From Singularities to Algebras to Pure Yang–Mills with Matter

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Abstract Since the advent of dualities in string theory, it has been well-known that codimension 4 orbifold singularities that appear in extra-dimensional spaces, such as Calabi–Yau or G_2 spaces, may be interpreted as ADE gauge theories. As to orbifold singularities of higher codimension, there has not been an analog of this interpretation. Here we show how the search for such an analog led us from the singularities to the creation of Lie Algebras of the Third Kind ("LATKes"). We introduce an example of a LATKe that arises from the singularity $\mathbf{C}^3/\mathbf{Z}_3$, and prove it to be simple and unique. We explain that the uniqueness of the LATKe serves as a vacuum selection mechanism. We also show how the LATKe leads to a new kind of gauge theory in which the matter field arises naturally and which is tantalizingly close to the Standard Model of particle physics.

1 Introduction and Motivation

One of the outcomes of the "string revolution" of the mid-1990s was an interpretation of ADE singularities in Calabi–Yau spaces as gauge theories with ADE gauge groups [1, 2]. This interpretation arose via string dualities, and later on was applied to the same singularities within manifolds of G_2 holonomy in the context of M-theory compactifications [3–5]. The usefulness of this interpretation lies in the fact that it enhances our understanding of the four-dimensional theory that is obtained when string/M theory is compactified on Calabi–Yau or G_2 spaces which have those ADE singularities. A particularly encouraging result of this interpretation was the first manifestation from M-theory [6–8] of Georgi–Glashow grand unification [9], where the $SU(5)$ grand unified group is obtained from an A_4

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singularity in a G_2 manifold, and is then naturally broken by Wilson lines precisely to $SU(3)\times SU(2)\times U(1)$, the gauge group of the standard model of particle physics.

Singularities other than ADE have arisen in the same context [6], but an analogous interpretation of those other singularities in terms of gauge groups and gauge theories was not available. For example, take orbifold singularities of codimension 6, of the form $\mathbf{C}^3/\Gamma$, where Γ is a discrete finite subgroup of $SU(3)$; these are direct generalizations of ADE singularities which are codimension 4 orbifold singularities of the form $\mathbf{C}^2/\Gamma$, where Γ is a discrete finite subgroup of $SU(2)$. For the codimension 6 singularities we ask: what is the four-dimensional physical theory that arises from string/M theory compactifications on CY or G_2 manifolds that have these codimension 6 singularities?

As it turns out, string dualities do not provide for a generalization of the interpretation of codimension 4 orbifold singularities to one for codimension 6 orbifold singularities. Instead, we address this question by turning to the mathematical roots of these dualities. Our approach is to analyze the mathematical aspects of codimension 4 singularities in a way that will allow us to generalize to codimension 6, and then obtain an interpretation of the results for the physical theory.

On the mathematical side, we introduce a new set of relations, which we call the Commutator-Intersection Relations, that illuminate the connection between codimension 4 singularities and Lie algebras. These relations pave the way to construct Lie Algebras of the Third Kind, or LATKes, a kind of algebras that arise from codimension 6 orbifold singularities. We also learn and prove the existence and uniqueness of a simple LATKe.

On the physics side, we discover a new kind of Yang–Mills theory, called "LATKe Yang–Mills," which arises from the LATKe. Unlike any known Yang–Mills theory, the LATKe Yang–Mills theory in its purest form automatically contains matter. We also propose that the uniqueness of the simple LATKe is a vacuum selection mechanism. The selected vacuum theory is an $SU(2)\times SU(2)$ gauge theory with matter in the $(2,2)$ representation, and the corresponding singularity is $\mathbf{C}^3/\mathbf{Z}_3$. The algebra $\mathfrak{su}(2)\times\mathfrak{su}(2)$ is protected by the LATKe from being broken. The selected singularity $\mathbf{C}^3/\mathbf{Z}_3$ is one of those which arose in the G_2 spaces of [6], and which at the time we put on hold in anticipation of the outcome of this investigation.

This paper is based on [10]; due to space constraints, we leave out many details and references which may be found there.

