) \mid x \in X\}$, $n \in \mathbb{N}$, of a metric space $X = (X, d)$, we have the following corollary:

Corollary 2.3.2. *Every metrizable space has a σ-discrete open basis.* □

Lemma 2.3.3. *A regular space X with a σ-locally finite open basis is perfectly normal.*

Proof. Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be an open basis for X where each $\mathcal{B}_n$ is locally finite in X. Instead of proving that every closed set in X is a G_δ set, we show that every open set $W \subset X$ is F_σ. For each $x \in W$, choose $k(x) \in \mathbb{N}$ and $B(x) \in \mathcal{B}_{k(x)}$ so that $x \in B(x) \subset \operatorname{cl} B(x) \subset W$. For each $n \in \mathbb{N}$, let

$$W_n = \bigcup \big\{B(x) \bigm| x \in W,\ k(x) = n\big\}.$$

Because of the local finiteness of $\mathcal{B}_n$, we have

$$\operatorname{cl} W_n = \bigcup \big\{\operatorname{cl} B(x) \bigm| x \in W,\ k(x) = n\big\} \subset W.$$

Since $W = \bigcup_{n\in\mathbb{N}} W_n$, it follows that $W = \bigcup_{n\in\mathbb{N}} \operatorname{cl} W_n$, which is F_σ in X.

To prove normality, let A and B be disjoint closed sets in X. As seen above, we have open sets $V_n, W_n \subset X$, $n \in \mathbb{N}$, such that $X \setminus A = \bigcup_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} \operatorname{cl} V_n$ and $X \setminus B = \bigcup_{n\in\mathbb{N}} W_n = \bigcup_{n\in\mathbb{N}} \operatorname{cl} W_n$. For each $n \in \mathbb{N}$, let

$$G_n = W_n \setminus \bigcup_{m\leq n} \operatorname{cl} V_m \quad \text{and} \quad H_n = V_n \setminus \bigcup_{m\leq n} \operatorname{cl} W_m.$$

Then, $G = \bigcup_{n\in\mathbb{N}} G_n$ and $H = \bigcup_{n\in\mathbb{N}} H_n$ are disjoint open sets in X such that $A \subset G$ and $B \subset H$. □

Theorem 2.3.4 (Bing; Nagata–Smirnov). *For a regular space X, the following conditions are equivalent:*

(a) *X is metrizable;*
(b) *X has a σ-discrete open basis;*
(c) *X has a σ-locally finite open basis.*

Proof. The implication (a) ⇒ (b) is Corollary 2.3.2 and (b) ⇒ (c) is obvious. It remains to show the implication (c) ⇒ (a).

(c) ⇒ (a): Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be an open basis for X where each $\mathcal{B}_n$ is locally finite in X. Since X is perfectly normal by Lemma 2.3.3, we have maps $f_B : X \to \mathbf{I}$,

$B \in \mathcal{B}$, such that $f_B^{-1}(0) = X \setminus B$ (Theorem 2.2.6). For each $n \in \mathbb{N}$, since $\mathcal{B}_n$ is locally finite, we can define a map $f_n : X \to \ell_1(\mathcal{B}_n)$ by $f_n(x) = (f_B(x))_{B \in \mathcal{B}_n} \in \ell_1(\mathcal{B}_n)$. Let $f : X \to \prod_{n\in\mathbb{N}} \ell_1(\mathcal{B}_n)$ be the map defined by $f(x) = (f_n(x))_{n\in\mathbb{N}}$. Since $\prod_{n\in\mathbb{N}} \ell_1(\mathcal{B}_n)$ is metrizable, it suffices to show that f is an embedding.

For each $x \neq y \in X$, choose $B \in \mathcal{B}_n \subset \mathcal{B}$ so that $x \in B$ and $y \notin B$. Then, $f_B(x) > 0 = f_B(y)$, so $f_n(x) \neq f_n(y)$. Hence, f is an injection.

For each $n \in \mathbb{N}$ and $B \in \mathcal{B}_n$, $V_B = \{y \in \ell_1(\mathcal{B}_n) \mid y(B) > 0\}$ is open in $\ell_1(\mathcal{B}_n)$. Observe that for $x \in X$,

$$x \in B \Leftrightarrow f_n(x)(B) = f_B(x) > 0 \Leftrightarrow f_n(x) \in V_B.$$

Then, it follows that $f(B) = \mathrm{pr}_n^{-1}(V_B) \cap f(X)$ is open in $f(X)$, where $\mathrm{pr}_n : \prod_{n\in\mathbb{N}} \ell_1(\mathcal{B}_n) \to \ell_1(\mathcal{B}_n)$ is the projection. Thus, f is an embedding. □

The equivalence of (a) and (b) in Theorem 2.3.4 is called the BING METRIZATION THEOREM, and the equivalence of (a) and (c) is called the NAGATA–SMIRNOV METRIZATION THEOREM. As a corollary, we have the URYSOHN METRIZATION THEOREM:

Corollary 2.3.5. *A space is separable and metrizable if and only if it is regular and second countable.* □

For a metrizable space X, let Γ be an infinite set with $w(X) \leq \operatorname{card} \Gamma$. In the proof of Theorem 2.3.4, note that $\operatorname{card} \mathcal{B}_n \leq \operatorname{card} \Gamma$ because of the local finiteness of $\mathcal{B}_n$ in X. Then, every $\ell_1(\mathcal{B}_n)$ can be embedded into $\ell_1(\Gamma)$. Therefore, we can state the following corollary:

Corollary 2.3.6. *Let X be a metrizable space and Γ an infinite set such that $w(X) \leq \operatorname{card} \Gamma$. Then, X can be embedded in the completely metrizable topological linear space*[2] $\ell_1(\Gamma)^{\mathbb{N}}$. □

Here, $w(\ell_1(\Gamma)^{\mathbb{N}}) = w(\ell_1(\Gamma)) = \operatorname{card} \Gamma$. In fact, $w(\ell_1(\Gamma)) \geq \operatorname{card} \Gamma$ because $\ell_1(\Gamma)$ has a discrete open collection with the same cardinality as Γ. Let

$$D = \Big\{x \in \ell_1(\Gamma) \;\Big|\; x(\gamma) \in \mathbb{Q} \text{ for all } \gamma \in \Gamma \text{ and } x(\gamma) = 0 \text{ except for finitely many } \gamma \in \Gamma\Big\}.$$

Then, $\{\mathrm{B}(x, n^{-1}) \mid x \in D, n \in \mathbb{N}\}$ is an open basis for $\ell_1(\Gamma)$ with the same cardinality as Γ, hence $w(\ell_1(\Gamma)) \leq \operatorname{card} \Gamma$.

The **hedgehog** $J(\Gamma)$ is the closed subspace of $\ell_1(\Gamma)$ defined as follows:

$$J(\Gamma) = \bigcup_{\gamma\in\Gamma} \mathbf{I}\mathbf{e}_\gamma = \Big\{x \in \ell_1(\Gamma) \;\Big|\; x(\gamma) \in \mathbf{I} \text{ for all } \gamma \in \Gamma \text{ and } x(\gamma) \neq 0 \text{ at most one } \gamma \in \Gamma\Big\},$$

[2] For topological linear spaces, refer to Sect. 3.4.

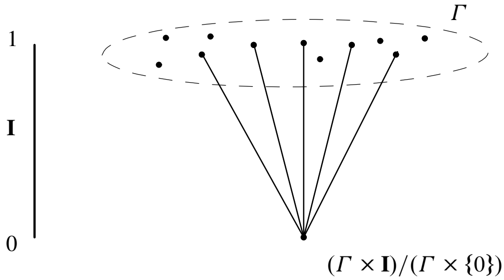

Fig. 2.4 The hedgehog $J(\Gamma)$

where $\mathbf{e}_\gamma \in \ell_1(\Gamma)$ is the unit vector defined by $\mathbf{e}_\gamma(\gamma) = 1$ and $\mathbf{e}_\gamma(\gamma') = 0$ for $\gamma' \neq \gamma$ (Fig. 2.4). The hedgehog $J(\Gamma)$ can also be defined as the space $(\Gamma \times \mathbf{I})/(\Gamma \times \{0\})$ with the metric induced from the pseudo-metric ρ on $\Gamma \times \mathbf{I}$ defined as follows:

$$\rho((\gamma,t),(\gamma',s)) = \begin{cases} |t-s| & \text{if } \gamma = \gamma', \\ t+s & \text{if } \gamma \neq \gamma'. \end{cases}$$

Note that $w(J(\Gamma)^{\mathbb{N}}) = \operatorname{card}\Gamma$. In the proof of Theorem 2.3.4, if each $\mathcal{B}_n$ is discrete in X, then $f_n(X) \subset J(\mathcal{B}_n)$. Similar to Corollary 2.3.6, we have the following:

Corollary 2.3.7. *Let X be a metrizable space and Γ an infinite set such that $w(X) \leq \operatorname{card}\Gamma$. Then, X can be embedded in $J(\Gamma)^{\mathbb{N}}$.* □

In the second countable case, X can be embedded in $\mathbf{I}^{\mathbb{N}}$, since we can take $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ in the proof of Theorem 2.3.4 so that each $\mathcal{B}_n$ contains only one open set. Thus, we have the following embedding theorem for separable metrizable spaces:

Corollary 2.3.8. *Every separable metrizable space can be embedded in the Hilbert cube $\mathbf{I}^{\mathbb{N}}$, and hence in $\mathbb{R}^{\mathbb{N}}$.* □

In association with Corollary 2.3.6, we state the following theorem:

Theorem 2.3.9. *Every metric space $X = (X,d)$ can be isometrically embedded into the Banach space $C^B(X)$.*

Sketch of Proof. Fix $x_0 \in X$ and define $\varphi : X \to C^B(X)$ as follows:

$$\varphi(x)(z) = d(x,z) - d(x_0,z), \ z \in X.$$

It is easy to see that $\|\varphi(x)\| = d(x,x_0)$ and $\|\varphi(x) - \varphi(y)\| = d(x,y)$.

The (**metric**) **completion** of a metric space $X = (X,d)$ is a complete metric space $\widetilde{X} = (\widetilde{X},\tilde{d})$ containing X as a dense set and as a metric subspace, that is, d is the restriction of $\tilde{d}$. Since a closed set in a complete metric space is also complete, Theorem 2.3.9 implies the following:

Corollary 2.3.10. *Every metric space has a completion.* □

2.4 Sequences of Open Covers and Metrization

In this section, we characterize metrizable spaces via sequences of open covers. Given a cover $\mathcal{V}$ of a space X and $A \subset X$, we define

$$\mathrm{st}(A, \mathcal{V}) = \bigcup \mathcal{V}[A],$$

which is called the **star** of A with respect to $\mathcal{V}$. When $A = \{x\}$, we write $\mathrm{st}(\{x\}, \mathcal{V}) = \mathrm{st}(x, \mathcal{V})$.

Theorem 2.4.1 (ALEXANDROFF–URYSOHN; FRINK). *For a space X, the following conditions are equivalent:*

(a) *X is metrizable;*
(b) *X has open covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ such that $\{\mathrm{st}(x, \mathcal{U}_n) \mid n \in \mathbb{N}\}$ is a neighborhood basis of each $x \in X$ and*

$$U, U' \in \mathcal{U}_{n+1},\ U \cap U' \neq \emptyset \Rightarrow \exists U'' \in \mathcal{U}_n \text{ such that } U \cup U' \subset U'';$$

(c) *Each $x \in X$ has an open neighborhood basis $\{V_n(x) \mid n \in \mathbb{N}\}$ satisfying the condition that, for each $x \in X$ and $i \in \mathbb{N}$, there exists a $j(x,i) \geq i$ such that*

$$V_{j(x,i)}(x) \cap V_{j(x,i)}(y) \neq \emptyset \Rightarrow V_{j(x,i)}(y) \subset V_i(x).$$

Proof. (a) $\Rightarrow$ (c): A metric space $X = (X, d)$ satisfies (c) because

$$\mathrm{B}(x, 3^{-n}) \cap \mathrm{B}(y, 3^{-n}) \neq \emptyset \Rightarrow \mathrm{B}(y, 3^{-n}) \subset \mathrm{B}(x, 3^{-n+1}).$$

(c) $\Rightarrow$ (b): For each $x \in X$, let $k(x,1) = 1$ and inductively define

$$k(x,n) = \max\{n, j(x,i) \mid i = 1, \dots, k(x, n-1)\} \geq n.$$

For each $n \in \mathbb{N}$, let $U_n(x) = \bigcap_{i=1}^{k(x,n)} V_i(x)$. Then, $\{U_n(x) \mid n \in \mathbb{N}\}$ is an open neighborhood basis of x and

$$\begin{aligned} U_n(x) \cap U_n(y) \neq \emptyset \ \Rightarrow U_n(x) \cup U_n(y) &\subset U_{n-1}(x) \text{ or} \\ U_n(x) \cup U_n(y) &\subset U_{n-1}(y). \end{aligned}$$

In fact, assume that $U_n(x) \cap U_n(y) \neq \emptyset$. In the case $k(x,n) \leq k(y,n)$, $V_{j(x,i)}(y) \subset V_i(x)$ for each $i = 1, \dots, k(x, n-1)$ because $V_{j(x,i)}(x) \cap V_{j(x,i)}(y) \neq \emptyset$. Then, it follows that

$$U_n(y) \subset \bigcap_{i=1}^{k(x,n)} V_i(y) \subset \bigcap_{i=1}^{k(x,n-1)} V_{j(x,i)}(y) \subset \bigcap_{i=1}^{k(x,n-1)} V_i(x) = U_{n-1}(x).$$

Since $U_n(x) \subset U_{n-1}(x)$ by definition, we have $U_n(x) \cup U_n(y) \subset U_{n-1}(x)$. As above, $k(y,n) \le k(x,n)$ implies $U_n(x) \cup U_n(y) \subset U_{n-1}(y)$.

For each $n \in \mathbb{N}$, we have $\mathcal{U}_n = \{U_n(x) \mid x \in X\} \in \operatorname{cov}(X)$. It remains to be prove that $\{\operatorname{st}(x,\mathcal{U}_n) \mid n \in \mathbb{N}\}$ is a neighborhood basis of $x \in X$. Evidently, each $\operatorname{st}(x,\mathcal{U}_n)$ is a neighborhood of $x \in X$. Then, it suffices to show that $\operatorname{st}(x,\mathcal{U}_{j(x,n)}) \subset V_n(x)$. If $x \in U_{j(x,n)}(y)$, then

$$V_{j(x,n)}(x) \cap V_{j(x,n)}(y) \supset V_{j(x,n)}(x) \cap U_{j(x,n)}(y) \neq \emptyset,$$

and hence $U_{j(x,n)}(y) \subset V_{j(x,n)}(y) \subset V_n(x)$.

(b) $\Rightarrow$ (a): First, note that $\mathcal{U}_{i+1} \prec \mathcal{U}_i$ for each $i \in \mathbb{N}$. Let $\mathcal{U}_0 = \{X\} \in \operatorname{cov}(X)$. For each $x, y \in X$, define

$$\delta(x,y) = \inf\left\{2^{-i} \;\middle|\; \exists U \in \mathcal{U}_i \text{ such that } x, y \in U\right\}.$$

Note that if $\delta(x,y) > 0$, then $\delta(x,y) = 2^{-n}$ for some $n \ge 0$. As can easily be shown, the following hold for each $x, y, z \in X$:

(1) $\delta(x,y) = 0$ if and only if $x = y$;
(2) $\delta(x,y) = \delta(y,x)$;
(3) $\delta(x,y) \le 2\max\{\delta(x,z), \delta(z,y)\}$.

Furthermore, we claim that

(4) for every $n \ge 3$ and each $x_1, \dots, x_n \in X$,

$$\delta(x_1,x_n) \le 2(\delta(x_1,x_2) + \delta(x_{n-1},x_n)) + 4\sum_{i=2}^{n-2} \delta(x_i,x_{i+1}).$$

In fact, when $n = 3$, the inequality follows from (3). Assuming claim (4) holds for any $n < k$, we show (4) for $n = k$. Then, we may assume that $x_k \neq x_1$. For each $x_1, \dots, x_k \in X$, let

$$m = \min\left\{i \;\middle|\; \delta(x_1,x_k) \le 2\delta(x_1,x_i)\right\} \ge 2.$$

Then, $\delta(x_1,x_k) \le 2\delta(x_1,x_m)$. From (3) and the minimality of m, we have $\delta(x_1,x_k) \le 2\delta(x_{m-1},x_k)$. If $m = 2$ or $m = k$, then the inequality in (4) holds for $n = k$. In the case $2 < m < k$,

$$\delta(x_1,x_k) = \frac{1}{2}\delta(x_1,x_k) + \frac{1}{2}\delta(x_1,x_k) \le \delta(x_1,x_m) + \delta(x_{m-1},x_k).$$

By the inductive assumption, we have

$$\delta(x_1,x_m) \le 2(\delta(x_1,x_2) + \delta(x_{m-1},x_m)) + 4\sum_{i=2}^{m-2} \delta(x_i,x_{i+1}) \quad \text{and}$$

$$\delta(x_{m-1},x_k) \le 2(\delta(x_{m-1},x_m) + \delta(x_{k-1},x_k)) + 4\sum_{i=m}^{k-2} \delta(x_i,x_{i+1}),$$

so the desired inequality is obtained. By induction, (4) holds for all $n \in \mathbb{N}$.

Now, we can define $d \in \operatorname{Metr}(X)$ as follows:

$$d(x, y) = \inf\left\{ \sum_{i=1}^{n-1} \delta(x_i, x_{i+1}) \;\middle|\; n \in \mathbb{N},\ x_i \in X,\ x_1 = x,\ x_n = y \right\}.$$

In fact, $d(x, y) = d(y, x)$ by (2) and the above definition. The triangle inequality follows from the definition of d. Since $\delta(x, y) \leq 4d(x, y)$ by (4), it follows from (1) that $d(x, y) = 0$ implies $x = y$. Obviously, $x = y$ implies $d(x, y) = 0$. Moreover, it follows that

$$d(x, y) \leq 2^{-n-2} \Rightarrow \exists U \in \mathcal{U}_n \text{ such that } x, y \in U,$$

which means that $\overline{\mathrm{B}}_d(x, 2^{-n-2}) \subset \operatorname{st}(x, \mathcal{U}_n)$ for each $x \in X$ and $n \in \mathbb{N}$. Since $d(x, y) \leq \delta(x, y)$, we have $\operatorname{mesh}_d \mathcal{U}_n \leq 2^{-n}$, so $\operatorname{st}(x, \mathcal{U}_n) \subset \overline{\mathrm{B}}_d(x, 2^{-n})$. Therefore, $\{\mathrm{B}_d(x, 2^{-n}) \mid n \in \mathbb{N}\}$ is a neighborhood basis of $x \in X$. □

Remark 3. In the above proof of (b) ⇒ (a), the obtained metric $d \in \operatorname{Metr}(X)$ has the following property:

$$\operatorname{st}(x, \mathcal{U}_{n+2}) \subset \overline{\mathrm{B}}_d(x, 2^{-n-2}) \subset \operatorname{st}(x, \mathcal{U}_n).$$

Moreover, $d(x, y) \leq 1$ for every $x, y \in X$.

