On Twisted Modules for N=2 Supersymmetric Vertex Operator Superalgebras

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Abstract The classification of twisted modules for N=2 supersymmetric vertex operator superalgebras with twisting given by vertex operator superalgebra automorphisms which are lifts of a finite automorphism of the N=2 Neveu–Schwarz Lie superalgebra representation is presented. These twisted modules include the Ramond-twisted sectors and mirror-twisted sectors for N=2 vertex operator super algebras, as well as twisted modules related to more general "spectral flow" representations of the N=2 Neveu–Schwarz algebra.

1 Introduction

We give an expository presentation following [6] of the classification of twisted modules for N=2 superconformal vertex operator superalgebras for the case of vertex operator superalgebra automorphisms that arise from finite Virasoro-preserving automorphisms of the underlying N=2 Neveu–Schwarz algebra.

For g an automorphism of a vertex operator superalgebra (VOSA), V, we have the notion of "g-twisted V-module". Twisted vertex operators were discovered and used in [24], and twisted modules arose in [19] in the course of the construction of the moonshine module vertex operator algebra. This structure came to be understood as an "orbifold model" in the sense of conformal field theory and string theory. Twisted modules are the mathematical counterpart of "twisted sectors", which are the basic building blocks of orbifold models in conformal field theory and string theory. The notion of twisted module for VOSAs was developed in [25]. In general, it is an open problem as to how to construct a g-twisted V-module.

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An automorphism g of a VOSA, in particular, fixes the Virasoro vector, and thus also fixes the corresponding endomorphisms giving the representation of the Virasoro algebra. A VOSA is said to be "N=2 supersymmetric", if in addition to being a positive energy representation for the Virasoro algebra, it is a representation of the N=2 Neveu–Schwarz Lie superalgebra, an extension of the Virasoro algebra; see, for instance, [5, 11]. The group of automorphisms of the N=2 Neveu–Schwarz algebra over $\mathbb{C}$, which preserve the Virasoro algebra, is isomorphic to $\mathbb{C}^\times \times \mathbb{Z}_2$. It is generated by a continuous family of automorphisms, denoted by σ_ξ for $\xi \in \mathbb{C}^\times$, and an order two automorphism κ called the "mirror map". If ξ is a root of unity, then σ_ξ is of finite order.

Given an N=2 supersymmetric VOSA, V, some questions naturally arise: When does κ or σ_ξ lift to an automorphism of V, and when is this lift unique? When such an automorphism of the N=2 Neveu–Schwarz algebra does lift to an automorphism g of V, what is the structure of a g-twisted V module? In this paper, we present the answer to the second question, and in [6] we provide the details of this study and answer the first question for free and lattice N=2 VOSAs.

If the mirror automorphism κ of the N=2 Neveu–Schwarz algebra lifts to a VOSA automorphism of an N=2 VOSA, V, then a "mirror-twisted V-module" is naturally a representation of what we call the "mirror-twisted N=2 superconformal algebra", which is also referred to as the "twisted N=2 superconformal algebra" [16,26,29], or the "topological N=2 superconformal algebra" [21]. If the automorphism σ_ξ of the N=2 Neveu–Schwarz algebra, for ξ a root of unity, lifts to a VOSA automorphism of V, then we show that a "σ_ξ-twisted V-module" is naturally a representation of one of the algebras in the one-parameter family of Lie superalgebras we call "shifted N=2 superconformal algebras". If $\xi = -1$, then σ_ξ is the parity map, and such a shifted N=2 superconformal algebra is the N=2 Ramond algebra. The N=2 Ramond algebra and the other shifted N=2 algebras are isomorphic, as Lie superalgebras, to the N=2 Neveu–Schwarz algebra via the "spectral flow" operators, as was first realized in [29]. The mirror-twisted N=2 algebra is not isomorphic to the N=2 Neveu–Schwarz algebra.

The representation theory of the N=2 Neveu–Schwarz algebra has been studied in, for instance, [9–11, 13–15, 17, 18, 23, 27, 28, 30, 31] and from a VOSA theoretic point of view in [1]. The representation theory of the N=2 Ramond algebra has been studied in, e.g, [13, 18, 20, 21, 30], and of the mirror-twisted N=2 superconformal algebra in, e.g, [13, 16, 21, 22, 26].