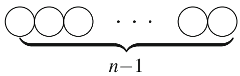
2 The Commutator-Intersection Relations (CIRs)

Let $\mathbf{C}^2/\Gamma$ be an orbifold singularity of codimension 4, with Γ a discrete, finite subgroup of $SU(2)$; the groups Γ were studied by Klein [11], who found them to have an ADE classification. Work of DuVal and of Artin [12–14] then provided a correspondence between these singularities and Lie algebras. We now present the correspondence in a way that will lead us to a new relation between intersection numbers of the blow-ups of the singularities and commutators of the Lie algebras. These relations, which we name the Commutator-Intersection Relations, will then be generalized to the $\mathbf{C}^3/\Gamma$ case.

We proceed via an example. Let $\Gamma = \mathbf{Z}_n \subset SU(2)$, which corresponds to A_{n-1} in the ADE classification, be generated by the $SU(2)$ matrix

$$\begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}. \tag{1}$$

Its action on $(x,y) \in \mathbf{C}^2$ is given by $(x,y) \longmapsto (e^{2\pi i/n}x, e^{-2\pi i/n}y)$. This action is free except at the origin where $\mathbf{C}^2/\mathbf{Z}_n$ has a singularity. The blow-up of this singularity has an exceptional divisor made up of $n-1$ spheres S^2 that intersect as follows:



When the spheres are replaced by nodes and their intersections are replaced by edges, we obtain the Dynkin diagram of the Lie algebra $\mathfrak{sl}_{n-1}$:

$$\bullet - \bullet - \bullet \quad \cdots \quad \bullet - \bullet \tag{2}$$

Furthermore, the intersection numbers between pairs of spheres of the exceptional divisor are exactly minus of the entries of the Cartan matrix of the Lie algebra:

$$I_{ij} = -C_{ij} \qquad i,j = 1,\ldots,n-1. \tag{3}$$

As shown in the work of Duval and of Artin, this relation between the $\mathbf{Z}_n$ singularity and the $\mathfrak{sl}_{n-1}$ Lie algebra generalizes to a correspondence between all the ADE singularities and ADE Lie algebras: in all cases, the blow-up of the ADE singularity corresponds to the Dynkin diagram of the ADE Lie algebra, and (3) holds.

Using the above correspondence, we now show how to obtain a direct relation between the intersection numbers of the blow-up of the ADE singularity and the commutators of the ADE Lie algebras.

Recall that a complex simple Lie algebra is generated by k triples $\{X_i, Y_i, H_i\}_{i=1}^{k}$ with their commutators determined by the following relations:

$$\begin{aligned} &[H_i, H_j] = 0\,; && [X_i, Y_j] = \delta_{ij} H_j; \\ &[H_i, X_j] = C_{ij} X_j\,; && [H_i, Y_j] = -C_{ij} Y_j; \\ &\mathrm{ad}(X_i)^{1-C_{ij}}(X_j) = 0\,; && \mathrm{ad}(Y_i)^{1-C_{ij}}(Y_j) = 0. \end{aligned} \tag{4}$$

Here, the H_i form the Cartan subalgebra, the X_i are simple positive roots, the Y_i are simple negative roots, k is the rank of the Lie algebra, C_{ij} is the Cartan matrix, and $\mathrm{ad}(X_i)(A) = [X_i, A]$. These equations are the familiar Chevalley–Serre relations.

Using (3), we can replace C_{ij} in (4) by $-I_{ij}$, giving a new set of relations:

$$
\begin{aligned}
&[H_i, H_j] = 0\,; && [X_i, Y_j] = \delta_{ij} H_j;\\
&[H_i, X_j] = -I_{ij} X_j\,; && [H_i, Y_j] = I_{ij} Y_j;\\
&\mathrm{ad}(X_i)^{1+I_{ij}}(X_j) = 0\,; && \mathrm{ad}(Y_i)^{1+I_{ij}}(Y_j) = 0.
\end{aligned}
\tag{5}
$$

These are the CIRs relations, which are central in what follows. They demonstrate that *the intersection numbers of the exceptional divisor completely determine the commutators of the corresponding Lie algebra*.

3 Lie Algebras of the Third Kind

Here we generalize the CIRs relations to the case of codimension 6 orbifold singularities.

