Springer Proceedings in Mathematics & Statistics

Vladimir Dobrev *Editor*

Lie Theory and Its Applications in Physics

IX International Workshop

Springer

Springer Proceedings in Mathematics & Statistics

Volume 36

For further volumes:
http://www.springer.com/series/10533

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of select contributions from workshops and conferences in all areas of current research in mathematics and statistics, including OR and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

Vladimir Dobrev
Editor

Lie Theory and Its Applications in Physics

IX International Workshop

Springer

Editor
Vladimir Dobrev
Institute for Nuclear Research
and Nuclear Energy
Bulgarian Academy of Sciences
72 Tsarigradsko Chaussee
Sofia, Bulgaria

ISSN 2194-1009 ISSN 2194-1017 (electronic)
ISBN 978-4-431-54269-8 ISBN 978-4-431-54270-4 (eBook)
DOI 10.1007/978-4-431-54270-4
Springer Tokyo Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013934220

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

The workshop series "Lie Theory and Its Applications in Physics" is designed to serve the community of theoretical physicists, mathematical physicists, and mathematicians working on mathematical models for physical systems based on geometrical methods and in the field of Lie theory.

The series reflects the trend towards a geometrization of the mathematical description of physical systems and objects. A geometric approach to a system yields in general some notion of symmetry which is very helpful in understanding its structure. Geometrization and symmetries are meant in their widest sense, i.e., classical geometry, differential geometry, groups and quantum groups, infinite-dimensional (super-)algebras, and their representations. Furthermore, we include the necessary tools from functional analysis and number theory. This is a big interdisciplinary and interrelated field.

The first three workshops were organized in Clausthal (1995, 1997, 1999), the 4th was part of the 2nd Symposium "Quantum Theory and Symmetries" in Cracow (2001), the 5th, 7th, and 8th were organized in Varna (2003, 2007, 2009), and the 6th was part of the 4th Symposium "Quantum Theory and Symmetries" in Varna (2005) but has its own volume of proceedings.

The 9th workshop of the series (LT-9) was organized by the Institute of Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences (BAS) in June 2011 (20–26), at the guest house of BAS near Varna on the Bulgarian Black Sea Coast.

The overall number of participants was 76 and they came from 21 countries.

The scientific level was very high as can be judged by the speakers. The plenary speakers were Anton Alekseev (Geneva), Loriano Bonora (Trieste), Branko Dragovich (Belgrade), Anthony Joseph (Rehovot), Toshiyuki Kobayashi (Tokyo), Jean-Louis Loday (Strasbourg), Ivan Penkov (Bremen), Karl-Henning Rehren (Gttingen), and Ivan Todorov (Sofia). A special plenary session, with the speakers Joris Van der Jeugt (Ghent), Ronald King (Southampton), and David Finkelstein (Atlanta), was devoted to the 75th-year Jubilee of Tchavdar Palev, Professor Emeritus at our Institute.

The topics covered the most modern trends in the field of the workshop: representation theory, quantum field theory, string theory, (super-)gravity theories, conformal field theory, supersymmetry, quantum groups, vertex algebras, and integrability.

The members of the International Organizing Committee were V.K. Dobrev (Sofia) and H.-D. Doebner (Clausthal), in collaboration with G. Rudolph (Leipzig).

The members of the Local Organizing Committee were V.K. Dobrev (Chairman), V.I. Doseva, A. Ganchev, S.G. Mihov, D. Nedanovski, T.V. Popov, T. Stefanova, M.N. Stoilov, and S.T. Stoimenov.

Acknowledgments

We express our gratitude to the Institute of Nuclear Research and Nuclear Energy and to the Abdus Salam International Centre for Theoretical Physics for financial help. We thank the Bulgarian Academy of Sciences for providing its guest house, which contributed very much to the stimulating and pleasant atmosphere during the workshop. We thank the publisher, Springer Japan, represented by Ms. Chino Hasebe (Executive Editor in Mathematics, Statistics, Business, Economics, Computer Science) and Mr. Masayuki Nakamura (Editorial Department), for assistance in this publication. Last but not least, we thank the members of the Local Organizing Committee, who, through their efforts, made the workshop run smoothly and efficiently.

Sofia and Geneva Vladimir Dobrev

Contents

Part VIII Various Mathematical Results

List of Participants

Naruhiko Aizawa Osaka Prefecture University, Sakai, Osaka, Japan

Anton Alekseev Department of Mathematics, University of Geneva, Switzerland

Iana Anguelova Department of Mathematics, College of Charleston, SC, USA

Bojko Bakalov Department of Mathematics, North Carolina State University, Raleigh, NC, USA

Katrina Barron Department of Mathematics, University of Notre Dame, Notre Dame, IN, USA

Elaine Beltaos Grant MacEwan University, Alberta, Canada

Nikolay Bobev Simons Center for Geometry and Physics – SUNY, Stony Brook, NY, USA

Loriano Bonora SISSA, Trieste, Italy

Cestmir Burdik Department of Mathematics, Czech Technical University in Prague, Trojanova, Prague, Czech Republic

Scott Carnahan University of Tokyo, IPMU, Kashiwa, Japan

Sultan Catto FIT-State University of New York, New York, NY, USA

Jean Francois Colombeau Institut Fourier, 100 rue des maths, St Martin d'Heres cedex, France

Kevin Coulembier Ghent University, Krijgslaan, Gent, Belgium

Thomas Creutzig Fachbereich Mathematik, Technische University of Darmstadt, Darmstadt, Germany

Domenico D'Alessandro Department of Mathematics, Iowa State University, IA, USA

Jennie Dambroise Bard College, NY, USA

Omer Faruk Dayi Physics Department, Istanbul Technical University, Faculty of Science and Letters, TR, Maslak-Istanbul, Turkey

Ivan Dimitrijevic Faculty of Mathematics, University of Belgrade, Belgrade, Serbia

Vladimir Dobrev Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Branko Dragovich Institute of Physics, Zemun Belgrade, Serbia

Laszlo Feher MTA KFKI RMKI and University of Szeged, Konkoly Thege Miklos ut 29-33, Budapest, Hungary

David Finkelstein Georgia Institute of Technology, Atlanta, GA, USA

Tamar Friedmann Department of Physics, University of Rochester, NY, USA

Alexander Ganchev Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Lachezar Georgiev Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Tatiana Gerasimova Taras Shevchenko National University, Kiev, Ukraine

Anastasia Golubtsova Peoples' Friendship University of Russia, Moscow, Russia

Ludmil Hadjiivanov Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Elitza Hristova Department of Mathematics, Jacobs University Bremen, Bremen, Germany

Jiri Hrivnak Czech Technical University in Prague, Praha, Czech Republic

Masao Jinzenji Hokkaido University, Kita-ku, Sapporo, Hokkaido, Japan

Anthony Joseph Mathematics Department, Weizmann Institute, Herzl St. Rehovot, Israel

Yuri Karadzhov Department of Applied Research, National Academy of Sciences of Ukraine, Institute of Mathematics, Kyiv, Ukraine

Ronald King School of Mathematics, University of Southampton, Southampton, England, UK

Ctirad Klimcik IML Luminy, Marseille, France

Miljan Knezevic Faculty of Mathematics, University of Belgrade, Belgrade, Serbia

Toshiyuki Kobayashi Graduate School of Mathematical Sciences, University of Tokyo, Meguro, Komaba, Tokyo, Japan

Takeo Kojima Graduate School of Science and Engineering, Yamagata University, Yonezawa, Japan

Rémi Léandre Laboratoire de Mathématiques, Université de Franche Comté, Besançon, France

Andrew Linshaw Technische University of Darmstadt, Darmstadt, Germany

Jean-Louis Loday Institut de Recherche Mathématique Avancée, CNRS et Université de Strasbourg, Strasbourg, France

Stefan Mihov Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Patrick Moylan The Abington College, Pennsylvania State University, Abington, PA, USA

Dimitar Nedanovski Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Christoph Neumann Goettingen University, Institute for Theoretical Physics, Goettingen, Germany

Nikolay Nikolov Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Emil Nissimov Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Petr Novotny Czech Technical University in Prague, FNSPE, Prague, Czech Republic

Svetlana Pacheva Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Tchavdar Palev Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Ivan Penkov Mathematics Department, Jacobs University Bremen, Bremen, Germany

Valentina Petkova Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Elena Poletaeva Department of Mathematics, University of Texas-Pan American, Edinburg, TX, USA

Todor Popov Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Severin Posta Czech Technical University in Prague, Prague, Czech Republic

Bela Gabor Pusztai Bolyai Institute, University of Szeged, Szeged, Hungary

Zoran Rakic Faculty of Mathematics, University of Belgrade, Beograd, Serbia

Karlhenning Rehren Institute for Theoretical Physics, Goettingen University, Goettingen, Germany

Adam Rej Imperial College London, London, UK

Maria Eugenia Rosado Universidad Politecnica de Madrid, ETSAM, Madrid, Spain

Igor Salom Institute of Physics, Pregrevica, Belgrade, Serbia

Michail Stoilov Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Stoimen Stoimenov Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Fumihiko Sugino Okayama Institute for Quantum Physics, Kita-ku, Okayama, Japan

Ivan Todorov Bulgarian Academy of Sciences, Institute for Nuclear Research and Nuclear Energy, BG Sofia, Bulgaria

Joris Van Der Jeugt Department of Applied Mathematics, Ghent University, Gent, Belgium

Olena Vaneeva Department of Applied Research, Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine

Joost Vercruysse Vrije Universiteit Brussel, Pleinlaan, Brussel, Belgium

Jirina Vodova Mathematical Institute, Silesian University in Opava, Na Rybnicku, Opava, Czech Republic

Petr Vojcak Mathematical Institute, Silesian University in Opava, Na Rybnicku, Opava, Czech Republic

Lena Wallenhorst Goettingen University, Institute for Theoretical Physics, Goettingen, Germany

Zhituo Wang Universite Paris XI, Laboratoire de Physique Theorique d‘Orsay, CNRS UMR, Orsay Cedex, France

Justin Wilson Department of Physics, University of Maryland at College Park, College Park, MD, USA

Asher Yahalom Ariel University Center of Samaria, Kiryat Hamada, Ariel, Israel

Milen Yakimov Department of Mathematics, Louisiana State University, Baton Rouge, LA, USA

Hiromichi Yamada Hitotsubashi University, Kunitachi, Tokyo, Japan

Part I
Plenary Talks

A Lump Solution in SFT

Loriano Bonora

Abstract A concrete example of lump solution in bosonic open string field theory is presented and discussed. It is shown that the solution satisfies the equation of motion and is not a pure gauge. The expression of its energy is written down explicitly. The value of the energy, calculated both numerically and analytically turns out to be in agreement with that of a D24 brane tension.

1 Introduction

The framework of this talk is tachyon condensation in bosonic open string field theory. Purely bosonic string theory is, of course, by itself theoretically incomplete, if anything because its spectrum does not contain fermions. However open string field theory is a simplified playground with respect to the corresponding superstring field theory versions. Exploiting the relative simplicity of the bosonic theory it has been possible in the last 10 years to make significant progress and, then, export some of the results to the superstring relatives. More precisely the framework of my talk is Witten's Open String Field Theory (OSFT) [16], and the guidelines for all its recent developments are represented by Sen's conjectures [12, 13].

The latter can be summarized as follows. Bosonic open string theory in D=26 dimensions is quantized on an unstable vacuum, an instability which manifests itself through the appearance of the open string tachyon. The effective tachyonic potential has, beside the local maximum where the theory is quantized, a local minimum. Sen's conjectures concern the nature of the theory around this local minimum. First of all, the energy density difference between the maximum and the minimum

L. Bonora (✉)
SISSA, International School for Advanced Studies, Via Bonomea 265, 34136 Trieste, Italy

INFN, Sezione di Trieste, Trieste, Italy
e-mail: bonora@sissa.it

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_1,

should exactly compensate for the D25-brane tension characterizing the unstable vacuum (first conjecture): this is a condition for the (relative) stability of the theory at the minimum. Therefore the theory around the minimum should not contain any quantum fluctuation pertaining to the original (unstable) open string theory (second conjecture). The minimum should therefore correspond to an entirely new theory, which can only be the bosonic closed string theory. If so, in the new theory one should be able to find in particular all the classical solutions characteristic of closed string theory, the D25-brane as well as all the solitonic solutions representing lower dimensional D-branes (third conjecture).

The evidence in favor of these conjectures has been accumulating over the years although not with a uniform degree of accuracy and reliability, until the first two conjectures were rigorously proved [6, 11]: an explicit analytic (non-perturbative) SFT solution was shown to exist, which links the initial vacuum to the final one and it was shown that this vacuum does not contain perturbative open string modes. As for the third conjecture a recent proposal has been put forward recently [1–3], see also [4, 7], as to how to construct analytic lump solutions. It is the purpose of this talk to illustrate this construction with an explicit example and compute (at least numerically) its energy.

Before plunging into the details of the construction, I would like to make a comment on the motivations for studying tachyon condensation is OSFT. The task of proving the three conjectures is not a merely academical (as ambitious as it may be) one, and the motivation is not exhausted once they are proved. In fact a very far-reaching consequence of Sen's conjectures has so far remained rather implicit in the literature: if the three conjectures are true and the new vacuum is the closed string vacuum, it is implicit that the closed string degrees of freedom can be represented (non-perturbatively) in terms of the open string ones [14, 15]. This is an exciting possibility which has not been methodically explored so far, and it is the real motivation behind this research.

2 The Analytic Tachyon Vacuum Solution

The open string field theory action proposed by Witten years ago [16], is

$$\mathscr{S}(\Psi) = -\frac{1}{g_o^2}\int\left(\frac{1}{2}\Psi * Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi\right). \tag{1}$$

with equation of motion

$$Q\Psi + \Psi * \Psi = 0 \tag{2}$$

The dimension of space-time is supposed to be the critical one, $D = 26$. The first analytic solution, representing the tachyon vacuum, was found by Martin Schnabl [11]. It allowed to prove first and second Sen's conjectures [6].

An interesting variant of the Schnabl solution has been proposed recently by Erler and Schnabl [8]. This solution lends itself to a generalization to lump solutions.

It has also the virtue of exploiting the simplification in the language of OSFT that has intervened in the last few years. To better describe it, it is in fact convenient to shift from the language of string fields and operators $K_L^1, B_L^1, c(z)$ used in the first Schnabl's solution [10,11] to an 'algebra with operator' language defined as follows. Let us first recall the definitions

$$\mathscr{L}_0 = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} L_{2k}, \qquad K_1 = L_1 + K_{-1}$$

$$\mathscr{B}_0 = b_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} b_{2k}, \qquad B_1 = b_1 + b_{-1}$$

and

$$B_1^L = \frac{1}{2}B_1 + \frac{1}{\pi}\left(\mathscr{B}_0 + \mathscr{B}_0^\dagger\right)$$

$$K_1^L = \frac{1}{2}K_1 + \frac{1}{\pi}\left(\mathscr{L}_0 + \mathscr{L}_0^\dagger\right)$$

where $c(z),\ b(z)$ are the ghost fields. We set

$$K = \frac{\pi}{2} K_1^L |I\rangle, \qquad B = \frac{\pi}{2} B_1^L |I\rangle, \qquad c = c\left(\frac{1}{2}\right)|I\rangle, \tag{3}$$

in the so-called sliver frame (obtained by mapping the UHP to an infinite cylinder C_2 of circumference 2, by the sliver map $f(z) = \frac{2}{\pi}\arctan z$). Then, with respect to the star product (which we will understand from now on), these states form the algebra

$$\{B,c\} = 1, \qquad KB = BK, \qquad [K,c] = \partial c, \qquad \{B, \partial c\} = 0, \tag{4}$$

where Q operates as follows

$$QB = K, \qquad Qc = c\partial c \tag{5}$$

In terms of this algebra with operator, the new solution [8], is given by

$$\psi_0 = \frac{1}{1+K} c(1+K)Bc = c - \frac{1}{1+K} Bc\partial c, \tag{6}$$

and can be formally obtained via a 'gauge transformation' of the perturbative vacuum

$$\psi_0 = U_0 Q U_0^{-1} \tag{7}$$

$$U_0 = 1 - \frac{1}{1+K} Bc \tag{8}$$

$$U_0^{-1} = 1 + \frac{1}{K} Bc. \tag{9}$$

This gauge transformation is in fact singular and this is the reason why the solution is nontrivial. The energy of this solution turns out to be the correct one (1st conjecture)

$$E = -\frac{S}{V} = \frac{1}{g_o^2 V}\left(\frac{1}{2}\langle \Psi, Q\Psi\rangle + \frac{1}{3}\langle \Psi, \Psi \star \Psi\rangle\right) = -\frac{1}{2\pi^2 g_0^2} \tag{10}$$

It is also possible to define a homotopy operator $A = \frac{B}{K+1}$, which satisfies the property $\mathscr{Q}A = 1$, where

$$\mathscr{Q}\psi \equiv Q\psi + \Psi_0 \star \psi - (-1)^{|\psi|}\psi \star \Psi_0 \tag{11}$$

is the BRST operator at the tachyon vacuum. This implies that the cohomology around the tachyon vacuum is trivial (2nd conjecture).

As we will see below, the Erler–Schnabl solution lends itself to a rather simple matter deformation, which turns out to be the searched for lump solution.

3 The Third Conjecture

The third conjecture predicts the existence of lower dimensional solitonic solutions or lumps, interpreted as Dp-branes, with $p < 25$. These solutions bring along the breaking of translational symmetry and background independence. The evidence for the existence of such solutions collected in the past years is overwhelming. It has been possible to find them with approximate methods or with exact methods in related theories. In the sequel I will present an explicit example of analytic lump solution in OSFT.

3.1 Analytic Lump Solutions

In a recent paper [1], a general method has been proposed to obtain new exact analytic solutions in open string field theory, and in particular solutions that describe inhomogeneous tachyon condensation. The method consists in translating an exact renormalization group (RG) flow generated in a two-dimensional world-sheet theory by a relevant operator, into the language of OSFT. The so-constructed solution is a deformation of the Erler–Schnabl solution described above. It has been shown in [1] that, if the operator has suitable properties, the solution will describe tachyon condensation in specific space directions, thus representing the condensation of a lower dimensional brane. In the following, after describing the general method, we will focus on a particular solution, generated by an exact RG flow analyzed first by Witten [17]. On the basis of the analysis carried out in the framework of 2D CFT in [9], we expect it to describe a D24 brane, with the correct ratio of tension with the starting D25 brane.

Let us see how to construct such kind of lump solutions. To start with we enlarge the K, B, c algebra by adding a (relevant) matter operator

$$\phi = \phi\left(\frac{1}{2}\right)|I\rangle. \tag{12}$$

with the properties

$$[c, \phi] = 0, \qquad [B, \phi] = 0, \qquad [K, \phi] = \partial\phi,$$

such that Q has the following action:

$$Q\phi = c\partial\phi + \partial c\delta\phi. \tag{13}$$

It can be easily proven that

$$\psi_\phi = c\phi - \frac{1}{K+\phi}(\phi - \delta\phi)Bc\partial c \tag{14}$$

does indeed satisfy (formally, see below) the OSFT equation of motion

$$Q\psi_\phi + \psi_\phi\psi_\phi = 0 \tag{15}$$

It is clear that (14) is a deformation of the Erler–Schnabl solution, which can be recovered for $\phi = 1$.

Much like in the Erler–Schnabl case, we can view this solution as a singular gauge transformation

$$\psi_\phi = U_\phi Q U_\phi^{-1} \tag{16}$$

where

$$U_\phi = 1 - \frac{1}{K+\phi}\phi Bc, \qquad U_\phi^{-1} = 1 + \frac{1}{K}\phi Bc, \tag{17}$$

In order to prove that (14) is a solution, one demands that $(c\phi)^2 = 0$, which requires the OPE of ϕ at nearby points to be not too singular.

It is instructive to write down the kinetic operator around this solution. With some manipulation, using the K, B, c, ϕ algebra it is possible to show that

$$\mathcal{Q}_{\psi_\phi}\frac{B}{K+\phi} = Q\frac{B}{K+\phi} + \left\{\psi_\phi, \frac{B}{K+\phi}\right\} = 1.$$

So, unless the string field $\frac{B}{K+\phi}$ is singular (as is the case for $\frac{B}{K}$ and the original Q), it defines a homotopy operator and the solution has trivial cohomology, which is the defining property of the tachyon vacuum [6]. On the other hand, in order for the solution to be well defined, the quantity $\frac{1}{K+\phi}(\phi - \delta\phi)$ should be well defined. Moreover, in order to be able to show that (14) satisfies the equation of motion, one needs $K + \phi$ to be invertible.

In full generality we thus have a new nontrivial solution if

1. $\frac{1}{K+\phi}$ is singular, but
2. $\frac{1}{K+\phi}(\phi - \delta\phi)$ is regular and
3. $\frac{1}{K+\phi}(K+\phi) = 1$

These conditions are in general impossible to satisfy: for instance, $K+\phi$ (like K) is not invertible. What is needed is a regularization. This problem was discussed in [2] (App.D), where an analytic regularization was used to satisfy the last condition above. In the following, as far as these conditions are concerned, we will limit ourselves to some heuristics. A detailed treatment can be found in the literature.

For concreteness we parametrize the worldsheet RG flow, referred to above, by a parameter u, where $u = 0$ represents the UV and $u = \infty$ the IR, and label ϕ by ϕ_u, with $\phi_{u=0} = 0$. Then we require for ϕ_u the following properties under the coordinate rescaling $f_t(z) = \frac{z}{t}$

$$f_t \circ \phi_u(z) = \frac{1}{t}\, \phi_{tu}\left(\frac{z}{t}\right). \tag{18}$$

and, most important, that the partition function

$$g(u) \equiv Tr[e^{-(K+\phi_u)}] = \left\langle e^{-\int_0^1 ds\, \phi_u(s)} \right\rangle_{C_1}, \tag{19}$$

satisfies the asymptotic finiteness condition

$$\lim_{u\to\infty} \left\langle e^{-\int_0^1 ds\, \phi_u(s)} \right\rangle_{C_1} = \texttt{finite}. \tag{20}$$

Barring subtleties, this should satisfy the first two conditions above (the third can be satisfied only with a regulator, as pointed out above), i.e. guarantee not only the regularity of the solution but also its 'non-triviality', in the sense that if this condition is satisfied, it cannot fall in the same class as the ES tachyon vacuum solution.

We will consider in the sequel a specific relevant operator ϕ_u and the corresponding SFT solution. This operator generates an exact RG flow studied by Witten in [17], see also [9], and is based on the operator (defined in the cylinder C_T of width T in the arctan frame)

$$\phi_u(s) = u(X^2(s) + 2\ln u + 2A) \tag{21}$$

where A is a constant first introduced in [5]. In C_1 we have

$$\phi_u(s) = u(X^2(s) + 2\ln Tu + 2A) \tag{22}$$

and on the unit disk D,

$$\phi_u(\theta) = u(X^2(\theta) + 2\ln\frac{Tu}{2\pi} + 2A) \tag{23}$$

If we set

$$g_A(u) = \langle e^{-\int_0^1 ds\, \phi_u(s)} \rangle_{C_1} \tag{24}$$

we have

$$g_A(u) = \langle e^{-\frac{1}{2\pi}\int_0^{4\pi} d\theta\, u\left(X^2(\theta)+2\ln\frac{u}{2\pi}+2A\right)} \rangle_D$$

We have

$$g_A(u) = Z(2u)e^{-2u(\ln\frac{u}{2\pi}+A)} \tag{25}$$

where $Z(u)$ is the partition function of the system on the unit disk computed by [17]. Requiring finiteness for $u \to \infty$ we get $A = \gamma - 1 + \ln 4\pi$, which implies

$$g_A(u) \equiv g(u) = \frac{1}{2\sqrt{\pi}}\sqrt{2u}\Gamma(2u)e^{2u(1-\ln(2u))} \tag{26}$$

and

$$\lim_{u\to\infty} g(u) = 1 \tag{27}$$

Moreover, as it turns out, $\delta\phi_u = -2u$, and so:

$$\phi_u - \delta\phi_u = u\partial_u\phi_u(s) \tag{28}$$

Therefore the ϕ_u just introduced satisfies all the requested properties. According to [9], the corresponding RG flow in BCFT reproduces the correct ratio of tension between D25 and D24 branes. Consequently $\psi_u \equiv \psi_{\phi_u}$ is expected to represent a D24 brane solution.

In SFT the most important gauge invariant quantity is of course the energy. Therefore in order to make sure that $\psi_u \equiv \psi_{\phi_u}$ is the expected solution we must prove that its energy equals a D24 brane energy.

The energy expression for the lump solution was determined in [1] by evaluating a three-point function on the cylinder C_T of circumference T in the arctan frame. It equals $-\frac{1}{6}$ times the following expression

$$\begin{aligned}\langle\psi_u\psi_u\psi_u\rangle = -\int_0^\infty dt_1dt_2dt_3\mathscr{E}_0(t_1,t_2,t_3)u^3g(uT)\Bigg\{&\left(-\frac{\partial_{2uT}g(uT)}{g(uT)}\right)^3\\ &+\frac{1}{2}\left(-\frac{\partial_{2uT}g(uT)}{g(uT)}\right)\left(G^2_{2uT}\left(\frac{2\pi t_1}{T}\right)+G^2_{2uT}\left(\frac{2\pi(t_1+t_2)}{T}\right)+G^2_{2uT}\left(\frac{2\pi t_2}{T}\right)\right)\\ &+G_{2uT}\left(\frac{2\pi t_1}{T}\right)G_{2uT}\left(\frac{2\pi(t_1+t_2)}{T}\right)G_{2uT}\left(\frac{2\pi t_2}{T}\right)\Bigg\}\end{aligned} \tag{29}$$

Here $T = t_1 + t_2 + t_3$ and $g(u)$ is as above, and represents the partition function of the underlying matter CFT t the boundary of the unit disk with definite boundary conditions at infinity. $G_u(\theta)$ represents the correlator on the boundary, first determined by Witten [17]:

$$G_u(\theta) = \frac{1}{u} + 2\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k+u} \tag{30}$$

Finally $\mathscr{E}_0(t_1,t_2,t_3)$ represents the ghost three-point function in C_T.

$$\begin{aligned} \mathrm{e}_0(t_1,t_2,t_3) &= \langle Bc\partial c(t_1+t_2)\partial c(t_1)\partial c(0)\rangle_{C_T} \\ &= -\frac{4}{\pi}\sin\frac{\pi t_1}{T}\sin\frac{\pi(t_1+t_2)}{T}\sin\frac{\pi t_2}{T} \end{aligned} \tag{31}$$

An absolutely remarkable property of (29) is that it does not depend on u. In fact u can be absorbed in a redefinition of variables $t_i \to ut_i$, $i = 1,2,3$, and disappears from the expression.

Taking into account that (29) contains up to three power of G_{2uT}, in order to evaluate it one must compute three infinite summations and three integrals. One can easily integrate over the *angular* variables $x = t_1/T, y = t_2/T$, after which the discrete summations are reduced to two. One can also carry out analytically one of the remaining summations. But this seems to be as far as one can get analytically. The rest of the computation has to rely on a numerical approximation. This has been done in [2]. An important remark is that this expression has an UV ($s \approx 0$, setting $s = 2uT$) singularity, which must be subtracted away. Once this done, the expression (29) has been analytically computed up to the point permitted by our present mathematical tools and continued with numerical means. The final result one gets is ≈ 0.0693926. This is 30% off the expected result (see below, (36)). However this failure is no surprise, for the result depends on the UV subtraction we have made. Therefore we cannot assign to it any physical significance.

At this point it seems that our calculation has been carried out for nothing, but this is not the case. In order to understand where we are we must return to the very meaning of third Sen's conjecture, which says that *the lump solution is a solution of the theory on the tachyon condensation vacuum*. Therefore we must measure the energy of our solution with respect to the tachyon condensation vacuum. Simultaneously the resulting energy must be a subtraction-independent quantity because only to a subtraction-independent quantity can a physical meaning be assigned. Both requirements have been satisfied in [2] in the following way.

First a new solution to the EOM, depending on a parameter ε, has been introduced

$$\psi_u^\varepsilon = c(\phi_u + \varepsilon) - \frac{1}{K + \phi_u + \varepsilon}(\phi_u + \varepsilon - \delta\phi_u)Bc\partial c. \tag{32}$$

and it has been shown that it is gauge equivalent to the tachyon vacuum solution, its energy (after the same UV subtraction as in the previous case) being (numerically) 0.

Then, using it, a solution to the EOM at the tachyon condensation vacuum has been obtained. The equation of motion at the tachyon vacuum is

$$\mathscr{Q}\Phi + \Phi\Phi = 0, \quad \text{where } \mathscr{Q}\Phi = Q\Phi + \psi_u^\varepsilon \Phi + \Phi \psi_u^\varepsilon. \tag{33}$$

One can easily show that

$$\Phi_0 = \psi_u - \psi_u^\varepsilon \tag{34}$$

is a solution to (33). The action at the tachyon vacuum is $-\frac{1}{2}\langle \mathscr{Q}\Phi, \Phi\rangle - \frac{1}{3}\langle \Phi, \Phi\Phi\rangle$. Thus the energy of Φ_0 is

$$\begin{aligned} E[\Phi_0] &= -\frac{1}{6}\langle \Phi_0, \Phi_0\Phi_0\rangle \\ &= -\frac{1}{6}\big[\langle \psi_u, \psi_u\psi_u\rangle - \langle \psi_u^\varepsilon, \psi_u^\varepsilon\psi_u^\varepsilon\rangle - 3\langle \psi_u^\varepsilon, \psi_u\psi_u\rangle + 3\langle \psi_u, \psi_u^\varepsilon\psi_u^\varepsilon\rangle\big]. \end{aligned} \tag{35}$$

The UV subtractions necessary for each correlator at the RHS of this equation are always the same, therefore they cancel out and the final result is subtraction-independent. A final bonus of this procedure is that the final result can be derived purely analytically and $E[\Phi_0]$ turns out to be precisely the D24-brane energy. With the conventions of [2], this is

$$T_{D24} = \frac{1}{2\pi^2} \tag{36}$$

In [3] the same result was extended to Dp-brane lump solutions for any p.

References

1. Bonora, L., Maccaferri, C., Tolla, D.D.: Relevant deformations in open string field theory: A simple solution for lumps. JHEP **1111**, 107 (2011)
2. Bonora, L., Giaccari, S., Tolla, D.D.: The energy of the analytic lump solution in SFT. JHEP **1108**, 158 (2011)
3. Bonora, L., Giaccari, S., Tolla, D.D.: Analytic solutions for Dp branes in SFT. [JHEP **12**, 033 (2011) arXiv:1105.5926 [hep-th]]
4. Bonora, L., Giaccari, S., Tolla, D.D.: Lump solutions in SFT. Complements. [POS, 044 (ICMP 2012) arXiv:1109.4336 [hep-th]]
5. Ellwood, I.: Singular gauge transformations in string field theory. JHEP **0905**, 037 (2009)
6. Ellwood, I., Schnabl, M.: Proof of vanishing cohomology at the tachyon vacuum. JHEP **0702**, 096 (2007)
7. Erler, T., Maccaferri, C.: Comments on lumps from RG flows. JHEP **1111**, 092 (2011) [arXiv:1105.6057 [hep-th]]
8. Erler, T., Schnabl, M.: A simple analytic solution for tachyon condensation. JHEP **0910**, 066 (2009)
9. Kutasov, D., Marino, M., Moore, G.W.: Some exact results on tachyon condensation in string field theory. JHEP **0010**, 045 (2000)
10. Okawa, Y.: Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory. JHEP **0604**, 055 (2006)

11. Schnabl, M.: Analytic solution for tachyon condensation in open string field theory. Adv. Theor. Math. Phys. **10**, 433 (2006)
12. Sen, A.: Tachyon condensation on the brane anti-brane system. JHEP **9808**, 012 (1998); BPS D-branes on nonsupersymmetric cycles. JHEP **9812**, 021 (1998)
13. Sen, A.: Descent relations among bosonic D-branes. Int. J. Mod. Phys. A **14**, 4061 (1999)
14. Sen, A.: Open closed duality: Lessons from matrix model. Mod. Phys. Lett. A **19**, 841 (2004)
15. Sen, A.: Open and closed strings from unstable D-branes. Phys. Rev. D **68**, 106003 (2003)
16. Witten, E.: Noncommutative geometry and string field theory. Nucl. Phys. B **268**, 253 (1986)
17. Witten, E.: Some computations in background independent off-shell string theory. Phys. Rev. D **47**, 3405 (1993)

Towards p-Adic Matter in the Universe

Branko Dragovich

Abstract Starting from p-adic string theory with tachyons, we introduce a new kind of non-tachyonic matter which may play an important role in evolution of the Universe. This matter retains nonlocal and nonlinear p-adic string dynamics, but does not suffer of negative square mass. In space-time dimensions $D = 2 + 4k$, what includes $D = 6, 10, \ldots, 26$, the kinetic energy term also maintains correct sign. In these spaces this p-adic matter provides negative cosmological constant and time-dependent scalar field solution with negative potential. Their possible cosmological role is discussed. We have also connected non-locality with string world-sheet in effective Lagrangian for p-adic string.

1 Introduction

Observational modern cosmology has achieved significant progress by high precision experiments in the last decades. One of the greatest cosmological achievements was discovery of accelerated expansion of the Universe in 1998. If General Relativity is theory of gravity for the Universe as a whole then about 96% of its energy content is of unknown nature. This dark side of the Universe consists of about 23% of *dark matter* and 73% of *dark energy*. Dark matter is supposed to be responsible for anomaly large rotational velocities in the spiral galaxies. Dark energy has negative pressure and should govern the Universe accelerated expansion (as a recent review see [1]). Thus, according to this point of view, there is now only about 4% of visible matter which is described by the Standard Model of particle physics. Although dark matter and dark energy are well adopted among majority of scientists, they are not directly verified in the laboratory and still remain hypothetical forms of matter. Also General Relativity has not been so far

B. Dragovich (✉)
Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Zemun, Belgrade, Serbia
e-mail: dragovich@ipb.ac.rs

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_2,

confirmed at the cosmic scale. For these reasons, there is not yet commonly accepted theoretical explanation of the Universe acceleration. This situation has influenced also alternative approaches, mostly related to modification of General Relativity (a recent review in [2]).

While observational cosmology is in an arising state, theoretical cosmology is facing a big challenge. An exotic matter and modification of gravity are two alternative theoretical approaches. Since string theory is the best candidate for unification of matter elementary constituents and fundamental interactions, some theoretical ideas come from string theory and we consider the following one. According to the adelic product formula for scalar string amplitudes it follows that p-adic strings are at equal footing with ordinary strings (reviews on p-adic strings and adelic product formula can be found in [3, 4]). Hence, if visible matter is composed of ordinary strings then there should be some matter made of p-adic strings. It is natural to assume that dark side of the Universe contains some kinds of p-adic matter. In p-adic string theory world-sheet has non-Archimedean (ultrametric) geometry and it should also somehow modify gravity. It is feasible that future theoretical cosmology will be a result of both modification of General Relativity and inclusion of new kinds of matter.

Inspired by p-adic string theory it has been already investigated some nonlocal modifications of General Relativity (see, e.g. [5] and references therein) and cosmology, see, e.g. [6, 7] and references therein. In this article we consider some modification of open scalar p-adic strings as candidates for a new kind of matter in the Universe. In Sect. 2 we present various aspects of p-adic string theory necessary for comprehensive exposition. It also contains introduction of non-tachyonic p-adic matter. Section 3 is related to some adelic approaches to cosmology.

2 *p*-Adic Strings

p-Adic strings are introduced in 1987 by Volovich in his paper [8]. p-Adic string theory is mainly related to strings which have only world-sheet p-adic and all other their properties are the same with theory of ordinary strings [9]. Having exact Lagrangian at the tree level, p-adic scalar strings have attracted significant attention in string theory and nonlocal cosmology. To be more comprehensive and self consistent we shall first give a brief review of p-adic numbers and their applicability in modern mathematical physics.

2.1 p-Adic Numbers and Their Applicability

p-Adic numbers are discovered by Kurt Hansel in 1897 as a new tool in number theory. In modern approach to introduce p-adic numbers one usually starts with the field $\mathbb{Q}$ of rational numbers. Recall that according to the Ostrowski theorem any non-trivial norm on $\mathbb{Q}$ is equivalent either to the usual absolute value $|\cdot|_\infty$ or to a p-adic norm (p-adic absolute value) $|\cdot|_p$. A rational number $x = p^\nu \frac{a}{b}$, where integers a and $b \neq 0$ are not divisible by prime number p, by definition has p-adic norm $|x|_p = p^{-\nu}$ and $|0|_p = 0$. Since $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, p-adic norm is a non-Archimedean (ultrametric) one. As completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_\infty$ gives the field $\mathbb{Q}_\infty \equiv \mathbb{R}$ of real numbers, by the same procedure using p-adic norm $|\cdot|_p$ one gets the field $\mathbb{Q}_p$ of p-adic numbers (for any prime number $p = 2, 3, 5 \cdots$). Any number $0 \neq x \in \mathbb{Q}_p$ has its unique canonical representation

$$x = p^\nu \sum_{n=0}^{+\infty} x_n p^n, \quad \nu \in \mathbb{Z}, \quad x_n \in \{0, 1, \cdots, p-1\}, \quad x_0 \neq 0. \tag{1}$$

$\mathbb{Q}_p$ is locally compact, complete and totally disconnected topological space. There is a rich structure of algebraic extensions of $\mathbb{Q}_p$.

There are many possibilities for mappings with $\mathbb{Q}_p$. The most elaborated is analysis related to mappings $\mathbb{Q}_p \to \mathbb{Q}_p$ and $\mathbb{Q}_p \to \mathbb{C}$. Usual complex valued functions of p-adic argument are additive $\chi_p(x) = e^{2\pi i \{x\}_p}$ and multiplicative $|x|^s$ characters, where $\{x\}_p$ is fractional part of x and $s \in \mathbb{C}$ (for many aspects of p-adic numbers and their analysis, we refer to [3, 4, 10, 11]).

The field $\mathbb{Q}$ of rational numbers, which is dense in $\mathbb{Q}_p$ and $\mathbb{R}$, is also important in physics. All values of measurements are rational numbers. Any measurement is comparison of two quantities of the same kind and it is in close connection with the Archimedean axiom. Set of rational numbers obtained in the process of repetition of measurement of the same quantity is naturally provided by usual absolute value. Hence, results of measurements are not rational numbers with p-adic norm but with real norm. It means that measurements give us real and not p-adic rational numbers. Then the following question arises: Being not results of measurements, what role p-adic numbers can play in description of something related to physical reality? Recall that we already have similar situation with complex numbers, which are not result of direct measurements but they are very useful. For example, in quantum mechanics wave function is basic theoretical tool which contains all information about quantum system but cannot be directly measured in experiments. p-Adic numbers should play unavoidable role where application of real numbers is inadequate. In physical systems such situation is at the Planck scale, because it is not possible to measure distances smaller than the Planck length. It should be also the case with very complex phenomena of living and cognitive systems. Thus, we expect inevitability of p-adic numbers at some deeper level in understanding of the content, structure and evolution of the Universe in its parts as well as a whole. The first steps towards probe of p-adic levels of knowledge is invention of relevant mathematical methods and construction of the corresponding physical models. A brief overview of p-adic mathematical physics is presented in [12]. It includes both p-adic valued and real (complex) valued functions of p-adic argument. In the sequel we shall consider p-adic strings, non-tachyonic p-adic matter and its some possible role in modern cosmology.

2.2 Scattering Amplitudes and Lagrangian for Open Scalar p-Adic Strings

Like ordinary strings, p-adic strings are introduced by construction of their scattering amplitudes. The simplest amplitude is for scattering of two open scalar strings. Recall the crossing symmetric Veneziano amplitude for ordinary strings

$$A_\infty(a,b) = g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1}\,|1-x|_\infty^{b-1}\,d_\infty x = g_\infty^2\,\frac{\zeta(1-a)}{\zeta(a)}\,\frac{\zeta(1-b)}{\zeta(b)}\,\frac{\zeta(1-c)}{\zeta(c)}, \qquad (2)$$

where $a+b+c=1$. The crossing symmetric Veneziano amplitude for scattering of two open scalar p-adic strings is direct analog of (2) [9], i.e.

$$A_p(a,b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1}\,|1-x|_p^{b-1}\,d_p x = g_p^2\,\frac{1-p^{a-1}}{1-p^{-a}}\,\frac{1-p^{b-1}}{1-p^{-b}}\,\frac{1-p^{c-1}}{1-p^{-c}}. \qquad (3)$$

Integral expressions in (2) and (3) are the Gel'fand-Graev-Tate beta functions on $\mathbb{R}$ and $\mathbb{Q}_p$, respectively [10]. Note that here, by definition, ordinary and p-adic strings differ only in description of their world-sheets—world-sheet of p-adic strings is presented by p-adic numbers. Kinematical variables contained in a,b,c are the same real numbers in both cases. It is worth noting that the final form of Veneziano amplitude for p-adic strings (3) is rather simple and presented by an elementary function.

It is remarkable that there is an effective field description of the above open p-adic strings. The corresponding Lagrangian is very simple and at the tree level describes not only four-point scattering amplitude but also all higher (Koba-Nielsen) ones. The exact form of this Lagrangian for effective scalar field φ, which describes open p-adic string tachyon, is

$$\mathcal{L}_p = \frac{m^D}{g^2}\,\frac{p^2}{p-1}\Big[-\frac{1}{2}\,\varphi\, p^{-\frac{\Box}{2m^2}}\,\varphi + \frac{1}{p+1}\,\varphi^{p+1}\Big], \qquad (4)$$

where p is any prime number, D—spacetime dimensionality, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and metric has signature $(-+\ldots+)$ [13, 14]. This is nonlocal and nonlinear Lagrangian. Nonlocality is in the form of infinite number of spacetime derivatives

$$p^{-\frac{\Box}{2m^2}} = \exp\Big(-\frac{\ln p}{2m^2}\,\Box\Big) = \sum_{k\ge 0}\Big(-\frac{\ln p}{2m^2}\Big)^k\,\frac{1}{k!}\,\Box^k \qquad (5)$$

and it is a consequence of the fact that strings are extended objects.

The corresponding potential $\mathcal{V}(\varphi)$ for Lagrangian (4) is $\mathcal{V}_p(\varphi) = -\mathcal{L}_p(\Box = 0)$, which the explicit form is

$$\mathcal{V}_p(\varphi) = \frac{m^D}{g^2}\Big[\frac{1}{2}\,\frac{p^2}{p-1}\,\varphi^2 - \frac{p^2}{p^2-1}\,\varphi^{p+1}\Big]. \qquad (6)$$

It has local minimum $\mathscr{V}_p(0) = 0$. If $p \neq 2$ there are two local maxima at $\varphi = \pm 1$ and there is one local maximum $\varphi = +1$ when $p = 2$.

The equation of motion for the field φ is

$$p^{-\frac{\Box}{2m^2}}\, \varphi = \varphi^p \tag{7}$$

and it has trivial solutions $\varphi = 0$ and $\varphi = 1$, and another $\varphi = -1$ for $p \neq 2$. There are also inhomogeneous solutions in any direction x^i resembling solitons

$$\varphi(x^i) = p^{\frac{1}{2(p-1)}} \exp\Big(- \frac{p-1}{2\,p \ln p}\, m^2\, (x^i)^2 \Big). \tag{8}$$

There is also homogeneous and isotropic time-dependent solution

$$\varphi(t) = p^{\frac{1}{2(p-1)}} \exp\Big(\frac{p-1}{2\,p \ln p}\, m^2\, t^2 \Big). \tag{9}$$

Solution (9) (and analogously (8)) can be obtained using identity

$$e^{A\, \partial_t^2}\, e^{B t^2} = \frac{1}{\sqrt{1-4AB}}\, e^{\frac{Bt^2}{1-4AB}}, \quad 1 - 4AB > 0. \tag{10}$$

Note that the sign of the above field solutions $\varphi(x^i)$ and $\varphi(t)$ can be also minus $(-)$ when $p \neq 2$. Various aspects of p-adic string theory with the above effective field have been pushed forward by papers [15, 16].

It is worth noting that Lagrangian (4) is written completely in terms of real numbers and there is no explicit dependence on the world sheet. However, (9) can be rewritten in the following form:

$$\begin{aligned} \mathscr{L}_p = \frac{m^D}{g^2}\, \frac{p^2}{p-1} \Big[\frac{1}{2}\, \varphi \int_{\mathbb{R}} \Big(\int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \chi_p(u) |u|_p^{\frac{k^2}{2m^2}}\, du \Big)\, \tilde{\varphi}(k)\, \chi_\infty(kx)\, d^4 k \\ + \frac{1}{p+1}\, \varphi^{p+1} \Big], \end{aligned} \tag{11}$$

where $\chi_\infty(kx) = e^{-2\pi i k x}$ is the real additive character. Since $\int_{\mathbb{Q}_p} \chi_p(u) |u|^{s-1} du = \frac{1-p^{s-1}}{1-p^{-s}} = \Gamma_p(s)$ and it is present in the scattering amplitude (3), one can say that expression $\int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \chi_p(u) |u|_p^{\frac{k^2}{2m^2}} du$ in (11) is related to the p-adic string world-sheet.

2.3 New Kind of Matter, Which Has Origin in p-Adic Strings

Since there are infinitely many primes p, in principle it can be infinitely many kinds of p-adic strings. We suppose that for all but a finite set $\mathscr{P}$ of primes p these p-adic strings are in their local potential minimum, i.e. $\varphi_p = 0$, and consequently $\mathscr{L}_p = 0$, for all $p \notin \mathscr{P}$. This can be a result of tachyon condensation. Further we suppose that in the remaining finite set of strings there was a transition $m^2 \to -m^2$ (transition from tachyon to no-tachyon state), what could be a result of some quantum effect which was happened before process of tachyon condensation was finished. For simplicity, we shall assume two kinds of such strings and denote their set by $\mathscr{P} = \{q, \ell\}$. In the sequel for these strings in the above expressions (4)–(11) one has to replace m^2 by $-m^2$ and $m^D \to (-1)^{\frac{D}{2}} m^D$, where spacetime dimensionality D is even. Note that there is transition $m^D \to -m^D$ for critical dimensions $D = 26$ and $D = 10$, but for $D = 4$ there is no change of sign. The corresponding potentials for $p = 2$ and $p = 3$, and $D = 2 + 4k$, are presented at Fig. 2. To make distinction with tachyons we denote these new scalar strings by ϕ_p, $p \in \mathscr{P}$.

The related new Lagrangian is

$$L_p = (-1)^{\frac{D}{2}} \frac{m^D}{g^2} \frac{p^2}{p-1} \Big[-\frac{1}{2} \phi \, p^{\frac{\Box}{2m^2}} \phi + \frac{1}{p+1} \phi^{p+1} \Big] \tag{12}$$

with the corresponding potential

$$V_p(\phi) = (-1)^{\frac{D}{2}} \mathscr{V}_p(\phi) = (-1)^{\frac{D}{2}} \frac{m^D}{g^2} \Big[\frac{1}{2} \frac{p^2}{p-1} \phi^2 - \frac{p^2}{p^2-1} \phi^{p+1} \Big]. \tag{13}$$

The equation of motion for scalar strings ϕ_p is

$$p^{\frac{\Box}{2m^2}} \phi_p = \phi_p^p \tag{14}$$

and it has trivial solutions $\phi_p = 0$ and $\phi_p = 1$, and another $\phi_p = -1$ for $p \neq 2$. What is local maximum and minimum depends on dimensionality D. When $D = 2 + 4k$, solution $\phi_p = 0$ is a local maximum, and $\phi_2 = +1$ and $\phi_p = \pm 1$, $p \neq 2$ are local minima. For any dimensionality, nontrivial solutions become now

$$\phi_p(x^i) = p^{\frac{1}{2(p-1)}} \exp\Big(\frac{p-1}{2\, p \ln p} m^2 (x^i)^2 \Big), \tag{15}$$

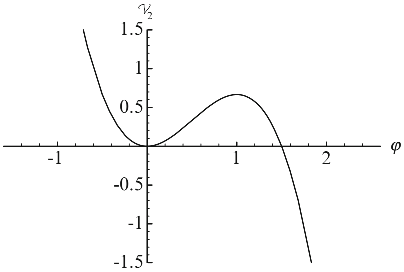
$$\phi_p(t) = p^{\frac{1}{2(p-1)}} \exp\Big(-\frac{p-1}{2\, p \ln p} m^2 t^2 \Big). \tag{16}$$

D-dimensional solution of (14) is product of solutions (15) and (16) (see also [17]), i.e.

$$\phi_p(x) = p^{\frac{D}{2(p-1)}} \exp\Big(\frac{p-1}{2\, p \ln p} m^2 x^2 \Big), \quad x^2 = -t^2 + \sum_{i=1}^{D-1} x_i^2. \tag{17}$$

Lagrangian for this collection of two distinct strings $\{\phi_q, \phi_\ell\}$ is of the form

$$L_{\mathscr{P}} = \sum_{p \in \mathscr{P}} L_p = \sum_{p \in \mathscr{P}} (-1)^{\frac{D}{2}} \frac{m^D}{g^2} \frac{p^2}{p-1} \Big[-\frac{1}{2} \phi_p \, p^{\frac{\Box}{2m^2}} \phi_p + \frac{1}{p+1} \phi_p^{p+1} \Big]. \tag{18}$$



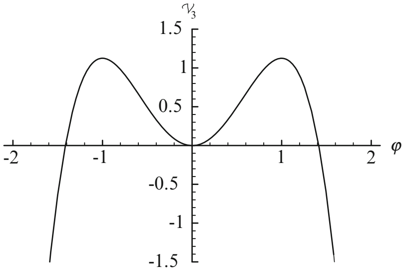


Fig. 1 The 2-adic string potential $\mathcal{V}_2(\varphi)$ (on the *left*) and 3-adic potential $\mathcal{V}_3(\varphi)$ (on the *right*) of standard Lagrangian (4), where potential is presented by expression (6) with $\frac{m^D}{g^2} = 1$

String field ϕ_ℓ in (18) we shall consider in its vacuum state $\phi_\ell = +1$ or -1 with Lagrangian

$$L_\ell = -V_\ell(\pm 1) = (-1)^{\frac{D}{2}+1} \frac{m^D}{g^2} \frac{\ell^2}{2(\ell+1)} \sim -\Lambda \tag{19}$$

related to the cosmological constant Λ (prime index ℓ may remind Λ). Note that vacuum state $\phi_\ell = \pm 1$ is stable only in spaces with dimension $D = 2 + 4k$.

Field ϕ_q corresponds to time-dependent solution (16) in dimensions which satisfy $(-1)^{\frac{D}{2}} = -1$, and the form of the corresponding potential is presented at Fig. 2. As a simple example, one can take $D = 6$ as respective solution of condition $(-1)^{\frac{D}{2}} = -1$. For the case $D = 6$, or any other $D = 2 + 4k$, we have (q may remind quintessence)

$$\phi_q(t) = q^{\frac{1}{2(q-1)}} \exp\Big(-\frac{q-1}{2\,q \ln q}\, m^2 t^2\Big) \tag{20}$$

which corresponds to potential of the form at Fig. 2. At the moment $t = 0$ (the big bang) the field ϕ_q has its maximum which is $\phi_q(0) = q^{\frac{1}{2(q-1)}}$ and it is a bit larger than 1. Then by increasing of time $\phi_q(t)$ is decreasing and $\phi_q(t) \to 0$ as $t \to +\infty$. The situation is symmetric with respect to transformation $t \to -t$.

If we consider $\phi_q(t)$ in spaces of dimension $D = 4k$, and in particular $D = 4$, then we face by two problems. First, the kinetic energy term has not correct sign. Second, the position of field at moment $t = 0$ is $\phi_q(0) = q^{\frac{1}{2(q-1)}} > 1$ and it should have rolling to $-\infty$, instead of to 0, what contradicts to the time dependence (20) of the field (see also Fig. 1).

3 Adelic Cosmological Modelling

In the preceding section we have seen that the fields of p-adic and real numbers can be obtained by completion of the field of rational numbers, and that $\mathbb{Q}$ is dense in $\mathbb{Q}_p$ as well as in $\mathbb{R}$. This gives rise to think that it should exist some way for unification

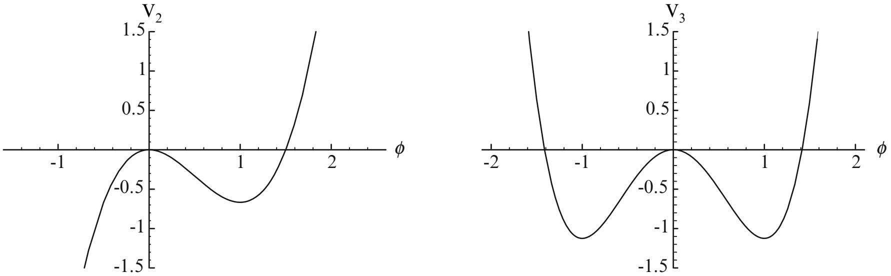


Fig. 2 New potentials $V_2(\phi)$ and $V_3(\phi)$, which are related to Lagrangian (12) with dimensionality D satisfying condition $(-1)^{\frac{D}{2}} = -1$, i.e. $D = 2+4k$, and $\frac{m^D}{g^2} = 1$

of p-adic and real numbers. A unified and simultaneous treatment of p-adic and real numbers is through concept of adeles. Adelic formalism is a mathematical method how to connect p-adic with ordinary real models.

3.1 Adeles and Their Applicability

An adele α is an infinite sequence made od real and p-adic numbers in the form

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \cdots, \alpha_p, \cdots), \quad \alpha_\infty \in \mathbb{R},\ \alpha_p \in \mathbb{Q}_p, \tag{21}$$

where for all but a finite set $\mathscr{P}$ of primes p it has to be $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. $\mathbb{Z}_p$ is ring of p-adic integers and they have $\nu \geq 0$ in (1). The set $\mathbb{A}_\mathbb{Q}$ of all completions of $\mathbb{Q}$ in the form of the above adeles can be presented as

$$\mathbb{A}_\mathbb{Q} = \bigcup_{\mathscr{P}} A(\mathscr{P}), \quad A(\mathscr{P}) = \mathbb{R} \times \prod_{p\in\mathscr{P}} \mathbb{Q}_p \times \prod_{p\notin\mathscr{P}} \mathbb{Z}_p\,. \tag{22}$$

Elements of $\mathbb{A}_\mathbb{Q}$ satisfy componentwise addition and multiplication and form the adele ring.

The multiplicative group of ideles $\mathbb{A}_\mathbb{Q}^\times$ is a subset of $\mathbb{A}_\mathbb{Q}$ with elements $\eta = (\eta_\infty, \eta_2, \eta_3, \cdots, \eta_p, \cdots)$, where $\eta_\infty \in \mathbb{R}^\times = \mathbb{R}\setminus\{0\}$ and $\eta_p \in \mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$ with the restriction that for all but a finite set $\mathscr{P}$ one has that $\eta_p \in \mathbb{U}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$, i.e. $\mathbb{U}_p$ is multiplicative group of p-adic units. The entire set of ideles, related to $\mathbb{Q}^\times = \mathbb{Q}\setminus\{0\}$, is

$$\mathbb{A}_\mathbb{Q}^\times = \bigcup_{\mathscr{P}} A^\times(\mathscr{P}), \quad A^\times(\mathscr{P}) = \mathbb{R}^\times \times \prod_{p\in\mathscr{P}} \mathbb{Q}_p^\times \times \prod_{p\notin\mathscr{P}} \mathbb{U}_p\,. \tag{23}$$

A principal adele (idele) is a sequence $(x, x, \cdots, x, \cdots) \in \mathbb{A}_\mathbb{Q}$, where $x \in \mathbb{Q}$ $\quad (x \in \mathbb{Q}^\times)$. $\mathbb{Q}$ and $\mathbb{Q}^\times$ are naturally embedded in $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}^\times$, respectively. This concept of principal adeles gives way to present rational numbers together with their nontrivial norms. Adeles are a generalization of principal adeles so that it takes into account all completions of $\mathbb{Q}$ and has well-defined mathematical structure.

Space of adeles (ideles) has its adelic (idelic) topology. With respect to their topology $\mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_{\mathbb{Q}}^{\times}$ are locally compact topological spaces. There are adelic-valued and complex-valued functions of adelic arguments. For various mathematical aspects of adeles and their functions we refer to books [10, 18] and for their applications in mathematical physics to [3, 4, 19].

Ideles and adeles are introduced in the 1930s by Claude Chevalley and André Weil, respectively. p-Adic numbers and adeles have many applications in mathematics. Since 1987, they have employed in *p-adic mathematical physics*.

Adelic connection of p-adic and real properties of the same rational quantity can be well illustrated by the following two simple examples:

$$|x|_\infty \times \prod_{p\in\mathbb{P}} |x|_p = 1\,, \text{ if } x \in \mathbb{Q}^{\times} \tag{24}$$

$$\chi_\infty(x) \times \prod_{p\in\mathbb{P}} \chi_p(x) = 1\,, \text{ if } x \in \mathbb{Q}, \tag{25}$$

where $\mathbb{P}$ is set of all primes and

$$\chi_\infty(x) = \exp(-2\pi i x), \quad \chi_p(x) = \exp(2\pi i \{x\}_p) \tag{26}$$

with $\{x\}_p$ as fractional part of x in expansion with respect to base p.

More complex connection, but also very significant, is the Freund-Witten product formula for string amplitudes [20]:

$$A(a,b) = A_\infty(a,b)\prod_p A_p(a,b) = g_\infty^2 \prod_p g_p^2 = const. \tag{27}$$

which connects p-adic Veneziano amplitudes (3) with their real analog (2). Formula (27) follows as a consequence of the Euler product formula for the Riemann zeta function applied to p-adic string amplitudes (3). Main significance of (27) is in the fact that scattering amplitude for real string $A_\infty(a,b)$, which is a special function, can be presented as product of inverse p-adic amplitudes, which are elementary functions. Also, this product formula treats p-adic and ordinary strings at the equal footing. It gives rise to suppose that if there exists an ordinary scalar string then it should exist also its p-adic analog. Moreover, p-adic strings seem to be simpler for theoretical investigation and useful for cosmological investigations.

3.2 Some Adelic Cosmological Investigations

The first consideration of p-adic gravity and adelic quantum cosmology was in [21]. It was introduced an idea of the fluctuating number fields at the Planck scale giving rise to p-adic valued as well as real valued gravity. Using Hartle-Hawking approach, it was shown that the wave function for the de Sitter minisuperspace model can be presented as an infinite product of its p-adic counterparts.

Since adelic generalization of the Hartle-Hawking proposal was serious problems in minisuperspace models with matter, further developments of adelic quantum cosmology were done (see [22] and references therein) using formalism of adelic quantum mechanics [23]. It was shown that p-adic effects in adelic approach yield some discreteness of the minisuperspace and cosmological constant.

Possibility that the universe is composed of real and some p-adic worlds was considered in [24]. In the present paper we adopted approach that p-adic worlds are made of non-tachyonic p-adic matter.

Let us also mention research on p-adic inflation [25], and investigation of nonlocal cosmology with tachyon condensation by rolling tachyon from a false local vacuum to a stable one (see, e.g., [26–29] and references therein).

4 Concluding Remarks

In the present article we have introduced a non-tachyonic p-adic matter which has origin in open scalar p-adic strings. Formally the corresponding Lagrangian was obtained replacing m^2 by $-m^2$ in Lagrangian for p-adic string. In space-time dimensions $D = 2 + 4k$ the kinetic energy term has correct sign and stable negative local vacua. For this case there is decreasing time dependent field solution of the equation of motion and negative cosmological constant.

This p-adic matter interacts with ordinary matter by gravity and should play some role in the dark side of the universe. In particular, the negative cosmological constant can change expansion to contraction and provide bouncing in cyclic universe evolution. These cosmological aspects are under consideration. If p-adic matter would be produced at the LHC experiment in CERN, then its first signature should be in the form of missing mass (energy) in the final state, because it interacts with ordinary matter only through gravitational interaction.

In the case of gravity with Friedmann-Lemaître-Robertson-Walker (FLRW) metric the d'Alembertian is $\Box = -\partial_t^2 - 3H\partial_t$, where H is the Hubble parameter $H(t) = \frac{\dot{a}}{a}$. Then equation of motion contains this operator $\Box$ and time dependent solution (20) for a constant H is

$$\begin{aligned}\phi_q(t) &= q^{\frac{1}{2(q-1)}} \exp\Big(-\frac{3\ln q}{2}\frac{H}{m^2}\partial_t\Big) \exp\Big(-\frac{q-1}{2q\ln q}m^2 t^2\Big) \\ &= q^{\frac{1}{2(q-1)}} \exp\Big(-\frac{q-1}{2q\ln q}m^2 \big(t - \frac{3\ln q}{2}\frac{H}{m^2}\big)^2\Big). \qquad (28)\end{aligned}$$

Note that equation of motion (14) can be formally obtained from (7) by partial replacement $p \to \frac{1}{p}$ in the following two ways. (i) In the LHS of (7) replace p by $\frac{1}{p}$ and φ by ϕ. (ii) In the RHS of (7) replace p by $\frac{1}{p}$ and φ by ϕ^p. In [30], the equation

$$e^{-\beta\Box}\Phi(x) = \sqrt{k}\Phi^k, \quad 0 < k < 1, \; (\beta > 0)$$

was considered and it corresponds to the case (ii) when $k = \frac{1}{p}$.