The realization of the N=2 Ramond algebra (i.e. the $\frac{1}{2}$-shifted N=2 superconformal algebra) and the mirror-twisted N=2 superconformal algebra as arising from twisting an N=2 VOSA (or comparable structure) has long been known, e.g. [9, 11, 29]. However to our knowledge, the other algebras related to the N=2 Neveu–Schwarz algebra—the shifted N=2 superconformal algebras other than the N=2 Ramond algebra—have only been studied through the spectral flow operators (which do not preserve the Virasoro algebra). We believe that the realization of these algebras as arising naturally as twisted modules for an N=2 VOSA is new.

Thus this complete classification of the twisted modules for an N=2 VOSA for finite automorphisms arising from Virasoro-preserving automorphisms of the N=2 Neveu–Schwarz algebra provides a uniform way of understanding and studying all of the N=2 superconformal algebras—the continuous one-parameter family of shifted N=2 Neveu–Schwarz algebras and the mirror-twisted N=2 superconformal algebra—in the context of the theory of VOSAs and their twisted modules.

2 The N=1 and N=2 Superconformal Algebras and Their Virasoro-Preserving Automorphisms

The *N=1 Neveu–Schwarz algebra* or *N=1 superconformal algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n for $n \in \mathbb{Z}$, and odd elements G_r for $r \in \mathbb{Z} + \frac{1}{2}$, and supercommutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\, d, \tag{1}$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \quad [G_r, G_s] = 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\, d, \tag{2}$$

for $m, n \in \mathbb{Z}$, and $r, s \in \mathbb{Z} + \frac{1}{2}$. The *N=1 Ramond algebra* is the Lie superalgebra with basis the central element d, even elements L_n for $n \in \mathbb{Z}$, and odd elements G_r for $r \in \mathbb{Z}$, and supercommutation relations given by (1)–(2), where now $r, s \in \mathbb{Z}$.

Note that the only nontrivial Lie superalgebra automorphism of the N=1 Neveu–Schwarz algebra which preserves the Virasoro algebra is the parity automorphism which is the identity on the even subspace (the Virasoro Lie algebra) and acts as -1 on the odd subspace (the subspace spanned by G_r for $r \in \mathbb{Z} + \frac{1}{2}$).

The *N=2 Neveu–Schwarz Lie superalgebra* or *N=2 superconformal algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n and J_n for $n \in \mathbb{Z}$, and odd elements $G_r^{(j)}$ for $j = 1, 2$ and $r \in \mathbb{Z} + \frac{1}{2}$, and such that the supercommutation relations are given as follows: L_n, d and $G_r^{(j)}$ satisfy the supercommutation relations for the N=1 Neveu–Schwarz algebra (1)–(2) for both $G_r = G_r^{(1)}$ and for $G_r = G_r^{(2)}$; the remaining relations are given by

$$[L_m, J_n] = -nJ_{m+n}, \quad [J_m, J_n] = \frac{1}{3}m\delta_{m+n,0}d, \tag{3}$$

$$\left[J_m, G_r^{(1)}\right] = -iG_{m+r}^{(2)}, \left[J_m, G_r^{(2)}\right] = iG_{m+r}^{(1)}, \left[G_r^{(1)}, G_s^{(2)}\right] = -i(r-s)J_{r+s}. \tag{4}$$

The *N=2 Ramond algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n and J_n for $n \in \mathbb{Z}$, and odd elements $G_r^{(j)}$ for $r \in \mathbb{Z}$ and $j = 1, 2$, and supercommutation relations given by those of the N=2 Neveu–Schwarz algebra but with $r, s \in \mathbb{Z}$, instead of $r, s \in \mathbb{Z} + \frac{1}{2}$.

More generally, there is an infinite family of algebras which includes the N=2 Neveu–Schwarz and Ramond algebras. However these are easier to express if we make a change of basis which is ubiquitous in superconformal field theory. So consider the substitutions $G_r^{(1)} = \frac{1}{\sqrt{2}}(G_r^+ + G_r^-)$, and $G_r^{(2)} = \frac{i}{\sqrt{2}}(G_r^+ - G_r^-)$, or equivalently $G_r^\pm = \frac{1}{\sqrt{2}}(G_r^{(1)} \mp iG_r^{(2)})$. This substitution is equivalent to the change of variables $\varphi^\pm = \frac{1}{\sqrt{2}}(\varphi^{(1)} \pm i\varphi^{(2)})$ in the variables $(x, \varphi^{(1)}, \varphi^{(2)})$ representing the one even and two odd local coordinates on an N=2 superconformal worldsheet representing superstrings propagating in space-time in N=2 superconformal field theory, cf. [4]. In terms of this basis (called the *homogeneous basis*), the N=2 Neveu–Schwarz (or Ramond) algebra supercommutation relations are given by (1), (3) and