For codimension $2n$ singularities, $n \geq 2$, the components of the exceptional divisor are $(2n-2)$-cycles, and the intersection of a pair of those has dimension

$$\dim(C_1 \cap C_2) = \dim C_1 + \dim C_2 - 2n = 2n - 4. \tag{6}$$

Therefore, when $n = 2$ (the codimension 4 case), $\dim C_1 = \dim C_2 = 2$ and a pair of cycles intersect in a zero-dimensional space, yielding a number. But for codimension 6 orbifolds, the components C_i of the exceptional divisor are 4-cycles, and the intersection of any pair C_1, C_2 of 4-cycles does not yield a number but a two-dimensional space:

$$\dim(C_1 \cap C_2) = 4 + 4 - 6 = 2. \tag{7}$$

To obtain intersection numbers, we consider instead intersections of *triples* of 4-cycles. By iterating (6), we see that such intersections are zero-dimensional. They yield intersection numbers I_{ijk} with three indices.

The triple intersection numbers enable us to generalize the CIRs to the codimension 6 case. Take the second line of (5)

$$[H_i, X_j] = -I_{ij} X_j\,; \qquad [H_i, Y_j] = I_{ij} Y_j\,. \tag{8}$$

Using I_{ijk}, we may generalize this to

$$[A_i, B_j, X_k] = -I_{ijk} X_k\,; \qquad [A_i, B_j, Y_k] = I_{ijk} Y_k\,. \tag{9}$$

The A_i, B_j, and X_k are as yet not defined. However, it is now clear how to generalize the original correspondence of Duval and of Artin to the codimension 6 case: there is a new algebraic object that takes the place of Lie algebras, and it involves a commutator of three objects.

Definition 1. A ***Lie Algebra of the Third Kind (a "LATKe")*** $\mathfrak{L}$ is a vector space equipped with a commutator of the third kind, which is a trilinear anti-symmetric map

$$[\,\cdot\,,\,\cdot\,,\,\cdot\,]:\Lambda^3\mathfrak{L}\to\mathfrak{L} \tag{10}$$

that satisfies the Jacobi identity of the third kind (or the LATKe Jacobi identity):

$$[X,Y,[Z_1,Z_2,Z_3]]=[[X,Y,Z_1],Z_2,Z_3]+[Z_1,[X,Y,Z_2],Z_3]+[Z_1,Z_2,[X,Y,Z_3]] \tag{11}$$

for $X,Y,Z_i\in\mathfrak{L}$.

We can now easily generalize this definition to an algebra that would correspond to codimension $2n$ orbifold singularities for any $n\geq 2$:

Definition 2. A ***Lie Algebra of the n-th Kind (a "LAnKe")*** $\mathfrak{L}$ is a vector space equipped with a commutator of the n-th kind, which is an n-linear, totally antisymmetric map

$$[\cdot,\cdot,\qquad,\cdot]:\wedge^n\mathfrak{L}\to\mathfrak{L}\,, \tag{12}$$

that satisfies the Jacobi identity of the n-th kind:

$$[X_1,\dots X_{n-1},[Z_1,\dots Z_n]]=\sum_{i=1}^{n}[Z_1,\dots,[X_1,\dots X_{n-1},Z_i],\dots Z_n]\,, \tag{13}$$

for $X_i,Z_j\in\mathfrak{L}$.

Since our original physical motivation involved singularities in the extra-dimensional manifolds of string and M-theory, and those are at most seven-dimensional, we will concentrate on codimension 6 orbifolds rather than higher dimensional ones.

4 Example of a LATKe

Before we go any further, we construct an explicit example of a LATKe arising from a singularity. We construct it directly from the singularity $\mathbf{C}^3/\mathbf{Z}_3$, where the $\mathbf{Z}_3$ action on $\mathbf{C}^3$ is given by

$$\varepsilon:(x,y,z)\longmapsto(\varepsilon x,\varepsilon y,\varepsilon z)\,,\ \text{where}\ \ \varepsilon^3=1,(x,y,z)\in\mathbf{C}^3. \tag{14}$$

The blow-up at the origin of this singularity is the 4-cycle $\mathbf{P}^2$. Recall that in the codimension-4 case, each component of the exceptional divisor corresponds to a node in the Dynkin diagram of the corresponding Lie algebra, and therefore to a simple root. Here too, the $\mathbf{P}^2$ corresponds to a "root" of the LATKe, which we now define.