In Theorem 2.4.1, the equivalence between (a) and (b) is called the ALEXANDROFF–URYSOHN METRIZATION THEOREM and the equivalence between (a) and (c) is called the FRINK METRIZATION THEOREM.

Let $\mathcal{U}$ and $\mathcal{V}$ be covers of X. When $\{\operatorname{st}(x, \mathcal{V}) \mid x \in X\} \prec \mathcal{U}$, we call $\mathcal{V}$ a **Δ-refinement** (or **barycentric refinement**) of $\mathcal{U}$ and denote

$$\mathcal{V} \overset{\Delta}{\prec} \mathcal{U} \quad (\text{or } \mathcal{U} \overset{\Delta}{\succ} \mathcal{V}).$$

The following corollary follows from the Alexandroff–Urysohn Metrization Theorem:

Corollary 2.4.2. *A space X is metrizable if and only if X has a sequence of open covers*

$$\mathcal{U}_1 \overset{\Delta}{\succ} \mathcal{U}_2 \overset{\Delta}{\succ} \mathcal{U}_3 \overset{\Delta}{\succ} \cdots$$

such that $\{\operatorname{st}(x, \mathcal{U}_n) \mid n \in \mathbb{N}\}$ is a neighborhood basis of each $x \in X$. □

For covers $\mathcal{U}$ and $\mathcal{V}$ of X, we define

$$\operatorname{st}(\mathcal{V}, \mathcal{U}) = \left\{ \operatorname{st}(V, \mathcal{U}) \;\middle|\; V \in \mathcal{V} \right\},$$

which is called the **star** of $\mathcal{V}$ with respect to $\mathcal{U}$. We denote $\operatorname{st}(\mathcal{V}, \mathcal{V}) = \operatorname{st} \mathcal{V}$, which is called the **star** of $\mathcal{V}$. When $\operatorname{st} \mathcal{V} \prec \mathcal{U}$, we call $\mathcal{V}$ a **star-refinement** of $\mathcal{U}$ and denote

$$\mathcal{V} \overset{*}{\prec} \mathcal{U} \quad (\text{or } \mathcal{U} \overset{*}{\succ} \mathcal{V}).$$

For each $n \in \mathbb{N}$, the ***n*-th star** of $\mathcal{V}$ is inductively defined as follows:

$$\mathrm{st}^n \mathcal{V} = \mathrm{st}(\mathrm{st}^{n-1} \mathcal{V}, \mathcal{V}),$$

where $\mathrm{st}^0 \mathcal{V} = \mathcal{V}$. Observe that $\mathrm{st}(\mathcal{V}, \mathrm{st}\,\mathcal{V}) = \mathrm{st}^3 \mathcal{V}$ and $\mathrm{st}(\mathrm{st}\,\mathcal{V}) = \mathrm{st}^4 \mathcal{V}$. When $\mathrm{st}^n \mathcal{V} \prec \mathcal{U}$, $\mathcal{V}$ is called an ***n*-th star-refinement** of $\mathcal{U}$. There is the following relation between Δ-refinements and star-refinements:

Proposition 2.4.3. *For every three open covers $\mathcal{U}, \mathcal{V}, \mathcal{W}$ of a space X,*

$$\mathcal{W} \overset{\Delta}{\prec} \mathcal{V} \overset{\Delta}{\prec} \mathcal{U} \;\Rightarrow\; \mathcal{W} \overset{*}{\prec} \mathcal{U}.$$

Sketch of Proof. For each $W \in \mathcal{W}$, take any $x \in W$ and choose $U \in \mathcal{U}$ so that $\mathrm{st}(x, \mathcal{V}) \subset U$. Then, we see that $\mathrm{st}(W, \mathcal{W}) \subset U$.

By virtue of this proposition, Δ-refinements in Corollary 2.4.2 can be replaced by star-refinements, which allows us to sate the following corollary:

Corollary 2.4.4. *A space X is metrizable if and only if X has a sequence of open covers*

$$\mathcal{U}_1 \overset{*}{\succ} \mathcal{U}_2 \overset{*}{\succ} \mathcal{U}_3 \overset{*}{\succ} \cdots$$

such that $\{\mathrm{st}(x, \mathcal{U}_n) \mid n \in \mathbb{N}\}$ is a neighborhood basis of each $x \in X$. □

Remark 4. By tracing the proof of Theorem 2.4.1, we can directly prove Corollary 2.4.4. This direct proof is simpler than that of Theorem 2.4.1, and the obtained metric $d \in \mathrm{Metr}(X)$ has the following, more acceptable, property than the previous remark:

$$\mathrm{st}(x, \mathcal{U}_{n+1}) \subset \overline{\mathrm{B}}_d(x, 2^{-n}) \subset \mathrm{st}(x, \mathcal{U}_n).$$

Similar to the previous metric, $d(x, y) \leq 1$ for every $x, y \in X$.

Sketch of the direct proof of Corollary 2.4.4. To see the "if" part, replicate the proof of (b) $\Rightarrow$ (a) in Theorem 2.4.1 to construct $d \in \mathrm{Metr}(X)$. Let $\mathcal{U}_0 = \{X\}$. For each $x, y \in X$, we define

$$\delta(x, y) = \inf\{2^{-i+1} \mid \exists U \in \mathcal{U}_i \text{ such that } x, y \in U\} \text{ and}$$
$$d(x, y) = \inf\left\{\textstyle\sum_{i=1}^n \delta(x_{i-1}, x_i) \;\middle|\; n \in \mathbb{N},\ x_0 = x,\ x_n = y\right\}.$$

The admissibility and additional property of d are derived from the inequality $d(x, y) \leq \delta(x, y) \leq 2d(x, y)$. To prove the right-hand inequality, it suffices to show the following:

$$\delta(x_0, x_n) \leq 2 \sum_{i=1}^n \delta(x_{i-1}, x_i) \text{ for each } x_0, x_1, \ldots, x_n \in X.$$

This is proved by induction on $n \in \mathbb{N}$. Set $\sum_{i=1}^n \delta(x_{i-1}, x_i) = \alpha$ and let k be the largest number such that $\sum_{i=1}^k \delta(x_{i-1}, x_i) \leq \alpha/2$. Then, $\sum_{i=k+2}^n \delta(x_{i-1}, x_i) < \alpha/2$. By the inductive assumption, $\delta(x_0, x_k) \leq \alpha$ and $\delta(x_{k+1}, x_n) < \alpha$. Note that $\delta(x_k, x_{k+1}) \leq \alpha$. Let $m = \min\{i \in \mathbb{N} \mid 2^{-i+1} \leq \alpha\}$. Since $\mathrm{st}\,\mathcal{U}_m \prec \mathcal{U}_{m-1}$, we can find $U \in \mathcal{U}_{m-1}$ such that $x_0, x_n \in U$, and hence $\delta(x_0, x_n) \leq 2^{-m+2} \leq 2\alpha$.

Additional Results on Metrizability 2.4.5.

(1) The perfect image of a metrizable space is metrizable, that is, if $f : X \to Y$ is a surjective perfect map of a metrizable space X, then Y is also metrizable.

Sketch of Proof. For each $y \in Y$ and $n \in \mathbb{N}$, let

$$U_n(y) = \mathrm{N}_d(f^{-1}(y), 2^{-n}) \text{ and } V_n(y) = Y \setminus f(X \setminus U_n(y)),$$

where d is an admissible metric for X. Show that $\{V_n(y) \mid n \in \mathbb{N}\}$ is a neighborhood basis of $y \in Y$ that satisfies condition 2.4.1(c). For each $y \in Y$ and $i \in \mathbb{N}$, since $f^{-1}(y)$ is compact, we can choose $j \geq i$ so that $U_j(y) \subset f^{-1}(V_{i+1}(y))$. Then, the following holds:

$$V_{j+1}(y) \cap V_{j+1}(z) \neq \emptyset \Rightarrow V_{j+1}(z) \subset V_i(y).$$

To see this, observe that

$$\begin{aligned} V_{j+1}(y) \cap V_{j+1}(z) \neq \emptyset &\Rightarrow U_j(y) \cap f^{-1}(z) \neq \emptyset \\ &\Rightarrow f^{-1}(z) \subset f^{-1}(V_{i+1}(y)) \subset U_{i+1}(y) \\ &\Rightarrow f^{-1}(V_{j+1}(z)) \subset U_{j+1}(z) \subset U_i(y). \end{aligned}$$

(2) A space X is metrizable if it is a locally finite union of metrizable closed subspaces.

Sketch of Proof. To apply (1) above, construct a surjective perfect map $f : \bigoplus_{\lambda \in \Lambda} X_\lambda \to X$ such that each X_λ is metrizable and $f|X_\lambda$ is a closed embedding. The metrizability of $\bigoplus_{\lambda \in \Lambda} X_\lambda$ easily follows from Theorem 2.3.4. (The metrizability of $\bigoplus_{\lambda \in \Lambda} X_\lambda$ can also be seen by embedding $\bigoplus_{\lambda \in \Lambda} X_\lambda$ into the product space $\Lambda \times \ell_1(\Gamma)^{\mathbb{N}}$ for some Γ, where we give Λ the discrete topology.)

2.5 Complete Metrizability

In this section, we consider complete metrizability. A space X has the **Baire property** or is a **Baire space** if the intersection of countably many dense open sets in X is also dense; equivalently, every countable intersection of dense G_δ sets in X is also dense. This property is very valuable. In particular, it can be used to prove various existence theorems. Observe that the Baire property can also be expressed as follows: if a countable union of closed sets has an interior point, then at least one of the closed sets has an interior point. The following statement is easily proved:

- Every open subspace and every dense G_δ subspace of a Baire space is also Baire.

Complete metrizability is preferable because it implies the Baire property.

Theorem 2.5.1 (Baire Category Theorem). *Every completely metrizable space X is a Baire space. Consequently, X cannot be written as a union of countably many closed sets without interior points.*

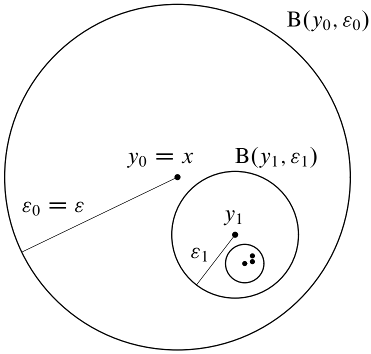

Fig. 2.5 Definition of $y_n \in X$ and $\varepsilon_n > 0$

Proof. For each $i \in \mathbb{N}$, let G_i be a dense open set in X and $d \in \mathrm{Metr}(X)$ be a complete metric. For each $x \in X$ and $\varepsilon > 0$, we inductively choose $y_i \in X$ and $\varepsilon_i > 0$, $i \in \mathbb{N}$, so that

$$y_i \in B(y_{i-1}, \tfrac{1}{2}\varepsilon_{i-1}) \cap G_i,\ B(y_i, \varepsilon_i) \subset G_i \text{ and } \varepsilon_i \leq \tfrac{1}{2}\varepsilon_{i-1},$$

where $y_0 = x$ and $\varepsilon_0 = \varepsilon$ (Fig. 2.5). Then, $(y_i)_{i \in \mathbb{N}}$ is d-Cauchy, hence it converges to some $y \in X$. For each $n \in \omega$,

$$d(y_n, y) \leq \sum_{i=n}^{\infty} d(y_i, y_{i+1}) < \sum_{i=n}^{\infty} \tfrac{1}{2}\varepsilon_i \leq \sum_{i=1}^{\infty} 2^{-i}\varepsilon_n = \varepsilon_n.$$

Thus, $y \in B(x, \varepsilon)$ and $y \in B(y_i, \varepsilon_i) \subset G_i$ for each $i \in \mathbb{N}$, that is, $y \in B(x, \varepsilon) \cap \bigcap_{i \in \mathbb{N}} G_i$. Therefore, $\bigcap_{i \in \mathbb{N}} G_i$ is dense in X. □

A metrizable space X is said to be **absolutely G_δ** if X is G_δ in an arbitrary metrizable space that contains X as a subspace. This concept characterizes complete metrizability, which leads us to the following:

Theorem 2.5.2. *A metrizable space is completely metrizable if and only if it is absolutely G_δ.*

This follows from Corollary 2.3.6 (or 2.3.10) and the following theorem:

Theorem 2.5.3. *Let $X = (X, d)$ be a metric space and $A \subset X$.*

(1) *If A is completely metrizable, then A is G_δ in X.*
(2) *If X is complete and A is G_δ in X, then A is completely metrizable.*

Proof. (1): Since $\mathrm{cl}\, A$ is G_δ in X, it suffices to show that A is G_δ in $\mathrm{cl}\, A$. Let $\rho \in \mathrm{Metr}(A)$ be a complete metric. For each $n \in \mathbb{N}$, let

$$G_n = \{x \in \operatorname{cl} A \mid x \text{ has a neighborhood } U \text{ in } X \text{ with}$$
$$\operatorname{diam}_d U < 2^{-n} \text{ and } \operatorname{diam}_\rho U \cap A < 2^{-n}\}.$$

Then, each G_n is clearly open in $\operatorname{cl} A$ and $A \subset \bigcap_{n\in\mathbb{N}} G_n$. Each $x \in \bigcap_{n\in\mathbb{N}} G_n$ has neighborhoods $U_1 \supset U_2 \supset \cdots$ in X such that $\operatorname{diam}_d U_n < 2^{-n}$ and $\operatorname{diam}_\rho U_n \cap A < 2^{-n}$. Since $x \in \operatorname{cl} A$, we have points $x_n \in U_n \cap A$, $n \in \mathbb{N}$. Then, $(x_n)_{n\in\mathbb{N}}$ converges to x. Since $(x_n)_{n\in\mathbb{N}}$ is ρ-Cauchy, it is convergent in A. Thus, we can conclude that $x \in A$. Therefore, $A = \bigcap_{n\in\mathbb{N}} G_n$, which is G_δ in $\operatorname{cl} A$.

(2): First, we show that any open set U in X is completely metrizable. We can define an admissible metric ρ for U as follows:

$$\rho(x,y) = d(x,y) + \left|d(x, X \setminus U)^{-1} - d(y, X \setminus U)^{-1}\right|.$$

Every ρ-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in U is d-Cauchy, so it converges to some $x \in X$. Since $(d(x_n, X \setminus U)^{-1})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, it is bounded. Then,

$$d(x, X \setminus U) = \lim_{n\to\infty} d(x_n, X \setminus U) > 0.$$

This means that $x \in U$, and hence $(x_n)_{n\in\mathbb{N}}$ is convergent in U. Thus, ρ is complete.

Next, we show that an arbitrary G_δ set A in X is completely metrizable. Write $A = \bigcap_{n\in\mathbb{N}} U_n$, where $U_1, U_2, \ldots$ are open in X. As we saw above, each U_n admits a complete metric $d_n \in \operatorname{Metr}(U_n)$. Now, we can define a metric $\rho \in \operatorname{Metr}(A)$ as follows:

$$\rho(x,y) = \sum_{n\in\mathbb{N}} \min\{2^{-n},\ d_n(x,y)\}.$$

Every ρ-Cauchy sequence in A is d_n-Cauchy, which is convergent in U_n. Hence, it is convergent in $A = \bigcap_{n\in\mathbb{N}} U_n$. Therefore, ρ is complete. □

Analogous to compactness, the completeness of metric spaces can be characterized by the finite intersection property (f.i.p.).

Theorem 2.5.4. *In order for a metric space $X = (X,d)$ to be complete, it is necessary and sufficient that, if a family $\mathcal{F}$ of subsets of X has the finite intersection property and contains sets with arbitrarily small diameter, then $\mathcal{F}^{\mathrm{cl}}$ has a non-empty intersection, which is a singleton.*

Proof. (*Necessity*) Let $\mathcal{F}$ be a family of subsets of X with the f.i.p. such that $\mathcal{F}$ contains sets with arbitrarily small diameter. For each $n \in \mathbb{N}$, choose $F_n \in \mathcal{F}$ so that $\operatorname{diam} F_n < 2^{-n}$, and take $x_n \in F_n$. For any $n < m$, $F_n \cap F_m \neq \emptyset$, hence

$$d(x_n, x_m) \leq \operatorname{diam} F_n + \operatorname{diam} F_m < 2^{-n} + 2^{-m} < 2^{-n+1}.$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, therefore it converges to a point $x \in X$. Then, $x \in \bigcap \mathcal{F}^{\mathrm{cl}}$. Otherwise, $x \notin \operatorname{cl} F$ for some $F \in \mathcal{F}$. Choose $n \in \mathbb{N}$ so that $d(x, x_n),\ 2^{-n} < \frac{1}{2} d(x, F)$. Since $F \cap F_n \neq \emptyset$, it follows that

$$d(x, F) \leq d(x, x_n) + \operatorname{diam} F_n < d(x, x_n) + 2^{-n} < d(x, F),$$

which is a contradiction.

(*Sufficiency*) Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in X. For each $n \in \mathbb{N}$, let $F_n = \{x_i \mid i \geq n\}$. Then, $F_1 \supset F_2 \supset \cdots$ and $\operatorname{diam} F_n \to 0$ $(n \to \infty)$. From this condition, we have $x \in \bigcap_{n \in \mathbb{N}} \operatorname{cl} F_n$. For each $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $\operatorname{diam} \operatorname{cl} F_n = \operatorname{diam} F_n < \varepsilon$. Then, $d(x_i, x) < \varepsilon$ for $i \geq n$, that is, $\lim_{n \to \infty} x_n = x$. Therefore, X is complete. □

Using compactifications, we can characterize complete metrizability as follows:

Theorem 2.5.5. *For a metrizable space X, the following are equivalent:*

(a) *X is completely metrizable;*
(b) *X is G_δ in an arbitrary compactification of X;*
(c) *X is G_δ in the Stone–Čech compactification βX;*
(d) *X has a compactification in which X is G_δ.*

Proof. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious. We show the converse (d) $\Rightarrow$ (c) $\Rightarrow$ (b) and the equivalence (a) $\Leftrightarrow$ (b).