We have also emphasized that results of measurements are rational numbers with norm in the form of the familiar absolute value, i.e. they are real rational numbers and not p-adic ones. In Lagrangian we have made some connection between nonlocality and p-adic valued world-sheet.

Acknowledgements This investigation is supported by Ministry of Education and Science of the Republic of Serbia, grant No 174012. This article is based on the author talk presented at the IX International Workshop "Lie Theory and its Applications in Physics", 20–26 June 2011, Varna, Bulgaria and the author thanks organizers for hospitality and creative scientific atmosphere. The author also thanks I.Ya. Aref'eva for useful discussions.

References

1. Li, M., Li, X.-D., Wang, S., Wang, Y.: Dark energy. Comm. Theor. Phys. **56**, 525–604 (2011) [arXiv:1103.5870v6 [astro-ph.CO]]
2. Clifton, T., Ferreira, P.G., Padilla, A., Skordis, C.: Modified gravity and cosmology. Phys. Rep. **513**(1), 1–189 (2012) [arXiv:1106.2476v2 [astro-ph.CO]]
3. Brekke, L., Freund, P.G.O.: p-Adic numbers in physics. Phys. Rep. JHEP. **12**(1), 1–66 (2011)
4. Vladimirov, V.S., Volovich, I.V., Zelenov, E.I.: p-Adic Analysis and Mathematical Physics.World Scientific, Singapore. POS. (ICHP 2012)
5. Dimitrijevic, I., Dragovich, B., Grujic, J., Rakic, Z.: On modified gravity [arXiv:1202.2352v1 [hep-th]]
6. Aref'eva, I.Ya.: Nonlocal string tachyon as a model for cosmological dark energy. AIP Conf. Proc. **826**, 301–311 (2006) [arXiv:astro-ph/0410443v2]
7. Koshelev, A.S., Vernov, S.Yu.: Analysis of scalar perturbations in cosmological models with a non-local scalar field. Class. Quant. Grav. **28**, 085019 (2011) [arXiv:1009.0746v2 [hep-th]]
8. Volovich, I.V.: p-Adic string. Class. Quant. Grav. **4**, L83–L87 (1987)
9. Freund, P.G.O., Olson, M.: Non-archimedean strings. Phys. Lett. B **199**, 186–190 (1987)
10. Gelf'and, I.M., Graev, M.I., Pyatetskii-Shapiro, I.I.: Representation Theory and Automorphic Functions. Saunders, Philadelphia (1969)
11. Schikhof, W.: Ultrametric Calculus. Cambridge University Press, Cambridge (1984)
12. Dragovich, B., Khrennikov, A.Yu., Kozyrev, S.V., Volovich, I.V.: On p-adic mathematical physics. p-Adic Numb. Ultr. Anal. Appl. **1**(1), 1–17 (2009)
13. Brekke, L., Freund, P.G.O., Olson, M., Witten, E.: Nonarchimedean string dynamics. Nucl. Phys. B **302**(3), 365–402 (1988)
14. Frampton, P.H., Okada, Y.: Effective scalar field theory of p-adic string. Phys. Rev. D **37**, 3077–3084 (1988)
15. Ghoshal, D., Sen, A.: Tachyon condensation and brane descent relations in p-adic string theory. Nucl. Phys. B **584**, 300–312 (2000) [arXiv:hep-th/0003278v1]
16. Moeller, N., Zwiebach, B.: Dynamics with infinitely many time derivatives and rolling tachyons. JHEP **0210**, 034 (2002) [arXiv:hep-th/0207107v2]
17. Vladimirov, V.S.: On some exact solutions in p-adic open-closed string theory. p-Adic Numb. Ultr. Anal. Appl. **4**(1), 57–63 (2012)
18. Weil, A.: Adeles and Algebraic Groups. Birkhauser, Basel (1982)
19. Dragovich, B.: Adeles in mathematical physics [arXiv:0707.3876v1 [math-ph]]
20. Freund, P.G.O., Witten, E.: Adelic string amplitudes. Phys. Lett. **199**, 191–194 (1987)

21. Aref'eva, A.Ya., Dragovich, B., Frampton, P.H., Volovich, I.V.: The wave function of the Universe and p-adic gravity. Int. J. Mod. Phys. A **6**, 4341–4358 (1991)
22. Djordjevic, G., Dragovich, B., Nešić, Lj., Volovich, I.V.: p-Adic and adelic minisuperspace quantum cosmology. Int. J. Mod. Phys. A **17**(10), 1413–1433 (2002) [arXiv:gr-qc/0105050v2]
23. Dragovich, B.: Adelic harmonic oscillator. Int. J. Mod. Phys. A **10**, 2349–2365 (1995) [arXiv:hep-th/0404160v1]
24. Dragovich, B.: p-Adic and adelic cosmology: p-adic origin of dark energy and dark matter. p-Adic Mathematical Physics, AIP Conf. Proc. **826** (2006) 25–42 [arXiv:hep-th/0602044v1]
25. Barnaby, N., Biswas, T., Cline, J.M.: p-Adic inflation. JHEP **0704**, 056 (2007) [arXiv:hep-th/0612230v1]
26. Aref'eva, I.Ya., Joukovskaya, L.V., Vernov, S.Yu.: Bouncing and accelerating solutions in nonlocal stringy models. JHEP **0707**, 087 (2007) [arXiv:hep-th/0701184v4]
27. Aref'eva, I.Ya., Koshelev, A.S.: Cosmological signature of tachyon condensation. JHEP **0809**, 068 (2008) [arXiv:0804.3570v2 [hep-th]]
28. Calcagni, G., Montobbio, M., Nardelli, G.: A route to nonlocal cosmology. Phys. Rev. D **76**, 126001 (2007) [arXiv:0705.3043v3 [hep-th]]
29. Dragovich, B.: Nonlocal dynamics of p-adic strings. Theor. Math. Phys. **164**(3), 1151–1155 (2010) [arXiv:1011.0912v1 [hep-th]]
30. Aref'eva, I.Ya., Joukovskaya, L.V.: Time lumps in nonlocal stringy models and cosmological applications. JHEP **0510**, 087 (2005) [arXiv:hep-th/0504200v2]

Palev Statistics and the Chronon

David Ritz Finkelstein

Abstract A finite relativistic quantum space-time is constructed. Its unit element is a spin pair with Palev statistics associated with an orthogonal group.

1 Modular Architecture

The goal is still a finite physical theory that fits our finite physical experiments. The classical space-time continuum led to singular (divergent) quantum field theories: Infinity in, infinity out. In ancient times, a continuum was the only way to understand the translational and rotational invariance of Euclid's geometry. Today there are quantum spaces with a finite number of quantum points that still have the continuous symmetries of gravity and the Standard Model, at least within experimental error. The strategy now is build up a cosmos from such elements, aiming at classical gravity and the Standard Model as singular limits.

Quantum spaces are statistically represented by vector spaces that define the lowest-order logic of their points. Modular architecture requires the higher-order logic classically dealt with in set theory. Classical space-time and field theory are formulated within classical set theory; perhaps quantum space-time and field theory call for a quantum set theory.

Classical set theory was invented by Cantor to represent the mind of the Eternal. Quantum set theory, more pragmatic [11], is intended to represent the system under study as a quantum computer. Like any quantum theory it statistically represents input/outtake (IO) beams of systems by what Heisenberg called *probability vectors*. Since it is standard to name vectors after their components, the full name is transition-probability-amplitude vectors. "IO vectors" is shorter, not actually wrong, and captures better the essential duality between bras and kets.

D.R. Finkelstein (✉)
Georgia Institute of Technology, Atlanta, GA, USA
e-mail: finkelstein@gatech.edu

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_3,

Quantum logic is a square root of classical logic: Transition probabilities in the classical sense are squares of components of IO vectors, the transition probability amplitudes. Call a quadratic space of IO vectors the IO space $\mathscr{P}$. In general $\mathscr{P}$, like the quantum space of Saller [9], includes both input (ket) and outtake (bra) vectors. Distinguish these by the signs of their norms. $\mathscr{P}$ is not a Hilbert space.

A quantum theory should describe populations as well as individuals. Enrich the quadratic IO space $\mathscr{P}$ to a probability or IO algebra $\mathscr{P}$ whose product *ab* represents successive application of IO operations. One-system IO vectors form a generating subspace $\mathscr{P}_1 \subset \mathscr{P}$. $\mathscr{P}$ consists of polynomials over $\mathscr{P}_1$, subject to constraints and identifications said to define the statistics.

A regular quantum theory is one with a finite-dimensional IO algebra [1].

The probability algebra $\mathscr{P}^-$ for fermions is a Clifford algebra, defined by anti-commutation relations among the one-fermion vectors:

$$\forall x \in \mathscr{P}_1^- : \quad x^2 = \|x\| = x \cdot x. \tag{1}$$

Its dimension is $\mathrm{Dim}\,\mathscr{P}^- = 2^{\mathrm{Dim}\,\mathscr{P}_1}$, so Fermi statistics is regular if the one-fermion IO space is.

The IO algebra $\mathscr{P}^+$ for even quanta is commonly assumed to be a Bose (Heisenberg, canonical) algebra whose generators obey

$$\forall x \in \mathscr{P}_1 : \quad xy - yx = \varepsilon(x, y) \tag{2}$$

with a given skew-symmetric bilinear form ε on $\mathscr{P}_1 \otimes \mathscr{P}_1$. This is not exactly right: Bose statistics is always singular. Pairs of fermions, however, obey a regular statistics whose probability algebra envelops a Lie algebra, with Bose statistics and the Heisenberg algebra as a singular limit. This is a special case of Palev statistics [6, 7], which is reviewed next.

2 Palev Statistics

For any semisimple Lie algebra $\mathfrak{p}$, a *Palev statistics of the* $\mathfrak{p}$ *class* is one whose probability algebra $\mathscr{P}$ is a finite-dimensional enveloping algebra of $\mathfrak{p}$ (has $\mathfrak{p}$ as commutator Lie algebra).

Fermi and Bose statistics have graded Lie algebras $\mathfrak{f}, \mathfrak{b}$ that specify their commutation relations. $\mathfrak{f}$ and $\mathfrak{b}$ as vector spaces are also one-quantum IO spaces. They have essentially unique irreducible unitary representations; these serve as many-quantum IO spaces. There are, however, an infinity of irreducible representations of a Palev algebra $\mathfrak{p}$ that might serve as many-quantum IO space. Empirical choices must be made for Palev statistics that are already decided for Fermi and Bose statistics.

Palev gives a representation of $\mathrm{sl}(n+1)$ on a Fock space W_p of symmetric tensors of degree p. W_p is a Hilbert space, appropriate for his applications and not for these.

The IO space of a hypothetical quantum event must have enough dimensions to allow for the observed quantum systems. It is not clear that events in space-time can be experimentally located relative to a macroscopic experimenter to within much less than a fermi, corresponding to a localization in time of about 10^{-25} s. The Planck time 10^{-43} s was initially a conjecture for the minimum time based on pure quantum gravity. Crystals have many length scales besides cell size, such as Debye shielding length, skin depth, coherence length, and mean free paths. The Planck time might correspond more closely to one of these than to a natural unit of time. To avoid a premature commitment, call the natural time unit the *chrone* X.

How many dimensions must the event IO space have? Suppose the lifetime of the four-dimensional universe is 10^{21} s; an error by a factor of 100 will not matter much. If $\mathsf{X} \sim T_{\mathrm{P}} \approx 10^{-43}$ s then the dimensionality of the history IO space of the cosmos—which we cannot observe maximally—is about 10^{1024}. The largest system that can be maximally observed by a co-system within such a cosmos—here we renounce the perspective of the Eternal—is much smaller. Its IO space might have no more than $\log_2 10^{256} \sim 3,000$ dimensions.

Here are two examples of Palev statistics:

2.1 Spins

The $\mathrm{so}(3)$ Lie algebra with commutation relations relation $\mathbf{L} \times \mathbf{L} = \mathbf{L}$ defines an aggregate of palevons of the $\mathrm{so}(3)$ kind, whose quanta may be called spins. Relative to an arbitrary component L_3 as generator of a Cartan subalgebra, the root vectors $L_{\pm} = L_1 \pm iL_2$ represent the input and outtake of spin-1 spins. An irreducible representation with extreme eigenvalues $\pm il$ for L_3 represents a Palev statistics with no more than $2l+1$ spins present in a single aggregation.

2.2 Di-Fermions

Di-fermions are palevons. If the fermion IO space is $2N\mathbb{R}$ then the di-fermion is a palevon of type $\mathrm{so}(N,N)$.

This refers to the elementary fact that the Fermi commutation relations for a fermion with N independent probability vectors define a Clifford algebra $\mathrm{Cliff}(N,N)$, and the second grade of $\mathrm{Cliff}(N,N)$ is both the IO space for a fermion pair, and the Lie algebra $\mathrm{spin}(N,N)$ defining a Palev statistics of class D_N.

2.3 Regular Space-Times

Call a spacetime regular if its coordinate algebra is regular. All its coordinates then have finite spectra. There are not many regular spacetimes in the literature.

Singular (non-semisimple) Lie algebras can be regularized by slightly changing some vanishing commutators, undoing the flattening contraction that led, presumably, from the regular to the singular.

The prototype of such regularization by decontraction is (special) relativization [5, 11]. This regularizes the Galilean Lie algebra $\mathfrak{g} = \mathfrak{g}(\mathbf{L}, \mathbf{K})$ of Euclidean rotations $\mathbf{L}$ and Galilean boosts $\mathbf{K}$, to the Lorentz Lie algebra $\mathrm{so}(3,1)$. Write such relations as

$$\mathrm{so}(3,1) \circ\!\!\to \mathfrak{g} \quad \text{or} \quad \mathfrak{g} \leftarrow\!\!\circ\ \mathrm{so}(3,1), \tag{3}$$

directed from the regular algebra to the singular.

The sole remaining culprit today is the Lie algebra $\mathfrak{h}$ of canonical quantization and Bose statistics. Its regularization requires adding new variables. This is the general case; special relativity and quantum theory were exceptional in this respect.

Some physical self-organization must then freeze these extra variables out near the singular limit. This can be tested experimentally in principle by disrupting this organization. Hopefully, a suitable regularization of the remaining singular theories will once again improve the fit with experiment.

The Killing form of classical observables is as singular as can be: identically 0. It is nearly regularized by canonical quantization:

$$a_{\mathrm{comm}}(\mathbf{x}, \mathbf{p}, i) \ \leftarrow\!\!\circ\ \mathfrak{h}(N), \tag{4}$$

where the canonical (Heisenberg) Lie algebra is

$$\mathfrak{h}(N) : [x^{\nu}, p_{\nu'}] = i\hbar\delta_{\nu}^{\nu'}, \quad \nu, \nu' = 1, \ldots, N, \tag{5}$$

other commutators vanishing. The solvable radical $\mathbb{C}$ generated by i survives canonical quantization. Recall that $\mathfrak{h}$ does not fit into any $\mathrm{sl}(N\mathbb{R})$. (The left-hand sides of the canonical commutation relations would have well-defined trace 0, and the right-hand side i would have non-zero trace.) Canonical quantization is a quantization interrupted by premature canonization.

2.4 Feynman Space-Time

Feynman (personal communication ca. 1961) did this in about 1941, before his work on the Lamb shift, and probably published this formula in a footnote, but we did not find the reference) seems to have constructed the first regular relativistic quantum space-time $\mathscr{F}$. Its positional coordinates are finite spin sums:

$$x^\mu \leftarrow\!\circ \; \widehat{x}^\mu = \mathsf{X}\,[\gamma^\mu(1) + \cdots + \gamma^\mu(N)], \quad \mu = 1,2,3,4. \tag{6}$$

The γ^μ have unit magnitudes, and X is the chrone. If the commutators $[\gamma^{\mu'}(n'), \gamma^\mu(n)]$ vanish for $n \neq n'$ then the probability vector space for $\mathscr{F}$ is a 16^N-dimensional Clifford algebra. Quantification theory abbreviates (6) to

$$\widehat{x}^\mu = \mathsf{X}\overline{\psi}\gamma^\mu\psi. \tag{7}$$

Each term in the sum represents a hypothetical quantum element of the space-time event; call it a chronon. The Feynman chronon has spin 0 or 1, because γ^μ has both a vector part $(\gamma^1, \gamma^2, \gamma^3)$ and a scalar part γ^4 with respect to spatial rotations.

Nature seems to have a unit of space-time size like X at every event, fixing the gauge in the original sense of Weyl. If each event has a space-time measure X^4, then the dimension of the one-event IO space is proportional to the space-time volume; as if the event statistics is extensive in the sense of Haldane [4, 7].

2.5 Yang Space-Time-Momentum-Energy

Let $\mathrm{hp}(N) \supset \mathfrak{h}(N)$ be the *Heisenberg-Poincaré Lie algebra*, made by adjoining to $\mathfrak{h}(N)$ the generators $L_{\mu'\mu}$ of the Lorentz-like group $\mathrm{SO}(N-1,1)$, with the usual commutation relations between them and the variables x^μ, p_μ and i. Yang [13] proposed the first regularization of $\mathrm{hp}(N)$, leaving the signature somewhat open:

$$\mathfrak{y}(5,1) = \mathrm{so}(5,1) \circ\!\!\rightarrow \mathrm{hp}(4) \leftarrow\!\circ \; \mathrm{so}(3,3) = \mathfrak{y}(3,3). \tag{8}$$

$\mathfrak{y}$ could represent a spinless colorless chargeless relativistic quantum, with only orbital variables. Yang did not require full regularity but used a Hilbert space representation of $\mathfrak{y}$.

2.6 Regular Space-Time-Momentum-Energy

A regular version of the Yang theory uses a finite-dimensional representation of the Yang Lie algebra $\mathrm{so}(3,3)$, whose IO space must then have an indefinite norm (Sect. 2.7). One example is the representation of the Yang Lie algebra by spin matrices in the Feynman manner, now using the six 8×8 spin matrices γ_y of $\mathrm{spin}(3,3)$, with vector index $y = 1, \ldots, 6$:

$$L_{y'y} = \overline{\psi}\,\gamma_{y'y}\,\psi, \tag{9}$$

a 6×6 matrix of 8×8 matrices. A regular theory might associate an elementary particle with an irreducible finite-dimensional isometric representation of $\mathfrak{y}$ instead of its singular limit, the Poincaré Lie algebra. Then all one-particle observables have finite spectra.

This is not a conformal theory, and the Yang Lie algebra $\mathfrak{y} \cong \mathrm{so}(3,3)$ is not the conformal Lie algebra $\mathrm{so}(3,3)$. The two algebras are isomorphic but act on different physical variables and have different physical effects. Yang's theory mixes space-time and momentum-energy space; the conformal group does not. We deal with groups of physical operations, not abstract groups.

2.7 Interpretation of the Indefinite Norm

In special relativity the sign of the Minkowski metric form $g_{\mu'\mu}$ distinguishes allowed (timelike) directions from forbidden (space-like) ones.

In a relativistic quantum theory of the Dirac kind, the probability-amplitude form β is neutral (of signature 0).

$$\overline{\Psi}\gamma^y\Psi = \Psi\beta\gamma^y\Psi \tag{10}$$

gives the flux of systems, positive or negative. The sign distinguishes input probability vectors from outtake probability vectors [2]. To fix the sign, let input vectors (kets) have positive norm and outtake vectors (bras) negative. This changes no physics in the usual quantum theory, which does not add bras and kets. Here it enlarges the group and must be tested by experiment.

The constant $i\hbar$ of the usual quantum physics is then another non-zero vacuum expectation value, a frozen variable like the Minkowski metric $g_{\mu'\mu}$ and the Higgs field. Centralizing a hypercomplex number by a condensation gives mass to any gauge boson that transports that number; this was shown for the quaternion case, for example. Such a frozen i must be assumed in the Yang and Segal space-time quantizations [11, 13] based on the Lie algebra

$$\mathfrak{y} = \mathrm{so}(3,3) \cong \mathrm{spin}(3,3) \cong \mathrm{sl}(4\mathbb{R}). \tag{11}$$

Infinitesimal generators of $\mathfrak{y}$ make up a tensor $[L_{y'y}]$ $(y, y' = 1, \ldots, 6)$ with 15 independent components, representing orbital variables of the Yang scalar quantum. The Feynman quantum space-time and the Penrose quantum space [8] can be regarded as spin representations of parts of the Yang Lie algebra.

$\widehat{\imath}$, the quantized i, is a normalized 2×2 sector $[L_{z'z}]$ $(z, z' = 5, 6)$ of $[L_{y'y}]$ in an adapted frame. The tensor $[L_{y'y}]$ then breaks up according to

$$[L_{y'y}] = \left[\begin{array}{c|cc} L_{\mu'\mu} & ix_{\mu'} & ip_{\mu'} \\ \hline ix_\mu & 0 & L_{56} \\ ip_\mu & L_{65} & 0 \end{array}\right] \sim \left[\begin{array}{c|c} 4\times 4 & 4\times 2 \\ \hline 2\times 4 & 2\times 2 \end{array}\right], \quad y, y' = 1, \ldots, 6, \tag{12}$$

which includes the Lorentz generator $L_{\mu'\mu}$ as a 4×4 block, position x^{μ} and momentum p^{μ} $(\mu, \mu' = 1,2,3,4)$ as 4×1 blocks, and $L_{z'z}$ $(z, z' = 1,2)$ as a 2×2 block.

Posit a self-organization, akin to ferromagnetism, that causes the extra component L_{65} to assume its maximum magnitude in the vacuum. Small first-order departures from perfect organization of i make second-order errors in $|i|$.

2.8 Locality

One more limit stands between the regular Yang Lie algebra $\mathfrak{y}$ and singular canonical field theories. The special-relativistic kinematics and $\mathfrak{y}$ have a canonical symmetry between the x^{μ} and p^{μ}. Yet there are great physical differences between these variables. Under the composition of systems, p_{μ} is extensive and x^{μ} is intensive. The fundamental gauge interactions of the Standard Model and gravity are local in x^{μ} and not in p_{μ}; unless asymptotic freedom can be regarded as a weak form of locality in p_{μ}.

This suggests that there is a richer class of regular quantum structures that have classical differential geometry and gauge field theories as organized singular limits, with at least three quantification levels: the chronon, the event, and the field.

Wigner proposed that an elementary particle corresponds to an irreducible unitary representation of the Poincaré group. Up-dates in this concept are called for by his later work. The Wigner concept of elementary particle gives no information about interactions between particles. Gauge theory requires an elementary particle to have a location in space-time where it interacts with a gaugeon. It would then seem useful to associate an elementary particle with an irreducible representation of the Heisenberg-Poincaré Lie algebra $\mathrm{hp}(3,1)$ instead. In a regular theory $\mathrm{hp}(3,1)$ must be regularized.

2.9 Quantification and Quantization

Quantification and quantization are related like architecture and archeology. They concern similar algebraic structures, but quantification synthesizes them from the bottom up, the order of formation, while quantization analyzes them from the top down, the order of discovery.

Quantization uncovers a quantum constant that the quantum individual carries and that the classical limit buries.

In particular, space-time quantization introduces a new quantum entity, the chronon, carrying a time unit, the chrone X, and an energy unit, the erge E. The chronon is no particle in the usual sense but a least part of the history of a particle.

Canonical quantization can also be interpreted as a quantification with a singular statistics. What is sometimes called "second quantization" is then more accurately a second quantification.

2.10 Gauge

A gauge is an arbitrarily fixed movable standard used in measurements. It is part of the co-system, the complement of the system in the cosmos. As part of the co-system, a gauge is normally studied under low resolution and treated classically. A field theory may postulate a replica of the gauge at every event in space-time, forming an infinite field of infinitesimal gauges. Weyl's original gauge was an infinitesimal analogue of a carpenter's gauge or a machinist's gauge block, a movable standard of length; hence the name.

A gauge transformation changes the gauges but fix the system. They form a Lie group, the gauge group, which indicates the arbitrariness of the gauges. It is customary to assume that the relevant dimensions of the gauge field are fixed during an experimental run, so that the experimental results can be compared meaningfully with each other. This means that the gauge must be stiff. For example, machinist's gauge blocks are often made of tungsten carbide. Idealized, such rigid constraints become a source of infinities.

In a simple quantum theory, however, all variables have discrete spectra. All eigenvalues can be defined by counting instead of by measuring. There is no need for arbitrary units, external gauges, or gauge group; Nature provides the gauge within the system. Thus a gauge group is another sign of interrupted quantization.

One well-known way to break a gauge group is by self-organization. It is often supposed that the Higgs field, which breaks an isospin group, is such a condensate.

A gauge group is also broken, however, when further quantization discovers a natural quantum unit, fixing a gauge. The quantized i that breaks $\mathrm{su}(2)$ and imparts mass in quaternion quantum gauge theory is of that kind. So is the quantized imaginary $\widehat{\imath}$ that breaks Yang $\mathrm{so}(3,3)$. The Higgs $\widehat{\eta}$ that breaks isospin $\mathrm{so}(3)$, and the gravitational $\widehat{g}_{\mu'\mu}$ that breaks $\mathrm{sl}(4\mathbb{R}) \cong \mathrm{so}(3,3)$ may also be such natural quantum gauges, to be recovered by regularizing the kinematical Lie algebra of the Standard Model through further quantization.

Notation: The one-quantum total momentum-energy vector is, up to a constant, the differentiator $[\partial_\mu]$, canonically conjugate to the space-time position vector $[x^\mu]$. $[\partial_\mu]$ reduces to a gauge-invariant differentiator $[D_\mu]$, also canonically conjugate to x^μ, related to kinetic energy, and a vector $\Gamma_\mu(x)$ that commutes with position, related to potential energy:

$$\begin{aligned} \partial_\mu &= D_\mu + \Gamma_\mu . \\ \text{Total} &= \text{Kinetic } + \text{Potential} \end{aligned} \tag{13}$$

The gauge commutator algebra $a(x^{\mu}, D_{\mu}(x), F_{\mu'\mu}, \dots)$ is generated by the space-time coordinates x^{μ} and the kinetic differentiator $D_{\mu}(x)$, and includes the field variables $F_{\mu'\mu} = [D_{\mu'}, D_{\mu}]$ and their higher covariant derivatives. Its radical includes all functions of the x^{μ}. This makes it singular too.

Gauging semi-quantizes. It converts the commutative operators ∂_{μ} into the non-commutative ones D_{μ}, and for individual quanta these are observables. Its contraction parameter is the coupling constant. Landau quantization in a magnetic field is of that kind.

Gauging also quantifies: It converts one finite-dimensional global gauge group G into many isomorphs of G, one at each space-time point.

Gauging introduces infinities because the number of gauges is assumed to be infinite. Thus quantum gauge physics can be regularized by regularizing its Lie algebras. This eliminates gauge groups as well as theory singularities. This will be taken up elsewhere.

3 Higher-Order Quantum Set Theory

Classical set theory iterates the power-set functor to form the space of all "regular" (ancestrally finite, hereditarily finite) sets. A regular set theory might therefore iterate the Fermi quantification functor [3], as follows.

The Peano ι, with

$$\{a,b,c,\dots\} := \{a\}\{b\}\{c\}\dots = \iota a\, \iota b\, \iota c\, \dots, \tag{14}$$

defines membership $a \in b$:

$$a \in b \quad :\equiv \quad \iota a \subset b \tag{15}$$

Let $\mathbb{S}$ designate the classical algebra of finite sets finitely generated from the empty set 1 by bracing $\iota x = \{x\}$ and the disjoint union $x \vee y$ (a group product with identity 1, the empty set). Sets of $\mathbb{S}$ are here called *perfinite* (elsewhere, ancestrally or hereditarily finite). They are finite, and so are their elements, and their elements, and so forth, all the way down to the empty set. Let $\bigvee s$ be the set of finite subsets of s. Then

$$\bigvee : \mathbb{S} \to \mathbb{S} = \bigvee \mathbb{S}. \tag{16}$$

An element of $\mathbb{S}$ is a set or simplex whose vertices may be sets or simplices. $\mathbb{S}$ is supposedly complex enough to represent any finite classical structure.

A quantum analogue $\widehat{\mathbb{S}}$ is a kind of linearization of $\mathbb{S}$:

For any quadratic space S, let $\bigsqcup S$ designate the Clifford algebra of finite-degree polynomials over S, modulo the exclusion principle

$$\forall s \in S : s \sqcup s = 0. \tag{17}$$

$\bigsqcup S$ and $\sqcup$ correspond to the classical power set and the symmetric union (XOR). If $\mathscr{P}_1$ is a one-fermion IO space then $\mathscr{P} = \text{Cliff}\,\mathscr{P}_1$ is the many-fermion probability algebra.

Each quantum subclass of a system is associated with a subspace $\mathscr{C} \subset \mathscr{P}$ in the IO space of the system, and so with a Clifford probability vector $e_{\mathscr{C}}$, a top vector of the Clifford algebra $\text{Cliff}\,\mathscr{C} \subset \text{Cliff}\,\mathscr{P}$.

Then define $\iota : \mathscr{P} \to \text{Cliff}\,\mathscr{P}$ as a Cantor brace, modulo linearity:

$$\forall p \in \mathscr{P} : \iota p := \{p\}, \quad \text{mod} \quad \iota(ax+by) \equiv a\,\iota x + b\,\iota y. \tag{18}$$

Take $\widehat{\mathbb{S}}$ (as a first trial) to be the least Clifford algebra that is its own Clifford algebra:

$$\bigsqcup : \widehat{\mathbb{S}} \to \bigsqcup \widehat{\mathbb{S}} = \widehat{\mathbb{S}}. \tag{19}$$

Call the quantum structures with probability vectors in $\widehat{\mathbb{S}}$ *quantum sets*. The quantum set is supposedly complex enough to represent any finite quantum structure.

Table 1 arranges basic probability vectors 1_n of $\widehat{\mathbb{S}}$ by rank r and serial number n.

3.1 Spin Structure of $\widehat{\mathbb{S}}$

For each rank r, $\widehat{\mathbb{S}}[r]$ is naturally a spinor space:

- $D[r] = \text{hexp}\,r$ is its dimension.
- $\widehat{\mathbb{S}}[r-1]$ is its Cartan semivector space.
- $\mathscr{W}[r-1] := \widehat{\mathbb{S}}[r-1] \oplus \text{Dual}\ \widehat{\mathbb{S}}[r-1]$ is its underlying quadratic space.
- $\text{SO}(D[r-1], D[r-1])$ is its orthogonal group.
- There is a neutral symmetric Pauli form $\beta[r] : \widehat{\mathbb{S}}[r] \to \text{Dual}\,\widehat{\mathbb{S}}[r]$ for which the first grade $\gamma^w \in \text{Cliff}[r]$ are hermitian symmetric.
- The Pauli form can be chosen to be a Berezin integral with respect to the top Grassmann element (or volume element) $\gamma^\top \in \mathscr{W}[r]$:

$$\beta[r-1] : \mathscr{W}[r-1] \otimes \mathscr{W}[r-1] \to \mathbb{R},$$
$$\forall \psi = w \oplus w' \in \mathscr{W}[r] : \quad \|\psi\|_{r-1} = \beta[r]\psi\psi := \int d\gamma^\top \psi^2 = \partial_{(\gamma^\top)} \psi^2. \tag{20}$$

- $\text{Cliff}(\mathscr{W}[r-1]) \cong \text{Endo}_{\text{Vec}}\,\widehat{\mathbb{S}}[r]$: The algebra of linear operators on the spinor space is isomorphic as algebra to the associated Clifford algebra $\text{Cliff}[r]$.

This β is just the β of Pauli and Chevalley expressed in the more powerful notation used by physicists. Since $L^2(\mathscr{M})$ designates a quadratic space defined by a quadratic Lebesgue integral over $\mathscr{M}$, write the quantum space defined by a quadratic Berezin integral over $\mathscr{W}$ as $B^2(\mathscr{W})$.

Table 1 Quantum and classical sets 1_n by rank r and serial number n

Every real Grassmann algebra $\mathscr{G} = \mathrm{Grass}\, N\mathbb{R}$ is a spinor space for the orthogonal group whose quadratic space $\mathscr{W}$ is the direct sum of the polar and axial vectors of $\mathscr{G}$, grades 1 and $N-1$ of $\mathscr{G}$:

$$\mathscr{W} = \mathrm{Grade}_1\, \mathscr{G} \oplus \mathrm{Grade}_{N-1}\, \mathscr{G}. \tag{21}$$

The norm on $\mathscr{W}$ is the quadratic Berezin form

$$\beta : \mathscr{W} \otimes \mathscr{W} \to \mathbb{R},$$
$$\forall \psi = w \oplus w' \in \mathscr{W} : \quad \|\psi\| = \beta\, \psi\psi := \int d\gamma^\top \psi^2 = \partial_{(\gamma^\top)}\, \psi^2. \tag{22}$$

This imbedding of the quadratic space $\mathscr{W}$ in its spinor space is isometric but not invariant under spin$(\mathscr{W})$, which mixes $\mathrm{Grade}_1\, G$ and $\mathrm{Grade}_{N-1}\, G$.

3.2 Schur Spinors

Naturally Cartan and Dirac based their spinor theories on classical space-times. There is none in nature, so the generators of the spin group should be interpreted in the earlier manner of Schur [10], not as rotations but as pair exchanges, now of quantum elements. The Clifford algebra Cliff(n) now represents a finite quantum group corresponding to the classical finite group 2^n.

4 Revised Quantum Set Theory

Here are some adaptations of $\widehat{\mathbb{S}}$ to current physics.

4.1 Bosons

The number of times one set belongs to another ($a \in b$) is either 0 or 1. In this respect classical sets have Fermi (odd) statistics. Classical thought did not allow for Bose (even) statistics, which grossly violates the Leibniz doctrine that indistinguishable objects are one. Nor does $\widehat{\mathbb{S}}$ describe elementary bosons. The Standard Model, however, requires them. Moreover, if a and b are monads (first-grade elements) of the space $\widehat{\mathbb{S}}$, hence fermionic, then $\{a,b\}$ is a fermionic monad too, although $a \vee b$ is an approximate boson and an exact palevon. Two odds make an odd in set theory, and an even in nature. $\widehat{\mathbb{S}}$ violates conservation of statistics.

This can be resolved within quantum set theory only by modeling the even quanta as pairs of odd quanta. These pairs can then be associated by dynamical binding. The spin-statistics relation refers to the particle rank. It is broken at the subparticle rank both in the Cartan theory of spin and the GUT theory of flavor. We should accept such violations in general.

4.2 Constituents of the Photon

The Standard Model uses the unreduced classical brace $\{\dots\}$, iterated as necessary, to assemble (say) a fermion probability vector from orbital, isospin, spin, and other probability vectors. The regular theory $\widehat{\mathbb{S}}$ does much the same.

Therefore the even quanta cannot be unit sets $\widehat{\mathbb{S}}$. They must be represented by tensors of even degree. This implies that the bosons of the regularized Standard Model are actually fermion pairs held together by binding rather than by unition.

Such a possibility was already raised by the de Broglie two-neutrino photon, and the four-neutrino graviton considered and rejected by Feynman. The main obstacle to such constituent theories of even vector quanta is that according to the Heisenberg indeterminacy principle, fermions near each other in position must be far apart in momentum. Then they require a correspondingly high interaction-energy for their binding. But experiment finds no such intense short-range interaction but only the asymptotic freedom implied by the Standard Model.

In a Feynman or Yang quantum space-time, however, the operator $\widehat{i\hbar}$ that replaces $i\hbar$ has a finite spectrum of magnitudes with extreme values $\pm\hbar$. Presumably a self-organization akin to ferromagnetization freezes $\widehat{i\hbar}$ to its maximum value $i\hbar$. The Heisenberg uncertainty principle is then weakened wherever a local disorganization reduces the magnitude of $\widehat{i\hbar}$. Such a weakening might allow two leptons to bind into a photon, say. This re-opens the question of a di-fermion theory of the gauge bosons.

Acknowledgements To Roger Penrose and Richard Feynman for kindly showing me their seminal spin quantizations of space and space-time before publication.
To O. B. Bassler, James Baugh, Walter L. Bloom, Jr., Dustin Burns, David Edwards, Shlomit Ritz Finkelstein, Andrei Galiautdinov, Dennis Marks, Zbigniew Oziewicz, Tchavdar Dimitrov Palev, Heinrich Saller, Stephen Selesnick, Abraham Sternlieb, Sarang Shah, and Frank (Tony) Smith, for helpful and enjoyable discussions.
To Lynn Margulis for introducing me to post-Darwinian symbiogenetic evolution in her Lindisfarne lectures.
To Cecylia Arszewski for leading me to Unger's most helpful radical pragmatism.
To Prof. V. K. Dobrev for inviting me to this retreat of the Bulgarian Academy of Science.

References

1. Bopp, F., Haag, R.: Über die Möglichkeit von Spinmodellen. Zeitschrift für Naturforschung **5a**, 644 (1950)
2. Dirac, P.A.M.: Spinors in Hilbert Space. Plenum, New York (1974)
3. Finkelstein, D.: Quantum Relativity. Springer, Heidelberg (1996)
4. Haldane, F.D.M.: "Fractional statistics" in arbitrary dimensions: A generalization of the Pauli principle. Phys. Rev. Lett. **67**, 937 (1991)
5. Inönü, E., Wigner, E.P.: On the contraction of groups and their representations. Proc. Natl. Acad. Sci. **39**, 510–525 (1952)
6. Palev, T.D.: Lie algebraical aspects of the quantum statistics. Unitary quantization (A-quantization). Joint Institute for Nuclear Research Preprint JINR E17-10550, Dubna (1977). hep-th/9705032
7. Palev, T.D., Van der Jeugt, J.: Jacobson generators, Fock representations and statistics of $sl(n+1)$. J. Math. Phys. **43**, 3850–3873 (2002)
8. Penrose, R.: Angular momentum: An approach to combinatorial space-time. In: Bastin, T. (ed.) Quantum Theory and Beyond, pp. 151–180. Cambridge University Press, Cambridge (1971). Penrose kindly shared much of this seminal work with me ca. 1960
9. Saller, H.: Operational Quantum Theory I. Nonrelativistic Structures. Springer, New York (2006)

10. Schur, I.: Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen. Journal für die reine und angewandte Mathematik 139:155–250 (1911)
11. Segal, I.E.: A class of operator algebras which are determined by groups. Duke Math. J. **18**, 221–265 (1951) Especially Sect. 6A
12. Unger, R.M.: The Self Awakened: Pragmatism Unbound. Harvard University Press, Cambridge (2007)
13. Yang, C.N.: On quantized space-time. Phys. Rev. **72**, 874 (1947)

Some Remarks on Weierstrass Sections, Adapted Pairs and Polynomiality

Anthony Joseph

Abstract Let A be a polynomial subalgebra of the algebra of regular functions on affine n-space k^n. A Weierstrass section for A is a translate $x+Y$ of a linear subspace of k^n such that the restriction of A to $x+Y$ induces an isomorphism of A onto the algebra $R[x+Y]$ of regular functions on $x+Y$. They arise notably in describing algebras of invariants both for reductive and non-reductive actions as well as in describing maximal Poisson commutative subalgebras of $R[k^n]$ in the case that the latter has a Poisson bracket structure. A Weierstrass section need not always exist and in any case can be very difficult to construct. A review of some known results and open problems is presented in an entirely elementary fashion.

Keywords Invariants • Weierstrass sections

AMS Classification: 17B35

1 Introduction

The base field k is assumed algebraically closed of characteristic zero throughout.

1.1. Let V be a finite dimensional vector space and $S(V)$ the symmetric algebra of V. Under the assumptions on the base field k we may identify $S(V)$ with the algebra of regular functions $R[V^*]$ on the dual V^* of V.

Work supported in part by Israel Science Foundation Grant, no. 710724.

A. Joseph (✉)
Donald Frey Professional Chair, Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: anthony.joseph@weizmann.ac.il

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_4,

Of course $S(V)$ and hence $R[V^*]$ is a polynomial algebra. In algebraic geometry one associates to an ideal I of this polynomial algebra a (closed) subvariety $\mathscr{V}$ of V^* defined as the zero locus of I, that is to say the set $\{v \in V^* | a(v) = 0, \forall a \in I\}$. Then the algebra of (regular) functions $R[\mathscr{V}]$ on $\mathscr{V}$ is defined to be the quotient algebra $S(V)/I$.

On the other hand we may start from a subalgebra A of $S(V)$ and ask if it can be presented as $R[\mathscr{V}]$ for some closed subvariety $\mathscr{V}$ of V^*. It is immediate that this can only be true if A is finitely generated. A more delicate question which makes sense if A has no zero divisors is to ask if there exists a closed irreducible subvariety $\mathscr{V}$ of V^* such the restriction of A to $\mathscr{V}$ is injective and the embedding $A \hookrightarrow R[\mathscr{V}]$ lifts to equality of fields of fractions. In this case one calls $\mathscr{V}$ a rational section for A.

A particularly interesting case of a rational section occurs when $\mathscr{V}$ is an affine translate of a vector subspace of V^*, that is a set of the form $x+Y$ with $x \in V^*$ and Y a vector subspace of V^*. Then the field $Q(A)$ of fractions of A must be a pure transcendental extension of k. In this case one calls $x+Y$ a linear section for A.

One calls a linear section $x+Y$ for A, a Weierstrass section if the injection $A \hookrightarrow R[x+Y]$ is an isomorphism. This is of course very special and can only arise if A is a polynomial algebra.

1.2. Particularly interesting examples of subalgebras of $R[V^*]$ arise as invariant subalgebras for group actions. Thus let G be an algebraic group acting linearly on V^*. Then by transport of structure G acts on $R[V^*]$, that is to say by the rule $(g.f)(\xi) := f(g^{-1}\xi) : g \in G, f \in R[V^*]$. We set $R[V^*]^G := \{f \in R[V^*] | g.f = f, \forall g \in G\}$. We may similarly define $R(V^*)^G$, where $R(V^*)$ denotes the field of rational functions on V^*.

There is a quite extensive theory of sections for invariant algebras. This has been reviewed in [29] and [31]. Notably if G is connected, it is not known [29, 1.5], if $R(V^*)^G$ is necessarily pure over k. Again a rather precise criterion has been given [29, Thm. 1.4.3] for $R[V^*]^G$ to admit a rational section. In particular if G is connected and solvable, then a theorem of Rosenlicht [32] asserts that this is always the case. It is not known if a linear section exists in this case. However recently Popov [30, Cor. 8] has shown this to hold if G is a unipotent subgroup of $GL(V^*)$.

One may hope to do a little better in the case that $V = \text{Lie } G$, that is to say when V^* is the co-adjoint module. More generally let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $K(\mathfrak{g})$ be the fraction field of $S(\mathfrak{g})$. The commutative version of a problem of Dixmier [6, Prob. 4] is that $K(\mathfrak{g})^{\mathfrak{g}}$ is pure over k. Here it is not suggested that $\mathfrak{g}$ is algebraic, that is of the form Lie G, with G an algebraic group. However in the latter case one can further ask [15, 7.11] if the co-adjoint action admits a linear section. Recall here that if $\mathfrak{g}$ is algebraic, then it can be written as a semi-direct product $\mathfrak{r}+\mathfrak{n}$, with $\mathfrak{r}$ reductive and $\mathfrak{n}$ a nilpotent ideal. When the action of $\mathfrak{r}$ on $\mathfrak{n}$ is trivial, then the existence of a linear section follows from Popov's result mentioned above and the result of Kostant in 1.3—see [30, Cor. 9].

1.3. For a linear group action, the first example of a Weierstrass section (see Sect. 2) occurs in the work of Weierstrass in his description of canonical form for elliptic curves. In the case of a semisimple Lie group G acting on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$, a Weierstrass section for $S(\mathfrak{g})^G$ was constructed by Kostant [22]. To do this Kostant used the principal s-triple (x,h,y) which forms an $\mathfrak{sl}(2)$ subalgebra of $\mathfrak{g}$ all of whose elements (except 0) are regular. It turns out that one can make do will rather less. Indeed suppose G is a connected algebraic group acting linearly on a vector space X and let $\mathfrak{g}$ denote the Lie algebra of G. Call $h \in \mathfrak{g}, x \in X$ an adapted pair if $hx = -x$ and x is regular. This last condition means that the stabilizer of $x \in X$ in $\mathfrak{g}$ has minimal possible dimension, denoted by $\iota_{X,G}$. Although "most" elements of X have this property, such elements would *not* be expected to be eigenvectors. In particular the existence of an adapted pair *is by no means assured.*

When an adapted pair exists one can assume h to be diagonalizable without loss of generality. Then the supposed injectivity implies that Y is a complement to $\mathfrak{g}.x$. Here we can assume Y to be h stable without loss of generality. The regularity hypothesis on x implies that Y has dimension $\iota_{X,G}$. Then surjectivity further implies a tight relation between the eigenvalues $\{m_i\}_{i=1}^{\iota_{X,G}}$, called the exponents, and the degrees $\{d_i\}_{i=1}^{\iota_{X,G}}$ of homogeneous generators of $R[X]^G$, namely $d_i = m_i + 1, \forall i$, up to ordering. In the case of a semisimple Lie algebra one may identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form and take x, h to be the first two members of a principal s-triple (x,h,y). Moreover Y can be taken to $\mathfrak{g}^y$, that is to say the centralizer of y in $\mathfrak{g}$.

1.4. The existence of an adapted pair, though hard to find is insufficient to produce a Weierstrass section for the invariant algebra $S(\mathfrak{g})^G$. Notably one must impose the additional condition that $S(\mathfrak{g})^G$ be polynomial. To some extent this is self-defeating and so one should look for a less hard to verify criterion. One useful criterion is to find sufficiently many algebraically independent elements of suitably small degree.

Conversely if the exponents are not all non-negative or if two different adapted pairs can be constructed with differing sets of exponents then under mild restrictions (see [21, 8.2]) the invariant algebra $R[X]^G$ cannot be polynomial, at least in the case of co-adjoint action and quite possibly in general.

Popov showed essentially by a case by case analysis, that for G simple and V a simple G module an adapted pair exists whenever $S(V)^G$ is polynomial. Popov thereby obtained a Weierstrass section in all such cases. However Popov also noted [29, 2.2.16.3] that a Weierstrass section may exist if G is simple but V is not a simple G module, without there being an adapted pair. We have described many other examples arising when G is not reductive [19].

1.5. Adapted pairs have also been constructed when G is not semisimple (nor reductive); with $V = \mathfrak{g}^*$, that is to say for the coadjoint module. Notable examples occur for truncated (see 3.2) biparabolic subalgebras [18] as well as centralizer subalgebras [17] in $\mathfrak{sl}(n)$. These two classes of examples often lead to Weierstrass sections. In the latter case one may avoid knowing that the invariant algebra is polynomial [17]. A third class to which Panyushev and Yakimova [28] have brought attention are contractions of semisimple Lie algebras.

1.6. Weierstrass sections may be constructed for certain algebras [19, 20] which are related to invariant algebras but which are not themselves invariant algebras. One quite general family of algebras obtains from an invariant algebra by shift of argument. Outside the reductive case, this does not necessarily lead to a Weierstrass section and for this additional conditions must be imposed (4.6).

1.7. In this paper we review some of the constructions of Weierstrass sections and of adapted pairs. We start with the example coming from the work of Weierstrass noted above. That this fits into the general theory of adapted pairs was noted by Popov [29, 2.2.2.1]. Nevertheless we go over the details as we believe this example is the best way of giving the reader an impression of what is involved in general. Here it is very easy to write down the adapted pair. On the other hand especially for truncated biparabolics in $\mathfrak{sl}(n)$ it is both difficult to guess such pairs and of particular difficulty to verify regularity.

1.8. In the case of co-adjoint action a more systematic approach to the construction of adapted pairs was suggested in [15]. This is discussed briefly in 3.8.

2 Weierstrass Canonical Form for Elliptic Curves

2.1. By definition a plane elliptic curve is given as the zero locus of an irreducible polynomial of degree 3 in two variables (a,b). However it is better to add the "points at infinity" by viewing the curve as a subset of the projective plane. To this effect the variables a,b are replaced by $a/c, b/c$ in which case the polynomial becomes homogeneous of degree three in the three variables a,b,c.

Since the number of linearly independent homogeneous polynomials of degree 3 in 3 variables is 10, an elliptic curve is described by ten parameters. However not all of these will be distinct curves. Indeed a given curve will be unchanged if we make a linear change of variables. Discounting multiplying each variable by the same scalar, which would result in the same polynomial, this means that we may permit an action of the special linear group $SL(3)$, which has dimension 8. This gives us the possibility of cutting down the number of parameters which describe the elliptic curve to just $10-8=2$. However it is not at all obvious how to describe these two parameters, nor in this if certain curves will be omitted by such a reduction.

2.2. Set $\mathfrak{s}=\mathfrak{sl}(3)$ which is the Lie algebra of $S:=SL(3)$. The elements of $\mathfrak{s}$ are 3×3 trace zero matrices. Thus $\dim\mathfrak{s}=8$. Let $\mathfrak{h}$ denote its two dimensional subspace of diagonal trace zero matrices. It is a Cartan subalgebra for $\mathfrak{s}$. The set of column matrices becomes a three dimensional module for $\mathfrak{s}$. Identifying a,b,c as a basis for this module, the space of homogeneous cubic polynomials in these three variables forms a ten dimensional module V for $\mathfrak{s}$, which is in fact a simple module. Set $V_{reg}=\{v\in V \mid \dim\mathfrak{s}.v=8\}$. It is a quite easy fact that V_{reg} is non-empty (see 2.3) and as a consequence (Zariski) dense in V. It is then immediate that V_{reg} is a union of S orbits and by definition all of these have the maximum dimension, namely 8.

Below we shall construct an adapted pair, namely a pair $(x \in V_{reg}, h \in \mathfrak{h})$ satisfying $hx = -x$. It turns out that $\mathfrak{s}.x$ admits a unique h stable complement Y. This is spanned by two eigenvectors with eigenvalues $3,5$. It will be enough to admit[1] that there are invariant homogeneous polynomials q_4, q_6 of degrees $4,6$ to show that $R[V]^S$ is the polynomial algebra generated by q_4, q_6 and then to deduce that every S orbit meeting $x+Y$ meets this set at exactly one point. Then we show that $V_{reg} = S(x+Y)$, equivalently that V_{reg} is an irreducible variety, which in general does not follow merely from adapted pairs and polynomiality, nor indeed is it a consequence of the existence of a Weierstrass section. Finally we relate the description of $x+Y$ to Weierstrass canonical form.

The above is mainly detailing the example given in [29, 2.2.2.1]. The only novelty (Lemma 2.5) is showing that V_{reg} is irreducible. Although this holds for the adjoint case studied by Kostant and in the case when the invariant algebra has a single generator, it is otherwise a *very rare phenomenon*. Our analysis is intended to give some insight into the reasoning for more general situations.

2.3. Let $\{\alpha, \beta\}$ denote a choice of simple roots for the pair $\mathfrak{s}, \mathfrak{h}$. We can then take V to have highest weight $2\beta + \alpha$. All weight spaces are one-dimensional and can be arranged on the plane to what may be recognized as an arrangement familiar in ten-pin bowling. This is illustrated in Fig. 1 in which the weights form the vertices and are joined by unbroken lines which correspond to the action of the root vectors. The small triangle in the centre with vertices joined by broken lines gives a similar presentation of the three dimensional representation spanned by (a,b,c).

For every weight ϖ of V let x_ϖ denote the corresponding weight vector (determined up to a non-zero scalar). Consider the vector $x := x_\beta + x_{-\beta+\alpha}$. The arrows in Fig. 1 indicate the action of the 6 root vectors of $\mathfrak{s}$ on x_β and on $x_{-\beta+\alpha}$. Each root vector is non-zero on just one of them and each of the resulting vectors are distinct. Set $Y = kx_{-\alpha-\beta} + kx_{-2\alpha-\beta}$. Taking account of the action of $\mathfrak{h}$ one obtains, practically by inspection of Fig. 1, the following direct sum decomposition

$$\mathfrak{s}.x \oplus Y = V. \tag{1}$$

Since $\dim Y = 2$, this means in particular that $x \in V_{reg}$, which is hence non-empty.

In order to obtain $hx = -x$, we require that $h(\beta) = -1, h(-\beta+\alpha) = -1$. There is exactly one h in the two dimensional space $\mathfrak{h}$ with this property. This unique h has eigenvalues $3,5$ on Y.

The reader should appreciate that this result was obtained in such an easy fashion only by good fortune and of course the rather small dimensions involved. Constructing an adapted pair for truncated biparabolics in $\mathfrak{sl}(n)$ which in number grow rather rapidly to infinity as n goes to infinity, is *much harder* [18]. However

[1] For nineteenth century mathematicians constucting invariants was a popular exercise. The modern mathematician would no doubt prefer to use the Weyl character formula which can be adjusted to compute the character of a given symmetric power of V and then to show it has a non-zero scalar product with the trivial character. Despite good intentions the author was too lazy to illustrate the recovery of the required invariants by these means.

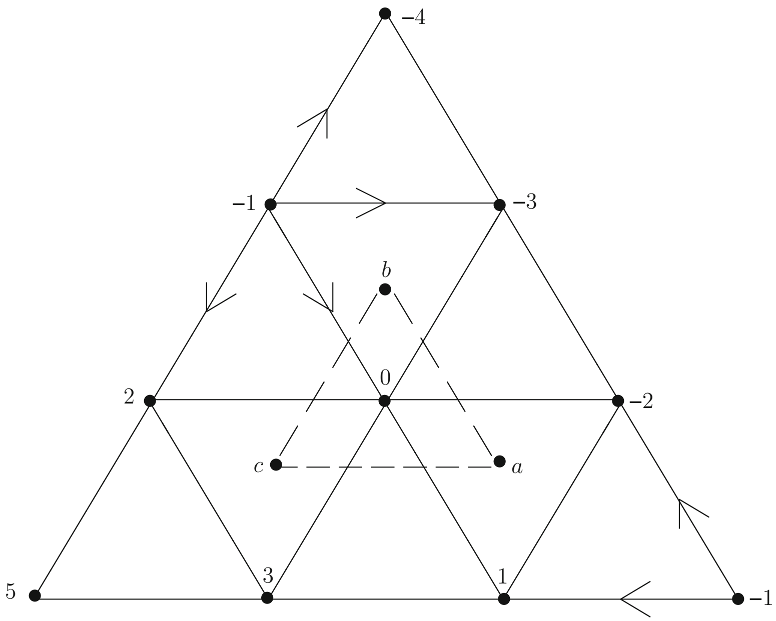


Fig. 1 The ten vertices joined by the unbroken lines describe the weight diagram of one of the simple ten dimensional modules for $\mathfrak{sl}(3)$. The numerical data over a given vertex gives the eigenvalue of h for the corresponding weight vector. Then x is the sum of the eigenvectors of eigenvalue -1. The action of the root vectors on x is described by the arrows. The numerical data over the two vertices not reached by the arrows give the exponents. The three vertices joined by the broken lines describe the weight diagram of one of the simple three dimensional modules for $\mathfrak{sl}(3)$, labelled by the root vectors a, b, c

there is one common feature, namely x was expressed as a linear combination of weight vectors whose weights are linearly independent elements of the dual of the Cartan subalgebra of the Levi factor of the truncated biparabolic and likewise this makes the choice of the second element h of the adapted pair unique [18]. Again proving the polynomiality of the invariant algebras also requires some major efforts [7, 15].

2.4. One may show that as a consequence of (1), the subvariety $S(x+Y)$ is dense in V. Consequently a non-zero invariant function $f \in R[V]^S$ cannot vanish on $x+V$ and so restriction defines an injection φ of $R[V]^S$ into $R[x+Y]$. A fortiori the restriction map $\hat{\varphi}$ of $R[V]^S$ to the subspace $kx+Y$ of V is also injective. Since this latter subspace is spanned by h eigenvectors having eigenvalues $-1, 3, 5$, we may choose a basis $\xi_1, \xi_{-3}, \xi_{-5}$ of its dual partly spanned by h eigenvectors having eigenvalues $1, -3, -5$. Complete to a basis of V^* formed from eigenvectors of h.

Now admit that we can find invariant polynomials q_4, q_6 of degrees $4, 6$ respectively. Of course these must be eigenvectors of h having zero eigenvalue. Comparing eigenvalues and degrees, we conclude that $\hat{\varphi}(q_4) = \xi_1^3\xi_{-3}, \hat{\varphi}(q_6) = \xi_1^5\xi_{-5}$ up to non-zero multiples. Consequently $\varphi(q_4) = \xi_{-3}, \varphi(q_6) = \xi_{-5}$, up to non-zero multiples. Yet the latter are generators of $R[v]$ and so φ is surjective. (In particular $R[V]^S$ is polynomial on generators q_4, q_6.) Geometrically this has the consequence that the invariant functions separate the points of $x+Y$ and so every S orbit which meets $x+Y$ meets it at *exactly one point*.

For the above argument to work in general it is not enough to find invariants whose degrees equal the exponents plus 1, one has to also know that they are algebraically independent. This can be rather difficult. In the above case we were rather lucky.

2.5. Since S acts linearly on V, the natural gradation on $R[V]$ passes to $R[V]^S$. Let $R[V]^S_+$ denote the subspace of $R[V]^S$ of elements of positive degree and J denote the ideal of $R[V]$ it generates. A very hard question is to show that J is a prime ideal. Remarkably there is an argument due to Kostant [22] which proves, in the presence of a Weierstrass section, that it is enough to show that the zero locus $\mathscr{N}$ of J is irreducible. An exposition of this argument is given in [6, 8.1.3]. A key point is the use of a theorem of Macaulay.

In the case when $\mathfrak{g}$ is semisimple and $V = \mathfrak{g}^*$ (the coadjoint module), the proof of $\mathscr{N}$ being irreducible is deceptively simple. Indeed fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and let $\mathfrak{n}$ be its nilradical. We claim that $\mathscr{N} = G\mathfrak{n}$. Since G is connected and $\mathfrak{n}$ is a subspace and so both are irreducible as varieties, it follows that $G\mathfrak{n}$ being an image of $G \times \mathfrak{n}$ is irreducible.

The above claim follows by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form, which identifies $\mathscr{N}$ with the cone of ad-nilpotent elements and applying the Jacobson-Morosov theorem. However let us give a second more involved proof which adapts better to the present situation.

Consider $\mathfrak{b}^*$. It is $\mathfrak{b}$ stable (for co-adjoint action) and hence stable for the corresponding Borel subgroup B. Since G/B is a complete variety it follows that $G\mathfrak{b}^*$ is closed in $\mathfrak{g}^*$. On the other hand a dimensionality argument shows that $G\mathfrak{b}^*$ is dense in $\mathfrak{g}^*$ and consequently $G\mathfrak{b}^* = \mathfrak{g}^* \supset \mathscr{N}$. Take $z \in R[\mathfrak{g}^*]^G$ and $\lambda \in \mathfrak{h}^*$. Then $z(\lambda+\mathfrak{n}) = z(\lambda) = \varphi(z)(\lambda)$, where φ is the Chevalley restriction map. After Chevalley the image of φ is just the Weyl group invariants in $R[\mathfrak{h}^*]$. Thus there is some $z \in R[\mathfrak{g}^*]^G_+$ which is non-zero on $\lambda+\mathfrak{n}$, unless $\lambda = 0$. This gives the desired conclusion.

Let us try to see what part of this argument goes over when we just have an adapted pair. Consider the gradation on V and on $\mathfrak{s}$ defined by eigenspace decomposition with respect to h. Precisely set

$$V_i := \{v \in V | h.v = iv\}, \quad \mathfrak{s}_i := \{x \in \mathfrak{s} | [h,x] = ix\},$$

and define

$$V_< := \sum_{i<0} V_i, \quad V_\leq := \sum_{i\leq 0} V_i, \quad \mathfrak{s}_< := \sum_{i<0} \mathfrak{s}_i, \quad \mathfrak{s}_\leq := \sum_{i\leq 0} \mathfrak{s}_i. \tag{2}$$

As a consequence of a general fact, $\mathfrak{p} := \mathfrak{s}_\leq$ is a parabolic subalgebra of $\mathfrak{s}$ with nilradical $\mathfrak{n} := \mathfrak{s}_<$. Let P (resp. N) be the connected algebraic subgroup of S with Lie algebra $\mathfrak{p}$ (resp. $\mathfrak{n}$). (These exist because the Lie algebras in question are algebraic.) It is clear that both $V_<$ and $V_\leq$ are P stable. Since S/P is a complete variety, it follows that $SV_<$ (resp. $SV_\leq$) is a closed subvariety of V. Moreover it is irreducible and (as a variety) has dimension $\leq \dim S/P + \dim V_<$ (resp. $\leq \dim S/P + \dim V_\leq$), where equality holds if $P = \text{Stab}_G V_<$ (resp. $P = \text{Stab}_G V_\leq$).

In our special case, using Fig. 1 we may easily compute the objects defined in (2) and thereby show that these sums have values $3+5=8$ and $3+6=9$ respectively.

