Chapter 3
Basics of QCD Sum Rules

3.1 Introduction

The method of QCD sum rules was invented in the late seventies (Shifman et al. 1979a, b) and then further developed in the subsequent years (Ioffe 1981a, b; Reinders et al. 1985). It is a useful tool in hadron phenomenology, to obtain a qualitative (in some cases even quantitative) understanding of the properties of hadrons. To do this, one exploits the analytic properties of various correlation functions of certain interpolating fields to connect two separate limits of QCD. On one hand, we have the high energy (or short distance) limit, where asymptotic freedom makes it possible to use perturbative methods to describe the behavior of quarks and gluons, which are the active degrees of freedom in this regime. Concretely, the operator product expansion (OPE) can be employed to evaluate the correlators, explicitly calculating its perturbative parts, while the non-perturbative contributions are stored in the vacuum expectation values of gauge invariant operators. On the other hand, in the low energy (or large distance) limit, the relevant degrees of freedom are the hadrons, whose properties we aim to extract by the sum rules. By this procedure, we can then connect non-perturbative information of the QCD vacuum, parametrized as various vacuum condensates, with the physical quantities of actual hadrons. Discussions on early applications of QCD sum rules, their extensions to a broader range of systems as well as technical details of the actual calculations can be found in various reviews and books (Shifman 1998; Colangelo and Khodjamirian 2001; Leinweber 1997; Ioffe et al. 2010; Pascual and Tarrach 1984; Kojo 2008), on which I partly relied when writing this review.

As the most simple case, let us consider the two-point function (in the following referred to as "correlator") of an interpolating field $J(x)$, which is composed of quark- and/or gluon-fields and carries the quantum number of the hadronic state that we aim to investigate:

$$\Pi(q^2) = i \int d^4x e^{iqx} \langle 0|T[J(x)\bar{J}(0)]|0\rangle. \tag{3.1}$$

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Starting from this expression, the analyticity of the function $\Pi(q^2)$ in the whole imaginary plane of the variable q^2 except the positive part of the real axis allows one to write down the dispersion relation

$$\begin{aligned}\Pi(q^2) &= \frac{1}{\pi}\int_0^\infty ds \frac{\mathrm{Im}\Pi(s+i\varepsilon)}{s-q^2} \\ &= \int_0^\infty ds \frac{\rho^J(s)}{s-q^2},\end{aligned} \tag{3.2}$$

where the spectral function related to the interpolating field $J(x)$ has been defined as $\frac{1}{\pi}\Pi(s+i\varepsilon) = \rho^J(s)$. The above equation connects the region of the correlator where calculations using perturbation theory are, in principle, possible ($|q^2| \to \infty$) with the region which is of actual physical interest ($q^2 \sim E^2_{\text{ground state}}$). A detailed derivation of Eq. (3.2) can be found in Appendix A.

3.1.1 The Theoretical Side

For extracting information on the spectral function $\rho^J(s)$ from the sum rule of Eq. (3.2), one first needs to calculate its left hand side, $\Pi(q^2)$ in the deep Euclidean region, where $-q^2$ goes to ∞. This is done by perturbative methods, but simple minded perturbation theory is not enough. Even though, due to asymptotic freedom, the coupling constant approaches 0 at $|q^2| \to \infty$, there are sizable non-perturbative contributions, which have to be taken into account. For this purpose, the authors of Shifman et al. (1979a, b) proposed to use the OPE, originally developed by Wilson Wilson (1969), to incorporate the non-perturbative effects of the low-energy vacuum fields interacting with the high-energy quarks and gluons induced by the external current $J(x)$.

This treatment leads to an expansion of the correlator into a various local operators O_d, ordered by their mass dimension d and the corresponding Wilson coefficients:

$$\Pi(q^2) = \sum_d C_d(q^2)\langle 0|O_d|0\rangle. \tag{3.3}$$

As the local operators are sandwiched between the vacuum state, only Lorentz- and Gauge-invariant operators with positive parity are allowed in the above expansion. The leading operator is the unit operator ($O_0 = 1$) and its Wilson coefficient stands for the correlator of Eq. (3.1), evaluated by standard perturbation theory. The next operators in the expansion can be constructed from all possible Lorentz- and Gauge-invariant local combinations of quark and gluon fields:

$$
\begin{aligned}
&\text{dim. 3}: \quad \langle 0|\overline{q}q|0\rangle, \\
&\text{dim. 4}: \quad \langle 0|G^a_{\mu\nu}G^{a\mu\nu}|0\rangle, \\
&\text{dim. 5}: \quad \langle 0|\overline{q}\sigma_{\mu\nu}\frac{\lambda^a}{2}G^{a\mu\nu}q|0\rangle, \\
&\text{dim. 6}: \quad \langle 0|\overline{q}q\overline{q}q|0\rangle, \langle 0|\overline{q}\gamma_5 q\overline{q}\gamma_5 q|0\rangle, \\
&\qquad\quad \langle 0|\overline{q}\frac{\lambda^a}{2}q\overline{q}\frac{\lambda^a}{2}q|0\rangle, \langle 0|\overline{q}\gamma^\mu\frac{\lambda^a}{2}q\overline{q}\gamma_\mu\frac{\lambda^a}{2}q|0\rangle, \ldots \\
&\qquad\quad \langle 0|g^3 f^{abc}G^{a\nu}_{\mu}G^{b\lambda}_{\nu}G^{c\mu}_{\lambda}|0\rangle, \\
&\qquad\quad \ldots
\end{aligned}
\tag{3.4}
$$

The vacuum expectation values of these operators are usually referred to as "condensates". Among these shown above, only the ones up to dimension 5 have been thoroughly studied and more or less reliable estimates on their actual values exist. As for the many possible versions of the four-quark condensates, they can all be related to the square of the chiral condensate of dimension 3, if one uses Fierz transformation in combination with the vacuum saturation approximation (which will be discussed later). The dimension 6 gluon condensate can also be related to the dimension 4 gluon condensate, with the help of the dilute instanton gas model (Novikov et al. 1979). For condensates with even higher dimension, the uncertainty of their values is much larger, as the validity of the above mentioned approximations used for the dimension 6 condensates is not at all established. Therefore, to obtain reliable results it is important that the OPE is dominated by the first few dimensional terms, and that the contributions from condensates larger that 6 only give small corrections.

3.1.2 The Phenomenological Side

Coming back to the dispersion relation in Eq. (3.2), we now consider the right hand side. Using the optical theorem and inserting a complete set of intermediate hadronic states, we can write

$$
\mathrm{Im}\Pi(q^2) = \frac{1}{2}\sum_n \langle 0|J|n(p_n)\rangle\langle n(p_n)|\overline{J}|0\rangle d\tau_n (2\pi)^4\delta^{(4)}(q-p_n), \tag{3.5}
$$

where n is summed over all hadronic states coupling to J, including sums over polarizations. $d\tau_n$ denotes the integration over the phase space of the states $|n\rangle$. To calculate the spectral function $\rho(s) = 1/\pi\,\mathrm{Im}\Pi(s+\varepsilon)$ and to completely understand its behavior is one of the major goals of hadron physics and can not be done easily.

As the sum rule of Eq. (3.2) only provides information on an integral over the spectral function, one can only hope to extract from it some bulk properties of the spectrum, but not all its detailed features. Therefore, traditionally it has been the custom in practical sum rule analyses to make a deliberated guess on the concrete structure

of the spectral function, parametrize this structure with a small number of parameters and finally fit these parameters with the help of the sum rule. In this thesis, we will adopt a different approach, trying to obtain the spectral function directly from the sum rule with the help of the maximum entropy method. The details of this procedure are explained in Chapter 4 of this thesis.

Here, the conventional approach to the sum rules will be explained, which is important for understanding the motivation for the Bayesian approach, proposed in the present work. As a first step, Eq. (3.5) is divided into two energy regions:

$$\mathrm{Im}\Pi(s) \equiv \theta(s_{th} - s)\mathrm{Im}\Pi^{<}(s) + \theta(s - s_{th})\mathrm{Im}\Pi^{>}(s). \tag{3.6}$$

s_{th} is the so-called threshold parameter and marks the boundary of the two domains. The high energy part of the correlator is then approximated using the quark-hadron duality (Poggio et al. 1976), which connects the hadronic spectral function with the perturbative calculations using the OPE technique, for which the degrees of freedom are quarks and gluons:

$$\mathrm{Im}\Pi^{>}(s) \approx \Pi^{OPE}(s). \tag{3.7}$$

This approximation is only valid at sufficiently high energies, where the perturbative terms dominate over the low-energy correlations.