Recall that for a Lie algebra $\mathfrak{g}$, a root is an element of the dual space of the Cartan subalgebra $\mathfrak{h}$, where $H \in \mathfrak{h}$ acts as an operator on $\mathfrak{g}$ via

$$H : X_\alpha \longmapsto [H, X_\alpha] = \alpha(H) X_\alpha, \tag{15}$$

where $X_\alpha \in \mathfrak{g}$ is a root vector. For a LATKe, there is no natural action of a subalgebra. However, given a subalgebra $\mathfrak{h}_{\mathfrak{L}} \subset \mathfrak{L}$, there *is* a natural action on $\mathfrak{L}$ of a pair $H_1, H_2 \in \mathfrak{h}_{\mathfrak{L}}$ given by

$$H_1 \wedge H_2 : X \longmapsto [H_1, H_2, X] \,, \tag{16}$$

where $X \in \mathfrak{L}$. Therefore, if we define a Cartan subalgebra $\mathfrak{h}_{\mathfrak{L}}$ to be a maximal commuting subalgebra of $\mathfrak{L}$ such that $\Lambda^2 \mathfrak{h}_{\mathfrak{L}}$ acts diagonally on $\mathfrak{L}$, then we can define a root as follows:

Definition 3. Let $\mathfrak{L}$ be a LATKe and let $\mathfrak{h}_{\mathfrak{L}}$ be a Cartan subalgebra of $\mathfrak{L}$. A ***root*** α of $\mathfrak{L}$ is a map in the dual space of $\Lambda^2 \mathfrak{h}_{\mathfrak{L}}$:

$$\alpha : \Lambda^2 \mathfrak{h}_{\mathfrak{L}} \longrightarrow \mathbf{C} \,. \tag{17}$$

Since we have a single cycle in our exceptional divisor (i.e. the $\mathbf{P}^2$), our root space is one-dimensional. From the definition of a root, we see this means that the Cartan subalgebra is two-dimensional. So we have, so far, four elements in the LATKe: H_1 and H_2 (making up the Cartan subalgebra), a positive root X, and a negative root Y. We also have

$$[H_1, H_2, X] = \alpha(H_1 \wedge H_2) X \,, \tag{18}$$

$$[H_1, H_2, Y] = -\alpha(H_1 \wedge H_2) Y \,, \tag{19}$$

where α is a simple root. Note that $\alpha(H_1 \wedge H_2) = -I_{111}$, the triple intersection of the exceptional divisor of our singularity, but we can normalize H_1 and H_2 so that

$$\begin{aligned} &[H_1, H_2, X] = X \,; \\ &[H_1, H_2, Y] = -Y \,. \end{aligned} \tag{20}$$

We have but two commutators left to determine: $[H_i, X, Y]$, $i = 1, 2$. To do so, we use the LATKe Jacobi identity, which can be shown to require, among other things, that $[H_i, X, Y] \in \mathfrak{h}_{\mathfrak{L}}$. We also restrict our attention to "simple" LATKes, which we now define.

Definition 4. An ***ideal*** of $\mathfrak{L}$ is a subalgebra $\mathscr{I}$ that satisfies

$$[\mathfrak{L}, \mathfrak{L}, \mathscr{I}] \subset \mathscr{I} \,. \tag{21}$$

Definition 5. A LATKe is ***simple*** if it is non-Abelian and has no non-trivial ideals.

Now our example of a simple LATKe, which we name $\mathfrak{L}_3$, is fully determined as follows:

Example 1. The simple LATKe $\mathfrak{L}_3$ corresponding to the singularity $\mathbf{C}^3/\mathbf{Z}_3$, with $\mathbf{Z}_3$ action given by (14), is four dimensional, with commutators

$$[H_1, H_2, X] = X, \qquad [H_1, H_2, Y] = -Y,$$
$$[H_1, X, Y] = H_2, \qquad [H_2, X, Y] = H_1,$$

where H_1, H_2 form the Cartan subalgebra $\mathfrak{h}_{\mathfrak{L}_3}$ and X, Y are positive and negative root vectors, respectively.

One may easily check that the LATKe Jacobi identity is satisfied. Note that with an appropriate change of basis [10], one can see that this algebra is given by the cross product in four dimensions, or equivalently by the algebra of differential forms in four dimensions with the triple commutator given by the Hodge dual of the exterior product of three 1-forms.