On the other hand $x \in V_<$, so $Sx \subset SV_<$. Yet the regularity of x implies that $\dim Sx = 8$.

We conclude from the above that $SV_<$ has dimension 8 and is hence the closure of Sy.

So much for general arguments. Now let us prove a lemma which requires the details of our very special situation.

Lemma. $SV_< = \mathscr{N}$.

Proof. Via eigenvector decomposition it follows easily that $V_< \subset \mathscr{N}$ and so $SV_< \subset \mathscr{N}$.

On the other hand $SV_\leq$ is also irreducible, contains $SV_<$ strictly and has dimension ≤ 9. Consequently $\dim SV_\leq = 9$, that is to say it has codimension 1 in V. Thus by Krull's theorem it is the zero locus of a polynomial $p \in R[V]$. On the other hand $\dim SV_\leq$ is irreducible, conical and S stable. Thus p must be irreducible, homogeneous (of say degree $d > 0$) and semi-invariant. Since S is simple and non-commutative $R[V]$ admits no proper semi-invariants and so $p \in R[V]^S$.

We can write $V_0 = kv_0$. Define a dual basis of V^* as in 2.4 using the same conventions. In particular ξ_0 is the dual basis vector corresponding to v_0.

Recall that by 2.4, the invariant algebra $R[V]^S$ is generated by q_4, q_6. By eigenspace decomposition it follows that the restriction of q_4 (resp. q_6) to $V_\leq$ takes the form $c_1\xi_0^4$ (resp. $c_2\xi_0^6$) $: c_1, c_2 \in k$. These scalars cannot both be zero for otherwise both q_4 and q_6 would vanish on $SV_\leq$ forcing both to lie in $pR[V]$, hence to be divisible by p in $R[V]^S$. Yet since they are generators of this polynomial algebra, hence irreducible in $R[V]^S$, this would force both to be proportional to p, which is clearly impossible.

Since p is an invariant polynomial homogeneous of positive degree it follows that $SV_\leq \supset \mathscr{N}$. Consider an arbitrary element $z := cx_0 + y : c \in k, y \in V_<$ in $\mathscr{N}$. Since both q_4 and q_6 vanish on $\mathscr{N}$, hence on z, the form of their restrictions given in the paragraph above implies that $c = 0$. Thus $z \in V_<$. This proves the assertion of the lemma. □

2.6. As noted in 2.5 it follows from Lemma 2.5, that $J = R[V]R[V]^S_+$ is a prime ideal. Again by general arguments [16, 8.7] this has the following consequence

Proposition. $V_{reg} = S(x+Y)$. *Thus every regular S orbit in V meets $x+Y$ at exactly one point.*

Remark 1. Although the arguments in [16, Sect. 8] are fairly standard we should like to take this opportunity of noting that the proof of [16, Thm. 8.2 (iii)] is better presented in [19, 7.8].

Remark 2. Just given an adapted pair (x,h), we could only have obtained the inclusion $V_{reg} \supset S(x+Y)$ with both closures equal to V. If $x+Y$ is just a Weierstrass section for the invariant ring $R[V]^S$ (with respect to linear action of algebra group S on a vector space V and $R[V]^S$ having no proper semi-invariants) then still both closures lie equal V; but $S(x+Y)$ may not lie entirely in V_{reg}. In neither [29], nor in [31] do the authors appear to address themselves to *equality* in the conclusion of the proposition. In particular we do not know if equality holds in all the remaining cases when S is simple, V and a simple S module and $R[V]^S$ is polynomial for which Popov [29] verified that $R[V]^S$ admits a Weierstrass section $x+Y$. However for the coadjoint action of a truncated biparabolic in type A (which always admits an adapted pair) equality very seldom holds [18]. This is related to the fact that they may be non-equivalent adapted pairs. Indeed suppose (x,h) is an adapted pair. Let $\mathscr{N}$ be the subvariety of all zeros of the homogeneous invariant polynomials of positive degree. Then $x \in V_{reg}$, by definition of an adapted pair and $x \in \mathscr{N}$, via the action of h. By [16, Lemma 8.8] (which extends easily to present situation) one has $S(x+Y) \cap \mathscr{N} = Sx$. Then if given a second adapted pair (x',h'), the equality $S(x+V) = V_{reg}$ forces $x' \in Sx$. In the (rather common) situation when the first member of the pair determines the second, this further implies $(x',h') \in S(x,h)$. When this holds we shall say that the adapted pairs in question are equivalent.

We may summarize the above by saying that the existence of inequivalent adapted pairs forces the inclusion $S(x+Y) \subset V_{reg}$ to be strict.

The simplest example of inequivalent adapted pairs occurs for the truncated Borel in $\mathfrak{sl}(3)$. Despite earlier optimism [16, 8.12, Remark] we found [18] embarrassingly many inequivalent adapted pairs for truncated biparabolics in type A. Their classification is a wide open problem.

2.7. Retain the notation of 2.3. One may note that the highest weight $2\beta+\alpha$ of V is just $3\varpi_\beta$, where ϖ_β is the fundamental weight corresponding to β. The latter is the highest weight of the fundamental module spanned by $\{a,b,c\}$, which we shall take to have weights $\varpi_\beta - \beta, \varpi_\beta, \varpi_\beta - (\alpha+\beta)$ respectively. Then in the presentation of V as the span of the homogeneous polynomials of degree 3 in a,b,c we have in particular that $x = a^3 - b^2c, V = kac^2 + kc^3$.

A non-zero homogeneous polynomial p of degree three in three variables viewed as an element of V necessarily belongs to V_{reg}. Indeed for every trace zero matrix with entries $a_{i,j} : i,j = 1,2,3$, we obtain an element of $\mathfrak{s}$ acting on a space of homogeneous polynomials in three variables through the differential operator $\sum_{i,j=1}^3 a_{i,j} x_i \partial/\partial x_j$. Then if $p \notin V_{reg}$, it must be annihilated by a subspace of such differential operators of dimension > 2 forcing $p = 0$. Then Proposition 2.5 exactly states that up to a linear transformation every elliptic curve can be defined in just one fashion by the equation $a^3 = b^2c + sac^2 + tc^3$ for some $s,t \in k$. This is Weierstrass canonical form.

A well-known application of Weierstrass canonical form is to determine which elliptic curves are singular. We describe this below.

Let $\mathscr{V}(p)$ be the variety (in projective space $\mathbb{P}^2$) of zeros of a homogeneous polynomial p. Then the tangent space $T_{\mathscr{V}(p),a}$ to $\mathscr{V}(p)$ at $a \in \mathscr{V}(p)$ is defined to be the points (in $\mathbb{P}^2$) defined by the non-zero solutions to the equation $\sum_{i=1}^{3} z_i(\partial p/\partial x_i)(a) = 0$. For most (called generic) points a this space is one dimensional (in $\mathbb{P}^2$). A point $a \in \mathscr{V}(p)$ is said to be singular if $\dim T_{\mathscr{V}(p),a}$ is strictly greater than its value at a generic point. In this case this just means that $(\partial p/\partial x_i)(a) = 0 : i = 1,2,3$. If $\mathscr{V}(p)$ admits such a point it is said to be singular. Taking $p = x_1^3 - x_2^2 x_3 - s x_1 x_3^2 - t x_3^3$, one checks that the corresponding elliptic curve $\mathscr{V}(p)$ is singular exactly when $4s^3 = 27t^2$.

Weierstrass canonical form plays a small role in the proof of Fermat's "last" theorem. Returning to affine coordinates (that is putting $x_3 = 1$) Weierstrass canonical form describes an elliptic curve through the equation $x_2^2 = x_1^3 - s x_1 - t$, or $y^2 = x(x-d)(x+e)$, making a slight change of variables. Hellegouarch came up with the radical idea of associating an elliptic curve to a (non-trivial) solution to Fermat's equation $a^n + b^n = c^n$ by taking $d = a^n, e = b^n$. Work of Frey, Serre and Ribet showed that such a curve would not be modular violating the Shimura-Taniyama-Weil conjecture. Yet mainly through the work of Wiles this conjecture has now been shown to be true, the contradiction proving Fermat's theorem.

2.8. Return to the framework of 1.2 that is of an algebraic group G acting linearly on a finite dimensional vector space V. In this context one may ask what aspects of preceding example remain true in general?

A necessary condition for the existence of a Weierstrass section $x + Y \subset V$ (for the subalgebra $R[V]^G$ of invariants) is of course that $R[V]^G$ be polynomial on $\dim Y$ variables. However this is far from sufficient. In [29, 2,2.4], Popov gives an example where even a rational section need not exist. In this case V is not the co-adjoint module, though this may not be significant. In any event in [19, 11.4.2] an example is given (the coadjoint module for the nilradical of a Borel in $\mathfrak{sp}(4)$) when a rational section exists but a Weierstrass section does not.

Although this is not strictly speaking necessary one tends to assume in the above situation, that the size of the invariant algebra $R[V]^G$ is comparable to the minimal orbit codimension, more precisely that the Gelfand Kirillov dimensional (equivalently the transcendence degree of its fraction field $Q(R[V]^G)$ equals $\iota_{V,G}$. By a general result of Chevalley-Rosenlicht, the transcendence degree of the possibly larger field $R(X)^G$ of G invariants in the fraction field $R(V)$ of $R[V]$ always equals $\iota_{V,G}$. By a further remark of Chevalley-Dixmier one has $Q(R[V]^G) = R(V)^G$, if $R[V]^G$ admits no proper semi-invariants, for example if G admits no non-trivial one dimensional modules. Again one can always eliminate proper semi-invariants by replacing G by a slightly smaller group G'. However this can also increase the size of invariant field, that is one may have $R(V)^G \subsetneq R(V)^{G'}$, so it is not always appropriate to replace G by G'. This is particularly true if G is the centralizer of a nilpotent element in a semisimple Lie group with $V = (\text{Lie } G)^*$. In this case $R[V]^G$ already has the required size, namely $\iota_{V,G}$ even though $R[V]$ can admit proper semi-invariants [21, 4.6(ii)].

One can ask if a Weierstrass section $x+Y$ exists without there being an adapted pair. A condition which pops up in this circumstance (and in several others) is the following. Set $V_{sing} = V \setminus V_{reg}$. Since V_{reg} is Zariski dense in V, it follows that $\text{codim}_V V_{sing} \geq 1$. We call the pair (G,V) singular if equality holds and non-singular otherwise.

If (G,V) is singular, then as in 2.5 there exists a unique up to scalars homogeneous semi-invariant polynomial on V of minimal degree whose zero set is V_{sing}. It is called the fundamental semi-invariant $p_{V,G}$. It can be a challenging problem to compute this polynomial.

Suppose (G,V) is non-singular, with G connected and semisimple. then after Popov [29, Thm. 2.2.15] every element of a Weierstrass section $x+Y$ must be regular. By contrast, in [29, 2.2.16.3], Popov notes that if $G = SL(n)$ and V is $n \times n$ matrices under left multiplication, then (G,V) is singular and a Weierstrass section exists admitting non-regular elements. In this case the fundamental semi-invariant is just the determinant of the matrix algebra.

There are also (rather more complicated) examples of the above phenomenon when G is not semisimple, more precisely when G is a truncated Borel B (of a simple Lie algebra $\mathfrak{g}$) acting on the dual of its Lie algebra $\mathfrak{b}^*$. We showed in [19] that $(B,\mathfrak{b}^*)$ is singular outside types A, C and that with the further exception outside types B_{2n}, F_4, that the invariant algebra admits a Weierstrass section and this contains non-regular elements. This result is also true (rather trivially) of the Heisenberg Lie algebra (defined by the only non-zero relation $[x,y] = z$).

An interesting situation occurs in the "Feigin" contraction $\mathfrak{a}$ of a semisimple Lie algebra. In this $\mathfrak{a}$ is a semi-direct product of a Borel subalgebra $\mathfrak{b}$ of a semisimple Lie algebra $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{b}$, the latter viewed as a commutative ideal. One has index $\mathfrak{a} =$ index $\mathfrak{g} =: \ell$ and clearly $\mathfrak{a}$ is unimodular. Panyushev and Yakimova [28] have shown that the invariant algebra $Y(\mathfrak{a})$ is polynomial on ℓ generators.

By Panyushev–Yakimova [28] $\mathfrak{a}$ is a singular Lie algebra (cf 3.6) if $\mathfrak{g}$ has factors not of type A. Yet it may still admit an adapted pair because $S(\mathfrak{a})$ admits proper semi-invariants. Indeed let $y \in (\mathfrak{g}/\mathfrak{b})^*$ be the sum of the simple root vectors and extend y to an element of $\mathfrak{a}^*$ by setting $y(\mathfrak{b}) = 0$. Let h be the element of the Cartan subalgebra of $\mathfrak{b}$ whose value on every simple root equals -1. Then (h,y) is an adapted pair for $\mathfrak{a}$. Let X be an h stable complement for $(\text{ad}\,\mathfrak{a})y$ in $\mathfrak{a}^*$. Then $y+X$ is a Weierstrass section for $Y(\mathfrak{a})$.

3 A Joke and a Misunderstanding

3.1. Let $\mathfrak{g}$ be a simple Lie algebra with $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$, a triangular decomposition. (Here $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}^\pm$ the span of positive (negative) root vectors.) Then $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}^+$ is a Borel subalgebra. Set $Y(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$. It identifies with the algebra of invariant functions on $\mathfrak{g}^*$ under co-adjoint action. Let P^+ denote the dominant weights of $\mathfrak{g}$ relative to this triangular decomposition and w_0 the unique longest element of the Weyl group.

It turns out (rather remarkably) that $S(\mathfrak{b})^{\mathfrak{n}^+}$ is a polynomial algebra [12] on rank $\mathfrak{g}$ generators. On the other hand it consists of highest weight vectors for the adjoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$, whilst $S(\mathfrak{n}^-)^{\mathfrak{n}^-}$ consists of lowest weight vectors. A piece of good fortune is that, in the latter, weights have multiplicity ≤ 1 and those of multiplicity one are exactly the negatives of the weights of $S(\mathfrak{b})^{\mathfrak{n}^+}$. This circumstance gives a natural way [8, 4.17] to construct a linear map ψ from $S(\mathfrak{b})^{\mathfrak{n}^+}$ to $Y(\mathfrak{g})$. In [8, 4.19] we suggested that ψ is surjective.

The surjectivity of ψ means that the invariant functions on $\mathfrak{g}^*$ are spanned by elements constructed as follows. Tensor the highest weight $\mathfrak{g}$ module generated by a weight vector of $S(\mathfrak{b})^{\mathfrak{n}^+}$ with the unique lowest weight $\mathfrak{g}$ module generated by a weight vector of opposite weight from $S(\mathfrak{n}^-)^{\mathfrak{n}^-}$ and take the image (possibly zero) in $S(\mathfrak{g})$ of the unique up to scalars $\mathfrak{g}$ invariant in their tensor product.

This was to have been explained in [7]; but the referee felt that it seemed too much of a joke to relate these invariant algebras. Nevertheless in [8] we showed that ψ is bijective if $\mathfrak{g}$ is of type A or C. The proof derived partly from a general injectivity property of the *enveloping algebra analogue* of ψ *restricted* to a subspace of $U(\mathfrak{b})^{\mathfrak{n}^+}$ generated by weight vectors whose weights are rather special, precisely lie in the set $D := \varpi - w_0\varpi : \varpi \in P^+$. Exactly in types A and C all the weights are of this form. Otherwise additional weights are arise. These are generated over D by including *some of the* $\frac{\varpi - w_0\varpi}{2}$, with ϖ a fundamental weight. Moreover ψ is not injective in general [8, 4.13].

3.2. Given M a $\mathfrak{g}$ module and $\mu \in \mathfrak{h}^*$, set $M_\mu := \{m \in M | hm = \mu(h)m, \forall h \in \mathfrak{h}\}$. Define

$$A(\mathfrak{g}) = \bigoplus_{\mu \in \mathfrak{h}^*} S(\mathfrak{b})_\mu^{\mathfrak{n}^+} S(\mathfrak{n}^-)_{-\mu}^{\mathfrak{n}^-}.$$

By what is stated in 3.1, $A(\mathfrak{g})$ is a polynomial subalgebra on rank $\mathfrak{g}$ generators. The conjectured surjectivity of ψ can be rephrased as follows.

Conjecture 1. $(\operatorname{ad} U(\mathfrak{g})A(\mathfrak{g}))^{\mathfrak{g}} = Y(\mathfrak{g})$.

3.3. An obvious question that arises from the above is whether this holds before taking $\mathfrak{g}$ invariants, that is does one have

$$(\operatorname{ad} U(\mathfrak{g})A(\mathfrak{g})) = S(\mathfrak{g}) \ ?$$

This is false even for $\mathfrak{sl}(2)$, however it is not excluded that it may hold when the symmetric algebra is replaced throughout by the enveloping algebra. Again Conjecture 1 would be implied by the truth of

Conjecture 2.

$$\operatorname{ad} U(\mathfrak{g})\Big(\bigoplus_{\mu,\nu \in \mathfrak{h}^*} S(\mathfrak{b})_\mu^{\mathfrak{n}^+} S(\mathfrak{n}^-)_{-\nu}^{\mathfrak{n}^-}\Big) = S(\mathfrak{g}).$$

3.4. In analyzing these questions it seemed appropriate to give $A(\mathfrak{g})$ some geometric interpretation. In this we were motivated by the following observation.

Recall that the "algebraic" fact that $\operatorname{ad} U(\mathfrak{g})S(\mathfrak{h}) = S(\mathfrak{g})$. Now $S(\mathfrak{h})$ (resp. $S(\mathfrak{g})$) is the algebra of regular functions on $\mathfrak{h}^*$ (resp. $\mathfrak{g}^*$) and it is a "geometric" fact that $\overline{G\mathfrak{h}^*} = \mathfrak{g}^*$.

This suggested the following programme. Construct a Weierstrass section $y + V \subset \mathfrak{g}^*$ for $A(\mathfrak{g})$ and show that $\overline{G(y+V)} = \mathfrak{g}^*$. Unfortunately here there was a misunderstanding. It is not that $\overline{G\mathfrak{h}^*} = \mathfrak{g}^*$ implies that $\operatorname{ad} U(\mathfrak{g})S(\mathfrak{h}) = S(\mathfrak{g})$; but that the proof of the two statements rely on the same fact namely that $\mathfrak{h} + (\operatorname{ad}\mathfrak{g})\mathfrak{h} = \mathfrak{g}$.

3.5. In spite of this joke and misunderstanding we did in fact manage to prove [19] that $A(\mathfrak{g})$ admits a Weierstrass section $y+V$, which is moreover a close cousin of the Weierstrass section $y + \mathfrak{g}^x$ for $Y(\mathfrak{g})$ obtained by Kostant [22] using a principal s-triple (x,h,y). Nevertheless there are some important differences which in particular make our construction much more difficult and technically complicated. First of all choosing a triangular decomposition of $\mathfrak{g}$ selects both a Cartan subalgebra and a choice of positive roots. Thus we cannot expect y in the description of the Weierstrass section for $A(\mathfrak{g})$ to be *any* principal nilpotent element. Indeed it must be very carefully chosen [19, Sect. 3]. Secondly in the situation treated by Kostant one may replace $\mathfrak{g}^x$ by any complement V to $\mathfrak{g}.y$ in $\mathfrak{g}$, the essential point being to ensure that $G(y+V)$ is dense in $\mathfrak{g}^*$. By contrast this consideration does not govern the choice of V for $A(\mathfrak{g})$ though ultimately we found a "canonical" procedure. A further distinction is that we found that $G(y+V)$ was dense in $\mathfrak{g}^*$ exactly when $\mathfrak{g}$ is of type A or type C, that is to say exactly when the weights of $S(\mathfrak{b})^{\mathfrak{n}^+}$ lie in D. Yet so far we do not know if this affects the status of Conjecture 1.

One of the most satisfying aspects of the construction of a Weierstrass section for $A(\mathfrak{g})$ is that it depended on the set of weights of $S(\mathfrak{b})^{\mathfrak{n}^+}$ being larger than D. Thus it "explained" the appearance of the additional weights without which the restriction map $A(\mathfrak{g}) \to R[y+V]$ would not be surjective.

3.6. Let $\mathfrak{a}$ be a finite dimensional Lie algebra. In general $S(\mathfrak{a})$ may admit semi-invariants, that is to say vectors spanning non-trivial one dimensional representations of $\mathfrak{a}$ under adjoint action. This occurs for example if $\mathfrak{a}$ is a Borel subalgebra of a simple Lie algebra, indeed in this case $S(\mathfrak{a})$ admits no non-trivial invariants. However if $\mathfrak{a}$ is algebraic, that is to say the Lie algebra of a connected algebraic group A, then there is a canonical construction of an ideal $\mathfrak{a}_E$ of $\mathfrak{a}$ such that the algebra generated by the semi-invariants of $S(\mathfrak{a})$ is just the invariant algebra $Y(\mathfrak{a}_E)$. (This is also true in general; except that the construction is not canonical [25]). We call $\mathfrak{a}_E$ the (canonical) truncation of $\mathfrak{a}$.

Specializing the terminology of 2.8 to the coadjoint case, an element $\xi \in \mathfrak{a}^*$ is called regular if $\operatorname{codim} \mathfrak{a}.\xi$ takes its minimal value, denoted $\iota_\mathfrak{a}$. The set $\mathfrak{a}^*_{sing} := \mathfrak{a}^* \setminus \mathfrak{a}^*_{reg}$ has codimension ≥ 1. We say that $\mathfrak{a}$ is *singular* if this codimension is exactly one. In the adjoint case there is an explicit way to compute (see [21, 4.1], for example) a polynomial $p_\mathfrak{a}$ whose zero set is $\mathfrak{a}^*_{sing}$. However it may not be the minimal polynomial with this property (for example if one takes a Heisenberg algebra of dimension $2n+1$ with $n > 1$). Nevertheless we shall still call it the fundamental semi-invariant of $\mathfrak{a}$.

In [19, 13.3] we calculated $p_{\mathfrak{b}_E}$ for every truncated Borel subalgebra $\mathfrak{b}_E$ of a simple Lie algebra $\mathfrak{g}$. It turns out that $\mathfrak{b}_E$ is singular exactly when $\mathfrak{g}$ is of types B,D,E,F,G. It would be interesting to calculate $p_{\mathfrak{p}_E}$ for every truncated biparabolic subalgebra $\mathfrak{p}_E$ of $\mathfrak{g}$, though outside the Borel case we know of no examples for which $\mathfrak{p}_E$ is non-singular.

Incidentally I believe that Yakimova has recently determined $p_{\mathfrak{a}}$ and $p_{\mathfrak{a}_E}$ for a Feigin contraction $\mathfrak{a}$ of a semisimple Lie algebra (cf 2.8).

3.7. Recall the notation of 3.5. There are some choices in the Weierstrass section $y+V$ for $A(\mathfrak{g})$. We found [19] that in all cases outside types B_{2n},C_n,F_4 that it is possible to choose $y+V$ to be a Weierstrass section for the invariant algebra $S(\mathfrak{b})^{\mathfrak{n}^+}=Y(\mathfrak{b}_E)$. Moreover if we further exclude type A then the restriction of y to $\mathfrak{b}_E$ is not a regular element of $\mathfrak{b}_E^*$. Conversely since $\mathfrak{b}_E$ is singular (3.6) and $S(\mathfrak{b}_E)$ admits no proper semi-invariants, it follows (cf [21, 1.7]) that $\mathfrak{b}_E$ cannot admit an adapted pair. This extends considerably the example of Popov discussed in 2.8.

3.8. Since the extraction of an adapted pair (y,h) for a Lie algebra $\mathfrak{a}$ (with respect to co-adjoint action) is very difficult particularly in the truncated biparabolic case, one should attempt to find some rationale behind their construction. In the successful attempts in the centralizer and truncated Borel case, a central theme is that $\mathfrak{a}$ occurs naturally as a subalgebra of a semisimple Lie algebra $\mathfrak{g}$, that y is the restriction of a *regular* nilpotent element y_0 of $\mathfrak{g}$ with h uniquely determined by y. Moreover in the truncated Borel case we found in nearly all cases that the invariant algebra of $\mathfrak{a}$ for coadjoint action admits a Weierstrass section of the form $y+V$ with again y the restriction of y_0, even when $\mathfrak{a}$ admits no adapted pair. This is made possible by the fact that y need not be regular in $\mathfrak{a}^*$.

Another feature is that in all cases we have examined one may identify $\mathfrak{a}^*$ with $\kappa(\mathfrak{g})$ for some Chevalley antiautomorphism of $\mathfrak{g}$. However so far this observation has played no significant role.

This has led us to re-examine the construction of adapted pairs of truncated biparabolics in type A described in [18]. In [9] we examined the "simplest" case, namely when the truncated biparabolic has index one. This can only arise in type A and when the biparabolic is a parabolic defined by two blocks of sizes p,q with p,q coprime. In this case there is just one generator of $Y(\mathfrak{a})$ which nevertheless has a rather large degree, namely $\frac{p^2+q^2+pq-1}{2}$. It is already rather difficult to construct an adapted pair (y,h) in this case. Indeed at first it seemed that we would need an explicit solution to the Euler (alias, the Bezout) equation! Ultimately we found a way to construct an adapted pair [14] which avoiding needing an explicit solution to the Euler equation at the same time found an algorithm (unfortunately hopelessly slow) describing its solution. In this we can view y as a nilpotent element of $\mathfrak{g}$ by identifying $\mathfrak{a}^*$ with $\kappa(\mathfrak{a})$. However then y is never regular in $\mathfrak{g}$.

Of course this last described construction is not the only way we can lift $y\in\mathfrak{a}^*$ to an element of $\mathfrak{g}^*$. If we let $\mathfrak{m}$ denote the kernel of the map $\mathfrak{g}^*\to\mathfrak{a}^*$ identified through the Killing form with a subspace of $\mathfrak{g}$, then our question becomes if $y+\mathfrak{m}$ contains a regular nilpotent element of $\mathfrak{g}$. There seems absolutely no reason why this should be so. Thus its proof must require a large amount of blind faith which will need to be even more fervent in the general case.

In fact we did manage [9] to establish the existence of such a regular nilpotent element in the "easy" index one case, with the proof a long drawn out analysis of meanders. It now seems that this has the potential to handle the general case. One interesting feature of our proof is that it gave an invariant on the set of all coprime pairs computed using meanders.

Although we showed [7,15] that the invariant algebra $Y(\mathfrak{a})$ is always polynomial for "most" truncated parabolics (all in type A) the description of their generators seems a quite impossible task. Even describing their number and their degrees can only be carried out algorithmically. Again for the derived algebra of a parabolic of $\mathfrak{sl}(n)$ defined by two blocks of sizes (p,q), the number of generators is the largest common divisor of (p,q). Yet even when this equals one, there is no known formula for the generator, except when $p=1$. This last case is what is now being called the mirabolic subalgebra of $\mathfrak{sl}(n)$. In this case the invariant (or semi-invariant in the parabolic) has a reasonably simple description [5, 13, 7.4].

4 The Hessenberg and Its Generalizations

4.1. Let $\mathfrak{a}$ be a finite dimensional Lie algebra and fix $f\in\mathfrak{a}$. The bilinear form on $\mathfrak{a}$ defined by $B_f := f([x,y]) : x,y\in\mathfrak{a}$ is alternating. It induces a non-degenerate bilinear form on $\mathfrak{a}/\ker B_f$, which is hence, even dimensional. Assume to keep things purely algebraic, that $\mathfrak{a}$ is the Lie algebra of a connected algebraic group A. Then $\ker B_f$ is just the Lie algebra of the stabilizer A_f of f in A. Consequently any co-adjoint orbit is even dimensional. This observation with its remarkably simple proof is due to Kirillov (as late as 1962!). It quickly implies that the algebra of regular functions $R[\overline{Af}]$ on the orbit closure $\overline{Af}$, the structure of Poisson algebra. Indeed let $J_{\overline{Af}}$ denote the ideal of definition of $\overline{Af}$. It is stable under the Poisson bracket on $S(\mathfrak{a})$ obtained from the Lie bracket on $\mathfrak{a}$. Hence the Poisson algebra structure of $S(\mathfrak{a})$ induces a Poisson algebra structure on $S(\mathfrak{a})/J_{\overline{Af}} = R[\overline{Af}]$. In addition it was later noted by Kirillov (and independently by Kostant) that the resulting two-form is closed and A invariant. Thus a co-adjoint orbit admits an invariant symplectic structure. It is practically impossible to overemphasize the impact these simple observations made.

4.2. Retain the above notation. One may view the enveloping algebra $U(\mathfrak{a})$ of $\mathfrak{a}$ as a deformation of $S(\mathfrak{a})$ which lifts the Poisson bracket to commutators in $U(\mathfrak{a})$. More precisely there is a filtration $\mathscr{F}$ on $U(\mathfrak{a})$ (the canonical filtration [6, 2.3]) such that $\mathrm{gr}_{\mathscr{F}}\, U(\mathfrak{a}) \xrightarrow{\sim} S(\mathfrak{a})$ from which the Poisson bracket on $S(\mathfrak{a})$ can be recovered as follows. Let $\overline{a},\overline{b}\in S(\mathfrak{a})$ be homogeneous of degree r,s respectively. Let a,b be their inverse images in $U(\mathfrak{a})$. Since $S(\mathfrak{a})$ is commutative, the image of $[a,b]$ in $S(\mathfrak{a})$ has degree at most $r+s-1$.The term of degree exactly $r+s-1$ (which can be zero) is exactly the Poisson bracket $\{\overline{a},\overline{b}\}$ of the pair $(\overline{a},\overline{b})$.

Just describing how to find a corresponding algebra U which similarly deforms the algebra of functions on an orbit closure, has occupied much of the theory of primitive ideals of $U(\mathfrak{a})$. This question is still of great interest especially for the case when $\mathfrak{a}$ is semisimple.

4.3. Retain the above notation. The invariant subalgebra $Y(\mathfrak{a}) := S(\mathfrak{a})^{\mathfrak{a}}$ of $S(\mathfrak{a})$ identifies with the centre of $S(\mathfrak{a})$ as a Poisson algebra. It is polynomial (rather infrequently) and admits a Weierstrass section (even less frequently). Nevertheless both hold if $\mathfrak{a}$ is semisimple and for certain truncated biparabolics [7, 15], certain centralizer subalgebras of semisimple Lie algebras [3, 17, 26] and even Feigin contractions of semisimple Lie algebras [28] and 2.8.

An almost immediate step (which nevertheless took about 40 years to make!) is to consider the corresponding questions for a maximal Poisson commutative subalgebra of $S(\mathfrak{a})$. Of course such algebras are less canonical, but nevertheless of some importance.

4.4. It is easy to show that the Gelfand-Kirillov dimension (that is growth rate [23]) of $Y(\mathfrak{a})$ is at most $\iota_{\mathfrak{a}}$. It is similarly easy to show that the Gelfand-Kirillov dimension of a Poisson commutative subalgebra of $S(\mathfrak{a})$ is at most $c(\mathfrak{a}) := \frac{1}{2}(\dim \mathfrak{a} + \iota_{\mathfrak{a}})$. By results of Borho-Chevalley-Rosenlicht equality holds in the former case if $\mathfrak{a}$ is algebraic and $S(\mathfrak{a})$ has no proper semi-invariants. (The first condition may be dropped if the second is retained [25].) Rather surprisingly Sadetov [33] has shown equality can always be assured in the second case for some Poisson commutative subalgebra, though the Sadetov construction has not drawn much attention.

Another construction of a large Poisson commutative subalgebra of $S(\mathfrak{a})$ can be obtained from $Y(\mathfrak{a})$ by "shift of argument". This goes back to Mishchenko and Fomenko [24]. Indeed fix $\eta \in \mathfrak{a}^*$. Given $f \in Y(\mathfrak{a})$ one obtains a subspace f_η of $S(\mathfrak{a})$ by taking the linear span of the coefficients in powers of λ of the function $\xi \mapsto f(\xi + \lambda\eta)$. The space $Y_\eta(\mathfrak{a})$ spanned by the $f_\eta : f \in Y(\mathfrak{a})$ is a Poisson commutative subalgebra of $S(\mathfrak{a})$. If η belongs to a suitable dense open subset $\mathfrak{a}^*_{wreg}$ (cf [21, 7.2 $(*)$]) of $\mathfrak{a}^*$ then the Gelfand-Kirillov dimension of $Y_\eta(\mathfrak{a})$ is exactly $c(\mathfrak{a})$ as long as $\mathfrak{a}$ is non-singular. This remarkably general result is due to Bolsinov [2]. When $\mathfrak{a}$ is singular, a similar result holds [21, Sect. 7], but with $c(\mathfrak{a})$ replaced $c(\mathfrak{a}) - d(\mathfrak{a})$, where $d(\mathfrak{a})$ denotes the degree of the fundamental invariant $p_{\mathfrak{a}}$ of $\mathfrak{a}$. Thus in the singular case shift of argument does not give a large enough algebra.

One can ask when is $Y_\eta(\mathfrak{a}) : \eta \in \mathfrak{a}^*_{wreg}$ polynomial? One answer is that it suffices for $Y(\mathfrak{a})$ to be polynomial on $\iota_{\mathfrak{a}}$ homogeneous polynomials whose degrees sum to $c(\mathfrak{a}) - d(\mathfrak{a})$. For $\mathfrak{a}$ semisimple such properties hold when $\mathfrak{a}$ is semisimple after the celebrated work of Chevalley. Outside the semisimple case truncated biparabolics or centralizer subalgebras gave the first few *general* examples for which $Y(\mathfrak{a})$ was known to be polynomial and in most of these cases the above sum rule could be simply checked [8, 17, 26]. Shortly afterwards Ooms and Van den Bergh [25] obtain this sum rule by a general argument in [25]. A slight generalization of their result, following partly a very simple argument due to Panyushev, can be found in [21, Thm. 2.2].

In general $Y_\eta(\mathfrak{a})$ need not be *maximal* Poisson commutative even when its Gelfand-Kirillov dimension equals $c(\mathfrak{a})$. However recently Panyushev and

Yakimova [27] have shown that this does hold if $\mathfrak{a}^*_{sing}$ has at least codimension 3. Conversely Ooms and Van den Bergh [25] showed that if $S(\mathfrak{a})$ has no proper semi-invariants and $Y(\mathfrak{a})$ is polynomial (hence on $\iota_{\mathfrak{a}}$ generators) then $\mathfrak{a}^*_{sing}$ has at most codimension 3. In the semisimple case $\mathfrak{a}^*_{sing}$ has at least codimension 3 by the existence of a principal s-triple and has at most codimension 3 by the "subregular sheet" S having codimension 3.

4.5. Retain the above notation. One can ask when does $Y_\eta(\mathfrak{a})$ admit a Weierstrass section? In proving that $Y_\eta(\mathfrak{a})$ is maximal commutative in the semisimple case Tarasov [34] effectively constructed a Weierstrass section for $Y_\eta(\mathfrak{a})$ (with $\mathfrak{a}$ semisimple). Indeed a simple argument (cf [20, Prop. 16]) shows that when $Y_\eta(\mathfrak{a})$ has Gelfand-Kirillov dimension $c(\mathfrak{a})$ and admits a Weierstrass section, then $Y_\eta(\mathfrak{a})$ is a maximal Poisson commutative subalgebra of $S(\mathfrak{a})$.

4.6. Tarasov's construction of a Weierstrass section in the semisimple case uses the principal s-triple in a seemingly essential way. However it turns out we can do with much less and thereby obtain many other examples when $Y_\eta(\mathfrak{a})$ admits a Weierstrass section. Let us review briefly what is needed and what is still unknown. Here our presentation follows [20] with some improvements (imposing less restrictive conditions) following [21, Sects. 2, 3].

Let $\mathfrak{a}$ be a finite dimensional Lie algebra (algebraic is not imposed though this will be true in all examples). Let ∂ be a semisimple derivation of $\mathfrak{a}$. For all $i \in k$ set $\mathfrak{a}_i := \{a \in \mathfrak{a} | \partial x = ix\}$.

A_1. Assume that $Y(\mathfrak{a})$ is polynomial on $\iota_{\mathfrak{a}}$ generators.
We need not assume that $S(\mathfrak{a})$ admits no proper semi-invariants, but only the weaker condition (cf [21, 1.3]) that $\mathfrak{a}$ is unimodular. This weakening is particularly relevant to centralizer subalgebras of a semisimple Lie algebra [21, 4.3].

A_2. Assume that $\mathfrak{a}$ admits a pair $(h,y) \in (\mathfrak{a} \times \mathfrak{a}^*_{reg})$ with $\operatorname{ad} h$ a semisimple endomorphism of $\mathfrak{a}^*$ and $(\operatorname{ad} h)y = -y$.

A_3. Assume that there exists $h^* \in \mathfrak{a}^*$ such that its stabilizer $\mathfrak{a}^{h^*}$ in $\mathfrak{a}$ equals $\mathfrak{a}^h$.

By a standard deformation argument (cf [19, 7.8]) A_2, A_3 imply that $\mathfrak{a}$ is nonsingular. Combined with A_1 this implies [21, 2.2], that the sums of the degrees of the homogeneous generators equals $c(\mathfrak{a})$. Combined with A_2 again, this implies [20, Lemma 11], [21, Lemma 3.4], that the eigenvalues of $\operatorname{ad} h$ on $\mathfrak{a}$ are integer, that $\dim \mathfrak{a}_i = \dim \mathfrak{a}_{-i}, \forall i \in \mathbb{Z}$ (here we say that $\mathfrak{a}$ is *balanced*) and that h^* is regular.

We call (h, y, h^*) an adapted triple.

Assume that $A_1 - A_3$ hold. It follows from the above that the (much larger set) of hypotheses $H1 - H5$ of [20] are satisfied. By [20, Lemma], it follows that $\mathfrak{a}^*_{\geq} := \sum_{i \in \mathbb{N}} \mathfrak{a}_i$ has dimension $c(\mathfrak{a})$ which of course is just the Gelfand-Kirillov of the translated algebra $Y_{h^*}(\mathfrak{a})$. However the most interesting point is the truth of [20, Thm. 25] holds. This can be expressed as follows.

Theorem. *$y + \mathfrak{a}^*_{\geq}$ is a Weierstrass section for $Y_{h^*}(\mathfrak{a})$.*

4.7. Retain the above notation and assume that $\mathfrak{a}$ satisfies $A_1 - A_3$ above. Since $\mathfrak{a}$ is balanced, it is natural to guess that $\mathfrak{a}_1^*$ admits a regular element of $\mathfrak{a}^*$ (under the embedding defined by h eigenvalue decomposition). In this case the standard deformation argument (cf [19, 7.8]) applied to the triple of regular elements (x, h^*, y) having eigenvalues $1, 0, -1$ respectively implies that $\mathfrak{a}_{sing}^*$ has codimension at least three. This is the condition of Panyushev and Yakimova [27] which implies that $Y_\eta(\mathfrak{a})$ is maximal Poisson commutative. Of course we already know this to be true since it admits a Weierstrass section. Nevertheless we were unable to prove this needed property of $\mathfrak{a}_1^*$. Conversely it is not obvious (perhaps improbable) that the hypotheses of Panyushev and Yakimova [27] are sufficient to imply that $Y_\eta(\mathfrak{a})$ admits a Weierstrass section.

We saw in 2.8, that a Feigin contraction $\mathfrak{a}$ of a semisimple Lie algebra $\mathfrak{g}$ does admit an adapted pair (h, y) with $y \in \mathfrak{a}_{-1}^*$ regular. However it does not admit an adapted triple. Moreover although $\mathfrak{a}$ is balanced, $\mathfrak{a}_1^*$ does not in general admit a regular element. Indeed the co-adjoint orbit generated by an element of $\mathfrak{a}_1^*$ has dimension at most $2\,\mathrm{rank}\,\mathfrak{g}$, whilst a regular co-adjoint orbit has dimension $\dim\mathfrak{g} - \mathrm{rank}\,\mathfrak{g}$. Yet $\dim\mathfrak{g} - \mathrm{rank}\,\mathfrak{g} > 2\,\mathrm{rank}\,\mathfrak{g}$, except if $\mathfrak{g}$ has only $\mathfrak{sl}(2)$ factors.

4.8. $A_1 - A_3$ hold if $\mathfrak{a}$ is reductive and then the above theorem recovers Tarasov's result. In [20] and [17] several families of examples are given for $\mathfrak{a}$ non-reductive when $A_1 - A_3$ hold. Our favourite example is when $\mathfrak{g}$ is simple of type B_n and $\mathfrak{a}$ is the derived algebra of a parabolic subalgebra whose Levi factor is simple of type B_{n-1}. For $n = 3$ the details of this example are worked out in [16, 8.16]. When $\mathfrak{a} = \mathfrak{sl}(n)$, the above Weierstrass section is known as the Hessenberg of the matrix algebra.

4.9. Let $\mathfrak{a}$ be a finite dimensional Lie algebra. One can ask if there exists a "quantization" of the translated algebra $Y_{h^*}(\mathfrak{a})$. Precisely find a commutative subalgebra of the enveloping algebra $U(\mathfrak{a})$ whose associated graded algebra is $Y_{h^*}(\mathfrak{a})$. Such algebras are particularly interesting from the point of view of representation theory especially in the semisimple case. Here the eigenvectors (under reasonable conditions) have multiplicity one and therefore define a distinguished basis of the module. In theoretical physics the joint eigenvalues of the generators of the maximal commutating subalgebras are thereby interpreted as giving a "complete set of variables".

Recently the above problem was resolved for $\mathfrak{g}$ semisimple through the work of many authors (see [4, Sect. 2] for example and references therein). The construction can be briefly described as follows.

Define the current algebra $\mathfrak{g}_- := \mathfrak{g} \otimes t^{-1}k[t^{-1}]$. (This is not quite standard terminology.) It is an $\mathfrak{g}$ module by action on the first factor. Consider the $\mathfrak{g}$ module map $x \mapsto x \otimes t^{-1}$ of $\mathfrak{g}$ into $\mathfrak{g} \otimes t^{-1}$. Extend this to an $\mathfrak{g}$ algebra map φ of $S(\mathfrak{g})$ into $S(\mathfrak{g}_-)$, that is to say an $\mathfrak{g}$ module map in which the $\mathfrak{g}$ action is by derivations with respect to the algebra structure. The derivation $\partial := \partial/\partial t$ of $S(\mathfrak{g}_-)$ commutes with this action. Thus $\varphi(Y(\mathfrak{g}))$ and its translates under ∂ generate a subalgebra $\mathscr{C}(\mathfrak{g}_-)$ of $S(\mathfrak{g}_-)$ in which $\mathfrak{g}$ acts by zero. (However this does *not* mean that $\mathscr{C}(\mathfrak{g}_-)$ lies in the Poisson centre of $S(\mathfrak{g}_-)$.)

A remarkable fact is that $\mathscr{C}(\mathfrak{g}_-)$ is Poisson commutative [11]. This result is referred to as the Hitchin integrable system [1, Thm. 2.2.4 (i)]. A second even more remarkable fact is that there exists a commutative subalgebra $\mathscr{Z}(\mathfrak{g}_-)$ of $U(\mathfrak{g}_-)$ whose associated graded algebra is $\mathscr{C}(\mathfrak{g}_-)$. In other words the Hitchin integrable system can be quantized [1]. This can be achieved through the centre of the affine Lie algebra at the critical level [10]. Let $z_1, z_2 \in k$ be distinct and non-zero. Let $\mathscr{Z}_{(z_1,z_2)}(\mathfrak{g})$ denote the image of $\mathscr{Z}(\mathfrak{g}_-)$ in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ under the coproduct and evaluation of t is the first (resp. second) factor at z_1 (resp. z_2). It is an $\mathfrak{g}$ invariant commutative subalgebra of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Finally the required quantization of $Y_{h^*}(\mathfrak{g})$ is achieved by taking the associated graded algebra of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ corresponding to the canonical filtration on just the second factor and evaluation at h^*.

The n-fold analogue of $\mathscr{Z}_{(z_1,z_2)}(\mathfrak{g})$ has been of much recent interest (see [4, Sect. 2] for example and references therein). For physicists this is because they contain the Gaudin Hamiltonians. For mathematicians this is because it is conjecturally capable of separating the components of the tensor product of simple finite dimensional $U(\mathfrak{g})$ modules.

The Gaudin Hamiltonians are constructed using an orthonormal basis of $\mathfrak{g}$ and then their commutativity obtains using the fact that the structure constants are totally antisymmetric with respect to this basis. It does not seem that this construction can be extended to the case when $\mathfrak{g}$ is any finite dimensional Lie algebra. However it is not excluded the Hitchin integrable system could be defined for certain non-reductive Lie algebras.

4.10. In view of the above one can ask if it is possible to carry out an analogous quantization of $Y_{h^*}(\mathfrak{a})$ when $\mathfrak{a}$ is *not* reductive.

It is appropriate to choose a (non-reductive) Lie algebra $\mathfrak{a}$ for which $Y_{h^*}(\mathfrak{a})$ admits a Weierstrass section. In 4.8 we saw that examples exist in which $\mathfrak{a}$ is not reductive with nevertheless admits a large Levi factor. Presumably their theory will be no less difficult than the semisimple case which is already formidable. However here we just stick to a simple example to see if any of the semisimple theory can go through.

In this let us consider the first step, namely to show that $\mathscr{C}(\mathfrak{a}_-)$ is Poisson commutative in a given example.

For this we first note the following simple (and of course well-known) formula valid for any vector space V and $v_1, v_2, \dots, v_m \in V$ not necessarily distinct.

$$\frac{(-1)^s}{s!}\partial^s \prod_{j=1}^{m}(v_j \otimes t^{-1}) = \sum \prod_{j=1}^{m}(v_j \otimes t^{-(1+s_j)}),$$

where the sum is over non-negative integer values of $s_j : j = 1, 2 \dots, m$ satisfying $\sum_{j=1}^m s_j = s$.

Again a general fact is that since ∂ is a derivation of the Poisson structure we need only show that $\{a, \partial^r b\} = 0$, for all $a, b \in \varphi(Y(\mathfrak{a})), r \in \mathbb{N}$.

As an example suppose that $\mathfrak{a}$ is the truncated Borel for $\mathfrak{sl}(3)$. This algebra has four basis elements x, y, z, h with the only non-zero commutation relations being

$[x,y]=z,[h,x]=x,[h,y]=-y$. Its centre is the polynomial algebra generated by $z, xy+hz$. Taking h^* to be the linear functional on $\mathfrak{a}$ which is zero on x,y,h and 1 on z, we obtain $Y_{h^*}(\mathfrak{a})$ to be the polynomial algebra generated by z,h,xy. Of course this can already be interpreted as a maximal commutative subalgebra of $U(\mathfrak{a})$ so in that sense there is nothing more to do. Nevertheless we present the following result which is analogous to and having the same level of non-triviality as the commutativity of that part of $\mathscr{C}(\mathfrak{a}_-)$, for $\mathfrak{a}$ semisimple, coming from the Casimir invariant.

Lemma. *Suppose $\mathfrak{a}$ is the truncated Borel in $\mathfrak{sl}(3)$. Then $\mathscr{C}(\mathfrak{a}_-)$ is Poisson commutative.*

Proof. Set $a:=\varphi(xy+hz), b=\partial^r a$. Use $\{\ ,\ \}$ to denote Poisson bracket. We obtain

$$\{a,b\}=\sum_{s=0}^{r}(A_s+B_s+C_s),$$

where

$$\begin{aligned}A_s &= \{(x\otimes t^{-1})(y\otimes t^{-1}),(x\otimes t^{-(s+1)})(y\otimes t^{-(r-s+1)})\}\\ &= (z\otimes t^{-(r-s+2)})[(y\otimes t^{-1})(x\otimes t^{-(s+1)})-(x\otimes t^{-1})(y\otimes t^{-(s+1)})],\end{aligned}$$

$$\begin{aligned}B_s &= \{(x\otimes t^{-1})(y\otimes t^{-1}),(h\otimes t^{-(s+1)})(z\otimes t^{-(r-s+1)})\}\\ &= (z\otimes t^{-(r-s+1)})[-(y\otimes t^{-1})(x\otimes t^{-(s+2)})+(x\otimes t^{-1})(y\otimes t^{-(s+2)})],\\ &= -A_{s+1}, \forall s | r>s\geq 0,\end{aligned}$$

$$\begin{aligned}C_s &= \{(h\otimes t^{-1})(z\otimes t^{-1}),(x\otimes t^{-(s+1)})(y\otimes t^{-(r-s+1)})\}\\ &= (z\otimes t^{-1})[(x\otimes t^{-(s+2)})(y\otimes t^{-(r-s+1)})-(x\otimes t^{-(s+1)})(y\otimes t^{-(r-s+2)})].\end{aligned}$$

It follows that

$$\sum_{s=0}^{r}(A_s+B_s)=A_0+B_r=B_r=-\sum_{s=0}^{r}C_s.$$

The cases when $a=\varphi(z)$ or $b=\partial^r\varphi(z)$, are trivial. Hence the assertion of the lemma. □

Acknowledgements I would like to thank Vladimir Dobrev for the kind invitation to speak at the 9-th International Workshop “Lie Theory and Its Applications in Physics” held during 20–26 June 2011 in Varna, Bulgaria. I would also like to thank my former student Anna Melnikov who served as an advisor in preparing this manuscript.

References

1. Beilinson, A., Drinfeld, V.: Quantization of Hitchin's integrable system and Hecke eigensheaves (unpublished). Available from 2003 as http://www.ma.utexas.edu/~benzvi/BD
2. Bolsinov, A.V.: Commutative families of functions related to consistent Poisson brackets. Acta Appl. Math. **24**(3), 253–274 (1991)
3. Brown, J., Brundan, J.: Elementary invariants for centralizers of nilpotent matrices. J. Aust. Math. Soc. **86**(1), 1–15 (2009)
4. Chervov, A., Falqui, G., Rybnikov, L.: Limits of Gaudin algebras, quantization of bending flows, Jucys-Murphy elements and Gelfand-Tsetlin bases. Lett. Math. Phys. **91**(2), 129–150 (2010)
5. Dixmier, J.: Sur les algèbres enveloppantes de $\mathfrak{sl}(n,\mathbb{C})$ et $\mathfrak{af}(n,\mathbb{C})$. Bull. Sci. Math. (2) **100**(1), 57–95 (1976)
6. Dixmier, J.: Algèbres enveloppantes (French) [Enveloping algebras] Reprint of the 1974 original. Les Grands Classiques Gauthier-Villars [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Paris (1996)
7. Fauquant-Millet, F., Joseph, A.: Semi-centre de l'algèbre enveloppante d'une sous-algèbre parabolique d'une algèbre de Lie semi-simple. Ann. Sci. École Norm. Sup. (4) **38**(2), 155–191 (2005)
8. Fauquant-Millet, F., Joseph, A.: La somme des faux degrés—un mystère en théorie des invariants (French) [The sum of the false degrees - a mystery in the theory of invariants]. Adv. Math. **217**(4), 1476–1520 (2008)
9. Fauquant-Millet, F., Joseph, A.: Slices for biparabolics of index 1. Transform. Groups **16**(4), 1081–1113 (2011)
10. Feigin, B., Frenkel, E., Reshetikhin, N.: Gaudin model, Bethe ansatz and critical level. Comm. Math. Phys. **166**(1), 27–62 (1994)
11. Hitchin, N.: Stable bundles and integrable systems. Duke Math. J. **54**(1), 91–114 (1987)
12. Joseph, A.: A preparation theorem for the prime spectrum of a semisimple Lie algebra. J. Algebra **48**(2), 241–289 (1977)
13. Joseph, A.: Second commutant theorems in enveloping algebras. Am. J. Math. **99**(6), 1167–1192 (1977)
14. Joseph, A.: A slice theorem for truncated parabolics of index one and the Bezout equation. Bull. Sci. Math. **131**(3), 276–290 (2007)
15. Joseph, A.: On semi-invariants and index for biparabolic (seaweed) algebras II. J. Algebra **312**(1), 158–193 (2007)
16. Joseph, A.: Parabolic actions in type A and their eigenslices. Transform. Groups **12**(3), 515–547 (2007)
17. Joseph, A.: Compatible adapted pairs and a common slice theorem for some centralizers. Transform. Groups **13**(3–4), 637–669 (2008)
18. Joseph, A.: Slices for biparabolic coadjoint actions in type A. J. Algebra **319**(12), 5060–5100 (2008)
19. Joseph, A.: An Algebraic Slice in the Coadjoint Space of the Borel and the Coxeter Element. Adv. Math. **227**, 522–585 (2011)
20. Joseph, A., Lamprou, P.: Maximal Poisson commutative subalgebras for truncated parabolic subalgebras of maximal index in $\mathfrak{sl}(n)$. Transform. Groups **12**(3), 549–571 (2007)
21. Joseph, A., Shafrir, D.: Polynomiality of invariants, unimodularity and adapted pairs. Transform. Groups **15**(4), 851–882 (2010)
22. Kostant, B.: Lie group representations on polynomial rings. Am. J. Math. **85**, 327–404 (1963)
23. Krause, G.R., Lenagan, T.H.: Growth of Algebras and Gelfand-Kirillov Dimension. Research Notes in Mathematics, vol. 116. Pitman (Advanced Publishing Program), Boston (1985)
24. Mishchenko, A.S., Fomenko, A.T.: Euler equation on finite-dimensional Lie groups. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **42**(2), 396–415, 471 (1978)

25. Ooms, A., Van den Bergh, M.: A degree inequality for Lie algebras with a regular Poisson semi-center. J. Algebra **323**(2), 305–322 (2010)
26. Panyushev, D., Premet, A., Yakimova, O.: On symmetric invariants of centralizers in reductive Lie algebras. J. Algebra **313**(1), 343–391 (2007)
27. Panyushev, D.I., Yakimova, O.S.: The argument shift method and maximal commutative subalgebras of Poisson algebras. Math. Res. Lett. **15**(2), 239–249 (2008)
28. Panyushev, D.I., Yakimova, O.S.: A remarkable contraction of semisimple Lie algebras. arXiv:1107.0702
29. Popov, V.L.: Sections in invariant theory. In: The Sophus Lie Memorial Conference, Oslo, 1992, pp. 315–361. Scandinavian University Press, Oslo (1994)
30. Popov, V.L.: Some subgroups of the Cremona groups. arXiv:1110.2410
31. Popov, V.L., Vinberg, E.B.: Invariant Theory. In: Parshin, A.N., Shafarevich, I.R. (eds.) Algebraic Geometry IV. Encyclopaedia of Mathematical Sciences, vol. 55. Springer, Berlin (1994)
32. Rosenlicht, M.: Another proof of a theorem on rational cross sections. Pac. J. Math. **20**, 129–133 (1967)
33. Sadètov, S.T.: A proof of the Mishchenko-Fomenko conjecture (1981). (Russian) Dokl. Akad. Nauk **397**(6), 751–754 (2004)
34. Tarasov, A.A.: The maximality of some commutative subalgebras in Poisson algebras of semisimple Lie algebras. (Russian) Uspekhi Mat. Nauk **57**5(347), 165–166 (2002); translation in Russian Math. Surv. **57**(5), 1013–1014 (2002)

From Palev's Study of Wigner Quantum Systems to New Results on Sums of Schur Functions

Ronald C. King

Dedicated to Tchavdar Palev on the occasion of his 75th birthday.

Abstract Palev's study of Wigner quantum oscillators led directly to the construction of unitary irreducible Fock space representations $V(p)$ of the Lie superalgebras $osp(1|2n)$. In the hands of Van der Jeugt, Lievens and Stoilova this yielded for all positive integers n and p an explicit formula for the corresponding character $\mathrm{ch}\,V(p)$. It was expressed as a sum of Schur functions specified by partitions of length no greater than p. They conjectured that this infinite sum could in turn be expressed as a quotient of certain signed sums of Schur functions. The validity of this conjecture is first established by relating it to a known result of Macdonald for the sum of Schur functions specified by partitions whose parts are no greater than p. It is then shown that the origin of these distinct restrictions on the size and number of parts of partitions can be traced back to the existence of certain Howe dual pairs of groups associated with the spin representation of $O(2n+1)$ and the metaplectic representation of $Sp(2n)$, respectively.

1 Introduction

My interactions with Tchavdar Palev started when I met him for the first time at the "Symmetries in Science III" conference in Bregenz, Austria, 1988. We met again at successive "International Colloquia on Group Theoretical Methods in Physics" in Moscow, 1990, Salamanca, 1993 and Goslar 1996. Then thanks to meeting Neli Stoilova in Brisbane in 2000, the three of us jointly submitted an application to the

R.C. King (✉)
School of Mathematics, University of Southampton, Southampton SO17 1BJ, England
e-mail: r.c.king@soton.ac.uk

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_5,

Royal Society for a UK/Bulgaria Joint Project Grant. This allowed two exchange visits in each direction during the period 2001–2003: Tchavdar Palev and Neli Stoilova to Southampton in the UK and myself to Sofia in Bulgaria.

To date I have had only two further meetings with Tchavdar Palev, both in Varna, Bulgaria, at the International Symposium "Quantum Theory and Symmetries IV" and the concurrent International Workshop "Lie Theory and its Applications in Physics VI" in 2005, together with the present meeting now in 2011. He introduced me to Wigner Quantum Systems, in particular to n-particle $3D$ Wigner quantum oscillators. In doing so he taught me about paraboson operators, non-commutative geometry, and Fock space unirreps of Lie superalgebras. Together with Neli Stoilova, he helped renew my interaction with Joris Van der Jeugt, with a concrete outcome by way of joint papers [9, 10] and a number of conference publications. My impressions of Tchavdar Palev were that he was immensely stimulating, with a great breadth of knowledge and expertise in group theoretical methods and their application to physics, characterised by great attention to detail. But perhaps above all he was great fun to work with, and it is both a delight and a privilege to contribute here to the celebration of his 75th birthday.

One totally unexpected outcome of this interaction with Tchavdar Palev was to receive a query from Joris Van der Jeugt, working with Stijn Lievens and Neli Stoilova in Gent. They had been studying unirreps of the Lie superalgebra $osp(1|2n)$ built using Ganchev and Palev's parabosons [4]. They had identified Fock space modules $\overline{V}(p)$ for any $p \in \mathbb{N}$, and found that $\overline{V}(p)$ is irreducible for $p \geq n$, but reducible for $p < n$. They went further, and constructed quite explicitly for all $p \in \mathbb{N}$ the unitary irreducible infinite-dimensional representations $V(p) = \overline{V}(p)/M(p)$, where $M(p)$ is the maximal submodule of $\overline{V}(p)$. They also calculated the characters of both $\overline{V}(p)$ and $V(p)$. In particular they established [12]:

Proposition 1. *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. *Then for all positive integers* p

$$\mathrm{ch}\, V(p) = (x_1 x_2 \cdots x_n)^{p/2} \sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(\mathbf{x}), \tag{1}$$

where the sum is over all partitions λ *and* $s_\lambda(\mathbf{x})$ *is the corresponding Schur function, with* $\ell(\lambda)$ *denoting the length of* λ.

In addition they made the following [12]:

Conjecture 1.

$$\sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(\mathbf{x}) = \frac{\sum_\eta \; (-1)^{c_\eta} \; s_\eta(\mathbf{x})}{\prod_{1 \leq i \leq n} (1 - x_i) \; \prod_{1 \leq j < k \leq n} \; (1 - x_i x_j)}, \tag{2}$$

with the sum taken over all those partitions η *which in Frobenius notation take the form*

$$\eta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1+p & a_2+p & \cdots & a_r+p \end{pmatrix} \tag{3}$$

with $c_\eta = (|\eta| - rp + r)/2$*, where* $|\eta|$ *denotes the weight of* η.

It is this conjecture that Joris Van der Jeugt sent me along with a query as to whether or not it was a known result, and if not could I supply a proof. At that time I was visiting Angèle Hamel in Waterloo, Canada, and she immediately reminded me of the following result due to Macdonald [14]:

Theorem 1. *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$*. Then for all* $p \in \mathbb{N}$

$$\sum_{\lambda:\ell(\lambda')\le p} s_\lambda(\mathbf{x}) = \frac{\left| x_i^{p+2n-j} - x_i^{j-1} \right|}{\left| x_i^{2n-j} - x_i^{j-1} \right|} \tag{4}$$

where λ' *is the partition conjugate to* λ*, and the determinants are both* $n \times n$*, with their* (i,j)*th elements displayed.*

The strategy was then to try to recast Macdonald's formula (4) in terms of Schur functions and to use conjugacy to recover the Lievens, Stoilova and Van der Jeugt conjecture (2). In addition it would be of interest to identify the origin of both the row length restriction $\ell(\lambda') \le p$ in (4) and the column length restriction $\ell(\lambda) \le p$ in (2).

In the next section we present some preliminaries on partitions, Young diagrams, Frobenius notation, Schur functions and Schur function series, along with some determinantal formulae for these series due to Littlewood [13]. These allow us in Sect. 3 to reformulate Macdonald's result (4) in a manner that in Sect. 4 leads, by means of a lemma generalising Littlewood's determinantal formulae and a simple conjugacy argument, to a proof of the validity of the Lievens, Stoilova and Van der Jeugt conjecture(2).

