A New Diffeomorphism Symmetry Group of Magnetohydrodynamics

Asher Yahalom

Abstract Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Yahalom (A four function variational principle for Barotropic magnetohydrodynamics, EPL 89, 34005 (2010) has shown that barotropic magnetohydrodynamics is mathematically equivalent to a four function field theory defined a by a Lagrangian for some topologies. The four functions include two surfaces whose intersections consist the magnetic field lines, the part of the velocity field not defined by the comoving magnetic field and the density. This Lagrangian admits a newly discovered group of Diffeomorphism Symmetry. I discuss the symmetry group and derive the related Noether current.

1 Introduction

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Eulerian variational principles for non-magnetic fluid dynamics were first introduced by Davydov [2]. Following the work of Davydov, Zakharov and Kuznetsov [10] suggested an Eulerian variational principle for magnetohydrodynamics. However, the variational principle suggested by Zakharov and Kuznetsov contained *two* more functions than the standard formulation of magnetohydrodynamics with a total sum of *nine* variational variables. Another Eulerian variational principle for magnetohydrodynamics was introduced independently by Calkin [1] in a work that preceded Zakharov and Kuznetsov paper by seven years. However, Calkin's variational principle also depends on as much as eleven variational variables. The situation was somewhat

A. Yahalom
Ariel University Center of Samaria, Ariel, Israel
e-mail: asya@ariel.ac.il

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_33,

improved when Vladimirov and Moffatt [6] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. Their variational principle contained only three more functions in addition to the seven variables which appear in the standard equations of magnetohydrodynamics which are the magnetic field $\mathbf{B}$ the velocity field $\mathbf{v}$ and the density ρ. Kats [3] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. The current paper will discuss only continuous flows due to space limitations, possible extensions of the current formalism to discontinuous flows will be discussed in a future paper. Sakurai [4] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [5]. In a work Yahalom and Lynden-Bell [7, 9] have combined the Lagrangian of Sturrock [5] with the Lagrangian of Sakurai [4] to obtain an Eulerian variational principle depending on only six functions. The vanishing of the variational derivatives of this Lagrangian entail all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints.

The non-singlevaluedness of the functions appearing in the reduced representation of barotropic magnetohydrodynamics was discussed in particular with connection to the topological invariants of magnetic and cross helicities. It was shown that flows with non trivial topologies which have non zero magnetic or cross helicities can be adequately described by the functions of the reduced representation provided that some of them are non-single valued [7, 9]. The cross helicity per unit flux was shown to be equal to the discontinuity of the function ν, this discontinuity was shown to be a conserved quantity along the flow. The magnetic helicity per unit flux was shown to be equal to the discontinuity of another function ζ.

In a more recent work [8] the number of needed functions was further reduced and it was shown that magnetohydrodynamics is mathematically equivalent to a four function field theory defined a by a Lagrangian.

In the current paper I show that the four function Lagrangian [8] admits a newly discovered group of diffeomorphism symmetry. The relevant Noether current and conservation laws of the newly discovered group of diffeomorphism symmetry are discussed.

The plan of this paper is as follows. First I introduce the standard notations and equations of barotropic magnetohydrodynamics. Next I introduce the potential representation of the magnetic field $\mathbf{B}$ and the velocity field $\mathbf{v}$. This is followed by a review of the Eulerian variational principle developed by Yahalom and Lynden-Bell [7, 9]. After those introductory sections I will present the four function Eulerian variational principles for non-stationary magnetohydrodynamics [8]. Finally I will discuss the newly discovered group of diffeomorphism symmetry and the relevant Noether current and conservation laws associated with the group of diffeomorphism symmetry.

2 The Standard Formulation of Barotropic Magnetohydrodynamics

The standard set of equations solved for barotropic magnetohydrodynamics are given below:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho (\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = -\nabla p(\rho) + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}. \tag{4}$$

The following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and ∇ has its standard meaning in vector calculus. $\mathbf{B}$ is the magnetic field vector, $\mathbf{v}$ is the velocity field vector and ρ is the fluid density. Finally $p(\rho)$ is the pressure which we assume depends on the density alone (barotropic case). The justification for those equations and the conditions under which they apply can be found in standard books on magnetohydrodynamics (see for example [5]). Equation (1) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), (2) describes the fact that the magnetic field is solenoidal, (3) describes the conservation of mass and equation (4) is the vector Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term:

$$\mathbf{J} = \frac{\nabla \times \mathbf{B}}{4\pi}, \tag{5}$$

is the electric current density which is not connected to any mass flow. The number of independent variables for which one needs to solve is seven $(\mathbf{v}, \mathbf{B}, \rho)$ and the number of equations (1), (3), (4) is also seven. Notice that (2) is a condition on the initial $\mathbf{B}$ field and is satisfied automatically for any other time due to (1). Also notice that $p(\rho)$ is not a variable rather it is a given function of ρ.