(d) $\Rightarrow$ (c): Let γX be a compactification of X and $X = \bigcap_{n \in \mathbb{N}} G_n$, where each G_n is open in γX. Then, by Theorem 2.1.4, we have a map $f : \beta X \to \gamma X$ such that $f|X = \mathrm{id}$, where $X = f^{-1}(X)$ by Theorem 2.1.8. Consequently, $X = \bigcap_{n \in \mathbb{N}} f^{-1}(G_n)$ is G_δ in βX.

(c) $\Rightarrow$ (b): By condition (c), we can write $\beta X \setminus X = \bigcup_{n \in \mathbb{N}} F_n$, where each F_n is closed in βX. For any compactification γX of X, we have a map $f : \beta X \to \gamma X$ such that $f|X = \mathrm{id}$ (Theorem 2.1.4). From Theorem 2.1.8, $\gamma X \setminus X = f(\beta X \setminus X) = \bigcup_{n \in \mathbb{N}} f(F_n)$ is F_σ in γX, hence X is G_δ in γX.

(b) $\Rightarrow$ (a): To prove the complete metrizability of X, we show that X is absolutely G_δ (Theorem 2.5.2). Let X be contained in a metrizable space Y. Since $\operatorname{cl}_{\beta Y} X$ is a compactification of X, it follows from (b) that X is G_δ in $\operatorname{cl}_{\beta Y} X$, and hence it is G_δ in $Y \cap \operatorname{cl}_{\beta Y} X = \operatorname{cl}_Y X$, where $\operatorname{cl}_Y X$ is also G_δ in Y. Therefore, X is G_δ in Y.

(a) $\Rightarrow$ (b): Let γX be a compactification of X and d an admissible complete metric for X. For each $n \in \mathbb{N}$ and $x \in X$, let $G_n(x)$ be an open set in γX such that $G_n(x) \cap X = \mathrm{B}_d(x, 2^{-n})$. Then, $G_n = \bigcup_{x \in X} G_n(x)$ is open in γX and $X \subset G_n$. We will show that each $y \in \bigcap_{n \in \mathbb{N}} G_n$ is contained in X. This implies that $X = \bigcap_{n \in \mathbb{N}} G_n$ is G_δ in γX.

For each $n \in \mathbb{N}$, choose $x_n \in X$ so that $y \in G_n(x_n)$. Since $y \in \operatorname{cl}_{\gamma X} X$ and $G_n(x_n) \cap X = \mathrm{B}_d(x_n, 2^{-n})$, it follows that $\{\mathrm{B}_d(x_n, 2^{-n}) \mid n \in \mathbb{N}\}$ has the f.i.p. By Theorem 2.5.4, we have $x \in \bigcap_{n \in \mathbb{N}} \operatorname{cl}_X \mathrm{B}_d(x_n, 2^{-n})$, where $\lim_{n \to \infty} x_n = x$ because $d(x_n, x) \leq 2^{-n}$. Thus, we have $y = x \in X$. Otherwise, there would be disjoint open sets U and V in γX such that $x \in U$ and $y \in V$. Since $y \in \bigcap_{n\in\mathbb{N}} G_n \cap V$, $\{\mathrm{B}_d(x_n, 2^{-n}) \cap V \mid n \in \mathbb{N}\}$ has the f.i.p. Again, by Theorem 2.5.4, we have

$$x' \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}_X(\mathrm{B}_d(x_n, 2^{-n}) \cap V) \subset \mathrm{cl}_X V.$$

Since $\lim_{n\to\infty} x_n = x'$ is the same as x, it follows that $x' = x \in U$, which is a contradiction. □

Note that conditions (b)–(d) in Theorem 2.5.5 are equivalent without the metrizability of X, but X should be assumed to be Tychonoff in order that X has a compactification. A Tychonoff space X is said to be **Čech-complete** if X satisfies one of these conditions.

Every compact metric space is complete. Since a non-compact locally compact metrizable space X is open in the one-point compactification $\alpha X = X \cup \{\infty\}$, X is completely metrizable because of Theorem 2.5.5. Thus, we have the following corollary:

Corollary 2.5.6. *Every locally compact metrizable space is completely metrizable.* □

We now state and prove the LAVRENTIEFF G_δ-EXTENSION THEOREM:

Theorem 2.5.7 (LAVRENTIEFF). *Let $f : A \to Y$ be a map from a subset A of a space X to a completely metrizable space Y. Then, f extends over a G_δ set G in X such that $A \subset G \subset \mathrm{cl}\, A$.*

Proof. We may assume that Y is a complete metric space. The oscillation of f at $x \in \mathrm{cl}\, A$ is defined as follows:

$$\mathrm{osc}_f(x) = \inf\big\{\mathrm{diam}\, f(A \cap U) \bigm| U \text{ is an open neighborhood of } x\big\}.$$

Let $G = \{x \in \mathrm{cl}\, A \mid \mathrm{osc}_f(x) = 0\}$. Then, $A \subset G$ because f is continuous. Since each $\{x \in \mathrm{cl}\, A \mid \mathrm{osc}_f(x) < 1/n\}$ is open in $\mathrm{cl}\, A$, it follows that G is G_δ in X. For each $x \in G$,

$$\mathcal{F}_x = \big\{f(A \cap U) \bigm| U \text{ is an open neighborhood of } x\big\},$$

has the f.i.p. and contains sets with arbitrarily small diameter. By Theorem 2.5.4, we have $\bigcap \mathcal{F}_x^{\mathrm{cl}} \neq \emptyset$, which is a singleton because $\mathrm{diam} \bigcap \mathcal{F}_x^{\mathrm{cl}} = 0$. The desired extension $\tilde{f} : G \to Y$ of f can be defined by $\tilde{f}(x) \in \bigcap \mathcal{F}_x^{\mathrm{cl}}$. □

If A is a subspace of a metric space X and Y is a complete metric space, then every uniformly continuous map $f : A \to Y$ extends over $\mathrm{cl}\, A$. This result can be obtained by showing that $G = \mathrm{cl}\, A$ in the above proof. However, a direct proof is easier.

We will modify Theorem 2.5.7 into the following, known as the LAVRENTIEFF HOMEOMORPHISM EXTENSION THEOREM:

Theorem 2.5.8 (LAVRENTIEFF). *Let X and Y be completely metrizable spaces and let $f : A \to B$ be a homeomorphism between $A \subset X$ and $B \subset Y$. Then, f extends to a homeomorphism $\tilde{f} : G \to H$ between G_δ sets in X and Y such that $A \subset G \subset \operatorname{cl} A$ and $B \subset H \subset \operatorname{cl} B$.*

Proof. By Theorem 2.5.7, f and f^{-1} extend to maps $g : G' \to Y$ and $h : H' \to X$, where $A \subset G' \subset \operatorname{cl} A$, $B \subset H' \subset \operatorname{cl} B$ and G', H' are G_δ in X and Y, respectively. Then, we have G_δ sets $G = g^{-1}(H')$ and $H = h^{-1}(G')$ that contain A and B as dense subsets, respectively. Consider the maps $h(g|G) : G \to X$ and $g(h|H) : H \to Y$. Since $h(g|G)|A = \mathrm{id}_A$ and $g(h|H)|B = \mathrm{id}_B$, it follows that $h(g|G) = \mathrm{id}_G$ and $g(h|H) = \mathrm{id}_H$. Then, as is easily observed, we have $g(G) \subset H$ and $h(H) \subset G$. Hence, $\tilde{f} = g|G : G \to H$ is a homeomorphism extending f. □

In the above, when $X = Y$ and $A = B$, we can take $G = H$, that is, we can show the following:

Corollary 2.5.9. *Let X be a completely metrizable space and $A \subset X$. Then, every homeomorphism $f : A \to A$ extends to a homeomorphism $\tilde{f} : G \to G$ over a G_δ set G in X with $A \subset G \subset \operatorname{cl} A$.*

Proof. Using Theorem 2.5.8, we extend f to a homeomorphism $g : G' \to G''$ between G_δ sets $G', G'' \subset X$ with $A \subset G' \cap G''$ and $G', G'' \subset \operatorname{cl} A$. We inductively define a sequence of G_δ sets $G' = G_1 \supset G_2 \supset \cdots$ in X as follows:

$$G_{n+1} = G_n \cap g(G_n) \cap g^{-1}(G_n).$$

Then, $G = \bigcap_{n \in \mathbb{N}} G_n$ is G_δ in X and $g(x), g^{-1}(x) \in G$ for each $x \in G$. Indeed, for each $n \in \mathbb{N}$, since $x \in G_{n+1}$, it follows that $g(x) \in G_n$ and $g^{-1}(x) \in G_n$. Thus, $\tilde{f} = g|G : G \to G$ is the desired extension of f. □

Additional Results on Complete Metrizability 2.5.10.

(1) Let $f : X \to Y$ be a surjective perfect map between Tychonoff spaces. Then, X is Čech-complete if and only if Y is Čech-complete. When X is metrizable, X is completely metrizable if and only if Y is completely metrizable.

Sketch of Proof. See Theorem 2.1.8.

(2) A space X is completely metrizable if it is a locally finite union of completely metrizable closed subspaces.

Sketch of Proof. Emulate 2.4.5(2). To prove the complete metrizability of the topological sum $\bigoplus_{\lambda \in \Lambda} X_\lambda$ of completely metrizable spaces, embed $\bigoplus_{\lambda \in \Lambda} X_\lambda$ into the product space $\Lambda \times \ell_1(\Gamma)^{\mathbb{N}}$ for some Γ.

2.6 Paracompactness and Local Properties

A space X is **paracompact** if each open cover of X has a locally finite open refinement.[3] According to Stone's Theorem 2.3.1, every metrizable space is paracompact. A space X is **collectionwise normal** if, for each discrete collection $\mathcal{F}$ of closed sets in X, there is a pairwise disjoint collection $\{U_F \mid F \in \mathcal{F}\}$ of open sets in X such that $F \subset U_F$ for each $F \in \mathcal{F}$. Obviously, every collectionwise normal space is normal. In the definition of collectionwise normality, $\{U_F \mid F \in \mathcal{F}\}$ can be discrete in X. Indeed, choose an open set V in X so that $\bigcup \mathcal{F} \subset V \subset \operatorname{cl} V \subset \bigcup_{F \in \mathcal{F}} U_F$. Then, $F \subset V \cap U_F$ for each $F \in \mathcal{F}$, and $\{V \cap U_F \mid F \in \mathcal{F}\}$ is discrete in X.

Theorem 2.6.1. *Every paracompact space X is collectionwise normal.*

Proof. To see the regularity of X, let A be a closed set in X and $x \in X \setminus A$. Each $a \in A$ has an open neighborhood U_a in X so that $x \notin \operatorname{cl} U_a$. Let $\mathcal{U}$ be a locally finite open refinement of

$$\{U_a \mid a \in A\} \cup \{X \setminus A\} \in \operatorname{cov}(X).$$

Then, $V = \operatorname{st}(A, \mathcal{U}) = \bigcup \mathcal{U}[A]$ is an open neighborhood of A. Since $\mathcal{U}$ is locally finite, it follows that $\operatorname{cl} V = \bigcup \mathcal{U}[A]^{\mathrm{cl}}$. Since each $U \in \mathcal{U}[A]$ is contained in some U_a, it follows that $x \notin \operatorname{cl} U$, and hence $x \notin \operatorname{cl} V$.

We now show that X is collectionwise normal. Let $\mathcal{F}$ be a discrete collection of closed sets in X. Since X is regular, each $x \in X$ has an open neighborhood V_x in X such that $\operatorname{card} \mathcal{F}[\operatorname{cl} V_x] \leq 1$. Let $\mathcal{U}$ be a locally finite open refinement of $\{V_x \mid x \in X\} \in \operatorname{cov}(X)$. For each $F \in \mathcal{F}$, we define

$$W_F = X \setminus \bigcup \{\operatorname{cl} U \mid U \in \mathcal{U},\ F \cap \operatorname{cl} U = \emptyset\}.$$

Then, W_F is open in X and $F \subset W_F \subset \operatorname{st}(F, \mathcal{U}^{\mathrm{cl}})$ (Fig. 2.6). Since $\operatorname{card} \mathcal{F}[\operatorname{cl} U] \leq 1$ for each $U \in \mathcal{U}$, it follows that $\operatorname{st}(F, \mathcal{U}^{\mathrm{cl}}) \cap W_{F'} = \emptyset$ if $F' \neq F \in \mathcal{F}$. Therefore, $\{W_F \mid F \in \mathcal{F}\}$ is pairwise disjoint. □

Lemma 2.6.2. *If X is regular and each open cover of X has a locally finite refinement (consisting of arbitrary sets), then for any open cover $\mathcal{U}$ of X there is a locally finite closed cover $\{F_U \mid U \in \mathcal{U}\}$ of X such that $F_U \subset U$ for each $U \in \mathcal{U}$.*

Proof. Since X is regular, we have $\mathcal{V} \in \operatorname{cov}(X)$ such that $\mathcal{V}^{\mathrm{cl}} \prec \mathcal{U}$. Let $\mathcal{A}$ be a locally finite refinement of $\mathcal{V}$. There exists a function $\varphi : \mathcal{A} \to \mathcal{U}$ such that $\operatorname{cl} A \subset \varphi(A)$ for each $A \in \mathcal{A}$. For each $U \in \mathcal{U}$, define

$$F_U = \bigcup \{\operatorname{cl} A \mid A \in \varphi^{-1}(U)\} \subset U.$$

[3] Recall that spaces are assumed to be **Hausdorff**.

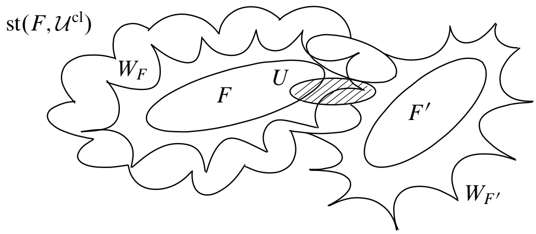

Fig. 2.6 The pairwise disjoint collection $\{W_F \mid F \in \mathcal{F}\}$

Since each $x \in X$ is contained in some $A \in \mathcal{A}$ and $A \subset F_{\varphi(A)}$, $\{F_U \mid U \in \mathcal{U}\}$ is a cover of X. Since $\mathcal{A}$ is locally finite, each F_U is closed in X and $\{F_U \mid U \in \mathcal{U}\}$ is locally finite. □

We have the following characterizations of paracompactness:

Theorem 2.6.3. *For a space X, the following conditions are equivalent:*

(a) *X is paracompact;*
(b) *Each open cover of X has an open Δ-refinement;*
(c) *Each open cover of X has an open star-refinement;*
(d) *X is regular and each open cover of X has a σ-discrete open refinement;*
(e) *X is regular and each open cover of X has a locally finite refinement.*

Proof. (a) ⇒ (b): Let $\mathcal{U} \in \mathrm{cov}(X)$. From Lemma 2.6.2, it follows that X has a locally finite closed cover $\{F_U \mid U \in \mathcal{U}\}$ such that $F_U \subset U$ for each $U \in \mathcal{U}$. For each $x \in X$, define

$$W_x = \bigcap \{U \in \mathcal{U} \mid x \in F_U\} \setminus \bigcup \{F_U \mid U \in \mathcal{U},\ x \notin F_U\}.$$

Then, W_x is an open neighborhood of x in X, hence $\mathcal{W} = \{W_x \mid x \in X\} \in \mathrm{cov}(X)$. For each $x \in X$, choose $U \in \mathcal{U}$ so that $x \in F_U$. If $x \in W_y$ then $y \in F_U$, which implies that $W_y \subset U$. Therefore, $\mathrm{st}(x, \mathcal{W}) \subset U$ for each $x \in X$, which means that $\mathcal{W}$ is a Δ-refinement of $\mathcal{U}$.

(b) ⇒ (c): Due to Proposition 2.4.3, for $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathrm{cov}(X)$,

$$\mathcal{W} \overset{\Delta}{\prec} \mathcal{V} \overset{\Delta}{\prec} \mathcal{U} \ \Rightarrow\ \mathcal{W} \overset{*}{\prec} \mathcal{U}.$$

This gives (b) ⇒ (c).

(c) ⇒ (d): To prove the regularity of X, let A $\subset X$ be closed and $x \in X \setminus A$. Then, $\{X \setminus A, X \setminus \{x\}\} \in \mathrm{cov}(X)$ has an open star-refinement $\mathcal{W}$. Choose $W \in \mathcal{W}$ so that $x \in W$. Then, $\mathrm{st}(W, \mathcal{W}) \subset X \setminus A$, i.e., $W \cap \mathrm{st}(A, \mathcal{W}) = \emptyset$. Hence, X is regular.

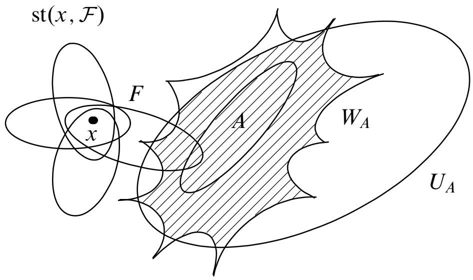

Fig. 2.7 Definition of W_A

Next, we show that each $\mathcal{U} \in \operatorname{cov}(X)$ has a σ-discrete open refinement. We may assume that $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$, where $\Lambda = (\Lambda, \leq)$ is a well-ordered set. By condition (c), we have a sequence of open star-refinements:

$$\mathcal{U} \overset{*}{\succ} \mathcal{U}_1 \overset{*}{\succ} \mathcal{U}_2 \overset{*}{\succ} \cdots .$$

For each $(\lambda, n) \in \Lambda \times \mathbb{N}$, let

$$U_{\lambda,n} = \bigcup \left\{U \in \mathcal{U}_n \;\middle|\; \operatorname{st}(U, \mathcal{U}_n) \subset U_\lambda\right\} \subset U_\lambda.$$

Then, we have

$(*)$ $\operatorname{st}(U_{\lambda,n}, \mathcal{U}_{n+1}) \subset U_{\lambda,n+1}$ for each $(\lambda, n) \in \Lambda \times \mathbb{N}$.

