Notes on Complete Book of Mathematics Vol. 10: Geometry

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Abstract The Complete Book of Mathematics is the most comprehensive treatise of mathematics in the Edo Period of Japan. The 20 volume book is about 900 sheets or 1800 pages long. Seki Takakazu (1642?–1708), Takebe Kataakira (1661–1721) and Takebe Katahiro (1664–1739) spent 28 years (1683–1711) in writing it. Unfortunately, the Book has never been published as a whole. Except for Volume 4 we reproduce here only one volume, Volume 10 Geometry, which is the first volume on geometry in the treatise. They develop here the basics of the plain geometry as algebraic relations among line segments which specify the geometric objects in question. This is exactly the same standpoint as René Descartes' (1596–1650) in his *Géométrie* (1637). At the end of the volume they discuss algebraic relations among the sides and diagonals in a general pentagon and hexagon by making use of Seki's theory of resultants.

1 Introduction

The Complete Book of Mathematics [大成算経 Taisei Sankei] is the most comprehensive treatise of mathematics in the Edo Period (1603–1868). Seki Takakazu [關孝和] (1642?–1708) and his pupils Takebe Kataakira [建部賢明] (1661–1721) and Takebe Katahiro [建部賢弘] (1664–1739) spent 28 years from Tenna 3 [天和 3] (1683) to Hōei 8 [宝永 8] (1711) until they completed it. The most important results Seki expounded in his Trilogy [三部抄] and Septenary [七部書] are all included in

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the Book. The known dates of volumes in Trilogy and Septenary are concentrated at the period when Seki and his pupils started to write the Book. That will probably mean that those which are in Trilogy or Septenary are the notes Seki prepared as sketches for the Book. It consists of 20 volumes and each volume about 45 sheets or 90 pages. Unfortunately the Book has never been published as a whole. The only one exception is Volume 13 Measurements [求積], which is included in *Collected Works* [2] as a volume in Septenary. There remain more than twenty copies as manuscripts (see Komatsu [9]), but most of them are not of good quality.

2 Outlines of Volume 10

In Volume 4 Three Essentials [三要 san'yō] the authors classified the objects in mathematics into abstract Images [象 shō] and visible Appearances [形 kei]. This Volume 10 is named Algorithms for Appearances [形法 keihō] as the first of five volumes on geometry. It consists of four chapters. Chapter 1 Algorithms for Squares [方法 houhou] deals with squares. The main result is the fact that the diagonal [斜 sha] of a square [方 hō] is $\sqrt{2}$ times a side [方面 hōmen].

Chapter 2 is entitled Algorithms for Rectangles [直法 chokuhō]. The longer side of a rectangle [直 choku] is called [縦 tate] or length [長 chō] and the shorter side [横 yoko] or [平 hei] or width [闊 katsu]. Their product is the area [積 seki]. Given the product ab and the sum $a+b$ or the difference $a-b$, the difference $a-b$ or the sum $a+b$ is obtained as the square root of $(a\pm b)^2 - \pm 4ab$. In the following Chapter 3 Algorithms for Right Triangles [勾股法 kōko hō] this is used to prove the Pythagorean theorem as in the Chinese classic Mathematics of Zhou Gnomons [周髀算經 Zhoubi Suanjing]. The shorter leg hook [勾 kō] and the longer leg leg [股 ko] of a right triangle stand for the right triangle itself. The hypothenuse is called chord [弦 gen]. Seki had given another proof of the Pythagorean theorem in Methods of Solving Explicit Problems [解見題之法 Kai Kendai no Hō], which is also reproduced. Then, many problems are solved by quadratic equations.

There are no formal treatments of proportions as in Euclid's Elements but it is remarked that rectangles or right triangles with a fixed ratio of heights to widths make a straight line. They say that this fact is very useful and also that a commentary writes that this is the most valuable in arithmetic [伝曰算術之極致也].

