Chapter 1
Preliminaries

The reader should have finished a first course in Set Theory and General Topology; basic knowledge of Linear Algebra is also a prerequisite. In this chapter, we introduce some terminology and notation. Additionally, we explain the concept of Banach spaces contained in the product of real lines.

1.1 Terminology and Notation

For the standard sets, we use the following notation:

- $\mathbb{N}$ — the set of natural numbers (i.e., positive integers);
- $\omega = \mathbb{N} \cup \{0\}$ — the set of non-negative integers;
- $\mathbb{Z}$ — the set of integers;
- $\mathbb{Q}$ — the set of rationals;
- $\mathbb{R} = (-\infty, \infty)$ — the real line with the usual topology;
- $\mathbb{C}$ — the complex plane;
- $\mathbb{R}_+ = [0, \infty)$;
- $\mathbf{I} = [0, 1]$ — the unit closed interval.

A (topological) **space** is assumed to be **Hausdorff** and a **map** is a **continuous** function. A **singleton** is a space consisting of one point, which is also said to be **degenerate**. A space is said to be **non-degenerate** if it is not a singleton. Let X be a space and $A \subset X$. We denote

- $\mathrm{cl}_X A$ (or $\mathrm{cl}\, A$) — the closure of A in X;
- $\mathrm{int}_X A$ (or $\mathrm{int}\, A$) — the interior of A in X;
- $\mathrm{bd}_X A$ (or $\mathrm{bd}\, A$) — the boundary of A in X;
- id_X (or id) — the identity map of X.

For spaces X and Y,

- $X \approx Y$ means that X and Y are homeomorphic.

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Given subspaces $X_1, \dots, X_n \subset X$ and $Y_1, \dots, Y_n \subset Y$,

- $(X, X_1, \dots, X_n) \approx (Y, Y_1, \dots, Y_n)$ means that there exists a homeomorphism $h : X \to Y$ such that $h(X_1) = Y_1, \dots, h(X_n) = Y_n$;
- $(X, x_0) \approx (Y, y_0)$ means $(X, \{x_0\}) \approx (Y, \{y_0\})$.

We call (X, x_0) a **pointed space** and x_0 its **base point**.

For a set Γ, the cardinality of Γ is denoted by $\operatorname{card}\Gamma$. The **weight** $w(X)$, the **density** $\operatorname{dens} X$, and the **cellularity** $c(X)$ of a space X are defined as follows:

- $w(X) = \min\{\operatorname{card}\mathcal{B} \mid \mathcal{B}$ is an open basis for $X\}$;
- $\operatorname{dens} X = \min\{\operatorname{card} D \mid D$ is a dense set in $X\}$;
- $c(X) = \sup\{\operatorname{card}\mathcal{G} \mid \mathcal{G}$ is a pair-wise disjoint open collection$\}$.

As is easily observed, $c(X) \leq \operatorname{dens} X \leq w(X)$ in general. If X is metrizable, all these cardinalities coincide.

Indeed, let D be a dense set in X with $\operatorname{card} D = \operatorname{dens} X$, and $\mathcal{G}$ be a pairwise disjoint collection of non-empty open sets in X. Since each $G \in \mathcal{G}$ meets D, we have an injection $g : \mathcal{G} \to D$, hence $\operatorname{card}\mathcal{G} \leq \operatorname{card} D = \operatorname{dens} X$. It follows that $c(X) \leq \operatorname{dens} X$. Now, let $\mathcal{B}$ be an open basis for X with $\operatorname{card}\mathcal{B} = w(X)$. By taking any point $x_B \in B$ from each $B \in \mathcal{B}$, we have a dense set $\{x_B \mid B \in \mathcal{B}\}$ in X, which implies $\operatorname{dens} X \leq w(X)$.

When X is metrizable, we show the converse inequality. The case $\operatorname{card} X < \aleph_0$ is trivial. We may assume that $X = (X, d)$ is a metric space with $\operatorname{diam} X \geq 1$ and $\operatorname{card} X \geq \aleph_0$. Let D be a dense set in X with $\operatorname{card} D = \operatorname{dens} X$. Then, $\{\mathrm{B}(x, 1/n) \mid x \in D,\ n \in \mathbb{N}\}$ is an open basis for X, which implies $w(X) \leq \operatorname{dens} X$. For each $n \in \mathbb{N}$, using Zorn's Lemma, we can find a maximal 2^{-n}-discrete subset $X_n \subset X$, i.e., $d(x, y) \geq 2^{-n}$ for every pair of distinct points $x, y \in X_n$. Then, $\mathcal{G}_n = \{\mathrm{B}(x, 2^{-n-1}) \mid x \in X_n\}$ is a pairwise disjoint open collection, and hence we have $\operatorname{card} X_n = \operatorname{card}\mathcal{G}_n \leq c(X)$. Observe that $X_* = \bigcup_{n\in\mathbb{N}} X_n$ is dense in X, which implies $\sup_{n\in\mathbb{N}} \operatorname{card} X_n = \operatorname{card} X_* \geq \operatorname{dens} X$. Therefore, $c(X) \geq \operatorname{dens} X$.

For the product space $\prod_{\gamma\in\Gamma} X_\gamma$, the γ-coordinate of each point $x \in \prod_{\gamma\in\Gamma} X_\gamma$ is denoted by $x(\gamma)$, i.e., $x = (x(\gamma))_{\gamma\in\Gamma}$. For each $\gamma \in \Gamma$, the projection $\mathrm{pr}_\gamma : \prod_{\gamma\in\Gamma} X_\gamma \to X_\gamma$ is defined by $\mathrm{pr}_\gamma(x) = x(\gamma)$. For $\Lambda \subset \Gamma$, the projection $\mathrm{pr}_\Lambda : \prod_{\gamma\in\Gamma} X_\gamma \to \prod_{\lambda\in\Lambda} X_\lambda$ is defined by $\mathrm{pr}_\Lambda(x) = x|\Lambda\ (= (x(\lambda))_{\lambda\in\Lambda})$. In the case that $X_\gamma = X$ for every $\gamma \in \Gamma$, we write $\prod_{\gamma\in\Gamma} X_\gamma = X^\Gamma$. In particular, $X^{\mathbb{N}}$ is the product space of countable infinite copies of X. When $\Gamma = \{1, \dots, n\}$, $X^\Gamma = X^n$ is the product space of n copies of X. For the product space $X \times Y$, we denote the projections by $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$.

A compact metrizable space is called a **compactum** and a connected compactum is called a **continuum**.[1] For a metrizable space X, we denote

- $\mathrm{Metr}(X)$ — the set of all admissible metrics of X.

Now, let $X = (X, d)$ be a metric space, $x \in X$, $\varepsilon > 0$, and $A, B \subset X$. We use the following notation:

[1] Their plurals are **compacta** and **continua**, respectively.

- $\mathrm{B}_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$ — the ε-neighborhood of x in X (or the open ball with center x and radius ε);
- $\overline{\mathrm{B}}_d(x,\varepsilon) = \{y \in X \mid d(x,y) \leq \varepsilon\}$ — the closed ε-neighborhood of x in X (or the closed ball with center x and radius ε);
- $\mathrm{N}_d(A,\varepsilon) = \bigcup_{x \in A} \mathrm{B}_d(x,\varepsilon)$ — the ε-neighborhood of A in X;
- $\operatorname{diam}_d A = \sup\{d(x,y) \mid x,y \in A\}$ — the diameter of A;
- $d(x,A) = \inf\{d(x,y) \mid y \in A\}$ — the distance of x from A;
- $\operatorname{dist}_d(A,B) = \inf\{d(x,y) \mid x \in A,\ y \in B\}$ — the distance of A and B.

It should be noted that $\mathrm{N}_d(\{x\},\varepsilon) = \mathrm{B}_d(x,\varepsilon)$ and $d(x,A) = \operatorname{dist}_d(\{x\},A)$. For a collection $\mathcal{A}$ of subsets of X, let

- $\operatorname{mesh}_d \mathcal{A} = \sup\{\operatorname{diam}_d A \mid A \in \mathcal{A}\}$ — the mesh of $\mathcal{A}$.

If there is no possibility of confusion, we can drop the subscript d and write $\mathrm{B}(x,\varepsilon)$, $\overline{\mathrm{B}}(x,\varepsilon)$, $\mathrm{N}(A,\varepsilon)$, $\operatorname{diam} A$, $\operatorname{dist}(A,B)$, and $\operatorname{mesh} \mathcal{A}$.

The standard spaces are listed below:

- $\mathbb{R}^n$ — the n-dimensional Euclidean space with the norm

$$\|x\| = \sqrt{x(1)^2 + \cdots + x(n)^2},$$

  $\mathbf{0} = (0,\ldots,0) \in \mathbb{R}^n$ — the origin, the zero vector or the zero element,
  $\mathbf{e}_i \in \mathbb{R}^n$ — the unit vector defined by $\mathbf{e}_i(i) = 1$ and $\mathbf{e}_i(j) = 0$ for $j \neq i$;
- $\mathbf{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ — the unit $(n-1)$-sphere;
- $\mathbf{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ — the unit closed n-ball;
- $\Delta^n = \{x \in (\mathbb{R}_+)^{n+1} \mid \sum_{i=1}^{n+1} x(i) = 1\}$ — the standard n-simplex;
- $\boldsymbol{Q} = [-1,1]^{\mathbb{N}}$ — the Hilbert cube;
- $\boldsymbol{s} = \mathbb{R}^{\mathbb{N}}$ — the space of sequences;
- $\mu^0 = \{\sum_{i=1}^{\infty} 2x_i/3^i \mid x_i \in \{0,1\}\}$ — the Cantor (ternary) set;
- $\nu^0 = \mathbb{R} \setminus \mathbb{Q}$ — the space of irrationals;
- $\mathbf{2} = \{0,1\}$ — the discrete space of two points.

Note that $\mathbf{S}^{n-1}$, $\mathbf{B}^n$, and Δ^n are not product spaces, even though the same notations are used for product spaces. The indexes $n-1$ and n represent their dimensions (the indexes of μ^0 and ν^0 are identical).