3 Potential Representation of Vector Quantities of Magnetohydrodynamics

It was shown in [9] that $\mathbf{B}$ and $\mathbf{v}$ can be represented in terms of five scalar functions $\alpha, \beta, \chi, \eta, \nu$. Following Sakurai [4] the magnetic field takes the form:

$$\mathbf{B} = \nabla \chi \times \nabla \eta. \tag{6}$$

Hence **B** satisfies automatically (2) for co-moving χ and η surfaces and is orthogonal to both $\nabla\chi$ and $\nabla\eta$. The above expression can also describe a magnetic field with non-zero magnetic helicity as was demonstrated in [9]. Moreover, the velocity $\mathbf{v}$ can be represented in the following form:

$$\mathbf{v} = \nabla\nu + \alpha\nabla\chi + \beta\nabla\eta. \tag{7}$$

this is a generalization of the Clebsch representation for magnetohydrodynamics.

4 The Action of Barotropic Magnetohydrodynamics

It was shown in [9] (Eq. (4.15) of [9], notice also the change in notation $\mathcal{L}$ here instead of $\hat{\mathcal{L}}$ used previously) that the action of barotropic magnetohydrodynamics takes the form:

$$\begin{aligned} A &\equiv \int \mathcal{L} d^3x dt, \\ \mathcal{L} &\equiv -\rho\left[\frac{\partial\nu}{\partial t} + \alpha\frac{\partial\chi}{\partial t} + \beta\frac{\partial\eta}{\partial t} + \varepsilon(\rho) + \frac{1}{2}(\nabla\nu + \alpha\nabla\chi + \beta\nabla\eta)^2\right] \\ &\quad - \frac{1}{8\pi}(\nabla\chi\times\nabla\eta)^2, \end{aligned} \tag{8}$$

in which $\varepsilon(\rho)$ is the specific internal energy. Taking the variational derivatives to zero for arbitrary variations leads to the following set of six equations:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{9}$$

$$\frac{d\chi}{dt} = 0, \tag{10}$$

$$\frac{d\eta}{dt} = 0, \tag{11}$$

$$\frac{d\nu}{dt} = \frac{1}{2}\mathbf{v}^2 - w, \tag{12}$$

in which w is the specific enthalpy.

$$\frac{d\alpha}{dt} = \frac{\nabla\eta\cdot\mathbf{J}}{\rho}, \tag{13}$$

$$\frac{d\beta}{dt} = -\frac{\nabla\chi\cdot\mathbf{J}}{\rho}. \tag{14}$$

In all the above equations $\mathbf{B}$ is given by (6) and $\mathbf{v}$ is given by (7). The mass conservation equation (3) is readily obtained. Now one needs to show that also (1) and (4) are satisfied.

It can be easily shown that provided that $\mathbf{B}$ is in the form given in (6), and (10), (11) are satisfied, then (1) are satisfied.

It was shown in [9] that a velocity field given by (7), such that the equations for $\alpha, \beta, \chi, \eta, \nu$ satisfy the corresponding equations (9), (10), (11), (12), (13), (14) must satisfy Euler's equations. This proves that the barotropic Euler equations can be derived from the action given in (8) and hence all the equations of barotropic magnetohydrodynamics can be derived from the above action without restricting the variations in any way except on the relevant boundaries and cuts.

5 A Simpler Action for Barotropic Magnetohydrodynamics

Can we obtain a further reduction of barotropic magnetohydrodynamics? Can we formulate magnetohydrodynamics with less than the six functions $\alpha, \beta, \chi, \eta, \nu, \rho$? The answer is yes, in fact four functions χ, η, ν, ρ will suffice. To see this we may write the two equations (10), (11) as equations for α, β that is:

$$\frac{d\chi}{dt} = \frac{\partial \chi}{\partial t} + \mathbf{v} \cdot \nabla \chi = \frac{\partial \chi}{\partial t} + (\nabla \nu + \alpha \nabla \chi + \beta \nabla \eta) \cdot \nabla \chi = 0,$$
$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla \eta = \frac{\partial \eta}{\partial t} + (\nabla \nu + \alpha \nabla \chi + \beta \nabla \eta) \cdot \nabla \eta = 0, \quad (15)$$

in which we have used (7). Solving for α, β we obtain:

$$\alpha[\chi, \eta, \nu] = \frac{(\nabla \eta)^2 (\frac{\partial \chi}{\partial t} + \nabla \nu \cdot \nabla \chi) - (\nabla \eta \cdot \nabla \chi)(\frac{\partial \eta}{\partial t} + \nabla \nu \cdot \nabla \eta)}{(\nabla \eta \cdot \nabla \chi)^2 - (\nabla \eta)^2 (\nabla \chi)^2}$$
$$\beta[\chi, \eta, \nu] = \frac{(\nabla \chi)^2 (\frac{\partial \eta}{\partial t} + \nabla \nu \cdot \nabla \eta) - (\nabla \eta \cdot \nabla \chi)(\frac{\partial \chi}{\partial t} + \nabla \nu \cdot \nabla \chi)}{(\nabla \eta \cdot \nabla \chi)^2 - (\nabla \eta)^2 (\nabla \chi)^2}. \quad (16)$$