2 Preliminaries

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of weight $|\lambda|$ and length $\ell(\lambda) \le n$ is a sequence of non-negative integers satisfying the condition $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$, such that their sum is $|\lambda|$, and $\lambda_i > 0$ if and only if $i \le \ell(\lambda)$.

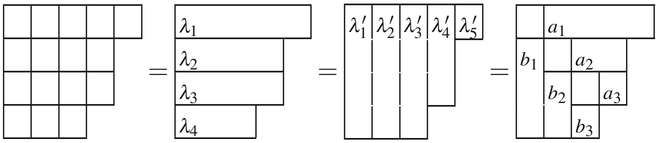
Each such partition specifies a corresponding Young or Ferrers diagram, F^λ, that consists of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left-adjusted rows of lengths λ_i for $i = 1, 2, \ldots \ell(\lambda)$. If the lengths of the columns of F^λ are λ'_j for $j = 1, 2, \ldots \lambda_1$, then these lengths define the partition λ' that is conjugate to λ.

If F^{λ} has r boxes on the main diagonal, with arm and leg lengths a_k and b_k for $k = 1,2,\ldots,r$, then λ is said to have rank $r(\lambda) = r$ and in Frobenius notation we write

$$\lambda = \begin{pmatrix} a_1\ a_2 \cdots a_r \\ b_1\ b_2 \cdots b_r \end{pmatrix} \quad \text{with conjugate} \quad \lambda' = \begin{pmatrix} b_1\ b_2 \cdots b_r \\ a_1\ a_2 \cdots a_r \end{pmatrix}, \tag{5}$$

where $a_1 > a_2 > \cdots > a_r \geq 0$ and $b_1 > b_2 > \cdots > b_r \geq 0$.

By way of illustration, we have



It is convenient to make use of the notation $\lambda \subseteq \mu$ if F^{λ} is contained wholly within F^{μ}, that is to say $\lambda_i \leq \mu_i$ for all i, or equivalently $\lambda'_j \leq \mu'_j$ for all j, and to define $\mu \pm \lambda = (\mu_1 \pm \lambda_1, \mu_2 \pm \lambda_2, \ldots, \mu_n \pm \lambda_n)$.

There are a number of special families of partitions. First let $\mathscr{P}$ denote the set of all partitions. This includes the zero partition $\lambda = 0 = (0,0,\ldots,0)$ which has weight, length and rank zero, i.e. $|0| = \ell(0) = r(0) = 0$. Then for any integer t let

$$\mathscr{P}_t = \left\{ \lambda = \begin{pmatrix} a_1\ a_2 \cdots a_r \\ b_1\ b_2 \cdots b_r \end{pmatrix} \in \mathscr{P} \,\middle|\, a_k - b_k = t \begin{array}{l} \text{for } k = 1,2,\ldots,r \\ \text{and } r = 0,1,\ldots \end{array} \right\}, \tag{6}$$

It is to be noted that the zero partition belongs to $\mathscr{P}_t$ for all integers t. The case $t = 0$ yields

$$\mathscr{P}_0 = \left\{ \lambda = \begin{pmatrix} a_1\ a_2 \cdots a_r \\ a_1\ a_2 \cdots a_r \end{pmatrix} \in \mathscr{P} \,\middle|\, r = 0,1,\ldots \right\}, \tag{7}$$

that is the set of all self-conjugate partitions $\{\lambda \in \mathscr{P} \,|\, \lambda = \lambda'\}$. More generally $\mathscr{P}_{-t}$ is the set of partitions conjugate to those in $\mathscr{P}_t$ for all integers t.

Partitions are required for the introduction of Schur functions, and the above sets of partitions arise naturally in what follows in dealing with certain infinite series of Schur functions.

Definition 1. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a sequence of n indeterminates, and let λ be a partition of length $\ell(\lambda) \leq n$. Then the Schur function $s_\lambda(\mathbf{x})$ is defined by:

$$s_\lambda(\mathbf{x}) = \left| x_i^{\lambda_j + n - j} \right| \Big/ \left| x_i^{n-j} \right|, \tag{8}$$

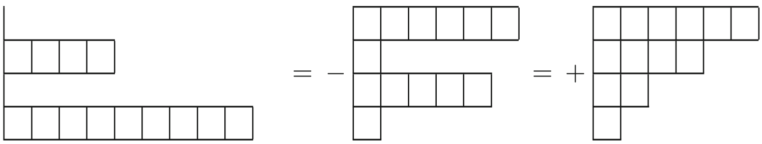
where the determinants are $n \times n$ and only the (i,j)th elements have been displayed. The denominator is just the Vandermonde determinant for which $\left| x_i^{n-j} \right| = \prod_{1 \leq i < j \leq n} (x_i - x_j)$.

These Schur functions form a $\mathbb{Z}$-basis of Λ_n, the ring of polynomial symmetric functions of $x_1,\dots,x_n$. It is important to note that the definition (8) is still meaningful if the partition λ is replaced by any sequence $\kappa = (\kappa_1,\kappa_2,\dots,\kappa_n)$ of n integers κ_i for $i = 1,2,\dots,n$. However, certain modification rules [13] may then be invoked to show that either $s_\kappa(\mathbf{x}) = 0$ or $s_\kappa(\mathbf{x}) = \pm(x_1x_2\cdots x_n)^k s_\lambda(\mathbf{x})$ for some partition λ and some integer k. The first case arises if $\kappa_i - i = \kappa_j - j$ for any $i \neq j$ as can be seen by noting that two columns of the determinant in the numerator of (8) will coincide if λ is replaced by any such κ. Quite generally, by permuting columns of this determinant $s_\kappa(\mathbf{x}) = -s_\mu(\mathbf{x})$ with $\mu = (\kappa_1,\dots,\kappa_{j+1}-1,\kappa_j+1,\dots,\kappa_n)$ for any j, and iterating this gives $s_\kappa(\mathbf{x}) = (-1)^{j-1}s_\nu(\mathbf{x})$ with $\nu = (\kappa_{j+1}-j,\kappa_1+1,\dots,\kappa_j+1,\kappa_{j+2}\dots,\kappa_n)$.

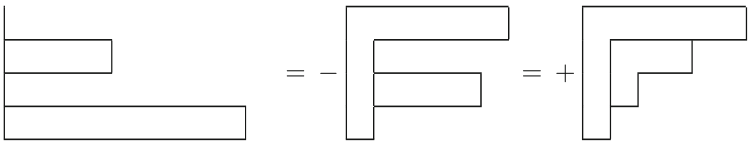
An example relevant to what follows is provided by the case $n = 4$ and $\kappa = (0,4,0,9)$, for which $s_\kappa(\mathbf{x}) = +s_\lambda(\mathbf{x})$ with $\lambda = (6,4,2,1)$. This can be seen from the identity

$$s_{(0409)}(\mathbf{x}) = \frac{\left|\, x_i^3\ x_i^6\ x_i\ x_i^9 \,\right|}{\left|\, x_i^3\ x_i^2\ x_i\ 1 \,\right|} = \frac{\left|\, x_i^9\ x_i^6\ x_i^3\ x_i \,\right|}{\left|\, x_i^3\ x_i^2\ x_i\ 1 \,\right|} = s_{(6421)}(\mathbf{x}) \tag{9}$$

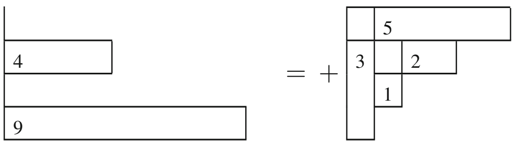
where just the ith row of each determinant has been displayed. Alternatively, one can proceed iteratively using the previous identities which give $s_{(0409)}(\mathbf{x}) = -s_{(6151)}(\mathbf{x}) = +s_{(6421)}(\mathbf{x})$. Diagrammatically, these modifications are illustrated by



These modifications can also be illustrated in the following manner



which corresponds to



Hence we have $s_\kappa(\mathbf{x}) = s_{(0409)}(\mathbf{x}) = s_{(6421)}(\mathbf{x}) = s_\lambda(\mathbf{x})$ with $\lambda = (6,4,2,1) = \begin{pmatrix} 5\ 2 \\ 3\ 1 \end{pmatrix}$ in Frobenius notation.

Finally, by way of preliminaries it should be noted that Littlewood [13] obtained the following generating functions for two important Schur function series:

$$\prod_{1\leq i\leq n}(1-x_i)^{-1}\prod_{1\leq j<k\leq n}(1-x_jx_k)^{-1}=\sum_{\lambda\in\mathscr{P}}s_\lambda(\mathbf{x});\tag{10}$$

$$\prod_{1\leq i\leq n}(1-x_i)\prod_{1\leq j<k\leq n}(1-x_jx_k)=\sum_{\lambda\in\mathscr{P}_0}(-1)^{(|\lambda|+r(\lambda))/2}s_\lambda(\mathbf{x}),\tag{11}$$

with $s_\lambda(x)=0$ if $\ell(\lambda)>n$. As can be seen from the expressions on the left, these two series are mutually inverse to one another. The first is infinite. However the second is finite since there are only a finite number of partitions λ for which $\ell(\lambda)=\ell(\lambda')\leq n$, and it forms the denominator of the formula (2).

Furthermore Littlewood derived the following determinantal expansions in terms of Schur function series [13]:

$$\left|x_i^{n-j}-x_i^{n+j-1}\right|\Big/\left|x_i^{n-j}\right|=\sum_{\lambda\in\mathscr{P}_0}(-1)^{(|\lambda|+r(\lambda))/2}\,s_\lambda(\mathbf{x});\tag{12}$$

$$\left|x_i^{n-j}-x_i^{n+j}\right|\Big/\left|x_i^{n-j}\right|=\sum_{\lambda\in\mathscr{P}_1}(-1)^{|\lambda|/2}\,s_\lambda(\mathbf{x});\tag{13}$$

$$\left|x_i^{n-j}+\chi_{j>1}\,x_i^{n+j-2}\right|\Big/\left|x_i^{n-j}\right|=\sum_{\lambda\in\mathscr{P}_{-1}}(-1)^{|\lambda|/2}\,s_\lambda(\mathbf{x}),\tag{14}$$

where as usual the determinants are all $n\times n$ with $i,j=1,2,\ldots,n$ and, for any proposition P, the truth symbol χ_P is such that $\chi_P=1$ if P is true, and 0 if P is false.

3 Macdonald's Formula

We are now in a position to reformulate Macdonald's formula (4). First we reorder columns by mapping $j\to n-j+1$, then change signs of all elements, and finally divide numerator and denominator by the Vandermonde determinant to give

$$\sum_{\lambda\in\mathscr{P}:\lambda\subseteq(p^n)}s_\lambda(\mathbf{x})=\frac{\left|x_i^{p+2n-j}-x_i^{j-1}\right|}{\left|x_i^{2n-j}-x_i^{j-1}\right|}=\frac{\left|x_i^{n-j}-x_i^{n+p+j-1}\right|\Big/\left|x_i^{n-j}\right|}{\left|x_i^{n-j}-x_i^{n+j-1}\right|\Big/\left|x_i^{n-j}\right|},\tag{15}$$

where without loss of generality we may replace $\lambda\supseteq(p^n)$ by $\ell(\lambda')=\lambda_1\leq p$ since $s_\lambda(\mathbf{x})=0$ if $\ell(\lambda)=\lambda_1'>n$.

To deal with both the numerator and the denominator of this expression it is helpful to exploit the following identity:

Lemma 1. *For all $n \geq 1$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$*

$$\frac{\left| x_i^{n-j} + q\,\chi_{j>-t}\; x_i^{n+t+j-1} \right|}{\left| x_i^{n-j} \right|} = \sum_{\lambda \in \mathscr{P}_t} (-1)^{(|\lambda| - r(\lambda)(t+1))/2}\, q^{r(\lambda)}\, s_\lambda(\mathbf{x}), \qquad (16)$$

where t is any integer, q is arbitrary, and the determinants are all $n \times n$, so that $i, j = 1, 2, \ldots, n$. The special cases $q = -1$, $t = 0$, $q = -1$, $t = 1$ and $q = 1$, $t = -1$ correspond to Littlewood's previous formulae (12), (13) and (14), respectively.

Proof. First the left hand side of (16) can be rewritten and expanded in terms of Schur functions as follows:

$$\frac{\left| x_i^{n-j} + q\,\chi_{j>-t}\; x_i^{(2j-1+t)+n-j} \right|}{\left| x_i^{n-j} \right|} = \sum_{r=0}^{n} \sum_{\kappa} q^r\, s_\kappa(\mathbf{x}) \qquad (17)$$

where κ is summed over all sequences of the form

$$(0, \ldots, 0, 2j_r - 1 + t, 0, \ldots, 0, 2j_2 - 1 + t, 0, \ldots, 0, 2j_1 - 1 + t, 0, \ldots, 0), \qquad (18)$$

where $n \geq j_1 > j_2 > \cdots > j_r \geq 1 - \chi_{t<0} t$ and the non-zero elements appear in positions $j_r, \ldots, j_2, j_1$. Then, by rearranging the columns of the determinant in the numerator of

$$s_\kappa(\mathbf{x}) = \left| x_i^{\kappa_j + n - j} \right| \Big/ \left| x_i^{n-j} \right|, \qquad (19)$$

one obtains, precisely, as in the illustrative example of the previous section,

$$s_\kappa(\mathbf{x}) = (-1)^{(j_r - 1) + \cdots + (j_2 - 1) + (j_1 - 1)}\, s_\lambda(\mathbf{x}), \qquad (20)$$

where

$$\lambda = \begin{pmatrix} j_1 - 1 + t & j_2 - 1 + t & \cdots & j_r - 1 + t \\ j_1 - 1 & j_2 - 1 & \cdots & j_r - 1 \end{pmatrix} \in \mathscr{P}_t, \qquad (21)$$

and $r = r(\lambda)$.

If we now set $b_k = j_k - 1$ for $k = 1, 2, \ldots, r$, then

$$\lambda = \begin{pmatrix} b_1 + t & b_2 + t & \cdots & b_r + t \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in \mathscr{P}_t, \qquad (22)$$

where now $n - 1 \geq b_1 > b_2 \cdots > b_r \geq -\chi_{t<0} t$, This is precisely the condition required to give all $\lambda \in \mathscr{P}_t$ of length $\ell(\lambda) \leq n$ for any integer t: positive, zero or negative. Finally, $|\lambda| = 2((j_1-1)+(j_2-1)+\cdots+(j_r-1))+r(t+1)$ so that $(-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)} = (-1)^{(|\lambda|-r(t+1))/2}$, as required to complete the proof of the Lemma. □

If we apply this Lemma to the numerator of (15) in the case $q=-1, t=p$ and to the denominator in the case $q=-1, t=0$, we find

$$\sum_{\lambda\in\mathscr{P}:\ell(\lambda')\leq p} s_\lambda(\mathbf{x}) = \frac{\sum_{\mu\in\mathscr{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2} s_\mu(\mathbf{x})}{\sum_{\varepsilon\in\mathscr{P}_0} (-1)^{[|\varepsilon|+r(\varepsilon)]/2} s_\varepsilon(\mathbf{x})}. \tag{23}$$

This reformulation of Macdonald's formula for the generating function of the sum of Schur functions specified by row length restricted partitions, is just what is required to derive, by means of conjugacy, the conjecture formulated by Lievens, Stoilova and Van der Jeugt.

4 Derivation of the LSV Result

There exists an automorphism ω on the ring of symmetric functions which acts as a conjugacy involution. On Schur functions $\omega : s_\lambda(\mathbf{x}) \mapsto s_{\lambda'}(\mathbf{x})$ for all $\lambda \in \mathscr{P}$. Since $|\lambda'| = |\lambda|$, $r(\lambda') = r(\lambda)$ and $\lambda \in \mathscr{P}_t$ implies that $\lambda' \in \mathscr{P}_{-t}$, it follows that

$$\begin{aligned}\omega : &\sum_{\lambda\in\mathscr{P}_t} (-1)^{(|\lambda|-r(\lambda)(t+1))/2}\, q^{r(\lambda)}\, s_\lambda(\mathbf{x}) \\ &\mapsto \sum_{\lambda'\in\mathscr{P}_{-t}} (-1)^{(|\lambda'|-r(\lambda')(t+1))/2}\, q^{r(\lambda')}\, s_{\lambda'}(\mathbf{x}) \\ &= \sum_{\lambda\in\mathscr{P}_{-t}} (-1)^{(|\lambda|-r(\lambda)(t+1))/2}\, q^{r(\lambda)}\, s_\lambda(\mathbf{x}),\end{aligned} \tag{24}$$

where, in the last step the summation partition label has without loss of generality been changed from λ' back to λ.

Since

$$\omega : \sum_{\lambda\in\mathscr{P}:\ell(\lambda')\leq p} s_\lambda(\mathbf{x}) = \sum_{\lambda\in\mathscr{P}:\ell(\lambda')\leq p} s_{\lambda'}(\mathbf{x}) = \sum_{\lambda\in\mathscr{P}:\ell(\lambda)\leq p} s_\lambda(\mathbf{x}), \tag{25}$$

it follows, by exploiting (24) in the cases $q=-1, t=p$ and $q=-1, t=0$, that the application of ω to (23) gives

$$\sum_{\lambda\in\mathscr{P}:\ell(\lambda)\leq p} s_\lambda(\mathbf{x}) = \frac{\sum_{\mu\in\mathscr{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2} s_\mu(\mathbf{x})}{\sum_{\varepsilon\in\mathscr{P}_0} (-1)^{[|\varepsilon|+r(\varepsilon)]/2} s_\varepsilon(\mathbf{x})}. \tag{26}$$

Thanks to (11) this yields

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(\mathbf{x}) = \frac{\sum_{\mu\in\mathscr{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2} s_\mu(\mathbf{x})}{\prod_{1\leq i\leq n}(1-x_i) \prod_{1\leq j<k\leq n}(1-x_j x_k)}, \tag{27}$$

which is precisely the conjectured result (2) of Lievens, Stoilova and Van der Jeugt.

It should be pointed out that the formulae (23) and (26) apply equally well to all $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ without restriction on the integer n. They thus apply in the ring Λ of symmetric functions of infinitely many indeterminates $\mathbf{x}=(x_1,x_2,\ldots)$. Taking advantage of the fact that the right hand sides of (10) and (11) remain mutually inverse for all $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, again without restriction on n, it follows that (23) and (26) can be rewritten in the form:

$$\sum_{\lambda\in\mathscr{P}:\ell(\lambda')\leq p} s_\lambda(\mathbf{x}) = \sum_{\lambda\in\mathscr{P}} s_\lambda(\mathbf{x}) \sum_{\mu\in\mathscr{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2} s_\mu(\mathbf{x}); \tag{28}$$

$$\sum_{\lambda\in\mathscr{P}:\ell(\lambda)\leq p} s_\lambda(\mathbf{x}) = \sum_{\lambda\in\mathscr{P}} s_\lambda(\mathbf{x}) \sum_{\mu\in\mathscr{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2} s_\mu(\mathbf{x}). \tag{29}$$

Since $\ell(\lambda')=\lambda_1$ and $\ell(\lambda)=\lambda_1'$, these are essentially inclusion-exclusion formulae for the sums of row-length restricted and column length restricted Schur functions, each expressed as a multiplicative amendment to the sum of all unrestricted Schur functions.

5 Classical Group Characters

We have recast the determinants in Macdonald's formula as signed sums of certain Schur functions, and then used conjugacy to prove the Lievens, Stoilova and Van der Jeugt conjecture. We have not explained why the various determinantal expressions lead to row or column length restrictions. To do this we may exploit the fact that they define characters of particular representations of classical groups, and then look for an alternative way of evaluating these characters through the use of Howe dual pairs of groups.

First it should be noted that for any m and any partition λ of length $\ell(\lambda)\leq n$ the character of the irreducible representation, $V_{GL(n)}^{m^n+\lambda}$, of highest weight $(m+\lambda_1, m+\lambda_2.\ldots,m+\lambda_n)$ is given by

$$\mathrm{ch}\, V_{GL(n)}^{m^n+\lambda}(\mathbf{x}) = \frac{\left|x_i^{m+\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|} = (x_1x_2\cdots x_n)^m s_\lambda(\mathbf{x}), \tag{30}$$

where the components of $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ are the eigenvalues of any group element of $GL(n)$.

Similarly, the character of the irreducible representation, $V^{\mu}_{SO(2n+1)}$, of highest weight μ, with either $\mu = \lambda$ or $\mu = \Delta$; $\lambda = (\frac{1}{2}^n + \lambda)$ for some partition λ of length $\ell(\lambda) \leq n$, is given by

$$\mathrm{ch}\, V^{\mu}_{SO(2n+1)}(\mathbf{x}, \overline{\mathbf{x}}, 1) = \frac{\left| x_i^{\mu_j+n-j+\frac{1}{2}} - \overline{x}_i^{\mu_j+n-j+\frac{1}{2}} \right|}{\left| x_i^{n-j+\frac{1}{2}} - \overline{x}_i^{n-j+\frac{1}{2}} \right|}, \tag{31}$$

where the components of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$, with $\overline{\mathbf{x}}_i = x_i^{-1}$ for $i = 1, 2, \ldots, n$, together with 1 are the $(2n+1)$ eigenvalues of any group element of $SO(2n+1)$.

It was in terms of these characters that Bracken and Green [2] first wrote down the following formula for the row length restricted sum of Schur functions:

Theorem 2. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ with $\overline{x}_k = x_k^{-1}$ for $k = 1, 2, \ldots, n$, and let p be any positive integer. Then*

$$\sum_{\lambda \in \mathscr{P}:\ell(\lambda') \leq p} s_\lambda(\mathbf{x}) = (x_1 x_2 \cdots x_n)^{p/2}\, \mathrm{ch}\, V^{(p/2)^n}_{SO(2n+1)}(\mathbf{x}, \overline{\mathbf{x}}, 1). \tag{32}$$

This same result was derived more recently by Stoilova and Van der Jeugt [18] through their explicit construction of unitary irreducible parafermion Fock space representations $W(p)$, for which they obtained the following character formula:

Theorem 3. *Let p be a positive integer. Then the character of the parafermion Fock space module $W(p)$ is given by*

$$\mathrm{ch}\, W(p) = (x_1 x_2 \cdots x_n)^{-p/2} \sum_{\lambda \in \mathscr{P}:\ell(\lambda') \leq p} s_\lambda(\mathbf{x}). \tag{33}$$

They pointed out that this coincided with the Bracken and Green formula (32), since it was well known that n pairs of parafermion operators generate the Lie algebra of $SO(2n+1)$, and with their construction they were able to identify $W(p)$ with $V^{(p/2)^n}_{SO(2n+1)}$.

Both results, Theorems 2 and 3, are then corollaries to Macdonald's Theorem 1, and *vice versa*, since we may rewrite and manipulate the determinants in (4) as follows:

$$\sum_{\lambda \in \mathscr{P}:\ell(\lambda') \leq p} s_\lambda(\mathbf{x}) = \frac{\left| x_i^{p+2n-j} - x_i^{j-1} \right|}{\left| x_i^{2n-j} - x_i^{j-1} \right|}$$

$$= \frac{\prod_{i=1}^{n} x_i^{p/2+n-1/2} \left| x_i^{p/2+n-j+1/2} - \overline{x}_i^{p/2+n-j+1/2} \right|}{\prod_{i=1}^{n} x_i^{n-1/2} \left| x_i^{n-j+1/2} - \overline{x}_i^{n-j+1/2} \right|}$$

$$= \prod_{i=}^{n} x_i^{p/2} \operatorname{ch} V_{SO(2n+1)}^{(p/2)^n}(\mathbf{x}, \overline{\mathbf{x}}, 1).$$

by virtue of the character formula (31).

6 Howe Dual Pairs

To proceed to an independent derivation of the Bracken and Green formula (32) we want to exploit the notion of Howe dual pairs of groups [5–7, 17]:

Definition 2. Let groups G and H act on a linear vector space V in such a way that their actions mutually commute and centralize one another. As a representation of $G \times H$, let

$$V = \bigoplus_{k \in K} V_G^{\lambda(k)} \otimes V_H^{\mu(k)} \tag{34}$$

as k varies over some index set K, where $V_G^{\lambda(k)}$ and $V_H^{\mu(k)}$ are irreducible representations of G and H, respectively, with $V_G^{\lambda(k)}$ and $V_H^{\mu(k)}$ each varying without repetition. In such a case we say that G and H form a Howe dual pair with respect to V.

If V carries a representation of a group $F \supseteq G \times H$, then on restriction to the subgroup $G \times H$ (34) implies

$$\operatorname{ch} V{\downarrow}^F_{G\times H} = \sum_{k \in K} \operatorname{ch} V_G^{\lambda(k)} \operatorname{ch} V_H^{\mu(k)}. \tag{35}$$

The great merit of this formula is that the character $\operatorname{ch} V_G^{\lambda(k)}$ is just the coefficient of $\operatorname{ch} V_H^{\mu(k)}$ in any formula we can devise for $\operatorname{ch} V^F_{G\times H}$.

In particular this is the case if $V = V^{\Delta}_{O(2np+p)}$, the spin representation of the group $O(2np+p)$ for some positive integers n and p, and this group is restricted to its subgroup $SO(2n+1) \times O(p)$. In this case [5, 15, 16]

$$\operatorname{ch} V^{\Delta}_{O(2np+p)} = \sum_{\lambda \in \mathscr{P} : \lambda \subseteq (p/2)^n} \operatorname{ch} V_{SO(2n+1)}^{(p/2)^n - \lambda_{\text{rev}}} \operatorname{ch} V_{O(p)}^{\Delta;\lambda'} \tag{36}$$

where if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ then $\lambda_{\text{rev}} = (\lambda_n, \ldots, \lambda_2, \lambda_1)$. Our required character, $\operatorname{ch} V_{SO(2n+1)}^{(p/2)^n}$ is therefore the coefficient of $\operatorname{ch} V^{\Delta}_{O(p)}$ in the expansion of $\operatorname{ch} V^{\Delta}_{O(2np+p)}$.

To effect this expansion one evaluates step by step the restriction of $\mathrm{ch}\, V^{\Delta}_{O(2np+p)}$ with respect to the group-subgroup chain

$$\begin{aligned} O(2np+p) &\supset O(2np)\times O(p) \;\supset\; GL(np)\times O(p) \\ &\supset GL(n)\times GL(p)\times O(p) \;\supset\; GL(n)\times O(p)\times O(p) \\ &\supset GL(n)\times O(p). \end{aligned} \tag{37}$$

Proceeding in this way one finds

$$\mathrm{ch}\, V^{\Delta}_{O(2np+p)} \to \sum_{\zeta,\eta\in\mathscr{P}:\zeta\cdot\eta\subseteq(p^n)} \mathrm{ch}\, V^{-(p/2)^n+\zeta\cdot\eta}_{GL(n)}\; \mathrm{ch}\, V^{\Delta;\eta'}_{O(p)}, \tag{38}$$

where our notation is such that

$$\mathrm{ch}\, V^{-(p/2)^n+\zeta\cdot\eta}_{GL(n)} = \mathrm{ch}\, V^{-(p/2)^n}_{GL(n)}\; \mathrm{ch}\, V^{\zeta}_{GL(n)}\; V^{\eta}_{GL(n)} \tag{39}$$

and $\zeta\cdot\eta \subseteq (p^n)$ signifies that in the product of the last two factors one only retains those terms $\mathrm{ch}\, V^{\nu}_{GL(n)}$ such that $\nu\subseteq(p^n)$.

Comparison of (36) and (38) yields

$$\mathrm{ch}\, V^{(p/2)^n-\lambda_{\mathrm{rev}}}_{SO(2n+1)} = \sum_{\zeta,\eta\in\mathscr{P}:\zeta\cdot\eta\subseteq(p^n)} \pm^{\eta',\lambda'}_{O(p)}\; \mathrm{ch}\, V^{-(p/2)^n+\zeta\cdot\eta}_{GL(n)} \tag{40}$$

for any $\lambda\subseteq(p/2)^n$, where the sum over η is restricted to those partitions $\eta\subseteq(p^n)$ such that under the modification rules for $O(p)$ [1]:

$$\mathrm{ch}\, V^{\Delta;\eta'}_{O(p)} = \pm^{\eta',\lambda'}_{O(p)}\; \mathrm{ch}\, V^{\Delta;\lambda'}_{O(p)}, \tag{41}$$

with $\pm^{\eta',\lambda'}_{O(p)} = \pm$. Fortunately in the special case of interest here, $\lambda=0$, the $O(p)$ modification rules are such that only one solution exists, namely $\eta=0$. Hence we have our required result

$$\begin{aligned} \mathrm{ch}\, V^{(p/2)^n}_{SO(2n+1)} &= \sum_{\zeta\subseteq(p^n)} \mathrm{ch}\, V^{-(p/2)^n+\zeta}_{GL(n)} \\ &= (x_1x_2\ldots x_n)^{-p/2} \sum_{\zeta:\ell(\zeta')\le p} s_{\zeta}(x), \end{aligned} \tag{42}$$

which is precisely the result (32) of Bracken and Green [2], that we have shown is equivalent to Macdonald's identity (4).

It can thus be seen that the origin of the row length restriction $\ell(\zeta') \leq p$ can be traced directly back to the restriction $\zeta \subseteq (p^n)$ in the Howe dual pair identity (36) originally due to Morris [15, 16].

7 Metaplectic Representations

We would like to identify some other Howe dual pair that might lead to characters expressible as a simple multiple of the sum of column length restricted Schur functions that appears in the LSV identity (2). Such characters are necessarily infinite dimensional. We therefore need an infinite-dimensional analogue of the spin representation of the orthogonal group. This is provided by the metaplectic representation of the symplectic group.

Let $V = V^{\tilde{\Delta}}_{Sp(2np)}$, the metaplectic representation of the symplectic $Sp(2np)$ with character $\mathrm{ch}\, V^{\tilde{\Delta}}$ of lowest weight $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Then a Howe dual pair is provided by restriction to the subgroup $Sp(2n) \times O(p)$. This pair was first identified much earlier by Moshinsky and Quesne [17] as what they called a complementary pair of subgroups of $Sp(2np)$. The restriction to this subgroup is such that [7, 8, 11]

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(2np)} \to \sum_{\lambda \in \mathscr{P}: \lambda'_1 + \lambda'_2 \leq p, \lambda'_1 \leq n} \mathrm{ch}\, V^{(p/2)^n + \lambda_{\mathrm{rev}}}_{Sp(2n)}\ \mathrm{ch}\, V^{\lambda}_{O(p)}. \tag{43}$$

This expansion may be determined by evaluating the restriction of the character $\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(2np)}$ with respect to the group-subgroup chain

$$Sp(2np) \supset GL(np) \supseteq GL(n) \times GL(p) \supseteq GL(n) \times O(p), \tag{44}$$

This yields [11]

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(2np)} \to \sum_{\delta \in 2\mathscr{P}, \eta \in \mathscr{P}: \ell(\delta \cdot \eta) \leq \min(p,n)} \mathrm{ch}\, V^{(p/2)^n + \delta \cdot \eta}_{GL(n)}\ \mathrm{ch}\, V^{\eta}_{O(p)} \tag{45}$$

where $\delta \in 2\mathscr{P}$ signifies that all the parts of the partition δ are even, and

$$\mathrm{ch}\, V^{(p/2)^n + \delta \cdot \eta}_{GL(n)} = \mathrm{ch}\, V^{(p/2)^n}_{GL(n)}\ \mathrm{ch}\, V^{\delta}_{GL(n)} \mathrm{ch}\, V^{\eta}_{GL(n)} \tag{46}$$

while $\ell(\delta \cdot \eta) \leq \min(p,n)$ signifies that in the product of the last two factors one only retains those terms $\mathrm{ch}\, V^{\nu}_{GL(n)}$ such that $\ell(\nu) \leq \min(p,n)$.

Comparing the coefficients of $\mathrm{ch}\, V^{\lambda}_{O(p)}$ in (43) and (45) one finds

$$\mathrm{ch}\, V^{(p/2)^n + \lambda_{\mathrm{rev}}}_{Sp(2n)} = \sum_{\delta \in 2\mathscr{P}, \eta \in \mathscr{P}: \ell(\delta \cdot \eta) \leq \min(p,n)} (\pm)^{\eta, \lambda}_{O(p)}\ \mathrm{ch}\, V^{(p/2)^n + \delta \cdot \eta}_{GL(n)}, \tag{47}$$

where the sum over η is restricted to those η such that

$$\mathrm{ch}\, V^{\eta}_{O(p)} = (\pm)^{\eta,\lambda}_{O(p)}\, \mathrm{ch}\, V^{\lambda}_{O(p)}. \tag{48}$$

In each of the special cases $\lambda = (1^k)$ with $0 \leq k \leq n$ the only contribution is that from $\eta = (1^k)$. Hence

$$\begin{aligned}\mathrm{ch}\, V^{(p/2)^n + (1^k)_{\mathrm{rev}}}_{Sp(2n)} &= \sum_{\delta \in 2\mathscr{P}:\ell(\delta)\leq \min(p,n)} \mathrm{ch}\, V^{(p/2)^n + \delta\cdot 1^k}_{GL(n)} \\ &= (x_1 x_2 \ldots x_n)^{p/2} \sum_{\delta \in 2\mathscr{P}:\ell(\delta\cdot 1^k)\leq \min(p,n)} s_\delta(\mathbf{x})\, s_{(1^k)}(\mathbf{x}).\end{aligned} \tag{49}$$

It should now be recalled that the origin of the Lievens, Stoilova and Van der Jeugt conjecture was a character formula for the irreducible representations $V(p)$ of the orthosymplectic group $osp(1|2n)$. The connection with infinite dimensional lowest weight irreducible representations of $Sp(2n)$ comes about through the restriction from $osp(1|2n)$ to the subalgebra $sp(2n)$. Lievens, Stoilova and Van der Jeugt showed explicitly that [12]

$$\mathrm{ch}\, V_{osp(1|2n)}(p) = \sum_{k=0}^{p} \mathrm{ch}\, V^{(p/2)^n + (1^k)_{\mathrm{rev}}}_{sp(2n)}. \tag{50}$$

It then follows from (49) that

$$\begin{aligned}\mathrm{ch}\, V_{osp(1|2n)}(p) &= (x_1 x_2 \ldots x_n)^{p/2} \sum_{k=0}^{p} \sum_{\delta \in 2\mathscr{P}:\ell(\delta\cdot 1^k)\leq p} s_\delta(\mathbf{x})\, s_{(1^k)}(\mathbf{x}) \\ &= (x_1 x_2 \ldots x_n)^{p/2} \sum_{\zeta \in \mathscr{P}:\ell(\zeta)\leq p} s_\zeta(\mathbf{x}),\end{aligned} \tag{51}$$

thereby giving an alternative proof of the Proposition 1 originally derived by Lievens et al. [12].

8 Conclusions

The finite sum of Schur functions restricted with respect to row length has been shown to be a consequence of a character formula for $SO(2n+1)$. This character formula and the row length restriction are a direct consequence of the existence of the Howe dual pair $SO(2n+1) \times O(p)$ with respect to a finite-dimensional spin representation of $O(2np+p)$.

Similarly, the infinite sum of Schur functions restricted with respect to column lengths is a consequence of a character formula for $osp(1|2n)$. This character formula and the column length restriction are a direct consequence of the existence of the Howe dual pair $Sp(2n) \times O(p)$ with respect to an infinite-dimensional metaplectic representation of $Sp(2np)$.

More recently Cheng et al. [3] have exploited superalgebra-algebra Howe dual pairs such as $osp(2m+1|2n) \times O(p)$ to obtain $osp(2m+1|2n)$ character formulae. These allow one to rederive and generalise some of the above results.

Acknowledgements It is a pleasure to thank Joris Van der Jeugt for sending me the query that led to the analysis presented here, and to acknowledge that it was the seminal work of Tchavdar Palev on parastatistics and Wigner quantum systems that stimulated this whole area of enquiry.

References

1. Black, G.R.E., King, R.C., Wybourne, B.G.: J. Phys. A **16**, 1555–1589 (1983)
2. Bracken, A.J., Green, H.S.: Nuovo Cimento **9A**, 349–365 (1972)
3. Cheng, S.-J., Kwon, J.-H., Wang, W.: Adv. Math. **224**, 1548–1588 (2010)
4. Ganchev, A.Ch., Palev, T.D.: J. Math. Phys. **21**, 797–799 (1980)
5. Hasegawa, K.: Publ. RIMS, Kyoto Univ. **25**, 741–828 (1989)
6. Howe, R.: Lect. in Applied Math., vol. 21, pp. 179–207. Amer. Math. Soc., Providence (1985)
7. Howe, R.: Trans. Am. Math. Soc. **313**, 539–570 (1989)
8. Kashiwara, M., Vergne, M.: Invent. Math. **31**, 1–48 (1978)
9. King, R.C., Palev, T.D., Stoilova, N.I., Van der Jeugt, J.: J. Phys. A **36**, 4337–4362 (2003)
10. King, R.C., Palev, T.D., Stoilova, N.I., Van der Jeugt, J.: J. Phys. A **36**, 11999–12019 (2003)
11. King, R.C., Wybourne, B.G.: J. Phys. A **18**, 3113–3139 (1985)
12. Lievens, S., Stoilova, N.I., Van der Jeugt, J.: Comm. Math. Phys. **281**, 805–826 (2008)
13. Littlewood, D.E.: The Theory of Group Characters. Clarendon Press, Oxford (1940)
14. Macdonald, I.G.: Symmetric Functions and Hall Polynomials. Oxford University Press, Oxford (1979)
15. Morris, A.O.: J. Lond. Math. Soc. **33**, 326–333 (1958)
16. Morris, A.O.: Quart. J. Math. Oxford (2) **12**, 169–176 (1961)
17. Moshinsky, M., Quesne, C.: J. Math. Phys. **12**, 1772–1780 (1971)
18. Stoilova, N.I., Van der Jeugt, J.: J. Phys. A **41**, 075202 (13pp) (2008)

Varna Lecture on L^2-Analysis of Minimal Representations

Toshiyuki Kobayashi

Abstract Minimal representations of a real reductive group G are the 'smallest' irreducible unitary representations of G. The author suggests a program of global analysis built on minimal representations from the philosophy:

small representation of a group = **large** symmetries in a representation space.

This viewpoint serves as a driving force to interact algebraic representation theory with geometric analysis of minimal representations, yielding a rapid progress on the program. We give a brief guidance to recent works with emphasis on the Schrödinger model.

1 What are Minimal Representations?

Minimal representations of reductive groups G are the 'smallest' infinite dimensional irreducible unitary representations.

The *Weil* (metaplectic, oscillator, the Segal–Shale–Weil, harmonic) *representation*, known by a prominent role in number theory, consists of two minimal representations of the metaplectic group $Mp(n,\mathbb{R})$. The minimal representation of a conformal group $SO(4,2)$ arises on the Hilbert space of bound states of the Hydrogen atom.

Minimal representations are distinguished among other (continuously many) irreducible unitary representations of G by the following properties that I state loosely.

- 'Smallest' infinite dimensional representations of G.
- One of the 'building blocks' of unitary representations of Lie groups.

T. Kobayashi (✉)
Kavli IPMU and Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, 153-8914 Tokyo, Japan
e-mail: toshi@ms.u-tokyo.ac.jp

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_6,

- 'Closest' to the trivial one dimensional representation of G.
- 'Quantization' of minimal nilpotent coadjoint orbits of G.
- Matrix coefficients have a 'slow decay' at infinity.

In algebraic representation theory, there is a distinguished ideal $\mathcal{J}$ introduced by Joseph [14] in the enveloping algebra of a complex simple Lie algebra other than type A (see also [8]). An irreducible representation of a real reductive Lie group G is called *minimal* if its infinitesimal representation is annihilated by $\mathcal{J}$. Thus the terminology 'minimal representations' is defined *inside* representation theory. We remark that not all reductive groups admit minimal representations. Further, minimal representations are not always highest weight modules. Beyond the case of highest weight modules, there has been an active study on minimal representations of reductive groups, in particular, by algebraic approaches, see e.g., [8, 14, 15, 30, 37, 39, 42, 43].

In contrast, my program focuses on global analysis inspired by minimal representations. For this, we switch the viewpoint, led by

Guiding principle 1.1 ([23]).

$$\begin{aligned} &\boldsymbol{small}\ \textit{representations of a group} \\ &= \boldsymbol{large}\ \textit{symmetries in a representation space}. \end{aligned}$$

An extremal case of 'large symmetries' might be stated as

$$\text{dimension of } \Xi < \text{dimension of any non-trivial } G\text{-space} \tag{1}$$

when the representation of G is realized on the space of functions on the geometry Ξ. An obvious implication of (1) is that G cannot act on Ξ.

The latter point of view, served as a driving force, has brought us to a new line of investigation of geometric analysis modeled on minimal representations. In this program we are trying to dig out new interactions with other areas of mathematics even *outside* representation theory:

- Conformal geometry for general pseudo-Riemannian manifolds [21, 31],
- Dolbeault cohomologies on open complex manifolds [25, 30].
- Conservative quantities for PDEs [21, 33],
- Breaking symmetries and discrete branching laws [32, 35, 36, 38, 39],
- Schrödinger model and the unitary inversion operator [11, 27, 28],
- Deformation of the Fourier transform [3],
- Geometric quantization of nilpotent orbits [11, 28],
- Holomorphic semigroup with a generalized Mehler kernel [3, 26, 27],
- New orthogonal polynomials for fourth order differential operators [9, 10, 29],
- A generalization of the Fock model and Bargmann transforms [12].

The aim of this article is to provide a brief guidance to the rapid progress on our program, [1, 3, 9–12, 23, 25, 28, 29, 36, 38]. We should mention that in order to avoid an overlap with a recent publication [24], we do not include here some other constructions such as a conformal model of minimal representations (e.g. the construction of the intrinsic conservative quantities for the conformally invariant differential equations). Instead, we highlight an L^2-model (*Schrödinger model*) of the minimal representations and its variant. We apologize for not being able to mention some other important works on minimal representations, e.g., see [8] and references therein. For a comparison of the L^2-model with the conformal model, we refer to [28, Introduction].

2 More Symmetric Than Symmetric Spaces

The traditional geometric construction of representations of Lie groups G is given in the following two steps:

Step 1. The group G acts on a geometry X.
Step 2. By the translation, G acts linearly on the space $\Gamma(X)$ of functions (sections of equivariant bundles, or cohomologies, ...).

Naïvely, the Gelfand–Kirillov dimension of the representation on $\Gamma(X)$ is supposed to be the dimension of X. Thus we may expect that the representation on the function space $\Gamma(X)$ is 'small' if the geometry X itself is small.

First of all, we ask when the geometry X is 'small'.

For this we may begin with the case when G acts transitively on X, or equivalently, X is a homogeneous space G/H. Further, if we compare two homogeneous spaces $X_1 = G/H_1$ and $X_2 = G/H_2$ with $H_1 \subset H_2$, we may think that X_2 is smaller than X_1. Hence 'smaller' representations on $\Gamma(X)$ should be attained if $X = G/H$ where H is a maximal subgroup of G.

Here are two typical settings for real reductive Lie groups G:

- (G,H) is a symmetric pair.
  In this case, the Lie algebra $\mathfrak{h}$ of H is maximal reductive in $\mathfrak{g}$. Analysis on reductive symmetric spaces G/H has been largely developed in particular, since 1950s by the Gelfand school, Harish-Chandra, Shintani, Helgason, Takahashi, Molchanov, Faraut, Flensted-Jensen, Matsuki–Oshima–Sekiguchi, Delorme, van den Ban, Schlichtkrull, among others.
- H is a Levi subgroup of G.
  In this case, there exists a G-invariant polarization on G/H, and its geometric quantization obtained by the combination of the Mackey induction (real polarization) and the Dolbeault cohomologies (complex polarization) produces a 'generic part' of irreducible unitary representations of G. The resulting representations are the 'smallest' if H is a maximal Levi subgroup.

These two typical examples are related: Tempered representations for reductive symmetric spaces (i.e. irreducible unitary representations that contribute to $L^2(G/H)$) are given by the combination of the ordinary and cohomological parabolic inductions. A missing picture in the above two settings is so called 'unipotent representations' including minimal representations.

On the other hand, it is rare but still happens that the representation of G on the function space $\Gamma(X)$ extends to a representation of a group $\widetilde{G}$ which contains G, even when the G-action on the geometry X does not extend to $\widetilde{G}$ (in particular, Step 1 does not work for the whole group $\widetilde{G}$). We discuss this phenomenon in the Schrödinger model of minimal representations when G is a maximal parabolic subgroup (the notation $(G,\widetilde{G})$ here will be replaced by (P,G) in Sect. 3). Such a phenomenon also occurs when G is reductive. Thus the analysis of minimal representations may be thought of as 'analysis with more symmetries' than the traditional analysis on homogeneous spaces. Here is a typical example:

Example 2.1 ([21, Theorem 5.3]). The minimal representation of the indefinite orthogonal group $\widetilde{G} = O(p,q)$ ($p+q$:even) is realized in function spaces on symmetric spaces of the subgroups $G = O(p-1,q)$ or $O(p,q-1)$ on which the whole group $\widetilde{G}$ cannot act geometrically.

Example 2.2 ([36]). The restriction of the most degenerate principal series representations of $\widetilde{G} = GL(n,\mathbb{R})$ to the subgroup $G = O(p,q)$ ($p+q=n$) reduces to the analysis of the symmetric space of G on which the whole group $\widetilde{G}$ cannot act transitively.

Further examples and explicit branching rules can be found in [21, 32, 36] where the restriction of minimal representations to subgroups (*broken symmetries*) reduce to analysis on certain semisimple symmetric spaces.

3 Schrödinger Model of Minimal Representations

Any coadjoint orbit of a Lie group is naturally a symplectic manifold endowed with the Kirillov–Kostant–Souriau symplectic form. For a reductive Lie group G, 'geometric quantization' of semisimple coadjoint orbits has been considerably well-understood—this corresponds to the ordinary or cohomological parabolic induction in representation theory, whereas 'geometric quantization' of nilpotent coadjoint orbits is more mysterious (see [4, 12, 25]).

In this section we explain a recent work [11] with Hilgert and Möllers on the L^2-construction of minimal representations built on a Lagrangian subvariety of a real minimal nilpotent orbit, which continues a part of the earlier works [33] with Ørsted, and [28] with Mano.

Suppose that V is a simple Jordan algebra over $\mathbb{R}$. We assume that its maximal Euclidean Jordan subalgebra is also simple. Let G and L be the identity components of the conformal group and the structure group of the Jordan algebra V, respectively. Then the Lie algebra $\mathfrak{g}$ is a real simple Lie algebra and has a Gelfand–Naimark decomposition $\mathfrak{g} = \overline{\mathfrak{n}} + \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{n} \simeq V$ is regarded as an Abelian Lie algebra, $\mathfrak{l} \simeq \mathfrak{str}(V)$ the structure algebra, and $\overline{\mathfrak{n}}$ acts on V by quadratic vector fields.

Let $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$ be a (real) minimal nilpotent coadjoint orbit. By identifying $\mathfrak{g}$ with the dual $\mathfrak{g}^*$, we consider the intersection $V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}}$, which may be disconnected (this happens in the case (3) below). Let Ξ be any connected component of $V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}}$. We note that the group L acts on Ξ but G does not. There is a natural L-invariant Radon measure on Ξ, and we write $L^2(\Xi)$ for the Hilbert space consisting of square integrable functions on Ξ. Then we can define a unitary representation on $L^2(\Xi)$ (Schrödinger model) built on a Lagrangian submanifold Ξ in this generality [11], see also [5, 33].

Theorem 3.1 (Schrödinger model). *Suppose $V \not\simeq \mathbb{R}^{p,q}$ with $p+q$ odd.*

1) *Ξ is a Lagrangian submanifold of $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$.*
2) *There is a finite covering group $\tilde{G}$ of G such that $\tilde{G}$ acts on $L^2(\Xi)$ as an irreducible unitary representation.*
3) *The Gelfand–Kirillov dimension of π attains its minimum among all infinite dimensional representations of $\tilde{G}$, i.e. $\mathrm{DIM}(\pi) = \frac{1}{2}\dim \mathbb{O}_{\min}^{G_{\mathbb{R}}}$.*
4) *The annihilator of the differential representation $d\pi$ is the Joseph ideal in the enveloping algebra $U(\mathfrak{g})$ if V is split and $\mathfrak{g}$ is not of type A.*

The simple Lie algebras arisen in Theorem 3.1 are listed as follows:

$$\mathfrak{sl}(2k,\mathbb{R}), \mathfrak{so}(2k,2k), \mathfrak{so}(p+1,q+1), \mathfrak{e}_{7(7)}, \tag{2}$$

$$\mathfrak{sp}(k,\mathbb{R}), \mathfrak{su}(k,k), \mathfrak{so}^*(4k), \mathfrak{so}(2,k), \mathfrak{e}_{7(-25)}, \tag{3}$$

$$\mathfrak{sp}(k,\mathbb{C}), \mathfrak{sl}(2k,\mathbb{C}), \mathfrak{so}(4k,\mathbb{C}), \mathfrak{so}(k+2,\mathbb{C}), \mathfrak{e}_7(\mathbb{C}), \tag{4}$$

$$\mathfrak{sp}(k,k), \mathfrak{su}^*(4k), \mathfrak{so}(k,1). \tag{5}$$

Remark 1. In the case where V is an Euclidean Jordan algebra, G is the automorphism group of a Hermitian symmetric space of tube type (see (3)) and there are two real minimal nilpotent orbits. The resulting representations π are highest (or lowest) weight modules.

Remark 2. If the complex minimal nilpotent orbit $\mathbb{O}_{\min}^{G_{\mathbb{C}}}$ intersects with $\mathfrak{g}$, then $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$ is equal to $\mathbb{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ or its connected component. We notice that $\mathbb{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ may be an empty set depending on the real form $\mathfrak{g}$. In the setting of Theorem 3.1, this occurs for (5). In this case, the representation π in Theorem 3.1 is not a minimal representation as the annihilator of $d\pi$ is not the Joseph ideal, but π is still one of the 'smallest' infinite dimensional representations in the sense that the Gelfand–Kirillov dimension attains its minimum.

Remark 3. There is no minimal representation for any group with Lie algebra $\mathfrak{o}(p+1,q+1)$ with $p+q$ odd, p, $q \geq 3$ (see [43, Theorem 2.13]).

Example 3.2. Let $V = \mathrm{Sym}(m,\mathbb{R})$. Then $G = Sp(m,\mathbb{R})$ and

$$V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}} = \{X \in M(m,\mathbb{R}) : X = {}^tX, \mathrm{rank} X = 1\}. \tag{6}$$

Let $\Xi := \{X \in V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}} : \operatorname{Trace} X > 0\}$. Then the double covering map (*folding map*)

$$\mathbb{R}^m \setminus \{0\} \to \Xi, \quad v \mapsto v^t v$$

induces an isomorphism between $L^2(\Xi)$ and the Hilbert space $L^2(\mathbb{R}^m)_{\text{even}}$ of even square integrable functions on $\mathbb{R}^m$. Thus our representation π on $L^2(\Xi)$ is nothing but the Schrödinger model of the even part of the Segal–Shale–Weil representation of the metaplectic group $Mp(m,\mathbb{R})$ [7, 13].

Example 3.3. Let $V = \mathbb{R}^{p,q}$ with $p+q$ even. Then $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$, and

$$V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}} = \left\{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\right\} \setminus \{0\}. \tag{7}$$

If $p = 1$, $V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}}$ consists of two connected components according to the signature of ξ_1, i.e. the past and future cones. They yield highest/lowest weight modules. For $p, q \geq 2$, $V \cap \mathbb{O}_{\min}^{G_{\mathbb{R}}}$ is connected, and our representation π on $L^2(\Xi)$ is the Schrödinger model of the minimal representation of $O(p+1, q+1)$ constructed in [33], which is a neither highest nor lowest weight module.

As we discussed in Sect. 2 in contrast to traditional analysis on homogeneous spaces, the group G in our setting is too large to act geometrically on Ξ. This very feature in the Schrödinger model is illustrated by the fact that the Lie algebra $\overline{\mathfrak{n}}$ acts as differential operators on Ξ of second order. They are *fundamental differential operators* [28] in the setting of Example 3.3 (see also Bargmann–Todorov [2]). In [11], these differential operators are said to be *Bessel operators*, and serve as a basic tool to study the Schrödinger model π in the setting of Theorem 3.1.

4 Special Functions to 4th Order Differential Operators

Guiding Principle 1.1 suggests that there should exist plentiful functional equations in the representation spaces for minimal representations. Classically, it is well-known that Hermite polynomials form an orthogonal basis for the radial part of the Schrödinger model of the Weil representation [7], whereas Laguerre polynomials arise in the minimal representation of the conformal group $SO(n,2)$ ([41]).

These classical minimal representations are highest weight modules. However, for more general reductive groups, minimal representations do not always have highest weight vectors, and the corresponding 'special functions' do not necessarily satisfy second order differential equations. We found in [28] that Meijer's G-functions $G_{04}^{20}(x|b_1,b_2,b_3,b_4)$ play an analogous role in the minimal representation of $O(p,q)$. Here Meijer's G-functions $G_{04}^{20}(x|b_1,b_2,b_3,b_4)$ satisfy a fourth order ordinary differential equation

$$\prod_{j=1}^{4}(x\frac{d}{dx} - b_j)u(x) = xu(x).$$

More generally, the following fourth order differential operators

$$\mathcal{D}_{\mu,\nu} := \frac{1}{x^2}((\theta+\nu)(\theta+\mu+\nu)-x^2)(\theta(\theta+\mu)-x^2) - \frac{(\mu-\nu)(\mu+\nu+2)}{2}$$

appear naturally in the Schrödinger model of minimal representations in the setting of Theorem 3.1. Here $\theta = x\frac{d}{dx}$.

The subject of [9, 10, 29] is the study of eigenfunctions of $\mathcal{D}_{\mu,\nu}$ including

- Generating functions for eigenfunctions of $\mathcal{D}_{\mu,\nu}$,
- Asymptotic behavior near the singularities,
- L^2-eigenfunctions and concrete formulas of L^2-norms,
- Integral representations of eigenfunctions,
- Recurrence relations among eigenfunctions,
- (Local) monodromy.

The L^2-eigenfunctions of $\mathcal{D}_{\mu,\nu}$ arise as K-finite vectors in the Schrödinger model of the minimal representations constructed in Theorem 3.1 in a uniform fashion. These 'special functions' with certain integral parameters yield orthogonal polynomials (the *Mano polynomials* $M_j^{\mu,l}(x)$) satisfying again fourth order differential equations [9], which include Hermite polynomials and Laguerre polynomials as special cases. We note that the fourth order differential equation $\mathcal{D}_{\lambda,\mu}f = \nu f$ reduces to a differential equation of second order when G/K is a tube domain (see (3)). See also Kowata–Moriwaki [38] for further analysis of the fundamental differential operators on Ξ.

5 Broken Symmetries and Branching Laws

As indicated in Guiding Principle 1.1, the 'large symmetries' in representation spaces of minimal representations produce also fruitful examples of branching laws which we can expect a simple and detailed study.

Suppose π is a unitary representation of a real reductive Lie group G. We consider π as a representation of a subgroup G' of G, referring it as the restriction $\pi|_{G'}$. In general, the restriction $\pi|_{G'}$ decomposes into a direct integral of irreducible representations of G' (*branching law*). It often happens that the branching law contains continuous spectrum if G' is non-compact. Even worse, each irreducible representation of G' may occur in the branching law with infinite multiplicities. See [20] for such wild examples even when (G,G') is a symmetric pair. In [16, 17], we raised the following:

Program 5.1. *1) Determine the triple (G,G',π) for which the restriction $\pi|_{G'}$ decompose discretely with finite multiplicities.*
2) Find branching laws for (1).

Program 5.1 intends to single out a nice framework of branching problems for which we can expect a detailed and explicit study of the restriction. Concerning Program 5.1 (1) for Zuckerman's derived functor modules π, a necessary and sufficient condition for *discrete decomposition with finite multiplicities* was proved in [17, 19], and a complete classification was given with Oshima [34] when (G, G') is a reductive symmetric pair.

As such, the local theta correspondence with respect to compact dual pairs is a classic example for minimal representations π:

Example 5.2. Suppose that π is the Weil representation, and that $G' = G_1' \cdot G_2'$ is a dual pair in $G = Mp(n, \mathbb{R})$ with G_2' compact. Then the restriction $\pi|_{G'}$ decomposes discretely and multiplicity-freely. The resulting branching laws yield a large part of unitarizable highest weight modules of G_1' (Enright–Howe–Wallach [6]).

In order to discuss Program 5.1 for minimal representations, we recall from [17–19] the general theory. Let K be a maximal compact subgroup of G, T a maximal torus of K, and $\mathfrak{t}$, $\mathfrak{k}$ the Lie algebras of T, K, respectively. We choose the set $\Delta^+(\mathfrak{k}, \mathfrak{t})$ of positive roots, and denote by $\mathfrak{t}_+$ the dominant Weyl chamber in $\sqrt{-1}\mathfrak{t}^*$. We also fix a K-invariant inner product on $\mathfrak{k}$, and regard $\sqrt{-1}\mathfrak{t}^*$ as a subspace of $\sqrt{-1}\mathfrak{k}^*$.

First, suppose that K' is a closed subgroup of K. The group K acts on the cotangent bundle $T^*(K/K')$ of the homogeneous space K/K' in a Hamiltonian fashion. We write

$$\mu : T^*(K/K') \to \sqrt{-1}\mathfrak{k}^*$$

for the momentum map, and define the following closed cone by

$$C_K(K') := \operatorname{Image} \mu \cap \mathfrak{t}_+.$$

Second, let $\operatorname{Supp}_K(\pi)$ be the set of highest weights of finite dimensional irreducible representations of K occurring in a K-module π. The asymptotic K-support $\operatorname{AS}_K(\pi)$ is defined to be the asymptotic cone of $\operatorname{Supp}_K(\pi)$. It is a closed cone in $\mathfrak{t}_+$. There are only finitely many possibilities of $\operatorname{AS}_K(\pi)$ for the restriction $\pi|_K$ of irreducible representations π of G.

The asymptotic cone $\operatorname{AS}_K(\pi)$ tends to be a 'small' subset in $\mathfrak{t}_+$ if π is a 'small' representation. For example,

$$\begin{aligned} \operatorname{AS}_K(\pi) &= \{0\} && \text{if } \dim \pi < \infty, \\ \operatorname{AS}_K(\pi) &= \mathbb{R}_+ \beta && \text{if } \pi \text{ is a minimal representation,} \end{aligned} \tag{8}$$

where β is the highest root of the K-module $\mathfrak{p}_{\mathbb{C}} := \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}$. The formula (8) holds in a slightly more general setting where the associated variety of π is the closure of a single minimal nilpotent $K_{\mathbb{C}}$-orbit on $\mathfrak{p}_{\mathbb{C}}$ [35]. Concerning Program 5.1, we established an easy-to-check criterion in [18]:

Theorem 5.3. *Suppose G' is a reductive subgroup of G such that $K' := G' \cap K$ is a maximal compact subgroup of G'. If*

$$C_K(K') \cap \mathrm{AS}_K(\pi) = \{0\}, \tag{9}$$

then the restriction $\pi|_{G'}$ decomposes discretely into a direct sum of irreducible unitary representations of G' with finite multiplicities.

As was observed in [22], we can expect from the formula (8) and from the criterion (9) that there is plenty of subgroups G' for which the restriction of the minimal representation of G decomposes discretely and with finite multiplicities. Reductive symmetric pairs (G, G') for which the restriction $\pi|_{G'}$ is (infinitesimally) discretely decomposable for a minimal representation π of G has been recently classified in [35].

6 Generalized Fourier Transform as a Unitary Inversion

In the L^2-model of the minimal representation π of G on $L^2(\Xi)$, the action of the maximal parabolic subgroup P with Lie algebra $\mathfrak{l} + \mathfrak{n}$ is simple, namely, it is given just by translations and multiplications. Let w be the conformal inversion of the Jordan algebra. In light of the Bruhat decomposition

$$G = P \amalg PwP,$$

it is enough to find $\pi(w)$ in order to give a global formula of the G-action on $L^2(\Xi)$. We highlight this specific unitary operator, and set

$$\mathcal{F}_{\Xi} := c\pi(w), \tag{10}$$

where c is a complex number of modulus one (the *phase factor*). We call $\mathcal{F}_{\Xi}$ the *unitary inversion operator*. We studied in a series of papers [26–28] with Mano the following:

Problem 6.1. Find an explicit formula of the integral kernel of $\mathcal{F}_{\Xi}$.

The kernel of the Euclidean Fourier transform is given by $e^{-i\langle x,\xi\rangle}$, which is locally integrable. It is plausible that this analytic feature happens if and only if the corresponding minimal representation is of highest weight. Thus we raise the following:

Question 6.2. Let $(\pi, L^2(\Xi))$ be the L^2-model of a minimal representation π of a simple Lie group G constructed on a Lagrangian submanifold Ξ of $\mathbb{O}^G_{\min}$ as in Theorem 3.1 [11]. Are the following two conditions equivalent?