As is well-known, the countable product $\mathbf{2}^{\mathbb{N}}$ of the discrete space $\mathbf{2} = \{0,1\}$ is homeomorphic to the Cantor set μ^0 by the correspondence:

$$x \mapsto \sum_{i \in \mathbb{N}} \frac{2x(i)}{3^i}.$$

On the other hand, the countable product $\mathbb{N}^{\mathbb{N}}$ of the discrete space $\mathbb{N}$ of natural numbers is homeomorphic to the space ν^0 of irrationals. In fact, $\mathbb{N}^{\mathbb{N}} \approx (0,1) \setminus \mathbb{Q} \approx (-1,1) \setminus \mathbb{Q} \approx \nu^0$. These three homeomorphisms are given as follows:

$$x \mapsto \cfrac{1}{x(1) + \cfrac{1}{x(2) + \cfrac{1}{x(3) + \cfrac{1}{\ddots}}}}; \quad t \mapsto 2t - 1; \quad s \mapsto \frac{s}{1 - |s|}.$$

That the first correspondence is a homeomorphism can be verified as follows: for each $n \in \mathbb{N}$, let $a_n : \mathbb{N}^{\mathbb{N}} \to \mathbf{I}$ be a map defined by

$$a_n(x) = \cfrac{1}{x(1) + \cfrac{1}{x(2) + \cfrac{1}{\ddots + \cfrac{1}{x(n)}}}}.$$

Then, $0 < a_2(x) < a_4(x) < \cdots < a_3(x) < a_1(x) \leq 1$. Using the fact shown below, we can conclude that the first correspondence $\mathbb{N}^{\mathbb{N}} \ni x \mapsto \alpha(x) = \lim_{n\to\infty} a_n(x) \in (0, 1)$ is well-defined and continuous.

Fact. *For every $m > n$, $|a_n(x) - a_m(x)| < (n + 1)^{-1}$.*

This fact can be shown by induction on $n \in \mathbb{N}$. First, observe that

$$|a_1(x) - a_2(x)| = \frac{1}{x(1)(x(1)x(2) + 1)} < 1/2,$$

which implies the case $n = 1$. When $n > 1$, for each $x \in \mathbb{N}^{\mathbb{N}}$, define $x^* \in \mathbb{N}^{\mathbb{N}}$ by $x^*(i) = x(i + 1)$. By the inductive assumption, $|a_{n-1}(x^*) - a_{m-1}(x^*)| < n^{-1}$ for $m > n$, which gives us

$$\begin{aligned} |a_n(x) - a_m(x)| &= \frac{|a_{n-1}(x^*) - a_{m-1}(x^*)|}{(x(1) + a_{n-1}(x^*))(x(1) + a_{m-1}(x^*))} \\ &\leq \frac{|a_{n-1}(x^*) - a_{m-1}(x^*)|}{(1 + a_{n-1}(x^*))(1 + a_{m-1}(x^*))} \\ &< \frac{|a_{n-1}(x^*) - a_{m-1}(x^*)|}{1 + |a_{n-1}(x^*) - a_{m-1}(x^*)|} \\ &\leq \frac{n^{-1}}{1 + n^{-1}} = \frac{1}{n+1}. \end{aligned}$$

Let $t = q_1/q_0 \in (0, 1) \cap \mathbb{Q}$, where $q_1 < q_0 \in \mathbb{N}$. Since $q_0/q_1 = t^{-1} > 1$, we can choose $x_1 \in \mathbb{N}$ so that $x_1 \leq q_0/q_1 < x_1 + 1$. Then, $1/(x_1 + 1) < t \leq 1/x_1$. If $t \neq 1/x_1$, then $x_1 < q_0/q_1$, and hence $t^{-1} = q_0/q_1 = x_1 + q_2/q_1$ for some $q_2 \in \mathbb{N}$ with $q_2 < q_1$. Now, we choose $x_2 \in \mathbb{N}$ so that $x_2 \leq q_1/q_2 < x_2 + 1$. Thus, $x_1 + 1/(x_2 + 1) < x_1 + q_2/q_1 \leq x_1 + 1/x_2$, so $1/(x_1 + 1/x_2) \leq t < 1/(x_1 + 1/(x_2 + 1))$. If $t \neq 1/(x_1 + 1/x_2)$, then $x_2 < q_1/q_2$. Similarly, we write $q_1/q_2 = x_2 + q_3/q_2$, where $q_3 \in \mathbb{N}$ with $q_3 < q_2$ $(< q_1)$, and choose $x_3 \in \mathbb{N}$ so that $x_3 \leq q_2/q_3 < x_3 + 1$. Then, $1/(x_1 + 1/(x_2 + 1/x_3)) \leq t < 1/(x_1 + 1/(x_2 + 1/(x_3 + 1)))$. This process has only a finite number of steps (at most q_1 steps). Thus, we have the following unique representation:

$$t = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots + \cfrac{1}{x_n}}}}, \quad x_1, \dots, x_n \in \mathbb{N}.$$

It follows that $\alpha(\mathbb{N}^{\mathbb{N}}) \subset (0,1) \setminus \mathbb{Q}$.

For each $t \in (0,1) \setminus \mathbb{Q}$, choose $x_1 \in \mathbb{N}$ so that $x_1 < t^{-1} < x_1 + 1$. Then, $1/(x_1+1) < t < 1/x_1$ and $t^{-1} = x_1 + t_1$ for some $t_1 \in (0,1) \setminus \mathbb{Q}$. Next, choose $x_2 \in \mathbb{N}$ so that $x_2 < t_1^{-1} < x_2 + 1$. Thus, $x_1 + 1/(x_2+1) < x_1 + t_1 < x_1 + 1/x_2$, and so $1/(x_1 + 1/x_2) < t < 1/(x_1 + 1/(x_2+1))$. Again, write $t_1^{-1} = x_2 + t_2$, $t_2 \in (0,1) \setminus \mathbb{Q}$, and choose $x_3 \in \mathbb{N}$ so that $x_3 < t_2^{-1} < x_3 + 1$. Then, $1/(x_1 + 1/(x_2 + 1/(x_3+1))) < t < 1/(x_1 + 1/(x_2 + 1/x_3))$. We can iterate this process infinitely many times. Thus, there is the unique $x = (x_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $a_{2n}(x) < t < a_{2n+1}(x)$ for each $n \in \mathbb{N}$, where $\alpha(x) = \lim_{n \to \infty} a_n(x) = t$. Therefore, $\alpha : \mathbb{N}^{\mathbb{N}} \to (0,1) \setminus \mathbb{Q}$ is a bijection.

In the above, let $a_{2n}(x) < s < a_{2n-1}(x)$ and define $y = (y_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ for this s similar to x for t. Then, $\alpha(y) = s$ and $x_i = y_i$ for each $i \leq 2n-1$, i.e., the first $2n-1$ coordinates of x and y are all the same. This means that α^{-1} is continuous.

Let $f : A \to Y$ be a map from a closed set A in a space X to another space Y. The **adjunction space** $Y \cup_f X$ is the quotient space $(X \oplus Y)/\sim$, where $X \oplus Y$ is the topological sum and $\sim$ is the equivalence relation corresponding to the decomposition of $X \oplus Y$ into singletons $\{x\}$, $x \in X \setminus A$, and sets $\{y\} \cup f^{-1}(y)$, $y \in Y$ (the latter is a singleton $\{y\}$ if $y \in Y \setminus f(A)$). In the case that Y is a singleton, $Y \cup_f X \approx X/A$. One should note that, in general, the adjunction spaces are *not Hausdorff*. Some further conditions are necessary for the adjunction space to be Hausdorff.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X and $Y \subset X$. We define

- $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$;
- $\mathcal{A}|Y = \{A \cap Y \mid A \in \mathcal{A}\}$;
- $\mathcal{A}[Y] = \{A \in \mathcal{A} \mid A \cap Y \neq \emptyset\}$.

When each $A \in \mathcal{A}$ is contained in some $B \in \mathcal{B}$, it is said that $\mathcal{A}$ **refines** $\mathcal{B}$ and denoted by:

$$\mathcal{A} \prec \mathcal{B} \text{ or } \mathcal{B} \succ \mathcal{A}.$$

It is said that $\mathcal{A}$ **covers** Y (or $\mathcal{A}$ is a **cover** of Y in X) if $Y \subset \bigcup \mathcal{A}$ ($= \bigcup_{A \in \mathcal{A}} A$). When $Y = X$, a cover of Y in X is simply called a cover of X. A cover of Y in X is said to be **open** or **closed** in X depending on whether its members are open or closed in X. If $\mathcal{A}$ is an open cover of X then $\mathcal{A}|Y$ is an open cover of Y and $\mathcal{A}[Y]$ is an open cover of Y in X. When $\mathcal{A}$ and $\mathcal{B}$ are open covers of X, $\mathcal{A} \wedge \mathcal{B}$ is also an open cover of X. For covers $\mathcal{A}$ and $\mathcal{B}$ of X, it is said that $\mathcal{A}$ is a **refinement** of $\mathcal{B}$ if $\mathcal{A} \prec \mathcal{B}$, where $\mathcal{A}$ is an **open** (or **closed**) **refinement** if $\mathcal{A}$ is an open (or closed) cover. For a space X, we denote

- $\operatorname{cov}(X)$ — the collection of all open covers of X.