Hence α and β are not free variables any more, but depend on χ, η, ν. Moreover, the velocity $\mathbf{v}$ now depends on the same three variables χ, η, ν:

$$\mathbf{v} = \nabla \nu + \alpha[\chi, \eta, \nu] \nabla \chi + \beta[\chi, \eta, \nu] \nabla \eta. \quad (17)$$

Since $\mathbf{v}$ is given now by (17) it follows that the two equations (10), (11) are satisfied identically and need not be derived from a variational principle. The above equation can be somewhat simplified resulting in:

$$\mathbf{v} = \nabla \nu + \frac{1}{\mathbf{B}^2} \left[\frac{\partial \eta}{\partial t} \nabla \chi - \frac{\partial \chi}{\partial t} \nabla \eta + \nabla \nu \times \mathbf{B} \right] \times \mathbf{B}$$
$$= \frac{1}{\mathbf{B}^2} \left[\left(\frac{\partial \eta}{\partial t} \nabla \chi - \frac{\partial \chi}{\partial t} \nabla \eta \right) \times \mathbf{B} + \mathbf{B} (\nabla \nu \cdot \mathbf{B}) \right] \equiv \mathbf{v}_\perp + \mathbf{v}_\parallel \quad (18)$$

Hence the velocity $\mathbf{v}$ is partitioned naturally into two components one which is parallel to the magnetic field and another one which is perpendicular to it. Inserting the velocity representation (18) into (16) will lead to the result:

$$\alpha = \frac{\vec\nabla \eta \cdot (\mathbf{B} \times (\mathbf{v} - \vec\nabla \nu))}{\mathbf{B}^2}$$
$$\beta = -\frac{\vec\nabla \chi \cdot (\mathbf{B} \times (\mathbf{v} - \vec\nabla \nu))}{\mathbf{B}^2}. \tag{19}$$

The reader should notice that the above quantities become singular for $\mathbf{B} = 0$, hence the formalism is only adequate for describing flows for which $\mathbf{B} \neq 0$. If the magnetic field vanishes at infinity this is not an obstacle from the present formalism point of view since the velocity field need not be defined at infinity where there is no flow. For flow without magnetic fields the present formalism is not appropriate and other variational economic formalisms may be suggested. Finally (16) should be substituted into (8) to obtain a Lagrangian density $\mathcal{L}$ in terms of χ, η, ν, ρ:

$$\mathcal{L}[\chi, \eta, \nu, \rho] = \rho \left[\frac{1}{2}\mathbf{v}^2 - \frac{d\nu}{dt} - \varepsilon(\rho)\right] - \frac{\mathbf{B}^2}{8\pi} \tag{20}$$

where $\mathbf{v}$ is given by (18) and $\mathbf{B}$ by (6). Or more explicitly as:

$$\mathcal{L}[\chi, \eta, \nu, \rho] = \frac{1}{2}\frac{\rho}{(\vec\nabla\chi \times \vec\nabla\eta)^2}\left[\vec\nabla\eta \frac{\partial\chi}{\partial t} - \vec\nabla\chi \frac{\partial\eta}{\partial t} + (\vec\nabla\chi \times \vec\nabla\eta) \times \vec\nabla\nu\right]^2$$
$$- \rho\left[\frac{\partial\nu}{\partial t} + \frac{1}{2}(\vec\nabla\nu)^2 + \varepsilon(\rho)\right] - \frac{(\vec\nabla\chi \times \vec\nabla\eta)^2}{8\pi}. \tag{21}$$

It is shown in [8] by variational analysis that indeed all the needed equations can be derived from the above Lagrangian.

6 Diffeomorphism Symmetry and Noether Currents

This Lagrangian density admits an infinite symmetry group of transformations of the form:

$$\hat\eta = \hat\eta(\chi, \eta), \quad \hat\chi = \hat\chi(\chi, \eta), \tag{22}$$

provided that the absolute value of the Jacobian of these transformation is unity:

$$\left|\frac{\partial(\hat\eta, \hat\chi)}{\partial(\eta, \chi)}\right| = 1. \tag{23}$$

In particular the Lagrangian density admits an exchange symmetry:

$$\hat\eta = \chi, \quad \hat\chi = \eta. \tag{24}$$

Consider the following transformations:

$$\hat{\eta} = \eta + \delta\eta(\chi, \eta), \qquad \hat{\chi} = \chi + \delta\chi(\chi, \eta), \tag{25}$$

in which $\delta\eta, \delta\chi$ are considered small in some sense. Inserting the above quantities into (23) and keeping only first order terms we arrive at:

$$\partial_\eta \delta\eta + \partial_\chi \delta\chi = 0. \tag{26}$$

This equation can be solved as follows:

$$\delta\eta = \partial_\chi \delta f, \quad \delta\chi = -\partial_\eta \delta f, \tag{27}$$

in which $\delta f = \delta f(\chi, \eta)$ is an arbitrary small function.