(i) The kernel of the unitary inversion operator $\mathcal{F}_\Xi$ is locally integrable.
(ii) π is a highest/lowest weight module.

Here we have excluded the case where the simple Lie algebra $\mathfrak{g}$ is of type A_n (the Joseph ideal is not defined for $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}_n(\mathbb{C})$). In the case $G = O(p+1, q+1)$ with $p+q$ even > 2, it was proved in [28] that (i) holds if and only if either $\min(p,q) = 1$ (equivalently, (ii) holds) or $(p,q) = (3,3)$ (equivalently, $\mathfrak{g} = \mathfrak{o}(3,3) \simeq \mathfrak{sl}(4,\mathbb{R})$ is of type A_3). The implication (ii) $\Rightarrow$ (i) was proved in [12] for tube type, see (17). The implication (i) $\Rightarrow$ (ii) is an open problem except for the above mentioned case $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$.

When $G = O(p+1, q+1)$ (see Example 3.3), $\mathcal{F}_\Xi$ intertwines the multiplication of coordinate functions ξ_j $(1 \le j \le p+q)$ with the operators R_j $(1 \le j \le p+q)$ which are mutually commuting differential operators of second order on Ξ (see Bargmann–Todorov [2], [28, Chap. 1]).

This algebraic feature is similar to the classical fact that the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^m}$ intertwines the multiplication operators ξ_j and the differential operators $\sqrt{-1}\partial_j$ $(1 \le j \le m)$ (see Example 3.2). In the setting of Theorem 3.1, $\mathcal{F}_\Xi$ intertwines the multiplication of coordinate functions with Bessel operators. Actually, this algebraic feature determines uniquely $\mathcal{F}_\Xi$ up to a scalar [11, 28].

Concerning Problem 6.1, the first case is well-known (see [7] for example):

1) $\mathfrak{g} = \mathfrak{sp}(m,\mathbb{R})$.
   $\mathcal{F}_\Xi =$ the Euclidean Fourier transform on $\mathbb{R}^m$.
   Here are some recent results on a closed formula of the integral kernel:
2) $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$ (with Mano [27]).
3) The associated Riemannian symmetric space G/K is of tube type (see (17)).

We note that minimal representations in the cases (1) and (3) are highest (or lowest) weight modules, whereas minimal representations in the case (2) do not have highest weights when $p, q \ge 2$ and $p+q$ is odd.

Problem 6.1 is open for other cases, in particular, for minimal representations without highest weights except for the case $G = O(p+1, q+1)$.

7 SL_2-Triple in the Schrödinger Model

On $\mathbb{R}^m$, we set $|x| := (\sum_{j=1}^m x_j^2)^{\frac{1}{2}}$, $E := \sum_{j=1}^m x_j \frac{\partial}{\partial x_j}$ (Euler operator) and $\Delta = \sum_{j=1}^m \frac{\partial^2}{\partial x_j^2}$ (Laplacian). Then it is classically known (e.g., [7, 13]) that the operators

$$\widetilde{h}' := E + \frac{m}{2}, \quad \widetilde{e}' := \frac{\sqrt{-1}}{2}|x|^2, \quad \widetilde{f}' := \frac{\sqrt{-1}}{2}\Delta \tag{11}$$

form an $\mathfrak{sl}_2$-triple, namely, the following commutation relation holds:

$$[\widetilde{h}', \widetilde{e}'] = 2\widetilde{e}', \quad [\widetilde{h}', \widetilde{f}'] = -2\widetilde{f}', \quad [\widetilde{e}', \widetilde{f}'] = \widetilde{h}'.$$

On the other hand, we showed in [27] that the following operators

$$\widetilde{h} := 2E + m - 1, \quad \widetilde{e} := 2\sqrt{-1}|x|, \quad \widetilde{f} := \frac{\sqrt{-1}}{2}|x|\Delta \tag{12}$$

also forms an $\mathfrak{sl}_2$-triple, i.e., $[\widetilde{h},\widetilde{e}] = 2\widetilde{e}, \quad [\widetilde{h},\widetilde{f}] = -2\widetilde{f}, \quad [\widetilde{e},\widetilde{f}] = \widetilde{h}.$

Further the differential operator

$$D := \frac{1}{2\sqrt{-1}}(-\widetilde{e} + \widetilde{f}) = |x|\left(\frac{\Delta}{4} - 1\right)$$

extends to a self-adjoint operator and has only discrete spectra on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ which are given by $\{-(j + \frac{m-1}{2}) : j = 0,1,2,\cdots\}$ (see [27]), whereas the *Hermite operator*

$$\mathcal{D} := \frac{1}{2\sqrt{-1}}(-\widetilde{e}' + \widetilde{f}') = \frac{1}{4}(\Delta - |x|^2)$$

extends to a self-adjoint operator and has only discrete spectra on $L^2(\mathbb{R}^m, dx)$ which are given by $\{-\frac{1}{2}(j + \frac{m}{2}) : j = 0,1,2,\cdots\}$ (see [7, 13]). Hence, one can define for $\operatorname{Re} t \geq 0$:

$$e^{tD} := \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k \quad \text{on } L^2\left(\mathbb{R}^m, \frac{dx}{|x|}\right),$$

$$e^{t\mathcal{D}} := \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{D}^k \quad \text{on } L^2(\mathbb{R}^m, dx).$$

They are holomorphic one-parameter semigroups consisting of Hilbert–Schmidt operators for $\operatorname{Re} t > 0$, and are unitary operators for $\operatorname{Re} t = 0$.

A closed formula for both e^{tD} and $e^{t\mathcal{D}}$ is known. That is, the holomorphic semigroup $e^{t\mathcal{D}}$ has the classical Mehler kernel given by the Gaussian kernel $e^{-|x|^2}$ and reduces to the Euclidean Fourier transform when $t = \sqrt{-1}\pi$ ([13, §5]), whereas the integral kernel of the holomorphic semigroup e^{tD} is given by the I-Bessel function and the special value at $t = \sqrt{-1}\pi$ is by the J-Bessel function (see [27, Theorem A and Corollary B] for concrete formulas).

We can study these holomorphic semigroups by using the theory of discretely decomposable unitary representations (e. g. [16–18]). Actually, the aforementioned $\mathfrak{sl}_2$-triple arises as the differential action of the Schrödinger model of the minimal representations of $Mp(m,\mathbb{R})$ on $L^2(\mathbb{R}^m, dx)$ and $SO_0(m+1,2)$ on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, respectively via

$$\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{sp}(1,\mathbb{R}) \subset \mathfrak{sp}(m,\mathbb{R}),$$

$$\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(1,2) \ \subset \mathfrak{so}(m+1,2),$$

for which we write as $d\iota : \mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{g}$.

In both cases, the Lie algebra $\mathfrak{g}$ contains a subalgebra commuting with $\iota(\mathfrak{sl}(2,\mathbb{R}))$, which is isomorphic to $\mathfrak{o}(m)$. Then the minimal representations decompose as the representation of the direct product group $SL(2,\mathbb{R}) \times O(m)$ (up to coverings and connected groups) as follows:

$$L^2(\mathbb{R}^m, \frac{dx}{|x|}) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi_{2j+m-1}^{SL(2,\mathbb{R})} \boxtimes \mathcal{H}^j(\mathbb{R}^m).$$

$$L^2(\mathbb{R}^m, dx) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi_{j+\frac{m}{2}}^{SL(2,\mathbb{R})} \boxtimes \mathcal{H}^j(\mathbb{R}^m),$$

where $\mathcal{H}^j(\mathbb{R}^m)$ denotes the natural representation of $O(m)$ (or $SO(m)$) on the space of harmonic polynomials on $\mathbb{R}^m$ of degree j and $\pi_b^{SL(2,\mathbb{R})}$ stands for the irreducible unitary lowest weight representation of $SL(2,\mathbb{R})$ (or its covering group) with minimal K-type b.

These considerations bring us to interpolate operators occurring two minimal representations of $SO_0(m+1,2)^\sim$ and $Sp(m,\mathbb{R})$. For this, we take $a > 0$ to be a deformation parameter, and define

$$\widetilde{h}_a := \frac{2}{a}E + \frac{m+a-2}{a}, \quad \widetilde{e}_a := \frac{\sqrt{-1}}{a}|x|^a, \quad \widetilde{f}_a := \frac{\sqrt{-1}}{a}|x|^{2-a}\Delta.$$

The operators (11) in the Weil representation corresponds to the case $a = 2$, and the operators (12) for $SO_0(m+1,2)^\sim$ corresponds to the case $a = 1$. They extend to self-adjoint operators on the Hilbert space $L^2(\mathbb{R}^m, |a|^{a-2}dx)$, form an $\mathfrak{sl}_2$-triple, and lift to a unitary representation of the universal covering group $SL(2,\mathbb{R})^\sim$ of $SL(2,\mathbb{R})$ for every $a > 0$. The Hilbert space decomposes into a multiplicity-free discrete sum of irreducible unitary representations of $SL(2,\mathbb{R})^\sim \times O(m)$ as follows:

$$L^2(\mathbb{R}^m, |x|^{a-2}dx) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi_{\frac{2j+m-2}{a}+1}^{SL(2,\mathbb{R})} \boxtimes \mathcal{H}^j(\mathbb{R}^m).$$

The discrete decomposition of $\mathfrak{sl}_2$-modules becomes a tool to generalize the study of the unitary inversion operator $\mathcal{F}_\Xi$ and the holomorphic semigroup in [26, 27] to the following settings:

- Dunkl operators (with Ben Saïd and Ørsted [3]),
- Conformal group of Euclidean Jordan algebras (with Hilgert and Möllers [12]).

8 Quantization of Kostant–Sekiguchi Correspondence

In this section we discuss Theorem 3.1 in a special case where V is Euclidean, equivalently, G/K is a tube domain, and explain a recent work [12] with Hilgert, Möllers, and Ørsted on the construction of a new model (a Fock-type model) of minimal representations with highest weights and a generalization of the classical Segal–Bargmann transform, which we called a 'geometric quantization' of the Kostant–Sekiguchi correspondence. In the underlying idea, the discretely decomposable restriction of $\mathfrak{sl}(2,\mathbb{R})$, which appeared in [26], plays again an important role.

We recall (e.g., [7, 13]) that the classical Fock space $\mathcal{F}(\mathbb{C}^m)$ is a Hilbert space in the space $\mathcal{O}(\mathbb{C}^m)$ of holomorphic functions defined by

$$\mathcal{F}(\mathbb{C}^m) := \left\{ f \in \mathcal{O}(\mathbb{C}^m) : \int_{\mathbb{C}^m} |f(z)|^2 e^{-|z|^2} dz < \infty \right\},$$

and that the Segal–Bargmann transform is a unitary operator

$$\mathcal{B} : L^2(\mathbb{R}^m) \xrightarrow{\sim} \mathcal{F}(\mathbb{C}^m), \quad u \mapsto (\mathcal{B}u)(z) := \int_{\mathbb{R}^m} K_{\mathcal{B}}(x,z) f(x) dx,$$

with the kernel

$$K_{\mathcal{B}}(x,z) := \exp\left(-\frac{1}{2}\langle z,z\rangle + 2\langle z,x\rangle - \langle x,x\rangle \right).$$

From a representation theoretic viewpoint, the classical Segal–Bargmann transform intertwines the two models of the Weil representation of the metaplectic group $Mp(m,\mathbb{R})$, namely, the Schrödinger model on $L^2(\mathbb{R}^m)$ and the Fock model on $\mathcal{F}(\mathbb{C}^m)$.

In order to find a natural generalization of this classical theory, we begin by examining how one may rediscover the classical Fock model. Our idea is to use the action of $\mathfrak{sl}_2$, more precisely, a 'holomorphically extended representation' of an open semigroup of $SL(2,\mathbb{C})$ rather than a unitary representation of $SL(2,\mathbb{R})$ itself. For this, we take a standard basis of $\mathfrak{sl}(2,\mathbb{R})$ as

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

They satisfy the following Lie bracket relations: $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$. We set

$$k := i(-e+f) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$c_1 := \begin{pmatrix} 1 & -i \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & \frac{1}{2i} \end{pmatrix} \begin{pmatrix} 1 & -\frac{i}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}. \tag{14}$$

By a simple matrix computation we have:

$$\exp\left(-\frac{t}{2}k\right)|_{t=i\pi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{R}). \tag{15}$$

The formula $\mathrm{Ad}(c_1)k = h$ shows that $c_1 \in SL(2,\mathbb{C})$ gives a Cayley transform. Correspondingly, the Bargmann transform may be interpreted as

$$\mathcal{B} = `\pi \circ \iota(c_1)'.$$

The right-hand side is not well-defined. We need an analytic continuation in the Schrödinger model and a lift in the diagram below:

$$\begin{array}{ccc} & SL(2,\mathbb{R})^{\sim} \xrightarrow{\iota} G \xrightarrow{\pi} GL(L^2(\Xi)) \\ & \downarrow \\ c_1 \in SL(2,\mathbb{C}) \supset SL(2,\mathbb{R}) & \end{array}$$

To be more precise, we write $w \in G$ for the lift of (15) via $d\iota : \mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{g}$. Since the action of the maximal parabolic subgroup P on $L^2(\Xi)$ is given by the translation and the multiplication of functions, it is easy to see what '$\pi(p)$' should look like for $p \in P_{\mathbb{C}}$. Therefore, we could give an explicit formula for the (generalized) Bargmann transform $\mathcal{B} =$ '$\pi \circ \iota(c_1)$' if we know the closed formula of the unitary inversion:

$$\mathcal{F}_{\Xi} = \mathcal{F}(w) \equiv \mathcal{F} \circ \iota \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Of course, this is not a rigorous argument, and $\pi(p)$ does not leave $L^2(\Xi)$ invariant. However, the formula (14) suggests what the function space $\pi \circ \iota(c_1)(L^2(\Xi))$ ought to be, and led us to an appropriate generalization of the classical Fock space as follows:

$$\mathcal{F}(\mathbb{O}_{\min}^{K_{\mathbb{C}}}) := \left\{ F \in \mathcal{O}(\mathbb{O}_{\min}^{K_{\mathbb{C}}}) : \int_{\mathbb{O}_{\min}^{K_{\mathbb{C}}}} |F(z)|^2 \widetilde{K}_{\lambda-1}(|z|) d\nu(z) < \infty \right\}. \tag{16}$$

Here $\mathbb{O}_{\min}^{K_{\mathbb{C}}}$ is the minimal nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ which is the counterpart of the minimal (real) nilpotent coadjoint orbit $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$ in $\mathfrak{g}^* \simeq \mathfrak{g}$ under the Kostant–Sekiguchi correspondence [40], see Fig. 1. Thus the generalized Fock space $\mathcal{F}(\mathbb{O}_{\min}^{K_{\mathbb{C}}})$ is a Hilbert space consisting of L^2-holomorphic functions on the complex manifold $\mathbb{O}_{\min}^{K_{\mathbb{C}}}$ against the measure given by a renormalized K-Bessel function $\widetilde{K}_{\lambda-1}(|z|)d\nu(z)$ (see the comments after (17)).

We recall that Ξ is a Lagrangian submanifold of $\mathbb{O}_{\min}^{G_{\mathbb{R}}}$, and $K_{\mathbb{C}}$ acts holomorphically on $\mathbb{O}_{\min}^{K_{\mathbb{C}}}$. Then as a 'quantization' of the Kostant–Sekiguchi correspondence,

$$\begin{array}{ccc} \mathfrak{g} & & \mathfrak{p}_{\mathbb{C}} \\ \cup & & \cup \\ \mathbb{O}_{\mathrm{min}}^{G_{\mathbb{R}}} & \overset{\text{Kostant–Sekiguchi}}{\longleftrightarrow} & \mathbb{O}_{\mathrm{min}}^{K_{\mathbb{C}}} \\ \cup \text{ Lagrangian} & & \\ \Xi & & \end{array}$$

Fig. 1 Minimal nilpotent orbits in $\mathfrak{g}$ and $\mathfrak{p}_{\mathbb{C}}$

we define the generalized Bargmann transform $\mathcal{B} : L^2(\Xi) \to \mathcal{F}(\mathbb{O}_{\mathrm{min}}^{K_{\mathbb{C}}})$ by

$$f \mapsto \Gamma(\lambda) e^{-\frac{1}{2}\operatorname{tr}(z)} \int_{\Xi} \widetilde{I}_{\lambda-1}(2\sqrt{(z|x)}) e^{-\operatorname{tr}(x)} f(x) d\mu(x),$$

whereas the unitary inversion operator $\mathcal{F}_{\Xi}$ is given by

$$(\mathcal{F}_{\Xi} f)(y) = 2^{-r\lambda} \Gamma(\lambda) \int_{\Xi} \widetilde{J}_{\lambda-1}(2\sqrt{(x|y)}) f(x) d\mu(x). \tag{17}$$

Here $r = \operatorname{rank} G/K$, $(\;|\;)$ denotes the trace form of the Jordan algebra V, and $\lambda = \frac{1}{2}\dim_{\mathbb{R}} \mathbb{F}$ if $V = \mathrm{Herm}(k, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$, quaternion $\mathbb{H}$, or the octonion $\mathbb{O}$ (and $k = 3$) or $\lambda = \frac{1}{2}(k-2)$ if $V = \mathbb{R}^{1,k-1}$. $\widetilde{J}(t)$, $\widetilde{I}(t)$, and $\widetilde{K}(t)$ are the renormalization of the J-, I-, and K-Bessel function, respectively, following the convention of [28].

Acknowledgements The author warmly thanks Professor Vladimir Dobrev for his hospitality during the ninth International Workshop: Lie Theory and its Applications in Physics in Varna, Bulgaria, 20–26 June 2011. Thanks are also due to anonymous referees for careful readings.

The author was partially supported by Grant-in-Aid for Scientific Research (B) (22340026), Japan Society for the Promotion of Sciences.

References

1. Achab, D., Faraut, J.: Analysis of the Brylinski–Kostant model for spherical minimal representations. Can. J. Math. **64**, 721–754 (2012) arXiv:1101.4402 (to appear)
2. Bargmann, V., Todorov, I.T.: Spaces of analytic functions on a complex cone as carriers for the symmetric tensor representations of SO(n). J. Math. Phys. **18**, 1141–1148 (1977)
3. Ben Saïd, S., Kobayashi, T., Ørsted, B.: Laguerre semigroup and Dunkl operators. Compos. Math. **148**, 1265–1336 (2012) (cf. arXiv:0907.3749). http://dx.doi.org/10.1112/S0010437X11007445
4. Brylinski, R., Kostant, B.: Minimal representations, geometric quantization, and unitarity. Proc. Natl. Acad. Sci. USA **91**, 6026–6029 (1994)
5. Dvorsky, A., Sahi, S.: Explicit Hilbert spaces for certain unipotent representations. II. Invent. Math. **138**(1), 203–224 (1999)

6. Enright, T., Howe, R., Wallach, N.: A classification of unitary highest weight modules. Representation Theory of Reductive Groups, Park City, Utah, 1982, pp. 97–143. Progr. Math., vol. 40. Birkhäuser, Boston (1983)
7. Folland, G.B.: Harmonic analysis in phase space. In: Annals of Mathematics Studies, vol. 122. Princeton University Press, Princeton (1989)
8. Gan, W.-T., Savin, G.: On minimal representations definitions and properties. Represent. Theor. **9**, 46–93 (2005)
9. Hilgert, J., Kobayashi, T., Mano, G., Möllers, J.: Orthogonal polynomials associated to a certain fourth order differential equation. Ramanujan J. **26**, 295–310 (2011) (cf. http://dx.doi.org/10.1007/s11139-011-9338-6arXiv:0907.2612)
10. Hilgert, J., Kobayashi, T., Mano, G., Möllers, J.: Special functions associated to a certain fourth order differential operator. Ramanujan J. **26**, 1–34 (2011) (cf. http://arxiv.org/abs/0907.2608arXiv:0907.2608). http://dx.doi.org/10.1007/s11139-011-9315-0
11. Hilgert, J., Kobayashi, T., Möllers, J.: Minimal representations via Bessel operators, to appear in J. Math. Soc. Japan, p. 72 (2012) (cf. arXiv:1106.3621)
12. Hilgert, J., Kobayashi, T., Möllers, J., Ørsted, B.: Segal–Bargmann transform and Fock space realization for minimal holomorphic representations. J. Funct. Anal. **263**, 3492–3563 (2012) (cf. http://dx.doi.org/10.1016/j.jfa.2012.08.026arXiv:1203.5462) (to appear)
13. Howe, R.: The oscillator semigroup. Proc. Sympos. Pure Math., vol. 48, pp. 61–132. Amer. Math. Soc., Providence (1988)
14. Joseph, A.: The minimal orbit in a simple lie algebra and its associated maximal ideal. Ann. Scient. Ec. Norm. Sup. **9**, 1–30 (1976)
15. Kazhdan, D.: The minimal representation of D_4. In: Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Paris, 1989. Progr. Math., vol. 92, pp. 125–158. Birkhäuser, Boston (1990)
16. Kobayashi, T.: The restriction of $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups. Proc. Jpn. Acad. **69**, 262–267 (1993). http://projecteuclid.org/euclid.pja/1195511349
17. Kobayashi, T.: Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its application. Invent. Math. **117**, 181–205 (1994). http://dx.doi.org/10.1007/BF01232239
18. Kobayashi, T.: Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$, II. — micro-local analysis and asymptotic K-support. Ann. Math. **147**, 709–729 (1998). http://dx.doi.org/10.2307/120963
19. Kobayashi, T.: Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$, III. — restrcition of Harish-Chandra modules and associated varieties. Invent. Math. **131**, 229–256 (1998). http://dx.doi.org/10.1006/jfan.1997.3128
20. Kobayashi, T.: Discretely Decomposable Restrictions of Unitary Representations of Reductive Lie Groups — Examples and Conjectures. Advanced Study in Pure Math., vol. 26, pp. 98–126 (2000). Math. Soc. Japan http://www.ms.u-tokyo.ac.jp/~toshi/pub/57.html
21. Kobayashi, T.: Conformal geometry and global solutions to the Yamabe equations on classical pseudo-Riemannian manifolds. In: Proceedings of the 22nd Winter School “Geometry and Physics”, Srní, 2002. Rend. Circ. Mat. Palermo (2) Suppl. 71, pp. 15–40 (2003). http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1365.pdf
22. Kobayashi, T.: Restrictions of Unitary Representations of Real Reductive Groups. Progress in Mathematics, vol. 229, pp. 139–207. Birkhäuser, Basel (2005). http://dx.doi.org/10.1007/b139076
23. Kobayashi, T.: Algebraic analysis of minimal representations. Publ. Res. Inst. Math. Sci. **47**, 585–611 (2011); Special issue in commemoration of the golden jubilee of algebraic analysis (cf. arXiv:1001.0224). http://dx.doi.org/10.2977/PRIMS/45
24. Kobayashi, T.: Branching problems of Zuckerman derived functor modules. In: Adams, J., Lian, B., Sahi, S. (eds.) Representation Theory and Mathematical Physics. Contemporary Mathematics, vol. 557, pp. 23–40. Amer. Math. Soc., Providence (2011). http://arxiv.org/abs/1104.4399 (cf. arXiv:1104.4399)
25. Kobayashi, T.: Geometric quantization, limits, and restrictions —some examples for elliptic and nilpotent orbits. In: Jeffrey, L., Ma, X., Vergne, M. (eds.) Geometric Quantization in

the Non-compact Setting, Oberwolfach Reports, vol. 8, Issue 1, pp. 39–42. European Mathematical Society Publishing House (2011). http://dx.doi.org/10.4171/OWR/2011/09. DOI: 10.4171/OWR/2011/09
26. Kobayashi, T., Mano, G.: Integral formulas for the minimal representation of $O(p,2)$. Acta Appl. Math. **86**, 103–113 (2005) http://www.springerlink.com/content/pt65717043175035/fulltext.pdf
27. Kobayashi, T., Mano, G.: The inversion formula and holomorphic extension of the minimal representation of the conformal group. In: Li, J.-S., Tan, E.-C., Wallach, N., Zhu, C.-B. (eds.) Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory: In honor of Roger E. Howe, pp. 159–223. Singapore University Press and World Scientific Publishing, Singapore (2007) (cf. http://uk.arxiv.org/abs/math.DG/0607007math.RT/0607007)
28. Kobayashi, T., Mano, G.: The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p,q)$. Mem. Amer. Math. Soc. **212**, vi+132 (2011). http://dx.doi.org/10.1090/S0065-9266-2011-00592-7 no. 1000
29. Kobayashi, T., Möllers, J.: An integral formula for L^2-eigenfunctions of a fourth order Bessel-type differential operator. Integr. Transforms Spec. Functions **22**, 521–531 (2011). http://dx.doi.org/10.1080/10652469.2010.533270
30. Kobayashi, T., Ørsted, B.: Conformal geometry and branching laws for unitary representations attached to minimal nilpotent orbits. C. R. Acad. Sci. Paris, vol. 326, 925–930 (1998)
31. Kobayashi, T., Ørsted, B.: Analysis on the minimal representation of $O(p,q)$. I. Realization via conformal geometry. Adv. Math. **180**, 486–512 (2003). http://dx.doi.org/10.1016/S0001-8708(03)00012-4
32. Kobayashi, T., Ørsted, B.: Analysis on the minimal representation of $O(p,q)$. II. Branching laws. Adv. Math. **180**, 513–550 (2003). http://dx.doi.org/10.1016/S0001-8708(03)00013-6
33. Kobayashi, T., Ørsted, B.: Analysis on the minimal representation of $O(p,q)$. III. Ultrahyperbolic equations on $\mathbb{R}^{p-1,q-1}$. Adv. Math. **180**, 551–595 (2003). http://dx.doi.org/10.1016/S0001-8708(03)00014-8
34. Kobayashi, T., Oshima, Y.: Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs. Adv. Math. **231**, 2013–2047 (2012) (cf. http://arxiv.org/abs/1104.4400 arXiv:1104.4400). http://dx.doi.org/10.1016/j.aim.2012.07.006
35. Kobayashi, T., Oshima, Y.: Classification of symmetric pairs that admit discretely decomposable $(\mathfrak{g},K)$-modules, 18 pp. (cf. http://arxiv.org/abs/1202.5743arXiv:1202.5743), preprint
36. Kobayashi, T., Ørsted, B., Pevzner, M.: Geometric analysis on small unitary representations of $GL(N,\mathbb{R})$. J. Funct. Anal. **260**, 1682–1720 (2011)
37. Kostant, B.: The vanishing of scalar curvature and the minimal representation of $SO(4,4)$. In: Connes, A., Duflo, M., Joseph, A., Rentschler, R. (eds.) Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Paris, 1989. Progr. Math., vol. 92, pp. 85–124. Birkhäuser, Boston (1990)
38. Kowata, A., Moriwaki, M.: Invariant differential operators on the Schrödinger model for the minimal representation of the conformal group. J. Math. Sci. Univ. Tokyo **18**, 355–395 (2011)
39. Moriwaki, M.: Multiplicity-free decompositions of the minimal representation of the indefinite orthogonal group. Int. J. Math. **19**, 1187–1201 (2008). http://dx.doi.org/10.1142/S0129167X08005084
40. Sekiguchi, J.: Remarks on real nilpotent orbits of a symmetric pair. J. Math. Soc. Jpn. **39**, 127–138 (1987)
41. Todorov, I.T.: Derivation and Solution of an Infinite-Component Wave Equation for the Relativistic Coulomb Problem. Lecture Notes in Physics, Group Representations in Mathematics and Physics, Rencontres, Battelle Res. Inst., Seattle, 1969, vol. 6, pp. 254–278. Springer, Berlin (1970)
42. Torasso, P.: Méthode des orbites de Kirillov–Duflo et représentations minimales des groupes simples sur un corps local de caractéristique nulle. Duke Math. J. **90**, 261–377 (1997)
43. Vogan, D.A., Jr.: Singular unitary representations. In: Noncommutative Harmonic Analysis and Lie Groups, Marseille, 1980. Lecture Notes in Math., vol. 880, pp. 506–535. Springer, Berlin (1981)

Exponential Series Without Denominators

Jean-Louis Loday†

Abstract For a commutative algebra which comes from a Zinbiel algebra the exponential series can be written without denominators. When lifted to dendriform algebras this new series satisfies a functional equation analogous to the Baker-Campbell-Hausdorff formula. We make it explicit by showing that the obstruction series is the sum of the brace products. In the multilinear case we show that the role the Eulerian idempotent is played by the iterated pre-Lie product.

1 Introduction

The classical exponential series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

can be written without denominators provided one assumes some properties on the commutative algebra. For instance in a Zinbiel algebra the term $\frac{x^n}{n!}$ can be replaced by the iterated product $x^{\prec n} := x \prec (x^{\prec n-1})$, with $x^{\prec 1} = x$. Recall that a Zinbiel algebra is a commutative algebra in which the product splits as $xy = x \prec y + y \prec x$. Matrices with coefficients in a commutative algebra (resp. Zinbiel algebra) are endowed with a structure of associative algebra (resp. dendriform algebra). Recall that a dendriform algebra is an associative algebra whose product is the sum of two binary operations: $xy = x \prec y + x \succ y$, supposed to satisfy some relations. There is a unique way to extend the exponential series from commutative algebras to associative algebras. However there are several ways to extend the exponential series

†Professor Jean-Louis Loday passed away on 6 June 2012.

J.-L. Loday (✉)
Institut de Recherche Mathématique Avancée, CNRS et Université de Strasbourg,
Zinbiel Institute of Mathematics, France

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_7,

from Zinbiel algebras to dendriform algebras: either take $\frac{x^n}{n!}$ or $x^{\prec n}$ or $x^{\succ n}$. These choices are equal in a Zinbiel algebra, but different in a dendriform algebra. We adopt the notation

$$e(x) := 1 + x + x^{\prec 2} + \cdots + x^{\prec n} + \cdots, \quad e'(x) := 1 + x + x^{\succ 2} + \cdots + x^{\succ n} + \cdots$$

for the exponential series without denominators. The classical functional equation

$$\exp(x)\exp(y) = \exp(x+y)$$

holds only when we are in the commutative or in the Zinbiel setting. In this last case it can be written:

$$e(x)e(y) = e(x+y).$$

In the associative setting there is an error term for exp which is called the Baker-Campbell-Hausdorff formula. We prove that, in the dendriform context, the functional equation for the exponential without denominators is:

$$e(x)e(y) = e\big(x + e(x) \succ y \prec e'(-x)\big),$$

which can be written

$$e(x)e(y) = e(x + y + \sum_{n\geq 1} \{\underbrace{x,\ldots,x}_{n};y\}),$$

where $\{-,\cdots,-;-\}$ is the brace product.

In the associative framework the BCH-formula is best understood by looking at the multilinearized case. Then, the multilinear polynomial involved in the functional equation of the exponential turns out to be the Eulerian idempotent. In the dendriform case, we show that the analogue of the Eulerian idempotent for the exponential without denominators is played by the iterated pre-Lie product

$$h(x_1,\ldots,x_n) = \{x_1,\{x_2,\{\cdots\{x_{n-1},x_n\}\cdot\}\}\},$$

where the pre-Lie product is $\{x,y\} := x \succ y - y \prec x$.

Here is the content of this paper. In the first section we recall the notions of Zinbiel algebras, dendriform algebras and some of their properties. In the second section we introduce and study the exponential without denominators. In particular we compute its inverse, the logarithm without denominators, which is closely related to the $1\frac{1}{2}$-logarithm of Kontsevich. In Sect. 4 we prove the BCH-type formula for the exponential series without denominators. As a corollary we show that the multilinear obstruction to the additivity of the exponential is the iterated pre-Lie product.

In this paper $\mathbb{K}$ denotes a unital commutative ring (for instance $\mathbb{Z}$) which is the ground ring. We sometimes need to suppose that $\mathbb{K}$ contains $\mathbb{Q}$. The tensor product over $\mathbb{K}$ is simply denoted by $\otimes$.

2 Zinbiel and Dendriform Algebras

We introduce the notion of Zinbiel algebra (called dual Leibniz algebra in [7]) and its relationship with commutative algebras. We also introduce the notion of dendriform algebras because matrices over a Zinbiel algebra bear the structure of a dendriform algebra.

2.1 Definition

A *Zinbiel algebra* is a module A over $\mathbb{K}$ equipped with a binary operation $x \prec y$ such that

$$(x \prec y) \prec z = x \prec (y \prec z + z \prec y).$$

Symmetrizing the Zinbiel operation, that is defining $xy := x \prec y + y \prec x$, we get a product on A which is commutative and associative. Indeed, one gets

$$\begin{aligned}(xy)z &= (x \prec y + y \prec x) \prec z + z \prec (xy) = x \prec (yz) + y \prec (xz) + z \prec (xy)\\ &= x \prec (yz) + y \prec (zx) + z \prec (yx) = x \prec (yz) + (y \prec z) \prec x + (z \prec y) \prec x = x(yz).\end{aligned}$$

Hence there is a forgetful functor between categories of algebras:

$$\mathit{Zinb}\text{-}\mathsf{alg} \to \mathit{Com}\text{-}\mathsf{alg}.$$

2.2 Free Zinbiel Algebra

It is shown in [8] that the free Zinbiel algebra over the vector space V is the reduced tensor module

$$\bar{T}(V) := V \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

where the generic element $v_1 \cdots v_n \in V^{\otimes n}$ corresponds to the product $v_1 \prec (v_2 \prec (\cdots \prec v_n))$. As a consequence the space of n-ary operations of the operad *Zinb* is the $\mathbb{S}_n$-module $\mathit{Zinb}(n) = \mathbb{K}[\mathbb{S}_n]$. More precisely we have:

Theorem 2.3. *Let V be a vector space and let $\bar{T}(V)$ be the reduced tensor module over V. The half-shuffle*:

$$v_1 \cdots v_p \prec v_{p+1} \cdots v_{p+q} := \sum_{\sigma} v_1 v_{\sigma^{-1}(2)} \cdots v_{\sigma^{-1}(p+q)}$$

where σ is a $(p-1,q)$-shuffle acting on $\{2,\dots,p+q\}$, makes $\bar{T}(V)$ into the free Zinbiel algebra on V, also denoted by $Zinb(V)$. The associated commutative algebra is the (nonunital) shuffle algebra, denoted by $\bar{T}^{sh}(V)$.

Proof. See [8]. □

2.4 Example

If V is one-dimensional spanned by x, then $Zinb(\mathbb{K}x)$ is spanned by $x^{\prec n}$ for $n \geq 1$. The operations are given by

$$x^{\prec p} \prec x^{\prec q} = \binom{p+q-1}{q-1} x^{\prec p+q} \quad \text{and} \quad x^{\prec p} x^{\prec q} = \binom{p+q}{q} x^{\prec p+q}.$$

Therefore it follows that $x^n = n!x^{\prec n}$.

2.5 Dendriform Algebra

A *dendriform algebra* is a $\mathbb{K}$-module A equipped with two linear maps (binary operations)

$$\prec: A \otimes A \to A \quad \text{and} \quad \succ: A \otimes A \to A$$

called the *left operation* and the *right operation* respectively, satisfying the following three relations

$$\begin{cases} (x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z), \\ (x \succ y) \prec z = x \succ (y \prec z), \\ (x \prec y) \succ z + (x \succ y) \succ z = x \succ (y \succ z). \end{cases}$$

From these axioms it follows readily that the binary operation

$$xy := x \prec y + x \succ y.$$

is associative. Under this notation the first relation becomes

$$(x \prec y) \prec z = x \prec (yz).$$

In the proof of the main Theorem we allow ourselves to write $x \prec yz$ in place of $x \prec (yz)$.

Let us mention that numerous combinatorial Hopf algebras come with a dendriform structure, cf. for instance [1, 2, 6, 9, 10].

2.6 Commutative Dendriform Algebra

By definition a commutative dendriform algebra is a dendriform algebra whose left and right operations are related by the following symmetry relation

$$x \prec y = y \succ x, \text{ for any } x \text{ and } y.$$

By direct inspection we see that a Zinbiel algebra is a commutative dendriform algebra and vice-versa (cf. [8]).

Proposition 2.7. *The module of $n \times n$-matrices $\mathscr{M}_n(A)$ with coefficients in the Zinbiel algebra A is a dendriform algebra.*

Proof. It is straightforward to check that the matrices over a dendriform algebra is still a dendriform algebra. The formulas are like in the classical case. Since a Zinbiel algebra is a particular case of a dendriform algebra, we are done. □

2.8 Dendriform Calculus

We recall some results from [8] about computation in unital dendriform algebras. By definition a *unital dendriform algebra* A is a module of the form $A = \mathbb{K}1 \oplus I$ where I is a dendriform algebra. The left and right operations of I are partially extended to A by the formulas:

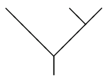
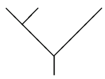
$$x \prec 1 = x, \qquad 1 \prec x = 0,$$
$$x \succ 1 = 0, \qquad 1 \succ x = x,$$

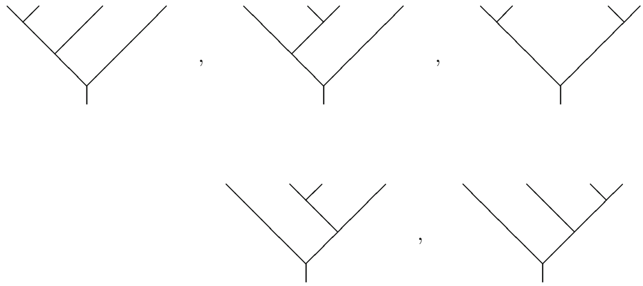
for any element $x \in I$.

The element 1 is a unit for the associative product $xy = x \prec y + x \succ y$. However the products $1 \prec 1$ and $1 \succ 1$ are not defined, so, in order for an expression like $(1+u)(1+v)$ to make sense, we have to write it as

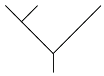
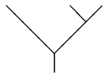
$$(1+u)(1+v) = 1 + u + v + uv = 1 + u + v + u \prec v + u \succ v.$$

The *free unital dendriform algebra* on one generator is spanned by the planar rooted binary trees. We denote by PBT_n the set of planar rooted binary trees t with n leaves. In low dimension we have:

$PBT_1 = \{ | \}$, $PBT_2 = \{$  $\}$, $PBT_3 = \{$  ,  $\}$,

$PBT_4 = \{$  $\}.$

When t has $n+1$ leaves (i.e. $t \in PBT_{n+1}$), it determines an n-ary operation. Applied to the generic n-tuple $x_1 \cdots x_n$ we denote the result by $t(x_1 \cdots x_n)$. For instance:

$|() = 1,$  $(x) = x,$  $(xy) = x \succ y,$  $(xy) = x \prec y.$

More generally, if the tree t is the grafting of t^l and t^r, denoted by $t = t^l \vee t^r$, then

$$t(x_1 \cdots x_n) = t^l(x_1 \cdots x_{i-1}) \succ x_i \prec t^r(x_{i+1} \cdots x_n).$$

The free dendriform algebra over the vector space V is $Dend(V) = \bigoplus_n Dend_n \otimes V^{\otimes n}$ where $Dend_n = \mathbb{K}[PBT_{n+1}]$. In order to check an equality of multilinear elements in the free dendriform algebra over the set $\{x_1, \ldots, x_n\}$ it is equivalent to check the equality for each permutation of the variables individually (in other terms the operad *Dend* is a nonsymmetric operad).

It is sometimes helpful to adopt the notation $x^t := t(x \cdots x)$ or simply t when there is only one variable into play. For instance we have: $x^t = x^{\prec n} = x \prec (x \prec (\cdots \prec x))$ for t the right comb with $n+1$ leaves. It can be shown that

$$x^n = \sum_{t \in PBT_{n+1}} x^t$$

in any dendriform algebra.

We already introduced the notation $x^{\prec n}$. Similarly we define $x^{\succ n} := (x^{\succ n-1}) \succ x$, with $x^{\succ 1} := x$.

If we now work in a Zinbiel algebra, then it can be shown that $x^t = \#\varphi^{-1}(t)x^{\prec n}$ where $\varphi : \mathbb{S}_n \to PBT_{n+1}$ is the surjective map constructed in *loc.cit.* As a consequence we have

$$x^n = \sum_{t \in PBT_{n+1}} x^t = \sum_{t \in PBT_{n+1}} \#\varphi^{-1}(t)x^{\prec n} = \sum_{\sigma \in \mathbb{S}_n} x^{\prec n} = n!x^{\prec n},$$

as we already know.

2.9 Pre-Lie Product and Brace Products

Recall that a binary operation is said to be *left pre-Lie* if its associator is symmetric in the first two variables. In a dendriform algebra the binary operation

$$\{x,y\} := x \succ y - y \prec x$$

is a left pre-Lie product. It is a direct consequence of the dendriform axioms. More generally, one can form the following $(n+1)$-ary operation:

$$\{x_1,\dots,x_n;y\} := \sum_{i=0}^{n} (-1)^{n-i} (x_1 \prec (x_2 \prec (\cdots \prec x_i))) \succ y \prec (((x_{i+1} \succ \cdots) \succ x_{n-1}) \succ x_n).$$

In [11] Ronco showed that these operations are primitive for the Hopf structure of the free dendriform algebra and that they satisfy the axioms of a *brace algebra*. They are called *brace products*. Observe that $\{x;y\} = \{x,y\}$ is the left pre-Lie product.

2.10 A Commutative Diagram of Algebras

The relationship between all these types of algebras can be summarized by the existence of a commutative diagram of categories of algebras [8, 11]:

$$\begin{array}{ccccc}
\mathit{Zinb}\text{-alg} & \longrightarrow & \mathit{Dend}\text{-alg} & \longrightarrow & \mathit{Brace}\text{-alg} \\
\downarrow & & \downarrow & & \downarrow \\
\mathit{Com}\text{-alg} & \longrightarrow & \mathit{Ass}\text{-alg} & \longrightarrow & \mathit{Lie}\text{-alg}
\end{array}$$

Considering free algebras on one generator x we get the polynomials $\mathbb{K}[x]$ for *Zinb*, *Com* and *Ass*, and we get $\bigoplus_n \mathbb{K}[PBT_{n+1}]$ for *Dend*:

$$\begin{array}{ccc}
\mathbb{K}[x] & \twoheadleftarrow & \bigoplus_n \mathbb{K}[PBT_{n+1}] \\
\uparrow & & \uparrow \\
\mathbb{K}[x] & \xleftarrow{\cong} & \mathbb{K}[x]
\end{array}
\qquad
\begin{array}{ccc}
n!\, x^{\prec n} & \longleftarrow\!\!\shortmid & \sum_t x^t \\
\uparrow & & \uparrow \\
x^n & \longleftarrow\!\!\shortmid & x^n
\end{array}$$

Observe that there are various possibilities to lift the element $\frac{x^n}{n!}$ to the free dendrifrom algebra, for instance $\frac{x^t}{\#\varphi^{-1}(t)}$ for any $t \in PBT_{n+1}$.

3 Exponential Series

We introduce the exponential series without denominators and we compute their inverse for the associative product and for composition (logarithm without denominators).

3.1 Exponential Series Without Denominators

Let V be a module over $\mathbb{K}$ and let $Dend(V)^\wedge$ be the infinite product

$$Dend(V)^\wedge := \prod_n Dend_n \otimes V^{\otimes n}.$$

For any $x \in V$ a series in x is an element of $Dend(V)^\wedge$ made out of products of the only variable $x \in V$.

Since in a Zinbiel algebra $x^n = n! x^{\prec n}$, there are several different ways to extend the exponential series $\exp(x) = e(x) = e'(x)$ from Zinbiel algebras to dendriform algebras. For instance we have

1. $\exp(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$, the classical exponential series when $\mathbb{Q} \subset \mathbb{K}$,
2. $e(x) := 1 + \sum_{n\geq 1} x^{\prec n}$, $e'(x) := 1 + \sum_{n\geq 1} x^{\succ n}$, the exponential series without denominators,
3. $ee(x) := 1 + \sum_{n\geq 1} \frac{1}{2}(x^{\prec n} + x^{\succ n})$, when 2 is invertible in $\mathbb{K}$.

Observe that we have $e(x) = 1 + x \prec e(x)$.

Lemma 3.2. *In the dendriform context we have*:

$$e'(-x)e(x) = 1 = e(x)e'(-x).$$

Proof. Let us first prove that $x^{\succ i} \succ x^{\prec j} = x^{\succ i+1} \prec x^{\prec j-1}$. We have

$$\begin{aligned} x^{\succ i} \succ x^{\prec j} &= x^{\succ i} \succ (x \prec x^{\prec j-1}) \\ &= (x^{\succ i} \succ x) \prec x^{\prec j-1} \\ &= x^{\succ i+1} \prec x^{\prec j-1} \ . \end{aligned}$$

Expanding the product $e'(-x)e(x)$ we get

$$x^{\prec n} + \cdots + (-1)^j (x^{\succ i} \succ x^{\prec j} + x^{\succ i} \prec x^{\prec j}) + \cdots + (-1)^n x^{\prec n}$$

in degree $n > 0$. This element is 0 as a consequence of the preceding formula. The second formula is an immediate consequence of the first one. $\square$

Proposition 3.3. *In the dendriform context, the inverse of the exponential series* $E(x) := e(x) - 1$ *for composition is the series*

$$L(x) := x \prec (1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots),$$

called the logarithm without denominators.

Proof. Let us write $L(x) = x + \varphi_2(x) + \cdots + \varphi_n(x) + \cdots$ for the inverse of $E(x)$. Since $E(x) = x \prec (1 + E(x))$, by replacing x by $L(x)$ in this equality we get

$$x = L(x) \prec (1 + x).$$

Hence we have $\varphi_2(x) = -x \prec x$ and $\varphi_{n+1}(x) + \varphi_n(x) \prec x = 0$. By induction we suppose that $\varphi_n(x) = (-1)^{n-1} x \prec x^{n-1}$. We compute:

$$\begin{aligned}
\varphi_{n+1}(x) &= -\varphi_n(x) \prec x, \\
&= -(-1)^{n-1}(x \prec x^{n-1}) \prec x, \\
&= (-1)^n x \prec (x^{n-1} \prec x + x \succ x^{n-1}), \\
&= (-1)^n x \prec (x^n).
\end{aligned}$$

□

This proposition gives a quick proof of the fact that exp and log are inverse to each other. This proof is similar to the proof which uses the integral definition of the logarithm: $\log(1+x) = \int \frac{dx}{1+x}$.

3.4 Zinbiel Algebras in Characteristic p

Let us suppose that $\mathbb{K}$ is a characteristic p field and let us work in the Zinbiel framework. Since $x^n = n! x^{\prec n}$ the logarithm becomes a polynomial which can be written

$$\log(1+x) = L(x) := \sum_{i=1}^{p-1} (-1)^{i-1} \frac{x^i}{i} + (-1)^{p-1}(p-1)! x^{\prec p}.$$

First, the element $x^{\prec p}$ is a divided power, that is $(A, \gamma(x) := x^{\prec p})$ is a divided power algebra, cf. for instance [3]. Second, the first part of the logarithm is the so-called "one and a half logarithm" introduced by Maxim Kontsevich in [4]. In other words we have the following result.

Proposition 3.5. *In a characteristic p divided power algebra (resp. Zinbiel algebra) the one and a half logarithm plus* $(-1)^{p-1}(p-1)!\,\gamma(x)$ *is an invertible series whose inverse is the exponential series.* □

4 BCH-Type Formula for the Exponential Series Without Denominators

In a Zinbiel algebra we have the functional equation $e(x)e(y) = e(x+y)$. But, in a dendriform algebra, $x \succ y \neq y \prec x$ (even for $x = y$), hence there are nontrivial dendriform polynomials $H_n(x,y)$ in x and y such that

$$e(x)e(y) = e(x+y+\cdots+H_n(x,y)+\cdots).$$

For instance we obviously have $H_2(x,y) = x \succ y - y \prec x$, which is the (left) pre-Lie product $\{x,y\}$. Our aim is to compute $H_n(x,y)$, in fact to show that it is a brace product.

Theorem 4.1. *In a dendriform algebra the following equalities hold*:

$$e(x)e(y) = e\big(x + e(x) \succ y \prec e'(-x)\big),$$

$$e'(x)e'(y) = e'\big(e(-y) \succ x \prec e'(y) + y\big),$$

for $e(x) := 1 + \sum_{n\geq 1} x^{\prec n}$ *and* $e'(x) := 1 + \sum_{n\geq 1} x^{\succ n}$.

Corollary 4.2. *The dendriform polynomial* H_n *is the brace product*:

$$H_n(x,y) = \sum_{i+j=n-1} (-1)^j (x^{\prec i}) \succ y \prec (x^{\succ j}) = \{\underbrace{x,\ldots,x}_{n-1};y\}\ .$$

Proof (of Theorem 4.1). We prove the first relation. Let us define

$$R(x,y) := e(x)e(y) - e\big(x + e(x) \succ y \prec e'(-x)\big).$$

The aim is to prove that $R(x,y) = 0$.

We use the following two relations: $e(z) = 1 + z \prec e(z)$ and $e'(-x)e(x) = 1$ (Lemma 3.2). On one hand we have

$$\begin{aligned} e\big(x + e(x) \succ y \prec e'(-x)\big) &= 1 + \big(x + e(x) \succ y \prec e'(-x)\big) \prec (e(x)e(y) - R(x,y)) \\ &= 1 + x \prec (e(x)e(y)) + (e(x) \succ y) \prec (e'(-x)e(x)e(y)) \\ &\quad - \Phi(x,y), \end{aligned}$$

where $\Phi(x,y) := (x + e(x) \succ y \prec e'(-x)) \prec R(x,y)$. Then we have

$$\begin{aligned} e\big(x + e(x) \succ y \prec e'(-x)\big) &= 1 + x \prec (e(x)e(y)) + e(x) \succ y \prec e(y) - \Phi(x,y) \\ &= 1 + x \prec (e(x)e(y)) + e(x) \succ E(y) - \Phi(x,y), \end{aligned}$$

by Lemma 3.2. On the other hand we have:

$$\begin{aligned} e(x)e(y) &= e(x)(1+E(y)) \\ &= e(x)+e(x) \prec E(y)+e(x) \succ E(y). \end{aligned}$$

Hence we compute

$$\begin{aligned} R(x,y) &= e(x)+e(x) \prec E(y)) - 1 - x \prec (e(x)e(y)) + \Phi(x,y) \\ &= e(x)+e(x) \prec E(y) - 1 - x \prec (e(x)(1+E(y)) + \Phi(x,y) \\ &= e(x)+e(x) \prec E(y) - 1 - x \prec e(x) - x \prec (e(x)E(y)) + \Phi(x,y) \\ &= e(x) \prec E(y) - x \prec (e(x)E(y)) + \Phi(x,y) \\ &= e(x) \prec E(y) - (x \prec e(x)) \prec E(y) + \Phi(x,y) \\ &= 1 \prec E(y) + \Phi(x,y) \\ &= \Phi(x,y). \end{aligned}$$

So we have proved that $R(x,y)$ satisfies the functional equation:

$$R(x,y) = (x+e(x) \succ y \prec e'(-x)) \prec R(x,y)\ .$$

Since $R(0,0)=0$ we see from this equation that the degree 1 part of $R(x,y)$ is also 0, and, by induction, the degree n part of $R(x,y)$ is 0 for any n. So $R(x,y)=0$ and we are done.

The second formula follows from the fact that the involution $\tau : Dend(V) \to Dend(V)$ which sends $\prec$ to $\succ$, $\succ$ to $\prec$ and $v_1 \cdots v_n$ to $v_n \cdots v_1$ is an isomorphism of dendriform algebras.

The Corollary is an immediate consequence. □

4.3 Functional Equation in One Variable

In the associative case, if $x=y$, then the functional equation of the classical exponential series is the same as in the commutative case. However in the dendriform case, if $x=y$, then the functional equation is different from the Zinbiel case because $x \prec x \neq x \succ x$. As an immediate corollary of the main result the term $H_n(x,x)$ in the formula

$$e(x)e(x) = e\Big(2x + \sum_{n\geq 2} H_n(x,x)\Big)$$

is as follows. Let us denote by lc^i (resp. rc^i) the left comb (resp. right comb) with i leaves and let us identify $\mathbb{K}[PBT_{n+1}]$ with the degree n part of the free dendriform algebra $Dend(\mathbb{K}x)$.

Corollary 4.4. *In the dendriform context we have*:

$$H_n(x,x) = \sum_{\substack{i+j=n+1 \\ i\geq 1, j\geq 1}} (-1)^j rc^i \vee lc^j \in \mathbb{K}[PBT_{n+1}] \ for\ n \geq 2.$$

Proof. It follows from Corollary 4.2 and from the following equalities in the free dendriform algebra on one generator:

$$rc^i(x,\ldots,x) = x^{\prec i-1}, \quad lc^j(x,\ldots,x) = x^{\succ j-1}.$$

□

4.5 The Multilinear Case

For the classical exponential series in the associative case, the Baker-Campbell-Hausdorff formula takes the form

$$\exp(x)\exp(y) = \exp(x+y+\cdots+BCH_m(x,y)+\cdots).$$

One of the way to compute the Lie polynomial $BCH_m(x,y)$ is to consider the multilinear version

$$\exp(x_1)\cdots\exp(x_n) = \exp(x_1+\cdots+x_n+\cdots+BCH_m(x_1,\ldots,x_n)+\cdots).$$

It is a result of Dynkin that $BCH_m(x,y)$ can be computed out of the multilinear part of $BCH_n(x_1,\ldots,x_n)$ denoted $eul(x_1,\ldots,x_n) \in \mathbb{Q}[\mathbb{S}_n]$. It is known, cf. for instance [5], that $eul(x_1,\ldots,x_n)$ is the Eulerian idempotent.

Let us multilinearize similarly the functional equation of the series $e(x)$ in the dendriform case:

$$e(x_1)\cdots e(x_n) = e(x_1+\cdots+x_n+\cdots+H_m(x_1,\ldots,x_n)+\cdots).$$

and denote by $h(x_1,\ldots,x_n)$ the multilinear part of $H_n(x_1,\ldots,x_n)$.

Proposition 4.6. *The dendriform polynomial $h(x_1,\ldots,x_n)$ is the iterated pre-Lie product*:

$$h(x_1,\ldots,x_n) = \{x_1,\{x_2,\{\cdots\{x_{n-1},x_n\}\cdot\}\}\},$$

where $\{x,y\} := x \succ y - y \prec x$.

Proof. In order to compute $e(x_1)\cdots e(x_n)$ we can apply the functional equation of Theorem 4.1 iteratively. It comes immediately that the polynomial $h(x_1,\ldots,x_n)$ is the multilinear part of

$$e(x_1) \succ \Big(e(x_2) \succ \big(\cdots(e(x_{n-1}) \succ x_n \prec e'(-x_{n-1}))\cdots\big) \prec e'(-x_2)\Big) \prec e'(-x_1).$$

It comes

$$h(x_1,\ldots,x_n) = \sum(-1)^\ell x_{i_1} \succ \Big(x_{i_2} \succ \big(\cdots(x_{i_k} \succ x_n \prec x_{j_\ell}) \prec \cdots\big) \prec x_{j_2}\Big) \prec x_{j_1},$$

where $\{i_1 \,\cdots\, i_k \mid j_1 \,\cdots\, j_\ell\}$ is a (k,l)-shuffle of $\{1,\ldots,n-1\}$.

It is clear that for $n=2$ we get $h(x_1,x_2) = x_1 \succ x_2 - x_2 \prec x_1 = \{x_1,x_2\}$. Then, by induction we get the expected result. □

Acknowledgements This work was partially supported by the French-Bulgarian project RILA under the contract Egide-Rila N112.

References

1. Aguiar, M.: Infinitesimal bialgebras, pre-Lie and dendriform algebras. In: Hopf Algebras, pp. 1–33. Lecture Notes in Pure and Applied Mathematics, vol. 237. Dekker, New York (2004)
2. Burgunder, E., Ronco, M.: Tridendriform structure on combinatorial Hopf algebras (English summary). J. Algebra **324**(10), 2860–2883 (2010)
3. Dokas, I.: Zinbiel algebras and commutative algebras with divided powers. Glasg. Math. J. **52**(2), 303–313 (2010)
4. Kontsevich, M.: The $1\frac{1}{2}$-logarithm. Appendix to: "On poly(ana)logs. I" [Compos. Math. **130**(2), 161–210 (2002)] by P. Elbaz-Vincent and H. Gangl. Compos. Math. **130**(2), 211–214 (2002)
5. Loday, J.-L.: Série de Hausdorff, idempotents eulériens et algèbres de Hopf. Exposition. Math. **12**(2), 165–178 (1994)
6. Loday, J.-L.: Algèbres ayant deux opérations associatives (digèbres). C. R. Acad. Sci. Paris Sér. I Math. **321**(2), 141–146 (1995)
7. Loday, J.-L.: Cup-product for Leibniz cohomology and dual Leibniz algebras. Math. Scand. **77**(2), 189–196 (1995)
8. Loday, J.-L.: Dialgebras. In: Dialgebras and Related Operads, pp. 7–66. Lecture Notes in Mathematics, vol. 1763. Springer, Berlin (2001)
9. Loday, J.-L., Ronco, M.: Combinatorial Hopf algebras. In: Quanta of Maths, pp. 347–383. Clay Math. Proc., vol. 11. American Mathematical Society, Providence (2010)
10. Novelli, J.-Ch., Thibon, J.-Y.: Hopf algebras and dendriform structures arising from parking functions. Fund. Math. **193**(3), 189–241 (2007)
11. Ronco, M.: Primitive elements in a free dendriform algebra. In: New Trends in Hopf Algebra Theory, La Falda, 1999, pp. 245–263. Contemp. Math., vol. 267. American Mathematical Society, Providence (2000)

New Methods in Conformal Partial Wave Analysis

Christoph Neumann, Karl-Henning Rehren, and Lena Wallenhorst

Abstract We report on progress towards the partial wave analysis of higher correlation functions in conformal quantum field theory.

1 Introduction

Partial wave analysis (PWA) is a powerful tool in conformal quantum field theory. It gives not only information about the field content and the operator product expansion (OPE) of a model [7, 8], but can also be used for the analysis whether the inner product induced by the correlation functions is positive (Wightman positivity) [11].

Positivity is difficult to establish because it is a nonlinear property. It also necessarily involves correlation functions of any number of fields [16]. The most prominent example is the classification of central charges below 1 of the Virasoro algebra. An example in four spacetime dimensions (4D) is the result that conformal scalar fields with global conformal invariance (GCI, [10]) are necessarily Wick squares of free fields [13], and cannot couple in a nontrivial manner to other fields [2].

C. Neumann • L. Wallenhorst
Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1, 37077 Göttingen, Germany
e-mail: christoph.neumann@theorie.physik.uni-goettingen.de; lena.wallenhorst@theorie.physik.uni-goettingen.de

K.-H. Rehren (✉)
Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1, 37077 Göttingen, Germany
and
Courant Research Centre "Higher Order Structures in Mathematics", Universität Göttingen, Bunsenstr. 3–5, 37073 Göttingen
e-mail: rehren@theorie.physik.uni-goettingen.de

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_8,

While conformal PWA for 4-point functions is well understood [6], we intend to develop methods for higher correlation functions. The basic task is to decompose a correlation function of conformally covariant fields into a sum over partial waves

$$(\Omega,\phi_1(x_1)\dots\phi_n(x_n)\Omega) = \sum_\lambda (\Omega,\phi_1(x_1)\dots\phi_{k-1}(x_{k-1})\Pi_\lambda\phi_k(x_k)\cdots\phi_n(x_n)\Omega), \quad (1)$$

where Π_λ is the projection to the subspace of the Hilbert space which carries the irreducible representation λ of the conformal group. A projection can be inserted in any position within the correlation, so that the n-point partial waves depend on $n-1$ representations, where the first and last projections are redundant because they are fixed by the first and the last field.

In principle, the non-vanishing partial waves give information about the contributions to the OPE of two or more fields [8]. Since a projection is a positive operator, each partial wave contribution of the form

$$(\Omega,\phi'(x_1)\dots\phi(x_n)\Pi_\lambda\phi(x_{n+1})\dots\phi'(x_{2n})\Omega)$$

must separately satisfy Wightman positivity (i.e., after smearing with test functions $\overline{f(x_n,\dots,x_1)}f(x_{n+1},\dots,x_{2n})$ it must yield a non-negative number which is the norm square of the vector $\Pi(\phi\otimes\cdots\otimes\phi')(f)\Omega$, assuming the fields to be hermitean). More generally, partial waves are subject to Cauchy-Schwartz type inequalities.

Now, partial waves are to a large extent determined by conformal symmetry, being solutions to eigenvalue equations for the Casimir operators of the conformal group. Therefore, the positivity requirement reduces to the positivity of a numerical coefficient, the partial wave amplitude, which multiplies a model-independent partial wave function [11].

Conformal PWA is by now mostly limited to 4-point functions, because the higher partial waves are not sufficiently well known. Even for 4 points, the determination of partial waves in 4D required a considerable effort [6]. Moreover, the decomposition of a given correlation function into a known system of partial waves may not be a straight-forward task without a suitable notion of orthogonality between the partial waves. Some progress was made in [11] giving a systematic expansion formula for scalar 4-point partial waves, and in [14] for a suitable notion of orthogonality.