$$\left[L_m, G_r^\pm\right] = \left(\frac{m}{2} - r\right) G_{m+r}^\pm, \quad \left[J_m, G_r^\pm\right] = \pm G_{m+r}^\pm, \quad \left[G_r^\pm, G_s^\pm\right] = 0, \quad (5)$$

$$\left[G_r^+, G_s^-\right] = 2L_{r+s} + (r-s)J_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\, d, \quad (6)$$

for $m, n \in \mathbb{Z}$, and $r, s \in \mathbb{Z} + \frac{1}{2}$, or $r, s \in \mathbb{Z}$, respectively.

Observe then that there is also the notion of a Lie superalgebra generated by even elements L_n and J_n for $n \in \mathbb{Z}$ and by odd elements $G_{r\pm t}^\pm$, for $r \in \mathbb{Z} + \frac{1}{2}$ and for $t \in \mathbb{C}$. We shall call this algebra the *t-shifted N=2 superconformal algebra* or *t-shifted N=2 Neveu–Schwarz algebra*. Thus the t-shifted N=2 Neveu–Schwarz algebra is the N=2 Neveu–Schwarz algebra if $t \in \mathbb{Z}$, and is the N=2 Ramond algebra if $t \in \mathbb{Z} + \frac{1}{2}$. As was first shown in [29], the t-shifted N=2 Neveu–Schwarz algebras are all isomorphic under the continuous family of *spectral flow* maps, denoted $\mathscr{D}(t)$, for $t \in \mathbb{C}$, but which fix the Virasoro algebra only for $t = 0$. These are given by

$$\mathscr{D}(t): \quad \begin{array}{lll} L_n \mapsto L_n + tJ_n + \frac{t^2}{6}\delta_{n,o} d, & d \mapsto d, \\ J_n \mapsto J_n + \frac{t}{3}\delta_{n,0} d, & G_r^\pm \mapsto G_{r\pm t}^\pm. \end{array} \quad (7)$$

The group of automorphisms of the N=2 Neveu–Schwarz algebra (or more generally the t-shifted N=2 superconformal algebras) which preserve the Lie subalgebra generated by L_n and J_n for $n \in \mathbb{Z}$ are given by:

$$\sigma_\xi: \quad G_r^\pm \mapsto \xi^{\pm 1} G_r^\pm, \quad J_n \mapsto J_n, \quad L_n \mapsto L_n, \quad d \mapsto d, \quad (8)$$

for $\xi \in \mathbb{C}^\times$. In addition, we have the *mirror map*, given by:

$$\kappa: \quad G_r^\pm \mapsto G_r^\mp, \quad J_n \mapsto -J_n, \quad L_n \mapsto L_n, \quad d \mapsto d. \quad (9)$$

The family σ_ξ along with κ generate the Virasoro-preserving automorphisms of the N=2 Neveu–Schwarz algebra, and thus this group is isomorphic to $\mathbb{Z}_2 \times \mathbb{C}^\times$, cf. [5].

The *mirror-twisted N=2 Neveu–Schwarz algebra* is defined to be the Lie superalgebra with basis consisting of even elements L_n, and J_r and central element d, odd elements $G_r^{(1)}$ and $G_n^{(2)}$, for $n \in \mathbb{Z}$ and $r \in \mathbb{Z}+\frac{1}{2}$, and supercommutation relations given as follows: The L_n and $G_r^{(1)}$ satisfy the supercommutation relations for the N=1 Neveu–Schwarz algebra with central charge d; the L_n and $G_n^{(2)}$ satisfy the supercommutation relations for the N=1 Ramond algebra with central charge d; and the remaining supercommutation relations are

$$[L_n, J_r] = -rJ_{n+r}, \quad [J_r, J_s] = \frac{1}{3}r\delta_{r+s,0}d \tag{10}$$

$$\left[J_r, G_s^{(1)}\right] = -iG_{r+s}^{(2)}, \quad \left[J_r, G_n^{(2)}\right] = iG_{r+n}^{(1)}, \quad \left[G_r^{(1)}, G_n^{(2)}\right] = -i(r-n)J_{r+n}. \tag{11}$$

Note that this mirror-twisted N=2 Neveu–Schwarz algebra is not isomorphic to the ordinary N=2 Neveu–Schwarz algebra [29].