Usually, one is interested in the ground state or lowest resonance coupling to $J(x)$, which lies below the threshold parameter s_{th}. Here, QCD sum rules often use the one pole approximation, completely ignoring the possible width of the state and potentially occurring scattering states. Such scattering states may be present even below the lowest peak, if it is a resonance. The one pole approximation is given as

$$\mathrm{Im}\Pi^{<}(s) \approx \pi|\lambda|^2\delta(s - m^2), \tag{3.8}$$

where m is the mass of the perceived ground state and $|\lambda|^2$ is the coupling strength of the pole to the interpolating field $J(x)$. Equations (3.7) and (3.8) are the basic assumptions of the usual QCD sum rule analysis and especially in the case of Eq. (3.8), one in principle does not know if it is accurate without any additional information. Thus, the validity of these assumptions has to be carefully checked in the course of the calculation.

3.1.3 Practical Versions of the Sum Rules

Even though we have already obtained a sum rule in Eq. (3.2), it is for several reasons not very useful in this form. First of all, the integral over the spectral function most likely does not converge to a finite value. As explained in Appendix A, this problem can be cured by introducing subtraction terms, which are polynomials in q^2 with infinite coefficients. However, at this point, both sides of Eq. (3.2) still diverge and the sum rule is therefore quite useless in practice. Furthermore, experience of actual

OPE calculations show that the convergence of the expansion in the present form is slow and high-order terms can be quite important.

To improve the situation, it was proposed in Shifman et al. (1979a, b) to use the Borel transformation $\widehat{L}_M$, which is defined as

$$\Pi(M^2) \equiv \widehat{L}_M \Pi(q^2) \equiv \lim_{\substack{-q^2,n\to\infty,\\ -q^2/n=M^2}} \frac{(-q^2)^n}{(n-1)!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2). \tag{3.9}$$

After applying this operator to the dispersion relation of Eq. (3.2), one obtains

$$\Pi(M^2) = \frac{1}{M^2}\int_0^\infty ds e^{-s/M^2}\rho^J(s). \tag{3.10}$$

As one can see in Eq. (3.9), the Borel transformation includes an infinite number of derivatives, therefore all subtraction terms automatically vanish. At the same time, the high energy part of the dispersion integral is exponentially suppressed, meaning that the integral now converges to a finite value (as long as the spectral function itself does not grow exponentially, which is not the case in practice).

Some typical and useful examples of the Borel transformation are shown below,

$$\widehat{L}_M(q^2)^k = 0, \tag{3.11}$$

$$\widehat{L}_M(q^2)^k \ln(-q^2) = -k!(M^2)^k, \tag{3.12}$$

$$\widehat{L}_M\left(\frac{1}{q^2}\right)^k = \frac{(-1)^k}{(k-1)!}\left(\frac{1}{M^2}\right)^k, \tag{3.13}$$

$$\widehat{L}_M\left(\frac{1}{s-q^2}\right)^k = \frac{1}{(k-1)!}\left(\frac{1}{M^2}\right)^k e^{-s/M^2}. \tag{3.14}$$

Here, k is a positive integer, and M the so-called Borel mass. As will be discussed later, higher dimensional terms are usually proportional to $(1/q^2)^k$, the power growing with dimension. Hence, it is observed in Eq. (3.13) that higher dimensional terms of the OPE are suppressed by an additional factor of $\frac{1}{(k-1)!}$, which considerably improves the convergence of the OPE.

The Borel transformed sum rule is, however, not the only way to improve the behavior of Eq. (3.2). As we will see in Chap. 6 of this thesis, there are cases in which one can do better. For instance, in Bertlmann et al. (1985), a Gaussian kernel was derived instead of the exponential one in Eq. (3.10). This leads to the "Gaussian sum rules", which can be given as,

$$\Pi(s,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_0^\infty dt e^{-\frac{(t-s)^2}{4\tau}}\rho^J(t), \tag{3.15}$$

where s and τ are free parameters and correspond to the Borel mass M in the Borel sum rule case. The advantage of this sum rule is that two parameters can be varied,

allowing one to extract more information on the spectral function $\rho^J(t)$ from the sum rule. Furthermore, the kernel of the Gaussian sum rule has a distinct peak at $t = s$, which means that the various peak structures that might be present in the spectral function are more likely to be preserved in $\Pi(s, \tau)$, rather than washed out as it is often the case for the Borel sum rule.

To summarize, Eqs. (3.10) and (3.15) are the final forms of the sum rules that will be repeatedly appear in this work. The goal is now to analyze these equations and to extract as much information as possible from them with the least amount of artificial assumptions. Our proposed approach of how to deal with this task will be discussed in Chap. 4 of this thesis.

3.2 More on the Operator Product Expansion

The operator product expansion (OPE) has already appeared in Eq. (3.3) of the preceding section. As this expansion plays a key role for the formulation of QCD sum rules, we will discuss it in more detail below.

3.2.1 Theoretical Foundations

The original idea of the OPE, proposed by Wilson (1969), can be formulated as follows:

$$\mathrm{T}\big[\widehat{A}(x)\widehat{B}(y)\big] \xrightarrow{x\to y} \sum_n C_n(x-y)\,\widehat{O}_n\Big(\frac{x+y}{2}\Big). \tag{3.16}$$

Spelled out in words, the above statement means that when two local operators are located at nearby space-time points, they can be rewritten as a series of local operators, defined at the average point between the two original operators, with respective numerical coefficients (the "Wilson coefficients"), which can be calculated perturbatively, due to asymptotic freedom of QCD. Fourier transforming the relation of Eq. (3.16) into momentum space and assuming that the local operators $\widehat{O}_n$ do not depend on the space-time coordinates, we get,

$$i\int d^4x e^{iq(x-y)}\mathrm{T}\big[\widehat{A}(x)\widehat{B}(y)\big] \xrightarrow{|q^2|\to\infty} \sum_n C_n(q^2)\,\widehat{O}_n. \tag{3.17}$$

It can be explicitly shown that the correlator at $|q^2| \to \infty$ corresponds to the short distance limit of Eq. (3.16) (Colangelo and Khodjamirian 2001). This Fourier transformed OPE makes it more apparent that one here essentially carries out a division of scales (also referred to as "factorization"), meaning that the perturbative high energy part goes into the Wilson coefficients, while the low energy parts contribute to the local operators.

The OPE is formulated to be a relation between operators, therefore it does not depend on what kind of matrix elements one calculates from Eqs. (3.16) and (3.17). Thus, the OPE provides some universal relations between various different matrix elements, which belong to different processes. These relations are exploited especially in the analysis of deep inelastic scattering, where the OPE has proven to be a very useful tool (Muta 1998). However, one should keep in mind that, even today, the formulation of the OPE is in fact not yet complete. Even though the correctness of the OPE was proven by Zimmermann in a completely perturbative regime (Zimmermann 1970), no proof for it is given for the case when non-perturbative effects are included. This situation is not very satisfactory and a proof for the general non-perturbative case is certainly desirable. However, the success of the OPE in its many places of application does at least provide convincing evidence for its validity.

3.2.1.1 Ordering of Operators According to Their Dimension

To understand why the OPE is a valid expansion and why usually only the first few terms suffice to obtain reliable results, one can refer to an easy argument by dimensional analysis. In the short distance (or large energy) limit $x \to y$, quark masses and other low energy scales can be ignored, meaning that the only dimensionful quantity appearing in the Wilson coefficients is $(x-y)^2$. Thus, in this limit, the functional dependence of $C_n(x-y)$ is fixed by its dimension. Denoting the mass dimensions of $\widehat{A}(x)$, $\widehat{B}(y)$ and $\widehat{O}_n$ as d_A, d_B and d_n, respectively, $C_n(x-y)$ will behave in the following way:

$$C_n(x-y) \xrightarrow{x\to y} \left(\frac{1}{(x-y)^2}\right)^{(d_A+d_B-d_n)/2}. \tag{3.18}$$

As the dimension of the condensates becomes larger, the corresponding singularities are weakened, until at a certain dimension they vanish completely. In momentum space the above relation is rewritten as

$$\begin{aligned} C_n(q^2) &\xrightarrow{|q^2|\to\infty} (q^2)^{(d_A+d_B-d_n-4)/2}\log(-q^2) \\ &\qquad (d_A+d_B-d_n-4 \geq 0), \\ C_n(q^2) &\xrightarrow{|q^2|\to\infty} (q^2)^{(d_A+d_B-d_n-4)/2} \\ &\qquad (d_A+d_B-d_n-4 < 0). \end{aligned} \tag{3.19}$$

Here, it is seen that the in the limit of $|q^2| \to \infty$, the terms corresponding to the operators with the lowest dimension d_n dominate the expansion.