Let $(X_\gamma)_{\gamma\in\Gamma}$ be a family of (topological) spaces and $X = \bigcup_{\gamma\in\Gamma} X_\gamma$. The **weak topology** on X with respect to $(X_\gamma)_{\gamma\in\Gamma}$ is defined as follows:

$$U \subset X \text{ is open in } X \;\Leftrightarrow\; \forall\gamma \in \Gamma, U \cap X_\gamma \text{ is open in } X_\gamma$$
$$\Big(A \subset X \text{ is closed in } X \;\Leftrightarrow\; \forall\gamma \in \Gamma, A \cap X_\gamma \text{ is closed in } X_\gamma\Big).$$

Suppose that X has the weak topology with respect to $(X_\gamma)_{\gamma\in\Gamma}$, and that the topologies of X_γ and $X_{\gamma'}$ agree on $X_\gamma \cap X_{\gamma'}$ for any $\gamma, \gamma' \in \Gamma$. If $X_\gamma \cap X_{\gamma'}$ is closed (resp. open) in X_γ for any $\gamma, \gamma' \in \Gamma$ then each X_γ is closed (resp. open) in X and the original topology of each X_γ is a subspace topology inherited from X. In the case that $X_\gamma \cap X_{\gamma'} = \emptyset$ for $\gamma \neq \gamma'$, X is the **topological sum** of $(X_\gamma)_{\gamma\in\Gamma}$, denoted by $X = \bigoplus_{\gamma\in\Gamma} X_\gamma$.

Let $f : X \to Y$ be a map. For $A \subset X$ and $B \subset Y$, we denote

$$f(A) = \big\{f(x) \bigm| x \in A\big\} \text{ and } f^{-1}(B) = \big\{x \in X \bigm| f(x) \in B\big\}.$$

For collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of X and Y, respectively, we denote

$$f(\mathcal{A}) = \big\{f(A) \bigm| A \in \mathcal{A}\big\} \text{ and } f^{-1}(\mathcal{B}) = \big\{f^{-1}(B) \bigm| B \in \mathcal{B}\big\}.$$

The restriction of f to $A \subset X$ is denoted by $f|A$. It is said that a map $g : A \to Y$ **extends over** X if there is a map $f : X \to Y$ such that $f|A = g$. Such a map f is called an **extension** of g.

Let $[a, b]$ be a closed interval, where $a < b$. A map $f : [a, b] \to X$ is called a **path** (from $f(a)$ to $f(b)$) in X, and we say that two points $f(a)$ and $f(b)$ are connected by the path f in X. An embedding $f : [a, b] \to X$ is called an **arc** (from $f(a)$ to $f(b)$) in X, and the image $f([a, b])$ is also called an **arc**. Namely, a space is called an **arc** if it is homeomorphic to $\mathbf{I}$. It is known that each pair of distinct points $x, y \in X$ are connected by an arc if and only if they are connected by a path.[2]

For spaces X and Y, we denote

- $\mathrm{C}(X, Y)$ — the set of (continuous) maps from X to Y.

For maps $f, g : X \to Y$ (i.e., $f, g \in \mathrm{C}(X, Y)$),

- $f \simeq g$ means that f and g are **homotopic** (or f is **homotopic** to g),

that is, there is a map $h : X \times \mathbf{I} \to Y$ such that $h_0 = f$ and $h_1 = g$, where $h_t : X \to Y$, $t \in \mathbf{I}$, are defined by $h_t(x) = h(x, t)$, and h is called a **homotopy** from f to g (between f and g). When g is a constant map, it is said that f is **null-homotopic**, which we denote by $f \simeq 0$. The relation $\simeq$ is an equivalence relation on $\mathrm{C}(X, Y)$. The equivalence class $[f] = \{g \in \mathrm{C}(X, Y) \mid g \simeq f\}$ is called the **homotopy class** of f. We denote

[2]This will be shown in Corollary 5.14.6.

- $[X, Y] = \{[f] \mid f \in \mathrm{C}(X, Y)\} = \mathrm{C}(X, Y)/\simeq$
  — the set of the homotopy classes of maps from X to Y.

For each $f, f' \in \mathrm{C}(X, Y)$ and $g, g' \in \mathrm{C}(Y, Z)$, we have the following:

$$f \simeq f',\ g \simeq g' \Rightarrow gf \simeq g'f'.$$

Thus, we have the composition $[X, Y] \times [Y, Z] \to [X, Z]$ defined by $([f], [g]) \mapsto [g][f] = [gf]$. Moreover,

- $X \simeq Y$ means that X and Y are **homotopy equivalent** (or X is **homotopy equivalent** to Y),[3]

that is, there are maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$, where f is called a **homotopy equivalence** and g is a **homotopy inverse** of f.

Given subspaces $X_1, \dots, X_n \subset X$ and $Y_1, \dots, Y_n \subset Y$, a map $f : X \to Y$ is said to be a map from $(X, X_1, \dots, X_n)$ to $(Y, Y_1, \dots, Y_n)$, written

$$f : (X, X_1, \dots, X_n) \to (Y, Y_1, \dots, Y_n),$$

if $f(X_1) \subset Y_1, \dots, f(X_n) \subset Y_n$. We denote

- $\mathrm{C}((X, X_1, \dots, X_n), (Y, Y_1, \dots, Y_n))$
  — the set of maps from $(X, X_1, \dots, X_n)$ to $(Y, Y_1, \dots, Y_n)$.

A homotopy h between maps $f, g \in \mathrm{C}((X, X_1, \dots, X_n), (Y, Y_1, \dots, Y_n))$ requires the condition that $h_t \in \mathrm{C}((X, X_1, \dots, X_n), (Y, Y_1, \dots, Y_n))$ for every $t \in \mathbf{I}$, i.e., h is regarded as the map

$$h : (X \times \mathbf{I}, X_1 \times \mathbf{I}, \dots, X_n \times \mathbf{I}) \to (Y, Y_1, \dots, Y_n).$$

Thus, $\simeq$ is an equivalence relation on $\mathrm{C}((X, X_1, \dots, X_n), (Y, Y_1, \dots, Y_n))$. We denote

- $[(X, X_1, \dots, X_n), (Y, Y_1, \dots, Y_n)] = \mathrm{C}((X, X_1, \dots, X_n), (Y, Y_1, \dots, Y_n))/\simeq$.

When there exist maps

$$f : (X, X_1, \dots, X_n) \to (Y, Y_1, \dots, Y_n),$$
$$g : (Y, Y_1, \dots, Y_n) \to (X, X_1, \dots, X_n)$$

such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$, we denote

- $(X, X_1, \dots, X_n) \simeq (Y, Y_1, \dots, Y_n)$.

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[3]It is also said that X and Y have the **same homotopy type** or X has the **homotopy type** of Y.

Similarly, for each pair of pointed spaces (X, x_0) and (Y, y_0),

- $C((X, x_0), (Y, y_0)) = C((X, \{x_0\}), (Y, \{y_0\}))$;
- $[(X, x_0), (Y, y_0)] = C((X, x_0), (Y, y_0))/\simeq$;
- $(X, x_0) \simeq (Y, y_0)$ means $(X, \{x_0\}) \simeq (Y, \{y_0\})$.

For $A \subset X$, a homotopy $h : X \times \mathbf{I} \to Y$ is called a **homotopy relative to** A if $h(\{x\} \times \mathbf{I})$ is degenerate (i.e., a singleton) for every $x \in A$. When a homotopy from f to g is a homotopy relative to A (where $f|A = g|A$), we denote

- $f \simeq g$ rel. A.

Let $f, g : X \to Y$ be maps and $\mathcal{U}$ a collection of subsets of Y (in usual, $\mathcal{U} \in \operatorname{cov}(Y)$). It is said that f and g are **$\mathcal{U}$-close** (or f is **$\mathcal{U}$-close** to g) if

$$\big\{\{f(x), g(x)\} \bigm| x \in X\big\} \prec \mathcal{U} \cup \big\{\{y\} \bigm| y \in Y\big\},$$

which implies that $\mathcal{U}$ covers the set $\{f(x), g(x) \mid f(x) \neq g(x)\}$. A homotopy h is called a **$\mathcal{U}$-homotopy** if

$$\big\{h(\{x\} \times \mathbf{I}) \bigm| x \in X\big\} \prec \mathcal{U} \cup \big\{\{y\} \bigm| y \in Y\big\},$$

which implies that $\mathcal{U}$ covers the set

$$\bigcup \big\{h(\{x\} \times \mathbf{I}) \bigm| h(\{x\} \times \mathbf{I}) \text{ is non-degenerate}\big\}.$$

We say that f and g are **$\mathcal{U}$-homotopic** (or f is **$\mathcal{U}$-homotopic** to g) and denoted by $f \simeq_{\mathcal{U}} g$ if there is a $\mathcal{U}$-homotopy $h : X \times \mathbf{I} \to Y$ such that $h_0 = f$ and $h_1 = g$.

When $Y = (Y, d)$ is a metric space, we define the distance between $f, g \in C(X, Y)$ as follows:

$$d(f, g) = \sup \big\{d(f(x), g(x)) \bigm| x \in X\big\}.$$

In general, it may be possible that $d(f, g) = \infty$, in which case d is not a metric on the set $C(X, Y)$. If Y is bounded or X is compact, then this d is a metric on the set $C(X, Y)$, called the **sup-metric**. For $\varepsilon > 0$, we say that f and g are **ε-close** or f is **ε-close** to g if $d(f, g) < \varepsilon$. A homotopy h is called an **ε-homotopy** if $\operatorname{mesh}\{h(\{x\} \times \mathbf{I}) \mid x \in X\} < \varepsilon$, where $f = h_0$ and $g = h_1$ are said to be **ε-homotopic** and denoted by $f \simeq_\varepsilon g$.

In the above, even if d is not a metric on $C(X, Y)$ (i.e., $d(f, g) = \infty$ for some $f, g \in C(X, Y)$), it induces a topology on $C(X, Y)$ such that each f has a neighborhood basis consisting of

$$B_d(f, \varepsilon) = \big\{g \in C(X, Y) \bigm| d(f, g) < \varepsilon\big\}, \ \varepsilon > 0.$$

This topology is called the **uniform convergence topology**.