3 The Notions of VOSA, Supersymmetric VOSA and Twisted Module

In this section, we recall the notions of VOSA, and N=1 or N=2 Neveu–Schwarz VOSA, following the notation and terminology of [2, 3] and [5]. We also recall the notion of g-twisted V-module for a VOSA, V, and an automorphism g of V of finite order following the notation of, e.g. [7, 8, 12, 25].

Let x, x_0, x_1, x_2, denote commuting independent formal variables. Let $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$. Expressions such as $(x_1 - x_2)^n$ for $n \in \mathbb{C}$ are to be understood as formal power series expansions in nonnegative integral powers of the second variable.

Definition 3.1. A *vertex operator superalgebra* is a $\frac{1}{2}\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n$, satisfying $\dim V < \infty$ and $V_n = 0$ for n sufficiently negative, that is also $\mathbb{Z}_2$-graded by *sign*, $V = V^{(0)} \oplus V^{(1)}$, and equipped with a linear map

$$V \longrightarrow (\mathrm{End}\, V)[[x, x^{-1}]], \quad v \mapsto Y(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}, \tag{12}$$

and with two distinguished vectors $\mathbf{1} \in V_0$, (the *vacuum vector*) and $\omega \in V_2$ (the *conformal element*) satisfying the following conditions for $u, v \in V$: $u_n v = 0$ for n sufficiently large; $Y(\mathbf{1},x)v = v$; $Y(v,x)\mathbf{1} \in V[[x]]$, and $\lim_{x\to 0} Y(v,x)\mathbf{1} = v$;

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) - (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v,x_2)Y(u,x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u,x_0)v,x_2) \tag{13}$$

(the *Jacobi identity*), where $|v| = j$ if $v \in V^{(j)}$ for $j \in \mathbb{Z}_2$; writing $Y(\omega,x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$, i.e. $L(n) = \omega_{n+1}$, for $n \in \mathbb{Z}$, then the $L(n)$ give a representation of the Virasoro algebra with central charge $c \in \mathbb{C}$ (the *central charge* of V); for $n \in \frac{1}{2}\mathbb{Z}$ and $v \in V_n$, $L(0)v = nv = (\mathrm{wt}\, v)v$; and the $L(-1)$*-derivative property* holds: $\frac{d}{dx}Y(v,x) = Y(L(-1)v,x)$.

If a VOSA, $(V,Y,\mathbf{1},\omega)$, contains an element $\tau \in V_{3/2}$ such that writing $Y(\tau,z) = \sum_{n\in\mathbb{Z}} \tau_n x^{-n-1} = \sum_{n\in\mathbb{Z}} G(n+1/2)x^{-n-2}$, the $G(n+1/2) = \tau_{n+1} \in (\mathrm{End}\, V)^{(1)}$ generate a representation of the N=1 Neveu–Schwarz Lie superalgebra, then we call $(V,Y,\mathbf{1},\tau)$ an *N=1 Neveu–Schwarz VOSA*, or an *N=1 supersymmetric VOSA*, or just an N=1 VOSA for short.

If a VOSA $(V,Y,\mathbf{1},\omega)$ has two vectors $\tau^{(1)}$ and $\tau^{(2)}$ such that $(V,Y,\mathbf{1},\tau^{(j)})$ is an N=1 VOSA for both $j = 1$ and $j = 2$, and the $\tau^{(j)}_{n+1} = G^{(j)}(n+1/2)$ generate a representation of the N=2 Neveu–Schwarz Lie superalgebra, then we call such a VOSA an *N=2 Neveu–Schwarz VOSA* or an *N=2 supersymmetric VOSA*, or for short, an N=2 VOSA. If V is an N=2 VOSA, then there exists a vector $\mu = \frac{i}{2}G^{(1)}(1/2)\tau^{(2)} = -\frac{i}{2}G^{(2)}(1/2)\tau^{(1)} \in V_{(1)}$ such that writing $Y(\mu,x) = \sum_{n\in\mathbb{Z}} \mu_n x^{-n-1} = \sum_{n\in\mathbb{Z}} J(n)x^{-n-1}$, we have that the $J(n) \in (\mathrm{End}\, V)^0$ along with the $G^{(j)}(n+1/2)$ and $L(n) = \omega_{n+1}$ for $\omega = \frac{1}{2}G^{(j)}(-1/2)\tau^{(j)}$ satisfy the supercommutation relations for the N=2 Neveu–Schwarz Lie superalgebra.

