Aggregate "Calculation" in Economic Phenomena: Distributions and Fluctuations

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Abstract. I review recent studies on distributions and fluctuations for personal-income and firm-size in real-economy by using recently available large-data. Specifically explained are Pareto-Zipf laws, Gibrat's law and detailed-balance, and the fact that they are mutually related in a simple way. These patterns and shapes are not "laws" in physics, but can break down in abnormal situations such as bubble-collapse. These findings provide an important foundation of phenomenology for real-economy. The expression "aggregate calculation" nicely fits into the paradigm of this workshop.

Keywords: Pareto-Zipf laws, Gibrat's law, detailed-balance, income distributions, firm-size distributions, growth and fluctuation.

1 Introduction

Phenomenology is crucial for our understanding of natural and human systems. Thermodynamics in physics is a beautiful example of phenomenology on macroscopic systems such as gas, water, light, all materials around us. It took more than a century to establish the phenomenological laws of thermodynamics for systems in non-equilibrium as well as in equilibrium, which later played a crucial role for constructing statistical mechanics based on microscopic principles for equilibrium systems. The role is crucial — statistical mechanics does not give a microscopic foundation of thermodynamics; rather, statistical mechanics is founded on the consistency and compatibility with thermodynamics[1].

Economic phenomena have phenomenological laws. Even if the economy consistes of millions of individuals, firms, banks and other agents at microscopic levels, the aggregate behaviors quite often display patterns and shapes at macroscopic levels. The presence of such patterns and shapes illuminate the validity of a certain phenomenology, although they are not be "laws" in physics. In social science, the emergence of macroscopic patterns from microscopic levels have been lucidly illustrated in many papers and books, for example, "Micromotives and

[1] It took me a long time after learning statistical mechanics in university class to recognize the important role of thermodynamics. Thanks to recent textbooks by Professors Yoshi Oono and Hal Tasaki.

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Macrobehavior" by Thomas Schelling. Recently, thanks to abundant electronic data of economic and social systems, researchers able to establish phenomenological findings in more quantitative ways. A good example is high-frequency data in financial markets which have been studies to a great extent. I shall focus on real-economy in this paper.

Specifically I shall show that

– distributions and fluctuations of personal income and firm-size have interesting phenomenological patterns and shapes;
– they are Pareto-Zipf laws, Gibrat's law and detailed-balance, which are mutually related in a simple way;
– but the patterns and shapes break down under abnormal situations in economy,

by using large-data of personal income and firm-size.

These patterns and shapes of distributions and fluctuations are quite robust in the sense that they can be observed for different choice of variables, for various industrial sectors and countries, even if they can break down. This phenomenological fact implies that the finding is linked to the dynamics of aggregate behavior of individuals and firms. Each individual or firm may be in growth, shrink and in birth-death (entry-exit); but, stochastically, the aggregated system is relatively stable unless it is subject to large shocks. In my opinion, it looks as if the system were "calculating" in a self-organized way. Honestly I do not know how to define calculation, say in biological system or in social system, but I think that the word "aggregate and stochastic calculation" nicely expresses the phenomenology I shall explain in what follows.

The figures in this paper were reproduced from the publication [1, pp.24–33]. Interested readers are also suggested to read the book [2] further.

2 Distributions

The distributions for personal income or firm-size are strikingly different from those for human heights, for example. One has no chance to observe persons 1km tall nor 1mm small. The universe of personal income or firm-size is dominated by giants and dwarfs — *a few giants and many dwarfs* — in the terminology of economics. Two examples of such distributions are shown in Fig. 1 and Fig. 2.

Let x be personal income or firm-size. Cumulative probability distribution $P_>(x)$ is the probability that a given individual or firm has income or size equal to, or greater than x. It is obvious from Fig. 1 and Fig. 2 that for large x one has a power-law:

$$P_>(x) \propto x^{-\mu} , \tag{1}$$

where μ is a constant, often called Pareto index. This phenomenon, now known as Pareto law, has been widely observed. μ is typically around 2 for personal income and around 1 for firm size distribution. The latter case is often referred to as Zipf law. We call the distribution (1) Pareto-Zipf law in this paper.