The **compact-open topology** on $\mathrm{C}(X, Y)$ is generated by the sets

$$\langle K; U\rangle = \{f \in \mathrm{C}(X, Y) \mid f(K) \subset U\},$$

where K is any compact set in X and U is any open set in Y. With respect to this topology, we have the following:

Proposition 1.1.1. *Every map $f : Z \times X \to Y$ (or $f : X \times Z \to Y$) induces the map $\bar{f} : Z \to \mathrm{C}(X, Y)$ defined by $\bar{f}(z)(x) = f(z, x)$ (or $\bar{f}(z)(x) = f(x, z)$).*

Proof. For each $z \in Z$, it is easy to see that $\bar{f}(z) : X \to Y$ is continuous, i.e., $\bar{f}(z) \in \mathrm{C}(X, Y)$. Thus, $\bar{f}$ is well-defined.

To verify the continuity of $\bar{f} : Z \to \mathrm{C}(X, Y)$, it suffices to show that $\bar{f}^{-1}(\langle K; U\rangle)$ is open in Z for each compact set K in X and each open set U in Y. Let $z \in \bar{f}^{-1}(\langle K; U\rangle)$, i.e., $f(\{z\} \times K) \subset U$. Using the compactness of K, we can easily find an open neighborhood V of z in Z such that $f(V \times K) \subset U$, which means that $V \subset \bar{f}^{-1}(\langle K; U\rangle)$. □

With regards to the relation $\simeq$ on $\mathrm{C}(X, Y)$, we have the following:

Proposition 1.1.2. *Each $f, g \in \mathrm{C}(X, Y)$ are connected by a path in $\mathrm{C}(X, Y)$. When X is metrizable or locally compact, the converse is also true, that is, $f \simeq g$ if and only if f and g are connected by a path in $\mathrm{C}(X, Y)$ if $f \simeq g$.*[4]

Proof. By Proposition 1.1.1, a homotopy $h : X \times \mathbf{I} \to Y$ from f to g induces the path $\bar{h} : \mathbf{I} \to \mathrm{C}(X, Y)$ defined as $\bar{h}(t)(x) = h(x, t)$ for each $t \in \mathbf{I}$ and $x \in X$, where $\bar{h}(0) = f$ and $\bar{h}(1) = g$.

For a path $\varphi : \mathbf{I} \to \mathrm{C}(X, Y)$ from f to g, we define the homotopy $\tilde{\varphi} : X \times \mathbf{I} \to Y$ as $\tilde{\varphi}(x, t) = \varphi(t)(x)$ for each $(x, t) \in X \times \mathbf{I}$. Then, $\tilde{\varphi}_0 = \varphi(0) = f$ and $\tilde{\varphi}_1 = \varphi(1) = g$. It remains to show that $\tilde{\varphi}$ is continuous if X is metrizable or locally compact.

In the case that X is locally compact, for each $(x, t) \in X \times \mathbf{I}$ and for each open neighborhood U of $\tilde{\varphi}(x, t) = \varphi(t)(x)$ in Y, x has a compact neighborhood K in X such that $\varphi(t)(K) \subset U$, i.e., $\varphi(t) \in \langle K; U\rangle$. By the continuity of φ, t has a neighborhood V in $\mathbf{I}$ such that $\varphi(V) \subset \langle K; U\rangle$. Thus, $K \times V$ is a neighborhood of $(x, t) \in X \times \mathbf{I}$ and $\tilde{\varphi}(K \times V) \subset U$. Hence, $\tilde{\varphi}$ is continuous.

In the case that X is metrizable, let us assume that $\tilde{\varphi}$ is not continuous at $(x, t) \in X \times \mathbf{I}$. Then, $\tilde{\varphi}(x, t)$ has some open neighborhood U in Y such that $\tilde{\varphi}(V) \not\subset U$ for any neighborhood V of (x, t) in $X \times \mathbf{I}$. Let $d \in \mathrm{Metr}(X)$. For each $n \in \mathbb{N}$, we have $x_n \in X$ and $t_n \in \mathbf{I}$ such that $d(x_n, x) < 1/n$, $|t_n - t| < 1/n$ and $\tilde{\varphi}(x_n, t_n) \notin U$. Because $x_n \to x$ $(n \to \infty)$ and $\varphi(t)$ is continuous, we have $n_0 \in \mathbb{N}$ such that $\varphi(t)(x_n) \in U$ for all $n \geq n_0$. Note that $K = \{x_n, x \mid n \geq n_0\}$ is compact and $\varphi(t)(K) \subset U$. Because $t_n \to t$ $(n \to \infty)$ and φ is continuous at t, $\varphi(t_{n_1})(K) \subset U$ for some $n_1 \geq n_0$. Thus, $\tilde{\varphi}(x_{n_1}, t_{n_1}) \in U$, which is a contradiction. Consequently, $\tilde{\varphi}$ is continuous. □

Remark 1. It is easily observed that Proposition 1.1.2 is also valid for

[4]More generally, this is valid for every k-space X, where X is a ***k*-space** provided U is open in X if $U \cap K$ is open in K for every compact set $K \subset X$. A k-space is also called a **compactly generated space**.

$$\mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n)).$$

Some Properties of the Compact-Open Topology 1.1.3.

The following hold with respect to the compact-open topology:

(1) For $f \in \mathrm{C}(Z, X)$ and $g \in \mathrm{C}(Y, Z)$, the following are continuous:

$$f^* : \mathrm{C}(X, Y) \to \mathrm{C}(Z, Y), \ f^*(h) = h \circ f;$$
$$g_* : \mathrm{C}(X, Y) \to \mathrm{C}(X, Z), \ g_*(h) = g \circ h.$$

(2) When Y is locally compact, the following (composition) is continuous:

$$\mathrm{C}(X, Y) \times \mathrm{C}(Y, Z) \ni (f, g) \mapsto g \circ f \in \mathrm{C}(X, Z).$$

Sketch of Proof. Let K be a compact set in X and U an open set in Z with $f \in \mathrm{C}(X, Y)$ and $g \in \mathrm{C}(Y, Z)$ such that $g \circ f(K) \subset U$. Since Y is locally compact, we have an open set V in Y such that $\mathrm{cl}\, V$ is compact, $f(K) \subset V$ and $g(\mathrm{cl}\, V) \subset U$. Then, $f'(K) \subset V$ and $g'(\mathrm{cl}\, V) \subset U$ imply $g' \circ f'(K) \subset U$.

(3) For each $x_0 \in X$, the following (evaluation) is continuous:

$$\mathrm{C}(X, Y) \ni f \mapsto f(x_0) \in Y.$$

(4) When X is locally compact, the following (evaluation) is continuous:

$$\mathrm{C}(X, Y) \times X \ni (f, x) \mapsto f(x) \in Y.$$

In this case, for every map $f : Z \to \mathrm{C}(X, Y)$, the following is continuous:

$$Z \times X \ni (z, x) \mapsto f(z)(x) \in Y.$$

(5) In the case that X is locally compact, we have the following inequalities:

$$w(Y) \leq w(\mathrm{C}(X, Y)) \leq \aleph_0 w(X) w(Y).$$

Sketch of Proof. By embedding Y into $\mathrm{C}(X, Y)$, we obtain the first inequality. For the second, we take open bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ for X and Y, respectively, such that $\mathrm{card}\, \mathcal{B}_X = w(X)$, $\mathrm{card}\, \mathcal{B}_Y = w(Y)$, and $\mathrm{cl}\, A$ is compact for every $A \in \mathcal{B}_X$. The following is an open sub-basis for $\mathrm{C}(X, Y)$:

$$\mathcal{B} = \big\{ \langle \mathrm{cl}\, A, B \rangle \bigm| (A, B) \in \mathcal{B}_X \times \mathcal{B}_Y \big\}.$$

Indeed, let K be a compact set in X, U be an open set in Y, and $f \in \mathrm{C}(X, Y)$ with $f(K) \subset U$, i.e., $f \in \langle K, U \rangle$. First, find $B_1, \ldots, B_n \in \mathcal{B}_Y$ so that $f(K) \subset B_1 \cup \cdots \cup B_n \subset U$. Next, find $A_1, \ldots, A_m \in \mathcal{B}_X$ so that $K \subset A_1 \cup \cdots \cup A_m$ and each $\mathrm{cl}\, A_i$ is contained in some $f^{-1}(B_{j(i)})$. Then, $f \in \bigcap_{i=1}^m \langle \mathrm{cl}\, A_i, B_{j(i)} \rangle \subset \langle K, U \rangle$.

(6) If X is compact and $Y = (Y, d)$ is a metric space, then the sup-metric on $C(X, Y)$ is admissible for the compact-open topology on $C(X, Y)$.

Sketch of Proof. Let K be a compact set in X and U be an open set in Y with $f \in C(X, Y)$ such that $f(K) \subset U$. Then, $\delta = \operatorname{dist}(f(K), Y \setminus U) > 0$, and $d(f, f') < \delta$ implies $f'(K) \subset U$. Conversely, for each $\varepsilon > 0$ and $f \in C(X, Y)$, we have $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^n f^{-1}(B(f(x_i), \varepsilon/4)$. Observe that

$$f'(f^{-1}(\overline{B}(f(x_i), \varepsilon/4))) \subset B(f(x_i), \varepsilon/2) \ (\forall i = 1, \ldots, n)$$
$$\Rightarrow d(f, f') < \varepsilon.$$

(7) Let $X = \bigcup_{n \in \mathbb{N}} X_n$, where X_n is compact and $X_n \subset \operatorname{int} X_{n+1}$. If $Y = (Y, d)$ is a metric space, then $C(X, Y)$ with the compact-open topology is metrizable.

Sketch of Proof. We define a metric ρ on $C(X, Y)$ as follows:

$$\rho(f, g) = \sup_{n \in \mathbb{N}} \min \left\{ n^{-1}, \sup_{x \in X_n} d(f(x), g(x)) \right\}.$$

Then, ρ is admissible for the compact-open topology on $C(X, Y)$. To see this, refer to the proof of (6).

1.2 Banach Spaces in the Product of Real Lines

Throughout this section, let Γ be an infinite set. We denote

- $\operatorname{Fin}(\Gamma)$ — the set of all non-empty finite subsets of Γ.