In this note, we report some further intermediate progress. In Sect. 2, we present a power series representation (5) for general n-point partial waves in two spacetime dimensions (2D) for all n, extending known formulae for $n \le 4$. In 4D, however, finding such an expansion seems unrealistic because of the complicated structure of the higher-order Casimir operators which the partial waves must diagonalize, and because the partial waves are no longer unique.

In Sect. 3 we therefore present an alternative to the actual decomposition (1), which is applicable also in 4D. The idea is a successive reduction of n-point functions to $n-1$-functions, in terms of local linear maps $\phi_1(x_1)\phi_2(x_2)\Omega \to \phi_\lambda(x)\Omega$ selecting each contribution to the OPE of the last two (or the first two) fields in the correlation. Our main result is the characterization of these linear maps as partial differential operators that intertwine the respective representations of the conformal group. This property is encoded in (10), which is subsequently solved. Acting on the correlation functions, the intertwiners effectuate the desired reduction. As we shall see, this method is applicable only for representations of integer scaling dimension (otherwise, the differential operators would have to be replaced by integral kernels [5], and locality would become a nontrivial issue).

This method is therefore well-suited for QFT with global conformal invariance, where all correlation functions are rational functions [10]. We shall apply it in Sect. 4 to address the problem of positivity of a class of "exotic" higher ($n \geq 6$) correlation structures of twist 2. The motivation is the following.

Twist-2 contributions in free field theories above the unitarity bound arise from quadratic Wick products such as $:\varphi^*(x_1)\varphi(x_2):$ or $x_{12\mu}:\overline{\psi}(x_1)\gamma^\mu\psi(x_2):$, in which each factor can be contracted "only once", so that both variables can only have poles w.r.t. one other variable. In contrast, the exotic structures contain so-called double poles, thus indicating a nontrivial theory. These are strongly constrained by the conservation laws for twist-2 fields [13], allowing for a classification [4]. In particular, they cannot arise in correlations of less than six fields. While the exotic structures satisfy all linear properties, it remains an open problem whether they are compatible with positivity.

First steps of the positivity analysis of the simplest exotic structure will be reported in Sect. 4.

2 Higher Chiral Partial Waves

Irreducible representations λ of the conformal group are eigenspaces of the Casimir operators. Thus, correlation functions with projections onto irreducible subrepresentations inserted:

$$\langle \Omega, \phi_1(x_1)\Pi_{\lambda_1}\phi_2(x_2)\cdots\Pi_{\lambda_{i-1}}\phi_i(x_i)\Pi_{\lambda_i}\cdots\phi_{n-1}(x_{n-1})\Pi_{\lambda_{n-1}}\phi_n(x_n)\Omega\rangle \tag{2}$$

are eigenvectors of the corresponding differential operators arising by commuting the conformal generators with the fields. Partial waves are, by definition, solutions to the same eigenvalue differential equations, with some standard normalization. These are "universal" in the sense that they are completely determined by conformal symmetry. They depend on the sequence of representations $(\mu_i)_{i=1\ldots n}$ of the fields ϕ_i in (2), and on the sequence of representations $(\lambda_i)_{i=1\ldots n-1}$ of the projections, where $\lambda_1 = \overline{\mu}_1$ and $\lambda_{n-1} = \mu_n$ are redundant.

The projected correlations (2) are multiples of the partial waves. The coefficients contain model-specific information, and Wightman positivity can be formulated as a system of numerical inequalities on the partial wave coefficients [11].

The conformal Lie algebra in 4D, $so(4,2)$, has three Casimir operators (quadratic, cubic and quartic in the generators). In contrast, the conformal Lie algebra in 2D factorizes: $so(2,2) \sim sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$, and each $sl(2,\mathbb{R})$ has one quadratic Casimir operator. For this reason, the Casimir eigenvalue differential equations are much simpler (both, to write down and to solve) in 2D.

The relevant positive-energy representations of $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ are parameterized by the chiral scaling dimensions $d_\pm$, such that $d_+ + d_-$ is the total scaling dimension, and $d_+ - d_-$ the helicity.

Because of the chiral factorization of the conformal group, also the partial waves factorize. In the sequel, we display only chiral partial waves as functions of either $x_+ = t + x$ or $x_- = t - x$, and suppress the subscript. Thus, a general projected correlation function in 2D has the form of a product of two chiral functions

$$\langle \Omega, \phi_1(x_1) \Pi_{a_1} \phi_2(x_2) \cdots \Pi_{a_{i-1}} \phi_i(x_i) \Pi_{a_i} \cdots \phi_{n-1}(x_{n-1}) \Pi_{a_{n-1}} \phi_n(x_n) \Omega \rangle \tag{3}$$

where the x_i are chiral variables, the chiral fields ϕ_i have chiral dimensions d_i, and Π_a are the projections onto the chiral representations with chiral scaling dimension a. In particular, $a_1 = d_1$ and $a_{n-1} = d_n$ are fixed.

The Casimir eigenvalue equation for the projector insertion Π_{a_i} reads

$$\Big(\sum_{i<j<k} x_{jk}^2 \partial_j \partial_k + 2 \sum_{i<j,k} d_j (x_{jk} \partial_k) + \sum_{i<k} d_k - \Big(\sum_{i<k} d_k \Big)^2 \Big) \langle \dots \Pi_{a_i} \phi_{i+1}(x_{i+1}) \dots \rangle$$
$$= (a_i - a_i^2) \langle \dots \Pi_{a_i} \phi_{i+1}(x_{i+1}) \dots \rangle,$$

which is equivalent by conformal invariance to

$$\Big(\sum_{j<k\le i} x_{jk}^2 \partial_j \partial_k + 2 \sum_{j,k\le i} d_j (x_{jk} \partial_k) + \sum_{k\le i} d_k - \Big(\sum_{k\le i} d_k \Big)^2 \Big) \langle \dots \phi_i(x_i) \Pi_{a_i} \dots \rangle$$
$$= (a_i - a_i^2) \langle \dots \phi_i(x_i) \Pi_{a_i} \dots \rangle.$$

In principle, these equations can be reformulated in terms of $n-3$ independent conformal cross ratios. It turns out convenient to choose

$$u_k = \frac{x_{kk+1} x_{k+2k+3}}{x_{kk+2} x_{k+1k+3}}.$$

We have worked out the invariant differential equations for $n \le 6$ points: Let

$$(3) = \frac{f(u_1,u_2,u_3)}{x_{12}^{d_1+d_2-d_3} x_{13}^{d_1+d_3-d_2} x_{23}^{d_2+d_3-d_1} \cdot x_{45}^{d_4+d_5-d_6} x_{46}^{d_4+d_6-d_5} x_{56}^{d_5+d_6-d_4}}.$$

Then (with the Euler operators $E_i = u_i \partial_{u_i}$)

$$\begin{aligned}(E_1+d_3-a_2)(E_1+d_3+a_2-1)f &= u_1(E_1+E_2)(E_1+d_1-d_2+d_3)f,\\ (E_2-a_3)(E_2+a_3-1)f &= u_2(E_2+E_1)(E_2+E_3)f,\\ (E_3+d_4-a_4)(E_3+d_4+a_4-1)f &= u_3(E_3+E_2)(E_3+d_6-d_5+d_4)f. \end{aligned} \tag{4}$$

(The cases $n<6$ are covered by admitting the trivial field **1** of dimension 0.) This system is obviously symmetric under hermitian conjugation $1,2,\dots,6 \to 6,5,\dots,1$. It can be recursively solved as a power series with leading powers $u_1^{a_2-d_3}u_2^{a_3}u_3^{a_4-d_4}$.

From the solution with $n \le 6$, we have extrapolated the general power series expansion for all n, as follows. By default, we put $a_0 = a_n := 0$, and $\ell_0 = \ell_{n-2} := 0$.

Proposition 1. *The general chiral n-point partial wave is*

$$\sum_{\ell_1,\dots,\ell_{n-3}\ge 0} \frac{\prod_{j=1}^{n-2} x_{jj+2}^{d_{j+1}-a_j-a_{j+1}} (a_j+a_{j+1}-d_{j+1})_{\ell_{j-1}+\ell_j}}{\prod_{i=1}^{n-1} x_{ii+1}^{d_i+d_{i+1}-a_{i-1}-a_{i+1}}} \cdot \prod_{k=1}^{n-3} \frac{u_k^{\ell_k}}{\ell_k!(2a_{k+1})_{\ell_k}}. \tag{5}$$

This formula has a remarkable "short-range" feature: It involves only coordinate distances x_{ij} with $j=i+1$ or $i+2$. The powers of x_{ii+1} and x_{ii+2} depend only on the dimensions of the fields ϕ_i, ϕ_{i+1}, respectively ϕ_{i+1}, and their adjacent projections, apart from the summation indices ℓ. The same is true for the numerical coefficients.

For $n=3$ points, this is just the 3-point function. For $n=4,5,6$ points, we have derived this formula by solving the differential equations (4) for the Casimir eigenvalues. For $n=4$, the sum is a hypergeometric series, and (5) coincides with well-known formulas.

One way to prove (5) for all n is an application of the method discussed in the next section. There, we introduce "intertwining" differential operators $\iota \circ \widehat{E}_h$ with the distinguishing property that they annihilate all partial waves carrying the "wrong" representation $a \neq h$, and reduce the n-point partial wave carrying the representation $a=h$ to an $(n-1)$-point partial wave with the first pair of fields replaced by ϕ_0 of dimension h.

Therefore, it is sufficient to show that this is true for our "candidate" partial waves (5). With (24), we have to apply the differential operator

$$\widehat{E}_h \equiv E_h \circ x_{12}^{d_1+d_2} = \Big(\sum_{p+q=h} \frac{(q-b)_p}{p!} \frac{(p+b)_q}{q!}\, \partial_1^p(-\partial_2)^q \Big) \circ x_{12}^{d_1+d_2},$$

where $b=d_1-d_2$, to (5), and then equate $x_1=x_2$. The result must be δ_{ha_2} times the reduced partial wave.

To do this, we have to exhibit all terms that involve x_1 or x_2. Equation (5) can be arranged as $x_{12}^{-d_1-d_2}$ times the sum $\sum_{\ell_2,\ldots,\ell_{n-3}}$ over

$$\left(\frac{x_{12}}{x_{13}x_{24}}\right)^a \left(\frac{x_{23}}{x_{13}}\right)^b \left(\frac{x_{23}}{x_{24}}\right)^c \sum_{\ell \geq 0} \frac{(a+b)_\ell (a+c)_\ell u^\ell}{\ell!(2a)_\ell} \times \text{remaining factors}, \tag{6}$$

where $a \equiv a_1$, $b \equiv d_1 - d_2$, $c \equiv a_3 - d_3 + \ell_2$. Notice that for each ℓ_2, the sum over ℓ is a 4-point partial wave where the 4$^{\text{th}}$ field has dimension $a_3 + \ell_3$. Thus, knowing that (5) correctly reproduces the 4-point partial waves, and that $\iota \circ \widehat{E}_h$ reduces 4-point partial waves to 3-point partial waves, the same must be true for the higher partial waves.

However, we have not been able to evaluate the result of $E_h^{d_1,d_2}$ on the power series (6), and to verify this conclusion by a direct computation. Only for $n = 3$ this can be done by the following argument. For $n = 3$, one has $c = 0$ in (6), only $\ell = 0$ contributes, and there are no "remaining factors". Then

(i) Because $E_h^{d_1,d_2}$ is a differential operator of order h, it annihilates the 3-point function whenever $h < a$, due to the surviving factors of x_{12}.
(ii) Writing $\frac{x_{12}}{x_{13}x_{23}} = \frac{1}{x_{23}} - \frac{1}{x_{13}}$ and performing the binomial expansion of its powers, $E_h^{d_1,d_2}$ can easily be applied. It is then seen by inspection that the resulting series is symmetric under the exchange $a \leftrightarrow h$. Therefore, it also vanishes whenever $h > a$.
(iii) When $h = a$, all derivatives must hit the factor x_{12}^a. That the result is the 2-point function, is then obvious.

For $n > 3$, the Leibniz rule produces multiples sums which are not easy to handle. But a trick helps: The sum in (6) equals ${}_2F_1(a+b, a+c; 2a; u)$. We then use the identity

$$\frac{x_{34}^{2a-1}}{(x_{13}x_{24})^a}\left(\frac{x_{23}}{x_{13}}\right)^b \left(\frac{x_{23}}{x_{24}}\right)^c \cdot {}_2F_1(a+b, a+c; 2a; u)$$
$$= \frac{\Gamma(2a)}{\Gamma(a+c)\Gamma(a-c)} \int_{x_3}^{x_4} dx\,(x_1 - x)^{-a-b}(x_2 - x)^{-a+b}(x_3 - x)^{a+c-1}(x - x_4)^{a-c-1},$$

which can be established by direct computation: namely, the change of variables $t = \frac{x_{24}(x_3-x)}{x_{34}(x_2-x)}$ yields precisely the standard integral representation of the hypergeometric function.

Therefore, each term (6) is, as far as its dependence on x_1 and x_2 is concerned, an integral over a 3-point function. Thus, we only have to evaluate $E_h^{d_1,d_2}$ on a 3-point function, which can be done as before. The remaining integral is again of the hypergeometric type (after the change of variables $t = -\frac{(x_3-x)}{x_{34}}$), and reproduces precisely the necessary "leading" factors for the $(n-1)$-point partial wave (5).

From this, we conclude that (5) indeed is the correct power series expansion of general n-point chiral partial waves.

3 Intertwining Differential Operators

We return to 4D. Let ϕ_1 and ϕ_2 be two conformal fields transforming in representations μ_1 and μ_2. We shall determine differential operators $\widehat{E}_\lambda$ w.r.t. x_1 and x_2 such that

$$\phi_\lambda(x) := \iota_x \circ \widehat{E}_\lambda\, \phi_1(x_1)\phi_2(x_2) \tag{7}$$

transforms like a conformal field in the representation λ. Here, ι_x is the evaluation map $\iota_x(f) = f(x_1,x_2)|_{x_1=x_2=x} = f(x,x)$.

It will become clear below that such operators exist only when the scaling dimensions satisfy $d_\lambda - d_1 - d_2 \in \mathbb{Z}$. They can therefore be expected to be exhaustive (w.r.t. λ) only in a globally conformal invariant (GCI) theory.

Such operators have been presented previously [9, Sect. VI.B] for the special case of ϕ_1 and ϕ_2 being two (complex conjugate) canonical scalar massless Klein-Gordon fields of dimension 1, in order to extract the current, the stress-energy tensor and higher conserved symmetric traceless tensor fields from $:\phi^*\phi:$. The same operators actually can be used also for scalar biharmonic bifields $V(x_1,x_2)$ which collect the twist-2 contribution in any product of two scalar fields of equal dimension [13], where biharmonicity, i.e., the wave equation w.r.t. both arguments is exploited in an essential way. We shall reproduce these operators, but there will be additional terms including the wave operators, so that (7) is true without using the equation of motion, or biharmonicity.

By conformal covariance, the assumed transformation behaviour of (7) implies

$$\iota_x \circ \widehat{E}_\lambda(\phi_\mu(y)\Omega, \phi_1(x_1)\phi_2(x_2)\Omega) = \delta_{\lambda\mu}(\phi_\mu(y)\Omega, \phi_\lambda(x)\Omega),$$

i.e., the operator annihilates all 3-point functions with fields in the "wrong" representation. In particular, if applied to the vacuum operator product expansion [8]

$$\phi_1(x_1)\phi_2(x_2)\Omega = \sum_\mu \int dx\, K_\mu^{\mu_1\mu_2}(x_1,x_2;x)\phi_\mu(x)\Omega\,,$$

it will annihilate all contributions $\mu \neq \lambda$, and if applied to a correlation function, it will annihilate all partial waves with $\mu \neq \lambda$ in the 1-2-channel, and reduce the contribution with $\mu = \lambda$ to an $n-1$-point partial wave. Thanks to the latter feature, one can perform a partial wave analysis without actually knowing the partial waves, cf. Sect. 4.

Let us now proceed to determine the differential operators.

For definiteness, we specialize to $\mu_1 = \mu_2$ being scalar representations of dimension $d_1 = d_2 = d$. In this case, only symmetric traceless tensor representations λ can occur [8]. It is convenient to write $\lambda = (\kappa, L)$ where L is the tensor rank, and 2κ the "twist", such that the scaling dimension is $d = 2\kappa + L$. The unitarity bound requires $\kappa \geq 0$ for $L = 0$, and $\kappa \geq 1$ for $L > 0$. We write a symmetric traceless tensor as $T(v) = T^{\mu_1 \cdots \mu_L} v_{\mu_1} \dots v_{\mu_1}$ which is a homogeneous polynomial of degree L in the polarization vector v. Tracelessness is equivalent to the harmonic equation $\square_v T(v) = 0$. Equation (7) implies that $\widehat{E}_{\kappa L}$ is a harmonic homogeneous polynomial of degree L in the polarization vector v. The harmonic part of any polynomial in v is uniquely determined [3], so it is sufficient to know $\widehat{E}_{\kappa L}$ up to terms involving v^2.

Let $T = P_\mu, D, M_{\mu\nu}, K_\mu$ be the generators of translations, dilations, Lorentz and special conformal transformations, respectively, and

$$i[T, \phi(x)] = t_x^\lambda \ \phi(x)$$

the commutation relations with covariant ("quasiprimary") fields, where $t^{\kappa L} = \partial$ for the translations, $= (x\partial + d_\lambda)$ for the scale transformations, $= x \wedge \partial + v \wedge \partial_v$ for the Lorentz transformations, and $= 2x(x\partial) - x^2\partial + 2(v(x\partial_v) - (xv)\partial_v) + 2d_\lambda x$ for the special conformal transformations. For the tensor representations, $d_\lambda = 2\kappa + L$, while for the scalar representations $\mu_1 = \mu_2$ the v-terms are absent and $d_\mu = d$.

Commuting the generators with (7), the assumption that ϕ_λ transforms in the representation λ is equivalent to the intertwining relations

$$\iota_x \circ \widehat{E}_\lambda \circ \left(t_{x_1}^{\mu_1} + t_{x_2}^{\mu_2}\right) = t_x^\lambda \circ \iota_x \circ \widehat{E}_\lambda . \tag{8}$$

In the case at hand, we make an Ansatz

$$\widehat{E}_\lambda = E_{\kappa L}(x_i, \partial_i, v) \circ (x_{12}^2)^d . \tag{9}$$

Notice that by virtue of the pole bounds [10], any correlation function of $\phi_1(x_1)\phi_2(x_2)$ is not more singular than $(x_{12}^2)^{-d}$, so that the differential operators $E_{\kappa L}$ act on a regular function, and the subsequent evaluation ι_x is possible (provided $E_{\kappa L}$ is regular).

Next, we evaluate the intertwining relations (8). They tell us in turn:

Translations: $(\partial_1 + \partial_2)E_{\kappa L} = 0$. Thus the differential operators do not involve the coordinate $x_1 + x_2$. Since $E_{\kappa L}$ is followed by the evaluation map ι_x, we may also assume that it does not involve the difference coordinate $x_1 - x_2$, hence $E_{\kappa L}$ involves only derivatives and the polarization vector v. Let us denote by ∇_i the derivatives with respect to the "variables" ∂_i of $E_{\kappa L}(\partial_1, \partial_2, v)$.

Scale transformations: $(\partial_1 \nabla_1 + \partial_2 \nabla_2)E_{\kappa L} = (2\kappa + L)E_{\kappa L}$. Thus, $E_{\kappa L}$ is homogeneous of degree $2\kappa + L$ in the derivatives ∂_i.

Lorentz transformations: $(\partial_1 \wedge \nabla_1 + \partial_2 \wedge \nabla_2 + v \wedge \partial_v)E_{\kappa L} = 0$. Thus, $E_{\kappa L}$ is a Lorentz scalar. It is therefore a function of $(\partial_i \partial_j)$, $(v\partial_i)$ and v^2. Together with the known homogeneities in v and in ∂_i, it can be a polynomial in the derivatives only if κ is an integer. This is in perfect agreement with GCI because tensor-scalar-scalar 3-point functions are rational only if the twist 2κ is even.

Special conformal transformations: While the previous intertwining conditions gave information about the gross structure of $E_{\kappa L}$, the special conformal transformations yield a differential equation that specifies the operators completely.

Proposition 2. *Given the previous specifications of $E_{\kappa L}(\partial_1,\partial_2,v)$ in (9) as homogeneous polynomials (of degrees depending on the parameters κ and L), the intertwining condition (8) is equivalent to*

$$\Big(2(\partial_1\nabla_1)\nabla_1-\partial_1\nabla_1^2+2(\partial_2\nabla_2)\nabla_2-\partial_2\nabla_2^2\Big)E_{\kappa L}(\partial_1,\partial_2,v)=0. \tag{10}$$

One may directly solve these equations with a polynomial Ansatz for $E_{\kappa L}$ with the specified homogeneities. A more systematic way is to write

$$E_{\kappa L}(\partial_1,\partial_2,v)=(\partial_1\partial_2)^\kappa\cdot\big[\big((v\partial_1)+(v\partial_2)\big)^L\cdot e_{\kappa L}(p,q,r)\big]_0 \tag{11}$$

where $p=\frac{\partial_1^2}{(\partial_1\partial_2)}$, $q=\frac{\partial_2^2}{(\partial_1\partial_2)}$, and $r=\frac{(v\partial_1)-(v\partial_2)}{(v\partial_1)+(v\partial_2)}$. Clearly, $e_{\kappa L}$ must be a polynomial of degree at most L in r, and degree of at most κ in p and q. The notation $[P(v)]_0$ stands for the harmonic part of the polynomial $P(v)$. The variable v^2 does not appear explicitly, because the harmonic part $[v^2Q(v)]_0=0$ for any polynomial Q [3].

With this Ansatz, the differential equation (10) turns into the system of three PDE for $e_{\kappa L}(p,q,r)$:

$$\big(L(L-1)+(1-r^2)\partial_r^2+2\kappa(L-r\partial_r)+2(p\partial_p-q\partial_q)\partial_r\big)\,e_{\kappa L}=0, \tag{12}$$

$$\begin{aligned}\big[4\big(p\partial_p-1\big)\partial_p-q(\kappa-p\partial_p-q\partial_q)(\kappa-1-p\partial_p-q\partial_q)\\ +2(\kappa-p\partial_p-q\partial_q)\big(\kappa-1-p\partial_p+q\partial_q+(r-1)\partial_r\big)\big]\,e_{\kappa L}=0,\end{aligned} \tag{13}$$

$$\begin{aligned}\big[4\big(q\partial_q-1\big)\partial_q-p(\kappa-p\partial_p-q\partial_q)(\kappa-1-p\partial_p-q\partial_q)\\ +2(\kappa-p\partial_p-q\partial_q)\big(\kappa-1+p\partial_p-q\partial_q+(r+1)\partial_r\big)\big]\,e_{\kappa L}=0.\end{aligned} \tag{14}$$

One may repeat the same strategy in 2D. In this case, the intertwining operators factorize into two chiral operators, labelled by the chiral dimensions $h_\pm$. These are polynomial functions in the chiral (one-dimensional) partial derivatives ∂_1 and ∂_2. Following the same line of arguments as in 4D, one finds the chiral intertwining condition

$$\big(\partial_1\nabla_1^2+\partial_2\nabla_2^2\big)E_h(\partial_1,\partial_2)=0,$$

where $E_h(\partial_1,\partial_2)$ is a homogeneous polynomial of degree h. Writing $E_h=(\partial_1+\partial_2)^h\cdot e_h\big(\frac{\partial_1-\partial_2}{\partial_1+\partial_2}\big)$, this reduces to the differential equation for $e_h(r)$

$$\big(h(h-1)+(1-r^2)\partial_r^2\big)e_h(r)=0,$$

which is exactly the same as the case $\kappa=0$, $L=h$ of (12).

Notice that in 4D, representations $(0,L)$ with $L\neq 0$ are below the unitarity bound. Such representations must not contribute to a correlation function. Thus, any admissible correlation function must be annihilated by the operators $\imath \circ \widehat{E}_{0L}$.

The solution for $\kappa = 0$ is

$$e_{0L}(r) = (1-r^2)\partial_r P_{L-1}(r),$$

where P_n are the Legendre polynomials. Using (11), this gives

$$E_{0L}(\partial_1, \partial_2, v) = \sum_{p+q=L} \frac{(q)_p}{p!} \frac{(p)_q}{q!} \Big[(v\partial_1)^p(-v\partial_2)^q\Big]_0, \tag{15}$$

or (in the chiral case)

$$E_h(\partial_1, \partial_2) = \sum_{p+q=h} \frac{(q)_p}{p!} \frac{(p)_q}{q!} \, \partial_1^p(-\partial_2)^q. \tag{16}$$

For $\kappa > 0$, we may expand

$$e_{\kappa L}(p,q,r) = \sum_{m,n\geq 0, m+n\leq \kappa} p^m q^n e_{\kappa L;mn}(r).$$

Then (12) must hold for each term $p^m q^n e_{\kappa L;mn}(r)$ separately, giving

$$\big((1-r^2)\partial_r^2 - 2\kappa r\partial_r + 2(m-n)\partial_r + L(L+2\kappa-1)\big)e_{\kappa L;mn}(r) = 0. \tag{17}$$

This equation involves only the difference $m-n =: \delta$. It is solved by polynomials of degree L with the symmetry $f_{\kappa L;\delta}(r) = (-1)^L f_{\kappa L;-\delta}(-r)$:

$$f_{\kappa L;\delta}(r) = (\kappa-\delta)_L \cdot {}_2F_1\Big(-L, L+2\kappa-1; \kappa-\delta; \frac{1-r}{2}\Big). \tag{18}$$

Thus, to solve (17) it remains to determine only the coefficients in

$$e_{\kappa L;mn}(r) = c_{\kappa L;mn} \cdot f_{\kappa L;m-n}(r). \tag{19}$$

Indeed, the remaining (13) and (14) turn into the recursive system

$$4(m^2-1)c_{\kappa L;m+1,n} + 2(\kappa-m-n)(L+\kappa-1-m+n)c_{\kappa L;m,n} \\ -(\kappa-m-n)(\kappa-m-n+1)c_{\kappa L;m,n-1} = 0, \tag{20}$$

$$4(n^2-1)c_{\kappa L;m,n+1} + 2(\kappa-m-n)(L+\kappa-1+m-n)c_{\kappa L;m,n} \\ -(\kappa-m-n)(\kappa-m-n+1)c_{\kappa L;m-1,n} = 0. \tag{21}$$

Here, we have used the fact [1, Eqs. (15.2.14) and (15.2.16)] that the differential operators

$$A^{\pm}_{\kappa L,\delta} := \frac{(r \mp 1)\partial_r + \kappa - 1 \mp \delta}{L + \kappa - 1 \mp \delta}$$

act as raising and lowering operators for the parameter δ:

$$A^{\pm}_{\kappa L,\delta}\, f_{\kappa L;\delta} = f_{\kappa L;\delta\pm 1}. \tag{22}$$

We conclude:

Proposition 3. *The intertwining differential operators in (7) are given by*

$$\widehat{E}_{\kappa L} = \sum_{m+n\le\kappa} c_{\kappa L,mn}(\partial_1\partial_2)^{\kappa-m-n}\Box_1^m\Box_2^n\Big[(v\partial_1+v\partial_2)^L f_{\kappa L;m-n}\big(\frac{v\partial_1-v\partial_2}{v\partial_1+v\partial_2}\big)\Big]_0 \circ (x_{12}^2)^d$$

where $[\dots]_0$ *stands for the harmonic part with respect to* $v\in\mathbb{R}^{1,3}$*, the polynomials* $f_{\kappa L;m-n}$ *are given by (18), and the coefficients* $c_{\kappa L;mn}$ *solve the recursion (20), (21).*

It may be interesting to note that $f_{\kappa L;0}$ are multiples of derivatives of Legendre polynomials (cf. [1, Eqs. (15.2.2), (15.4.4)]):

$$f_{\kappa L;0}(r) = \frac{2^{\kappa-1}L!}{(L+\kappa)_{\kappa-1}}\cdot\partial_r^{\kappa-1}P_{L+\kappa-1}(r). \tag{23}$$

so that, by (22), all functions $f_{\kappa L;mn}(r)$ are derivatives of the Legendre polynomials $P_{L+\kappa-1}(r)$. E.g., for twist 2 ($\kappa=1$), we have

$$e_{1L}(p,q,r) = \big(1+\frac{p}{2}(r-1)\partial_r+\frac{q}{2}(1+r)\partial_r\big)P_L(r).$$

The next task is to relax the assumption $\mu_1=\mu_2=$ scalar, and to find and solve the analogue of (10) in the general case. This will be necessary in order to compute the contributions from all insertions of projectors as in (1) by successive reduction according to (7).

For two scalar fields of different dimensions, $d_1\neq d_2$, the Ansatz $\widehat{E}_\lambda = E_{\kappa L}\circ (x_{12}^2)^{(d_1+d_2)/2}$ is solved by a scalar polynomial $E_{\kappa L}(\partial_1,\partial_2,v)$, homogeneous of degree $2\kappa+L$ in ∂_i, homogeneous of degree L and harmonic in v, as before, but now satisfying the differential equation

$$\big(2(\partial_1\nabla_1)\nabla_1-\partial_1\nabla_1^2+2(\partial_2\nabla_2)\nabla_2-\partial_2\nabla_2^2+(d_1-d_2)(\nabla_1-\nabla_2)\big)E_{\kappa L}(\partial_1,\partial_2,v)=0.$$

Note that the homogeneity conditions require that κ is an integer, and that in a GCI theory, fields with even twist 2κ can arise in the OPE only if d_1-d_2 is even. One would therefore have to modify the Ansatz when d_1-d_2 is odd.

Similarly, in the chiral case, the Ansatz $\widehat{E}_h^{d_1,d_2} = E_h^{d_1,d_2}\circ(x_{12})^{d_1+d_2}$ implies that $E_h^{d_1,d_2}$ is homogeneous of degree h in ∂_i and satisfies the differential equation

$$\big(\partial_1\nabla_1^2+\partial_2\nabla_2^2+(d_1-d_2)(\nabla_1-\nabla_2)\big)E_h(\partial_1,\partial_2)=0.$$

This is solved by

$$E_h^{d_1,d_2}(\partial_1,\partial_2) = \sum_{p+q=h} \frac{(q-d_1+d_2)_p}{p!} \frac{(p+d_1-d_2)_q}{q!} \, \partial_1^p(-\partial_2)^q. \tag{24}$$

4 Application: Test of Positivity of a 6-Point Structure

Recall the positivity problem for the exotic scalar 6-point structures addressed in the introduction. We consider here only the simplest example of such a structure, which has double poles and is consistent with the nontrivial constraints due to the requirement that the OPE in both the first and last pair of fields starts with twist 2 [13]. More general double pole structures have been classified in [4].

In [13], the leading part of this structure was displayed. In [12], its "tetraharmonic completion" (i.e., the biharmonic completion in both pairs of variables x_1, x_2 and x_5, x_6) was presented in terms of a transcendental function $g(s,t)$. The tetraharmonic completion is precisely the twist-2 part in both channels. Unfortunately, however, due to a wrong resummation factor, this function $g(s,t)$ was incorrectly computed in [12]. We shall display the correct function below.

The leading part of the exotic structure for four scalar fields $\phi_1, \phi_2, \phi_5, \phi_6$ of dimension d and two scalar fields ϕ_3, ϕ_4 of dimension d' is given by

$$E(x_1,\cdots,x_6) = \frac{\big(x_{15}^2 x_{26}^2 x_{34}^2 - 2x_{15}^2 x_{23}^2 x_{46}^2 - 2x_{15}^2 x_{24}^2 x_{36}^2\big)_{[1,2][5,6]}}{(x_{12}^2)^{d-1} \cdot x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 \cdot (x_{34}^2)^{d'-3} \cdot x_{35}^2 x_{45}^2 x_{36}^2 x_{46}^2 \cdot (x_{56}^2)^{d-1}}\,, \tag{25}$$

where $(\cdot)_{[k,l]}$ stands for antisymmetrization. Without loss of generality, we choose $d = d' = 3$. For comparison, we also introduce the following 6-point structure with the same symmetries as E, but which has no double poles and appears as part of the 6-point function of six cubic Wick products of a complex massless scalar free field:

$$B(x_1,\cdots,x_6) = \frac{1}{(x_{12}^2)^2} \cdot \left(\frac{1}{x_{14}^2 x_{23}^2}\right)_{[1,2]} \cdot \frac{1}{x_{34}^2} \cdot \left(\frac{1}{x_{36}^2 x_{45}^2}\right)_{[5,6]} \cdot \frac{1}{(x_{56}^2)^2}.$$

The structure B is separately biharmonic in both the 1-2 and 5-6 channels. It turns out that the tetraharmonic completion H of $B - \frac{1}{2}E$ can be written more compactly than that of E given in [12], namely

$$H(x_1,\cdots,x_6) = \big(B - \frac{E}{2}\big) \cdot g(s,t) g(s',t'), \tag{26}$$

where $s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $t = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$, and $s' = \frac{x_{34}^2 x_{56}^2}{x_{35}^2 x_{46}^2}$, $t' = \frac{x_{36}^2 x_{45}^2}{x_{35}^2 x_{46}^2}$. The condition of biharmonicity amounts to the differential equation [12]

$$\big[(1-t\partial_t)(1+t\partial_t+s\partial_s)-\big((1-t\partial_t)+t(2+t\partial_t+s\partial_s)\big)\partial_s\big]\,g=0$$

for the function $g(s,t)$. The expansion in a power series in s, $g(s,t)=\sum_n \frac{s^n}{n!}g_n(t)$, gives the recursion $(1+(n+1)t-t(1-t)\partial_t)g_n=(1-t\partial_t)(n+t\partial_t)g_{n-1}$ with $g_0(t)=1$. This can be solved in terms of hypergeometric functions, giving

$$g(s,t)=\sum_n s^n\frac{n!(n+1)!}{(2n+1)!}\cdot{}_2F_1(n,n+1;2n+2;1-t). \tag{27}$$

The sum can be performed when the integral representation of the hypergeometric functions [1, Eq. (15.3.1)] is inserted, and s,t are expressed in terms of the "chiral variables" $u_\pm$ such that $s=u_+u_-$ and $t=(1-u_+)(1-u_-)$. Then

$$\begin{aligned}
g(s,t) &= \sum_n (n+1)s^n\int_0^1 dx x^n(1-x)^n(1-(1-t)x)^{-n}\\
&= \int_0^1 dx\Big[\frac{1-(u_++u_--u_+u_-)x}{(1-u_+x)(1-u_-x)}\Big]^2\\
&= 1+2u_+u_-\cdot\frac{(1-u_+)(1-u_-)\cdot\log\frac{1-u_+}{1-u_-}+u_+-u_--\frac12u_+^2+\frac12u_-^2}{(u_+-u_-)^3}\\
&= 1+\sum_{a,b\ge1}\frac{2ab}{(a+b)((a+b)^2-1)}u_+^au_-^b\\
&= \frac{(1-u_+)(1-u_-)}{u_+-u_-}\cdot\sum_{a,b\ge0,a+b>0}\frac{a-b}{a+b}u_+^au_-^b\,.
\end{aligned} \tag{28}$$

(In the first line, we corrected a wrong factor of $n!$, whose presence in [12] spoiled the subsequent expressions.)

Because the twist-2 part is obtained by inserting projections, it must separately satisfy Wightman positivity. Of course, we would like to apply the twist-2 intertwiners E_{1L} of Sect. 3 in both channels, so that the issue reduces to the positivity of tensor-scalar-scalar-tensor 4-point functions. Applying successively the unknown intertwiners for the resulting tensor-scalar channels, the problem would be reduced to the positivity of the resulting 2-point function, i.e., to the positivity of the numerical amplitude.

Since we know the intertwiners E_{1L}, the first step can in principle be done. Notice that it is sufficient to act on the leading part, because it differs from the twist-2 part by contributions of higher twist, that are annihilated by E_{1L}. Notice also that B has the form of a product of two 4-point functions in the variables x_1,x_2,x_3,x_4 and in the variables x_3,x_4,x_5,x_6. Therefore, the application of the intertwining differential operators in the 1-2 channel and in the 5-6 channel also factorizes. The same, however, is not true for E.

Thus, even the first step at present seems to be too involved to be carried out in practice. The second step is at present not possible because we have not yet determined the tensor-scalar intertwiners.

For this reason, we decided to perform only a weaker test of positivity. Namely, we restrict the twist-2 structure to 2D, by setting two spatial coordinates to 0. Since this essentially amounts to a smaller class of test functions, Wightman positivity must still be preserved; but notice that 2D positivity after the restriction is necessary but not sufficient to ensure positivity in 4D.

The intertwining operators in 2D are at our disposal (16), and we have computed all coefficients (see below). It turns out that the partial wave amplitudes of the restricted exotic twist-2 structure $B - \frac{1}{2}E$ differ from those of the non-exotic structure B only by certain signs. This means that E has the same partial wave amplitudes as $4B$, except that some of them are absent.

The non-exotic structure B may itself be indefinite, but we know that it occurs in a free-field model, and therefore can be dominated by other positive free-field structures, because free fields are manifestly positive. This seems to indicate that the restricted exotic structure as well can be dominated by positive free-field structures. Thus positivity at the 6-point level alone would not forbid the appearance of this structure as part of a 6-point correlation function.

Let us indicate some details of the actual computations.

Upon restriction to 2D, u_+ and u_- turn into the chiral cross ratios $u = \frac{x_{12}x_{34}}{x_{13}x_{24}}$. Moreover, the function $B - \frac{1}{2}E$ drastically simplifies:

$$B - \frac{1}{2}E \overset{2D}{=} \frac{1}{(x_{12}^2)^2 x_{13}^2 x_{24}^2 \cdot x_{34}^2 \cdot x_{35}^2 x_{46}^2 (x_{56}^2)^2} \cdot \frac{(u_+ - u_-)}{(1-u_+)(1-u_-)} \cdot \frac{(u'_+ - u'_-)}{(1-u'_+)(1-u'_-)}.$$

After multiplication with $g(s,t)g(s',t')$, using (28), we have

$$H \overset{2D}{=} \frac{1}{(x_{12}^2)^2 x_{13}^2 x_{24}^2 \cdot x_{34}^2 \cdot x_{35}^2 x_{46}^2 (x_{56}^2)^2} \cdot \sum_{a,b\geq 0, a+b>0} \frac{a-b}{a+b} u_+^a u_-^b \cdot \sum_{a,b\geq 0, a+b>0} \frac{a-b}{a+b} u_+'^a u_-'^b.$$

For the non-exotic structure B, one has instead

$$B \overset{2D}{=} \frac{1}{(x_{12}^2)^2 x_{13}^2 x_{24}^2 \cdot x_{34}^2 \cdot x_{35}^2 x_{46}^2 (x_{56}^2)^2} \cdot \sum_{a,b\geq 0, a+b>0} u_+^a u_-^b \cdot \sum_{a,b\geq 0, a+b>0} u_+'^a u_-'^b.$$

Because the sums factorize, the evaluations of the chiral intertwining differential operators $\iota \circ E_{h_\pm}(\partial_k, \partial_l) \circ (x_{kl,\pm})^d$ in the 1-2 channel ($k,l = 1,2$) and in the 5-6 channel, with E_h given by (16), completely decouple. Actually, because all structures of interest are of order $x_{kl}^{\geq 1-d}$, and therefore only chiral dimensions $h \geq 1$ will occur, we found it more efficient to work with chiral intertwining operators $\iota \circ D_{h_\pm} \circ (x_{kl,\pm})^{d-1}$ where $D_h(\partial_k, \partial_l) = (\nabla_k - \nabla_l)E_h(\partial_k, \partial_l)$, and adopt a normalization different from (16):

$$D_h(\partial_1, \partial_2) = \frac{1}{(h-1)!} \sum_{p+q=h-1} \frac{\partial_1^p(-\partial_2)^q}{p!^2 q!^2}.$$

Thus, we apply $\iota \circ D_{h_+,h_-} \circ (x_{12}^2)^2 = \iota \circ \left[D_{h_+} \circ (x_{12,+})^2 \otimes D_{h_-} \circ (x_{12,-})^2\right]$. We find

$$\iota \circ D_h \left[\frac{1}{x_{13}x_{24}} u^a\right] = x_{34}^a \cdot \iota \circ D_h \left[\frac{x_{12}^a}{(x_{13}x_{24})^{a+1}}\right] = (-1)^{h-1} c_{a,h} \cdot \frac{x_{34}^{h-1}}{(x-x_3)^h (x-x_4)^h}$$

where $c_{a,h} = \frac{(h)_a(1-h)_a}{a!^2}$. Multiplying the two chiral factors and performing the sum over a and b gives for the structure B

$$\iota \circ D_{h_+,h_-} \left[\frac{1}{x_{13}^2 x_{24}^2} \sum_{a,b\geq 0, a+b>0} u_+^a u_-^b\right] = C_B(h_+,h_-) \cdot \frac{x_{34,+}^{h_+-1}}{(x-x_3)_+^{h_+}(x-x_4)_+^{h_+}} \left[+ \to -\right]$$

where, by virtue of $F(z) := \sum_a c_{a,h} z^a = P_{h-1}(1-2z)$ and $P_L(-1) = (-1)^L$,

$$C_B(h_+,h_-) = 2\chi_{\text{odd}}(h), \tag{29}$$

where $\chi_{\text{odd}}(h) = 1$ if the helicity $h = h_+ - h_-$ is odd, and zero otherwise. To perform the corresponding computation for the sum weighted with $\frac{a-b}{a+b}$, as in the structure H, one may for $b > 0$ put $G_b(z) = \sum_a \frac{a-b}{a+b} c_{a,h} z^a$, solve the equation $zG' + bG = zF' - bF$ by $G(z) = F(z) - 2bz^{-b}\int_0^z t^{b-1}F(t)dt$, and use the orthogonality of the Legendre polynomials to conclude $G(1) = F(1) = (-1)^{h-1}$ if $h_+ > h_-$. One finds

$$C_H(h_+,h_-) = \text{sign}(h) \cdot 2\chi_{\text{odd}}(h). \tag{30}$$

The same factors arise in the 5-6 channels. Thus, when the 6-point structures B and H are reduced in both channels by means of $(\iota_x \circ D_{h_+,h_-} \circ (x_{12}^2)^2) \otimes (\iota_{x'} \circ D_{h'_+,h'_-} \circ (x_{56}^2)^2)$, the result is always a multiple of the same 4-point function

$$W_{h_+,h_-;h'_+,h'_-}(x,x_3,x_4,x') = \frac{x_{34,+}^{h_++h'_+-3}}{(x-x_3)_+^{h_+}(x-x_4)_+^{h_+}(x_3-x')_+^{h'_+}(x_4-x')_+^{h'_+}} \times \left[+ \to -\right].$$

The respective coefficients for the structures B and H are

$$\begin{aligned} C_B(h_+,h_-)C_B(h'_+,h'_-) &= 4\chi_{\text{odd}}(h)\chi_{\text{odd}}(h'), \\ C_H(h_+,h_-)C_H(h'_+,h'_-) &= \text{sign}(h)\text{sign}(h') \cdot 4\chi_{\text{odd}}(h)\chi_{\text{odd}}(h'), \end{aligned} \tag{31}$$

where $h = h_+ - h_-$, $h' = h'_+ - h'_-$ are the helicities.

Because H is the twist-2 part of $B - \frac{1}{2}E$, we conclude that (after 2D restriction) all partial waves with helicities of equal sign in the 1-2-channel and in the 5-6-channel, that are present in B, are absent in the twist-2 part of E, while those with helicities of opposite sign arise in the twist-2 part of E with 4 times the coefficient in B.

It remains to perform the partial wave expansion of the 4-point functions $W_{h_+,h_-;h'_+,h'_-}$. Here one may use standard methods, e.g., [6, 11, 15]. Namely,

$$W_{h_+,h_-;h'_+,h'_-}(x,x_3,x_4,x') = \sum_{k_+,k_-} B^{k_+,k_-}_{h_+,h_-;h'_+,h'_-} \cdot W^{k_+,k_-}_{h_+,h_-;h'_+,h'_-}(x,x_3,x_4,x'),$$

where W^{k_+,k_-} is the partial wave for the insertion of a projection on the representation with scaling dimensions (k_+,k_-). It turns out that only $k_\pm \in \frac{3}{2} + \mathbb{N}_0$ contribute. Because of chiral factorization of $W_{h_+,h_-;h'_+,h'_-}$, one has $B^{k_+,k_-}_{h_+,h_-;h'_+,h'_-} = B^{k_+}_{h_+,;h'_+} B^{k_-}_{h_-,;h'_-}$, where the chiral coefficients are determined by the expansion

$$1 = \sum_{k=\frac{3}{2}+n} B^k_{h,h'} \cdot u^n \, {}_2F_1(n+h, n+h'; 2n+3; u).$$

The problem of Wightman positivity of the (2D-restricted) structures B and H has now been reduced to the positivity of linear combinations of matrices of the form

$$P_\pm P_{\text{odd}} \left[B^{k_+} \otimes B^{k_-} \right] P_{\text{odd}} P_\pm,$$

where P_{odd} and $P_\pm$ are the projections on the odd resp. positive or negative helicities.

To be admissible in a QFT, the exotic structure does not need to be separately positive, but must only be dominated by other, non-exotic structures that contribute to a full 6-point function. Thus, if positivity should fail for H (it certainly does for the twist-2 part of E because in this case all diagonal matrix elements vanish), one would have to establish a bound for the negative part of the above matrices by positive matrices of partial wave amplitudes arising from other structures.

We have not completed this analysis yet.

To conclude: the tools are available to test Wightman positivity of 6-point correlation functions. If a 6-point function involving the exotic structure (25) passes the test, then it could be a candidate for a nontrivial 4D conformal QFT.

Acknowledgements KHR is grateful for helpful discussions with N.M. Nikolov and I. Todorov, and also with Ch. Mishra (IISER, Kolkata) in an early stage of this work. Supported in part by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen.

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1972)
2. Bakalov, B., Nikolov, N., Rehren, K.-H., Todorov, I.: Comm. Math. Phys. **271**, 223–246 (2007)
3. Bargmann, V., Todorov, I.T.: J. Math. Phys. **18**, 1141–1148 (1977)
4. Bischoff, M.: Über die Pol-Struktur höherer Korrelationsfunktionen in global konform-invarianter Quantenfeldtheorie, Univ. Göttingen Diploma Thesis 2009 (in German)
5. Dobrev, V.K., Mack, G., Petkova, V.B., Petrova, S.G., Todorov, I.T.: Lect. Notes Phys. **63**, 1–280 (1977)
6. Dolan, F.A., Osborn, H.: Nucl. Phys. **B 678**, 491–507 (2004)
7. Lang, K., Rühl, W.: Nucl. Phys. **B 402**, 573–603 (1993)
8. Mack, G.: Comm. Math. Phys. **53**, 155–184 (1977)
9. Mintchev, M.C., Petkova, V.B., Todorov, I.T.: Conformal Invariance in Quantum Field Theory, Pubblicazione della classe di science della Scuola Normale Superiore di Pisa, pp. 1–273 (1978)
10. Nikolov, N., Todorov, I.: Comm. Math. Phys. **218**, 417–436 (2001)
11. Nikolov, N., Rehren, K.-H., Todorov, I.: Nucl. Phys. **B 722**, 266–296 (2005)
12. Nikolov, N., Rehren, K.-H., Todorov, I.: Bulg. J. Phys. **35 (s1)**, 113–123 (2008) [arXiv:0711.0628]
13. Nikolov, N., Rehren, K.-H., Todorov, I.: Comm. Math. Phys. **279**, 225–250 (2008)
14. Rathlev, D.: Higher Order Correlation Functions in Globally Conformal QFT, University of Göttingen Bachelor Thesis 2010 (in German)
15. Rehren, K.-H., Schroer, B.: Nucl. Phys. **B 295**, 229–242 (1988)
16. Yngvason, J.: Comm. Math. Phys. **34**, 315–333 (1973)

Euclidean Configuration Space Renormalization, Residues and Dilation Anomaly

Nikolay M. Nikolov, Raymond Stora, and Ivan Todorov

Abstract Configuration (x-)space renormalization of Euclidean Feynman amplitudes in a massless quantum field theory is reduced to the study of local extensions of associate homogeneous distributions. Primitively divergent graphs are renormalized, in particular, by subtracting the residue of an analytically regularized expression. Examples are given of computing residues that involve zeta values. The renormalized Green functions are again associate homogeneous distributions of the same degree that transform under indecomposable representations of the dilation group.

N.M. Nikolov
Institute for Nuclear Research and Nuclear Energy, Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria
e-mail: mitov@inrne.bas.bg

R. Stora
Laboratoire d'Annecy-le-Vieux de Physique Théorique (LAPTH), 74941 Annecy-le-Vieux Cedex, France

Theory Division, Department of Physics, CERN, 1211 Geneva 23, Switzerland

I. Todorov (✉)
Institut des Hautes Études Scientifiques, 91440 Bures-sur-Yvette, France

Theory Division, Department of Physics, CERN, 1211 Geneva 23, Switzerland

Institute for Nuclear Research and Nuclear Energy, Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria
e-mail: todorov@inrne.bas.bg

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_9,

1 Introduction

Fourier transform is a prime example of the now fashionable notion of duality. It maps a problem of integrating large momenta into one of studying the short distance behaviour of correlation functions. Divergences were first discovered and renormalization theory was developed for momentum space integration. Stueckelberg and Petermann [61], followed by Bogolubov, a mathematician who set himself to master quantum field theory (QFT), realized that (perturbative) renormalization can be formulated as a problem of extending products of distributions, originally defined for non-coinciding arguments[1] and that such extensions are naturally restricted by locality or *micro-causality* (a concept introduced in QFT by Stueckelberg [44] and further developed by Bogolubov and collaborators—for a review and references see [6]). The idea was taken up and implemented systematically by Epstein and Glaser [8,18,19] (see also parallel work by Steinmann [57]; for later contributions and surveys see [52, 59, 60]). It is conceptually clear and represents a crucial step in turning QFT renormalization into a mathematically respectable theory. By the late 1990s when the problem of developing perturbative QFT and operator product expansions on a curved background became the order of the day, it was realized that it is just the x-space approach that offers a way to its solution [10, 11, 16, 35–38]. It is therefore not surprising that this approach attracts more attention now than half a century ago when it was originally conceived—see e.g. [1, 17, 22, 24, 29, 30, 40, 41, 49]. Papers like [2] reflect, surely, later developments in both renormalization theory (Kreimer's Hopf algebra structure—see e.g. [42]—and Connes-Kreimer's reduction to the Riemann-Hilbert's problem [14]) and the mathematical study of singularities in configuration space [15, 26]. Recent work on Feynman graphs and motives [3, 4] also generated a configuration space development [12, 48, 49].

A starting point in our work was the observation (cf. [10, 16, 29, 37]) that Hörmander's treatment of the extension of homogeneous distributions (Sect. 3.2 of [39]) is tailor-made for treating the ultraviolet (UV) renormalization problem, that is particularly transparent in a massless QFT. In order to explain the main ideas stripped of technicalities, we begin with the study of *dilation invariant Euclidean Green's functions* (the only case considered in [2]). Furthermore, we concentrate on the UV problem excluding integration in configuration space by considering all vertices as external.The validity of the results in the physically better motivated Minkowski space framework is established in [50]. It is, on the other hand, known that the leading UV singularities in a massive QFT are given by the corresponding massless limit. The full study of the renormalization problem in the massive case requires, however, additional steps and is relegated to future work.

[1] Whereas x-space renormalization was straightened out in all generality [5, 33, 34, 51, 58], it took some more time to settle the p-space problem [45, 46, 63, 64], resulting in what is now termed the BPHZ theory.

We begin with a framework that differs from standard QFT (cf. [48]). We separate the renormalization program from concrete (massless) QFT models and state it as a mathematical problem of extension of a class of homogeneous distributions. In Sect. 2 we formulate general axiomatic conditions for our construction, such that when combined with a given Lagrangian model it reproduces the result of Epstein-Glaser for the renormalized time ordered products (see [49]). To this end we introduce a universal algebra of rational translation invariant functions in $\mathbb{R}^{Dn}$, where n runs in $\mathbb{N}$ while D, the space-time dimension, is fixed ($D = 4$ being the case of chief interest). We assume that this algebra is generated by 2-point functions of the type

$$G_{ij}(x_{ij}) = \frac{P_{ij}(x_{ij})}{\rho_{ij}^{\mu_{ij}}}, \quad x_{ij} = x_i - x_j, \quad \mu_{ij} \in \mathbb{N},$$

$$\rho_{ij} = |x_{ij}| = (x_{ij}^2)^{1/2}, x^2 = \sum_{\alpha=1}^{D} (x^\alpha)^2 \tag{1}$$

(for a Minkowski space signature $\rho^2 = \mathbf{x}^2 - (x^0 - i0)^2)$, $\mathbf{x}^2 = \sum_{i=1}^{D-1}(x^i)^2$); here P_{ij} are homogeneous polynomials in the components of the D-vector x_{ij}. (For free massless fields in an odd dimensional spacetime the exponents μ_{ij} are odd.[2] For an even D one can assume that all μ_{ij} are even integers so that G_{ij} are rational functions.) We note that the renormalization of any massless QFT can be reduced to the extension of (a subspace of) rational functions $G = \prod_{i<j} G_{ij}(x_{ij})$ of this algebra to distributions on $\mathbb{R}^{D(n-1)}$. The correspondence between the rational functions and such distributions is called a *renormalization map*. Each expression

$$G_\Gamma = \prod_{(ij)\in\Gamma} G_{ij}(x_{ij}), \tag{2}$$

can be represented by a decorated graph Γ of n vertices and of lines connecting pairs of different vertices (i, j) whenever there is a (non-constant) factor G_{ij} in the product (2). Each $G_{ij} = G_{ij}(x_{ij})$ appears at most once in this expression, so that there are no multiple lines in the graph Γ. The presence of different powers μ and different polynomials P indicates the fact that we give room for composite fields in our theory such as normal products of derivatives of the basic fields. (Matrix valued vertices that enter the Feynman rules can be accounted for by admitting linear combinations of expressions of type (2).) A disconnected graph Γ corresponds to the (tensor) product of the distributions associated to its connected components. We shall restrict our attention to connected graphs.

We remark that a quantum field theorist may wish to replace the polynomial in x in (1) by a polynomial of derivatives acting on the scalar field propagator.

[2] In view of recent interest in 3D CFT [28, 47] we explicitly include here odd D.

The difference is not accidental: we shall impose the requirement, convenient for the subsequent analysis, that the renormalization map commutes with multiplication by polynomials in x_{ij}. On the other hand, derivatives typically yield anomalies independently of the above requirement (see [49], Sect. 8). Using the renormalization map we achieve the basic property of the time–ordered product: causality. Other constraints compatible with causality and power counting may be imposed—including a description of possible associated anomalies—by adjustment of additional finite renormalizations. An example of such a phenomenon, concerned with the behaviour of renormalized Feynman amplitudes under dilations, is considered in Sect. 4.

Thus, to any graph Γ in a given massless QFT there corresponds a bare Feynman amplitude G_Γ. It is a homogeneous rational function of degree $-d_\Gamma$ which depends on n-1 D-vector differences. We shall denote the arguments of G_Γ by $\mathbf{x}$, for short, and will introduce a uniform ordering $x^1,\ldots,x^N$ of their components, where $N = D(n-1)$ (for a connected graph). Then, the homogeneity of G_Γ is expressed as

$$G_\Gamma(\lambda\,\mathbf{x}) = \lambda^{-d_\Gamma} G_\Gamma(\mathbf{x})\,. \tag{3}$$

We shall call the difference $\kappa := d_\Gamma - N$ the *index of divergence*. It coincides with (minus) the degree of homogeneity of the density form

$$G_\Gamma(\mathbf{x})\,dx^1 \wedge dx^2 \wedge \ldots \wedge dx^N \equiv G_\Gamma(\mathbf{x})\,\mathrm{Vol.}. \tag{4}$$

(Whenever the orientation is not relevant we shall skip the wedge product sign. The use of densities rather than functions streamlines changes of variables and partial integration.) We say that G_Γ is *superficially divergent* if $\kappa \geq 0; G_\Gamma$ is called *divergent* if it is not locally integrable. The following easy to prove statement justifies the above terminology.

Proposition 1.1. *If the indices of divergence of a connected graph Γ and of all its connected subgraphs are negative then G_Γ is locally integrable and admits, as a consequence, a unique continuation as a distribution on $\mathbb{R}^{D(n-1)}$.*

The power counting index of divergence of standard renormalization theory is thus replaced by the degree of homogeneity of bare Green functions for a (classically dilation invariant) massless QFT.

Abusing the terminology we shall also speak of *(superficially) divergent graphs*. Each function G_Γ defines a tempered distribution (in the sense of Schwartz [55]) on test functions f with support

$$\mathrm{supp}\, f \subset \mathbb{R}^{D(n-1)} \backslash \Delta_2\,, \quad \Delta_2 = \{\mathbf{x};\, \exists\,(i,j)\, i<j,\ s.t.\ x_{ij} = 0\}\,. \tag{5}$$

One can, similarly, introduce the partial diagonals Δ_k involving k-tuples of coinciding points; we have $\Delta_n := \{\mathbf{x}; x_1 = \ldots = x_n\} \subset \Delta_{n-1} \subset \ldots \subset \Delta_2$. We shall be mostly using the *small* or *full* diagonal Δ_n in what follows. The problem of renormalization consists in extending all distributions G_Γ to $\mathscr{S}(\mathbb{R}^{D(n-1)})$ in such a way that a certain recursion relation, which reflects the causality condition, is satisfied. This condition is known as *causal factorization*. We give the precise formulation of its Euclidean version in Sect. 2 that follows from the more involved but physically motivated Minkowski space requirement (see [50]). We use an x-space counterpart of Speer's *analytic renormalization* in [56] to define the notion of *residue*[3] of G_Γ adapted, in particular, to primitively divergent graphs. It is based on the observation that if $r = r(x_{ij})$ is a norm in the (Euclidean) space of coordinate differences and $G(\mathbf{x})$ is primitively divergent of index κ then the *analytically regularized Feynman amplitude*

$$r^{\kappa+\varepsilon} G(\mathbf{x}) \ (\varepsilon > 0) \tag{6}$$

is locally integrable. It will be proven in Sect. 2 and Appendix A that (6) defines a distribution valued meromorphic function in ε which only has simple poles for non-positive integer values of ε. This will allow us to define the *renormalized Feynman distribution* G^R of a primitively divergent graph by just subtracting the pole term for $\varepsilon = 0$. The result will be enforced by one of our main requirements (see (MC2) of Sect. 2, below), namely that G^R is associate homogeneous of the same degree as G (its behaviour for small r only differing from G by log terms). More precisely, we say that G is an *associate homogeneous distribution of degree d and order k* if it obeys the (infinitesimal) indecomposable dilation law

$$(E+d)^{k+1} G(\mathbf{x}) = 0 \quad \text{where} \quad E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \left(x \frac{\partial}{\partial x} = \sum_{\alpha=1}^{D} x^\alpha \frac{\partial}{\partial x^\alpha} \right), \tag{7}$$

—i.e., if it is an associate eigenvector of the Euler operator E—see [27].

The study of divergent graphs with subdivergences is outlined in Sect. 4, where a global characterization of associate homogeneous distributions is also given. It is remarkable that in all cases renormalization is reduced to a 1-dimensional extension problem for associate homogeneous distributions. A construction that provides the solution to this problem is outlined in Appendix A.

One objective of our work is to demonstrate in a systematic fashion that x-space calculations are not only more transparent conceptually but also practical (especially in the Euclidean massless case—something noticed long ago by Chetyrkin et al. [13] (see also [43]) but only rarely appreciated afterwards—cf. [29]). To this end we consider (in Sects. 3 and 4) a number of examples (of 1-, 2- and 3-loop graphs) displaying the basic simplicity of the argument. A primitively divergent n-loop graph whose residue involves $\zeta(2n-3)$ is displayed as Example 3.2.

[3] A notion of residue of a Feynman graph has been introduced in the momentum space approach in terms of the *graph polynomial* [3, 4]. It would be interesting to establish the precise relationship between that notion and ours. The notion of *Poincaré residue* considered in [12], on the other hand, works in a straightforward manner for simple poles in x-space, a rather unnatural restriction for ultraviolet divergences.

2 General Requirements. Reduction to a One-Dimensional Problem

We shall define *ultraviolet (i.e. short distance) renormalization* by induction with respect to the number of vertices. Assume that all contributions of diagrams with less than n points are renormalized. If then Γ is an arbitrary connected n-point graph its renormalized contribution should satisfy the following inductive *factorization requirement*.