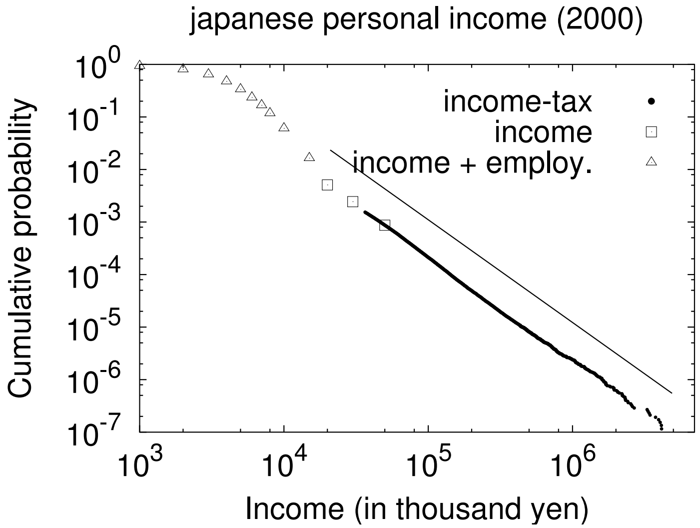


Fig. 1. Cumulative probability distribution of Japanese personal income in the year 2000. The line is simply a guide for eyes with $\mu = 1.96$ in (1). Note that the dots are income tax data of about 80,000 taxpayers. Reproduced from [1].

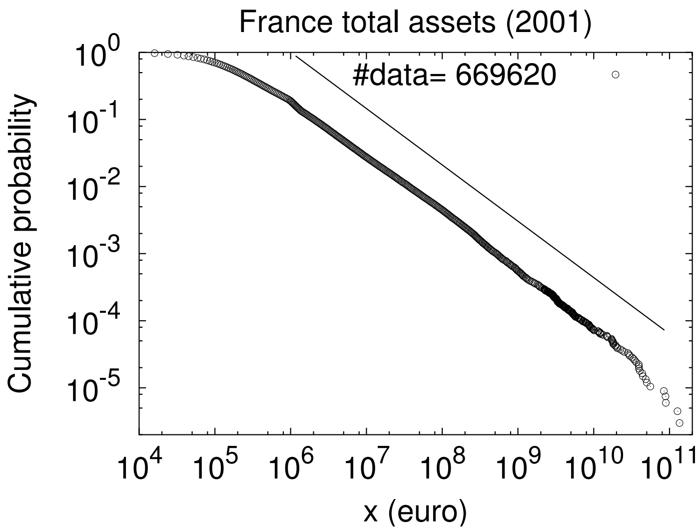


Fig. 2. Cumulative probability distribution of firm size (total-assets) in France in the year 2001. Data consist of 669,620 firms, which are exhaustive in the sense that firms exceeding a threshold are all listed. The line corresponds to $\mu = 0.84$ in (1). Reproduced from [1].

Note that $P_>(x)$ is a cumulative probability distribution function (cumulative PDF). The probability distribution (density) function (PDF), $P(x)$, is related to $P_>(x)$ as

$$P(x) = -\frac{dP_>(x)}{dx} \ , \tag{2}$$

by definition.

Understanding the origin of the law has importance in economics because of its linkage with consumption, business cycles, and other macro-economic activities. Also note that even if the range for which (1) is valid is a few percent in the upper tail of the distribution, it is often observed that such a small fraction of individuals (firms) occupies a large amount of total sum of income (size). Small idiosyncratic shock can make a considerable macro-economic impact.

3 Fluctuations

Pareto's power-law is commonly found in many phenomena observed within natural science, as well as the social and economic sciences. Most researchers in these fields would accept that there is no single mechanism giving a universal explanation for all these phenomena, but rather that different mechanisms would be found for each of a variety of classes of phenomena. Mechanisms explaining the origins of power-laws in natural and social sciences have been uncovered for more than a dozen classes of phenomena. See references given in [2, Chapter 3].

Personal income and firm-size obviously change over time: last year's sales are not equal to this year's. Thus we can see that observations on the distribution of company size are an instantaneous snapshot of the state of a collection of companies, each of which is individually subject to fluctuations.