Note that $\operatorname{card}\operatorname{Fin}(\Gamma) = \operatorname{card}\Gamma$. The product space $\mathbb{R}^\Gamma$ is a linear space with the following scalar multiplication and addition:

$$\mathbb{R}^\Gamma \times \mathbb{R} \ni (x, t) \mapsto tx = (tx(\gamma))_{\gamma \in \Gamma} \in \mathbb{R}^\Gamma;$$
$$\mathbb{R}^\Gamma \times \mathbb{R}^\Gamma \ni (x, y) \mapsto x + y = (x(\gamma) + y(\gamma))_{\gamma \in \Gamma} \in \mathbb{R}^\Gamma.$$

In this section, we consider various (complete) norms defined on linear subspaces of $\mathbb{R}^\Gamma$. In general, the unit closed ball and the unit sphere of a normed linear space $X = (X, \|\cdot\|)$ are denoted by $\mathbf{B}_X$ and $\mathbf{S}_X$, respectively. Namely, let

$$\mathbf{B}_X = \{x \in X \mid \|x\| \leq 1\} \text{ and } \mathbf{S}_X = \{x \in X \mid \|x\| = 1\}.$$

The zero vector (the zero element) of X is denoted by $\mathbf{0}_X$, or simply $\mathbf{0}$ if there is no possibility of confusion.

Before considering norms, we first discuss the product topology of $\mathbb{R}^\Gamma$. The scalar multiplication and addition are continuous with respect to the product topology. Namely, $\mathbb{R}^\Gamma$ with the product topology is a topological linear space.[5] Note that $w(\mathbb{R}^\Gamma) = \operatorname{card}\Gamma$.

Let $\mathcal{B}_0$ be a countable open basis for $\mathbb{R}$. Then, $\mathbb{R}^\Gamma$ has the following open basis:

$$\bigl\{ \bigcap_{\gamma\in F} \mathrm{pr}_\gamma^{-1}(B_\gamma) \bigm| F \in \mathrm{Fin}(\Gamma),\ B_\gamma \in \mathcal{B}_0\ (\gamma \in F) \bigr\}.$$

Thus, we have $w(\mathbb{R}^\Gamma) \leq \aleph_0 \operatorname{card} \mathrm{Fin}(\Gamma) = \operatorname{card}\Gamma$. Let $\mathcal{B}$ be an open basis for $\mathbb{R}^\Gamma$. For each $B \in \mathcal{B}$, we can find $F_B \in \mathrm{Fin}(\Gamma)$ such that $\mathrm{pr}_\gamma(B) = \mathbb{R}$ for every $\gamma \in \Gamma \setminus F_B$. Then, $\operatorname{card} \bigcup_{B\in\mathcal{B}} F_B \leq \aleph_0 \operatorname{card}\mathcal{B}$. If $\operatorname{card}\mathcal{B} < \operatorname{card}\Gamma$ then $\operatorname{card}\bigcup_{B\in\mathcal{B}} F_B < \operatorname{card}\Gamma$, so we have $\gamma_0 \in \Gamma \setminus \bigcup_{B\in\mathcal{B}} F_B$. The open set $\mathrm{pr}_{\gamma_0}^{-1}((0,\infty)) \subset \mathbb{R}^\Gamma$ contains some $B \in \mathcal{B}$. Then, $\mathrm{pr}_{\gamma_0}(B) \subset (0,\infty)$, which means that $\gamma_0 \in F_B$. This is a contradiction. Therefore, $\operatorname{card}\mathcal{B} \geq \operatorname{card}\Gamma$, and thus we have $w(\mathbb{R}^\Gamma) \geq \operatorname{card}\Gamma$.

For each $\gamma \in \Gamma$, we define the unit vector $\mathbf{e}_\gamma \in \mathbb{R}^\Gamma$ by $\mathbf{e}_\gamma(\gamma) = 1$ and $\mathbf{e}_\gamma(\gamma') = 0$ for $\gamma' \neq \gamma$. It should be noted that $\{\mathbf{e}_\gamma \mid \gamma \in \Gamma\}$ is not a Hamel basis for $\mathbb{R}^\Gamma$, and the linear span of $\{\mathbf{e}_\gamma \mid \gamma \in \Gamma\}$ is the following:[6]

$$\mathbb{R}^\Gamma_f = \bigl\{ x \in \mathbb{R}^\Gamma \bigm| x(\gamma) = 0 \text{ except for finitely many } \gamma \in \Gamma \bigr\},$$

which is a dense linear subspace of $\mathbb{R}^\Gamma$. The subspace $\mathbb{R}^\mathbb{N}_f$ of $\boldsymbol{s} = \mathbb{R}^\mathbb{N}$ is also denoted by $\boldsymbol{s}_f$, which is the space of finite sequences (with the product topology). When $\operatorname{card}\Gamma = \aleph_0$, the space $\mathbb{R}^\Gamma$ is linearly homeomorphic to the space of sequences $\boldsymbol{s} = \mathbb{R}^\mathbb{N}$, i.e., there exists a linear homeomorphism between $\mathbb{R}^\Gamma$ and $\boldsymbol{s}$, where the linear subspace $\mathbb{R}^\Gamma_f$ is linearly homeomorphic to $\boldsymbol{s}_f$ by the same homeomorphism. The following fact can easily be observed:

Fact. *The following are equivalent:*

(a) $\mathbb{R}^\Gamma$ *is metrizable;*
(b) $\mathbb{R}^\Gamma_f$ *is metrizable;*
(c) $\mathbb{R}^\Gamma_f$ *is first countable;*
(d) $\operatorname{card}\Gamma \leq \aleph_0$.

The implication (c) $\Rightarrow$ (d) is shown as follows: Let $\{U_i \mid i \in \mathbb{N}\}$ be a neighborhood basis of $\mathbf{0}$ in $\mathbb{R}^\Gamma_f$. Then, each $\Gamma_i = \{\gamma \in \Gamma \mid \mathbb{R}\mathbf{e}_\gamma \not\subset U_i\}$ is finite. If Γ is uncountable, then $\Gamma \setminus \bigcup_{i\in\mathbb{N}} \Gamma_i \neq \emptyset$, i.e., $\mathbb{R}\mathbf{e}_\gamma \subset \bigcap_{i\in\mathbb{N}} U_i$ for some $\gamma \in \Gamma$. In this case, $U_i \not\subset \mathrm{pr}_\gamma^{-1}((-1,1))$ for every $i \in \mathbb{N}$, which is a contradiction.

Thus, every linear subspace L of $\mathbb{R}^\Gamma$ containing $\mathbb{R}^\Gamma_f$ is non-metrizable if Γ is uncountable, and it is metrizable if Γ is countable. On the other hand, due to the following proposition, every linear subspaces L of $\mathbb{R}^\Gamma$ containing $\mathbb{R}^\Gamma_f$ is non-normable if Γ is infinite.

Proposition 1.2.1. *Let Γ be an infinite set. Any norm on $\mathbb{R}^\Gamma_f$ does not induce the topology inherited from the product topology of $\mathbb{R}^\Gamma$.*

[5] For topological linear spaces, refer Sect. 3.4.

[6] The linear subspace generated by a set B is called the **linear span** of B.

Proof. Assume that the topology of $\mathbb{R}_f^\Gamma$ is induced by a norm $\|\cdot\|$. Because $U = \{x \in \mathbb{R}_f^\Gamma \mid \|x\| < 1\}$ is an open neighborhood of $\mathbf{0}$ in $\mathbb{R}_f^\Gamma$, we have a finite set $F \subset \Gamma$ and neighborhoods V_γ of $0 \in \mathbb{R}$, $\gamma \in F$, such that $\mathbb{R}_f^\Gamma \cap \bigcap_{\gamma \in F} \mathrm{pr}_\gamma^{-1}(V_\gamma) \subset U$. Take $\gamma_0 \in \Gamma \setminus F$. As $\mathbb{R}\mathbf{e}_{\gamma_0} \subset U$, we have $\|\mathbf{e}_{\gamma_0}\|^{-1}\mathbf{e}_{\gamma_0} \in U$ but $\big\| \|\mathbf{e}_{\gamma_0}\|^{-1}\mathbf{e}_{\gamma_0} \big\| = \|\mathbf{e}_{\gamma_0}\|^{-1}\|\mathbf{e}_{\gamma_0}\| = 1$, which is a contradiction. □

The Banach space $\ell_\infty(\Gamma)$ and its closed linear subspaces $\boldsymbol{c}(\Gamma) \supset \boldsymbol{c}_0(\Gamma)$ are defined as follows:

- $\ell_\infty(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \sup_{\gamma \in \Gamma} |x(\gamma)| < \infty\}$ with the sup-norm

$$\|x\|_\infty = \sup_{\gamma \in \Gamma} |x(\gamma)|;$$

- $\boldsymbol{c}(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \exists t \in \mathbb{R}$ such that $\forall \varepsilon > 0,\ |x(\gamma) - t| < \varepsilon$ except for finitely many $\gamma \in \Gamma\}$;
- $\boldsymbol{c}_0(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \forall \varepsilon > 0,\ |x(\gamma)| < \varepsilon$ except for finitely many $\gamma \in \Gamma\}$.

These are linear subspaces of $\mathbb{R}^\Gamma$, but are not topological subspace according to Proposition 1.2.1. The space $\boldsymbol{c}(\Gamma)$ is linearly homeomorphic to $\boldsymbol{c}_0(\Gamma) \times \mathbb{R}$ by the correspondence

$$\boldsymbol{c}_0(\Gamma) \times \mathbb{R} \ni (x, t) \mapsto (x(\gamma) + t)_{\gamma \in \Gamma} \in \boldsymbol{c}(\Gamma).$$

This correspondence and its inverse are Lipschitz with respect to the norm $\|(x,t)\| = \max\{\|x\|_\infty, |t|\}$. Indeed, let $y = (x(\gamma)+t)_{\gamma \in \Gamma}$. Then, $\|y\|_\infty \le \|x\|_\infty + |t| \le 2\|(x,t)\|$. Because $x \in \boldsymbol{c}_0(\Gamma)$ and $|t| \le |y(\gamma)| + |x(\gamma)| \le \|y\|_\infty + |x(\gamma)|$ for every $\gamma \in \Gamma$, it follows that $|t| \le \|y\|_\infty$. Moreover, $|x(\gamma)| \le |y(\gamma)| + |t| \le 2\|y\|_\infty$ for every $\gamma \in \Gamma$. Hence, $\|x\|_\infty \le 2\|y\|_\infty$, and thus we have $\|(x,t)\| \le 2\|y\|_\infty$.