It is, however, important to note that in reality, the behavior of the OPE is not as simple as the above argument suggests. Most importantly, the numerical factors appearing in front of the expressions of Eq. (3.19) can be large and thus distort the ordering of the various terms according to their dimension. As was pointed out in

Ioffe (1981a, b), the lowest order terms usually contain a number of loops, which are numerically suppressed due to the momentum integral related to the loops. At higher orders of the OPE, these loops are cut, which leads to an enhancement of higher order terms compared to the leading order. Therefore, as a simple rule, one should include all orders in the OPE until the loops are cut, to make sure the expansion converges.

3.2.1.2 Accurate Treatment of Factorization

After the OPE was applied to low energy QCD in the framework of QCD sum rules, some criticism was raised by several authors about the validity of this expansion (Quinn and Gupta 1982; David 1984), pointing out that there are problems for the OPE in a theory with spontaneously broken symmetries and that there are ambiguities in the definition of condensates due to IR singularities. Later, these issues were addressed in Novikov et al. (1985), in which an unambiguous definition of the OPE was given. Essentially, the idea is that the simple expression of Eq. (3.3) is in fact not completely correct, but should be written down including a scale μ, specifying the energy at which one separates the high- and low-energy contributions:

$$\Pi(q^2) = \sum_d C_d(q^2, \mu)\langle 0|O_d|0\rangle(\mu). \tag{3.20}$$

Here, both the Wilson coefficients $C_d(q^2, \mu)$ and the condensates $\langle 0|O_d|0\rangle(\mu)$ depend on the factorization scale μ. However, as a whole, this expression should not depend on μ and hence these dependencies of the different parts cancel.

What this means is that the factorization of the OPE should *not* be considered to be a division between perturbative and non-perturbative contributions, but rather between high-energy ("hard") and low-energy ("soft") parts. How this works has been beautifully shown in Shifman (1998), where, as an example, the gluonic contribution to a mesonic correlator was discussed and the division between the hard and soft domain was explicitly carried out. Here, we will only explain the general idea of this procedure. Consider for instance the integral,

$$\int_0^\infty dq^2 G_{\text{full}}(q^2), \tag{3.21}$$

in which $G_{\text{full}}(q^2)$ is a full propagator (or a combination of several of them) contributing to some correlator. One can now easily divide this integral into a hard and a soft part, the division scale being μ:

$$\int_0^\infty dq^2 G_{\text{full}}(q^2) = \int_{\mu^2}^\infty dq^2 G_{\text{full}}(q^2) + \int_0^{\mu^2} dq^2 G_{\text{full}}(q^2). \tag{3.22}$$

The hard part can be replaced by the corresponding perturbative propagator, while the soft part is parametrized by the appropriate condensates $\langle O_G\rangle(\mu)$:

$$\int_0^\infty dq^2 G_{\text{full}}(q^2) \simeq \int_{\mu^2}^\infty dq^2 G_{\text{pert}}(q^2) + \langle O_G \rangle(\mu). \tag{3.23}$$

Formally, this is the correct way to implement the factorization in some OPE calculation. However, in practice such a clear division is not always feasible and one usually follows a much simpler method. Namely, one carries out the full perturbative calculation without the lower boundary μ^2:

$$\begin{aligned}\int_0^\infty dq^2 G_{\text{full}}(q^2) &\simeq \int_0^\infty dq^2 G_{\text{pert}}(q^2) + \left\{ \langle O_G \rangle(\mu) - \int_0^{\mu^2} dq^2 G_{\text{pert}}(q^2) \right\} \\ &= \int_0^\infty dq^2 G_{\text{pert}}(q^2) + \langle O_G^{'} \rangle(\mu).\end{aligned} \tag{3.24}$$

In this way, the perturbative part of the low energy domain is included into the effective condensates $\langle O_G^{'} \rangle(\mu)$. Of course, this treatment is only possible if the perturbative part below μ contains no divergences. If it does, one has to go back to the original form of the factorization, given in Eq. (3.23).

In Novikov et al. (1985), some arguments are given that in QCD, in contrast to other cases such as the $\lambda\phi^4$ theory or the $O(N)$ sigma model at low N, the perturbative effects on the effective condensates $\langle O_G^{'} \rangle(\mu)$ are small and can be essentially neglected. This situation can be related to the existence of a suitable "window" of the factorization scale μ (not to be confused with the "Borel window" appearing in conventional QCD sum rule analyses). Within this window, μ is large enough so that $\alpha_s(\mu)/\pi \ll 1$ and at the same time small enough so that the perturbative contributions to $\langle O_G^{'} \rangle(\mu)$ are small.

3.2.2 Calculation of Wilson Coefficients

Here, we show the fundamental steps of how to obtain the various Wilson coefficients. Derivations of some formulae appearing in this section can be found in the appendices. We here concentrate on the light quark sector, in which the quark masses can be treated as small perturbations. Calculations using heavy quark masses are discussed for instance in Novikov et al. (1984).

3.2.2.1 Fock-Schwinger Gauge

The Fock-Schwinger gauge (Fock 1937; Schwinger 1954) is defined as

$$(x - x_0)^\mu A_\mu^a(x) = 0, \tag{3.25}$$

where x_0 is a space-time point that can be freely chosen. For simplicity, it is usually taken to be 0, which will be done in the following discussion. The merit of using this gauge is that one can derivatively expand the gluon and quark fields in a covariant way, which was first pointed out in Cronstrom (1980) and further discussed in Dubovikov and Smilga (1981), Shifman (1980). Specifically, one obtains,

$$A_\mu^a(x) = \frac{1}{2}x^\nu G_{\nu\mu}^a(0) + \frac{1}{3}x^\nu x^\alpha [D_\alpha G_{\nu\mu}(0)]^a + \frac{1}{8}x^\nu x^\alpha x^\beta [D_\alpha D_\beta G_{\nu\mu}(0)]^a + \cdots \tag{3.26}$$

for the gluon fields and

$$q(x) = q(0) + x^\alpha D_\alpha q(0) + \frac{1}{2}x^\alpha x^\beta D_\alpha D_\beta q(0) + \cdots \tag{3.27}$$

for the quark fields. Note that even though we have used the same notation, the covariant derivatives in Eqs. (3.26) and (3.27) belong to the adjoint and fundamental representation, respectively. A detailed derivation of the above equations is given in Appendix B.

3.2.2.2 The Quark-Propagator in a Non-perturbative Background

As a first step of the actual OPE calculation, it is convenient to derive the quark propagator in the non-perturbative QCD vacuum, which can then be used repeatedly. For this, we will rely on the Fock-Schwinger gauge and treat the quark mass m_q as a small parameter, retaining only terms up to the first power of m_q. Furthermore, we give all the results in the x-representation, which will proof to be useful in later calculations.

Relegating the derivation to Appendix C, we here directly give the final form of the quark propagator in a non-perturbative background:

$$\begin{aligned}
&\langle 0|T[q_i^a(x)\overline{q}_j^b(0)]|0\rangle \\
&\approx \frac{i}{2\pi^2x^4}(\not{x})_{ij}\delta^{ab} - \frac{m_q}{2^2\pi^2x^2}\delta_{ij}\delta^{ab} - \frac{ig}{2^5\pi^2x^2}G_{\mu\nu}^k(\sigma^{\mu\nu}\not{x} + \not{x}\sigma^{\mu\nu})_{ij}\left(\frac{\lambda^k}{2}\right)^{ab} \\
&- \frac{gm_q}{2^5\pi^2}\ln\left(-\frac{x^2\Lambda^2}{4} + 2\gamma_{EM}\right)G_{\mu\nu}^k(\sigma^{\mu\nu})_{ij}\left(\frac{\lambda^k}{2}\right)^{ab} \\
&- \frac{1}{2^23}\delta_{ij}\delta^{ab}\langle\overline{q}q\rangle + \frac{im_q}{2^43}(\not{x})_{ij}\delta^{ab}\langle\overline{q}q\rangle - \frac{x^2}{2^63}\delta_{ij}\delta^{ab}\langle\overline{q}g\sigma Gq\rangle \\
&+ \frac{im_qx^2}{2^73^2}(\not{x})_{ij}\delta^{ab}\langle\overline{q}g\sigma Gq\rangle - \frac{\pi^2x^4}{2^83^3}\delta_{ij}\delta^{ab}\langle\overline{q}q\rangle\langle\frac{\alpha_s}{\pi}G^2\rangle \\
&+ \mathcal{O}(m_q^2, g^2)
\end{aligned} \tag{3.28}$$

Here, a, b are color indices, while i, j stand for the Dirac indices. Furthermore, γ_{EM} is the Euler-Mascheroni constant, which has the value $\gamma_{EM} = 0.57721\ldots$. The first term in the third line of Eq. (3.28) with $\ln\left(-\frac{x^2\Lambda^2}{4} + 2\gamma_{EM}\right)$ corresponds to an IR singularity, which arises because of the perturbative treatment of the quark mass m_q. Using the techniques explained in Appendix C, one can in principle systematically calculate higher orders, which is indeed necessary in certain cases.