There are two appendices to Chapter 3. The first one deals with Pythagorean triplets, i.e. the integral solutions of

$$x^2 + y^2 = z^2.$$

However, their definition of integral numbers [整数 seisū] is different from ours. In Volume 4, they classified numbers into four categories: A number is said to be entire [全 zen] or integral [整 sei] if it is represented by a finite number of counting rods, i.e. if it is a finite decimal fraction; duplex [繁 han] if it is the quotient of two entire numbers or if it is rational; multiplex [畸 ki] if it is a root of an algebraic equation represented by a finite number of counting rods or if it is algebraic; and rudimentary [零 rei] otherwise or if it is transcendental (Xu [4], Komatsu [10]).

Firstly it is shown that Pythagorean triplets with the chord $z = 1$ are obtained by a computation equivalent to the powers of complex numbers

$$\pm x + \pm iy \text{ or } \pm y + \pm ix = (0.6 + 0.8i)^n, \quad n = 1, 2, 3, \ldots .$$

There are no others, as is easily proved by the unique factorization theorem of the Gauss integers $\mathbf{Z} + i\mathbf{Z}$.

Another way is to find a rational solution x of $x^2 + n^2 = (x + m)^2$ for each pair (m, n) of integers with $0 < m < n$. Clearly any integral triple in our sense can be found in this way. Moreover, the famous formula $(n^2 - m^2, 2nm, n^2 + m^2)$ of Pythagorean triplets is obtained by multiplying the above solution by $2m$.

The second appendix is a brief introduction to the methods of survey described in Sea Islands Mathematics [海島算經 Haidao Suanjing] by Liu Hui. The last problem is taken from Nine Chapters of Mathematics [九章算術 Shizhang Suanshu].

In the last Chapter 4 Algorithms for Polygons [斜法] the authors develop a general theory of triangles [三斜 sansha], quadrilaterals [四斜 shisha], pentagons [五斜 gosha] and hexagons [六斜]. After the preparations of the cosine law and the Heron formula for triangles, they proceed to establish Algorithm for Quadrilaterals [四斜法 shishahō] which is the algebraic relation among the four sides and two diagonals of a general quadrilateral. The equation with 22 terms of total degree 8 and of degree 4 in each variable is derived from Pythagorean theorem. This became famous after Seki used it in his book Mathematical Methods that Clarify Subtleties [発微算法 Hatsubi Sanpō] (1674) in his solutions of Problems posed by Sawaguchi Kazuyuki [沢口一之] in 1670 (see Komatsu [12]).

To obtain similar results for a pentagon and a hexagon they employ Seki's theory of elimination [5]. In the case of a pentagon, it is decomposed into the sum of two quadrilaterals with a common side or diagonal. Then, they eliminate the common variable from two equations corresponding to the decomposed quadrilaterals. Since each term of the Algorithm for Quadrilaterals depends only on the squares of sides and diagonals, the elimination is actually done for two quadratic equations in the variable to be eliminated. This is the case already discussed by Zhu Shijie [朱世傑] in his Jadelike Examples of Four Unknowns [四元玉鑑 Siyuan Yujian] (1303) (see Hoe [3, pp. 133–134] and [11, p. 104]).

The principle is simple but the actual calculation of the resultant was hard. In his book Weaving Methods in Mathematics [綴術算經 Tetsujutu Sankei] Takebe Katahiro recalled of his brother Kataakira and wrote "He once tried to deparenthesize the algorithm for pentagon, which was complicated, and said that even if the number of terms was ten thousand, one could calculate it in one hundred days by calculating one hundred terms every day. He really did it in a month and a few days." A rough estimate of the number of terms of an intermediate expansion is about 4000, and we obtain an equation of degree 8 with 843 monomial terms as its complete expansion, which is not given in the text but a few main terms are calculated in order to get a formula of the algorithm for a general hexagon.

Similarly a hexagon is decomposed as the sum of a quadrilateral and a pentagon with a common segment. Thus, the algorithm for a hexagon is obtained as the resultant of a quadrilateral algorithm and a pentagonal algorithm, which is represented as a 4×4 determinant with complicated entries, is equal to 0. Its complete expansion has 273,123 terms by Kinji Kimura's calculation by a computer.

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