Furthermore, we denote $\mathbb{R}_f^\Gamma$ with this norm as $\ell_\infty^f(\Gamma)$. We then have the inclusions:

$$\ell_\infty^f(\Gamma) \subset \boldsymbol{c}_0(\Gamma) \subset \boldsymbol{c}(\Gamma) \subset \ell_\infty(\Gamma).$$

The topology of $\ell_\infty^f(\Gamma)$ is different from the topology inherited from the product topology. Indeed, $\{\mathbf{e}_\gamma \mid \gamma \in \Gamma\}$ is discrete in $\ell_\infty^f(\Gamma)$, but $\mathbf{0}$ is a cluster point of this set with respect to the product topology.

We must pay attention to the following fact:

Proposition 1.2.2. *For an arbitrary infinite set Γ,*

$$w(\ell_\infty(\Gamma)) = 2^{\operatorname{card}\Gamma} \quad \textit{but} \quad w(\boldsymbol{c}(\Gamma)) = w(\boldsymbol{c}_0(\Gamma)) = w(\ell_\infty^f(\Gamma)) = \operatorname{card}\Gamma.$$

Proof. The characteristic map $\chi_\Lambda : \Gamma \to \{0,1\} \subset \mathbb{R}$ of $\Lambda \subset \Gamma$ belongs to $\ell_\infty(\Gamma)$ ($\chi_\emptyset = \mathbf{0} \in \ell_\infty(\Gamma)$), where $\|\chi_\Lambda - \chi_{\Lambda'}\|_\infty = 1$ if $\Lambda \ne \Lambda' \subset \Gamma$. It follows that $w(\ell_\infty(\Gamma)) = c(\ell_\infty(\Gamma)) \ge 2^{\operatorname{card}\Gamma}$. Moreover, $\mathbb{Q}^\Gamma \cap \ell_\infty(\Gamma)$ is dense in $\ell_\infty(\Gamma)$, and hence we have

$$w(\ell_\infty(\Gamma)) = \operatorname{dens} \ell_\infty(\Gamma) \le \operatorname{card} \mathbb{Q}^\Gamma = \aleph_0^{\operatorname{card}\Gamma} = 2^{\operatorname{card}\Gamma}.$$

On the other hand, $\mathbf{e}_\gamma \in \ell_\infty^f(\Gamma)$ for each $\gamma \in \Gamma$ and $\|\mathbf{e}_\gamma - \mathbf{e}_{\gamma'}\|_\infty = 1$ if $\gamma \ne \gamma'$. Since $\ell_\infty^f(\Gamma) \subset \boldsymbol{c}_0(\Gamma)$, it follows that

$$w(\boldsymbol{c}_0(\Gamma)) \ge w(\ell_\infty^f(\Gamma)) = c(\ell_\infty^f(\Gamma)) \ge \operatorname{card}\Gamma.$$

Moreover, $\boldsymbol{c}_0(\Gamma)$ has the following dense subset:

$$\mathbb{Q}_f^\Gamma = \big\{x \in \mathbb{Q}^\Gamma \bigm| x(\gamma) = 0 \text{ except for finitely many } \gamma \in \Gamma\big\},$$

and so it follows that

$$w(\boldsymbol{c}_0(\Gamma)) = \operatorname{dens} \boldsymbol{c}_0(\Gamma) \le \operatorname{card} \mathbb{Q}_f^\Gamma \le \aleph_0 \operatorname{card} \operatorname{Fin}(\Gamma) = \operatorname{card}\Gamma.$$

Thus, we have $w(\boldsymbol{c}_0(\Gamma)) = w(\ell_\infty^f(\Gamma)) = \operatorname{card}\Gamma$. As already observed, $\boldsymbol{c}(\Gamma) \approx \boldsymbol{c}_0(\Gamma) \times \mathbb{R}$, hence $w(\boldsymbol{c}(\Gamma)) = w(\boldsymbol{c}_0(\Gamma))$. □

When $\Gamma = \mathbb{N}$, we write

- $\ell_\infty(\mathbb{N}) = \ell_\infty$ — the space of bounded sequences,
- $\boldsymbol{c}(\mathbb{N}) = \boldsymbol{c}$ — the space of convergent sequences,
- $\boldsymbol{c}_0(\mathbb{N}) = \boldsymbol{c}_0$ — the space of sequences convergent to 0, and
- $\ell_\infty^f(\mathbb{N}) = \ell_\infty^f$ — the space of finite sequences with the sup-norm,

where $\ell_\infty^f \ne \boldsymbol{s}_f$ as (topological) spaces. According to Proposition 1.2.2, $\boldsymbol{c}$ and $\boldsymbol{c}_0$ are separable, but ℓ_∞ is non-separable. When $\operatorname{card}\Gamma = \aleph_0$, the spaces $\ell_\infty(\Gamma)$, $\boldsymbol{c}(\Gamma)$, and $\boldsymbol{c}_0(\Gamma)$ are linearly isometric to these spaces ℓ_∞, $\boldsymbol{c}$ and $\boldsymbol{c}_0$, respectively.

Here, we regard $\operatorname{Fin}(\Gamma)$ as a directed set by $\subset$. For $x \in \mathbb{R}^\Gamma$, we say that $\sum_{\gamma\in\Gamma} x(\gamma)$ is **convergent** if $\big(\sum_{\gamma\in F} x(\gamma)\big)_{F\in\operatorname{Fin}(\Gamma)}$ is convergent, and define

$$\sum_{\gamma\in\Gamma} x(\gamma) = \lim_{F\in\operatorname{Fin}(\Gamma)} \sum_{\gamma\in F} x(\gamma).$$

In the case that $x(\gamma) \ge 0$ for all $\gamma \in \Gamma$, $\sum_{\gamma\in\Gamma} x(\gamma)$ is convergent if and only if $\big(\sum_{\gamma\in F} x(\gamma)\big)_{F\in\operatorname{Fin}(\Gamma)}$ is upper bounded, and then

$$\sum_{\gamma\in\Gamma} x(\gamma) = \sup_{F\in\operatorname{Fin}(\Gamma)} \sum_{\gamma\in F} x(\gamma).$$

By this reason, $\sum_{\gamma\in\Gamma} x(\gamma) < \infty$ means that $\sum_{\gamma\in\Gamma} x(\gamma)$ is convergent.

For $x \in \mathbb{R}^\mathbb{N}$, we should distinguish $\sum_{i\in\mathbb{N}} x(i)$ from $\sum_{i=1}^\infty x(i)$. When the sequence $\big(\sum_{i=1}^n x(i)\big)_{n\in\mathbb{N}}$ is convergent, we say that $\sum_{i=1}^\infty x(i)$ is **convergent**, and define

$$\sum_{i=1}^{\infty} x(i) = \lim_{n\to\infty} \sum_{i=1}^{n} x(i).$$

Evidently, if $\sum_{i\in\mathbb{N}} x(i)$ is convergent, then $\sum_{i=1}^{\infty} x(i)$ is also convergent and $\sum_{i=1}^{\infty} x(i) = \sum_{i\in\mathbb{N}} x(i)$. However, $\sum_{i\in\mathbb{N}} x(i)$ is not necessary convergent even if $\sum_{i=1}^{\infty} x(i)$ is convergent. In fact, due to Proposition 1.2.3 below, we have the following:

$$\sum_{i\in\mathbb{N}} x(i) \text{ is convergent } \Leftrightarrow \sum_{i=1}^{\infty} |x(i)| \text{ is convergent.}$$

Proposition 1.2.3. *For an infinite set Γ and $x \in \mathbb{R}^{\Gamma}$, $\sum_{\gamma\in\Gamma} x(\gamma)$ is convergent if and only if $\sum_{\gamma\in\Gamma} |x(\gamma)| < \infty$. In this case, $\Gamma_x = \{\gamma \in \Gamma \mid x(\gamma) \neq 0\}$ is countable, and $\sum_{\gamma\in\Gamma} x(\gamma) = \sum_{i=1}^{\infty} x(\gamma_i)$ for any sequence $(\gamma_i)_{i\in\mathbb{N}}$ in Γ such that $\Gamma_x \subset \{\gamma_i \mid i \in \mathbb{N}\}$ and $\gamma_i \neq \gamma_j$ if $i \neq j$.*

Proof. Let us denote $\Gamma_+ = \{\gamma \in \Gamma \mid x(\gamma) > 0\}$ and $\Gamma_- = \{\gamma \in \Gamma \mid x(\gamma) < 0\}$. Then, $\Gamma_x = \Gamma_+ \cup \Gamma_-$.

If $\sum_{\gamma\in\Gamma} x(\gamma)$ is convergent, we have $F_0 \in \mathrm{Fin}(\Gamma)$ such that

$$F_0 \subset F \in \mathrm{Fin}(\Gamma) \Rightarrow \left| \sum_{\gamma\in\Gamma} x(\gamma) - \sum_{\gamma\in F} x(\gamma) \right| < 1.$$

Then, for each $E \in \mathrm{Fin}(\Gamma_+) \cup \mathrm{Fin}(\Gamma_-)$ (i.e., $E \in \mathrm{Fin}(\Gamma_+)$ or $E \in \mathrm{Fin}(\Gamma_-)$),

$$\sum_{\gamma\in E\setminus F_0} |x(\gamma)| = \left| \sum_{\gamma\in E\setminus F_0} x(\gamma) \right| = \left| \sum_{\gamma\in E\cup F_0} x(\gamma) - \sum_{\gamma\in F_0} x(\gamma) \right| < 2.$$

Hence, $\sum_{\gamma\in F} |x(\gamma)| < \sum_{\gamma\in F_0} |x(\gamma)| + 4$ for every $F \in \mathrm{Fin}(\Gamma)$, which means that $\left(\sum_{\gamma\in F} |x(\gamma)|\right)_{F\in\mathrm{Fin}(\Gamma)}$ is upper bounded, i.e., $\sum_{\gamma\in\Gamma} |x(\gamma)| < \infty$.