3.2.2.3 Non-perturbative Coupling of Quarks and Gluons

As one further step, the non-perturbative contractions of a gluon and a quark must be calculated. These will, at least in leading order, be proportional to the mixed condensate $\langle \overline{q} g\sigma Gq \rangle$. For this we have to carry out the OPE of the following operator:

$$\begin{aligned}\langle 0|q_i^a(x) g G_{\mu\nu}^k(0) \overline{q}_j^b(0)|0\rangle &= \langle 0|q_i^a(0) g G_{\mu\nu}^k(0) \overline{q}_j^b(0)|0\rangle \\ &\quad + x^\alpha \langle 0|D_\alpha q_i^a(0) g G_{\mu\nu}^k(0) \overline{q}_j^b(0)|0\rangle + \cdots \end{aligned} \tag{3.29}$$

For these first two terms, the derivation can be found in the Appendix D, where their final form is obtained as

$$\begin{aligned}\langle 0|q_i^a(x) g G_{\mu\nu}^k(0) \overline{q}_j^b(0)|0\rangle \approx &-\frac{(\sigma_{\mu\nu})_{ij}}{2^6 3}\left(\frac{\lambda^k}{2}\right)^{ab} \langle \overline{q} g\sigma Gq\rangle \\ &+ \frac{i m_q}{2^8 3}(\not{x}\sigma^{\mu\nu} - \sigma^{\mu\nu}\not{x})_{ij}\left(\frac{\lambda^k}{2}\right)^{ab} \langle \overline{q} g\sigma Gq\rangle \\ &+ \mathcal{O}(m_q^2, g^2). \end{aligned} \tag{3.30}$$

3.2.2.4 A Simple Example

To show the procedure of an actual calculation, we demonstrate in this subsection the OPE of the vector current, composed of an (anti-) u and a d quark, which strongly couples to the ρ-meson in the low energy domain. The OPE of this current will be analyzed by the MEM technique in Chap. 5 of this thesis.

Our aim here is to carry out the OPE of the following correlator:

$$\Pi_{\mu\nu}(q) = i \int d^4 e^{iqx} \langle 0|\mathrm{T}[j_\mu(x) j_\nu^\dagger(0)]|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2). \tag{3.31}$$

Here the operator $j_\mu(x)$ is defined as $j_\mu(x) = \bar{u}(x)\gamma_\mu d(x)$. The last equality in the above equation follows from the fact that $j_\mu(x)$ is a conserved current. Therefore, to simplify the calculations, we can directly work with the contracted correlator,

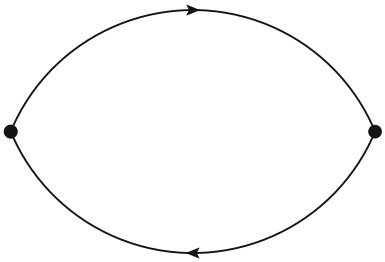

Fig. 3.1 The leading perturbative diagram contributing to the OPE of the vector current correlator of Eq. (3.31)

$$\Pi^{\mu}_{\mu}(q) = -3q^2\Pi(q^2). \tag{3.32}$$

For light quarks, it is most convenient to perform the OPE directly in x-space and to implement the Fourier transform only at the very end of the calculation, which we will do in the following.

The first step is to contract the quark operator by the Wick theorem. This gives

$$\langle 0|\mathrm{T}[j^{\mu}(x)j^{\dagger}_{\mu}(0)]|0\rangle = -\mathrm{Tr}[S(x)\gamma^{\mu}S(-x)\gamma_{\mu}], \tag{3.33}$$

where the trace Tr is taken for both spinor and color indices. $S(x)$ stands for the quark propagator of Eq. (3.28), which contains both perturbative and non-perturbative contributions.

Let us now calculate the leading perturbative term, which corresponds to the Feynman diagram of Fig. 3.1. For this, we simply have to substitute the first term of Eq. (3.28) into Eq. (3.33) and take the traces, which leads to

$$\langle 0|\mathrm{T}[j^{\mu}(x)j^{\dagger}_{\mu}(0)]|0\rangle\Big|_{\text{leading pert.}} = \frac{6}{\pi^4}\frac{1}{x^6}. \tag{3.34}$$

To calculate the first order α_s correction of the perturbative term, one needs to calculate the diagrams shown in Fig. 3.2. This calculation has to be done in momen-

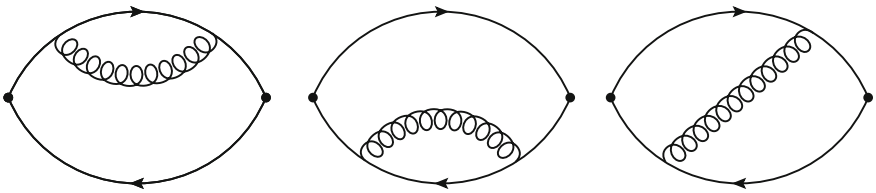

Fig. 3.2 The diagrams contributing to the first order α_s correction of the perturbative OPE term

tum space and is quite involved. Therefore, we do not discuss it here and refer the reader to literature (Colangelo and Khodjamirian 2001; Schwinger 1998) for details.

The next term would in principle be the one with the quark condensate with mass dimension 3. This term, however vanishes due to chiral symmetry. Therefore, we now consider the next order, which involves condensates with mass dimension 4. For demonstration of the method, we calculate the Wilson coefficient for the gluon condensate, for which we have to evaluate the diagrams shown in Fig. 3.3. Here we notice that in the limit of vanishing quark mass, the quark propagator with two attached gluons vanishes (Ioffe et al. 2010) and is therefore strongly suppressed for light quarks. Therefore, we only have to calculate the graph on the left side of Fig. 3.3. (Note, however, that for heavy quarks, all three graphs give contributions of comparable size and thus have to be taken into account (Novikov et al. 1984).)

The concrete evaluation of the relevant diagram of Fig. 3.3 is then quite simple. One substitutes the third term of Eq. (3.28) into Eq.(3.33), takes the traces and contracts the gluon operators according to

$$\langle G^k_{\mu\nu} G^l_{\rho\sigma} \rangle = \frac{\delta^{kl}}{2^5 3} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \langle G^2 \rangle. \tag{3.35}$$

At the end of the calculation, we get

$$\langle 0|\mathrm{T}[j^\mu(x)\, j^\dagger_\mu(0)]|0\rangle \Big|_{\text{gluon cond.}} = -\frac{1}{2^4} \frac{1}{x^2} \langle \frac{\alpha_s}{\pi} G^2 \rangle. \tag{3.36}$$

The calculation can be continued to include more higher order terms, up to the point when the expansion is expected to converge. As a last step, one then Fourier transforms the obtained results back to momentum space with the help of the formulae given in Appendix F.

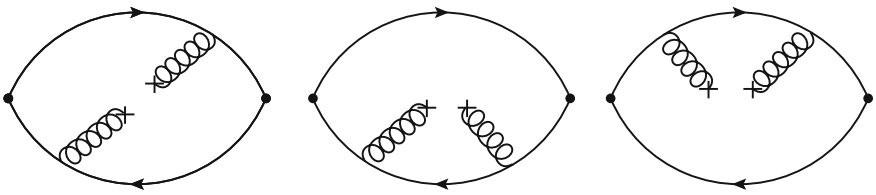

Fig. 3.3 The diagrams contributing to the Wilson coefficient of the gluon condensate. In the light quark case, the diagram on the *left* gives the dominant contribution, while the other two are suppressed

3.3 More on the QCD Vacuum

The vacuum expectation values of Lorentz- and gauge-invariant operators are fundamental parameters of QCD. As they are genuinely non-perturbative objects, their values cannot be calculated analytically and have to be determined by more involved methods. In the following, the values of the most important condensates and their derivation is briefly reviewed.