Indeed, each $U \in \mathcal{U}_{n+1}[U_{\lambda,n}]$ meets some $U' \in \mathcal{U}_n$ such that $\operatorname{st}(U', \mathcal{U}_n) \subset U_\lambda$. Since $U \subset \operatorname{st}(U', \mathcal{U}_{n+1})$, it follows that

$$\operatorname{st}(U, \mathcal{U}_{n+1}) \subset \operatorname{st}^2(U', \mathcal{U}_{n+1}) \subset \operatorname{st}(U', \operatorname{st}\mathcal{U}_{n+1}) \subset \operatorname{st}(U', \mathcal{U}_n) \subset U_\lambda,$$

which implies that $U \subset U_{\lambda,n+1}$. Thus, we have $(*)$.

Now, for each $(\lambda, n) \in \Lambda \times \mathbb{N}$, let

$$V_{\lambda,n} = U_{\lambda,n} \setminus \operatorname{cl} \bigcup_{\mu<\lambda} U_{\mu,n+1} \subset U_\lambda.$$

Then, each $\mathcal{V}_n = \{V_{\lambda,n} \mid \lambda \in \Lambda\}$ is discrete in X. Indeed, each $x \in X$ is contained in some $U \in \mathcal{U}_{n+1}$. If $U \cap V_{\mu,n} \neq \emptyset$, then $U \subset \operatorname{st}(U_{\mu,n}, \mathcal{U}_{n+1}) \subset U_{\mu,n+1}$ by $(*)$. Hence, $U \cap V_{\lambda,n} = \emptyset$ for all $\lambda > \mu$. This implies that U meets at most one member of $\mathcal{V}_n$ — Fig. 2.8.

It remains to be proved that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n \in \operatorname{cov}(X)$. Each $x \in X$ is contained in some $U \in \mathcal{U}_1$. Since $\operatorname{st}(U, \mathcal{U}_1) \subset U_\lambda$ for some $\lambda \in \Lambda$, it follows that $x \in U_{\lambda,1}$. Thus, we can define

$$\lambda(x) = \min\left\{\lambda \in \Lambda \;\middle|\; x \in U_{\lambda,n} \text{ for some } n \in \mathbb{N}\right\}.$$

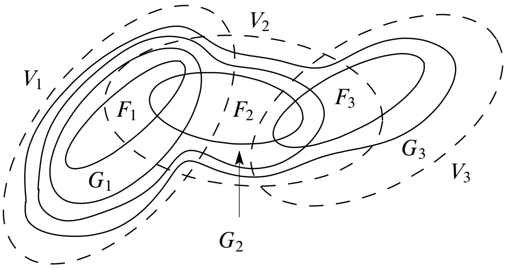

Fig. 2.8 Construction of G_n

Then, $x \in U_{\lambda(x),n}$ for some $n \in \mathbb{N}$. It follows from $(*)$ that

$$\begin{aligned}\mathrm{cl}\bigcup_{\mu<\lambda(x)} U_{\mu,n+1} &\subset \mathrm{st}\Big(\bigcup_{\mu<\lambda(x)} U_{\mu,n+1}, \mathcal{U}_{n+2}\Big)\\ &= \bigcup_{\mu<\lambda(x)} \mathrm{st}(U_{\mu,n+1}, \mathcal{U}_{n+2}) \subset \bigcup_{\mu<\lambda(x)} U_{\mu,n+2},\end{aligned}$$

hence $x \notin \mathrm{cl}\bigcup_{\mu<\lambda(x)} U_{\mu,n+1}$. Therefore, $x \in V_{\lambda(x),n}$, and hence $\mathcal{V} \in \mathrm{cov}(X)$. Consequently, $\mathcal{V}$ is a σ-discrete open refinement of $\mathcal{U}$.

(d) $\Rightarrow$ (e): It suffices to show that every σ-discrete open cover $\mathcal{U}$ of X has a locally finite refinement. Let $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, where each $\mathcal{U}_n$ is discrete in X and $\mathcal{U}_n \cap \mathcal{U}_m = \emptyset$ if $n \neq m$. For each $U \in \mathcal{U}_n$, let $A_U = U \setminus \bigcup_{m<n}(\bigcup \mathcal{U}_m)$. Then, $\mathcal{A} = \{A_U \mid U \in \mathcal{U}\}$ is a cover of X that refines $\mathcal{U}$. For each $x \in X$, choose the smallest $n \in \mathbb{N}$ such that $x \in \bigcup \mathcal{U}_n$ and let $x \in U_0 \in \mathcal{U}_n$. Then, U_0 misses A_U for all $U \in \bigcup_{m>n} \mathcal{U}_m$. For each $m \leq n$, since $\mathcal{U}_m$ is discrete, x has a neighborhood V_m in X such that $\mathrm{card}\,\mathcal{U}_m[V_m] \leq 1$. Then, $V = U_0 \cap V_1 \cap \cdots \cap V_n$ is a neighborhood of x in X such that $\mathrm{card}\,\mathcal{A}[V] \leq n$. Hence, $\mathcal{A}$ is locally finite in X — Fig. 2.9.

(e) $\Rightarrow$ (a): Let $\mathcal{U} \in \mathrm{cov}(X)$. Then $\mathcal{U}$ has a locally finite refinement $\mathcal{A}$. For each $x \in X$, choose an open neighborhood V_x of x in X so that $\mathrm{card}\,\mathcal{A}[V_x] < \aleph_0$. According to Lemma 2.6.2, $\{V_x \mid x \in X\} \in \mathrm{cov}(X)$ has a locally finite closed refinement $\mathcal{F}$. Then, $\mathrm{card}\,\mathcal{A}[F] < \aleph_0$ for each $F \in \mathcal{F}$. For each $A \in \mathcal{A}$, choose $U_A \in \mathcal{U}$ so that $A \subset U_A$ and define

$$W_A = U_A \setminus \bigcup \{F \in \mathcal{F} \mid A \cap F = \emptyset\}.$$

Then, $A \subset W_A \subset U_A$ and W_A is open in X, hence $\mathcal{W} = \{W_A \mid A \in \mathcal{A}\}$ is an open refinement of $\mathcal{U}$. Since $\mathcal{F}$ is a locally finite closed cover of X, $\mathrm{st}(x, \mathcal{F})$ is a neighborhood of $x \in X$. For each $F \in \mathcal{F}$ and $A \in \mathcal{A}$, $F \cap W_A \neq \emptyset$ implies $F \cap A \neq \emptyset$. Then, $\mathrm{card}\,\mathcal{W}[F] \leq \mathrm{card}\,\mathcal{A}[F] < \aleph_0$ for each $F \in \mathcal{F}$. Since $\mathrm{card}\,\mathcal{F}[x] < \aleph_0$, $\mathrm{st}(x, \mathcal{F})$ meets only finitely many members of $\mathcal{W}$. Hence, $\mathcal{W}$ is locally finite in X. □

A space X is **Lindelöf** if every open cover of X has a countable open refinement. By verifying condition (d) above, we have the following:

Corollary 2.6.4. *Every regular Lindelöf space is paracompact.* □

Let $\mathcal{P}$ be a property of subsets of a space X. It is said that X has property $\mathcal{P}$ **locally** if each $x \in X$ has a neighborhood U in X that has property $\mathcal{P}$. Occasionally, we need to determine whether X has some property $\mathcal{P}$ if X has property $\mathcal{P}$ locally. Let us consider this problem now. A property $\mathcal{P}$ of *open* sets in X is said to be ***G*-hereditary** if the following conditions are satisfied:

(G-1) If U has property $\mathcal{P}$, then every open subset of U has $\mathcal{P}$;
(G-2) If U and V have property $\mathcal{P}$, then $U \cup V$ has property $\mathcal{P}$;
(G-3) If $\{U_\lambda \mid \lambda \in \Lambda\}$ is discrete in X and each U_λ has property $\mathcal{P}$, then $\bigcup_{\lambda \in \Lambda} U_\lambda$ has property $\mathcal{P}$.

The following theorem is very useful to show that a space has a certain property:

Theorem 2.6.5 (E. Michael). *Let $\mathcal{P}$ be a G-hereditary property of open sets in a paracompact space X. If X has property $\mathcal{P}$ locally, then X itself has property $\mathcal{P}$.*

Proof. Since X has property $\mathcal{P}$ locally, there exists $\mathcal{U} \in \operatorname{cov}(X)$ such that each $U \in \mathcal{U}$ has property $\mathcal{P}$. According to Theorem 2.6.3, $\mathcal{U}$ has an open refinement $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ such that each $\mathcal{V}_n$ is discrete in X. Each $V \in \mathcal{V}$ has property $\mathcal{P}$ by (G-1). For each $n \in \mathbb{N}$, let $V_n = \bigcup \mathcal{V}_n$. Then, each V_n has property $\mathcal{P}$ by (G-3), hence $V_1 \cup \cdots \cup V_n$ has property $\mathcal{P}$ by (G-2). From Lemma 2.6.2, it follows that X has a closed cover $\{F_n \mid n \in \mathbb{N}\}$ such that $F_n \subset V_n$ for each $n \in \mathbb{N}$.[4] Inductively choose open sets G_n $(n \in \mathbb{N})$ so that

$$F_n \cup \operatorname{cl} G_{n-1} \subset G_n \subset \operatorname{cl} G_n \subset V_1 \cup \cdots \cup V_n,$$

where $G_0 = \emptyset$ (Fig. 2.7). For each $n \in \mathbb{N}$, let $W_n = G_n \setminus \operatorname{cl} G_{n-2}$, where $G_{-1} = \emptyset$. Then, each W_n also has property $\mathcal{P}$ by (G-1). Let $X_i = \bigcup_{n \in \omega} W_{3n+i}$, where $i = 1, 2, 3$. Since $\{W_{3n+i} \mid n \in \omega\}$ is discrete in X, each X_i has property $\mathcal{P}$ by (G-3). Hence, $X = X_1 \cup X_2 \cup X_3$ also has property $\mathcal{P}$ by (G-2). □

There are many cases where we consider properties of closed sets rather than open sets. In such cases, Theorem 2.6.5 can also be applied. In fact, let $\mathcal{P}$ be a property of *closed* sets of X. We define the property $\mathcal{P}^\circ$ of *open* sets in X as follows:

$$U \text{ has property } \mathcal{P}^\circ \underset{\text{def}}{\Longleftrightarrow} \operatorname{cl} U \text{ has property } \mathcal{P}.$$

It is said that $\mathcal{P}$ is ***F*-hereditary** if it satisfies the following conditions:

(F-1) If A has property $\mathcal{P}$, then every closed subset of A has property $\mathcal{P}$;

[4] Closed sets $F_n \subset X$, $n \in \mathbb{N}$ can be inductively obtained so that $X = \bigcup_{i \leq n} \operatorname{int} F_i \cup \bigcup_{i > n} V_i$.

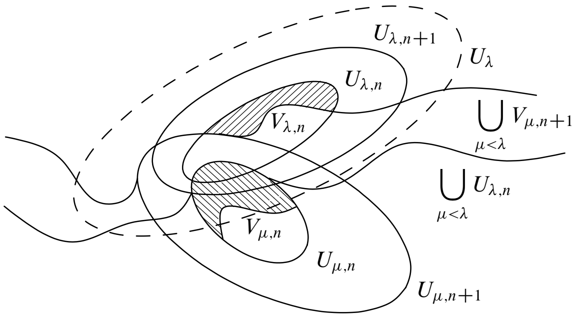

Fig. 2.9 Definition of $V_{\lambda,n}$

(F-2) If A and B have property $\mathcal{P}$, then $A \cup B$ has property $\mathcal{P}$;
(F-3) If $\{A_\lambda \mid \lambda \in \Lambda\}$ is discrete in X and each A_λ has property $\mathcal{P}$, then $\bigcup_{\lambda\in\Lambda} A_\lambda$ has property $\mathcal{P}$.

Evidently, if property $\mathcal{P}$ is F-hereditary, then $\mathcal{P}^\circ$ is G-hereditary. Therefore, Theorem 2.6.5 yields the following corollary:

Corollary 2.6.6 (E. Michael). *Let $\mathcal{P}$ be an F-hereditary property of closed sets in a paracompact space X. If X has property $\mathcal{P}$ locally, then X itself has property $\mathcal{P}$.* □

Additional Results on Paracompact Spaces 2.6.7.

(1) A space is paracompact if it is a locally finite union of paracompact closed subspaces.

Sketch of Proof. Let $\mathcal{F}$ be a locally finite closed cover of a space X such that each $F \in \mathcal{F}$ is paracompact. To prove regularity, let $x \in X$ and U an open neighborhood of x in X. Since each $F \in \mathcal{F}[x]$ is regular, we have an open neighborhood U_F of x in X such that $\mathrm{cl}(F \cap U_F) \subset U$. The following U_0 is an open neighborhood of x in X:

$$U_0 = \bigcap_{F\in\mathcal{F}[x]} U_F \setminus \bigcup(\mathcal{F} \setminus \mathcal{F}[x]) \left(\subset \bigcup \mathcal{F}[x] = \mathrm{st}(x, \mathcal{F}) \right).$$

Observe that $\mathrm{cl}_X U_0 = \mathrm{cl} \bigcup_{F\in\mathcal{F}[x]}(U_0 \cap F) = \bigcup_{F\in\mathcal{F}[x]} \mathrm{cl}(U_0 \cap F) \subset U$. Thus, it suffices to show that X satisfies condition 2.6.3(e).

(2) Every F_σ subspace A of a paracompact space X is paracompact.

Sketch of Proof. It suffices to show that A satisfies condition 2.6.3(d). Let $A = \bigcup_{n\in\mathbb{N}} A_n$, where each A_n is closed in X. For each $\mathcal{V} \in \mathrm{cov}(A)$ and $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{X \setminus A_n\} \cup \{\widetilde{V} \mid V \in \mathcal{V}\} \in \mathrm{cov}(X),$$

where each $\widetilde{V}$ is open in X with $\widetilde{V} \cap A = V$. Note that $\mathcal{V}_n \prec \mathcal{U}_n$ implies that $\mathcal{V}_n[A_n]|A \prec \mathcal{V}$.

(3) Let X be a paracompact space. If every open subspace of X is paracompact, then every subspace of X is also paracompact.

Sketch of Proof. To find a locally finite open refinement of $\mathcal{U} \in \operatorname{cov}(A)$, take an open collection $\widetilde{\mathcal{U}}$ in X such that $\widetilde{\mathcal{U}}|A = \mathcal{U}$ and use the paracompactness of $\bigcup \widetilde{\mathcal{U}}$.

(4) A paracompact space X is (completely) metrizable if it is locally (completely) metrizable.

Sketch of Proof. To apply 2.4.5(2) (2.5.10(2)), construct a locally finite cover of X consisting of (completely) metrizable closed sets.

A space X is **hereditarily paracompact** if every subspace of X is paracompact. The following theorem comes from (2) and (3).

Theorem 2.6.8. *Every perfectly normal paracompact space is hereditarily paracompact.* □

2.7 Partitions of Unity

A collection $\mathcal{A}$ of subsets of X is said to be **point-finite** if each point $x \in X$ is contained in only finitely many members of $\mathcal{A}$, that is, $\operatorname{card} \mathcal{A}[x] < \aleph_0$. Obviously, every locally finite collection is point-finite. We prove the following, which is called the Open Cover Shrinking Lemma.

Lemma 2.7.1. *Each point-finite open cover $\mathcal{U}$ of a normal space X has an open refinement $\{V_U \mid U \in \mathcal{U}\}$ such that $\operatorname{cl} V_U \subset U$ for each $U \in \mathcal{U}$.*

Proof. Let $\mathcal{T}$ be the topology of X (i.e., the collection of all open sets in X) and define an ordered set $\Phi = (\Phi, \leq)$ as follows:

$$\Phi = \left\{\varphi : \mathcal{U} \to \mathcal{T} \;\middle|\; \textstyle\bigcup_{U \in \mathcal{U}} \varphi(U) = X;\ \operatorname{cl}\varphi(U) \subset U \ \text{ if } \ \varphi(U) \neq U\right\},$$
$$\varphi_1 \leq \varphi_2 \underset{\text{def}}{\iff} \varphi_1(U) \neq U \ \text{ implies } \ \varphi_1(U) = \varphi_2(U).$$

Observe that if Φ has a maximal element φ_0 then $\operatorname{cl}\varphi_0(U) \subset U$ for each $U \in \mathcal{U}$. Then, the desired open refinement $\{V_U \mid U \in \mathcal{U}\}$ can be defined by $V_U = \varphi_0(U)$.

We apply Zorn's Lemma to show that Φ has a maximal element. It suffices to show that every totally ordered subset $\Psi \subset \Phi$ is upper bounded in Φ. For each $U \in \mathcal{U}$, let $\varphi(U) = \bigcap_{\psi \in \Psi} \psi(U)$. Then, $\varphi(U) \neq U$ implies $\psi_U(U) \neq U$ for some $\psi_U \in \Psi$, which means that $\varphi(U) = \psi_U(U)$ because $\psi(U) = \psi_U(U)$ or $\psi(U) = U$ for every $\psi \in \Psi$. Thus, we have $\varphi : \mathcal{U} \to \mathcal{T}$ such that $\operatorname{cl}\varphi(U) \subset U$ if $\varphi(U) \neq U$. To verify $X = \bigcup_{U \in \mathcal{U}} \varphi(U)$, let $x \in X$. If $\varphi(U) = U$ for some $U \in \mathcal{U}[x]$ then $x \in U = \varphi(U)$. When $\varphi(U) \neq U$ for every $U \in \mathcal{U}[x]$, by the same argument as above, we can see that $\varphi(U) = \psi_U(U)$ for each $U \in \mathcal{U}[x]$. Since $\mathcal{U}[x]$ is finite, we have $\psi_0 = \max\{\psi_U \mid U \in \mathcal{U}[x]\} \in \Psi$. Then, $\varphi(U) = \psi_U(U) = \psi_0(U)$ for each $U \in \mathcal{U}[x]$. Since $X = \bigcup_{U \in \mathcal{U}} \psi_0(U)$, it follows that $x \in \psi_0(U)$

($\subset U$) for some $U \in \mathcal{U}$, which implies $x \in \varphi(U)$ because $U \in \mathcal{U}[x]$. Consequently, $\varphi \in \Phi$. It follows from the definition that $\psi \leq \varphi$ for any $\psi \in \Psi$. $\square$

Remark 5. The above lemma can be proved using the transfinite induction instead of Zorn's Lemma.