Let us denote by x_1 and x_2 a firm's sizes at time t_1 and a succeeding time t_2 respectively (or a person's incomes). To examine the temporal change, we define growth-rate by

$$R = \frac{x_2}{x_1} \ . \tag{3}$$

The variable R represents the ratio at which a company increases or decreases its size, such as s ales and total-assets, in the time interval. In addition, we will define the logarithmic growth-rate as

$$r = \log_{10} R = \log_{10} x_2 - \log_{10} x_1 \ , \tag{4}$$

where $\log_{10}$ is a logarithm with base 10. Thus $r = 1$ and $r = -1$ correspond to $R = 10$ and $R = 0.1$ respectively.

By identifying each high-income earner and large-size firm, one can directly observe the temporal change from x_1 to x_2 at one year and the next year, say. The examples are shown in the scatter plots of Fig. 3 and Fig. 4.

The scatter plot represent the joint distribution $P_{12}(x_1, x_2)$ for the pair of variables x_1 and x_2. The plots in Fig. 3 and Fig. 4 are consistent with what is

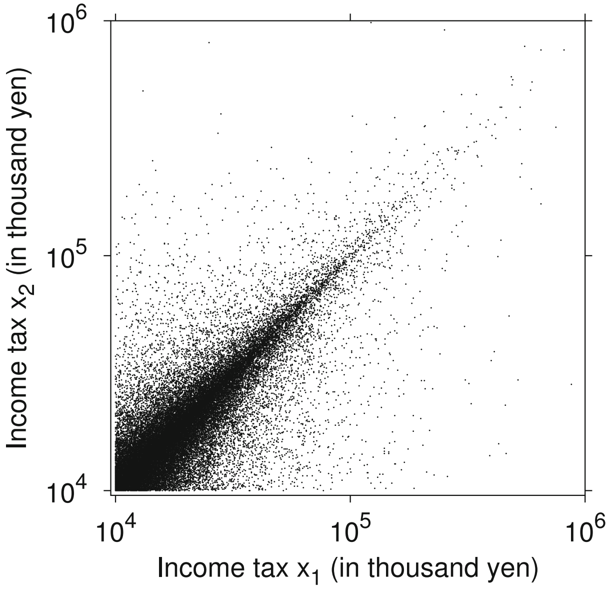


Fig. 3. Scatter-plot of all individuals whose income tax exceeds 10 million yen in both 1997 and 1998. These points (52,902) were identified from the complete list of high-income taxpayers in 1997 and 1998, with income taxes x_1 and x_2 in each year. Reproduced from [1].

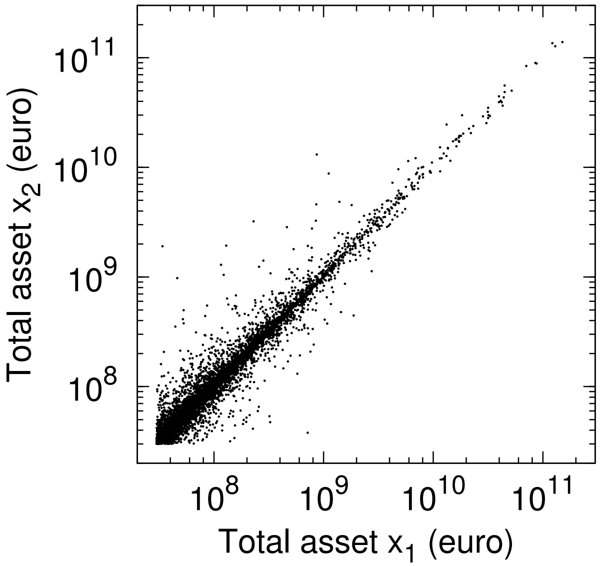


Fig. 4. Scatter-plot of the firms in France whose firm size (total-assets) exceeds a certain threshold in both 2000 and 2001

called *detailed-balance* in the sense that the joint distribution is invariant under the exchange of values x_1 and x_2, i.e.

$$P_{12}(x_1, x_2) = P_{12}(x_2, x_1) \ . \tag{5}$$

One can actually perform a statistical test for the symmetry in the two arguments of $P_{12}(x_1, x_2)$ by two-dimensional Kolmogorov-Smirnov test. Detailed-balance means that the empirical probability for an individual to change its income from a value to another is statistically the same as that for its reverse process in the ensemble.