Conversely, we assume that $\sum_{\gamma\in\Gamma} |x(\gamma)| < \infty$. Then, for each $n \in \mathbb{N}$, $\Gamma_n = \{\gamma \in \Gamma \mid |x(\gamma)| > 1/n\}$ is finite, and hence $\Gamma_x = \bigcup_{n\in\mathbb{N}} \Gamma_n$ is countable. Note that $\sum_{\gamma\in\Gamma_+} |x(\gamma)| < \infty$ and $\sum_{\gamma\in\Gamma_-} |x(\gamma)| < \infty$. We show that

$$\sum_{\gamma\in\Gamma} x(\gamma) = \sum_{\gamma\in\Gamma_+} |x(\gamma)| - \sum_{\gamma\in\Gamma_-} |x(\gamma)|.$$

For each $\varepsilon > 0$, we can find $F_+ \in \mathrm{Fin}(\Gamma_+)$ and $F_- \in \mathrm{Fin}(\Gamma_-)$ such that

$$F_\pm \subset E \in \mathrm{Fin}(\Gamma_\pm) \Rightarrow \sum_{\gamma\in\Gamma_\pm} |x(\gamma)| - \varepsilon/2 < \sum_{\gamma\in E} |x(\gamma)| \leq \sum_{\gamma\in\Gamma_\pm} |x(\gamma)|.$$

Then, it follows that, for each $F \in \mathrm{Fin}(\Gamma)$ with $F \supset F_+ \cup F_-$,

$$\begin{aligned}
&\left|\sum_{\gamma\in F} x(\gamma) - \left(\sum_{\gamma\in\Gamma_+} |x(\gamma)| - \sum_{\gamma\in\Gamma_-} |x(\gamma)|\right)\right| \\
&\qquad \le \left|\sum_{\gamma\in F\cap\Gamma_+} |x(\gamma)| - \sum_{\gamma\in\Gamma_+} |x(\gamma)|\right| + \left|\sum_{\gamma\in F\cap\Gamma_-} |x(\gamma)| - \sum_{\gamma\in\Gamma_-} |x(\gamma)|\right| \\
&\qquad < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Now, let $(\gamma_i)_{i\in\mathbb{N}}$ be a sequence in Γ such that $\Gamma_x \subset \{\gamma_i \mid i \in \mathbb{N}\}$ and $\gamma_i \ne \gamma_j$ if $i \ne j$. We define

$$n_0 = \max\{i \in \mathbb{N} \mid \gamma_i \in F_+ \cup F_-\}.$$

For each $n \ge n_0$, it follows from $F_+ \cup F_- \subset \{\gamma_1, \ldots, \gamma_n\}$ that

$$\left|\sum_{i=1}^{n} x(\gamma_i) - \left(\sum_{\gamma\in\Gamma_+} |x(\gamma)| - \sum_{\gamma\in\Gamma_-} |x(\gamma)|\right)\right| < \varepsilon.$$

Thus, we also have $\sum_{\gamma\in\Gamma} x(\gamma) = \sum_{i=1}^{\infty} x(\gamma_i)$. □

For each $p \ge 1$, the Banach space $\ell_p(\Gamma)$ is defined as follows:

- $\ell_p(\Gamma) = \left\{x \in \mathbb{R}^\Gamma \;\middle|\; \sum_{\gamma\in\Gamma} |x(\gamma)|^p < \infty\right\}$ with the norm

$$\|x\|_p = \left(\sum_{\gamma\in\Gamma} |x(\gamma)|^p\right)^{1/p}.$$

Similar to $\ell_\infty^f(\Gamma)$, we denote the space $\mathbb{R}_f^\Gamma$ with this norm by $\ell_p^f(\Gamma)$.

The triangle inequality for the norm $\|x\|_p$ is known as the Minkowski inequality, which is derived from the following Hölder inequality:

$$\sum_{\gamma\in\Gamma} a_\gamma b_\gamma \le \left(\sum_{\gamma\in\Gamma} a_\gamma^p\right)^{1/p} \left(\sum_{\gamma\in\Gamma} b_\gamma^{\frac{1}{1-1/p}}\right)^{1-1/p} \quad \text{for every } a_\gamma, b_\gamma \ge 0.$$

Indeed, for every $x, y \in \ell_p(\Gamma)$,

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{\gamma\in\Gamma} |x(\gamma)+y(\gamma)|^p \\
&\le \sum_{\gamma\in\Gamma} \big(|x(\gamma)| + |y(\gamma)|\big)|x(\gamma)+y(\gamma)|^{p-1} \\
&= \sum_{\gamma\in\Gamma} |x(\gamma)|\cdot|x(\gamma)+y(\gamma)|^{p-1} + \sum_{\gamma\in\Gamma} |y(\gamma)|\cdot|x(\gamma)+y(\gamma)|^{p-1}
\end{aligned}$$

$$\leq \left(\sum_{\gamma\in\Gamma}|x(\gamma)|^p\right)^{1/p}\left(\sum_{\gamma\in\Gamma}|x(\gamma)+y(\gamma)|^{(p-1)\frac{1}{1-1/p}}\right)^{1-1/p}$$
$$+\left(\sum_{\gamma\in\Gamma}|y(\gamma)|^p\right)^{1/p}\left(\sum_{\gamma\in\Gamma}|x(\gamma)+y(\gamma)|^{(p-1)\frac{1}{1-1/p}}\right)^{1-1/p}$$
$$= (\|x\|_p+\|y\|_p)\big(\|x+y\|_p^p\big)^{1-1/p} = (\|x\|_p+\|y\|_p)\frac{\|x+y\|_p^p}{\|x+y\|_p},$$

so it follows that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

As for $\boldsymbol{c}_0(\Gamma)$, we can show $w(\ell_p(\Gamma)) = \operatorname{card}\Gamma$. When $\operatorname{card}\Gamma = \aleph_0$, the Banach space $\ell_p(\Gamma)$ is linearly isometric to $\ell_p = \ell_p(\mathbb{N})$, which is separable. The space $\ell_2(\Gamma)$ is the Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{\gamma\in\Gamma} x(\gamma)y(\gamma),$$

which is well-defined because

$$\sum_{\gamma\in\Gamma}|x(\gamma)y(\gamma)| \leq \tfrac{1}{2}(\|x\|_2^2 + \|y\|_2^2) < \infty.$$

For $1 \leq p < q$, we have $\ell_p(\Gamma) \subsetneqq \ell_q(\Gamma) \subsetneqq \boldsymbol{c}_0(\Gamma)$ as sets (or linear spaces). These inclusions are continuous because $\|x\|_\infty \leq \|x\|_q \leq \|x\|_p$ for every $x \in \ell_p(\Gamma)$. When Γ is infinite, the topology of $\ell_p(\Gamma)$ is distinct from that induced by the norm $\|\cdot\|_q$ or $\|\cdot\|_\infty$ (i.e., the topology inherited from $\ell_q(\Gamma)$ or $\boldsymbol{c}_0(\Gamma)$). In fact, the unit sphere $\mathbf{S}_{\ell_p(\Gamma)}$ is closed in $\ell_p(\Gamma)$ but not closed in $\ell_q(\Gamma)$ for any $q > p$, nor in $\boldsymbol{c}_0(\Gamma)$. To see this, take distinct $\gamma_i \in \Gamma$, $i \in \mathbb{N}$, and let $(x_n)_{n\in\mathbb{N}}$ be the sequence in $\mathbf{S}_{\ell_p(\Gamma)}$ defined by $x_n(\gamma_i) = n^{-1/p}$ for $i \leq n$ and $x_n(\gamma) = 0$ for $\gamma \neq \gamma_1, \ldots, \gamma_n$. It follows that $\|x_n\|_\infty = n^{-1/p} \to 0$ $(n \to \infty)$ and

$$\|x_n\|_q = \big(n\cdot n^{-q/p}\big)^{1/q} = n^{(p-q)/pq} \to 0 \quad (n\to\infty)$$

because $(p-q)/pq < 0$.

For $1 \leq p \leq \infty$, we have $\mathbb{R}^\Gamma_f \subset \ell_p(\Gamma)$ as sets (or linear spaces). We denote by $\ell_p^f(\Gamma)$ this $\mathbb{R}^\Gamma_f$ with the topology inherited from $\ell_p(\Gamma)$, and we write $\ell_p^f(\mathbb{N}) = \ell_p^f$ (when $\Gamma = \mathbb{N}$). From Proposition 1.2.1, we know $\ell_p^f(\Gamma) \neq \mathbb{R}^\Gamma_f$ as spaces for any infinite set Γ. In the above, the sequence $(x_n)_{n\in\mathbb{N}}$ is contained in the unit sphere $\mathbf{S}_{\ell_p^f(\Gamma)}$ of $\ell_p^f(\Gamma)$, which means that $\mathbf{S}_{\ell_p^f(\Gamma)}$ is not closed in ℓ_q^f, hence $\ell_p^f \neq \ell_q^f$ as spaces for $1 \leq p < q \leq \infty$. Note that $\mathbf{S}_{\ell_p^f(\Gamma)}$ is a closed subset of ℓ_q^f for $1 \leq q < p$.