5 Classification of Simple LATKes

Having constructed a LATKe from a codimension 6 orbifold singularity, we turn to the task of classifying all simple finite dimensional LATKes. We present here only a brief sketch of the proof of the classification; the complete proof and any omitted details can be found in [10].

Let $\mathrm{Der}(\mathfrak{L}) = \mathfrak{g}_{\mathfrak{L}}$ be the Lie algebra of derivations of $\mathfrak{L}$, consisting of operators D satisfying

$$D[X, Y, Z] = [DX, Y, Z] + [X, DY, Z] + [X, Y, DZ]\,, \tag{22}$$

with the Lie bracket

$$[D_1, D_2] = D_1 D_2 - D_2 D_1\ . \tag{23}$$

Then $\mathfrak{L}$ itself is a representation space for $\mathfrak{g}_{\mathfrak{L}}$. In fact, it can be shown that if $\mathfrak{L}$ is simple, it is irreducible and faithful as a representation of $\mathfrak{g}_{\mathfrak{L}}$, so $g_{\mathfrak{L}}$ is reductive. Further, it can be shown that the center of $\mathfrak{g}_{\mathfrak{L}}$ is trivial, leading to:

Lemma 1. *If $\mathfrak{L}$ is simple then $\mathfrak{g}_{\mathfrak{L}}$ is semi-simple.*

The surjective morphism of representations of $\mathfrak{g}_{\mathfrak{L}}$

$$\mathrm{ad} : \Lambda^2 \mathfrak{L} \longrightarrow \mathfrak{g}_{\mathfrak{L}} \tag{24}$$

indicates a close relation between weights of $\Lambda^2\mathfrak{L}$ and roots of $\mathfrak{g}_{\mathfrak{L}}$. By studying this morphism and its kernel we obtain an equation relating highest roots of $\mathfrak{g}_{\mathfrak{L}}$ to the highest weight of $\mathfrak{L}$:

Lemma 2. *Let θ be a highest root of $\mathfrak{g}_{\mathfrak{L}}$, and let Λ be the highest weight of $\mathfrak{L}$ as a representation of $\mathfrak{g}_{\mathfrak{L}}$. Then*

$$\theta = 2\Lambda - \alpha \tag{25}$$

for some simple positive root α of $\mathfrak{g}_{\mathfrak{L}}$.

The two lemmas put together mean that for a Lie algebra $\mathfrak{g}$ to serve as $\mathfrak{g}_{\mathfrak{L}}$ for some LATKe $\mathfrak{L}$, it must be semisimple and it must admit a faithful, irreducible representation whose highest weight Λ satisfies (25). Interestingly, the condition in (25) was studied in an entirely different context by Kac [15].

For our purposes, the condition in (25) is necessary but not sufficient; an additional requirement is that the map $\omega : \Lambda^2 V \to \mathfrak{g}$ of representations of $\mathfrak{g}$ must yield a LATKe commutator via

$$[v_1, v_2, v_3] = (\omega(v_1 \wedge v_2)) \cdot v_3 \ , \qquad v_i \in V, \tag{26}$$

where the expression on the right hand side must be antisymmetric in all three variables.

As it happens, rather surprisingly, there is only one Lie algebra that satisfies all these conditions. It is $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, the Lie algebra of derivations of our example $\mathfrak{L}_3$. So we have

Theorem 1. *There is precisely one simple LATKe, namely $\mathfrak{L}_3$ of Example 1.*

6 The Physics of LATKes

Having constructed a LATKe directly from a codimension 6 singularity, and having discovered its uniqueness, we now turn to two physical applications: first, we describe LATKe gauge theory, which is a new kind of gauge theory that arises from codimension 6 orbifold singularities; and second, we interpret the very uniqueness of the LATKe as a new kind of vacuum selection mechanism for the string landscape.