For a map $f : X \to \mathbb{R}$, let

$$\operatorname{supp} f = \operatorname{cl}\left\{x \in X \;\middle|\; f(x) \neq 0\right\} \subset X,$$

which is called the **support** of f. A **partition of unity** on X is an indexed family $(f_\lambda)_{\lambda\in\Lambda}$ of maps $f_\lambda : X \to \mathbf{I}$ such that $\sum_{\lambda\in\Lambda} f_\lambda(x) = 1$ for each $x \in X$. It is said that $(f_\lambda)_{\lambda\in\Lambda}$ is **locally finite** if each $x \in X$ has a neighborhood U such that

$$\operatorname{card}\left\{\lambda \in \Lambda \;\middle|\; U \cap \operatorname{supp} f_\lambda \neq \emptyset\right\} < \aleph_0.$$

A partition of unity $(f_\lambda)_{\lambda\in\Lambda}$ on X is said to be (**weakly**) **subordinated** to $\mathcal{U} \in \operatorname{cov}(X)$ if $\{\operatorname{supp} f_\lambda \mid \lambda \in \Lambda\} \prec \mathcal{U}$ ($\{f_\lambda^{-1}((0,1]) \mid \lambda \in \Lambda\} \prec \mathcal{U}$).

Theorem 2.7.2. *Let $\mathcal{U}$ be a locally finite open cover of a normal space X. Then, there is a partition of unity $(f_U)_{U\in\mathcal{U}}$ on X such that* $\operatorname{supp} f_U \subset U$ *for each $U \in \mathcal{U}$.*

Proof. By Lemma 2.7.1, we have $\{V_U \mid U \in \mathcal{U}\}, \{W_U \mid U \in \mathcal{U}\} \in \operatorname{cov}(X)$ such that $\operatorname{cl} W_U \subset V_U \subset \operatorname{cl} V_U \subset U$ for each $U \in \mathcal{U}$. For each $U \in \mathcal{U}$, let $g_U : X \to \mathbf{I}$ be a Urysohn map with $g_U(\operatorname{cl} W_U) = 1$ and $g_U(X \setminus V_U) = 0$. Since $\mathcal{U}$ is locally finite and $\operatorname{supp} g_U \subset \operatorname{cl} V_U \subset U$ for each $U \in \mathcal{U}$, we can define a map $\varphi : X \to [1,\infty)$ by $\varphi(x) = \sum_{U\in\mathcal{U}} g_U(x)$. For each $U \in \mathcal{U}$, let $f_U : X \to \mathbf{I}$ be the map defined by $f_U(x) = g_U(x)/\varphi(x)$. Then, $(f_U)_{U\in\mathcal{U}}$ is the desired partition of unity. $\square$

Since every open cover of a paracompact space has a locally finite open refinement, we have the following corollary:

Corollary 2.7.3. *A paracompact space X has a locally finite partition of unity subordinated to each open cover of X.* $\square$

There exists a partition of unity which is not locally finite. For example, the hedgehog $J(\mathbb{N})$ has a non-locally finite partition of unity $(f_n)_{n\in\omega}$ defined as follows: $f_0(x) = 1 - \|x\|_1$ and $f_n(x) = x(n)$ for each $n \in \mathbb{N}$, where

$$\begin{aligned} J(\mathbb{N}) = \Big\{x \in \ell_1 \;\Big|\; & x(n) \in \mathbf{I} \text{ for all } n \in \mathbb{N} \text{ and} \\ & x(n) \neq 0 \text{ at most one } n \in \mathbb{N}\Big\} \subset \ell_1. \end{aligned}$$

However, the existence of a partition of unity implies the existence of a locally finite one.

Proposition 2.7.4. *If X has a partition of unity $(f_\lambda)_{\lambda\in\Lambda}$ then X has a locally finite partition of unity $(g_\lambda)_{\lambda\in\Lambda}$ such that* $\operatorname{supp} g_\lambda \subset f_\lambda^{-1}((0,1])$ *for each $\lambda \in \Lambda$.*

Proof. We define $h : X \to \mathbf{I}$ by $h(x) = \sup_{\lambda\in\Lambda} f_\lambda(x) > 0$. To see the continuity of h, for each $x \in X$, choose $\Lambda(x) \in \mathrm{Fin}(\Lambda)$ so that $\sum_{\lambda\in\Lambda(x)} f_\lambda(x) > 1 - \frac{1}{2}h(x)$. Then, $f_\lambda(x) < \frac{1}{2}h(x)$ for every $\lambda \in \Lambda \setminus \Lambda(x)$, so $h(x) = f_{\lambda(x)}(x)$ for some $\lambda(x) \in \Lambda(x)$. Since $\sum_{\lambda\in\Lambda(x)} f_\lambda$ and $f_{\lambda(x)}$ are continuous, x has a neighborhood U_x in X such that

$$\sum_{\lambda\in\Lambda(x)} f_\lambda(y) > 1 - \frac{1}{2}h(x) \text{ and } f_{\lambda(x)}(y) > \frac{1}{2}h(x) \text{ for all } y \in U_x.$$

Thus, $f_\lambda(y) < \frac{1}{2}h(x) < f_{\lambda(x)}(y)$ for $\lambda \in \Lambda \setminus \Lambda(x)$ and $y \in U_x$. Therefore,

$$h(y) = \max\left\{ f_\lambda(y) \mid \lambda \in \Lambda(x) \right\} \text{ for each } y \in U_x.$$

Hence, h is continuous.

For each $\lambda \in \Lambda$, let $k_\lambda : X \to \mathbf{I}$ be a map defined by

$$k_\lambda(x) = \max\left\{ 0, f_\lambda(x) - \frac{2}{3}h(x) \right\}.$$

Then, $\operatorname{supp} k_\lambda \subset f_\lambda^{-1}((0,1])$. Indeed, if $f_\lambda(x) = 0$ then x has a neighborhood U such that $f_\lambda(y) < \frac{2}{3}h(y)$ for every $y \in U$, which implies $x \notin \operatorname{supp} k_\lambda$. For each $x \in X$, take U_x and $\Lambda(x)$ as in the proof of the continuity of h. Choose an open neighborhood V_x of x in X so that $V_x \subset U_x$ and $h(y) > \frac{3}{4}h(x)$ for all $y \in V_x$. If $\lambda \in \Lambda \setminus \Lambda(x)$ and $y \in V_x$, then

$$f_\lambda(y) - \frac{2}{3}h(y) < f_\lambda(y) - \frac{1}{2}h(x) < 0,$$

which implies that $V_x \cap \operatorname{supp} k_\lambda = \emptyset$ for any $\lambda \in \Lambda \setminus \Lambda(x)$. Thus, $(k_\lambda)_{\lambda\in\Lambda}$ is locally finite. As in the proof of Theorem 2.7.2, for each $\lambda \in \Lambda$, let $g_\lambda : X \to \mathbf{I}$ be the map defined by $g_\lambda(x) = k_\lambda(x)/\varphi(x)$, where $\varphi(x) = \sum_{\lambda\in\Lambda} k_\lambda(x)$. Then, $(g_\lambda)_{\lambda\in\Lambda}$ is the desired partition of unity on X. □

The paracompactness can be characterized by the existence of a partition of unity as follows:

Theorem 2.7.5. *A space X is paracompact if and only if X has a partition of unity (weakly) subordinated to each open cover of X.*

Proof. The "only if" part is Corollary 2.7.3. The "if" part easily follows from Proposition 2.7.4. □

It is said that a real-valued function $f : X \to \mathbb{R}$ is **lower semi-continuous**, abbreviated as **l.s.c.** (or **upper semi-continuous**, **u.s.c.**) if $f^{-1}((t,\infty))$ (or $f^{-1}((-\infty,t))$) is open in X for each $t \in \mathbb{R}$. Then, $f : X \to \mathbb{R}$ is continuous if and only if f is l.s.c. and u.s.c.

Theorem 2.7.6. *Let $g, h : X \to \mathbb{R}$ be real-valued functions on a paracompact space X such that g is u.s.c., h is l.s.c. and $g(x) < h(x)$ for each $x \in X$. Then, there* *exists a map $f : X \to \mathbb{R}$ such that $g(x) < f(x) < h(x)$ for each $x \in X$. Moreover, given a map $f_0 : A \to \mathbb{R}$ of a closed set A in X such that $g(x) < f_0(x) < h(x)$ for each $x \in A$, the map f can be an extension of f_0.*

Proof. For each $q \in \mathbb{Q}$, let

$$U_q = g^{-1}((-\infty, q)) \cap h^{-1}((q, \infty)).$$

For each $x \in X$, we have $q \in \mathbb{Q}$ such that $g(x) < q < h(x)$, hence $\mathcal{U} = \{U_q \mid q \in \mathbb{Q}\} \in \operatorname{cov}(X)$. By Corollary 2.7.3, X has a locally finite partition of unity $(f_\lambda)_{\lambda \in \Lambda}$ subordinated to $\mathcal{U}$. For each $\lambda \in \Lambda$, choose $q(\lambda) \in \mathbb{Q}$ so that $\operatorname{supp} f_\lambda \subset U_{q(\lambda)}$. Then, we define a map $f : X \to \mathbb{R}$ as follows:

$$f(x) = \sum_{\lambda \in \Lambda} q(\lambda) f_\lambda(x).$$

For each $x \in X$, let $\{\lambda \in \Lambda \mid x \in \operatorname{supp} f_\lambda\} = \{\lambda_1, \cdots, \lambda_n\}$. Since $x \in \bigcap_{i=1}^n U_{q(\lambda_i)}$, we have $g(x) < q(\lambda_i) < h(x)$ for each $i = 1, \cdots, n$, hence it follows that

$$\begin{aligned} g(x) = \sum_{i=1}^n g(x) f_{\lambda_i}(x) < f(x) &= \sum_{i=1}^n q(\lambda_i) f_{\lambda_i}(x) \\ &< h(x) = \sum_{i=1}^n h(x) f_{\lambda_i}(x). \end{aligned}$$

To prove the additional statement, apply the Tietze Extension Theorem 2.2.2 to extend f_0 to a map $f' : X \to \mathbb{R}$. Then, we have an open neighborhood U of A in X such that $g(x) < f'(x) < h(x)$ for each $x \in U$. Let $k : X \to \mathbf{I}$ be a Urysohn map with $k(A) = 1$ and $k(X \setminus U) = 0$. We can define $\tilde{f} : X \to \mathbb{R}$ as follows:

$$\tilde{f}(x) = (1 - k(x)) f(x) + k(x) f'(x).$$

Therefore, $\tilde{f}|A = f_0$ and $g(x) < \tilde{f}(x) < h(x)$ for each $x \in X$. □

Refinements by Open Balls 2.7.7.

(1) Let X be a metrizable space and $\mathcal{U}$ an open cover of X. Then, X has an admissible metric ρ such that

$$\left\{\overline{\mathrm{B}}_\rho(x, 1) \;\middle|\; x \in X\right\} \prec \mathcal{U}.$$

Moreover, for a given $d \in \operatorname{Metr}(X)$, ρ can be chosen so that $\rho \geq d$ (hence, if d is complete then ρ is) and if d is bounded then ρ is also bounded.

Sketch of Proof. Take an open Δ-refinement $\mathcal{V}$ of $\mathcal{U}$ and a locally finite partition of unity $(f_\lambda)_{\lambda \in \Lambda}$ on X subordinated to $\mathcal{V}$. For a given $d \in \operatorname{Metr}(X)$, the desired metric $\rho \in \operatorname{Metr}(X)$ can be defined as follows:

$$\rho(x, y) = d(x, y) + \sum_{\lambda \in \Lambda} |f_\lambda(x) - f_\lambda(y)| \geq d(x, y).$$

If $\rho(x, y) \leq 1$ then $x, y \in f_\lambda^{-1}((0, 1]) \subset \operatorname{supp} f_\lambda$ for some $\lambda \in \Lambda$, otherwise we have

$$\sum_{\lambda \in \Lambda} |f_\lambda(x) - f_\lambda(y)| = \sum_{\lambda \in \Lambda} f_\lambda(x) + \sum_{\lambda \in \Lambda} f_\lambda(y) = 2 > 1.$$

Then, it follows that $\overline{\mathrm{B}}_\rho(x, 1) \subset \operatorname{st}(x, \mathcal{V})$.

Sketch of another Proof. The above can be obtained as a corollary of 2.6.3 and 2.4.2 (or 2.4.4) as follows: By 2.4.2 (or 2.4.4), X has a sequence of open covers

$$\mathcal{U}_1 \overset{\Delta}{\succ} \mathcal{U}_2 \overset{\Delta}{\succ} \mathcal{U}_3 \overset{\Delta}{\succ} \cdots \quad \left(\text{or } \mathcal{U}_1 \overset{*}{\succ} \mathcal{U}_2 \overset{*}{\succ} \mathcal{U}_3 \overset{*}{\succ} \cdots\right)$$

such that $\{\operatorname{st}(x, \mathcal{U}_n) \mid n \in \mathbb{N}\}$ is a neighborhood basis of each $x \in X$. By 2.6.3, we can inductively define $\mathcal{V}_n \in \operatorname{cov}(X)$, $n \in \mathbb{N}$, such that

$$\mathcal{V}_n \prec \mathcal{U}_n \text{ and } \mathcal{V}_n \overset{\Delta}{\prec} \mathcal{V}_{n-1} \quad \left(\mathcal{V}_n \overset{*}{\prec} \mathcal{V}_{n-1}\right),$$

where $\mathcal{V}_0 = \mathcal{U}$. Let $d' \in \operatorname{Metr}(X)$ be the bounded metric obtained by applying Corollary 2.4.2 (or 2.4.4) with Remark 3 (or 4). For a given $d \in \operatorname{Metr}(X)$, the desired $\rho \in \operatorname{Metr}(X)$ can be defined by $\rho = 8d' + d$ (or $\rho = 2d' + d$).

(2) Let $X = (X, d)$ be a metric space. For each open cover $\mathcal{U}$ of X, there is a map $\gamma : X \to (0, 1)$ such that

$$\left\{\overline{\mathrm{B}}(x, \gamma(x)) \;\middle|\; x \in X\right\} \prec \mathcal{U}.$$

Sketch of Proof. For each $x \in X$, let

$$r(x) = \sup_{U \in \mathcal{U}} \min\{1,\ d(x, X \setminus U)\} = \sup_{U \in \mathcal{U}} \bar{d}(x, X \setminus U),$$

where $\bar{d} = \min\{1, d\}$. Show that $r : X \to (0, \infty)$ is l.s.c. Then, we can apply Theorem 2.7.6 to obtain a map $\gamma : X \to (0, 1)$ such that $\gamma(x) < r(x)$ for each $x \in X$.

Remark. If $\mathcal{U}$ is locally finite, r is continuous (in fact, r is 1-Lipschitz), so we can define $\gamma = \frac{1}{2}r$.

2.8 The Direct Limits of Towers of Spaces

In this section, we consider the direct limit of a tower $X_1 \subset X_2 \subset \cdots$ of spaces, where each X_n is a subspace of X_{n+1}. The **direct limit** $\varinjlim X_n$ is the space $\bigcup_{n \in \mathbb{N}} X_n$ endowed with the weak topology with respect to the tower $(X_n)_{n \in \mathbb{N}}$, that is,

$$U \subset \varinjlim X_n \text{ is open in } \varinjlim X_n \ \Leftrightarrow\ \forall n \in \mathbb{N},\ U \cap X_n \text{ is open in } X_n$$

$$\left(\text{equiv. } A \subset \varinjlim X_n \text{ is closed in } \varinjlim X_n \ \Leftrightarrow\ \forall n \in \mathbb{N},\ A \cap X_n \text{ is closed in } X_n\right).$$

In other words, the topology of $\varinjlim X_n$ is the finest topology such that every inclusion $X_n \subset \varinjlim X_n$ is continuous; equivalently, every X_n is a subspace of $\varinjlim X_n$. For an arbitrary space Y,

$$f : \varinjlim X_n \to Y \text{ is continuous } \Leftrightarrow \forall n \in \mathbb{N},\ f|X_n \text{ is continuous.}$$

Remark 6. Each point $x \in \varinjlim X_n$ belongs to some $X_{n(x)}$. If V is a neighborhood of x in $\varinjlim X_n$, then $V \cap X_n$ is a neighborhood x in X_n for every $n \geq n(x)$. However, it should be noted that the converse does not hold. For example, consider the direct limit $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ of the tower $\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$, where each $\mathbb{R}^n$ is identified with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. Let $W = \bigcup_{n\in\mathbb{N}}(-2^{-n}, 2^{-n})^n \subset \mathbb{R}^\infty$. Then, every $W \cap \mathbb{R}^n$ is a neighborhood of $0 \in \mathbb{R}^n$ because it contains $(-2^{-n}, 2^{-n})^n$. Nevertheless, W is not a neighborhood of 0 in $\mathbb{R}^\infty$. Indeed,

$$(\operatorname{int}_{\mathbb{R}^\infty} W) \cap \mathbb{R}^n \subset \operatorname{int}_{\mathbb{R}^n}(W \cap \mathbb{R}^n) = (-2^{-n}, 2^{-n})^n \quad \text{for each } n \in \mathbb{N}.$$

Then, it follows that $(\operatorname{int}_{\mathbb{R}^\infty} W) \cap \mathbb{R} \subset \bigcap_{n\in\mathbb{N}}(-2^{-n}, 2^{-n}) = \{0\}$, which means that $(\operatorname{int}_{\mathbb{R}^\infty} W) \cap \mathbb{R} = \emptyset$, and hence $0 \notin \operatorname{int}_{\mathbb{R}^\infty} W$.

It should also be noted that the direct limit $\varinjlim X_n$ is T_1 but, in general, non-Hausdorff. Such an example is shown in 2.10.3.