For an N=2 VOSA, it follows from the definition that $\omega = L(-2)\mathbf{1}$, $\tau^{(j)} = G^{(j)}(-3/2)\mathbf{1}$, for $j = 1,2$, $\mu = J(-1)\mathbf{1}$, and

$$L(n)\mathbf{1} = G^{(j)}(n+1/2)\mathbf{1} = J(n+1)\mathbf{1} = 0, \quad \text{for } n \geq -1,\ j = 1,2. \tag{14}$$

If V is an N=2 VOSA such that V is not only $\frac{1}{2}\mathbb{Z}$ graded by $L(0)$ but also $\mathbb{Z}$-graded by $J(0)$ such that $J(0)v = nv$ with $n \equiv j \bmod 2$, for $v \in V^{(j)}$ for $j = 0,1$, then we say that V is $J(0)$*-graded* or *graded by charge*.

An *automorphism* of a VOSA, V, is a linear map g from V to itself, preserving $\mathbf{1}$ and ω such that the actions of g and $Y(v,x)$ on V are compatible in the sense that $gY(v,x)g^{-1} = Y(gv,x)$, for $v \in V$. Then $gV_n \subset V_n$ for $n \in \frac{1}{2}\mathbb{Z}$.

If g has finite order, V is a direct sum of the eigenspaces V^j of g, i.e., $V = \bigoplus_{j\in\mathbb{Z}/k\mathbb{Z}} V^j$, where $k \in \mathbb{Z}_+$ is a period of g (i.e., $g^k = 1$ but k is not necessarily the order of g) and $V^j = \{v \in V \mid gv = \eta^j v\}$, for η a fixed primitive k-th root of unity.

Definition 3.2. Let $(V,Y,\mathbf{1},\omega)$ be a VOSA and g an automorphism of V of period $k \in \mathbb{Z}_+$. A *weak g-twisted V-module* is a vector space M equipped with a linear map

$$V \longrightarrow (\mathrm{End}\, M)[[x^{1/k},x^{-1/k}]], \quad v \mapsto Y^g(v,x) = \sum_{n\in\frac{1}{k}\mathbb{Z}} v^g_n x^{-n-1}, \tag{15}$$

satisfying the following conditions for $u,v \in V$ and $w \in M$: $v^g_n w = 0$ for n sufficiently large; $Y^g(\mathbf{1},x)w = w$; $Y^g(v,x) = \sum_{n\in\mathbb{Z}+\frac{j}{k}} v^g_n x^{-n-1}$ for $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$;

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y^g(u,x_1)Y^g(v,x_2)-(-1)^{|u||v|}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y^g(v,x_2)Y^g(u,x_1)$$
$$=x_2^{-1}\frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}}\delta\left(\eta^j\frac{(x_1-x_0)^{1/k}}{x_2^{1/k}}\right)Y^g(Y(g^ju,x_0)v,x_2) \tag{16}$$

(the *twisted Jacobi identity*) where η is a fixed primitive k-th root of unity.

If we take $g=1$, then we obtain the notion of weak V-module. The term "weak" means we are making no assumptions about a grading on M.

As a consequence of the definition, we have the following supercommutator relation on M for $u\in V^j$:

$$[Y^g(u,x_1),Y^g(v,x_2)]$$
$$=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\left(\frac{x_1-x_0}{x_2}\right)^{-j/k}Y^g(Y(u,x_0)v,x_2). \tag{17}$$

It also follows that writing $Y^g(\omega,x)=\sum_{n\in\mathbb{Z}}L^g(n)x^{-n-2}$, i.e., setting $L^g(n)=\omega^g_{n+1}$, for $n\in\mathbb{Z}$, then the $L^g(n)$ satisfy the relations for the Virasoro algebra with central charge c the central charge of V. And the $L(-1)$-derivative property for the twisted vertex operators holds:

$$\frac{d}{dx}Y^g(u,x)=Y^g(L(-1)u,x). \tag{18}$$

4 Twisting by Automorphisms Arising from Virasoro-Preserving Automorphisms of the N=2 Neveu–Schwarz Algebra

In this section, we give the structure of weak g-twisted V-modules for V an N=2 VOSA and g any automorphism of V which is a lift of a Virasoro-preserving automorphism of the N=2 Neveu–Schwarz algebra of finite order.