3.3.1 The Quark Condensate

As already discussed in Sect. 2.3.2, a finite value of the quark condensate signals the spontaneous breaking of the chiral symmetry. $\langle \overline{q}q \rangle$ is therefore probably the most important and most well-known of all the condensates. Its properties have been discussed already many years ago, and its value (for the light quarks) can be obtained via the Gell-Mann-Oakes-Renner relation of Eq. (2.23), approximately giving

$$\langle \overline{q}q \rangle = -(240 \pm 10\,\text{MeV})^3. \tag{3.37}$$

In all the considerations above, the exact realization of the isospin symmetry has been assumed. Moreover, the value of (3.37) is in fact scale dependent due to logarithmical quantum corrections and thus, a renormalization scale has to be given to make it fully determined. The value given above corresponds to the scale of $\mu \sim 1\,\text{GeV}$.

For determining the condensate of the strange quark condensate $\langle \overline{s}s \rangle$, mainly the QCD sum rule analyses of the octet strange baryons have been used (Reinders et al. 1985). These give

$$\frac{\langle \overline{s}s \rangle}{\langle \overline{q}q \rangle} = 0.8 \pm 0.2. \tag{3.38}$$

This is one example, where the value of a condensate actually has been constrained by a QCD sum rule result.

3.3.2 The Gluon Condensate

The gluon condensate $\langle \frac{\alpha_s}{\pi} G^2 \rangle$ is another important quantity of QCD, closely connected to the trace of the energy-momentum tensor. The derivation of its value has first been given in the founding papers of the QCD sum rule approach (Shifman et al. 1979a, b). There, a calculation of the charmonium sum rules was used to extract the value of $\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle$. As a result

$$\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = (0.012 \pm 0.004)\,\text{GeV}^4 \tag{3.39}$$

was obtained. Some later evaluations lead to values quite close to the one above used (Reinders et al. 1985), while others have yielded values about two times larger (Narison 2004), four to five times larger (Marrow et al. 1987), about two times smaller (Ioffe et al. 2010) or even consistent with zero (Brodsky and Shrock 2011). Thus, it has to be said that the uncertainty in the value of $\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle$ is still very large.

3.3.3 The Mixed Condensate

The mixed condensate $\langle \overline{q} g\sigma Gq \rangle$ is a further quantity, which appears in the OPE of QCD sum rules. It is constructed from quarks and gluons and is therefore called the mixed condensate. Although its dimension is five and it thus appears only at relatively high orders compared to the above two condensates, it actually plays a quite important role in the determination of the masses of various baryons.

For the light u, d quarks, the mixed condensate is usually parametrized relative to the quark condensate:

$$\frac{\langle \overline{q} g\sigma Gq \rangle}{\langle \overline{q} q \rangle} \equiv m_0^2. \tag{3.40}$$

The value of m_0^2 has been estimated in the early days of the QCD sum rule approach (Belyaev and Ioffe 1982) and has since not much changed:

$$m_0^2 = (0.8 \pm 0.1)\,\mathrm{GeV}^2. \tag{3.41}$$

There is also an estimate of $\langle \overline{q} g\sigma Gq \rangle$ by lattice QCD (Doi et al. 2003), which gives a value much larger than the one obtained by QCD sum rules ($\sim$2.5 GeV^2). This disagreement between the results of these two approaches certainly needs further investigation. We will use the value of (3.41) in our calculations.

The mixed condensate for the strange quarks $\langle \overline{q} g\sigma Gq \rangle$ is parametrized similarly to Eq. (3.40):

$$\frac{\langle \overline{s} g\sigma Gs \rangle}{\langle \overline{s} s \rangle} \equiv m_1^2. \tag{3.42}$$

Its value has been estimated to be the same as for the light quarks (Beneke and Dosch 1992):

$$m_1^2 = (0.8 \pm 0.1)\,\mathrm{GeV}^2. \tag{3.43}$$

3.3.4 Higher Order Condensates

There are two sorts of condensates with mass dimension six. Firstly there is the three-gluon condensate $\langle g^3 f_{abc} G_\mu^{av} G_\nu^{b\lambda} G_\lambda^{c\mu} \rangle$, where f_{abc} are the structure constants

of the $SU(3)$ group. The value of this condensate is not very well known and only a crude estimate based on the dilute instanton gas model exists (Novikov et al. 1979). This model gives

$$\langle g^3 f_{abc} G_\mu^{a\nu} G_\nu^{b\lambda} G_\lambda^{c\mu} \rangle \approx 0.045\,\mathrm{GeV}^6. \tag{3.44}$$

There have also been attempts to calculate this quantity on the lattice (Panagopoulos and Vicari 1990), but the results are not yet conclusive and more research is needed for obtaining a reliable estimate.

Secondly, there are the condensates containing four quarks. Generally, these can be expressed as

$$\langle \overline{q}_\alpha^i \overline{q}_\beta^k q_\gamma^l g_\delta^m \rangle, \tag{3.45}$$

where the color and spinor indices have to be contracted to give a singlet, a task for which there exist various combinations. The resulting condensates are, however, only very poorly known and one usually resorts to the vacuum saturation approximation, which reads as follows:

$$\langle \overline{q}_\alpha^i \overline{q}_\beta^k q_\gamma^l q_\delta^m \rangle \simeq \frac{1}{144} (\delta^{im}\delta^{kl}\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta^{il}\delta^{km}\delta_{\alpha\gamma}\delta_{\beta\delta}) \langle \overline{q}q \rangle^2. \tag{3.46}$$

What this equation essentially means is that one inserts a complete set of states between the two pairs of quarks and assumes that the vacuum part gives the dominant contribution. This approximation can be justified in the limit of an infinite number of colors N_c (Novikov et al. 1984), but it is not clear to what degree it holds for QCD with $N_c = 3$. For parametrizing the violation of this approximation, a factor κ is sometimes introduced, which is simply multiplied to Eq. (3.46). Estimates for this parameter range from close to 1 (Ioffe et al. 2010), $2 \sim 3$ (Narison 2004), up to ~ 6 (Leinweber 1997).

For the operators containing even more quarks or gluons, the vacuum saturation approximation discussed above is frequently used, even though the accuracy of this approximation is not properly justified for the various operators that appear at higher orders. Therefore it has to be admitted that the OPE results at higher orders become quite ambiguous. Thus, it is desirable that the OPE converges sufficiently fast, so that these terms do not have to be taken into account. In all, it can be said that the OPE calculation of higher order terms leaves room for further improvement of the QCD sum rule method.

3.4 Parity Projection for Baryonic Sum Rules

The problem of how to accurately project out positive or negative states from baryonic sum rules has been discussed only quite recently and is still not completely resolved. In this section, we introduce an improved method for constructing the parity projected sum rules, with which we hope to clarify this issue.

3.4.1 The Problem of Parity Projection in Baryonic Sum Rules

In contrast to mesons, the operators with quantum numbers of baryons couple to states of both parity, and one can thus not simply fix this quantum number by choosing an appropriate interpolating field. For illustration, let us consider the spatial inversion operation applied to a general Dirac operator $\eta(t, \mathbf{x})$ for a spin $\frac{1}{2}$ state.

$$\eta(t, \mathbf{x}) \to \pm\gamma_0\eta(t, -\mathbf{x}) \tag{3.47}$$

The parity of this operator can be switched by simply multiplying γ_5:

$$\gamma_5\eta(t, \mathbf{x}) \to \pm\gamma_5\gamma_0\eta(t, -\mathbf{x}) = \mp\gamma_0\gamma_5\eta(t, -\mathbf{x}). \tag{3.48}$$

Therefore, $\eta(t, \mathbf{x})$ couples to both positive and negative parity states. Supposing that the intrinsic parity of $\eta(t, \mathbf{x})$ is positive, we thus get:

$$\begin{aligned}\langle 0|\eta(x)|n^+(q)\rangle &= \lambda_+^n u_+(p)e^{-iqx},\\ \langle 0|\eta(x)|n^-(q)\rangle &= \lambda_-^n \gamma_5 u_-(p)e^{-iqx}.\end{aligned} \tag{3.49}$$