As is easily observed, $\varinjlim X_{n(i)} = \varinjlim X_n$ for any $n(1) < n(2) < \cdots \in \mathbb{N}$. It is also easy to prove the following proposition:

Proposition 2.8.1. *Let $X_1 \subset X_2 \subset \cdots$ and $Y_1 \subset Y_2 \subset \cdots$ be towers of spaces. Suppose that there exist $n(1) < n(2) < \cdots$, $m(1) < m(2) < \cdots \in \mathbb{N}$ and maps $f_i : X_{n(i)} \to Y_{m(i)}$ and $g_i : Y_{m(i)} \to X_{n(i+1)}$ such that $g_i f_i = \mathrm{id}_{X_{n(i)}}$ and $f_{i+1} g_i = \mathrm{id}_{Y_{m(i)}}$, that is, the following diagram is commutative:*

$$\begin{tikzcd}
X_{n(1)} \arrow[r, phantom, "\subset"] \arrow[d, "f_1"'] & X_{n(2)} \arrow[r, phantom, "\subset"] \arrow[d, "f_2"'] & X_{n(3)} \arrow[r, phantom, "\subset"] \arrow[d, "f_3"'] & \cdots \\
Y_{m(1)} \arrow[r, phantom, "\subset"] \arrow[ur, "g_1"] & Y_{m(2)} \arrow[r, phantom, "\subset"] \arrow[ur, "g_2"] & Y_{m(3)} \arrow[r, phantom, "\subset"] & \cdots
\end{tikzcd}$$

Then, $\varinjlim X_n$ is homeomorphic to $\varinjlim Y_n$. □

Remark 7. It should be noted that $\varinjlim X_n$ is not a subspace of $\varinjlim Y_n$ even if each X_n is a closed subspace of Y_n. For example, let $Y_n = \mathbb{R}$ be the real line and

$$X_n = \{0\} \cup [n^{-1}, 1] \subset Y_n = \mathbb{R}.$$

Then, $\mathbf{I} = \bigcup_{n\in\mathbb{N}} X_n$, $\mathbb{R} = \varinjlim Y_n$, and 0 is an isolated point of $\varinjlim X_n$ but is not in the subspace $\mathbf{I} \subset \mathbb{R}$.

On the other hand, as is easily observed, if each X_n is an open subspace of Y_n then $\varinjlim X_n$ is an open subspace of $\varinjlim Y_n$.

The following proposition is also rather obvious:

Proposition 2.8.2. *Let $Y_1 \subset Y_2 \subset \cdots$ be a tower of spaces. If X is a closed (resp. open) subspace of $Y = \varinjlim Y_n$, then $X = \varinjlim(X \cap Y_n)$. Equivalently, if each $X \cap Y_n$ is closed (resp. open) in Y_n, then $\varinjlim(X \cap Y_n)$ is a closed (resp. open) subspace of Y.* □

Remark 8. In general, $X \neq \varinjlim(X \cap Y_n)$ for a subspace $X \subset \varinjlim Y_n$. For example, let Y_n be a subspace of the Euclidean plane $\mathbb{R}^2$ defined by

$$Y_n = \left\{(0,0),\ (i^{-1},0),\ (j^{-1},k^{-1}) \;\middle|\; i,k \in \mathbb{N},\ j = 1,\dots,n\right\}.$$

Observe that $A = \{(j^{-1},k^{-1}) \mid j,k \in \mathbb{N}\}$ is dense in $\varinjlim Y_n$, hence it is not closed in the following subspace X of $\varinjlim Y_n$:

$$X = \{(0,0)\} \cup \{(j^{-1},k^{-1}) \mid j,k \in \mathbb{N}\},$$

whereas A is closed in $\varinjlim(X \cap Y_n)$.

With regard to products of direct limits, we have:

Proposition 2.8.3. *Let $X_1 \subset X_2 \subset \cdots$ be a tower of spaces. If Y is locally compact then $(\varinjlim X_n) \times Y = \varinjlim(X_n \times Y)$ as spaces.*

Proof. First of all, note that

$$(\varinjlim X_n) \times Y = \varinjlim(X_n \times Y) = \bigcup_{n\in\mathbb{N}}(X_n \times Y) \text{ as sets.}$$

It is easy to see that $\mathrm{id} : \varinjlim(X_n \times Y) \to (\varinjlim X_n) \times Y$ is continuous. To see this is an open map, let W be an open set in $\varinjlim(X_n \times Y)$. For each $(x,y) \in W$, choose $m \in \mathbb{N}$ so that $x \in X_m$. Since Y is locally compact, there exist open sets $U_m \subset X_m$ and $V \subset Y$ such that $x \in U_m$, $y \in V$, $U_m \times \mathrm{cl}_Y V \subset W$ and $\mathrm{cl}_Y V$ is compact. Then, by the compactness of $\mathrm{cl}_Y V$, we can find an open set $U_{m+1} \subset X_{m+1}$ such that $U_m \subset U_{m+1}$ and $U_{m+1} \times \mathrm{cl}_Y V \subset W$. Inductively, we can obtain $U_m \subset U_{m+1} \subset U_{m+2} \subset \cdots$ such that each U_n is open in X_n and $U_n \times \mathrm{cl}_Y V \subset W$. Then, $U = \bigcup_{n\geq m} U_n$ is open in $\varinjlim X_n$, and hence $U \times V$ is an open neighborhood of (x,y) in $(\varinjlim X_n) \times Y$ with $U \times V \subset W$. Thus, W is open in $(\varinjlim X_n) \times Y$. □

Proposition 2.8.4. *Let $X_1 \subset X_2 \subset \cdots$ and $Y_1 \subset Y_2 \subset \cdots$ be towers of spaces. If each X_n and Y_n are locally compact, then*

$$\varinjlim X_n \times \varinjlim Y_n = \varinjlim(X_n \times Y_n) \text{ as spaces.}$$

Proof. First of all, note that

$$\varinjlim X_n \times \varinjlim Y_n = \varinjlim (X_n \times Y_n) = \bigcup_{n \in \mathbb{N}} (X_n \times Y_n) \text{ as sets.}$$

It is easy to see that $\mathrm{id} : \varinjlim (X_n \times Y_n) \to \varinjlim X_n \times \varinjlim Y_n$ is continuous. To see that this is open, let W be an open set in $\varinjlim (X_n \times Y_n)$. For each $(x, y) \in W$, choose $m \in \mathbb{N}$ so that $(x, y) \in X_m \times Y_m$. Since X_m and Y_m are locally compact, we have open sets $U_m \subset X_m$ and $V_m \subset Y_m$ such that

$$x \in U_m, \ y \in V_m, \ \mathrm{cl}_{X_m} U_m \times \mathrm{cl}_{Y_m} V_m \subset W$$

and both $\mathrm{cl}_{X_m} U_m$ and $\mathrm{cl}_{Y_m} V_m$ are compact. Then, by the compactness of $\mathrm{cl}_{X_m} U_m$ and $\mathrm{cl}_{Y_m} V_m$, we can easily find open sets $U_{m+1} \subset X_{m+1}$ and $V_{m+1} \subset Y_{m+1}$ such that

$$\mathrm{cl}_{X_m} U_m \subset U_{m+1}, \ \mathrm{cl}_{Y_m} V_m \subset V_{m+1}, \ \mathrm{cl}_{X_{m+1}} U_{m+1} \times \mathrm{cl}_{Y_{m+1}} V_{m+1} \subset W$$

and both $\mathrm{cl}_{X_{m+1}} U_{m+1}$ and $\mathrm{cl}_{Y_{m+1}} V_{m+1}$ are compact. Inductively, we can obtain $U_m \subset U_{m+1} \subset U_{m+2} \subset \cdots$ and $V_m \subset V_{m+1} \subset V_{m+2} \subset \cdots$ such that U_n and V_n are open in X_n and Y_n, respectively, $\mathrm{cl}_{X_n} U_n$ and $\mathrm{cl}_{Y_n} V_n$ are compact, and $\mathrm{cl}_{X_n} U_n \times \mathrm{cl}_{Y_n} V_n \subset W$. Then, $U = \bigcup_{n \geq m} U_n$ and $V = \bigcup_{n \geq m} V_n$ are open in $\varinjlim X_n$ and $\varinjlim Y_n$, respectively, and $(x, y) \in U \times V \subset W$. Therefore, W is open in $\varinjlim X_n \times \varinjlim Y_n$. □

A tower $X_1 \subset X_2 \subset \cdots$ of spaces is said to be **closed** if each X_n is closed in X_{n+1}; equivalently, each X_n is closed in the direct limit $\varinjlim X_n$. For a pointed space $X = (X, *)$, let

$$X_f^{\mathbb{N}} = \left\{ x \in X^{\mathbb{N}} \mid x(n) = * \text{ except for finitely many } n \in \mathbb{N} \right\} \subset X^{\mathbb{N}}.$$

Identifying each X^n with $X^n \times \{(*, *, \ldots)\} \subset X_f^{\mathbb{N}}$, we have a closed tower $X \subset X^2 \subset X^3 \subset \cdots$ with $X_f^{\mathbb{N}} = \bigcup_{n \in \mathbb{N}} X^n$. We write $X^\infty = \varinjlim X^n$, which is the space $X_f^{\mathbb{N}}$ with the weak topology with respect to the tower $(X^n)_{n \in \mathbb{N}}$. A typical example is $\mathbb{R}^\infty$, which appeared in Remark 6.

Proposition 2.8.5. *Let $X = (X, *)$ be a pointed locally compact space. Then, each $x \in X^\infty = \varinjlim X^n$ has a neighborhood basis consisting of $X^\infty \cap \prod_{n \in \mathbb{N}} V_n$, where each V_n is a neighborhood of $x(n)$ in X^n.*[5]

Sketch of Proof. Let U be an open neighborhood of x in X^∞. Choose $n_0 \in \mathbb{N}$ so that $x \in X^{n_0}$. For each $i = 1, \ldots, n_0$, each $x(i)$ has a neighborhood V_i in X such that $\mathrm{cl}\, V_i$ is

[5]In other words, the topology of $\varinjlim X^n$ is a relative (subspace) topology inherited from the box topology of $X^{\mathbb{N}}$.

compact and $\prod_{i=1}^{n_0} \operatorname{cl} V_i \subset U \cap X^{n_0}$. Recall that we identify $X^{n-1} = X^{n-1} \times \{*\} \subset X^n$. For $n > n_0$, we can inductively choose a neighborhood V_n of $x(n) = *$ in X so that $\operatorname{cl} V_n$ is compact and $\prod_{i=1}^{n} \operatorname{cl} V_i \subset U \cap X^n$, where we use the compactness of $\prod_{i=1}^{n-1} \operatorname{cl} V_i$ $\left(= \prod_{i=1}^{n-1} \operatorname{cl} V_i \times \{*\}\right)$. This is an excellent exercise as the first part of the proof of Wallace's Theorem 2.1.2.

Remark 9. Proposition 2.8.3 does not hold without the local compactness of Y even if each X_n is locally compact. For example, $(\varinjlim \mathbb{R}^n) \times \ell_2 \neq \varinjlim(\mathbb{R}^n \times \ell_2)$. Indeed, each $\mathbb{R}^n$ is identified with $\mathbb{R}^n \times \{\mathbf{0}\} \subset \mathbb{R}^{\mathbb{N}}_f \subset \ell_2$. Then, we regard

$$(\varinjlim \mathbb{R}^n) \times \ell_2 = \varinjlim(\mathbb{R}^n \times \ell_2) = \mathbb{R}^{\mathbb{N}}_f \times \ell_2 \text{ as sets.}$$

Consider the following set:

$$D = \left\{(k^{-1}\mathbf{e}_n, n^{-1}\mathbf{e}_k) \in \mathbb{R}^{\mathbb{N}}_f \times \ell_2 \;\middle|\; k, n \in \mathbb{N}\right\},$$

where each $\mathbf{e}_i \in \mathbb{R}^{\mathbb{N}}_f \subset \ell_2$ is the unit vector defined by $\mathbf{e}_i(i) = 1$ and $\mathbf{e}_i(j) = 0$ for $j \neq i$. For each $n \in \mathbb{N}$, let

$$D_n = \left\{(k^{-1}\mathbf{e}_n, n^{-1}\mathbf{e}_k) \;\middle|\; k \in \mathbb{N}\right\}.$$

Since $\{n^{-1}\mathbf{e}_k \mid k \in \mathbb{N}\}$ is discrete in ℓ_2, it follows that D_n is discrete (so closed) in $\mathbb{R}^n \times \ell_2$, hence it is also closed in $\mathbb{R}^m \times \ell_2$ for every $m \geq n$. Observe that $D \cap (\mathbb{R}^n \times \ell_2) = \bigcup_{i=1}^{n} D_i$. Then, D is closed in $\varinjlim(\mathbb{R}^n \times \ell_2)$. On the other hand, for each neighborhood U of $(0, 0)$ in $(\varinjlim \mathbb{R}^n) \times \ell_2$, we can apply Proposition 2.8.5 to take $\delta_i > 0$ ($i \in \mathbb{N}$) and $n \in \mathbb{N}$ so that

$$\left(\mathbb{R}^{\mathbb{N}}_f \cap \prod_{i \in \mathbb{N}} [-\delta_i, \delta_i]\right) \times n^{-1}\mathbf{B}_{\ell_2} \subset U,$$

where $\mathbf{B}_{\ell_2}$ is the unit closed ball of ℓ_2. Choose $k \in \mathbb{N}$ so that $k^{-1} < \delta_n$. Then, $(k^{-1}\mathbf{e}_n, n^{-1}\mathbf{e}_k) \in U$, which implies $U \cap D \neq \emptyset$. Thus, D is not closed in $(\varinjlim \mathbb{R}^n) \times \ell_2$.

Remark 10. In Proposition 2.8.4, it is necessary to assume that both X_n and Y_n are locally compact. Indeed, let $X_n = \mathbb{R}^n$ and $Y_n = \ell_2$ for every $n \in \mathbb{N}$. Then, $\varinjlim X_n \times \varinjlim Y_n \neq \varinjlim(X_n \times Y_n)$, as we saw in the above remark. Furthermore, this equality does not hold even if $X_n = Y_n$. For example, $\varinjlim(\ell_2)^n \times \varinjlim(\ell_2)^n \neq \varinjlim((\ell_2)^n \times (\ell_2)^n)$. Indeed, consider

$$\varinjlim(\ell_2)^n \times \varinjlim(\ell_2)^n = \varinjlim((\ell_2)^n \times (\ell_2)^n) = (\ell_2)^{\mathbb{N}}_f \times (\ell_2)^{\mathbb{N}}_f \text{ as sets.}$$

Identifying $\mathbb{R}^n = (\mathbb{R}\mathbf{e}_1)^n \subset (\ell_2)^n$ and $\ell_2 = \ell_2 \times \{\mathbf{0}\} \subset (\ell_2)^{\mathbb{N}}_f$, we can also consider

$$(\varinjlim \mathbb{R}^n) \times \ell_2 = \varinjlim(\mathbb{R}^n \times \ell_2) = \mathbb{R}^{\mathbb{N}}_f \times \ell_2 \subset (\ell_2)^{\mathbb{N}}_f \times (\ell_2)^{\mathbb{N}}_f \text{ as sets.}$$

By Proposition 2.8.2, $(\varinjlim \mathbb{R}^n) \times \ell_2$ and $\varinjlim(\mathbb{R}^n \times \ell_2)$ are closed subspaces of $\varinjlim(\ell_2)^n \times \varinjlim(\ell_2)^n$ and $\varinjlim((\ell_2)^n \times (\ell_2)^n)$, respectively. As we saw above, $(\varinjlim \mathbb{R}^n) \times \ell_2 \neq \varinjlim(\mathbb{R}^n \times \ell_2)$. Thus, $\varinjlim(\ell_2)^n \times \varinjlim(\ell_2)^n \neq \varinjlim((\ell_2)^n \times (\ell_2)^n)$.

Theorem 2.8.6. *For the direct limit $X = \varinjlim X_n$ of a tower $X_1 \subset X_2 \subset \cdots$ of spaces, the following hold:*

(1) *Every compact set $A \subset X$ is contained in some X_n.*
(2) *For each map $f : Y \to X$ from a first countable space Y to X, each point $y \in Y$ has a neighborhood V in Y such that the image $f(V)$ is contained in some X_n. In particular, if $A \subset X$ is a metrizable subspace then each point of A has a neighborhood in A that is contained in some X_n.*

Proof. (1): Assume that A is not contained in any X_n. For each $n \in \mathbb{N}$, take $x_n \in A \setminus X_n$ and let $D = \{x_n \mid n \in \mathbb{N}\} \subset A$. Then, D is infinite and discrete in $\varinjlim X_n$. Indeed, every $C \subset D$ is closed in $\varinjlim X_n$ because $C \cap X_n$ is finite for each $n \in \mathbb{N}$. This contradicts the compactness of A.

(2): Let $\{V_n \mid n \in \mathbb{N}\}$ be a neighborhood basis of y_0 in Y such that $V_n \subset V_{n-1}$. Assume that $f(V_n) \not\subset X_n$ for every $n \in \mathbb{N}$. Then, taking $y_n \in V_n \setminus f^{-1}(X_n)$, we have a compact set $A = \{y_n \mid n \in \omega\}$ in Y. Due to (1), $f(A)$ is contained in some X_m, and hence $f(y_m) \in X_m$. This is a contradiction. Therefore, $f(V_n) \subset X_n$ for some $n \in \mathbb{N}$. □

By Theorem 2.8.6(2), the direct limit of metrizable spaces is non-metrizable in general (e.g., $\varinjlim \mathbb{R}^n$ is non-metrizable). However, it has some favorable properties, which we now discuss.

Theorem 2.8.7. *For the direct limit $X = \varinjlim X_n$ of a closed tower $X_1 \subset X_2 \subset \cdots$ of spaces, the following properties hold:*

(1) *If each X_n is normal, then X is also normal;*
(2) *If each X_n is perfectly normal, then X is also perfectly normal;*
(3) *If each X_n is collectionwise normal, then X is also collectionwise normal;*
(4) *If each X_n is paracompact, then X is also paracompact.*

Proof. (1): Obviously, every singleton of X is closed, so X is T_1. Let A and B be disjoint closed sets in X. Then, we have a map $f_1 : X_1 \to \mathbf{I}$ such that $f_1(A \cap X_1) = 0$ and $f_1(B \cap X_1) = 1$. Using the Tietze Extension Theorem 2.2.2, we can extend f_1 to a map $f_2 : X_2 \to \mathbf{I}$ such that $f_2(A \cap X_2) = 0$ and $f_2(B \cap X_2) = 1$. Thus, we inductively obtain maps $f_n : X_n \to \mathbf{I}$, $n \in \mathbb{N}$, such that

$$f_n | X_{n-1} = f_{n-1}, \ f_n(A \cap X_n) = 0 \ \text{and} \ f_n(B \cap X_n) = 1.$$

Let $f : X \to \mathbf{I}$ be the map defined by $f | X_n = f_n$ for $n \in \mathbb{N}$. Evidently, $f(A) = 0$ and $f(B) = 1$. Therefore, X is normal.