We first consider the mirror map κ. If an N=2 VOSA, V, with central charge c, has a VOSA automorphism κ such that $\kappa(\mu)=-\mu$, $\kappa(\tau^{(1)})=\tau^{(1)}$ and $\kappa(\tau^{(2)})=-\tau^{(2)}$, then such a VOSA automorphism of V is called an *N=2 VOSA mirror map*. If such a map exists for V, and M is a weak κ-twisted module for V, then write Y^κ for the κ-twisted operators, and

$$Y^\kappa(\omega,x)=\sum_{n\in\mathbb{Z}}L^\kappa(n)x^{-n-2},\qquad Y^\kappa(\tau^{(1)},x)=\sum_{r\in\mathbb{Z}+\frac{1}{2}}G^{(1),\kappa}(r)x^{-r-\frac{3}{2}}$$
$$Y^\kappa(\mu,x)=\sum_{r\in\mathbb{Z}+\frac{1}{2}}J^\kappa(r)x^{-r-1},\quad Y^\kappa(\tau^{(2)},x)=\sum_{n\in\mathbb{Z}}G^{(2),\kappa}(n)x^{-n-\frac{3}{2}}. \tag{19}$$

That is, define $J^\kappa(n) = \mu_n^\kappa$ and $G^{(2),\kappa}(n-1/2) = \tau_n^{(2),\kappa}$, for $n \in \mathbb{Z} + \frac{1}{2}$. Then, using the supercommutator relation (17) for the κ-twisted vertex operators acting on M, using the $L(-1)$-derivative property (18), using the N=2 Neveu–Schwarz supercommutation relations on V, and using (14), we have that the supercommutation relations for the κ-twisted modes of ω, μ, $\tau^{(1)}$ and $\tau^{(2)}$, given by $L^\kappa(n), G^{(2),\kappa}(n)$, for $n \in \mathbb{Z}$, and $J^\kappa(r), G^{(1),\kappa}(r)$, for $r \in \mathbb{Z} + \frac{1}{2}$, satisfy the relations of the mirror-twisted N=2 Neveu–Schwarz algebra given by (10)–(11) with central charge c.

In particular, a weak κ-twisted module, M, for an N=2 VOSA reduces the N=2 Neveu–Schwarz algebra representation to an N=1 Neveu–Schwarz algebra representation coupled with an N=1 Ramond algebra representation.

Next we consider the automorphisms σ_ξ which are of finite order. Let $\eta = e^{2\pi i/k}$, for $k \in \mathbb{Z}_+$, and let $\xi = \eta^j$, for $j = 1, \ldots, k-1$. Let σ_ξ be a VOSA automorphism of an N=2 VOSA, V, such that $\sigma_\xi(\mu) = \mu$ and $\sigma_\xi(\tau^{(\pm)}) = \xi^{\pm 1}\tau^{(\pm)}$, for $\tau^{(\pm)} = \frac{1}{\sqrt{2}}(\tau^{(1)} \mp i\tau^{(2)})$. Then $\omega, \mu \in V^0$ and $\tau^{(\pm)} \in V^{\pm j}$. If such a map exists for V, and M is a weak σ_ξ-twisted module for V, then write Y^{σ_ξ} for the σ_ξ-twisted operators, and

$$Y^{\sigma_\xi}(\omega,x) = \sum_{n\in\mathbb{Z}} L^{\sigma_\xi}(n)x^{-n-2}, \quad Y^{\sigma_\xi}(\mu,x) = \sum_{n\in\mathbb{Z}} J^{\sigma_\xi}(n)x^{-n-1}$$
$$Y^{\sigma_\xi}(\tau^{(\pm)},x) = \sum_{r\in\mathbb{Z}-\frac{1}{2}\pm\frac{j}{k}} G^{\pm,\sigma_\xi}(r)x^{-r-\frac{3}{2}}. \tag{20}$$

Then using the supercommutator relation (17) for the σ_ξ-twisted vertex operators acting on M, using the $L(-1)$-derivative property (18), using the N=2 Neveu–Schwarz supercommutation relations on V, and using (14), we have that the supercommutation relations for the σ_ξ-twisted modes of ω, μ, and $\tau^{(\pm)}$, that is the $L^{\sigma_\xi}(n)$ and $J^{\sigma_\xi}(n)$ for $n \in \mathbb{Z}$, and $G^{\pm}(r)$ for $r \in \mathbb{Z} + \frac{1}{2} \pm \frac{j}{k}$, respectively, satisfy the relations for the $\frac{j}{k}$-shifted N=2 Neveu–Schwarz algebra (1), (3), (5)–(6) with central charge c.