The variational derivative of the Lagrangian $L = \int d^3x\mathcal{L}$ in which $\mathcal{L}$ is given in (21) with respect to χ and η was calculated in [8] (31) and (32) (in action form). Assuming that the relevant equations of motion and boundary conditions hold, we have:

$$\delta_\chi L = -\frac{d\int d^3x\rho\alpha\delta\chi}{dt}, \qquad \delta_\eta L = -\frac{d\int d^3x\rho\beta\delta\eta}{dt} \tag{28}$$

The sum of the above two expressions is:

$$\delta_\chi L + \delta_\eta L = -\frac{d}{dt}\int d^3x\rho(\alpha\delta\chi + \beta\delta\eta), \tag{29}$$

If we choose variations $\delta\chi, \delta\eta$ such that (27) hold, then the variations are symmetry variations and the total variation $\delta_\chi L + \delta_\eta L$ vanishes. For this case we have:

$$\frac{d}{dt}\int d^3x\rho(\alpha\partial_\eta \delta f - \beta\partial_\chi \delta f) = 0. \tag{30}$$

Hence the quantity $\delta G = \int d^3x\rho(\alpha\partial_\eta \delta f - \beta\partial_\chi \delta f)$ is conserved. Using the comoving magnetic metage μ defined in [9] (4.23), (6.25), this can be written as:

$$\delta G = \int d\chi d\eta d\mu(\alpha\partial_\eta \delta f - \beta\partial_\chi \delta f) = \int d\chi d\mu\alpha\delta f\Big|_{\eta 1}^{\eta 2} - \int d\eta d\mu\beta\delta f\Big|_{\chi 1}^{\chi 2} + \int d\chi d\eta d\mu\delta f(\partial_\chi \beta - \partial_\eta \alpha). \tag{31}$$

Since δf is arbitrary, we arrive at the following conserved current.

$$J = \int d\mu(\partial_\chi \beta - \partial_\eta \alpha). \tag{32}$$

7 Conclusion

We have discussed the significance of the magnetohydrodynamics diffeomorphism symmetry group and in particular have shown the existence of the related conserved Noether current. Future research will be oriented towards understanding the physical consequences of this new conservation law.

References

1. Calkin, M.: An action prinicple for magnetohydrodynamics. Can. J. Phys. **41**, 2241–2251 (1963)
2. Davydov, B.: Doklady Akad. Nauk **69**, 165–168 (1949) (in Russian)
3. Kats, A.V.: Variational principle in canonical variables, Weber transformation, and complete set of the local integrals of motion for dissipation-free magnetohydrodynamics. (Los Alamos Archives physics-0212023) *JETP Lett.* **77**, 657 (2003)
4. Sakurai, T.: A new approach to the force-free field and its application to the magnetic field of solar active regions. Pub. Ast. Soc. Jpn. **31**, 209 (1979)
5. Sturrock, P.A.: Plasma Physics. Cambridge University Press, Cambridge (1994)
6. Vladimirov, V.A., Moffatt, H.K.: On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part 1. fundamental principles. J. Fluid. Mech. **283**, 125–139 (1995)
7. Yahalom, A.: Using Magnetic Fields to Optimize Material Flow - a Variational Approach to Magnetohydrodynamics. In: Proceedings of the Israeli - Russian Bi-National Workshop, Jerusalem, Israel, June 2007
8. Yahalom, A.: A four function variational principle for barotropic magnetohydrodynamics. EPL **89**, 34005 (2010). doi: 10.1209/0295-5075/89/34005. [Los-Alamos Archives - arXiv:0811.2309]
9. Yahalom, A., Lynden-Bell, D.: Simplified variational principles for barotropic magnetohydrodynamics. (Los-Alamos Archives - physics/0603128). J. Fluid Mech. **607**, 235–265 (2008)
10. Zakharov, V.E., Kuznetsov, E.A.: Variational Principle and Canonical Variables in Magnetohydrodynamics. Doklady Akad. Nauk **194**(6), 1288–1289 (1970) (in Russian). The paper was translated to English: Zakharov, V.E., Kuznetsov, E.A.: Variational principle and canonical variables in magnetohydrodynamics. Sov. Phys. Dokl. **15**(10), 913–914 (1971)