Here, $u_\pm(q)$ are Dirac spinors with positive and negative parity and $\lambda_\pm^n$ parameters corresponding to the strength of the coupling of η to the state $|n\rangle$. Using these definitions, the correlator of η will have the following form (in which the continuum states are omitted for simplicity):

$$\begin{aligned}\Pi(q) &= i\int d^4x e^{iqx}\langle 0|T[\eta(x)\overline{\eta}(0)]|0\rangle\\ &= \sum_n\left\{-|\lambda_+^n|^2\frac{\not{q}+m_+^n}{q^2-(m_+^n)^2+i\varepsilon} - |\lambda_-^n|^2\frac{\not{q}-m_-^n}{q^2-(m_-^n)^2+i\varepsilon}\right\}\\ &\equiv \not{q}\Pi_1(q^2) + \Pi_2(q^2).\end{aligned} \tag{3.50}$$

When only the sum rule for $\Pi_1(q^2)$ is used, it is not possible to determine the parity of the state contributing to some specific pole, as both positive and negative parity states couple to its spectral function:

$$\frac{1}{\pi}\mathrm{Im}\Pi_1(q^2) = \sum_n\Big[\left|\lambda_+^n\right|^2\delta(q^2-(m_+^n)^2) + \left|\lambda_-^n\right|^2\delta(q^2-(m_-^n)^2)\Big]. \tag{3.51}$$

In contrast, $\Pi_2(q^2)$ contains the positive and negative parity states with different signs:

$$\frac{1}{\pi}\mathrm{Im}\Pi_2(q^2) = \sum_n\Big[\left|\lambda_+^n\right|^2 m_+^n\delta(q^2-(m_+^n)^2) - \left|\lambda_-^n\right|^2 m_-^n\delta(q^2-(m_-^n)^2)\Big]. \tag{3.52}$$

Thus, the problem of parity projection boils down to consistently disentangling the contributions of positive and negative parity to the spectral functions of Eqs. (3.51) and (3.52).

3.4.2 Use of the "Old Fashioned" Correlator

As a solution to the problem discussed above, the use of the "old fashioned" correlator in the rest frame ($\mathbf{q} = 0$) was proposed in Jido et al. (1996),

$$\begin{aligned}\Pi^{\rm of}(q_0) &= i\int d^4x e^{iqx}\theta(x_0)\langle 0|\eta(x)\bar\eta(0)|0\rangle\Big|_{\mathbf{q}=0}\\ &\equiv \gamma_0\Pi_1^{\rm of}(q_0)+\Pi_2^{\rm of}(q_0),\end{aligned}\tag{3.53}$$

where the essential difference to Eq. (3.50) is the insertion of the Heaviside step-function $\theta(x_0)$ before carrying out the Fourier transform. Let us examine the effect of this step-function. First, we note that $\eta(x)\bar\eta(0)$ can be replaced by the corresponding time-ordered product $T[\eta(x)\bar\eta(0)]$. (Alternatively, we could have started directly from Eq. (3.53) with $T[\eta(x)\bar\eta(0)]$ instead of $\eta(x)\bar\eta(0)$.) Next, we use Eq. (3.50) and the Fourier transform of the Heaviside step-function, giving

$$\begin{aligned}&\Pi^{\rm of}(q_0)\\ &= \int d^4x e^{iqx}\frac{1}{2\pi i}\int dk_0\frac{1}{k_0-i\varepsilon}e^{ik_0x_0}\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\\ &\quad\times\sum_n\left\{-|\lambda_+^n|^2\frac{\not p+m_+^n}{p^2-(m_+^n)^2+i\varepsilon}-|\lambda_-^n|^2\frac{\not p-m_-^n}{p^2-(m_-^n)^2+i\varepsilon}\right\}\Big|_{\mathbf{q}=0}\\ &= \frac{1}{2\pi i}\int dk_0\frac{1}{k_0-i\varepsilon}\sum_n\left\{-|\lambda_+^n|^2\frac{(k_0+q_0)\gamma_0-\mathbf{q}\cdot\gamma+m_+^n}{(k_0+q_0)^2-\mathbf{q}^2-(m_+^n)^2+i\varepsilon}\right.\\ &\quad\left.-|\lambda_-^n|^2\frac{(k_0+q_0)\gamma_0-\mathbf{q}\cdot\gamma-m_-^n}{(k_0+q_0)^2-\mathbf{q}^2-(m_-^n)^2+i\varepsilon}\right\}\Big|_{\mathbf{q}=0}.\end{aligned}\tag{3.54}$$

The integrand of the above result contains three poles, two in the upper half of the imaginary plane of k_0 and one in the lower half. Therefore, closing the contour of integration in the lower half and using the Cauchy theorem, we pick up the residue of one pole and get

$$\begin{aligned}&\Pi^{\rm of}(q_0)\\ &= \frac{1}{2}\sum_n\left\{-|\lambda_+^n|^2\frac{1}{q_0-\sqrt{\mathbf{q}^2+(m_+^n)^2}+i\varepsilon}\left[\gamma_0-\frac{\mathbf{q}\cdot\gamma-m_+^n}{\sqrt{\mathbf{q}^2+(m_+^n)^2}-i\varepsilon}\right]\right.\end{aligned}$$

$$- |\lambda_-^n|^2 \frac{1}{q_0 - \sqrt{\mathbf{q}^2 + (m_-^n)^2} + i\varepsilon} \Big[\gamma_0 - \frac{\mathbf{q}\cdot\boldsymbol{\gamma} + m_-^n}{\sqrt{\mathbf{q}^2 + (m_-^n)^2} - i\varepsilon}\Big]\Bigg\}\Bigg|_{\mathbf{q}=0} \tag{3.55}$$

$$= \frac{1}{2}\sum_n \left\{ -|\lambda_+^n|^2 \frac{1}{q_0 - m_+^n + i\varepsilon}(\gamma_0 + 1) - |\lambda_-^n|^2 \frac{1}{q_0 - m_-^n + i\varepsilon}(\gamma_0 - 1) \right\}.$$

From this result it can be understood that the functions $\Pi_1^{\text{of}}(q_0)$ and $\Pi_2^{\text{of}}(q_0)$ contain only poles in the positive q_0 region and that they are analytic for $\text{Im}\, q_0 \geq 0$.

Furthermore, by applying the projection operator $\frac{1}{2}(\gamma_0 \pm 1)$ to the "old fashioned" correlator of Eq. (3.53) and taking the trace over the spinor indices, we are able to construct functions that only contain positive or negative parity states, as

$$\begin{aligned} \frac{1}{2}\text{Tr}\Big[\frac{1}{2}(\gamma_0 \pm 1)\Pi^{\text{of}}(q_0)\Big] &= \Pi_1^{\text{of}}(q_0) \pm \Pi_2^{\text{of}}(q_0) \equiv \Pi^{\pm}(q_0) \\ &= -\sum_n |\lambda_\pm^n|^2 \frac{1}{q_0 - m_\pm^n + i\varepsilon}. \end{aligned} \tag{3.56}$$

The imaginary parts of $\Pi^{\pm}(q_0)$ defined above then give the desired parity projected spectral functions:

$$\frac{1}{\pi}\text{Im}\,\Pi^{\pm}(q_0) = \sum_n |\lambda_n^+|^2 \delta(q_0 - m_n^{\pm}). \tag{3.57}$$

These can, however, not be calculated directly because perturbation theory is not reliable in the low q_0 region, but only at $|q_0| \to \infty$. We thus have to rely on the analytic properties of $\Pi^{\pm}(q_0)$, which allows us to extract information on the spectral functions via certain sum rules, that we will discuss in the next subsection.

3.4.3 Construction of the Sum Rules

We now use the analyticity of the functions $\Pi^{\pm}(q_0)$ to construct the sum rules. To do that, we first have to remember that there are two distinct ways of expressing $\Pi^{\pm}(q_0)$. The first expression uses the OPE and is written down in the language of the elementary degrees of freedom of QCD. In the following discussion we denote it as $\Pi^{\pm}_{\text{OPE}}(q_0)$. This expression is only reliable at high energies. The second one employs the hadronic degrees of freedom, contained in the experimentally observable physical spectrum. We have already given its concrete form in Eq. (3.56) and in the following denote it as $\Pi^{\pm}_{\text{Phys.}}(q_0)$.