That is the sectors for N=2 supersymmetric VOSAs that arise under spectral flow $\mathscr{D}(t)$, for $t = j/k$, $k \in \mathbb{Z}_+$, $j = 1, \ldots, k-1$ are twisted sectors under the Virasoro-preserving automorphisms σ_ξ of the N=2 Neveu–Schwarz algebra.

If $\xi = -1$, then the map σ_ξ always extends to V, via the parity map $\sigma_{-1}(v) = (-1)^{|v|}v$, for $v \in V$. In this case, a weak σ_{-1}-twisted V-module is a representation of the N=2 Ramond algebra. If V is an N=2 VOSA which is also $J(0)$-graded such that the $J(0)$ eigenvalues are integral with $J(0)\omega = J(0)\mu = 0$, and $J(\tau^{(\pm)}) = \pm\tau^{(\pm)}$, then setting $\sigma_\xi(v) = \xi^n v$ if $J(0)v = nv$ gives a VOSA automorphism.

We summarize these results as follows:

Theorem 4.6. *If the Virasoro-preserving automorphisms of the N=2 Neveu–Schwarz algebra, κ, and σ_ξ, for ξ a root of unity, extend to VOSA automorphisms for an N=2 VOSA, V, then:*

(i) A weak κ-twisted V-module is a representation of the mirror-twisted N=2 superconformal algebra.

(ii) A weak σ_ξ-twisted V-module, for $\xi = e^{2j\pi i/k}$, is a representation of the $\frac{j}{k}$-shifted N=2 superconformal algebra. If $\xi = -1$, such a VOSA automorphism always exists (the parity map), and in this case, a weak σ_{-1}-twisted V-module is a representation of the N=2 Ramond algebra.

In [6], we also show, in particular, that if V is a free or lattice N=2 VOSA, then each Virasoro-preserving automorphism of the N=2 Neveu–Schwarz algebra extends to a VOSA automorphism of V, but not uniquely in the case of the mirror map. In fact, we show that there are two distinct mirror maps for free and lattice N=2 VOSAs and these mirror maps give nonisomorphic mirror-twisted V-modules. We also construct examples of g-twisted V-modules for $g = \kappa$ and $g = \sigma_\xi$ of finite order.

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References

1. Adamović, D.: Representations of the N=2 superconformal vertex algebra. Int. Math. Res. Notice **2**, 61–79 (1999). Vertex algebra approach to fusion rules for N=2 superconformal minimal models. J. Algebra **239**(2), 549–572 (2001)
2. Barron, K.: $N = 1$ Neveu–Schwarz vertex operator superalgebras over Grassmann algebras and with odd formal variables. In: Representations and Quantizations: Proceedings of the International Conference on Representation Theory, pp. 9–39. Shanghai China, Springer (2000)
3. Barron, K.: The notion of $N = 1$ supergeometric vertex operator superalgebra and the isomorphism theorem. Commun. Contemp. Math. **5**(4), 481–567 (2003)
4. Barron, K.: The moduli space of N=2 super-Riemann spheres with tubes. Commun. Contemp. Math. **9**, 857–940 (2007)
5. Barron, K.: Axiomatic aspects of N=2 vertex superalgebras with odd formal variables. Commun. Alg. **38**, 1199–1268 (2010)
6. Barron, K.: Twisted modules for N=2 supersymmetric vertex operator superalgebras arising from finite automorphisms of the N=2 Neveu–Schwarz algebra. arXiv: 1110.0229
7. Barron, K., Dong, C., Mason, G.: Twisted sectors for tensor products vertex operator algebras associated to permutation groups. Comm. Math. Phys. **227**, 349–384 (2002)
8. Barron, K., Huang, Y.-Z., Lepowsky, J.: An equivalence of two constructions of permutation-twisted modules for lattice vertex operator algebras. J. Pure Appl. Algebra **210**, 797–826 (2007)
9. Boucher, W., Friedan, D., Kent, A.: Determinant formulae and unitarity for the N=2 superconformal algebras in two dimensions or exact results on string compactification. Phys. Lett. B **172**(3–4), 316–322 (1986)
10. Di Vecchia, P., Petersen, J., Yu, M.: On the unitary representations of N=2 superconformal theory. Phys. Lett. B **172**(2), 211–215 (1986)
11. Di Vecchia, P., Petersen, J., Zheng, H.: N=2 extended superconformal theories in two dimensions. Phys. Lett. B **162**(4–6), 327–332 (1985)
12. Dong, C., Li, H., Mason, G.: Twisted representations of vertex operator algebras. Math. Ann. **310**, 571–600 (1998)