6.1 LATKe Gauge Theory

In analogy with the traditional treatment of Lie algebras and their applications in particle physics, we define a representation for LATKes. We begin with an example: the adjoint representation. This is a map that utilizes the commutator in a natural way:

$$\mathrm{ad} : \mathfrak{L} \wedge \mathfrak{L} \longrightarrow \mathrm{End}(\mathfrak{L}) \tag{27}$$

$$\mathrm{ad}(X \wedge Y) : Z \longmapsto [X, Y, Z] \ . \tag{28}$$

The map ad satisfies the condition

$$[\mathrm{ad}(X_1 \wedge X_2), \mathrm{ad}(X_3 \wedge X_4)] = \mathrm{ad}([X_1, X_2, X_3] \wedge X_4) + \mathrm{ad}(X_3 \wedge [X_1, X_2, X_4]), \quad (29)$$

which is equivalent to the LATKe Jacobi identity. If we generalize (27) and (29), we have

Definition 6. A ***representation*** of a LATKe $\mathfrak{L}$ is a map

$$\rho : \Lambda^2 \mathfrak{L} \longrightarrow \mathrm{End}(V) \quad (30)$$

for some vector space V, subject to the condition

$$[\rho(X_1 \wedge X_2), \rho(X_3 \wedge X_4)] = \rho([X_1, X_2, X_3] \wedge X_4) + \rho(X_3 \wedge [X_1, X_2, X_4]). \quad (31)$$

In traditional Yang–Mills theory, one studies matter fields ψ in certain representations ρ of the gauge group or Lie algebra, and the Yang–Mills Lagrangian contains terms in which the fields are transformed according to those representations. For the LATKe, we are able [10] to construct an analogous system, using the definition of representations of a LATKe rather than representations of an ordinary Lie algebra.

We end up with more than we could have hoped for: in conventional Yang Mills theory, we have what is known as "pure Yang–Mills theory," where the gauge fields, which live in the adjoint representation of the gauge group, are the only fields. There are no matter fields—that is, no field ψ appears—and the Lagrangian consists only of the kinetic term of the gauge field. In general, for physical theories to include matter fields they typically have to be put in by hand.

But in the LATKe Yang–Mills theory, this is not the case. Built into the theory is not just the adjoint representation $\Lambda^2 \mathfrak{L}$ of $\mathfrak{g}_{\mathfrak{L}}$, but also the adjoint representation of the LATKe itself, i.e. $\mathfrak{L}$. This representation is in fact a matter representation of $\mathfrak{g}_{\mathfrak{L}}$ and an inseparable part of *pure* LATKe Yang–Mills theory.

Therefore, unlike pure Yang–Mills theory, pure *LATKe* Yang–Mills theory *automatically includes matter*, without the need to put it in by hand. The fact that matter, which must of course be included in any physical theory, is intrinsic to LATKe gauge theory makes it all the more compelling.

6.2 Uniqueness of the LATKe as a Vacuum Selection Mechanism

One of the central outcomes of the "string revolution" of the mid-1980s [16–18] was that string theory came along with gauge theories. At the time, the gauge theories that arose were far larger than the Standard Model gauge group: anomaly cancellation dictated they may be only $E_8 \times E_8$ or $SO(32)$. However, the fact that gauge theories appeared at all was a triumph for string theory, as it gave hope for applications of string theory to the real world. It led physicists to believe for many years that upon searching further, the Calabi–Yau or G_2 manifold that results precisely in the Standard Model of Particle Physics would be found.

After a while it became apparent [19] that there is a staggering number of possible CY or G_2 manifolds, forming what is now known as the "string landscape." Therefore, the idea of a "vacuum selection mechanism," which is some principle that would single out one vacuum or at least narrow down the choices considerably, has been sought after.

The uniqueness of the LATKe *is* a vacuum selection mechanism. The selected compactification space is a Calabi–Yau or G_2 space with a $\mathbf{C}^3/\mathbf{Z}_3$ singularity, and the selected vacuum theory is a supersymmetric $\mathfrak{su}(2)\times\mathfrak{su}(2)$ gauge theory with matter in the $(2,2)$ representation.

While it has been accepted that no vacuum selection mechanisms have as yet been proposed [19], in retrospect we claim that before the present work, there did exist a vacuum selection mechanism: anomaly cancellation. It selected a string theory with gauge group either $E_8\times E_8$ or $SO(32)$.

While neither the uniqueness of the LATKe nor anomaly cancellation actually selects the standard model itself, our unique, simple LATKe Yang–Mills is tantalizingly close to the standard model.

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