To construct the sum rules, we consider the contour integral

$$\oint_C dq_0 \big[\Pi^{\pm}_{\text{OPE}}(q_0) - \Pi^{\pm}_{\text{Phys.}}(q_0)\big] W(q_0) = 0, \tag{3.58}$$

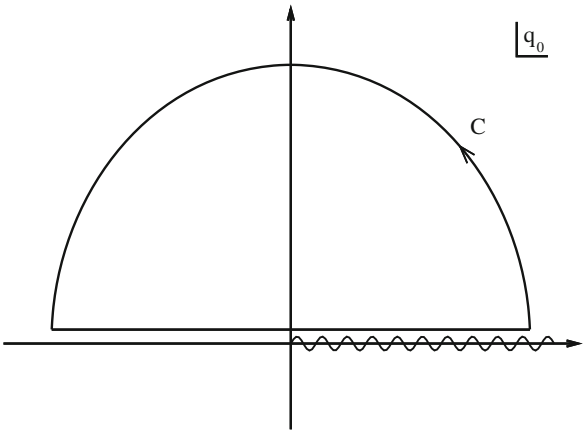

Fig. 3.4 The contour integral C on the complex plane of the variable q_0, used in Eq. (3.58). For the actual calculations, the radius of the half circle of C is taken to infinity. The wavy line denotes the non-analytic cut (or poles) of $\Pi^{\pm}(q_0)$ on the positive side of the real axis. Compared to the discussion in the main text, we have here slightly shifted the contour away from the real axis into the upper half of the imaginary plane for better visuality

where the contour C is given in Fig. 3.4. $W(q_0)$ must be analytic in the upper half of the imaginary plane and real on the real axis. As long as it satisfies these conditions, it can be arbitrarily chosen. That Eq. (3.58) gives 0 follows from the analyticity of both $\Pi^{\pm}_{\text{OPE}}(q_0)$ and $\Pi^{\pm}_{\text{Phys.}}(q_0)$. In other words, there are no poles or cuts inside of the contour C. We know from asymptotic freedom that the perturbative expression $\Pi^{\pm}_{\text{OPE}}(q_0)$ approaches the hadronic one at $|q_0| \to \infty$. Thus, the integrand of Eq. (3.58) vanishes on the half circle of C, whose radius is taken to infinity. Therefore, we are left with the section of C which runs along the real axis and can write down the sum rule as

$$\int_{-\infty}^{\infty} dq_0 \Pi^{\pm}_{\text{Phys.}}(q_0) W(q_0) = \int_{-\infty}^{\infty} dq_0 \Pi^{\pm}_{\text{OPE}}(q_0) W(q_0). \tag{3.59}$$

Finally, taking the imaginary part of the above equation we get

$$\int_{0}^{\infty} dq_0 \rho^{\pm}_{\text{Phys.}}(q_0) W(q_0) = \int_{-\infty}^{\infty} dq_0 \rho^{\pm}_{\text{OPE}}(q_0) W(q_0), \tag{3.60}$$

where the definitions $\rho^{\pm}_{\text{Phys.}}(q_0) \equiv \frac{1}{\pi}\text{Im}\Pi^{\pm}_{\text{Phys.}}(q_0)$ and $\rho^{\pm}_{\text{OPE}}(q_0) \equiv \frac{1}{\pi}\text{Im}\Pi^{\pm}_{\text{OPE}}(q_0)$ were used. Moreover, on the left-hand side, we have exploited the fact that $\Pi^{\pm}_{\text{Phys.}}(q_0)$ only has poles on the positive side of the real axis (see Eq. (3.56)) and have hence restricted the integral to this region.

The authors of Jido et al. (1996) have also restricted to integral on the right-hand side of Eq. (3.60) to positive values. Even though this procedure is roughly correct,

it leads to ambiguities in the higher order OPE terms, which in the chiral limit have poles at $q_0 = 0$. The problem here is that one does not know, whether these poles should be included into the integral of positive values, as they lie just on the border. Furthermore, in Jido et al. (1996) the imaginary part of the time ordered correlator was used instead of $\mathrm{Im}\Pi^{\pm}_{\mathrm{OPE}}(q_0)$, which in principle should be derived from the "old fashioned" correlator of Eq. (3.53). As was pointed out in Kondo et al. (2006), it is not entirely clear whether this prescription is justifiable. Therefore, in this study, we implement two essential improvements as compared to Jido et al. (1996): (1) We do not use the time ordered correlator, but derive all results directly from the "old fashioned" correlator of Eq. (3.53). (2) We do not restrict the region of integration of the OPE side of Eq. (3.60) to positive values and therefore remove the ambiguities that might occur for higher order OPE terms. How this is done will be explained in the next subsection.

3.4.4 General Analysis of the Sum Rules for Three-Quark Baryons

The OPE result of a time ordered correlator with interpolating fields containing three quarks can in coordinate space generally be expressed as

$$\begin{aligned}\Pi(x) = \big[&C_x^{(0)}\frac{1}{(x^2-i\varepsilon)^5} + C_x^{(4)}\frac{1}{(x^2-i\varepsilon)^3}\\ &+ C_x^{(6)}\frac{1}{(x^2-i\varepsilon)^2} + C_x^{(8)}\frac{1}{x^2-i\varepsilon} + \cdots\big] \not{x}\\ &+ C_x^{(3)}\frac{i}{(x^2-i\varepsilon)^4} + C_x^{(5)}\frac{i}{(x^2-i\varepsilon)^3}\\ &+ C_x^{(7)}\frac{i}{(x^2-i\varepsilon)^2} + C_x^{(9)}\frac{i}{x^2-i\varepsilon} + \cdots\end{aligned} \tag{3.61}$$

Here, $C_x^{(n)}$ are constants containing condensates with a total mass dimension n and dimensionless numerical factors. This equation is only correct as long as we work at leading order in α_s for the Wilson coefficients, because higher order corrections may involve additional logarithmic dependencies on x^2. In momentum space, this gives,

$$\begin{aligned}\Pi(q) = \big[&C_q^{(0)}q^4\ln(-q^2-i\varepsilon) + C_q^{(4)}\ln(-q^2-i\varepsilon)\\ &+ C_q^{(6)}\frac{1}{q^2+i\varepsilon} + C_q^{(8)}\frac{1}{(q^2+i\varepsilon)^2} + \cdots\big] \not{q}\\ &+ C_q^{(3)}q^2\ln(-q^2-i\varepsilon) + C_q^{(5)}\ln(-q^2-i\varepsilon)\\ &+ C_q^{(7)}\frac{1}{q^2+i\varepsilon} + C_q^{(9)}\frac{1}{(q^2+i\varepsilon)^2} + \cdots,\end{aligned} \tag{3.62}$$

where, as above, $C_q^{(n)}$ contains condensates and numerical factors. Furthermore, we here have neglected all polynomials in q^2, as they are not relevant for the further discussion. Also, note that we here have used the conventional Fourier transform, not the one with a step function as in Eq. (3.53).

Starting from these results, the question is now how to calculate the right-hand side of Eq. (3.60), from these results, which is all we need to get to the final form of the sum rules. This is a straightforward mathematical exercise: one plugs Eq. (3.61) into Eq. (3.53) and calculates the necessary integrals. One key ingredient for this calculation is the use of the Fourier transformed Heaviside step function of Eq. (F.6). Furthermore, we also mention that for the terms up to dimension 5, it is convenient to work in coordinate space while for terms with mass dimension 6 or larger, it is better to start from the momentum space expression of Eq. (3.62). We leave the detailed derivation to Appendix F and show here only the final result, which is obtained as

$$\begin{aligned}\Pi^{\text{of}}(q_0) = \Big[& C_q^{(0)} q_0^5 \ln(-q_0 - i\varepsilon) + C_q^{(4)} q_0 \ln(-q_0 - i\varepsilon) + \frac{1}{2} C_q^{(6)} \frac{1}{q_0 + i\varepsilon} \\ & + \frac{1}{4} C_q^{(8)} \frac{1}{\sqrt{\mathbf{q}^2}} \frac{1}{(q_0 - \sqrt{\mathbf{q}^2} + i\varepsilon)^2} \Big|_{\mathbf{q}=0} + \cdots \Big] \gamma_0 \\ & + C_q^{(3)} q_0^2 \ln(-q_0 - i\varepsilon) + C_q^{(5)} \ln(-q_0 - i\varepsilon) \\ & + \frac{1}{2} C_q^{(7)} \frac{1}{\sqrt{\mathbf{q}^2}} \frac{1}{q_0 - \sqrt{\mathbf{q}^2} + i\varepsilon} \Big|_{\mathbf{q}=0} \\ & + \frac{1}{4} C_q^{(9)} \Big[\frac{1}{(\sqrt{\mathbf{q}^2})^2} \frac{1}{(q_0 - \sqrt{\mathbf{q}^2} + i\varepsilon)^2} \\ & - \frac{1}{(\sqrt{\mathbf{q}^2})^3} \frac{1}{q_0 - \sqrt{\mathbf{q}^2} + i\varepsilon} \Big] \Big|_{\mathbf{q}=0} + \cdots \end{aligned} \tag{3.63}$$

Here, we have set $\sqrt{\mathbf{q}^2} = 0$ wherever this limit does not lead to divergencies. As one can see in the above equation, we are seemingly running into problems for terms with dimension 7 and higher, as the limit $\sqrt{\mathbf{q}^2} \to 0$ leads to a divergence for these terms. However, as we will see below, after substituting Eq. (3.63) into Eq. (3.60) and evaluating the integral over q_0, these divergences in fact vanish, leaving only finite expressions for the final form of the sum rules.