Concerning the convergence of sequences in $\ell_p(\Gamma)$, we have the following:

Proposition 1.2.4. *For each $p \in \mathbb{N}$ and $x \in \ell_p(\Gamma)$, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in $\ell_p(\Gamma)$ if and only if*

$$\|x\|_p = \lim_{n\to\infty} \|x_n\|_p \ \textit{ and } \ x(\gamma) = \lim_{n\to\infty} x_n(\gamma) \ \textit{ for every } \gamma \in \Gamma.$$

Proof. The "only if" part is trivial, so we concern ourselves with proving the "if" part for $\ell_p(\Gamma)$. For each $\varepsilon > 0$, we have $\gamma_1, \dots, \gamma_k \in \Gamma$ such that

$$\sum_{\gamma\neq\gamma_i} |x(\gamma)|^p = \|x\|_p^p - \sum_{i=1}^{k} |x(\gamma_i)|^p < 2^{-p}\varepsilon^p/4.$$

Choose $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then $\big|\|x_n\|_p^p - \|x\|_p^p\big| < 2^{-p}\varepsilon^p/8$,

$$\big||x_n(\gamma_i)|^p - |x(\gamma_i)|^p\big| < 2^{-p}\varepsilon^p/8k \ \text{ and } \ |x_n(\gamma_i) - x(\gamma_i)|^p < \varepsilon^p/4k$$

for each $i = 1, \dots, k$. Then, it follows that

$$\begin{aligned}
\sum_{\gamma\neq\gamma_i} |x_n(\gamma)|^p &= \|x_n\|_p^p - \sum_{i=1}^{k} |x_n(\gamma_i)|^p \\
&= \|x_n\|_p^p - \|x\|_p^p + \|x\|_p^p - \sum_{i=1}^{k} |x(\gamma_i)|^p \\
&\quad + \sum_{i=1}^{k} \big(|x(\gamma_i)|^p - |x_n(\gamma_i)|^p\big) \\
&< 2^{-p}\varepsilon^p/8 + 2^{-p}\varepsilon^p/4 + 2^{-p}\varepsilon^p/8 = 2^{-p}\varepsilon^p/2,
\end{aligned}$$

and hence we have

$$\begin{aligned}
\|x_n - x\|_p^p &\leq \sum_{i=1}^{k} |x_n(\gamma_i) - x(\gamma_i)|^p + \sum_{\gamma\neq\gamma_i} 2^p \max\big\{|x_n(\gamma)|,\ |x(\gamma)|\big\}^p \\
&< \varepsilon^p/4 + \sum_{\gamma\neq\gamma_i} 2^p|x_n(\gamma)|^p + \sum_{\gamma\neq\gamma_i} 2^p|x(\gamma)|^p \\
&< \varepsilon^p/4 + \varepsilon^p/2 + \varepsilon^p/4 = \varepsilon^p,
\end{aligned}$$

that is, $\|x_n - x\|_p < \varepsilon$. □

Remark 2. It should be noted that Proposition 1.2.4 is valid not only for sequences, but also for nets, which means that the unit spheres $\mathbf{S}_{\ell_p(\Gamma)}$, $p \in \mathbb{N}$, are subspaces of the product space $\mathbb{R}^\Gamma$, whereas $\mathbb{R}^\Gamma$ and $\mathbb{R}^\Gamma_f$ are not metrizable if Γ is uncountable. Therefore, if $1 \leq p < q \leq \infty$, then $\mathbf{S}_{\ell_p(\Gamma)}$ is also a subspace of $\ell_q(\Gamma)$, although, as we have seen, $\mathbf{S}_{\ell_p(\Gamma)}$ of $\ell_p(\Gamma)$ is not closed in the space $\ell_q(\Gamma)$. The unit sphere $\mathbf{S}_{\ell_p^f(\Gamma)}$ of $\ell_p^f(\Gamma)$ is a subspace of $\mathbb{R}_f^\Gamma$ ($\subset \mathbb{R}^\Gamma$), and also a subspace of $\ell_q(\Gamma)$ for $1 \leq q \leq \infty$.

Remark 3. The "if" part of Proposition 1.2.4 does not hold for the space $\boldsymbol{c}_0(\Gamma)$ (although the "only if" part obviously does hold), where Γ is infinite. For instance, take distinct $\gamma_n \in \Gamma$, $n \in \omega$, and let $(x_n)_{n\in\mathbb{N}}$ be the sequence in $\boldsymbol{c}_0(\Gamma)$ defined by $x_n = \mathbf{e}_{\gamma_n} + \mathbf{e}_{\gamma_0}$. Then, $\|x_n\|_\infty = 1$ for each $n \in \mathbb{N}$,

$$\lim_{n\to\infty} x_n(\gamma_0) = 1 = \mathbf{e}_{\gamma_0}(\gamma_0) \text{ and } \lim_{n\to\infty} x_n(\gamma) = 0 = \mathbf{e}_{\gamma_0}(\gamma) \text{ for } \gamma \neq \gamma_0,$$

but $\|x_n - \mathbf{e}_{\gamma_0}\|_\infty = 1$ for every $n \in \Gamma$. In addition, the unit sphere $\mathbf{S}_{\boldsymbol{c}_0(\Gamma)}$ of $\boldsymbol{c}_0(\Gamma)$ is not a subspace of $\mathbb{R}^\Gamma$, because $\mathbf{e}_{\gamma_n} \in \mathbf{S}_{\boldsymbol{c}_0(\Gamma)}$ but $(\mathbf{e}_{\gamma_n})_{n\in\mathbb{N}}$ converges to $\mathbf{0}$ in $\mathbb{R}^\Gamma$.

Concerning the topological classification of $\ell_p(\Gamma)$, we have the following:

Theorem 1.2.5 (Mazur). *For each $1 < p < \infty$, $\ell_p(\Gamma)$ is homeomorphic to $\ell_1(\Gamma)$. By the same homeomorphism, $\ell_p^f(\Gamma)$ is also homeomorphic to $\ell_1^f(\Gamma)$.*

Proof. We define $\varphi : \ell_1(\Gamma) \to \ell_p(\Gamma)$ and $\psi : \ell_p(\Gamma) \to \ell_1(\Gamma)$ as follows:

$$\varphi(x)(\gamma) = \operatorname{sign} x(\gamma) \cdot |x(\gamma)|^{1/p} \text{ for } x \in \ell_1(\Gamma),$$
$$\psi(x)(\gamma) = \operatorname{sign} x(\gamma) \cdot |x(\gamma)|^{p} \text{ for } x \in \ell_p(\Gamma),$$

where $\operatorname{sign} 0 = 0$ and $\operatorname{sign} a = a/|a|$ for $a \neq 0$. We can apply Proposition 1.2.4 to verify the continuity of φ and ψ. In fact, the following functions are continuous:

$$\ell_1(\Gamma) \ni x \mapsto \|\varphi(x)\|_p = \big(\|x\|_1\big)^{1/p} \in \mathbb{R},\ \ \ell_1(\Gamma) \ni x \mapsto \varphi(x)(\gamma) \in \mathbb{R},\ \gamma \in \Gamma;$$
$$\ell_p(\Gamma) \ni x \mapsto \|\psi(x)\|_1 = \big(\|x\|_p\big)^{p} \in \mathbb{R},\ \ \ell_p(\Gamma) \ni x \mapsto \psi(x)(\gamma) \in \mathbb{R},\ \gamma \in \Gamma.$$

Observe that $\psi\varphi = \mathrm{id}$ and $\varphi\psi = \mathrm{id}$. Thus, φ is a homeomorphism with $\varphi^{-1} = \psi$, where $\varphi(\ell_p^f(\Gamma)) \subset \ell_1^f(\Gamma)$ and $\psi(\ell_1^f(\Gamma)) \subset \ell_p^f(\Gamma)$. □

For each space X, we denote $\mathrm{C}(X) = \mathrm{C}(X, \mathbb{R})$. The Banach space $\mathrm{C}^B(X)$ is defined as follows:

- $\mathrm{C}^B(X) = \{f \in \mathrm{C}(X) \mid \sup_{x\in X} |f(x)| < \infty\}$ with the sup-norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

This sup-norm of $\mathrm{C}^B(X)$ induces the uniform convergence topology. If X is discrete and infinite, then we have $\mathrm{C}^B(X) = \ell_\infty(X)$, and so, in particular, $\mathrm{C}^B(\mathbb{N}) = \ell_\infty$. When X is compact, $\mathrm{C}^B(X) = \mathrm{C}(X)$ and the topology induced by the norm coincides with the compact-open topology.

The **uniform convergence topology** of $\mathrm{C}(X)$ is induced by the following metric:

$$d(f, g) = \sup_{x \in X} \min\{|f(x) - g(x)|,\ 1\}.$$

As can be easily observed, $\mathrm{C}^B(X)$ is closed and open in $\mathrm{C}(X)$ under the uniform convergence topology. Note that $\mathrm{C}^B(X)$ is a component of the space $\mathrm{C}(X)$ because $\mathrm{C}^B(X)$ is path-connected as a normed linear space.

Regarding $\mathrm{C}(X)$ as a subspace of the product space $\mathbb{R}^X$, we can introduce a topology on $\mathrm{C}(X)$, which is called the **pointwise convergence topology**. With respect to this topology,

$$\lim_{n\to\infty} f_n = f \quad \Leftrightarrow \quad \lim_{n\to\infty} f_n(x) = f(x) \text{ for every } x \in X.$$

The space $\mathrm{C}(X)$ with the pointwise convergence topology is usually denoted by $\mathrm{C}_p(X)$. The space $\mathrm{C}_p(\mathbb{N})$ is simply the space of sequences $\boldsymbol{s} = \mathbb{R}^{\mathbb{N}}$.

In this chapter, three topologies on $\mathrm{C}(X)$ have been considered — the compact-open topology, the uniform convergence topology, and the pointwise convergence topology. Among them, the uniform convergence topology is the finest and the pointwise convergence topology is the coarsest.

Notes for Chap. 1

Theorem 1.2.5 is due to Mazur [3]. Zhongqiang Yang pointed out that Proposition 1.2.4 can be applied to show the continuity of φ and ψ in the proof of Theorem 1.2.5. Related to Mazur's result, Anderson [1] proved that $\boldsymbol{s} = \mathbb{R}^{\mathbb{N}}$ is homeomorphic to the Hilbert space ℓ_2. For an elementary proof, refer to [2].

References

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