(2): From (1), it suffices to show that every closed set A in X is a G_δ set. Each X_n has open sets $G_{n,m}$, $m \in \mathbb{N}$, such that $A \cap X_n = \bigcap_{m \in \mathbb{N}} G_{n,m}$. For each $n, m \in \mathbb{N}$, let

$G^*_{n,m} = G_{n,m} \cup (X \setminus X_n)$. Since X_n is closed in X, each $G^*_{n,m}$ is open in X. Observe that $A = \bigcap_{n,m\in\mathbb{N}} G^*_{n,m}$. Hence, A is G_δ in X.

(3): Let $\mathcal{F}$ be a discrete collection of closed sets in X. By induction on $n \in \mathbb{N}$, we have discrete collections $\{U^F_n \mid F \in \mathcal{F}\}$ of open sets in X_n such that $(F \cap X_n) \cup \operatorname{cl} U^F_{n-1} \subset U^F_n$ for each $F \in \mathcal{F}$, where $U^F_0 = \emptyset$. For each $F \in \mathcal{F}$, let $U_F = \bigcup_{n\in\mathbb{N}} U^F_n$. Then, $F \subset U_F$ and U_F is open in X because $U_F \cap X_n = \bigcup_{i\geq n} U^F_i \cap X_n$ is open in X_n for each $n \in \mathbb{N}$. If $F \neq F'$, then $U_F \cap U_{F'} = \emptyset$ because

$$U^F_i \cap U^{F'}_j \subset U^F_{\max\{i,j\}} \cap U^{F'}_{\max\{i,j\}} = \emptyset \text{ for each } i, j \in \mathbb{N}.$$

Therefore, X is collectionwise normal.

(4): Since every paracompact space is collectionwise normal (Theorem 2.6.1), X is also collectionwise normal by (3), so it is regular. Then, due to Theorem 2.6.3, it suffices to show that each $\mathcal{U} \in \operatorname{cov}(X)$ has a σ-discrete open refinement. By Theorem 2.6.3, we have $\bigcup_{m\in\mathbb{N}} \mathcal{V}_{n,m} \in \operatorname{cov}(X_n)$, $n \in \mathbb{N}$, such that each $\mathcal{V}_{n,m}$ is discrete in X_n and $\mathcal{V}_{n,m} \prec \mathcal{U}$. For each $V \in \mathcal{V}_{n,m}$, choose $U_V \in \mathcal{U}$ so that $V \subset U_V$. Note that each $\mathcal{V}^{\mathrm{cl}}_{n,m}$ is discrete in X, and recall that X is collectionwise normal. So, X has a discrete open collection $\{W_V \mid V \in \mathcal{V}_{n,m}\}$ such that $\operatorname{cl} V \subset W_V$. Let $\mathcal{W}_{n,m} = \{W_V \cap U_V \mid V \in \mathcal{V}_{n,m}\}$. Then, $\mathcal{W} = \bigcup_{n,m\in\mathbb{N}} \mathcal{W}_{n,m} \in \operatorname{cov}(X)$ is a σ-discrete open cover refinement of $\mathcal{U}$. □

From Theorems 2.8.7 and 2.6.8, we conclude the following:

Corollary 2.8.8. *The direct limit of a closed tower of metrizable spaces is perfectly normal and paracompact, and so it is hereditarily paracompact.* □

2.9 The Limitation Topology for Spaces of Maps

Let X and Y be spaces. Recall that $\mathrm{C}(X, Y)$ denotes the set of all maps from X to Y. For each $f \in \mathrm{C}(X, Y)$ and $\mathcal{U} \in \operatorname{cov}(Y)$, we define

$$\mathcal{U}(f) = \{g \in \mathrm{C}(X, Y) \mid g \text{ is } \mathcal{U}\text{-close to } f\}.$$

Observe that if $\mathcal{V} \in \operatorname{cov}(Y)$ is a Δ-refinement (or a star-refinement) of $\mathcal{U}$ then $\mathcal{V}(g) \subset \mathcal{U}(f)$ for each $g \in \mathcal{V}(f)$. Then, in the case that Y is paracompact, $\mathrm{C}(X, Y)$ has a topology such that $\{\mathcal{U}(f) \mid \mathcal{U} \in \operatorname{cov}(Y)\}$ is a neighborhood basis of f. Such a topology is called the **limitation topology**.

The limitation topology is Hausdorff. Indeed, let $f \neq g \in \mathrm{C}(X, Y)$. Then $f(x_0) \neq g(x_0)$ for some $x_0 \in X$. Take disjoint open sets $U, V \subset Y$ with $f(x_0) \in U$ and $g(x_0) \in V$, and define

$$\mathcal{U} = \{U,\ Y \setminus \{f(x_0)\}\},\ \mathcal{V} = \{V,\ Y \setminus \{g(x_0)\}\} \in \operatorname{cov}(Y).$$

Then, $\mathcal{U}(f) \cap \mathcal{V}(g) = \emptyset$.

Remark 11. In the above, $\mathcal{U}(f)$ is not open in general. For example, consider the hedgehog $J(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} \mathbf{I}\mathbf{e}_n$ (see Sect. 2.3) and the map $f : \mathbb{N} \to J(\mathbb{N})$ defined by $f(n) = \mathbf{e}_n$ for each $n \in \mathbb{N}$, where $\mathbf{e}_n(n) = 1$ and $\mathbf{e}_n(i) = 0$ if $i \neq n$. For each $n \in \mathbb{N}$, let

$$U_n = \mathbf{I}\mathbf{e}_n \cup \mathrm{B}(\mathbf{0}, n^{-1}) \subset J(\mathbb{N}).$$

Then, $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\} \in \operatorname{cov}(J(\mathbb{N}))$. We show that $\mathcal{U}(f)$ is not open in $\mathrm{C}(\mathbb{N}, J(\mathbb{N}))$ with respect to the limitation topology. Indeed, $\mathcal{U}(f)$ contains the constant map f_0 with $f_0(\mathbb{N}) = \{\mathbf{0}\}$. For each $\mathcal{V} \in \operatorname{cov}(J(\mathbb{N}))$, choose $k \in \mathbb{N}$ so that $\mathrm{B}(\mathbf{0}, k^{-1}) \subset V_0$ for some $V_0 \in \mathcal{V}$. Then, $\mathcal{V}(f_0)$ contains the map $g : \mathbb{N} \to J(\mathbb{N})$ defined by $g(n) = (k+1)^{-1}\mathbf{e}_{n+1}$ for each $n \in \mathbb{N}$. Observe that $g(k+1) = (k+1)^{-1}\mathbf{e}_{k+2} \notin U_{k+1}$ but $f(k+1) = \mathbf{e}_{k+1} \notin U_n$ if $n \neq k+1$, which means that $g \notin \mathcal{U}(f)$. Thus, $\mathcal{V}(f_0) \not\subset \mathcal{U}(f)$. Hence, $\mathcal{U}(f)$ is not open.

The set of all admissible bounded metrics of a metrizable space Y is denoted by $\mathrm{Metr}^B(Y)$. If Y is completely metrizable, let $\mathrm{Metr}^c(Y)$ denote the set of all admissible bounded complete metrics of Y. The sup-metric on $\mathrm{C}(X,Y)$ defined by $d \in \mathrm{Metr}^B(Y)$ is denoted by the same notation d. For each $f \in \mathrm{C}(X,Y)$ and $d \in \mathrm{Metr}^B(Y)$, let

$$U_d(f) = \mathrm{B}_d(f,1) = \{g \in \mathrm{C}(X,Y) \mid d(f,g) < 1\}.$$

Then, $U_{n\cdot d}(f) = \mathrm{B}_d(f, n^{-1})$ for each $n \in \mathbb{N}$.

Proposition 2.9.1. *When Y is metrizable, $\{U_d(f) \mid d \in \mathrm{Metr}^B(Y)\}$ is a neighborhood basis of $f \in \mathrm{C}(X,Y)$ in the space $\mathrm{C}(X,Y)$ with the limitation topology. If Y is completely metrizable, then $\{U_d(f) \mid d \in \mathrm{Metr}^c(Y)\}$ is also a neighborhood basis of $f \in \mathrm{C}(X,Y)$.*

Proof. For each $d \in \mathrm{Metr}^B(Y)$, let

$$\mathcal{U} = \{\mathrm{B}_d(y, \tfrac{1}{3}) \mid y \in Y\} \in \operatorname{cov}(Y).$$

Then, clearly $\mathcal{U}(f) \subset U_d(f)$ for each $f \in \mathrm{C}(X,Y)$. Conversely, for each $\mathcal{U} \in \operatorname{cov}(Y)$, choose $d \in \mathrm{Metr}^B(Y)$ (or $d \in \mathrm{Metr}^c(Y)$) so that $\{\mathrm{B}_d(y,1) \mid y \in Y\} \prec \mathcal{U}$ (cf. 2.7.7(1)). Thus, $U_d(f) \subset \mathcal{U}(f)$ for each $f \in \mathrm{C}(X,Y)$. $\square$

For a space X, let $\mathrm{Homeo}(X)$ be the set of all homeomorphisms of X onto itself. The **limitation topology** on $\mathrm{Homeo}(X)$ is the subspace topology inherited from the space $\mathrm{C}(X,X)$ with the limitation topology. If X is metrizable, for each $f \in \mathrm{Homeo}(X)$ and $d \in \mathrm{Metr}^B(X)$, let

$$U_{d^*}(f) = \mathrm{B}_{d^*}(f,1) = \{g \in \mathrm{Homeo}(X) \mid d^*(f,g) < 1\},$$

where d^* is the metric on $\mathrm{Homeo}(X)$ defined as follows:

$$d^*(f,g) = d(f,g) + d(f^{-1}, g^{-1}).$$

The following is the homeomorphism space version of Proposition 2.9.1:

Proposition 2.9.2. *When X is metrizable, $\{U_{d^*}(f) \mid d \in \mathrm{Metr}^B(X)\}$ is a neighborhood basis of $f \in \mathrm{Homeo}(X)$ in the space $\mathrm{Homeo}(X)$ with the limitation topology. If X is completely metrizable, then $\{U_{d^*}(f) \mid d \in \mathrm{Metr}^c(X)\}$ is also a neighborhood basis of $f \in \mathrm{Homeo}(X)$.*

Proof. For each $f \in \mathrm{Homeo}(X)$ and $d \in \mathrm{Metr}^B(Y)$, let

$$\mathcal{U} = \left\{\mathrm{B}_d\,(x, 1/5) \cap f\left(\mathrm{B}_d(f^{-1}(x), 1/5)\right) \;\middle|\; x \in X\right\} \in \mathrm{cov}(X).$$

Then, $\mathcal{U}(f) \cap \mathrm{Homeo}(X) \subset U_{d^*}(f)$. Indeed, for each $g \in \mathcal{U}(f) \cap \mathrm{Homeo}(X)$ and $x \in X$, we can find $y \in X$ such that

$$f(g^{-1}(x)), x = g(g^{-1}(x)) \in f\left(\mathrm{B}_d\left(f^{-1}(y), 1/5\right)\right),$$

which means that $d(g^{-1}(x), f^{-1}(y)) < 1/5$ and $d(f^{-1}(x), f^{-1}(y)) < 1/5$, hence $d(f^{-1}(x), g^{-1}(x)) < 2/5$. Therefore, $d(f^{-1}, g^{-1}) \leq 2/5$. On the other hand, it is easy to see that $d(f, g) \leq 2/5$. Thus, we have $d^*(f, g) < 1$, that is, $g \in U_{d^*}(f)$.

Conversely, for each $f \in \mathrm{Homeo}(X)$ and $\mathcal{U} \in \mathrm{cov}(X)$, choose $d \in \mathrm{Metr}^B(X)$ (or $d \in \mathrm{Metr}^c(X)$) so that $\{\mathrm{B}_d(y, 1) \mid y \in Y\} \prec \mathcal{U}$ (cf. 2.7.7(1)). Then, $U_{d^*}(f) \subset \mathcal{U}(f)$. Indeed, for each $g \in U_{d^*}(f)$ and $x \in X$, $d(f(x), g(x)) < 1$ and $\mathrm{B}_d(f(x), 1)$ is contained in some $U \in \mathcal{U}$, hence $f(x), g(x) \in U$. Therefore, $g \in \mathcal{U}(f)$. □

If $Y = (Y, d)$ is a metric space, for each $f \in \mathrm{C}(X, Y)$ and $\alpha \in \mathrm{C}(Y, (0, \infty))$, let

$$N_\alpha(f) = \left\{g \in \mathrm{C}(X, Y) \;\middle|\; \forall x \in X,\ d(f(x), g(x)) < \alpha(f(x))\right\}.$$

Proposition 2.9.3. *When $Y = (Y, d)$ is a metric space, $\{N_\alpha(f) \mid \alpha \in \mathrm{C}(Y, (0, \infty))\}$ is a neighborhood basis of $f \in \mathrm{C}(X, Y)$ in the space $\mathrm{C}(X, Y)$ with the limitation topology.*

Proof. Let $\alpha \in \mathrm{C}(Y, (0, \infty))$. For each $y \in Y$, choose an open neighborhood U_y so that $\operatorname{diam} U_y \leq \frac{1}{2}\alpha(y)$ and $\alpha(y') > \frac{1}{2}\alpha(y)$ for all $y' \in U_y$. Thus, we have $\mathcal{U} = \{U_y \mid y \in Y\} \in \mathrm{cov}(Y)$. Let $f \in \mathrm{C}(X, Y)$ and $g \in \mathcal{U}(f)$. Then, for each $x \in X$, we have some $y \in Y$ such that $f(x), g(x) \in U_y$, which implies $d(f(x), g(x)) \leq \frac{1}{2}\alpha(y) < \alpha(f(x))$. Therefore, $\mathcal{U}(f) \subset N_\alpha(f)$.

Conversely, let $\mathcal{U} \in \mathrm{cov}(Y)$. For each $y \in Y$, let

$$\gamma(y) = \sup\left\{r > 0 \;\middle|\; \exists U \in \mathcal{U} \text{ such that } \mathrm{B}(y, r) \subset U\right\}.$$

Then, $\gamma : Y \to (0, \infty)$ is lower semi-continuous. Hence, by Theorem 2.7.6, we have $\alpha \in \mathrm{C}(Y, (0, \infty))$ such that $\alpha < \gamma$, which implies that $N_\alpha(f) \subset \mathcal{U}(f)$ for any $f \in \mathrm{C}(X, Y)$. □

The following two theorems are very useful to show the existence of some types of maps or homeomorphisms:

Theorem 2.9.4. *For a completely metrizable space Y, the space* $\mathrm{C}(X,Y)$ *with the limitation topology is a Baire space.*

Proof. Let G_n, $n \in \mathbb{N}$, be dense open sets in $\mathrm{C}(X,Y)$. To see that $\bigcap_{n\in\mathbb{N}} G_n$ is dense in $\mathrm{C}(X,Y)$, let $f \in \mathrm{C}(X,Y)$ and $d \in \mathrm{Metr}^c(Y)$. We can inductively choose $g_n \in \mathrm{C}(X,Y)$ and $d_n \in \mathrm{Metr}(Y)$, $n \in \mathbb{N}$, so that

$$g_n \in U_{2d_{n-1}}(g_{n-1}) \cap G_n,\ U_{d_n}(g_n) \subset G_n \text{ and } d_n \geq 2d_{n-1},$$

where $g_0 = f$ and $d_0 = d$. Observe $d_m \leq 2^{-n} d_{m+n}$ for each $m, n \in \omega$. Since $d(g_{n-1}, g_n) \leq 2^{-n+1} d_{n-1}(g_{n-1}, g_n) < 2^{-n}$ for each $n \in \mathbb{N}$, $(g_n)_{n\in\mathbb{N}}$ is d-Cauchy. From the completeness of d, $(g_n)_{n\in\mathbb{N}}$ converges uniformly to $g \in \mathrm{C}(X,Y)$ with respect to d. Since

$$d(f,g) \leq \sum_{n\in\mathbb{N}} d(g_{n-1}, g_n) < \sum_{n\in\mathbb{N}} 2^{-n} = 1,$$

we have $g \in U_d(f)$ and, for each $n \in \mathbb{N}$,

$$\begin{aligned} d_n(g_n, g) &\leq \sum_{i\in\mathbb{N}} d_n(g_{n+i-1}, g_{n+i}) \\ &\leq \sum_{i\in\mathbb{N}} 2^{-i+1} d_{n+i-1}(g_{n+i-1}, g_{n+i}) < \sum_{i\in\mathbb{N}} 2^{-i} = 1, \end{aligned}$$

hence $g \in U_{d_n}(g_n) \subset G_n$. Thus, $U_d(f) \cap \bigcap_{n\in\mathbb{N}} G_n \neq \emptyset$, hence $\bigcap_{n\in\mathbb{N}} G_n$ is dense in $\mathrm{C}(X,Y)$. □

In the above proof, replace $\mathrm{C}(X,Y)$ and U_{d_n} with $\mathrm{Homeo}(X)$ and $U_{d_n^*}$, respectively. Then, we can see that $(g_n)_{n\in\mathbb{N}}$ is d^*-Cauchy. From the completeness of d^*, we have $g \in \mathrm{Homeo}(X)$ with $\lim_{n\to\infty} d^*(g_n, g) = 0$. By the same calculation, we can see $d_n^*(g_n, g) < 1$, that is, $g \in U_{d_n^*}(g) \subset G_n$ for every $n \in \mathbb{N}$. Then, $U_{d^*}(f) \cap \bigcap_{n\in\mathbb{N}} G_n \neq \emptyset$. Therefore, we have:

Theorem 2.9.5. *For a completely metrizable space X, the space* $\mathrm{Homeo}(X)$ *with the limitation topology is a Baire space.* □

Now, we consider the space of proper maps.

Proposition 2.9.6. *Let $\mathcal{U}$ be a locally finite open cover of Y such that* $\mathrm{cl}\, U$ *is compact for every $U \in \mathcal{U}$ (so Y is locally compact). If a map $f : X \to Y$ is $\mathcal{U}$-close to a proper map g then f is also proper.*

Proof. For each compact set A in Y, $f^{-1}(A) \subset g^{-1}(\mathrm{st}(A, \mathcal{U}^{\mathrm{cl}}))$. Since $\mathcal{U}^{\mathrm{cl}}$ is locally finite, it follows that $\mathcal{U}^{\mathrm{cl}}[A]$ is finite, and hence $\mathrm{st}(A, \mathcal{U}^{\mathrm{cl}})$ is compact. Then, $g^{-1}(\mathrm{st}(A, \mathcal{U}^{\mathrm{cl}}))$ is compact because g is proper. Thus, its closed subset $f^{-1}(A)$ is also compact. □

Let $\mathrm{C}^P(X,Y)$ be the subspace of $\mathrm{C}(X,Y)$ consisting of all proper maps.[6] Then, Proposition 2.9.6 yields the following corollary:

Corollary 2.9.7. *If Y is locally compact and paracompact, then $\mathrm{C}^P(X,Y)$ is clopen (i.e., closed and open) in the space $\mathrm{C}(X,Y)$ with the limitation topology, where X is also locally compact if $\mathrm{C}^P(X,Y) \neq \emptyset$.* □

From Theorem 2.9.4 and Corollary 2.9.7, we have:

Theorem 2.9.8. *For every pair of locally compact metrizable spaces X and Y, the space $\mathrm{C}^P(X,Y)$ with the limitation topology is a Baire space.* □

Some Properties of the Limitation Topology 2.9.9.