Let us now carry out this last step and calculate the right hand side of Eq. (3.60). For this, we consider two classes of weight functions $W(q_0)$, one which is an even function of q_0, $W_{\text{e}}(q_0) = F(q_0^2)$, the other being an odd function, $W_{\text{o}}(q_0) = q_0 F(q_0^2)$. Here, $F(q_0^2)$ should be an analytic function on the upper half of the imaginary plane and take only real values on the real axis.

The contributions of the terms of dimension 0, 3, 4, 5, and 6 to the sum rules can be easily obtained. As can be observed from Eq. (3.63), their imaginary part is either proportional to a step function $\theta(q_0)$ (dimension 0–5) or to a δ function $\delta(q_0)$ (dimension 6). The calculation of the higher order terms is somewhat more involved and we thus explicitly show their evaluation here. First, we note that in the limit $\sqrt{\mathbf{q}^2} \to 0$, the imaginary parts of the terms of dimension $7, 9, \ldots$ are

odd functions of q_0, while the ones of dimension 6, 8, 10, ... are even functions of q_0. A proof for this statement is given in Sect. F.2.3 of Appendix F. We thus can immediately conclude that the terms with dimension 7, 9, ... vanish when the even weight function $W_{\rm e}(q_0)$ is used while the terms of dimension 6, 8, 10, ... vanish if the odd weight function $W_{\rm o}(q_0)$ is used. Therefore, all we have to do is to evaluate the remaining non-vanishing parts.

For dimension 7, we have

$$-\frac{1}{2\sqrt{\mathbf{q}^2}}C_q^{(7)}\int dq_0\delta(q_0-\sqrt{\mathbf{q}^2})q_0F(q_0^2)\Big|_{\mathbf{q}=0}=-\frac{1}{2}C_q^{(7)}F(0). \tag{3.64}$$

Next, dimension 8 gives

$$\begin{aligned}&\frac{1}{4\sqrt{\mathbf{q}^2}}C_q^{(8)}\int dq_0\delta'(q_0-\sqrt{\mathbf{q}^2})F(q_0^2)\Big|_{\mathbf{q}=0}\\&\quad=-\frac{1}{2\sqrt{\mathbf{q}^2}}C_q^{(8)}\int dq_0\delta(q_0-\sqrt{\mathbf{q}^2})q_0F'(q_0^2)\Big|_{\mathbf{q}=0}=-\frac{1}{2}C_q^{(8)}F'(0).\end{aligned} \tag{3.65}$$

Finally, dimension 9 is evaluated as

$$\begin{aligned}&\frac{1}{4}C_q^{(9)}\Bigg[\frac{1}{(\sqrt{\mathbf{q}^2})^2}\int dq_0\delta'(q_0-\sqrt{\mathbf{q}^2})q_0F(q_0^2)\\&\qquad+\frac{1}{(\sqrt{\mathbf{q}^2})^3}\int dq_0\delta(q_0-\sqrt{\mathbf{q}^2})q_0F(q_0^2)\Bigg]\Bigg|_{\mathbf{q}=0}\\&\quad=\frac{1}{4}C_q^{(9)}\Bigg[-\frac{1}{(\sqrt{\mathbf{q}^2})^2}\int dq_0\delta(q_0-\sqrt{\mathbf{q}^2})[F(q_0^2)+2q_0^2F'(q_0^2)]\\&\qquad+\frac{1}{(\sqrt{\mathbf{q}^2})^2}F(0)\Bigg]\Bigg|_{\mathbf{q}=0}=-\frac{1}{2}C_q^{(9)}F'(0).\end{aligned} \tag{3.66}$$

We have explicitly checked that for the dimension 10 term, all potential divergences vanish in a similar fashion. Thus, the same procedure could presumably be continued to even higher orders, but this is not of much practical use as the OPE is only available up to terms of dimension 9 at present.

Let us also mention that if α_s corrections to the Wilson coefficients are included, oddness (evenness) of the imaginary parts of the dimension 7, 9, ... (6, 8, 10, ...) terms no longer holds due to the additional logarithmic dependencies on q_0. Thus, the sum rules with an $W_{\rm e}(q_0)$ ($W_{\rm o}(q_0)$) weight function do get contributions from the dimension 7, 9, ... (6, 8, 10, ...) terms, but only at subleading order in α_s. The detailed form of these α_s corrections are given elsewhere (Ohtani et al. 2013), here we for simplicity concentrate only on the results at leading order.

Assembling the findings of the preceding paragraphs, we can now write down the sum rules. For the even weight function $W_{\rm e}(q_0)$, we get

$$\begin{aligned}\int_0^\infty dq_0 \rho^{\pm}_{\rm Phys.}(q_0) W_{\rm e}(q_0) &= \int_0^\infty dq_0 \rho^{\pm}_{\rm Phys.}(q_0) F(q_0^2) \\ &= -C_q^{(0)} \int_0^\infty dq_0 q_0^5 F(q_0^2) \mp C_q^{(3)} \int_0^\infty dq_0 q_0^2 F(q_0^2) \\ &\quad - C_q^{(4)} \int_0^\infty dq_0 q_0 F(q_0^2) \mp C_q^{(5)} \int_0^\infty dq_0 F(q_0^2) \\ &\quad - \frac{1}{2} C_q^{(6)} F(0) - \frac{1}{2} C_q^{(8)} F'(0) + \cdots ,\end{aligned} \tag{3.67}$$

while for the odd weight function $W_{\rm o}(q_0)$, the sum rules reads as follows:

$$\begin{aligned}\int_0^\infty dq_0 \rho^{\pm}_{\rm Phys.}(q_0) W_{\rm o}(q_0) &= \int_0^\infty dq_0 \rho^{\pm}_{\rm Phys.}(q_0) q_0 F(q_0^2) \\ &= -C_q^{(0)} \int_0^\infty dq_0 q_0^6 F(q_0^2) \mp C_q^{(3)} \int_0^\infty dq_0 q_0^3 F(q_0^2) \\ &\quad - C_q^{(4)} \int_0^\infty dq_0 q_0^2 F(q_0^2) \mp C_q^{(5)} \int_0^\infty dq_0 q_0 F(q_0^2) \\ &\quad \mp \frac{1}{2} C_q^{(7)} F(0) \mp \frac{1}{2} C_q^{(9)} F'(0) + \cdots\end{aligned} \tag{3.68}$$

The sum rule of Eq. (3.67) has been considered in Jido et al. (1996), while the one of Eq. (3.68) is to our knowledge formulated here for the first time. Eq. (3.68) has the disadvantage of an additional power of q_0 compared to Eq. (3.67), which will enhance the contributions from the continuum to the sum rules. On the other hand, the contribution of the four-quark condensate of dimension 6 vanishes (or, the be more accurate, is suppressed by α_s) for Eq. (3.68). As this term contains large uncertainties, Eq. (3.68) has the potential to be more reliable. To establish which sum rule is more useful, a detailed study of both Eqs. (3.67) and (3.68) is certainly necessary. Also note that if one employs the maximum entropy method (to be discussed in the next chapter) to analyze the sum rules, it even becomes possible to study both sum rules at the same time.

As a last point of this section, we briefly touch on the issue of what function should be chosen for the weight function $F(q_0^2)$. The traditionally favored choice has been the Borel weight function, given as

$$F(q_0^2, M) = e^{-q_0^2/M^2} . \tag{3.69}$$

However, our recent study (Ohtani et al. 2011) of the (non parity projected) nucleon sum rules suggests that rather the Gaussian weight function,

$$F(q_0^2, s, \tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-(q_0^2 - s)^2/(4\tau)}, \tag{3.70}$$

is in fact to be preferred. Again, a more detailed discussion of this question has to be relegated to another publication (Ohtani et al. 2013).

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