4 Distributions of Growth-Rates

Finally we shall take a look at the distributions of growth, R or equivalently r. We examine the PDF for the growth rate $P(r|x_1)$ on the condition that the income x_1 in the initial year is fixed. By doing this, we can see whether the growth of personal income or firm-size depends on the starting income or size of x_1. In other words, do giants grow at faster or slower pace than dwarfs do?

The distributions for such growth rates are shown in Fig. 5 (a) and Fig. 5 (b). Note that the different curves collapse into a single curve of distribution in each of the figures. This means that the distribution for growth rate r is statistically independent of the value of x_1. This is known as *law of proportionate effect* or *Gibrat's law* [3].

This can be stated mathematically as

$$P(r|x_1) = \text{a function of } r \text{ only} \equiv Q(r) \ , \tag{6}$$

where $Q(r)$ is a function of r, which has different functional forms for different variables.

The probability distribution for the growth rate, such as the one observed in Fig. 5, contains information of dynamics. One can notice that it has a skewed and heavy-tailed shape with a peak at $R = 1$. How is such a functional form consistent with the detailed-balance shown in Fig. 3? And how these phenomenological facts are consistent with Pareto's law in Fig. 1? Answers to these questions are given in the next section.

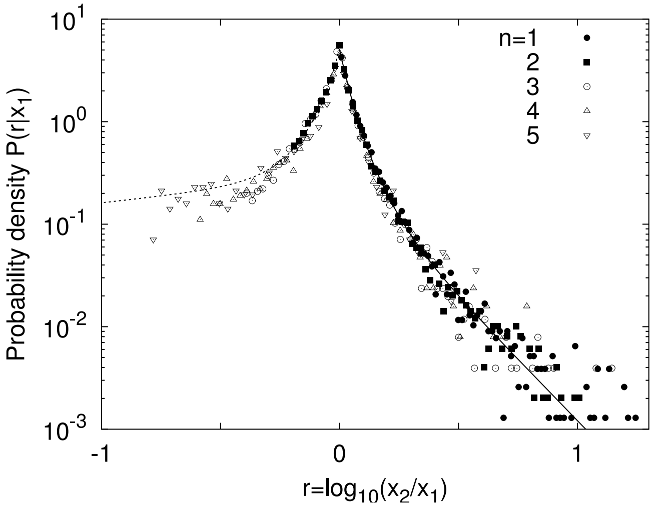
5 Relations among the Laws

To summarize the empirical findings, one has the Pareto's law in (1), the detailed-balance in (5), and the Gibrat's law in (6). We shall show that Gibrat's law and the detailed balance lead to Pareto's law.

Since the pair of variables (x_1, x_2) and that of (x_1, R) are related by the change of variable, $R = x_2/x_1$, one can easily see that the joint probability distribution $P_{1R}(x_1, R)$ is related to the joint probability distribution $P_{12}(x_1, x_2)$ by

$$P_{12}(x_1, x_2) = \frac{1}{x_1} P_{1R}(x_1, R) \ . \tag{7}$$

(a)



(b)