Let the index set $I(n) = \{1,\ldots,n\}$ of Γ be split into any two non-empty non-intersecting subsets

$$I(n) = I_1 \dot{\cup} I_2 \; (I_1 \neq \emptyset,\, I_2 \neq \emptyset,\, I_1 \cap I_2 = \emptyset)\,.$$

Let $\mathscr{U}_{I_1,I_2}$ be the open subset of $\mathbb{R}^{Dn} \equiv (\mathbb{R}^D)^{\times n}$ such that $(x_1,\ldots,x_n) \notin \mathscr{U}_{I_1,I_2}$ whenever there is a pair (i,j) such that $i \in I_1,\; j \in I_2$. Let further G_1^R and G_2^R be the contributions of the subgraphs of Γ with vertices in I_1 and I_2, respectively. For each such splitting our distribution G_Γ^R, defined on all partial diagonals, exhibits the *Euclidean factorization property* (see [48]):

$$G_\Gamma^R = G_1^R \left(\prod_{\substack{i\in I_1 \\ j\in I_2}} G_{ij} \right) G_2^R \quad \text{on} \quad \mathscr{U}_{I_1,I_2}\,, \tag{8}$$

where G_{ij} are factors (of type (1)) in the rational function G_Γ and are understood as *multipliers* on $\mathscr{U}_{I_1,I_2}$. This property is inspired by the Minkowski space causal factorization of Epstein-Glaser [18] considered in [50].

We shall add to this basic physical requirement a few more *mathematical conventions* (MC) which will substantially restrict the notion of renormalization used in this paper.

(MC1) *The renormalization commutes with permutation of indices* (which may stand for both position variables and discrete quantum numbers).

(MC2) *Renormalization maps rational homogeneous functions onto associate homogeneous distributions of the same degree of homogeneity; it extends associate homogeneous distributions defined off the small diagonal to associate homogeneous distributions of the same degree (but possibly of higher order) defined everywhere on* $\mathbb{R}^N$.

(MC3) *The renormalization map commutes with multiplication by (homogeneous) polynomials.* If we extend the class of our distributions allowing multiplication with smooth functions of no more than polynomial growth (in the domain of definition of the corresponding functionals), then this requirement will imply commutativity of the renormalization map with such multipliers.

(MC4) *In a Euclidean invariant theory the renormalization map commutes with Euclidean transformation in* $\mathbb{R}^D$.

The induction is based on the following *Euclidean diagonal lemma*.

Proposition 2.1. *The complement $C(\Delta_n)$ of the small diagonal is the union of all $\mathscr{U}_{I_1,I_2}$ for all pairs of disjoint I_1, I_2 with $I_1 \dot\cup I_2 = \{1,\ldots,n\}$, i.e.,*

$$C(\Delta_n) = \bigcup_{I_1 \dot\cup I_2 = \{1,\ldots,n\}} \mathscr{U}_{I_1,I_2}\,.$$

Proof. Let $(x_1,\ldots,x_n) \in C(\Delta_n)$. Then there are at least two different points $x_{i_1} \neq x_{j_1}$. We define I_1 as the set of all indices i of $I = I(n)$ for which $x_i \neq x_{j_1}$ and $I_2 := I \backslash I_1$. Hence, $C(\Delta_n)$ is included in the union of all such pairs. Each $\mathscr{U}_{I_1,I_2}$, on the other hand, is defined to belong to $C(\Delta_n)$. This completes the proof of our statement.

In order to apply and implement the inductive factorization property (8) one needs two steps:

(i) To renormalize all *primitively divergent* graphs, *i.e.* all divergent diagrams with no proper subdivergences, in particular, to extend all (superficially) divergent 2-point functions G_{ij} to distributions on $\mathscr{S}(\mathbb{R}^D)$;
(ii) To extend the resulting associate homogeneous distributions defined on the complement of the *full diagonal* $x_1 = x_2 = \ldots = x_n$ to distributions on $\mathscr{S}(\mathbb{R}^{D(n-1)})$.

We shall only elaborate on the first step in this exposé. Concerning step (ii), briefly reviewed in Sect. 4, we refer to our paper [50].

A primitively divergent graph gives rise to a homogeneous distribution $G^0(\mathbf{x})$ defined on $\mathbb{R}^N \backslash \{0\}$ (i.e. off the small diagonal, as $\mathbf{x}$ is expressed in terms of the coordinate differences). The following statement concerns more generally associate homogeneous distributions and thus applies to any graph with renormalized subdivergences.

Theorem 2.2. *Let Σ be any cone section—i.e., a smooth (compact) hypersurface in $\mathbb{R}^N \backslash \{0\}$ that intersects transversally every ray $\{\lambda\,\mathbf{x}\}_{\lambda>0}$ ($\mathbf{x} \neq 0$) and let $\rho_\Sigma(\mathbf{x})$ be a positive smooth function such that $\mathbf{u} := \rho(\mathbf{x})^{-1}\mathbf{x} \in \Sigma$. Then every associate homogeneous distribution of degree $-d$ and order n has an expansion of the form*[4]

$$G^0(r\mathbf{u}) = \sum_{m=0}^{n} G_m^\Sigma(\mathbf{u})\, L_{-dm}(r)\,, \quad r = \rho_\Sigma(\mathbf{x})\,, \tag{9}$$

$$L_{am}(r) = \theta(r)\, r^a \frac{(\ell n\, r)^m}{m!} \quad \left(= r^a \frac{(\ell n\, r)^m}{m!} \;\; for \;\; r > 0 \right). \tag{10}$$

The *proof* uses induction in n, based on the formula

$$(E+d)\,L_{-dn} = L_{-dn-1} \;\text{ for }\; E = \mathbf{x}\frac{\partial}{\partial \mathbf{x}}\,,\; n = 1,2,\ldots, \tag{11}$$

[4] A similar decomposition in an overall scale and angle variables is derived and used very recently in momentum space in [9].

along with the observation that for $n = 0$

$$\frac{\partial}{\partial r}\left(r^d\, G^0(r\mathbf{u})\right) = 0\,.$$

Thus the renormalization problem is reduced to the extension of 1-dimensional distributions of type (10). The latter is achieved by exploiting the simple pole structure of analytic regularization [56] and the resulting generating formula (see Appendix):

$$\theta(r)\, r^{\varepsilon-\kappa-1} - \frac{(-1)^\kappa}{\kappa!\varepsilon}\, \delta^{(\kappa)}(r) = \sum_{\kappa=0}^{\infty} L_{-\kappa-1n}(r)\, \varepsilon^n\,. \tag{12}$$

The distributions L_{-dn} can be then defined on the real line using (MC3) and (11); they depend on a single scale parameter hidden in the argument of the logarithm (see Appendix).

The following proposition may serve as a definition of both the notion of a *residue* Res and of a *primary renormalization map* $\mathscr{P}_N^\Sigma : \mathscr{S}'(\mathbb{R}^N\backslash\{0\}) \to \mathscr{S}'(\mathbb{R}^N)$.

Theorem 2.3. *If $G^0(\mathbf{x})$ is a homogeneous distribution of degree $-d$ on $\mathbb{R}^N\backslash\{0\}$ $(d = N+\kappa \geq N)$, then*

$$\rho_\Sigma(\mathbf{x})^\varepsilon\, G^0(\mathbf{x}) - \frac{1}{\varepsilon}\,(\mathrm{Res}\, G)(\mathbf{x}) = G^\Sigma(\mathbf{x}) + 0(\varepsilon) \quad (G^\Sigma = \mathscr{P}_N^\Sigma\, G^0)\,; \tag{13}$$

here Res G *is a distribution with support at the origin whose calculation is reduced to the case $d = N$ of a logarithmically divergent graph by using the identity*

$$\mathrm{Res}\, G = \frac{(-1)^\kappa}{\kappa!}\, \partial_{i_1} \ldots \partial_{i_\kappa} (\mathrm{Res}(x^{i_1} \ldots x^{i_\kappa}\, G))(\mathbf{x}) \tag{14}$$

where summation is understood over all repeated indices $i_1, \ldots, i_\kappa$ from 1 *to N. If $G^0(\mathbf{x})$ is homogeneous of degree $-N$ then*

$$\mathrm{Res}\, G(\mathbf{x}) = (\mathrm{res}\, G^0)\, \delta(\mathbf{x}) \qquad (\textit{for } (E+N)\, G^0(\mathbf{x}) = 0) \tag{15}$$

where

$$\mathrm{res}\, G^0 = \int_\Sigma G^0(\mathbf{x}) \sum_{j=1}^{N} (-1)^{j-1}\, x^j\, dx^1 \wedge \ldots d\widehat{x}^j \ldots \wedge dx^N \tag{16}$$

is independent of Σ since the form under the integral sign is closed. (A hat, $\widehat{\ }$, over an argument means, as usual, that this argument is omitted.)

Proof. The fact that the distribution valued function of ε $\rho_\Sigma^\varepsilon G^0$ is meromorphic and only has a simple pole at $\varepsilon = 0$ follows from Theorem 2.2 and (12). Equation (14) follows from the assumed homogeneity property $\partial_i x^i\, G^0 = -\kappa\, G^0$ of G^0. The integrand in (16) is a contraction of G^0 Vol with the Euler vector field:

$$i_E\, G^0\, \mathrm{Vol} = \sum_{j=1}^{N} G^0 (-1)^j\, x^j\, dx^1 \wedge \ldots d\widehat{x}^j \ldots \wedge dx^N \tag{17}$$

and it is a (homogeneous) form of maximal degree in the $(N-1)$-dimensional projective space for $\lambda^N G^0(\lambda\,\mathbf{x}) = G^0(\mathbf{x})$.

The residue (16) is a special case of the so called *Wodzicki residue* (see [29, 31] and references therein).

3 Residues and Renormalization of Primitively Divergent Graphs

For the (Euclidean covariant) 2-point function in a D-dimensional space-time $N = D$ ($\mathbf{x} = x = x_1 - x_2$) it is natural to choose for Σ the unit hypersphere $\mathbb{S}^{D-1}$, so that $\rho_\Sigma(x) = \sqrt{x^2} =: r$. For a scalar 2-point function of a composite field of dimension $\frac{D}{2}$ (D-even), we would have

$$G^0(x) = \frac{C}{(x^2)^{D/2}}, \quad \operatorname{Res} G = C\,|\mathbb{S}^{D-1}|\,\delta(x) \tag{18}$$

where $|\mathbb{S}^{2m-1}| = \frac{2\pi^m}{(m-1)!}$.

The renormalization map $\mathscr{R}_D^\Sigma : G^0 \to G^\Sigma$ (13) can be computed explicitly in terms of the radial coordinate r of (12) (see Appendix).

Here we shall compute it instead in Cartesian coordinates in two examples of 4-dimensional ($4D$) scalar field theory.

Example 3.1. The logarithmically divergent 2-point graph shown on Fig. 1a is ubiquitous as a (sub)divergence in any scalar field theory in $4D$: it appears as a self-energy graph in a φ^3 model and as a contribution to the 4-particle scattering amplitude in the φ^4 theory. The limit $\varepsilon \to 0$ in (13) for this 1-loop graph reads

$$\begin{aligned} G_1(x,\ell) &= \lim_{\varepsilon\to 0}\left[\frac{1}{(x^2)^2}\left(\frac{x^2}{\ell^2}\right)^{\varepsilon} - \frac{2\pi^2}{2\varepsilon}\,\delta(x)\right] \\ &= \frac{1}{2}\,\frac{\partial}{\partial x^\alpha}\left[\frac{x^\alpha}{(x^2)^2}\,\ell n\left(\frac{x^2}{\ell^2}\right)\right]\left(= \frac{1}{r^2}\,\frac{\partial}{\partial r^2}\left(\ell n\,\frac{r^2}{\ell^2}\right)_+\right., \\ (\ell n\,\rho)_+ &= \left.\begin{cases} \ell n\,\rho & \text{for } \rho > 0 \\ 0 & \text{for } \rho < 0 \end{cases}\right). \end{aligned} \tag{19}$$

Fig. 1 Logarithmically and quadratically divergent 2-point graphs

This is another instance of *differential renormalization* (cf. (A.4) and see [25,32,53]). Renormalized expressions of the type $\frac{\partial}{\partial x^\alpha}\left[\frac{x^\alpha}{(x^2)^2}\,\ell n\,\frac{x^2}{\ell^2}\right]$ (sum over α) are used systematically in [29].

Remark 3.1. Note that the double and the triple lines in Fig. 1 should both be viewed as a single line with a different decoration (corresponding to different powers, $\mu = 2$ and $\mu = 3$, in (1)). Thus, the self-energy graph on Fig. 1b, which displays overlapping divergences in momentum space, is primitively divergent in x-space according to our definition. Its renormalized expression is additionally restricted by the requirement of full Euclidean invariance. (In general, we require the presence of as much of the symmetry of the rational function in the renormalized expression as allowed by the existing anomalies.) Applying further requirement (MC3) which yields the identity $G_1(x,\ell) = x^2\,G_2(x,\ell)$, valid for the original rational functions away from the origin, we find

$$
\begin{aligned}
G_2(x,\ell) &= \lim_{\varepsilon\to 0}\left\{\frac{1}{(x^2)^3}\left(\frac{x^2}{\ell^2}\right)^{\varepsilon} - \frac{\pi^2}{8\,\varepsilon}\,\Delta\,\delta(x)\right\} \\
&= \frac{3\,\pi^2}{16}\,\Delta\,\delta(x) + \frac{\Delta}{8}\,G_1(x,\ell)\,.
\end{aligned}
\tag{20}
$$

In deriving (20) we have used the identities

$$
\Delta f = 4\,\frac{\partial^2}{\partial\rho^2}\,(\rho f) + \frac{1}{\rho}\,\Delta_\omega f \quad \text{for} \quad \rho = x^2 (= r^2)\,,\ x = r\omega\,;
$$

$$
\begin{aligned}
\frac{1}{\rho^{n+1}}\left(\frac{\rho}{\ell^2}\right)^{\varepsilon} &= \frac{1}{(n-\varepsilon)(n-1-\varepsilon)^2\ldots(1-\varepsilon)^2(-\varepsilon)}\left(\frac{\partial^2}{\partial\rho^2}\,\rho\right)^{n}\frac{1}{\rho}\left(\frac{\rho}{\ell^2}\right)^{\varepsilon} \\
&= \frac{1}{n!(n-1)!}\left(\frac{\Delta}{4}\right)^{n-1}\left(\frac{\pi^2}{\varepsilon}\,\delta(x) + \pi^2\,s_n\,\delta(x) + G_1(x,\ell)\right) + O(\varepsilon)\,,
\end{aligned}
$$

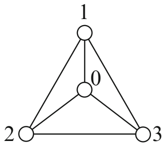
where s_n is a sum of partial harmonic series (cf. (A.5)):

$$
s_n = \sum_{j=1}^{n-1}\frac{1}{j} + \sum_{j=2}^{n}\frac{1}{j} \quad \left(s_1 = 0\,,\ s_2 = \frac{3}{2}\,,\ s_3 = \frac{7}{3}\,,\ \ldots\right).
$$

One can use a more general (homogeneous, $O(D)$-invariant) norm on the distances x_{ij}^2 instead of the ($O(N)$-invariant) radial coordinate for $N = D(n-1)$ in order to compute both the residue and the renormalized expression of a primitively divergent graph as illustrated on the following n-loop example.

Example 3.2. We consider the $4D$ n-loop ($n+1$-point) primitively divergent Feynman amplitude

$$
G_n = \Big(\prod_{i=1}^{n} x_{0i}^2 x_{ii+1}^2\Big)^{-1},\ x_{n+1} \equiv x_1\,,
\tag{21}
$$



$$G_3 = \frac{1}{x_{01}^2\, x_{02}^2\, x_{03}^2\, x_{12}^2\, x_{23}^2\, x_{13}^2}$$

Fig. 2 The tetrahedron graph in the $(\varphi^4)_4$-theory

which we shall parametrize by the spherical coordinates of the n independent 4-vectors x_{0i}:

$$x_{0i} = r_i\, \omega_i\,, \quad r_i \geq 0\,, \quad \omega_i^2 = 1\,, \quad i = 1, 2, \ldots, n. \tag{22}$$

An important special case is given by the complete 4-point graph on Fig. 2

Setting[5]

$$G_n^\varepsilon = \left(\frac{R^2}{\ell^2}\right)^\varepsilon G_n,\ R = max(r_1, \ldots, r_n), \tag{23}$$

we shall compute its residue by first integrating the corresponding analytically regularized density $G_n^\varepsilon \mathrm{Vol}$ over the angles ω_i using the identification of the propagators $\frac{1}{x_{ij}^2}$ with the generating functions for the Gegenbauer polynomials. Having in mind applications to a scalar field theory in D dimensions (see Example 4.2 below) we shall write down the corresponding more general formulas. The propagator $(x_{12}^2)^{-\lambda}$ of a free massless scalar field in $D = 2\lambda + 2$ dimensional space-time is expanded as follows in (hyperspherical) Gegenbauer polynomials:

$$(x_{ij}^2)^{-\lambda} = (r_i^2 + r_j^2 - 2 r_i r_j \omega_i \omega_j)^{-\lambda} = \frac{1}{R_{ij}^{2\lambda}} \sum_{n=0}^{\infty} \left(\frac{r_{ij}}{R_{ij}}\right)^n C_n^\lambda(\omega_i\, \omega_j)\,,$$

$$R_{ij} = \max(r_i, r_j)\,, \quad r_{ij} = \min(r_i, r_j)\,, \quad i \neq j,\ i, j = 1, 2, 3. \tag{24}$$

We shall also use the integral formula

$$\int_{\mathbb{S}^{2\lambda+1}} d\omega\, C_m^\lambda(\omega_1\, \omega)\, C_n^\lambda(\omega_2\, \omega) = \frac{\lambda\, |\mathbb{S}^{2\lambda+1}|}{n + \lambda}\, \delta_{mn}\, C_n^\lambda(\omega_1\, \omega_2)\,, \tag{25}$$

where $|\mathbb{S}^{2\lambda+1}| = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}$ is the volume of the unit hypersphere in $D = 2\lambda + 2$ dimensions.

Clearly, the expansion (24) requires an ordering of the lengths r_i. In general, one should consider separately $n!$ sectors, obtained from one of them, say

$$r_1 \geq r_2 \geq \ldots \geq r_n\ (\geq 0) \tag{26}$$

[5] The fact that the maximum function R, which replaces $\rho_\Sigma(\mathbf{x})$ of Theorem 2.2, does not depend smoothly on the coordinates, requires, in general, a special treatment of the lower dimensional manifolds of discontinuities of its derivatives. (See Example 4.1 below.)

by permutations of the indices. It is, in fact, sufficient to consider just the sector (26) (and multiply the result for the residue by $n!$). (Because of the symmetry of the tetrahedron graph (Fig. 2) this is obvious for $n = 3$ but it is actually true for any $n (\geq 3)$.) The result involves a polylogarithmic function:

$$\begin{aligned} \widetilde{G}_n^{\varepsilon} &:= \int_{\mathbb{S}^3} \cdots \int_{\mathbb{S}^3} G_n^{\varepsilon}(r_1\,\omega_1, \ldots, r_n\,\omega_n)\,\mathrm{Vol} \\ &= (2\pi^2)^n \left(\frac{r_1}{\ell}\right)^{2\varepsilon} \frac{dr_1 \wedge \ldots \wedge dr_n}{r_1 \ldots r_n}\, Li_{n-2}(\frac{r_n^2}{r_1^2}), \\ Li_{n-2}(\xi) &= \sum_{m=1}^{\infty} \frac{1}{m^{n-2}}\, \xi^m \, (\xi = \frac{r_n^2}{r_1^2}) \end{aligned} \tag{27}$$

($r_n = min(r_1, \ldots, r_n), r_1 = max(r_1, \ldots, r_n) (= R)$). To derive the last equation we have applied once more (25) and used

$$\left(C_m^1(\omega_1^2) =\right) C_m^1(1) = m + 1.$$

The residue distribution corresponding to the (integrated over the angles) density (27) is given by

$$\mathrm{Res}\, \widetilde{G}_n^{\varepsilon} = \mathrm{res}\, G_n^0\, \delta(r_1) \ldots \delta(r_n)\, dr_1 \wedge \ldots \wedge dr_n. \tag{28}$$

In order to compute the numerical residue $\mathrm{res}\, G_n^0$ in (28) we have to take the sum of $(n-1)$-dimensional radial integrals over the surface $R = 1$ for different orderings of r_i. The result is a multiple (with a binomial coefficient) of the integral corresponding to the sector (26):

$$\begin{aligned} \mathrm{res}\, G_n^0 &= \binom{2n-2}{n-1} (2\pi^2)^n \int_0^1 \frac{dr_2}{r_2} \cdots \int_0^{r_{n-1}} \frac{dr_n}{r_n} Li_{n-2}(r_n^2) \\ &= 2 \binom{2n-2}{n-1} \pi^{2n}\, \zeta(2n-3)\,. \end{aligned} \tag{29}$$

In particular, for the tetrahedron graph, $n = 3$, we reproduce the known result, $\mathrm{res}\, G_3^0 = 12\pi^6 \zeta(3)$—see, for instance, [29].

The integration technique based on the properties of Gegenbauer polynomials has been introduced in the study of x-space Feynman integrals in [13]. The appearance of ζ-values in similar computations has been detected in early work of Rosner [54] and Usyukina [62]. It was related to the non-trivial topology of graphs by Broadhurst and Kreimer [7, 42].

4 Dilation Anomaly. Examples of Graphs with Subdivergences

We now turn to the behaviour under dilations of a renormalized primitively divergent density $G(\mathbf{x})\mathrm{Vol}$ of index κ (≥ 0). By the definition of $G\,\mathrm{Vol}$ the *dilation anomaly*

$$A(\mathbf{x},\lambda) := \lambda^{\kappa}\, G(\lambda\,\mathbf{x})\,\mathrm{Vol} - G(\mathbf{x})\,\mathrm{Vol} \tag{30}$$

is a distribution valued density with support at the small diagonal, $x_1 = x_2 = \ldots = x_n$. Invoking the requirement (MC2), we can restrict it, following [39], by demanding that it is again homogeneous in $\mathbf{x}$ of degree $-\kappa$:

$$A(\mathbf{x},\lambda) = \sum_{\boldsymbol{\alpha},|\boldsymbol{\alpha}|=\kappa} a_{\boldsymbol{\alpha}}(\lambda)\, D_{\boldsymbol{\alpha}}\,\delta(\mathbf{x}) \prod_{i=1}^{n-1} d^D x_{in} \tag{31}$$

where $\delta(\mathbf{x})$ is the $D(n-1)$-dimensional δ-function,

$$D_{\boldsymbol{\alpha}} = \prod_{i=1}^{n-1}\prod_{\nu=1}^{D} (\partial_i^{\nu})^{\alpha_{i\nu}}\,, \quad |\boldsymbol{\alpha}| = \sum_{i,\nu} \alpha_{i\nu}\,.$$

Repeated application of the dilation law (30) yields the cocycle condition[6]

$$a_{\boldsymbol{\alpha}}(\lambda\mu) = a_{\boldsymbol{\alpha}}(\lambda) + a_{\boldsymbol{\alpha}}(\mu)\,. \tag{32}$$

The general form of $a_{\boldsymbol{\alpha}}$ satisfying (32) is

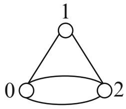
$$a_{\boldsymbol{\alpha}}(\lambda) = a_{\boldsymbol{\alpha}}(G)\,\ell n\,\lambda \tag{33}$$

where $a_{\boldsymbol{\alpha}}(G)$ is a linear functional of the Green function G (or the corresponding density GVol). It is important to note that the coefficient $a_{\boldsymbol{\alpha}}(G)$ in (33) is independent of the ambiguity in the definition of the renormalized Green function. Once the problem of renormalizing a primitively divergent graph is reduced to a 1-dimensional one (as in Sect. 2) this follows from the simple observation that the coefficient of $\ell n\,r$ in (A.5) is independent of the ambiguity reflected in the scale parameter ℓ (and of the transverse hypersurface Σ that enters (16)).

In fact, each renormalization of a subdivergence in a given graph increases by one the order—i.e. the maximal power of $\ell n\,\lambda$ in the associate homogeneity law. Since $r\frac{\partial}{\partial r}(\ell n\,r)^j = j(\ell n\,r)^{j-1}$, a general associate homogeneous renormalized Feynman amplitude G will satisfy (7), $(E+d)^{k+1}\,G(\mathbf{x}) = 0$. We can then characterize G

[6]Usually, in perturbation theory one is dealing with Lie algebra cohomology. Group cohomology has occurred in various contexts in the early 1980s [20, 21, 52, 59].

Fig. 3 Logarithmically divergent 3-point graph with a 2-point subdivergence



$$G_\Delta = \frac{1}{x_{01}^2 \; x_{12}^2 \; [x_{02}^2]^2} .$$

by a (column) vector $\boldsymbol{G} = (G_0 = G, G_1 = (E+d)\, G_0, \ldots, G_k = (E+d)\, G_{k-1})$ of distributions. It carries an *indecomposable representation of the dilation group*[7] *of degree* $-d$ *and order* k such that

$$\boldsymbol{G}(\mathbf{x}) \to \lambda^d \, \boldsymbol{G}(\lambda\, \mathbf{x}) = e^{\Delta \ell n \lambda} \, \boldsymbol{G}(\mathbf{x}) = \sum_{j=0}^{k} \frac{(\ell n \, \lambda)^j}{j!} \, G_j(\mathbf{x}) \tag{34}$$

where Δ is a nilpotent Jordan cell with k units above the diagonal. The nilpotency condition $\Delta^{k+1} = 0$ remains invariant under an arbitrary non-singular transformation $\boldsymbol{G} \to S\boldsymbol{G}, \Delta \to S^{-1} \Delta S$. One usually only uses this freedom to change the relative normalization of G_j.

It follows from the factorization property (8) that the dimension of the support of G_j is decreasing with j and

$$G_k(\mathbf{x}) = (\mathbf{x}\,\partial + d)\, G_0(\mathbf{x}) = \sum_{\alpha} a_\alpha(G)\, D_\alpha \, \delta(x) \, . \tag{35}$$

Following the terminology of Gelfand-Shilov [27] we call both $\boldsymbol{G}$ and its components *associate homogeneous distributions* (*cf.* (7)).

The following simple example of a graph with a subdivergence illustrate the complication (mentioned in connection with (23)) coming from the use of a non-smooth radial coordinate.

Example 4.1. Renormalization the 3-point two loop diagram displayed on Fig. 3.

We introduce as independent variables the spherical coordinates of the vectors x_{0i}, $i = 1,2$

$$x_{01} = r\omega_1 \, , \quad x_{02} = \rho\, \omega_2 \, , \quad r, \rho \geq 0 \, , \quad \omega_i^2 = 1 \; (i.e. \; \omega_i \in \mathbb{S}^3) \; i = 1,2 \tag{36}$$

and set

$$\omega_1 \cdot \omega_2 = \cos \vartheta \, , \quad x_{12}^2 = r^2 + \rho^2 - 2r\rho \cos \vartheta \, . \tag{37}$$

The renormalized 2-point Green function (19), corresponding to the subgraph of vertices $(0,2)$ is

$$G_1(x_{02}, \ell) = \frac{1}{2} \frac{\partial}{\partial x_{02}^\alpha} \left[\frac{x_{02}^\alpha}{(x_{02}^2)^2} \, \ell n \, \frac{x_{02}^2}{\ell^2} \right]_+ = \frac{1}{\rho^3} \frac{\partial}{\partial \rho} \left(\ell n \, \frac{\rho}{\ell} \right)_+ . \tag{38}$$

---

[7]Representations of this type have been considered back in the 1970's [23] within a study of a spontaneous breaking of dilation symmetry.

(The last expression only makes sense as a density after multiplying with the volume element $d^4x = \rho^3\, d\rho\, d^3\omega$ that cancels the $\frac{1}{\rho^3}$ factor and permits to transfer the derivative to the test function.)

Next we shall write down the density $G_\Delta\mathrm{Vol}$ with renormalized subdivergence integrated over the six angular variables ω_1 and ω_2

$$\begin{aligned} G_\Delta\,\mathrm{Vol} &:= \left[\int d^3\omega_1 \int d^3\omega_2\, G_\Delta(r\omega_1, \rho\,\omega_2; \ell)\right] r^3\, dr \rho^3 d\rho \\ &= 8\pi^3 \int_0^\pi \frac{\sin^2\vartheta\, d\vartheta}{r^2+\rho^2-2r\rho\cos\vartheta}\, \frac{\partial}{\partial\rho}\left(\ell n\,\frac{\rho}{\ell}\right)_+ r\,dr\,d\rho \\ &= 4\pi^4\, \frac{r\,dr\,d\rho}{r_\vee^2}\, \frac{\partial}{\partial\rho}\left(\ell n\,\frac{\rho}{\ell}\right)_+,\; r_\vee = \max(r,\rho) = \frac{r+\rho+|r-\rho|}{2}\,. \end{aligned} \quad (39)$$

Smearing $G_\Delta\mathrm{Vol}$ with a test function $f(r,\rho)$ we find that the *leading term*, $LT\, G_\Delta\mathrm{Vol}$, for $r_\vee \to 0$ (the only one that requires overall renormalization) corresponds to $r = \rho$

$$(LT\, G^R_\Delta\,\mathrm{Vol}, f) = -4\pi^4 \int_0^\infty dr\, \frac{\ell n^2\left(\frac{r}{\ell}\right)}{2}\, \frac{d}{dr} f(r,r)\,. \quad (40)$$

Here we have made use of the renormalized associate homogeneous distribution $L_{-11}(r)$ thus illustrating Theorem 2.2.

Somewhat symbolically we can write

$$G^R_\Delta(r,\rho;\ell)\,\mathrm{Vol} = 4\pi^4 L_{-11}\left(\frac{r}{\ell}\right)\delta(\rho-r)\,\frac{dr}{\ell}\,d\rho + G_0(r,\rho)\,\mathrm{Vol}\, L_{01}\left(\frac{\rho}{\ell}\right)d\rho \quad (41)$$

where $G_0\mathrm{Vol}$ is the regular part of the homogeneous 1-form $4\pi^4\,\frac{r\,dr}{r_\vee^2}$ (for $\rho \neq r$).

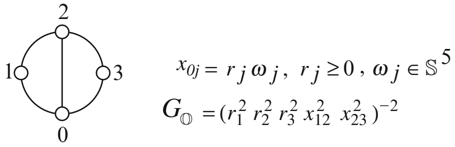
Displaying the associate homogeneity law for the renormalized density (41) we observe a manifestation of the general rule: only the coefficient of the highest log term ($\ell n\,\lambda$ for $L_{01}\,d\rho$ and $(\ell n\,\lambda)^2$ for $L_{-11}\,dr$) is independent of the ambiguity parametrized here by the scale ℓ in the renormalized subdivergence.

Remark 4.1. One could be tempted to replace the renormalization parameter ℓ in the expression (39) by the (external to the divergent 2-point subgraph) variable r for $r > \rho$. This would amount to subtracting a local in ρ term, $4\pi^4\,\frac{dr}{r}\,\ell n\,\frac{r}{\ell}\,\delta(\rho)\,d\rho$. It is straightforward to observe, however, that neglecting such a term in (39) would violate the causal factorization requirement (8).

The techniques developed in Example 4.1 also apply to more complicated graphs (*cf.* Example3.2 in [50]).

Example 4.2. As a last example we consider the graph displayed on Fig. 4 which exhibits overlapping divergences in 6-dimensional space-time.

Fig. 4 Quadratically divergent diagram in 6-dimensions



Applying the relations (24), (25) for $\lambda = 2$, we find the following expression for the analytically regularized integrated with respect to the angles Green function density

$$\widetilde{G}_{\mathbb{O}}^{\varepsilon_1\,\varepsilon_2} = \pi^9\, \frac{r_1\, r_2\, r_3}{(R_{12}\, R_{23})^4} \left(\frac{R_{12}^2}{\ell_1^2}\right)^{\varepsilon_1} \left(\frac{R_{23}^2}{\ell_2^2}\right)^{\varepsilon_2} dr_1\, dr_2\, dr_3\,, \tag{42}$$

where $R_{ij} = \max(r_i, r_j)$ (*cf.* (24)). The renormalized expression for $G_{\mathbb{O}}$ again depends, as in the preceding examples (see, in particular, Example 3.2) on the inequalities satisfied by the radial variables. For

$$r_1 < r_2 < r_3 \tag{43}$$

(and, similarly, for $r_3 < r_2 < r_1$) we have a case of nested singularities. One first renormalizes the logarithmicly divergent triangular subgraph with vertices $(0,1,2)$. Integrating first with respect to r_1 in the domain (43) we find

$$\begin{aligned} &\lim_{\varepsilon_1 \to 0} \left(\int_0^{r_2} \widetilde{G}_{\mathbb{O}}^{\varepsilon_1\,\varepsilon_2} - \frac{\pi^9}{4\,\varepsilon_1}\, \delta(r_2) \left(\frac{r_3}{\ell_2}\right)^{2\varepsilon_2} \frac{dr_2\, dr_3}{r_3^3} \right) \\ &\qquad = \frac{\pi^9}{2}\, d\left(\ell n\, \frac{r_2}{\ell_1}\right) \left(\frac{r_3}{\ell_2}\right)^{2\varepsilon_2} \frac{dr_3}{r_3^3}\,. \end{aligned} \tag{44}$$

The renormalization of the resulting quadratically divergent in r_3 associate homogeneous distribution follows the lines of Example 4.1. The case $r_1 < r_2 > r_3$, in which $R_{12} = R_{23} = r_2$ and "the divergences overlap", is actually simpler; it is reduced to a single radial renormalization. Setting $\varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2}$ and $\ell_1\, \ell_2 = \ell^2$ and integrating in r_1 and r_3, we find

$$\lim_{\varepsilon \to 0} \left(\int_{r_1=0}^{r_2} \int_{r_3=0}^{r_2} G_{\mathbb{O}}^{\varepsilon} - \frac{\pi^9}{8}\, \frac{\delta''(r_2)}{2\,\varepsilon}\, dr_2 \right) = \frac{\pi^9}{8} \left(\frac{d^3}{dr^3}\, \ell n\, \frac{r}{\ell} + \frac{3}{2}\, \delta''(r) \right). \tag{45}$$

5 Concluding Remarks

The work [50], surveyed here, is concerned with a mathematical reformulation of the problem of ultraviolet renormalization of massless QFT. The extension of rational homogeneous functions to associate homogeneous distributions of the same degree obeying (Euclidean) factorization, considered here, only partly resolves the physical problem (see [49]). It does not consider integration over internal vertices in concrete Lagrangian theories (like φ^4) and so does not control the corresponding adiabatic limit (which is separated in standard approaches from the study of on shell infrared singularities[8]).

The present survey is only confined to the part of [50] dealing with the Euclidean picture. The reader willing to understand the physical origin of the causal factorization and the way one goes around the light cone singularities should consult the original paper.

Acknowledgements The authors thank Detlev Buchholz, Maxim Kontsevich and Dirk Kreimer for discussions. N.N. acknowledges partial support by the French-Bulgarian Project Rila under the contract Egide-Rila N112. N.N. and I.T. acknowledge partial support by grant DO 02-257 of the Bulgarian National Science Foundation. N.N. and I.T. thank the organizers of the International Workshops LT-9 and SQS'2011 for the invitation and the IHÉS and the Theory Division of CERN for hospitality during the course of this work.

Appendix A. Radial Associate Homogeneous Distributions

The study of radial homogeneous distributions is based on the observation ([39, Sect. 3.2]) that the family of distributions ("divided powers")

$$\chi^a(r) := \frac{(r^a)_+}{\Gamma(a+1)}\,, \quad a \neq -1,-2,\dots \quad ((r^a)_+ \equiv \theta(r)\,r^a) \tag{A.1}$$

is uniquely extendable to a distribution valued entire analytic function in a. The property $\Gamma(a+1) = a\,\Gamma(a)$ gives

$$\frac{d}{dr}\chi^a(r) = \chi^{a-1}(r) \quad (r\,\chi^a(r) = (a+1)\,\chi^{a+1}(r))\,. \tag{A.2}$$

Combined with $\chi^0(r) = \theta(r)$ the (Heaviside) characteristic function of the positive semiaxis—we find

$$\chi^{-\kappa-1}(r) = \delta^{(\kappa)}(r)\,, \quad \kappa = 0,1,\dots \left(\int \delta^{(\kappa)}(r)\,f(r)\,dr = (-1)^\kappa f^{(\kappa)}(0)\right)\,. \tag{A.3}$$

From the known pole structure of $\Gamma(a)$ we deduce the formula (12) for the generating function of $L_{-\kappa-1n}$. The distributions $L_{-\kappa-1n}$ can be defined in terms of *differential renormalization* [25]:

[8]We thank Detlev Buchholz for stressing this point to us.

$$L_{-\kappa-1n}(r) = \lim_{\varepsilon\to 0} \frac{1}{n!} \frac{\partial^n}{\partial \varepsilon^n} \left(\theta(r)\, r^{\varepsilon-\kappa-1} - \frac{\delta^{(\kappa)}(-r)}{\varepsilon \kappa!} \right)$$

$$= \frac{(-1)^\kappa}{\kappa!} \left(\frac{d}{dr} \right)^{\kappa+1} \sum_{m=0}^{n+1} \sigma_{\kappa m} L_{0n+1-m}\,, \tag{A.4}$$

where $L_{0\nu}(r) = \theta(r)\, \frac{(\ell n\, r)^\nu}{\nu!}$ are (integrable) powers of logarithms and the constants $\sigma_{\kappa m}$ are given by

$$\sigma_{\kappa 0} = 1\,, \qquad \sigma_{0m} = 0 \qquad \text{for} \qquad m = 1, \ldots, n+1\,,$$

$$\sigma_{\kappa m} = \sigma_{\kappa-1m} + \frac{\sigma_{\kappa m-1}}{\kappa} = \sum_{1 \le j_1 \le \ldots \le j_m \le \kappa} \frac{1}{j_1 \ldots j_m}\,. \tag{A.5}$$

The freedom in the extension of the rational homogeneous function r^{-k} from the positive semiaxis to an associate homogeneous distribution on $\mathbb{R}$ is hidden in the scale of $\log r$. In fact, the general associate homogeneous distribution that coincides with $r^{-\kappa-1}$ for $r > 0$ involves a single scale parameter ℓ:

$$\ell^{-\kappa-1} L_{-\kappa-1,0}\left(\frac{r}{\ell}\right) = \frac{(-1)^\kappa}{\kappa!} \left\{ \frac{d^{\kappa+1}}{dr^{\kappa+1}} \left(\theta(r)\, \ell n\, \frac{r}{\ell} \right) + \sum_{j=1}^{\kappa} \frac{1}{j}\, \delta^{(\kappa)}(r) \right\}$$

$$= L_{-\kappa-1,0}(r) - \ell n\, \ell\, \frac{\delta^{(\kappa)}(-r)}{\kappa!}\,. \tag{A.6}$$

Once ℓ is fixed, say $\ell = 1$, all distributions $L_{kn}(r)$ ($k \in \mathbb{Z}$, $n = 0, 1, \ldots$) are uniquely determined.

Proposition A.1. *The distributions $L_{kn}(r)$, given for negative integer k by (A.4), satisfy*

(i) $L_{kn}(r) = \theta(r)\, r^k\, \frac{(\ell n\, r)^n}{n!}$ *for* $r \neq 0$;
(ii) $(E-k)\, L_{kn}(r) = L_{kn-1}(r)$ *for* $n = 1, 2, \ldots$, $(E-k)\, L_{k0}(r) = 0$;
(iii) $r\, L_{kn}(r) = L_{k+1n}(r)$.

Conversely, the properties (i) *and* (ii) *determine uniquely the system of distributions* L_{kn}.

Proof. Properties (i)–(iii) follow from the corresponding properties of $\theta(r)\, r^{\varepsilon+k}$ (and from (12)). To prove the uniqueness, assume that there are two sets of associate homogeneous distributions L_{kn} and L'_{kn} satisfying (i) and (ii). Then their differences $D_{kn} := L_{kn} - L'_{kn}$ would satisfy $D_{kn} = 0$ for $k \geq 0$ and $D_{-\kappa-1n}(r) = C_{\kappa n}\, \delta^{(\kappa)}(r)$ for $\kappa, n = 0, 1, \ldots$. It then follows from (ii) that

$$0 = (E + \kappa + 1)\, C_{\kappa n+1}\, \delta^{(\kappa)}(r) = C_{\kappa n}\, \delta^{(\kappa)}(r)\,,$$

hence $C_{\kappa n} = 0$ for all $n \geq 0$.

References

1. Bergbauer, C.: Notes on Feynman integrals and renormalization. arXiv: 1005.3960 [hep-th]
2. Bergbauer, C., Brunetti, R., Kreimer, D.: Renormalization and resolution of singularities. arXiv:0908.0633v2 [hep-th]
3. Bloch, S., Esnault, H., Kreimer, D.: On motives and graph polynomials. Comm. Math. Phys. **267**, 181–225 (2006). math/0510011
4. Bloch, S., Kreimer, D.: Mixed Hodge structures and renormalization in physics. Comm. Number Theor. Phys. **2**, 637–718 (2008). arXiv:0804.4399 [hep-th]; Feynman amplitudes and Landau singularities for 1-loop graphs. arXiv:1007.0338 [hep-th]
5. Bogoliubov, N.N., Parasiuk, O.S.: Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder. Acta Math. **97**, 227–266 (1957)
6. Bogoliubov, N.N., Shirkov, D.V.: Introduction to the Theory of Quantized Fields, 3rd edn. Wiley, New York (1980)
7. Broadhurst, D.J., Kreimer, D.: Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. Phys. Lett. **B393**, 403–412 (1997). hep-th/9609128
8. Bros, J., Epstein, H., Glaser, V., Stora, R.: Quelques aspects globaux des problèmes d'Edge-of-the-Wedge, Nice, 1973. In:Hyperfunctions and Theoretical Physics. Lecture Notes in Mathematics, vol. 449, pp. 185–218. Berlin (1975)
9. Brown, F., Kreimer, D.: Angles, scales and parametric renormalization. arXiv:1112.1180 [hep-th]
10. Brunetti, R., Fredenhagen, K.: Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. Comm. Math. Phys. **208**, 623–661 (2000). math-ph/990328
11. Brunetti, R., Fredenhagen, K., Verch, R.: The generally covariant locality principle: A new paradigm for local quantum physics. Comm. Math. Phys. **237**, 31–68 (2003). math-ph/0112041
12. Ceyhan, O., Marcolli, M.: Feynman integrals and motives of configuration space. arXiv:1012.5485 [math-ph]
13. Chetyrkin, K.G., Kataev, A.L., Tkachov, F.V.: New approach to evaluation of multiloop Feynman integarls: The Gegenbauer polynomial x-space technique. Nucl. Phys. **B174**, 345–377 (1980)
14. Connes, A., Kreimer, D.: Renormalization in quantum field theory and the Riemann-Hilbert problem I: The Hopf algebra structure of graphs and the main theorem, II: the β-function diffeomorphism and the renormalization group. Comm. Math. Phys. **210**, 249–273 (2000), **216**, 215–241 (2001). hep-th/9912092, hep-th/0003188; Insertion and elimination: The doubly infinite algebra of Feynman graphs. Ann. Inst. Henri Poincaré **3**, 411–433 (2002). hep-th/0201157
15. De Concini, C., Procesi, C.: Wonderful models of subspace arrangements. Selecta Math. **1**(3) 459–494 (1995)
16. Dütsch, M., Fredenhagen, K.: Causal perturbation theory in terms of retarded products, and a proof of the action Ward identity. Rev. Math. Phys. **16**(10) 1291–1348 (2004)
17. Ebrahimi-Fard, K., Gracia-Bondia, J.M., Patras, F.: A Lie algebra approach to renormalization. Comm. Math. Phys. **276**, 519549 (2007). hep-th/0609035
18. Epstein, H., Glaser, V.: The role of locality in perturbation theory. Ann. Inst. H. Poincaré **A19**(3), 211–295 (1973)
19. Epstein, H., Glaser, V., Stora, R.: General properties of the n-point functions in local quantum field theory. In: Balian, R., Iagolnitzer, D. (eds.) Structural Analysis of Collision Amplitudes, Les Houches, 1975. North-Holland, Amsterdam
20. Faddeev, L.D.: Operator anomaly for the Gauss law. Phys. Lett. **B145**, 81–84 (1984)
21. Faddeev, L.D., Shatashvili, S.L.: Algebraic and Hamiltonian methods in the theory of nonabelian anomalies. Teoret. Mat. Fiz. **60**(2) 206–217 (1984) [English translation: Theoret. Math. Phys. **60**(2) (1984) 770–778]

22. Falk, S., Häußling, R., Scheck, F.: Renormalization in quantum field theory: an improved rigorous method. J. Phys. A: Math. Theor. **43**, 035401 (15p) (2010). arXiv:0901.2252 [hep-th]
23. Ferrara, S., Gatto, R., Grillo, A.F.: Logarithmic scaling and spontaneous breaking. Phys. Lett. **42B** 264–266 (1972)
24. Figeroa, H., Gracia-Bondia, J.M.: Combinatorial Hopf algebras in quantum field theory. Rev. Math. Phys. **17** 881 (2005). hep-th/0408145
25. Freedman, D.Z., Johnson, K., Latorre, J.I.: Differential regularization and renormalization: A new method of calculation in quantum field theory. Nucl. Phys. **B371** 353–314 (1992)
26. Fulton, W., MacPherson, R.: A compactification of configuration space. Ann. Math. **139**, 183–225 (1994)
27. Gelfand, I.M., Shilov, G.E.: Generalized Functions I. Academic, New York (1968) (original Russian edition: Moscow, 1958)
28. Giombi, S., Prakash, S., Yin, X.: A note on CFT correlators in three dimensions. arXiv:1104.4317v3 [hep-th]
29. Gracia-Bondia, J.M.: Improved Epstein-Glaser renormalization in coordinate space I. Euclidean framework. Math. Phys. Anal. Geom. **6** 55–88 (2003). hep-th/0202023
30. Gracia-Bondia, J.M., Lazzarini, S.: Improved Epstein-Glaser renormalization in coordinate space II. Lorentz invariant framework. J. Math. Phys. **44** 3863–3875 (2003). hep-th/0212156
31. Gracia Bondía, J.M., Várilly, J.C., Figueroa, H.: Elements of Noncommutative Geometry. Birkhäuser, Basel (2000)
32. Haagensen, P.E., Latorre, J.I.: Differential renormalization of massive quantum field theoris. Phys. Lett. **B283** 293–297 (1992). hep-ph/9203207; A comprehensive coordinate space renormalization of quantum electrodynamics. Ann. Phys. **221**, 77–105 (1993). hep-ph/9206252
33. Hepp, K.: Proof of the Bogoliubov-Parasiuk theorem on renormalization. Comm. Math. Phys. **2**(4) 301–326 (1966)
34. Hepp, K.: La Théorie de la Renormalisation. Lecture Notes in Physics, vol. 2. Springer, Berlin (1969)
35. Hollands, S.: The operator product expansion for perturbative quantum field theory in curved spacetime. Comm. Math. Phys. **273** 1–36 (2007). gr-qc/0605072
36. Hollands, S.: Renormalized quantum Yang-Mills fields in curved spacetime. Rev. Math. Phys. **20** 1033–1172 (2008). arXiv:0705.3340 [gr-qc]
37. Hollands, S., Wald, R.M.: Existence of local covariant time ordered products of quantum fields in curved spacetime. Comm. Math. Phys. **231** 309–345 (2002). gr-qc/0111108
38. Hollands, S., Wald, R.M.: Quantum field theory in curved spacetime the operator product expansion and dark energy. Gen. Rel. Grav. **40** 2051–2059 (2008). arXiv:0805.3419 [gr-qc]
39. Hörmander, L.: The Analysis of Linear Partial Differential Operators, I. Distribution Theory and Fourier Analysis, 2nd edn. Springer, Berlin (1990) (see, in particular, Sect. 3.2 and Chap. VIII)
40. Keller, K.J.: Euclidean Epstein-Glaser renormalization. J. Math. Phys. **50** 103503 (2009). arXiv:0902.4789 [math-ph]
41. Keller, K.J.: Dimensional regularization in position space and a forest formula for regularized Epstein-Glaser renormalization. arXiv:1006.2148 [math-ph]
42. Kreimer, D.: Knots and Feynman Diagrams. Cambridge Lecture Notes in Physics, vol. 13. Cambridge University Press, Cambridge (2000)
43. Kuznetsov, A.N., Tkachov, F.V., Vlasov, V.V.: Techniques of distributions in perturbative quantum field theory I. hep-th/9612037
44. Lacki, J., Wanders, G., Ruegg, H. (eds.): Stueckelberg, an Unconventional Figure in Twentieth Century Physics. Birkhäuser Verlag AG (2009)
45. [10] Lam, Y.M.: Equivalence theorem on Bogolyubov-Parasiuk-Hepp-Zimmermann renormalized Lagrangian field theories. Phys. Rev. **D7**, 2943 (1973)
46. Lowenstein, J.H., Zimmermann, W.: On the formulation of theories with zero-mass propagators. Nuclear Phys. **B86**, 77–103 (1975)

47. Maldacena, J., Zhiboedov, A.: Constraining conformal field theories with higher spin symmetry. arXiv:1112.1016 [hep-th]
48. Nikolov, N.M.: Anomalies in quantum field theory and cohomologies in configuration spaces. arXiv:0903.0187 [math-ph]; Talk on anomaly in quantum field theory and cohomologies of configuration spaces. arXiv:0907.3735 [hep-th]
49. Nikolov, N.M.: Renormalization theory of Feynman amplitudes on configuration space. arXiv:0907.3734 [hep-th]
50. Nikolov, N.M., Stora, R., Todorov, I.: Configuration space renormalization of massless QFT as an extension problem for associatehomogeneous distributions, Bures-sur-Yvette preprint IHES/P/11/07 and work in progress
51. Parasiuk, O.S.: On the theory of Bogoliubov's R-operation (Russian). Ukr. Math. Zh. **12**, 287–307 (1960)
52. Popineau, G., Stora, R.: A Pedagogical Remark on the Main Theorem of Perturbative Renormalization Theory, unpublished preprint (1982)
53. Prange, D.: Causal perturbation theory and differential renormalization. J. Phys. **A32**, 2225–2238 (1999). hep-th/9710225
54. Rosner, J.: Sixth order contribution to Z_3 in finite quantum electrodynamics. Phys. Rev. Lett. **17**(23) 1190–1192 (1966); Higher order contributions to the divergent part of Z_3 in a model quantum electrodynamics. Ann. Phys. **44** 11–34 (1967)
55. Schwartz, L.: Théorie des distributions I, II. Hermann, Paris (1950–1951)
56. Speer, E.R.: On the structure of anlytic renormalization. Comm. Math. Phys. **23**, 23–36 (1971); Comm. Math. Phys. **25**, 336 (1972)
57. Steinmann, O.: Perturbative Expansion in Axiomatic Field Theory. Lecture Notes in Physics. Springer, Berlin (1971)
58. Stepanov, B.M.: Abstraktnaia teoriia R-operatsii. Izv. AN SSSR, ser. mat. **27**, 819 (1963); On the construction of S-matrix in accordance with perturbation theory. Izv. AN SSSR, ser. mat. **29** 1037–1054 (1965) (transl. Amer. Math. Soc. (2) **91** (1969))
59. Stora, R.: DifferentialAlgebras in Lagrangean Field Theory. ETHZ Zürich Lectures, January–February 1993
60. Stora, R.: Causalité et groupes de renormalisation perturbatifs. In: Boudjedaa, T., Makhlouf, A. (eds.) Théorie Quantique des Champs Méthode et Applications, Ecole de Physique Théorique de Jijel, Algérie, 2006. Collection Travaux en cours, vol. 68. Editions Hermann (2007/2008)
61. Stueckelberg, E.C.G., Petermann, A.: La normalization des constantes dansla théorie des quanta. Helv. Phys. Acta **26**, 499–520 (1953)
62. Usyukina, N.I.: Algorithm for calculating massless Feynman diagrams. Theor. Math. Phys. **79**, 385–391 (1989) (transl from Teor. Mat. Fiz. **79**, 63–71 (1989))
63. Zavyalov, O.I.: Renormalized Quantum Field Theory. Kluwer, Dordrecht (1990) (Russian original: Nauka, Moscow, 1979)
64. Zimmermann, W.: Convergence of Bogoliubov's method of renormalization in momentum space. Comm. Math. Phys. **15**, 208–234 (1969); Local operator products and renormalization in quantum field theory. In: Deser, S., et al. (eds.) Lectures on Elementary Particles and Quantum Field Theory. Proceedings of the 1970 Brandeis Summer Institute in Theoretical Physics, pp. 399–589. MIT, Cambridge

Wigner Quantization and Lie Superalgebra Representations

Joris Van der Jeugt

Dedicated to T.D. Palev on the occasion of his 75th birthday

Abstract T.D. Palev laid the foundations of the investigation of Wigner quantum systems through representation theory of Lie superalgebras. His work has been very influential, in particular on my own research. It is quite remarkable that the study of Wigner quantum systems has had some impact on the development of Lie superalgebra representations. In this review paper, I will present the method of Wigner quantization and give a short overview of systems (Hamiltonians) that have recently been treated in the context of Wigner quantization. Most attention will go to a system for which the quantization conditions naturally lead to representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$. I shall also present some recent work in collaboration with G. Regniers, where generating functions techniques have been used in order to describe the energy and angular momentum contents of 3-dimensional Wigner quantum oscillators.

1 Introduction and Some History

The main ideas of Wigner quantization go back to a short paper that Wigner published in 1950 [36]. Due to the fact that his method leads to algebraic relations for operators which are in general very difficult to solve, it took many years before his work was continued. About 30 years later, when Lie superalgebra theory was developed, it was T.D. Palev who realized that particular Lie superalgebra generators satisfy the algebraic relations appearing in the Wigner quantization of certain

J. Van der Jeugt (✉)
Department of Applied Mathematics and Computer Science, Ghent University,
Krijgslaan 281-S9, 9000 Gent, Belgium
e-mail: Joris.VanderJeugt@UGent.be

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_10,

systems. This was the real start of Wigner quantization, a program to which Palev contributed much of his scientific career. He also inspired many other scientists to work on the program, including myself. It has been a pleasure for me to collaborate with Tchavdar Palev and his former student Neli Stoilova, and to contribute to the theory.

In this review paper, I will give an introduction to the topic, first by presenting Wigner's original example in a contemporary context. In Sect. 2, Palev's general method of Wigner quantization is briefly presented, and then we give a short overview of his contributions to the field, and of some other papers on Wigner quantization. Our purpose is to include also some recent work, and therefore the Wigner quantization of the n-dimensional non-isotropic oscillator is discussed in Sect. 3. This problem stimulated the search for infinite-dimensional unitary representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$; a class of these representations was constructed only a few years ago. Using these representations, we present some interesting aspects of this Wigner quantum system in Sect. 3, and its angular momentum contents in Sect. 4. There is no new material in this paper: we only present and summarize some of the main ideas of Wigner quantization and some recent contributions.

In his seminal paper [36], Wigner asked the question: "Do the equations of motion determine the quantum mechanical commutation relations?" It was known at that time that, for a class of Hamiltonians written as analytic functions of the generalized position and momentum operators $\hat{q}_i$ and $\hat{p}_i$ $(i = 1, \ldots, n)$, the Heisenberg equations of motion together with the canonical commutation relations (CCRs) imply formally Hamilton's equations. Vice versa, starting from the operator form of Hamilton's equations and using the CCRs, one can derive the Heisenberg equations. Since Wigner believed that the Heisenberg equations of motion and the operator form of Hamilton's classical equations of motion have a deeper physical meaning than the mathematically imposed CCRs, he wondered whether requiring the compatibility of the Heisenberg equations with Hamilton's equations would automatically lead to the CCRs. Wigner investigated this question for the Hamiltonian of the one-dimensional harmonic oscillator, given by

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2) \tag{1}$$

under the convention $m = \omega = \hbar = 1$. The Heisenberg equations are:

$$\dot{\hat{q}} = i[\hat{H}, \hat{q}], \quad \dot{\hat{p}} = i[\hat{H}, \hat{p}], \tag{2}$$

and the operator form of Hamilton's equations read:

$$\dot{\hat{q}} = \mathrm{op}\left(\frac{\partial H}{\partial p}\right) = \hat{p}, \quad \dot{\hat{p}} = -\mathrm{op}\left(\frac{\partial H}{\partial q}\right) = -\hat{q}. \tag{3}$$

So for this example the compatibility conditions become:

$$\hat{p} = i\left[\frac{1}{2}(\hat{p}^2+\hat{q}^2),\hat{q}\right], \quad -\hat{q} = i\left[\frac{1}{2}(\hat{p}^2+\hat{q}^2),\hat{p}\right]. \tag{4}$$

The goal is to find (self-adjoint) operators $\hat{p}$, $\hat{q}$ satisfying these equations, without making any assumptions about the commutation relation between $\hat{p}$ and $\hat{q}$. Otherwise said, are there other operator solutions to (4) besides the canonical solution where $[\hat{q},\hat{p}] = i$? Wigner found that indeed there are other solutions. In order to describe these, let us use the language of Lie superalgebras (of course, Wigner used a different method, as Lie superalgebras were not known at that time).

Rewriting the operators $\hat{q}$ and $\hat{p}$ by the linear combinations

$$b^+ = \frac{\hat{q}-i\hat{p}}{\sqrt{2}}, \quad b^- = \frac{\hat{q}+i\hat{p}}{\sqrt{2}}, \tag{5}$$

the conditions (4) are equivalent to the two relations

$$[\{b^+,b^-\},b^\pm] = \pm 2b^\pm. \tag{6}$$

Note that these relations involve both commutators and anti-commutators. This is why it will be helpful to use Lie superalgebras. In fact, it is known that (6), the compatibility conditions to solve, are exactly the defining relations of the Lie superalgebra $\mathfrak{osp}(1|2)$ in terms of two odd generators b^+, b^- [5]. Moreover, it should hold that $\hat{p}^\dagger = \hat{p}$ and $\hat{q}^\dagger = \hat{q}$, or rewritten in terms of the new operators: $(b^\pm)^\dagger = b^\mp$. Thus, we are led to the unitary (or unitarizable) representations of $\mathfrak{osp}(1|2)$ i.e. Hilbert space representations in which $(b^\pm)^\dagger = b^\mp$ holds.

The unitary irreducible representations of $\mathfrak{osp}(1|2)$ were classified by Hughes [8]; see also [31] for a more comprehensive method. The unitary irreducible representations are labelled by a positive real number p ($p/2$ is the lowest weight); the orthonormal basis vectors are $|n\rangle$, with $n \geq 0$. The action of b^+ and b^- is given by:

$$b^+|n\rangle = \sqrt{\nu_{n+1}}\,|n{+}1\rangle, \quad b^-|n\rangle = \sqrt{\nu_n}\,|n{-}1\rangle; \quad \nu_n = n+(p-1)(1-(-1)^n)/2. \tag{7}$$

Using (5) and (7), one can deduce:

$$\hat{H}|n\rangle = \tfrac{1}{2}\{b^+,b^-\}|n\rangle = (n+\tfrac{p}{2})|n\rangle, \tag{8}$$

$$[\hat{q},\hat{p}]|2n\rangle = ip|2n\rangle, \quad [\hat{q},\hat{p}]|2n+1\rangle = i(2-p)|2n+1\rangle. \tag{9}$$

From this it is clear that only the case $p = 1$ corresponds to the CCRs. All other solutions (i.e. all other positive values of p) are non-canonical. Wigner concluded that requiring the equivalence of Hamilton's and Heisenberg's equations is a very natural approach that may lead to other quantizations besides the canonical one; and the canonical quantization solution appears as one of the more general solutions.

In the example of Wigner, the apparent difference with the canonical case is the shift in energy, as is clear from (8). It is interesting to have also a look at the wave functions for these non-canonical solutions. This was in fact not performed by Wigner, but only much later, when the above operators b^+ and b^- were studied as "parabosons" [22]. An alternative way of finding these wave functions is described in the Appendix of [9]. This is obtained by computing the (formal) eigenvectors of $\hat{q} = (b^+ + b^-)/\sqrt{2}$ in the above Hilbert space. Writing these formal eigenvectors of $\hat{q}$ as

$$v(q) = \sum_{n=0}^{\infty} \Psi_n^{(p)}(q)\,|n\rangle, \tag{10}$$

and expressing $\hat{q}\,v(q) = q\,v(q)$ by means of the action (7) yields a set of recurrence relation for the coefficients $\Psi_n^{(p)}(q)$. The solution leads to the conclusion that the spectrum of $\hat{q}$ is $\mathbb{R}$, and that

$$\Psi_{2n}^{(p)}(x) = (-1)^n \sqrt{\frac{n!}{\Gamma(n+p/2)}}\,|x|^{(p-1)/2}\,e^{-x^2/2} L_n^{(p/2-1)}(x^2),$$

$$\Psi_{2n+1}^{(p)}(x) = (-1)^n \sqrt{\frac{n!}{\Gamma(n+p/2+1)}}\,|x|^{(p-1)/2}\,e^{-x^2/2} x L_n^{(p/2)}(x^2), \tag{11}$$

in terms of generalized Laguerre polynomials. These coefficients $\Psi_n^{(p)}(q)$ have an interpretation as the position wave functions of the Wigner oscillator. Alternatively, one can work in the position representation, where the operator $\hat{q}$ is still represented by "multiplication by x", and the operator $\hat{p}$ has a realization as $-i\frac{d}{dx} + i\frac{p-1}{2x}R$, where $Rf(x) = f(-x)$ is a reflection operator [23, Chap. 23]. Using this realization the time-independent Schrödinger equation can be solved, also yielding the expressions (11) [22]. For $p = 1$, the Laguerre polynomials reduce to Hermite polynomials, and one gets the commonly known wave functions. It is interesting to compare the plots of the wave functions for $p \neq 1$ with those of the canonical case $p = 1$, see Fig. 1.