13. Dobrev, V.: Characters of the unitarizable highest weight modules over the N=2 superconformal algebras. Phys. Lett. B **186**, 43–51 (1987)
14. Dörrzapf, M.: Singular vectors of the N=2 superconformal algebra. Int. J. Modern Phys. A **10**(15), 2143–2180 (1995)
15. Dörrzapf, M., Gato-Rivera, B.: Singular dimensions of the N=2 superconformal algebras. I. Comm. Math. Phys. **206**(3), 493–531 (1999)
16. Dörrzapf, M., Gato-Rivera, B.: Singular dimensions of the N=2 superconformal algebras. II. The twisted N=2 algebra. Comm. Math. Phys. **220**(2), 263–292 (2001)
17. Feigin, B., Semikhatov, A., Sirota, V., Tipunin, I.: Resolutions and characters of irreducible representations of the N=2 superconformal algebra. Nuclear Phys. B **536**(3), 617–656 (1999)
18. Feigin, B., Semikhatov, A., Tipunin, I.: Equivalence between chain categories of representations of affine sl(2) and N=2 superconformal algebras. J. Math. Phys. **39**(7), 3865–3905 (1998)
19. Frenkel, I., Lepowsky, J., Meurman, A.: A natural representation of the Fischer-Griess Monster with the modular function J as character. Proc. Natl. Acad. Sci. USA **81**, 3256–3260 (1984). Vertex operator calculus. In: Yau, S.-T. (ed.) Mathematical Aspects of String Theory, Proc. 1986 Conference, San Diego, pp. 150–188. World Scientific, Singapore (1987). Vertex operator algebras and the monster. Pure Appl. Math., vol. 134. Academic Press, Boston, MA (1988)
20. Fu, J., Jiang, Q., Su, Y.: Classificiation of modules of the intermediate series over Ramond N=2 superconformal algebras. J. Math. Phys. **48**(043508), 17 (2007)
21. Gato-Rivera, B.: Construction formulae for singular vectors of the topological and of the Ramond N=2 superconformal algebras. Int. J. Modern Phys. A **17**, 4515–4541 (2002)
22. Iohara, K., Koga, Y.: Representation theory of $N = 2$ super Virasoro algebra: twisted sector. J. Funct. Anal. **214**, 450–518 (2004)
23. Kiritsis, E.: Character formulae and the structure of the representations of the N=1,N=2 superconformal algebras. Int. J. Modern Phys. A **3**, 1871–1906 (1988)
24. Lepowsky, J., Wilson, R.: Construction of the affine Lie algebra $A_1^{(1)}$. Comm. Math. Phys. **62**, 43–53 (1978)
25. Li, H.: Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, in: "Moonshine, the Monster, and related topics (South Hadley, MA, 1994). Contemp. Math **193**, 203–236 (1996)
26. Li, J., Su, Y., Zhu, L.: Classification of indecomposable modules of the intermediate series over the twisted N=2 superconformal algebra. J. Math. Phys. **51**(8), 083515, 17 (2010)
27. Matsuo, Y.: Character formula of c<1 unitary representation of N=2 superconformal algebra. Prog. Theor. Phys. **77**, 793–797 (1987)
28. Nam, S.: The Kac formula for the N=1 and the N=2 super-conformal algebras. Phys. Lett. B **172**(3–4), 323–327 (1986)
29. Schwimmer, A., Seiberg, N.: Comments on the $N = 2, 3, 4$ superconformal algebras in two dimensions. Phys. Lett. B, **184**, 191–196 (1986)
30. Semikhatov, A., Tipunin, I.: The structure of Verma modules over the N=2 superconformal algebra. Comm. Math. Phys. **195**(1), 129–173 (1998)
31. Semikhatov, A., Tupunin, I., Feigin, B.: Semi-infinite realization of unitary representations of the N=2 algebra and related constructions. Theor. Math. Phys. **126**, 1–47 (2001)