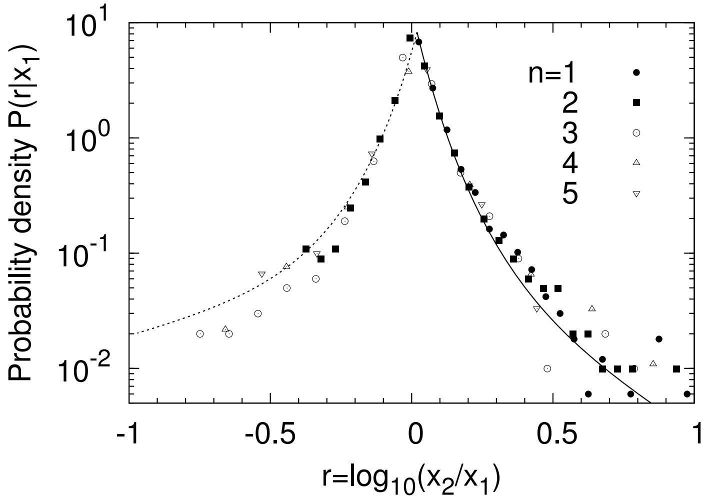


Fig. 5. (a) Probability density $P(r|x_1)$ of growth rate of individual income in Japan, $r \equiv \log_{10}(x_2/x_1)$, from 1997 to 1998 for income exceeding a threshold. Different bins of initial income-tax with equal size in logarithmic scale were taken to plot probability densities separately for each such bin $n = 1, \cdots, 5$. The solid line in the portion of positive growth $(r > 0)$ is an analytic fit. The dashed line $(r < 0)$ on the other side is calculated by the relation in (12). Reproduced from [1]. (b) Probability density $P(r|x_1)$ of growth rate of firm-size in France, $r \equiv \log_{10}(x_2/x_1)$, from 2000 to 2001 for size exceeding a threshold. Different bins of initial income-tax with equal size in logarithmic scale were taken to plot probability densities separately for each such bin $n = 1, \cdots, 5$. The solid line in the portion of positive growth $(r > 0)$ is an analytic fit. The dashed line $(r < 0)$ on the other side is calculated by the relation in (12).

Now, the conditional PDF $P(R|x_1)$ for the growth-rate satisfies

$$P_{1R}(x_1, R) = P(R|x_1)\, P(x_1) \ , \tag{8}$$

by definition, where $P(x_1)$ is the PDF for the size x_1.

Assume that detailed balance holds, then (7) yields

$$P_{1R}(x_1, R) = \frac{1}{R} P_{1R}(Rx_1, R^{-1}) \ , \tag{9}$$

as readily shown by a simple calculation. Therefore, (9) and (8) lead us to

$$\frac{P(R^{-1}|x_2)}{P(R|x_1)} = R\frac{P(x_1)}{P(x_2)} \ . \tag{10}$$

Note that this is a consequence from detailed balance alone.

With the additional assumption of Gibrat's law in (6), one can immediately rewrite (10) as

$$\frac{P(x_1)}{P(x_2)} = \frac{1}{R}\frac{Q(R^{-1})}{Q(R)} \ . \tag{11}$$

Note that the left-hand side of (11) is a function of x_1 and $x_2 = R\,x_1$, while the right-hand side of (11) is a function of R. The equality holds if and only if $P(\cdot)$ is a power-law function, $P(x) \propto x^{-\mu-1}$, where μ is a constant. (For example, expand (11) in terms of R around $R = 1$, and obtain a differential equation that $P(\cdot)$ has to satisfy, which can be solved easily.) By integrating (2), one has the Pareto's law in (1).

Furthermore, by inserting the power-law function $P(x) \propto x^{-\mu-1}$ into (11), one has

$$Q(R) = R^{-\mu-2} Q(R^{-1}) \ , \tag{12}$$

which relates the positive and negative growth rates, $R > 1$ and $R < 1$, through the Pareto index μ. This is a non-trivial consequence of our argument here, and can be checked for its validity in the real data. See Fig. 5 (a) and Fig. 5 (b).

Therefore, the phenomenological properties (A) detailed-balance, (B) Pareto-Zipf law, and (C) Gibrat's law are observed for firm size as well as for personal income.

6 Summary

We have shown the following stylized facts concerning distribution of personal income and firm size, their growth and fluctuations by studying exhaustive lists of high-income persons and firm sizes in Japan and in Europe.

- In power-law regime, detailed-balance and Gibrat's law hold.
- Under the condition of detailed-balance, Gibrat's law implies Pareto's law.
- Growth-rate distribution has a non-trivial relation between its positive and negative growth sides through Pareto index.

The empirical “laws” of Pareto, Gibrat, and detailed-balance are not laws in physics, but patterns and properties of economics. They do not hold in non-power-law regimes, or can break down when the economy is under abnormal conditions.

- Power-law, detailed-balance and Gibrat’s law break down according to abrupt change in risky asset market, such as Japanese “bubble” collapse of real estate and stock.
- For firm size in non-power-law regime corresponding to small and middle size firms, Gibrat’s law does not hold. Instead, there is a scaling relation of variance in the growth-rates of those firms with respect to firm size, which asymptotically approaches to non-scaling region as firm size comes to power-law regime.

See [1,2] and references therein.

The stylized facts that we described in this paper, however, serve as an established phenomenology, which any models for households and firms activities should satisfy.

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