(1) For each paracompact space Y, the evaluation map

$$\mathrm{ev} : X \times \mathrm{C}(X,Y) \ni (x,f) \mapsto f(x) \in Y$$

is continuous with respect to the limitation topology.

Sketch of Proof. For each $(x,f) \in X \times \mathrm{C}(X,Y)$ and each open neighborhood V of $f(x)$ in Y, take an open neighborhood W of $f(x)$ in Y so that $\mathrm{cl}\, W \subset V$ and let $\mathcal{V} = \{V, X \setminus \mathrm{cl}\, W\} \in \mathrm{cov}(Y)$. Show that $(x',f') \in f^{-1}(W) \times \mathcal{V}(f)$ implies $f'(x') \in V$.

(2) If both Y and Z are paracompact, the composition

$$\mathrm{C}(X,Y) \times \mathrm{C}(Y,Z) \ni (f,g) \mapsto g \circ f \in \mathrm{C}(X,Z)$$

is continuous with respect to the limitation topology.

Sketch of Proof. For each $(f,g) \in \mathrm{C}(X,Y) \times \mathrm{C}(Y,Z)$ and $\mathcal{U} \in \mathrm{cov}(Z)$, let $\mathcal{V} \in \mathrm{cov}(Z)$ be a star-refinement of $\mathcal{U}$. Show that $f' \in g^{-1}(\mathcal{V})(f)$ and $g' \in \mathcal{V}(g)$ implies $g' \circ f' \in \mathcal{U}(g \circ f)$.

(3) For every paracompact space X, the inverse operation

$$\mathrm{Homeo}(X) \ni h \mapsto h^{-1} \in \mathrm{Homeo}(X)$$

is continuous with respect to the limitation topology. Combining this with (1), the group $\mathrm{Homeo}(X)$ with the limitation topology is a topological group.

Sketch of Proof. Let $h \in \mathrm{Homeo}(X)$ and $\mathcal{U} \in \mathrm{cov}(X)$. Show that $g \in h(\mathcal{U})(h)$ implies $g^{-1} \in \mathcal{U}(h^{-1})$.

Remark 12. If $Y = (Y,d)$ is a metric space, for each $f \in \mathrm{C}(X,Y)$ and $\gamma \in \mathrm{C}(X,(0,\infty))$, let

$$V_\gamma(f) = \{g \in \mathrm{C}(X,Y) \mid \forall x \in X,\ d(f(x),g(x)) < \gamma(x)\}.$$

[6] If Y is locally compact, $\mathrm{C}^P(X,Y)$ is the subspace of $\mathrm{C}(X,Y)$ consisting of all perfect maps (Proposition 2.1.5).

We have the topology of $\mathrm{C}(X, Y)$ such that $\{V_\gamma(f) \mid \gamma \in \mathrm{C}(X, (0, \infty))\}$ is a neighborhood basis of f. This is finer than the limitation topology. In general, these topologies are not equal.

For example, let $\gamma \in \mathrm{C}(\mathbb{N}, (0, \infty))$ be the map defined by $\gamma(n) = 2^{-n}$ for $n \in \mathbb{N}$. Then, $V_\gamma(\mathbf{0})$ is not a neighborhood of $\mathbf{0} \in \mathrm{C}(\mathbb{N}, \mathbb{R})$ in the limitation topology. Indeed, for any $\alpha \in \mathrm{C}(\mathbb{R}, (0, \infty))$, we define $g \in \mathrm{C}(\mathbb{N}, \mathbb{R})$ by $g(n) = \frac{1}{2}\alpha(0)$ for every $n \in \mathbb{N}$. Then, $g \in N_\alpha(\mathbf{0})$ but $g \notin V_\gamma(\mathbf{0})$. Thus, $N_\alpha(\mathbf{0}) \not\subset V_\gamma(\mathbf{0})$. Moreover, the composition

$$\mathrm{C}(\mathbb{N}, \mathbb{R}) \times \mathrm{C}(\mathbb{R}, \mathbb{R}) \ni (f, g) \mapsto g \circ f \in \mathrm{C}(\mathbb{N}, \mathbb{R})$$

is not continuous with respect to this topology.

Indeed, let γ be the above map. For any $\alpha \in \mathrm{C}(\mathbb{R}, (0, \infty))$, we have $n \in \mathbb{N}$ such that $2^{-n} < \frac{1}{2}\alpha(0)$. Let $h = \mathrm{id} + \frac{1}{2}\alpha \in \mathrm{C}(\mathbb{R}, \mathbb{R})$. Then, $h \in V_\alpha(\mathrm{id})$ but $h \circ \mathbf{0} \notin V_\gamma(\mathrm{id} \circ \mathbf{0})$ because $h \circ \mathbf{0}(n) = h(0) = \frac{1}{2}\alpha(0) > 2^{-n} = \gamma(n)$. (Here, id can be replaced by any $g \in \mathrm{C}(\mathbb{R}, \mathbb{R})$.)

2.10 Counter-Examples

In this section, we show that the concepts of normality, collectionwise normality, and paracompactness are neither hereditary nor productive, and that the concepts of perfect normality and hereditary normality are not productive either. Moreover, we show that the direct limit of a closed tower of Hausdorff spaces need not be Hausdorff.

The following example shows that *the concepts of normality, collectionwise normality and paracompactness are not hereditary.*

The Tychonoff plank 2.10.1. *Let $[0, \omega_1)$ be the space of all countable ordinals with the order topology. The space $[0, \omega_1]$ is the one-point compactification of the space $[0, \omega_1)$. Let $[0, \omega]$ be the one-point compactification of the space $\omega = [0, \omega)$ of non-negative integers. The product space $[0, \omega_1] \times [0, \omega]$ is a compact Hausdorff space, hence it is paracompact. The following dense subspace of $[0, \omega_1] \times [0, \omega]$ is called the* ***Tychonoff plank****:*

$$T = [0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}.$$

We now prove that

– *The Tychonoff plank T is not normal.*

Proof. We have disjoint closed sets $\{\omega_1\} \times [0, \omega)$ and $[0, \omega_1) \times \{\omega\}$ in T. Assume that T has disjoint open sets U, V such that $\{\omega_1\} \times [0, \omega) \subset U$ and $[0, \omega_1) \times \{\omega\} \subset V$. For each $n \in \omega$, choose $\alpha_n < \omega_1$ so that $[\alpha_n, \omega_1] \times \{n\} \subset U$. Let $\alpha = \sup_{n \in \mathbb{N}} \alpha_n <$

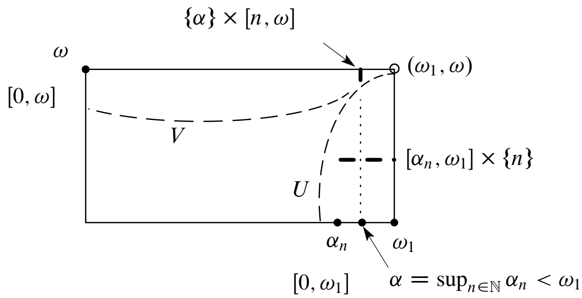

Fig. 2.10 Tychonoff plank

ω_1. Then, $[\alpha, \omega_1] \times \mathbb{N} \subset U$. On the other hand, we can choose $n \in \mathbb{N}$ so that $\{\alpha\} \times [n, \omega] \subset V$. Then, $U \cap V \neq \emptyset$, which is a contradiction (Fig. 2.10). □

The next example shows that *the concepts of normality, perfect normality, hereditary normality, collectionwise normality, and paracompactness are not productive.*

The Sorgenfrey Line 2.10.2. *The* ***Sorgenfrey line*** S *is the space* $\mathbb{R}$ *with the topology generated by* $[a, b)$, $a < b$. *The product* S^2 *is called the* ***Sorgenfrey plane****. These spaces have the following properties:*

(1) S *is a separable regular Lindelöf space, hence it is paracompact, and so is collectionwise normal;*
(2) S *is perfectly normal, and so is hereditarily normal;*
(3) S^2 *is not normal.*

Proof. (1): It is obvious that S is Hausdorff. Since each basic open set $[a, b)$ is also closed in S, it follows that S is regular. Clearly, $\mathbb{Q}$ is dense in S, hence S is separable. To see that S is Lindelöf, let $\mathcal{U} \in \mathrm{cov}(S)$. We have a function $\gamma : S \to \mathbb{Q}$ so that $\gamma(x) > x$ and $[x, \gamma(x)) \subset U$ for some $U \in \mathcal{U}$. Then, $\{[x, \gamma(x)) \mid x \in S\} \in \mathrm{cov}(S)$ is an open refinement of $\mathcal{U}$. For each $q \in \gamma(S)$, if there exists $\min \gamma^{-1}(q)$, let $R(q) = \{\min \gamma^{-1}(q)\}$. Otherwise, choose a countable subset $R(q) \subset \gamma^{-1}(q)$ so that $\inf R(q) = \inf \gamma^{-1}(q)$, where we mean $\gamma^{-1}(q) = -\infty$ if $\gamma^{-1}(q)$ is unbounded below. Then, the following is a subcover of $\{[x, \gamma(x)) \mid x \in S\} \in \mathrm{cov}(S)$:

$$\big\{[z, q) \bigm| q \in \gamma(S),\ z \in R(q)\big\} \in \mathrm{cov}(S),$$

which is a countable open refinement of $\mathcal{U}$.

(2): Let U be an open set in S. We have a function $\gamma : U \to \mathbb{Q}$ so that $\gamma(x) > x$ and $[x, \gamma(x)) \subset U$. Then, $U = \bigcup_{x \in U} [x, \gamma(x))$. By the same argument as the proof of (1), we can find a countable subcollection

$$\big\{[a_i, b_i) \bigm| i \in \mathbb{N}\big\} \subset \big\{[x, \gamma(x)) \bigm| x \in U\big\}$$

such that $U = \bigcup_{i \in \mathbb{N}} [a_i, b_i)$, hence U is F_σ in S. Thus, S is perfectly normal.

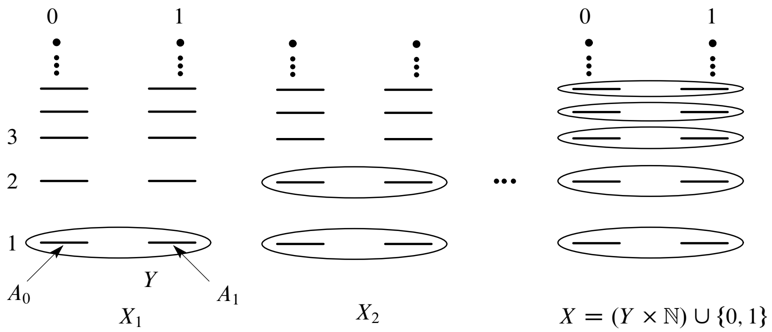

Fig. 2.11 Non-Hausdorff direct limit

(3): As we saw in the proof of (1), $\mathbb{Q}$ is dense in S, hence $\mathbb{Q}^2$ is dense in S^2. It follows that the restriction $\mathrm{C}(S^2, \mathbb{R}) \ni f \mapsto f|\mathbb{Q}^2 \in \mathbb{R}^{\mathbb{Q}^2}$ is injective. Therefore,

$$\operatorname{card} \mathrm{C}(S^2, \mathbb{R}) \leq \operatorname{card} \mathbb{R}^{\mathbb{Q}^2} = 2^{\aleph_0} = \mathfrak{c}.$$

On the other hand, $D = \{(x, y) \in S^2 \mid x + y = 0\}$ is a discrete set in S^2. Then, we have

$$\operatorname{card} \mathrm{C}(D, \mathbb{R}) = \operatorname{card} \mathbb{R}^D = 2^{\mathfrak{c}} > \mathfrak{c} \geq \operatorname{card} \mathrm{C}(S^2, \mathbb{R}).$$

If S^2 is normal, it would follow from the Tietze Extension Theorem 2.2.2 that the restriction $\mathrm{C}(S^2, \mathbb{R}) \ni f \mapsto f|D \in \mathrm{C}(D, \mathbb{R})$ is surjective, which is a contradiction. Consequently, S^2 is not normal. $\square$

Finally, we will construct a closed tower such that the direct limit is *not Hausdorff*.

A Non-Hausdorff Direct Limit 2.10.3. *Let Y be a space which is Hausdorff but non-normal, such as the Tychonoff plank. Let A_0, A_1 be disjoint closed sets in Y that have no disjoint neighborhoods. We define $X = (Y \times \mathbb{N}) \cup \{0, 1\}$ with the topology generated by open sets in the product space $Y \times \mathbb{N}$ and sets of the form*

$$\bigcup_{k>n} (U_k \times \{k\}) \cup \{i\},$$

where $i = 0, 1$ and each U_k is an open neighborhood of A_i. Then, X is not Hausdorff because 0 and 1 have no disjoint neighborhoods in X. For each $n \in \mathbb{N}$, let

$$X_n = Y \times \{1, \ldots, n\} \cup (A_0 \cup A_1) \times \{k \mid k > n\} \cup \{0, 1\}.$$

Then, $X_1 \subset X_2 \subset \cdots$ are closed in X and $X = \bigcup_{n \in \mathbb{N}} X_n$ (Fig. 2.11). As is easily observed, every X_n is Hausdorff. We will prove that $X = \varinjlim X_n$, that is,

– *X has the weak topology with respect to the tower $(X_n)_{n \in \mathbb{N}}$.*

Proof. Since $\mathrm{id} : \varinjlim X_n \to X$ is obviously continuous, it suffices to show that every open set V in $\varinjlim X_n$ is open in X. To this end, assume that $V \cap X_n$ is open in X_n for each $n \in \mathbb{N}$. Each $x \in V \setminus \{0, 1\}$ is contained in some $Y \times \{n\} \subset X_n$. Then, $V \cap (Y \times \{n\})$ is an open neighborhood of x in $Y \times \{n\}$, and so is an open neighborhood in X. When $0 \in V$, $A_0 \times \{k \mid k > n\} \subset V$ for some $n \in \mathbb{N}$ because $V \cap X_1$ is open in X_1. For each $k > n$, since $V \cap (Y \times \{k\})$ is open in $Y \times \{k\}$, there is an open set U_k in Y such that $V \cap (Y \times \{k\}) = U_k \times \{k\}$. Note that $A_0 \subset U_k$. Then, $\bigcup_{k>n}(U_k \times \{k\}) \cup \{0\} \subset V$, hence V is a neighborhood of 0 in X. Similarly, V is a neighborhood of 1 in X if $1 \in V$. Thus, V is open in X. □

Notes for Chap. 2

For more comprehensive studies on General Topology, see Engelking's book, which contains excellent historical and bibliographic notes at the end of each section.

- R. Engelking, *General Topology, Revised and complete edition*, Sigma Ser. in Pure Math. **6** (Heldermann Verlag, Berlin, 1989)

The following classical books are still good sources.

- J. Dugundji, *Topology*, (Allyn and Bacon, Inc., Boston, 1966)
- J.L. Kelly, *General Topology*, GTM **27** (Springer-Verlag, Berlin, 1975); Reprint of the 1955 ed. published by Van Nostrand

For counter-examples, the following is a good reference:

- L.A. Steen and J.A. Seebach, Jr., *Counterexamples in Topology, 2nd edition* (Springer-Verlag, New York, 1978)

Of the more recent publications, the following textbook is readable and seems to be popular:

- J.R. Munkres, *Topology, 2nd edition* (Prentice Hall, Inc., Upper Saddle River, 2000)

Most of the contents discussed in the present chapter are found in Chaps. 5–8 of this text, although it does not discuss the Frink Metrization Theorem (cf. 2.4.1) and Michael's Theorem 2.6.5 on local properties.

Among various proofs of the Tychonoff Theorem 2.1.1, our proof is a modification of the proof due to Wright [19]. Our proof of the Tietze Extension Theorem 2.2.2 is due to Scott [14]. Theorem 2.3.1 was established by Stone [16], but the proof presented here is due to Rudin [13]. The Nagata–Smirnov Metrization Theorem (cf. 2.3.4) was independently proved by Nagata [12] and Smirnov [15]. The Bing Metrization Theorem (cf. metrization) was proved in [2]. The Urysohn Metrization Theorem 2.3.5 and the Alexandroff–Urysohn Metrization Theorem (cf. 2.4.1) were established in [18] and [1], respectively. The Frink Metrization Theorem (cf. 2nd-metrization) was proved by Frink [5]. The Baire Category Theorem 2.5.1 was first proved by Hausdorff [6] (Baire proved the theorem for the real line in 1889). The equivalence of (a) and (b) in Theorem 2.5.5 was shown by Čech [3]. Theorems 2.5.7 and 2.5.8 were established by Lavrentieff [7].

The concept of paracompactness was introduced by Dieudonné [4]. In [2], Bing introduced the concept of collectionwise normality and showed the collectionwise normality of paracompact spaces (Theorem 2.6.1). The equivalence of (b) and (c) in Theorem 2.6.3 was proved by Tukey [17], where he called spaces satisfying condition (c) *fully normal spaces*. The equivalence of (a) and (c) and the equivalence of (a), (d), and (e) were respectively proved by Stone [16] and Michael [10]. Theorem 2.6.5 on local properties was established by Michael [11]. Lemma 2.7.1 appeared in [8]. Theorem 2.7.2 and Proposition 2.7.4 were also established by Michael [11]. The simple proof of Proposition 2.7.4 presented here is due to Mather [9]. Theorem 2.7.6 was proved by Dieudonné [4]. These notes are based on historical and bibliographic notes in Engelking's book, listed above.

In some literature, it is mentioned that the direct limit of a closed tower of Hausdorff spaces need not be Hausdorff. The author could not find such an example in the literature. Example 2.10.3 is due to H. Ohta.

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