2 Wigner Quantum Systems and Palev's Contributions

Wigner's work on this alternative quantization method for the one-dimensional oscillator did not receive much attention originally. This was mainly because of the mathematical difficulties when trying to apply it to a Hamiltonian different from (1). In fact, trying to solve Wigner's compatibility conditions for other systems leads to complicated operator relations, for which often no general solutions are known. By 1980 however, Lie superalgebra theory and their representations became well understood [10, 11]. T.D. Palev had worked with Lie superalgebras,

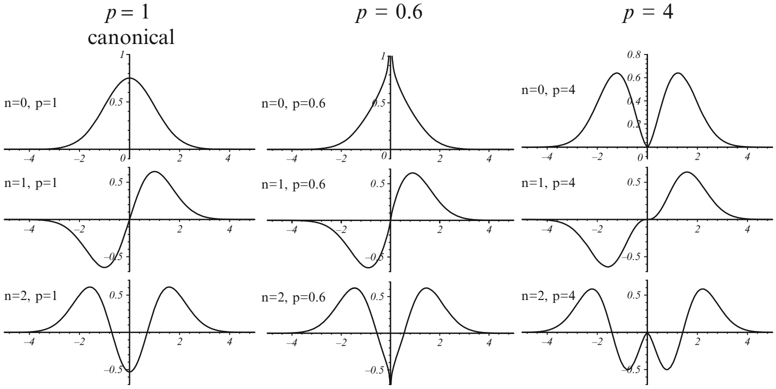


Fig. 1 Plots of the wave functions $\Psi_n^{(p)}(x)$. The three figures on the *left* are for $p=1$ and correspond to the canonical case; the figures in the *middle* are for $p=0.6$, and three figures on the *right* are for $p=4$. In each case, we plot the wave functions for $n=0,1,2$

mainly in the context of parabosons and parafermions [24]. He was the first to realize the importance of Lie superalgebra representations in the context of Wigner quantization. It is also to him that we owe the term "Wigner quantum system" or "Wigner quantization". In one of his first papers on the topic [25], however, he used the term "Dynamical quantization", referring to the fact that quantization follows from compatibility conditions related to the equations of motion.

Let us briefly summarize the main principles of Wigner quantization, as developed by Palev. Consider a quantum system with n degrees of freedom and a Hamiltonian of the form

$$\hat{H} = \sum_{j=1}^{n} \frac{\hat{p}_j^2}{2m_j} + \mathscr{V}(\hat{q}_1,\ldots,\hat{q}_n). \tag{12}$$

In Wigner quantization, one keeps all axioms of quantum mechanics, only the axiom on the CCRs is replaced. The canonical commutation relations

$$[\hat{q}_k,\hat{q}_l] = [\hat{p}_k,\hat{p}_l] = 0, \quad [\hat{q}_k,\hat{p}_l] = i\hbar\delta_{kl} \tag{13}$$

are *replaced* by a different set of operator relations between position and momentum operators. This set consists of the (operator) Compatibility Conditions (CC) between the Heisenberg equations and the operator form of Hamilton's equations.

So, in short, Wigner quantization for a system described by (12) consists of the following three steps:

1. Rewrite the Hamiltonian $\hat{H}$ appropriately in terms of operators $\hat{p}_k$ and $\hat{q}_k$ (in some symmetric form, not assuming any commutativity between the operators).

2. Determine the Compatibility Conditions (CC). This gives rise to a (non-linear) set of operator relations for the $\hat{p}_k$ and $\hat{q}_k$. The $\star$-algebra $\mathscr{A}$ is then defined as an algebra with generators $\hat{p}_k$ and $\hat{q}_k$ and defining relations (CC), subject to the $\star$-conditions $\hat{p}_k^\star = \hat{p}_k$ and $\hat{q}_k^\star = \hat{q}_k$.
3. Find $\star$-representations (unitary representations) of $\mathscr{A}$.

Very often, it is difficult to identify $\mathscr{A}$ as a known algebra, and hence it is too difficult to find all $\star$-representations. So instead of trying to work with $\mathscr{A}$, one looks for a *known* algebra $\mathscr{B}$ whose generators also satisfy (CC). Then it remains to construct the $\star$-representations of $\mathscr{B}$ and to determine physical properties (energy, spectrum of observables, ...) in these representations. This gives rise to a subset of solutions.

Note that this approach leads quite naturally to *non-commutative coordinate operators*, without any forced or external input as is sometimes done in other approaches of "non-commutative quantum mechanics".

In the first main paper on Wigner quantization [25], Palev investigated two particles interacting via a harmonic potential. After removal of the center of mass, the remaining Hamiltonian is essentially that of the 3-dimensional isotropic harmonic oscillator (HO). Palev investigated the CCs, and found that these were satisfied by certain generators of the Lie superalgebra $\mathfrak{gl}(1|3)$. In other words, he chose $\mathscr{B} = \mathfrak{gl}(1|3)$. Then, he went on to study properties in a particular class of $\star$-representations, namely the so-called Fock space representations. A remarkable feature here is the finite-dimensionality of these $\star$-representations, implying that all physical operators have a *discrete* spectrum. In the same year, Palev showed [26] that the CCs for the n-dimensional HO are satisfied by generators of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$; however, no representations were considered. Later, Kamupingene et al. [12] considered in more detail the 2-dimensional HO with $\mathscr{B} = \mathfrak{sl}(1|2)$. Interesting physical properties were obtained by Palev and Stoilova for the $\mathfrak{osp}(3|2)$ solutions of the 3-dimensional HO. Here, one could make use of a classification of the $\star$-representations of $\mathfrak{osp}(3|2)$ [35]. Palev and Stoilova [27, 28] later compared the solutions of the 3-dimensional isotropic Wigner HO provided by $\mathfrak{sl}(1|3)$, $\mathfrak{osp}(1|6)$ and $\mathfrak{osp}(3|2)$. The postulates of Wigner quantum systems were more carefully described in [29]. In this paper, the n-dimensional isotropic HO is revisited, and for the first time angular momentum operators are discussed (for $n = 3N$). In a review paper, Palev and Stoilova [30] describe the algebraic solutions for the n-particle 3-dimensional isotropic HO in terms of the Lie superalgebras $\mathfrak{sl}(1|3n)$, $\mathfrak{osp}(1|6n)$ and $\mathfrak{sl}(3|n)$. Further physical properties for the $\mathfrak{sl}(1|3)$ or $\mathfrak{sl}(1|3n)$ solutions, in particular related to the discrete spacial structure, were investigated in [14, 15]. Then, a few years ago, Stoilova and Van der Jeugt [34] made a quite general classification of Lie superalgebra solutions of the CCs for the n-dimensional isotropic HO.

Lievens et al. [17, 18] applied Wigner quantization to more complicated Hamiltonians, such as a linear chain of coupled particles. They show how this reduces to the Hamiltonian for an n-dimensional *non-isotropic* HO, and obtain new solutions in terms of $\mathfrak{gl}(1|n)$.

There appeared also a number of papers related to the fundamentals of Wigner quantization, or related algebraic quantizations. We mention here in particular the work of Man'ko et al. [21], Blasiak, Horzela, Kapuscik [4, 7, 13], and that of Atakishiyev, Wolf and collaborators [1–3] in the context of finite oscillator models.

More recently, Regniers and Van der Jeugt [32] investigated one-dimensional Hamiltonians with continuous energy spectra as Wigner quantum systems.

All these papers make it clear that Wigner quantization has given rise to challenging mathematical problems, and to interesting physical properties. Wigner quantization has also raised questions in Lie superalgebra representation theory, and stimulated further research into specific classes of Lie superalgebra representations.

In the following section we shall review the treatment of the n-dimensional non-isotropic harmonic oscillator in Wigner quantization as our main example. This has given rise to the study of a new class of representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$.

3 Main Example: The *n*-Dimensional Non-isotropic Harmonic Oscillator

For this example, we drop the previous convention with $m = \omega = \hbar = 1$, and consider the n-dimensional non-isotropic harmonic oscillator with Hamiltonian:

$$\hat{H} = \frac{1}{2m}\sum_{j=1}^{n}\hat{p}_j^2 + \frac{m}{2}\sum_{j=1}^{n}\omega_j^2\hat{q}_j^2, \tag{14}$$

where m stands for the mass of the oscillator and ω_j for the frequency in direction j. Let us construct the compatibility conditions CC. Clearly, the operator form of Hamilton's equations reads:

$$\dot{\hat{q}}_j = \text{op}\left(\frac{\partial H}{\partial p_j}\right) = \frac{1}{m}\hat{p}_j, \quad \dot{\hat{p}}_j = -\text{op}\left(\frac{\partial H}{\partial q_j}\right) = -m\omega_j^2\hat{q}_j, \quad j = 1,\ldots,n. \tag{15}$$

The Heisenberg equations are:

$$\dot{\hat{q}}_j = \frac{i}{\hbar}[\hat{H},\hat{q}_j], \qquad \dot{\hat{p}}_j = \frac{i}{\hbar}[\hat{H},\hat{p}_j], \qquad j = 1,\ldots,n. \tag{16}$$

So the compatibility conditions become:

$$[\hat{H},\hat{q}_j] = -i\frac{\hbar}{m}\hat{p}_j, \qquad [\hat{H},\hat{p}_j] = i\hbar m\omega_j^2\hat{q}_j, \qquad j = 1,\ldots,n, \tag{17}$$

where $\hat{H}$ is given by (14).

It is useful to write these compatibility conditions in a different form. For this purpose, introduce the following linear combinations of the operators $\hat{q}_j$ and $\hat{p}_j$:

$$a_j^{\mp} = \sqrt{\frac{m\omega_j}{2\hbar}}\hat{q}_j \pm \frac{i}{\sqrt{2m\hbar\omega_j}}\hat{p}_j, \quad j = 1,\ldots,n. \tag{18}$$

Now the expression of the Hamiltonian becomes

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j(a_j^+a_j^- + a_j^-a_j^+) = \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j\{a_j^+,a_j^-\}. \tag{19}$$

The new form of the compatibility conditions can be written as:

$$\left[\sum_{j=1}^{n}\omega_j\{a_j^+,a_j^-\},a_k^{\pm}\right] = \pm 2\omega_k a_k^{\pm}, \qquad k = 1,\ldots,n. \tag{20}$$

In terms of the notation of the previous section, $\mathscr{A}$ is the $\star$-algebra generated by $2n$ generators $a_j^{\pm}$ ($j = 1,\ldots,n$) with $\star$-relations $(a_j^{\pm})^{\star} = a_j^{\mp}$ and with defining relations (20).

Quite surprisingly, the structure of $\mathscr{A}$ and its unitary Hilbert space representations is known completely only for $n = 1$ (in which case it is just Wigner's example of Sect. 1). For $n > 1$, only some classes of unitary Hilbert space representations are known.

We shall now describe an algebraic solution for the conditions (20), in other words we shall determine an algebra $\mathscr{B}$ whose generators also satisfy (20) (but for which (20) are not the defining relations). This is provided by the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$. In fact, it were Ganchev and Palev [5] who discovered—in the context of parabosons—that $\mathfrak{osp}(1|2n)$ can be defined as an algebra with $2n$ generators $b_j^{\pm}$ subject to the following triple relations:

$$[\{b_j^{\xi},b_k^{\eta}\},b_l^{\varepsilon}] = (\varepsilon-\xi)\delta_{jl}b_k^{\eta} + (\varepsilon-\eta)\delta_{kl}b_j^{\xi}, \tag{21}$$

where $j,k,l \in \{1,\ldots,n\}$, and $\eta,\varepsilon,\xi \in \{+,-\}$ (to be interpreted as $+1$ or -1 in algebraic expressions such as $\varepsilon - \xi$). It is indeed very easy to verify that the operators

$$a_j^- = b_j^-, \quad a_j^+ = b_j^+ \tag{22}$$

satisfy the compatibility conditions (20). Otherwise said, the triple relations (21) imply the relations (20). Furthermore, the $\star$-relations for the generators of $\mathscr{A}$ imply the following $\star$-relations for the $\mathfrak{osp}(1|2n)$ generators:

$$(b_j^{\pm})^{\dagger} = b_j^{\mp}. \tag{23}$$

So we are led to investigating unitary representation of $\mathfrak{osp}(1|2n)$ for these $\star$-conditions.

In order to study the $\mathfrak{osp}(1|2n)$ solutions, it will be useful to identify some subalgebras of $\mathfrak{osp}(1|2n)$. First of all, note that due to the triple relations (21), a basis of $\mathfrak{osp}(1|2n)$ is given by the $2n$ *odd* elements $b_j^{\pm}$ and by the $2n^2+n$ *even* elements $\{b_j^{\xi}, b_k^{\eta}\}$ ($j,k \in \{1,\dots,n\}$; $\eta,\xi \in \{+,-\}$). The even subalgebra of $\mathfrak{osp}(1|2n)$ is the symplectic Lie algebra $\mathfrak{sp}(2n)$, so a basis of $\mathfrak{sp}(2n)$ consists of all even elements $\{b_j^{\xi}, b_k^{\eta}\}$ ($j,k \in \{1,\dots,n\}$; $\eta,\xi \in \{+,-\}$). A subalgebra of $\mathfrak{sp}(2n)$ is the general linear Lie algebra $\mathfrak{gl}(n)$, whose standard basis is given by the n^2 even elements $\frac{1}{2}\{b_j^+, b_k^-\}$ ($j,k \in \{1,\dots,n\}$). Finally, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(1|2n)$ is that of its even subalgebra $\mathfrak{sp}(2n)$. A basis of $\mathfrak{h}$ is given by the n elements $h_j = \frac{1}{2}\{b_j^-, b_j^+\}$ ($j = 1,\dots,n$). So we have, in this realization of $\mathfrak{osp}(1|2n)$, a natural chain of subalgebras:

$$\mathfrak{osp}(1|2n) \supset \mathfrak{sp}(2n) \supset \mathfrak{gl}(n) \supset \mathfrak{h}. \tag{24}$$

Note that for this algebraic solution the Hamiltonian is written as

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^{n} \omega_j \{a_j^-, a_j^+\} = \frac{\hbar}{2}\sum_{j=1}^{n} \omega_j \{b_j^-, b_j^+\} = \hbar \sum_{j=1}^{n} \omega_j h_j, \tag{25}$$

so it is an element of the Cartan subalgebra. This will facilitate the problem of determining the spectrum of $\hat{H}$.

It should be mentioned that a second algebraic solution of the conditions (20) can be given by means of generators of the Lie superalgebra $\mathfrak{gl}(1|n)$ [16]. This class of solutions also gives rise to many interesting properties, but these cannot be presented in this short review.

The algebraic $\mathfrak{osp}(1|2n)$ solution to (20) is easy to describe. In fact, it was already known since 1982 for the simpler isotropic case with $\omega_1 = \cdots = \omega_n = \omega$ [26]. The reason why it was not studied further was because no class of unitary representations was known (for the $\star$-condition (23)). This changed in 2008, when Lievens et al [19] managed to construct a class of unitary representations. These are the infinite-dimensional lowest weight representations $V(p)$ of $\mathfrak{osp}(1|2n)$ with lowest weight $(\frac{p}{2},\dots,\frac{p}{2})$. For these representations, the authors obtained an appropriate Gelfand-Zetlin basis, explicit actions of the generators on the basis vectors, and a character formula [19]. For these results, the subalgebra chain (24) plays an important role, in particular the decomposition with respect to the $\mathfrak{gl}(n)$ subalgebra. Irreducible characters of $\mathfrak{gl}(n)$ are given as a Schur function $s_\lambda(x_1,\dots,x_n)$, where λ is a partition of length $\ell(\lambda)$ at most n (see the standard book [20] for notations of partitions, Schur functions, etc.). In such character formulas, the exponents of $(x_1,\dots,x_n)$ carry the components of the corresponding weight of the representation according to the basis $(h_1,\dots,h_n)$ of the Cartan subalgebra $\mathfrak{h}$. In other words, a term $x_1^{\nu_1}\cdots x_n^{\nu_n}$ corresponds to the weight $(\nu_1,\dots,\nu_n)$.

The character determined in [19] can be described as follows:
The $\mathfrak{osp}(1|2n)$ representation $V(p)$ with lowest weight $(\frac{p}{2},\dots,\frac{p}{2})$ is a unitary irreducible representation if and only if $p \in \{1,2,\dots,n-1\}$ or $p > n-1$.

- For $p > n-1$, one has

$$\operatorname{char} V(p) = \frac{(x_1 \cdots x_n)^{p/2}}{\prod_i (1-x_i) \prod_{j<k} (1-x_j x_k)} = (x_1 \cdots x_n)^{p/2} \sum_\lambda s_\lambda(x_1, \ldots, x_n). \tag{26}$$

- For $p \in \{1, 2, \ldots, n-1\}$, the character of $V(p)$ is given by

$$\operatorname{char} V(p) = (x_1 \cdots x_n)^{p/2} \sum_{\lambda,\ \ell(\lambda) \leq p} s_\lambda(x_1, \ldots, x_n) \tag{27}$$

where $\ell(\lambda)$ is the *length* of the partition λ.

Such characters can be used to determine the spectrum of $\hat{H}$ in the $\mathfrak{osp}(1|2n)$ representation $V(p)$. Indeed, as noted earlier, the character is a weight generating function:

$$\operatorname{char} V(p) = \sum_{\nu_1, \ldots, \nu_n} d_{\nu_1, \ldots, \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}, \tag{28}$$

where $(\nu_1, \ldots, \nu_n)$ is a weight from the representation and $d_{\nu_1, \ldots, \nu_n}$ stands for the multiplicity of this weight. Recall that in the current solution

$$\hat{H} = \sum_{j=1}^{n} \hbar \omega_j h_j, \tag{29}$$

i.e. $\hat{H}$ is an element from the Cartan subalgebra $\mathfrak{h}$. Hence, to get a *spectrum generating function* one must make the substitution $x_j \to t^{\hbar\omega_j}$ in the character (27). Expressions like $s_\lambda(t^{\hbar\omega_1}, t^{\hbar\omega_2}, \ldots, t^{\hbar\omega_n})$ simplify a lot in the isotropic case $\omega_1 = \cdots = \omega_n = \omega$, when $x_j \to t^{\hbar\omega} \equiv z$, since

$$s_\lambda(z, \ldots, z) = z^{|\lambda|} s_\lambda(1, \ldots, 1). \tag{30}$$

Formulas for $s_\lambda(1, \ldots, 1)$ are well known [20]; after all, $s_\lambda(1, \ldots, 1)$ stands for the dimension of the $\mathfrak{gl}(n)$ representation characterized by the partition λ. So from (27) one obtains a "spectrum generating function" in the representation $V(p)$:

$$\operatorname{spec} \hat{H} = z^{np/2} \sum_{\lambda,\, \ell(\lambda) \leq p} s_\lambda(z, \ldots, z) = \sum_{k \geq 0} \sum_{\lambda,\, |\lambda|=k,\, \ell(\lambda) \leq p} s_\lambda(1, \ldots, 1) t^{\hbar\omega(np/2+k)}. \tag{31}$$

In this series expansion, the power of t gives the energy level E, and the coefficient in front of t^E gives the multiplicity $\mu(E)$ of the energy level E. Clearly, we have equidistant energy levels

$$E_k^{(p)} = \hbar\omega(np/2+k), \quad k = 0,1,2,3,\cdots \tag{32}$$

with spacing $\hbar\omega$ and with multiplicities (degeneracies)

$$\mu(E_k^{(p)}) = \sum_{\lambda,\, |\lambda|=k,\, \ell(\lambda)\le p} s_\lambda(1,\ldots,1). \tag{33}$$

In the representation $V(1)$ of $\mathfrak{osp}(1|2n)$, the CCRs are satisfied, so this is the representation corresponding to the canonical solution. One finds indeed that:

$$\mu(E_k^{(p=1)}) = \binom{n+k-1}{k}, \tag{34}$$

and (with $z = t^{\hbar\omega}$)

$$\operatorname{spec}\hat{H} = \frac{z^{n/2}}{(1-z)^n} = \sum_{k\ge 0}\binom{n+k-1}{k} t^{\hbar\omega(n/2+k)}, \tag{35}$$

which is a classical result.

For a more detailed analysis of spectrum generating functions for the other representations $V(p)$, we refer to [16]. Let us just give the results for the 3-dimensional oscillator, i.e. the case $n=3$. Then there are essentially three distinct cases to be considered for the $\mathfrak{osp}(1|6)$ representations $V(p)$, namely $p=1$, $p=2$ and $p>2$. The spectrum generating functions, the energy levels, and the energy multiplicities are given in the following table [16]:

	GF	Levels	Multiplicity
$p=1$	$\frac{z^{1/2}}{(1-z)^3}$	$\hbar\omega(\frac{3}{2}+k)$	$\mu(E_k^{(1)}) = \binom{k+2}{2}$
$p=2$	$\frac{z^3(1+z+z^2)}{(1-z^2)^3(1-z)^2}$	$\hbar\omega(3+k)$	$\mu(E_{2k}^{(2)}) = \binom{k+2}{2}^2$
			$\mu(E_{2k+1}^{(2)}) = \binom{k+2}{2}\binom{k+3}{2}$
$p>2$	$\frac{z^{3p/2}}{(1-z^2)^3(1-z)^3}$	$\hbar\omega(\frac{3p}{2}+k)$	$\mu(E_{2k}^{(p)}) = \frac{4k+5}{5}\binom{k+4}{4}$
			$\mu(E_{2k+1}^{(p)}) = \frac{4k+15}{5}\binom{k+4}{4}$

Expanding the above generating functions, or alternatively working out the above multiplicities, one finds for the first few energy levels the following results:

	$\mu(E_0^{(p)})$	$\mu(E_1^{(p)})$	$\mu(E_2^{(p)})$	$\mu(E_3^{(p)})$	$\mu(E_4^{(p)})$
$p=1$	1	3	6	10	15
$p=2$	1	3	9	18	36
$p>2$	1	3	9	19	39

So, just as for the one-dimensional Wigner oscillator, this $\mathfrak{osp}(1|6)$ approach to the 3-dimensional Wigner oscillator leads to a shift in energy compared to the canonical case ($p = 1$). Moreover, the degeneracies increase from the 3rd energy level onwards.

4 Recent Advances: Angular Momentum Operators and Their Spectrum

Now that the structure of the representations $V(p)$ of $\mathfrak{osp}(1|2n)$ is well known, we can also consider the angular momentum contents in the case that n is a multiple of 3 (i.e. if we work in 3-dimensional space). Let us first concentrate on the simple case that $n = 3$, i.e. a 3-dimensional harmonic oscillator.

In the canonical case, the components of the angular momentum operator $\mathbf{M}$ are determined by $\mathbf{M} = \mathbf{q} \times \mathbf{p}$, in other words, $M_j = \sum_{k,l} \varepsilon_{jkl} \hat{q}_k \hat{p}_l$ $(j = 1,2,3)$, where ε_{jkl} is the Levi-Civita symbol. Since the position and momentum operators cannot be assumed to commute in Wigner quantization, and since we want Wigner quantization to coincide with canonical quantization when the CCRs do hold, it is logical to define the angular momentum operators in Wigner quantization by

$$M_j = \frac{1}{2} \sum_{k,l} \varepsilon_{jkl} \{\hat{q}_k, \hat{p}_l\} = \frac{-i\hbar}{2} \sum_{k,l} \varepsilon_{jkl} \{a_k^+, a_l^-\} \qquad (j = 1,2,3). \tag{36}$$

The last expression follows from (18). Now one can investigate whether these operators satisfy any particular commutation relations. It turns out that using the CCs (20) do *not* lead to closed commutation relations between the operators M_1, M_2 and M_3. In other words, in the algebra $\mathscr{A}$, the commutation relations between the M_j do not close. Next, consider the $\mathfrak{osp}(1|6)$ solution with $a_j^\pm = b_j^\pm$ satisfying (21). Once again, the commutation relations between the M_j do not close, except when all ω_j are equal, i.e. except one works in the isotropic case. In that case, one finds:

$$[M_1, M_2] = i\hbar M_3 \quad (+ \text{ cyclic}), \tag{37}$$

just as in the canonical case. For this reason, we shall now continue with the isotropic case. The above relations imply that we have identified an $\mathfrak{so}(3)$ subalgebra in our chain of subalgebras:

$$\mathfrak{osp}(1|6) \supset \mathfrak{sp}(6) \supset \mathfrak{gl}(3) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1). \tag{38}$$

Herein $\mathfrak{so}(3)$ is generated by M_1, M_2 and M_3, and $\mathfrak{u}(1)$ by the Hamiltonian $\hat{H} = \hbar\omega(h_1 + h_2 + h_3)$. In other words, the $\mathfrak{so}(3) \oplus \mathfrak{u}(1)$ decomposition of a representation gives us the angular momentum and energy contents of the Wigner quantum system in that representation.

In the current situation, the question is: how does the representation $V(p)$ of $\mathfrak{osp}(1|6)$ decompose with respect to these subalgebras? As before, the answer follows from the expression of the character of $V(p)$,

$$\mathrm{char}\, V(p) = (x_1x_2x_3)^{p/2} \sum_{\lambda,\ell(\lambda)\leq\lceil p\rceil} s_\lambda(x_1,x_2,x_3) \tag{39}$$

where there are three distinct cases to be considered: $p=1$, $p=2$ or $p>2$. Since this character already gives in a straightforward way the decomposition of $V(p)$ with respect to $\mathfrak{gl}(3)$, it remains to determine the next step in the "branching", from $\mathfrak{gl}(3)$ to $\mathfrak{so}(3)\oplus\mathfrak{u}(1)$. This step has a well known solution and is known as the $U(3)$ to $SO(3)$ branching [6]. In our notation, where the $\mathfrak{gl}(3)$ representation is characterized by a partition $\lambda=(\lambda_1,\lambda_2,\lambda_3)$, this $\mathfrak{gl}(3)$ to $\mathfrak{so}(3)\oplus\mathfrak{u}(1)$ branching rule generating function reads

$$G = \frac{1+A_1^2A_2z^3J}{(1-A_1A_2A_3z^3)(1-A_1zJ)(1-A_1A_2z^2J)(1-A_1^2z^2)(1-A_1^2A_2^2z^4)}. \tag{40}$$

In the expansion of G as a power series, the coefficient of $A_1^{\lambda_1}A_2^{\lambda_2}A_3^{\lambda_3}$ is a polynomial $p_\lambda(J,z)=\sum\mu_{j,E}^{\lambda}J^jz^E$ in J and z. The coefficient $\mu_{j,E}^{\lambda}$ is the multiplicity of the $\mathfrak{so}(3)\oplus\mathfrak{u}(1)$ representation $(j,E)=(j)\oplus(E)$ in the decomposition of the $\mathfrak{gl}(3)$ representation (characterized by) λ.

Using (39) and (40), it now follows that we have generating functions for the angular momentum and energy contents for the representations $V(p)$ of $\mathfrak{osp}(1|6)$. For $V(1)$:

$$G_1 = \frac{z^{3/2}}{(1-zJ)(1-z^2)}.$$

For $V(2)$:

$$G_2 = \frac{z^3(1+z^3J)}{(1-zJ)(1-z^2J)(1-z^2)(1-z^4)}.$$

For $V(p)$, $p>2$:

$$G_p = \frac{z^{3p/2}(1+z^3J)}{(1-zJ)(1-z^2J)(1-z^2)(1-z^3)(1-z^4)}.$$

Clearly, one can use these generating functions to derive the $\mathfrak{so}(3)$ representations that emerge at energy level $E_k^{(p)}$. This information can be made accessible by means of a table in which the element in row $k+1$ and column $j+1$ (counted from the bottom) marks the number of representations (j) at energy level $E_k^{(p)}$ in the angular momentum decomposition of $\mathfrak{osp}(1|6)$. We call this the (j,E)-diagram of $\mathfrak{osp}(1|6)$ for $V(p)$. For G_1, the expansion gives

$$G_1 = z^{3/2} + Jz^{5/2} + (1+J^2)z^{7/2} + (J+J^3)z^{9/2} + (1+J^2+J^4)z^{11/2} + \cdots,$$

yielding the following (j,E)-diagram:

E_k \ j	0	1	2	3	4	$\cdots$
$\vdots$						$\iddots$
11/2	1		1		1	
9/2		1		1		
7/2	1		1			
5/2		1				
3/2	1					

Of course, this result is already known because $p = 1$ represents the canonical case. This (j,E)-diagram for instance appears in [37].

For $p = 2$, the expansion of G_2 gives rise to the following diagram:

E_k \ j	0	1	2	3	4	$\cdots$
$\vdots$						$\iddots$
7	2	1	3	1	1	
6		2	1	1		
5	1	1	1			
4		1				
3	1					

and for $p > 2$ the expansion of G_p gives:

E_k \ j	0	1	2	3	4	$\cdots$
$\vdots$						$\iddots$
$3p/2+4$	2	2	3	1	1	
$3p/2+3$	1	2	1	1		
$3p/2+2$	1	1	1			
$3p/2+1$		1				
$3p/2$	1					

Note that for the lower energy levels, the cases $p = 2$ and $p > 2$ do not differ very much from the canonical case. The larger discrepancies are found in higher energy regions.

So far, we have considered only the 3-dimensional Wigner harmonic oscillator and its angular momentum contents in the $\mathfrak{osp}(1|6)$ representations $V(p)$. In the more general case of $\mathfrak{osp}(1|2n)$ with $n = 3N$, the Hamiltonian $\hat{H}$ can be interpreted as an N-particle 3-dimensional oscillator. It is common to write the position operators and momentum operators by a multi-index: e.g. the position operators are

$\hat{q}_{j,\alpha}$, with $j = 1,2,3$ (referring to the 3 dimensions) and $\alpha = 1,2,\ldots,N$ (referring to the N particles). The angular momentum operators of particle α are given by

$$M_{j,\alpha} = \frac{1}{2}\sum_{k,l}\varepsilon_{jkl}\{\hat{q}_{k,\alpha},\hat{p}_{l,\alpha}\} = \frac{-i\hbar}{2}\sum_{k,l}\varepsilon_{jkl}\{a^+_{k,\alpha},a^-_{l,\alpha}\}$$

and then the components of the total angular momentum operator are

$$M_j = \sum_{\alpha=1}^{N} M_{j,\alpha} \quad (j = 1,2,3).$$

If we want these operators to satisfy the commutation relations (37), we need again to work in the $\mathfrak{osp}(1|6N)$ picture where $a^\pm_{k,\alpha} = b^\pm_{k,\alpha}$ and moreover in the case that all ω_j's are equal to ω (i.e. N identical isotropic oscillators). In $\mathfrak{osp}(1|6N) \supset \mathfrak{sp}(6N) \supset \mathfrak{gl}(3N)$, the $\mathfrak{gl}(3N)$ basis elements are $\{b^+_{j,\alpha},b^-_{k,\beta}\}$. The relevant subalgebras of $\mathfrak{gl}(3N)$ are $\mathfrak{gl}(3)$ and $\mathfrak{gl}(N)$, with basis elements:

$$\mathfrak{gl}(3): \quad E_{jk} = \frac{1}{2}\sum_{\alpha}\{b^+_{j,\alpha},b^-_{k,\alpha}\} \quad (j,k = 1,2,3),$$

$$\mathfrak{gl}(N): \quad \mathscr{E}_{\alpha,\beta} = \frac{1}{2}\sum_{j}\{b^+_{j,\alpha},b^-_{j,\beta}\} \quad (\alpha,\beta = 1,2,\ldots,N).$$

So the total angular momentum operators M_1, M_2, M_3 are the basis elements of an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{gl}(3)$, and one needs to decompose the representations $V(p)$ of $\mathfrak{osp}(1|6N)$ according to

$$\mathfrak{osp}(1|6N) \supset \mathfrak{sp}(6N) \supset \mathfrak{gl}(3N) \supset \mathfrak{gl}(3) \oplus \mathfrak{gl}(N) \supset \mathfrak{so}(3) \oplus \mathfrak{u}(1)$$

Although the decomposition of $\mathfrak{gl}(3N)$ representations according to $\mathfrak{gl}(3N) \supset \mathfrak{gl}(3) \oplus \mathfrak{gl}(N)$ is in principle known, it turns out to be computationally quite involved when $N \geq 2$. For details, and results with angular momentum and energy decompositions, see [33].

Acknowledgements This research was supported by project P6/02 of the Interuniversity Attraction Poles Programme (Belgian State—Belgian Science Policy).

References

1. Atakishiyev, N.M., Pogosyan, G.S., Vicent, L.E., Wolf, K.B.: Finite two-dimensional oscillator: I. The cartesian model, J. Phys. A **34**, 9381–9398 (2001)
2. Atakishiyev, N.M., Pogosyan, G.S., Vicent, L.E., Wolf, K.B.: Finite two-dimensional oscillator: I. The radial model, J. Phys. A **34**, 9399–9415 (2001)

3. Atakishiyev, N.M., Pogosyan, G.S., Wolf, K.B.: Finite models of the oscillator, Phys. Part. Nuclei **36**, 247–265 (2005)
4. Blasiak, P., Horzela, A., Kapuscik, E.: Alternative Hamiltonians and Wigner quantization, J. Optic. B **5**, S245–S260 (2003)
5. Ganchev, A.C., Palev, T.D.: A Lie superalgebraic interpretation of the para-Bose statistics, J. Math. Phys. **21**, 797–799 (1980)
6. Gaskell, R., Peccia, A., Sharp, R.T.: Generating functions for polynomial irreducible tensors, J. Math. Phys. **19**, 727–733 (1978)
7. Horzela, A., Kapuscik, E.: On time asymmetric Wigner quantization, Chaos, Solitons and Fractals **12**, 2801–2803 (2001)
8. Hughes, J.W.B.: Representations of osp(2,1) and the metaplectic representation, J. Math. Phys. **22**, 245–250 (1981)
9. Jafarov, E.I., Stoilova, N.I., Van der Jeugt, J.: Finite oscillator models: the Hahn oscillator, J. Phys. A: Math. Theor. **44**, 265203 (2011)
10. Kac, V.G.: Lie superalgebras, Adv. Math. **26**, 8–96 (1977)
11. Kac, V.G.: Representations of classical Lie superalgebras, Lect. Notes Math. **676**, 597–626 (1978)
12. Kamupingene, A.H., Palev, T.D., Tsavena, S.P.: Wigner quantum systems. Two particles interacting via a harmonic potential. I. Two-dimensional space, J. Math. Phys. **27**, 2067–2075 (1986)
13. Kapuscik, E.: Galilean covariant Lie algebra of quantum mechanical observables, Czech J. Phys. **50**, 1279–1282 (2000)
14. King, R.C., Palev, T.D., Stoilova, N.I., Van der Jeugt, J.: The non-commutative and discrete spatial structure of a 3D Wigner quantum oscillator, J. Phys. A: Math. Gen. **36**, 4337–4362 (2003)
15. King, R.C., Palev, T.D., Stoilova, N.I., Van der Jeugt, J.: A non-commutative n-particle 3D Wigner quantum oscillator, J. Phys. A: Math. Gen. **36**, 11999–12019 (2003)
16. Lievens, S., Van der Jeugt, J.: Spectrum generating functions for non-canonical quantum oscillators, J. Phys. A: Math. Theor. **41**, 355204 (2008)
17. Lievens, S., Stoilova, N.I., Van der Jeugt, J.: Harmonic oscillators coupled by springs: Discrete solutions as a Wigner quantum system, J. Math. Phys. **47**, 113504 (2006)
18. Lievens, S., Stoilova, N.I., Van der Jeugt, J.: Harmonic oscillator chains as Wigner Quantum Systems: periodic and fixed wall boundary conditions in gl(1|n) solutions, J. Math. Phys. **49**, 073502 (2008)
19. Lievens, S., Stoilova, N.I., Van der Jeugt, J.: The paraboson Fock space and unitary irreducible representations of the Lie superalgebra osp(1/2n), Comm. Math. Phys. **281**, 805–826 (2008)
20. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, Oxford (1995)
21. Man'ko, V.I., Marmo, G., Zaccaria, F., Sudarshan, E.C.G.: Wigner's problem and alternative commutation relations for quantum mechanics, Int. J. Mod. Phys. **B11**, 1281–1296 (1997)
22. Mukunda, N., Sudarshan, E.C.G., Sharma, J.K., Mehta, C.L.: Representations and properties of para-Bose oscillator operators. 1. Energy, position and momentum eigenstates, J. Math. Phys. **21**, 2386–2394 (1980)
23. Ohnuki, Y., Kamefuchi, S.: Quantum Field Theory and Parastatistics. Springer, New-York (1982)
24. Palev, T.D.: Lie-superalgebraical approach to the second quantization, Czech J. Phys. **29**, 91–98 (1979)
25. Palev, T.D.: On a dynamical quantization, Czech J. Phys. **32**, 680–687 (1982)
26. Palev, T.D.: Wigner approach to quantization. Noncanonical quantization of two particles interacting via a harmonic potential, J. Math. Phys. **23**, 1778–1784 (1982)
27. Palev, T.D., Stoilova, N.I.: Wigner quantum oscillators, J. Phys. A: Math. Gen. **27**, 977–983 (1994)
28. Palev, T.D., Stoilova, N.I.: Wigner quantum oscillators – osp(3/2) oscillators, J. Phys. A: Math. Gen. **27**, 7387–7401 (1994)

29. Palev, T.D., Stoilova, N.I.: Many-body Wigner quantum systems, J. Math. Phys. **38**, 2506–2523 (1997)
30. Palev, T.D., Stoilova, N.I.: Wigner quantum systems: Lie superalgebraic approach, Rep. Math. Phys. **49**, 395–404 (2002)
31. Regniers, G., Van der Jeugt, J.: The Hamiltonian H = xp and Classification of osp(1/2) Representations, AIP Conf. Proc. **1243**, 138–147 (2010)
32. Regniers, G., Van der Jeugt, J.: Wigner quantization of some one-dimensional Hamiltonians, J. Math. Phys. **51**, 123515 (2010)
33. Regniers, G., Van der Jeugt, J.: Angular momentum decomposition of the three-dimensional Wigner harmonic oscillator, J. Math. Phys. **52**, 113503 (2011)
34. Stoilova, N.I., Van der Jeugt, J.: Solutions of the compatibility conditions for a Wigner quantum oscillator, J. Phys. A: Math. Gen. **38**, 9681–9687 (2005)
35. Van der Jeugt, J.: Finite- and infinite-dimensional representations of the orthosymplectic superalgebra osp(3,2), J. Math. Phys. **25**, 3334–3349 (1984)
36. Wigner, E.P.: Do the equations of motion determine the quantum mechanical commutation relations? Phys. Rev. **77**, 711–712 (1950)
37. Wybourne, B.G.: Classical Groups for Physicists. Wiley, New York (1978)

Part II
Quantum Field Theory

Spontaneous Breaking of Supersymmetry, Localization and Nicolai Mapping in Matrix Models

Fumihiko Sugino

Abstract We consider supersymmetric matrix models of the type of the Wess-Zumino model, whose supersymmetry (SUSY) may be spontaneously broken. When SUSY is broken, the partition function vanishes since it is equivalent to the Witten index. We need some regularization to give a small value to the partition function in computing expectation values of observables in a well-defined way. Here, we employ twisted boundary condition to fermionic variables with a small angle α, and use this as the above regularization. Interestingly, the twist can be interpreted as an external field to detect spontaneous SUSY breaking, which is analogous to the magnetic field in Ising model whose Z_2 symmetry is spontaneously broken. Also, we discuss the SUSY breaking from the viewpoints of localization and Nicolai mapping, and find interesting localization phenomena specific to matrix models.

1 Introduction and Discussion

Spontaneous breaking of supersymmetry (SUSY) is one of the most interesting phenomena in quantum field theory. Since in general SUSY cannot be broken by radiative corrections at the perturbative level, its spontaneous breaking requires understanding of nonperturbative aspects of quantum field theory [12]. In particular, recent developments in nonperturbative aspects of string theory heavily rely on the presence of SUSY. Thus, in order to deduce predictions to the real world from string theory, it is important to investigate a mechanism of spontaneous SUSY breaking in a nonperturbative framework of strings. Since one of the most promising approaches

F. Sugino (✉)
Okayama Institute for Quantum Physics, Kyoyama 1-9-1, Okayama 700-0015, Japan
e-mail: fumihiko_sugino@pref.okayama.lg.jp

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_11,

of nonperturbative formulations of string theory is provided by large-N matrix models [1, 3, 5], it will be desirable to understand how SUSY can be spontaneously broken in the large-N limit of simple matrix models as a first step.

In the next section, we discuss a simple SUSY quantum mechanics of the type of the Wess-Zumino model, which include cases that SUSY is spontaneously broken. Analogously to the situation of ordinary spontaneous symmetry breaking, we introduce an external field to choose one of degenerate broken vacua to detect spontaneous SUSY breaking. The external field plays the same role as a magnetic field in the Ising model introduced to detect the spontaneous magnetization. For the supersymmetric system, we deform the boundary condition for fermions from the periodic boundary condition (PBC) to a twisted boundary condition (TBC) with twist α, which can be regarded as such an external field. If a supersymmetric system undergoes spontaneous SUSY breaking, the partition function with the PBC for all the fields, Z_{PBC}, which usually corresponds to the Witten index, is expected to vanish [13]. Then, the expectation values of observables, which are normalized by Z_{PBC}, would be ill-defined or indefinite. By introducing the twist, the partition function is regularized and the expectation values become well-defined.

In Sects. 3 and 4, we analyze a SUSY matrix model, which is a matrix-model analog of the model in the previous section, by the two methods of localization and Nicolai mapping. As for localization, we make change of integration variables in the path integral, which is always possible whether or not the SUSY is explicitly broken (the external field is on or off). In terms of eigenvalues of matrix variables, an interesting phenomenon for localization arises. Localization attracts the eigenvalues to the critical points of superpotential, while the square of the Vandermonde determinant arising from the measure factor prevents the eigenvalues from collapsing. The dynamics of the eigenvalues is governed by balance of attractive force from the localization and repulsive force from the Vandermonde determinant. In the case that the external field is turned on, computation based on the localization is still possible, but we find that a method by the Nicolai mapping is more effective. Interestingly, the Nicolai mapping works for SUSY matrix models even in the presence of the external field which explicitly breaks SUSY. It enables us to calculate the partition function at least in the leading nontrivial order of an expansion with respect to the small external field for finite N.

In Sect. 5, we can take the large-N limit of our result before turning off the external field and detect whether SUSY is spontaneously broken or not in the large-N limit. For illustration, we obtain large-N solutions for a SUSY matrix model with double-well potential. It is found that there is a phase transition of the third order between a SUSY phase and a SUSY broken phase.

For future directions, this kind of argument can be expected to be useful to investigate localization in various lattice models for supersymmetric field theories which realize some SUSYs on the lattice. Also, it will be interesting to investigate localization in models constructed in [8], which couple a supersymmetric quantum field theory to a certain large-N matrix model and cause spontaneous SUSY breaking at large N.

This article is mainly based on the collaboration with Tsunehide Kuroki [9, 10].

2 SUSY Quantum Mechanics: SUSY Breaking and External Field

Let us start with a system defined by the Euclidean (Wick-rotated) action:

$$S^{\rm QM} = \int_0^\beta dt \left[\frac{1}{2}B^2 + iB\left(\dot\phi + W'(\phi)\right) + \bar\psi\left(\dot\psi + W''(\phi)\psi\right)\right], \tag{1}$$

where ϕ is a real scalar field, $\psi, \bar\psi$ are fermions, and B is an auxiliary field. The dot means the derivative with respect to the Euclidean time $t \in [0, \beta]$. For a while, all the fields are supposed to obey the PBC. $W(\phi)$ is a real function of ϕ called superpotential, and the prime ($'$) represents the ϕ-derivative.

$S^{\rm QM}$ is invariant under one-dimensional $\mathcal{N} = 2$ SUSY transformations generated by Q and $\bar Q$. They act on the fields as

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar\psi = -iB, \quad QB = 0, \tag{2}$$

$$\bar Q\phi = -\bar\psi, \quad \bar Q\bar\psi = 0, \quad \bar Q\psi = -iB + 2\dot\phi, \quad \bar QB = 2i\dot{\bar\psi}, \tag{3}$$

with satisfying the algebra $Q^2 = \bar Q^2 = 0, \{Q, \bar Q\} = 2\partial_t$.

Next, let us consider quantum aspects of SUSY breaking in this model. We take $\langle B\rangle$ (or $\langle B^n\rangle$ $(n = 1, 2, \cdots)$) as an order parameter for SUSY breaking. Suppose the SUSY of the model is spontaneously broken, so the ground state energy E_0 is positive. Then, for each of the energy levels E_n $(0 < E_0 < E_1 < E_2 < \cdots)$, the SUSY algebra $\{Q, \bar Q\} = 2E_n$, $Q^2 = \bar Q^2 = 0$ leads to the SUSY multiplet formed by bosonic and fermionic states $|b_n\rangle = \frac{1}{\sqrt{2E_n}}\bar Q|f_n\rangle$, $|f_n\rangle = \frac{1}{\sqrt{2E_n}}Q|b_n\rangle$.

As a convention, we assume that $|b_n\rangle$ and $|f_n\rangle$ have the fermion number charges $F = 0$ and 1, respectively. Since the Q-transformation for B in (2) is expressed as $[Q, B] = 0$ in the operator formalism, we can see that $\langle b_n|B|b_n\rangle = \langle f_n|B|f_n\rangle$ holds for each n. Then, it turns out that the unnormalized expectation values of B^m $(m = 1, 2, \cdots)$ vanish: $\langle B^m\rangle' \equiv \int_{\rm PBC} d(\text{fields})\, B^m\, e^{-S^{\rm QM}} = \text{tr}\left[B^m(-1)^F e^{-\beta H}\right] = 0$. This observation shows that, in order to judge SUSY breaking from the expectation value of B, we should choose either of the SUSY broken ground states ($|b_0\rangle$ or $|f_0\rangle$) and see the expectation value with respect to the chosen ground state. The situation is somewhat analogous to the case of spontaneous breaking of ordinary (bosonic) symmetry.

However, differently from the ordinary case, when SUSY is broken, the supersymmetric partition function vanishes:

$$Z^{\rm QM}_{\rm PBC} = \int_{\rm PBC} d(\text{fields})\, e^{-S^{\rm QM}} = \text{tr}\left[(-1)^F e^{-\beta H}\right] = 0. \tag{4}$$

So, the expectation values normalized by $Z^{\rm QM}_{\rm PBC}$ could be ill-defined [6, 7].

2.1 Twisted Boundary Condition

To detect spontaneous breaking of ordinary symmetry, some external field is introduced so that the ground state degeneracy is resolved to specify a single broken ground state. The external field is turned off after taking the thermodynamic limit, then we can judge whether spontaneous symmetry breaking takes place or not, seeing the value of the corresponding order parameter.

We will do a similar thing also for the case of spontaneous SUSY breaking. For this purpose, let us change the boundary condition for the fermions to the TBC: $\psi(t+\beta)=e^{i\alpha}\psi(t)$, $\bar{\psi}(t+\beta)=e^{-i\alpha}\bar{\psi}(t)$, then the twist α can be regarded as an external field. Other fields remain intact. It turns out that the partition function with the TBC corresponds to the expression (4) with $(-1)^F$ replaced by $(-e^{-i\alpha})^F$:

$$Z_\alpha^{\rm QM} \equiv -e^{-i\alpha}\int_{\rm TBC} d(\text{fields})\, e^{-S^{\rm QM}} = {\rm tr}\left[(-e^{-i\alpha})^F e^{-\beta H}\right]$$
$$= \sum_{n=0}^{\infty}\left(\langle b_n|b_n\rangle - e^{-i\alpha}\langle f_n|f_n\rangle\right)e^{-\beta E_n} = \left(1-e^{-i\alpha}\right)\sum_{n=0}^{\infty} e^{-\beta E_n}. \qquad (5)$$

Then, the normalized expectation value of B under the TBC becomes

$$\langle B\rangle_\alpha \equiv \frac{1}{Z_\alpha^{\rm QM}}\,{\rm tr}\left[B(-e^{-i\alpha})^F e^{-\beta H}\right]$$
$$= \frac{1}{Z_\alpha^{\rm QM}}\sum_{n=0}^{\infty}\left(\langle b_n|B|b_n\rangle - e^{-i\alpha}\langle f_n|B|f_n\rangle\right)e^{-\beta E_n}$$
$$= \frac{\sum_{n=0}^{\infty}\langle b_n|B|b_n\rangle e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = \frac{\sum_{n=0}^{\infty}\langle f_n|B|f_n\rangle e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}}. \qquad (6)$$

Note that the factors $\left(1-e^{-i\alpha}\right)$ in the numerator and the denominator cancel each other, and thus $\langle B\rangle_\alpha$ does not depend on α even for finite β. As a result, $\langle B\rangle_\alpha$ is equivalent to the expectation value taken over one of the ground states and its excitations $\{|b_n\rangle\}$ (or $\{|f_n\rangle\}$). The normalized expectation value of B under the PBC was of the indefinite form $0/0$, which is now regularized by introducing the parameter α. The expression (6) is well-defined.

3 Localization in SUSY Matrix Models

In the following, we consider a matrix-model analog in the previous section, whose action is

$$S^{\mathrm{M}}_{\alpha} = N\,\mathrm{tr}\left[\frac{1}{2}B^2 + iBW'(\phi) + \bar{\psi}\left(e^{i\alpha}-1\right)\psi + \bar{\psi}QW'(\phi)\right], \tag{7}$$

where all variables are $N \times N$ Hermitian matrices, and the partition function is defined by

$$Z^{\mathrm{M}}_{\alpha} \equiv \int d^{N^2}B\, d^{N^2}\phi\, \left(d^{N^2}\psi\, d^{N^2}\bar{\psi}\right)\, e^{-S^{\mathrm{M}}_{\alpha}} \tag{8}$$

with the measure normalized as

$$\int d^{N^2}\phi\, e^{-N\mathrm{tr}\,(\frac{1}{2}\phi^2)} = \int d^{N^2}B\, e^{-N\mathrm{tr}\,(\frac{1}{2}B^2)} = 1, \quad \int \left(d^{N^2}\psi\, d^{N^2}\bar{\psi}\right)\, e^{-N\mathrm{tr}\,(\bar{\psi}\psi)} = 1. \tag{9}$$

When $\alpha = 0$, $S^{\mathrm{M}}_{\alpha=0}$ is invariant under Q and $\bar{Q}$ transformations:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \tag{10}$$

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \tag{11}$$

both of which become broken explicitly in S^{M}_{α} by introducing the external field α.

We make a change of variables

$$\phi = \tilde{\phi} + \bar{\varepsilon}\psi, \quad \bar{\psi} = \tilde{\bar{\psi}} - i\bar{\varepsilon}B, \tag{12}$$

where in the second equation, $\tilde{\bar{\psi}}$ satisfies

$$N\,\mathrm{tr}(B\tilde{\bar{\psi}}) = 0, \tag{13}$$

namely, $\tilde{\bar{\psi}}$ is orthogonal to B with respect to the inner product $(A_1, A_2) \equiv N\,\mathrm{tr}(A_1^\dagger A_2)$.

If we write (8) as

$$Z^{\mathrm{M}}_{\alpha} = \int d^{N^2}B\, \Xi_{\alpha}(B), \qquad \Xi_{\alpha}(B) \equiv \int d^{N^2}\phi\, \left(d^{N^2}\psi\, d^{N^2}\bar{\psi}\right)\, e^{-S^{\mathrm{M}}_{\alpha}}, \tag{14}$$

and consider the change of the variables in $\Xi_{\alpha}(B)$, B may be regarded as an external variable. The measure $d^{N^2}\bar{\psi}$ can be expressed by the measures associated with $\tilde{\bar{\psi}}$ and $\bar{\varepsilon}$ as

$$d^{N^2}\bar{\psi} = \frac{i}{\mathscr{N}_B}\, d\bar{\varepsilon}\, d^{N^2-1}\tilde{\bar{\psi}}, \tag{15}$$

where $\mathscr{N}_B \equiv ||B|| = \sqrt{N\,\mathrm{tr}(B^2)}$ is the norm of the matrix B, and $d^{N^2-1}\tilde{\bar{\psi}}$ is explicitly given by introducing the constraint (13) as a delta-function:

$$d^{N^2-1}\tilde{\bar{\psi}} \equiv (-1)^{N^2-1} d^{N^2}\tilde{\bar{\psi}}\, \delta\left(\frac{1}{\mathscr{N}_B}N\,\mathrm{tr}(B\tilde{\bar{\psi}})\right). \tag{16}$$

Notice that the measure on the RHS of (15) depends on B. When $B \neq 0$, we can safely change the variables as in (12) and in terms of them the action becomes

$$S_\alpha^{\rm M} = N\,{\rm tr}\left[\frac{1}{2}B^2 + iBW'(\tilde{\phi}) + \tilde{\bar{\psi}}\left((e^{i\alpha}-1)\psi + QW'(\tilde{\phi})\right) - (e^{i\alpha}-1)i\bar{\varepsilon}B\psi\right] \quad (17)$$

with $Q\tilde{\phi} = \psi$.

3.1 $\alpha = 0$ Case

Let us first consider the case of the PBC ($\alpha = 0$). $S_0^{\rm M}$ does not depend on $\bar{\varepsilon}$ as a consequence of its SUSY invariance, because (12) reads $\phi = \tilde{\phi} + \varepsilon Q\tilde{\phi}$, $\bar{\psi} = \tilde{\bar{\psi}} + \bar{\varepsilon}Q\tilde{\bar{\psi}}$. Therefore, the contribution to the partition function from $B \neq 0$ vanishes due to the integration over $\bar{\varepsilon}$ according to (15). Namely, when $\alpha = 0$, the path integral of the partition function (8) is localized to $B = 0$. Notice that the measure (15) is singular at $B = 0$.

However, for the unnormalized expectation values of $\prod_{i=1}^k \frac{1}{N}\,{\rm tr}\,B^{n_i}$ $(n_1, \cdots, n_k \geq 1, k \geq 1)$:

$$\left\langle \prod_{i=1}^k \frac{1}{N}\,{\rm tr}\,B^{n_i}\right\rangle' \equiv \int d^{N^2}B\left(\prod_{i=1}^k \frac{1}{N}\,{\rm tr}\,B^{n_i}\right)\Xi_0(B), \quad (18)$$

the singular behavior of the measure is cancelled by the integrand. Thus the argument of the change of variables is always applicable giving the result

$$\left\langle \prod_{i=1}^k \frac{1}{N}\,{\rm tr}\,B^{n_i}\right\rangle' = 0. \quad (19)$$

3.1.1 Localization to $W'(\phi) = 0$, and Localization Versus Vandermonde

Since (19) means

$$\left\langle e^{-N\,{\rm tr}\left(\frac{u-1}{2}B^2\right)}\right\rangle' = \sum_{n=0}^\infty \frac{1}{n!}\left(-N^2\frac{u-1}{2}\right)^n \left\langle\left(\frac{1}{N}\,{\rm tr}\,B^2\right)^n\right\rangle' = \langle 1\rangle' = Z_0^{\rm M} \quad (20)$$

for an arbitrary parameter u, we may compute $\left\langle e^{-N\,{\rm tr}\left(\frac{u-1}{2}B^2\right)}\right\rangle'$ to evaluate the partition function $Z_0^{\rm M}$. Taking $u > 0$ and integrating B first, we obtain

$$Z_0^{\rm M} = \int d^{N^2}\phi\left(\frac{1}{u}\right)^{\frac{N^2}{2}} e^{-N\,{\rm tr}\left[\frac{1}{2u}W'(\phi)^2\right]}\int\left(d^{N^2}\psi\, d^{N^2}\bar{\psi}\right) e^{-N\,{\rm tr}\left[\bar{\psi}QW'(\phi)\right]}. \quad (21)$$

Then, in the $u \to 0$ limit, localization to $W'(\phi) = 0$ takes place.

In terms of eigenvalues of ϕ, the partition function (21) becomes

$$Z_0^{\mathrm{M}} = \tilde{C}_N \int \left(\prod_{i=1}^{N} d\lambda_i\right) \left(\prod_{i=1}^{N} W''(\lambda_i)\right) \left\{\prod_{i>j} \frac{1}{u} \left(W'(\lambda_i) - W'(\lambda_j)\right)^2\right\}$$
$$\times \left(\frac{1}{u}\right)^{\frac{N}{2}} e^{-N\sum_{i=1}^{N} \frac{1}{2u} W'(\lambda_i)^2} \tag{22}$$

with $\tilde{C}_N^{-1} = \int \left(\prod_{i=1}^{N} d\lambda_i\right) \triangle(\lambda)^2 e^{-N\sum_{i=1}^{N} \frac{1}{2}\lambda_i^2}$. $\triangle(\lambda) = \prod_{i>j}(\lambda_i - \lambda_j)$ is the Vandermonde determinant. In this expression, the factor in the second line forces eigenvalues to be localized at the critical points of the superpotential as $u \to 0$, while the last factor in the first line, which is proportional to the square of the Vandermonde determinant of $W'(\lambda_i)$, gives repulsive force among eigenvalues which prevents them from collapsing to the critical points. The dynamics of eigenvalues is thus determined by balance of the attractive force to the critical points originating from the localization and the repulsive force from the Vandermonde determinant.

To proceed with the analysis, let us consider the situation of each eigenvalue λ_i fluctuating around the critical point $\phi_{c,i}$: $\lambda_i = \phi_{c,i} + \sqrt{u}\,\tilde{\lambda}_i$ $(i = 1, \cdots, N)$, where $\tilde{\lambda}_i$ is a fluctuation, and $\phi_{c,1}, \cdots, \phi_{c,N}$ are allowed to coincide with each other. Then, the partition function (22) is computed to be

$$Z_0^{\mathrm{M}} = \sum_{\phi_{c,i}} \prod_{i=1}^{N} \mathrm{sgn}\left(W''(\phi_{c,i})\right) = \left[\sum_{\phi_c: W'(\phi_c)=0} \mathrm{sgn}\left(W''(\phi_c)\right)\right]^N \tag{23}$$

in the limit $u \to 0$. The RHS of (23) tells that the total partition function is given by the N-th power of the degree of the map $\phi \to W'(\phi)$.

Furthermore, we consider a case that the superpotential $W(\phi)$ has K nondegenerate critical points $a_1, \cdots, a_K$. Namely, $W'(a_I) = 0$ and $W''(a_I) \neq 0$ for each $I = 1, \cdots, K$. The scalar potential $\frac{1}{2}W'(\phi)^2$ has K minima at $\phi = a_1, \cdots, a_K$. When N eigenvalues are fluctuating around the minima, we focus on the situation that

the first $\nu_1 N$ eigenvalues λ_i $(i = 1, \cdots, \nu_1 N)$ are around $\phi = a_1$,
the next $\nu_2 N$ eigenvalues $\lambda_{\nu_1 N + i}$ $(i = 1, \cdots, \nu_2 N)$ are around $\phi = a_2$,
$\cdots$,

and the last $\nu_K N$ eigenvalues $\lambda_{\nu_1 N + \cdots + \nu_{K-1} N + i}$ $(i = 1, \cdots, \nu_K N)$ are around $\phi = a_K$, where $\nu_1, \cdots, \nu_K$ are filling fractions satisfying $\sum_{I=1}^{K} \nu_I = 1$. Let $Z_{(\nu_1, \cdots, \nu_K)}$ be a contribution to the total partition function $Z_{\alpha=0}^{\mathrm{M}}$ from the above configuration. Then,

$$Z_0^{\mathrm{M}} = \sum_{\nu_1 N, \cdots, \nu_K N = 0}^{N} \frac{N!}{(\nu_1 N)! \cdots (\nu_K N)!} Z_{(\nu_1, \cdots, \nu_K)}, \tag{24}$$

(The sum is taken under the constraint $\sum_{I=1}^{K} \nu_I = 1$.) and $Z_{(\nu_1,\cdots,\nu_K)} = \prod_{I=1}^{K} Z_{G,\nu_I}$, $Z_{G,\nu_I} = (\text{sgn}(W''(a_I)))^{\nu_I N}$. Z_{G,ν_I} can be interpreted as the partition function of the Gaussian SUSY matrix model with the matrix size $\nu_I N \times \nu_I N$ describing contributions from Gaussian fluctuations around $\phi = a_I$.

3.2 $\alpha \neq 0$ *Case*

In the presence of the external field α, let us consider $\Xi_\alpha(B)$ in (14) with the action (17) obtained after the change of variables (12). Using the explicit form of the measure (15) and (16), we obtain

$$\Xi_\alpha(B) = (e^{i\alpha}-1)\frac{-1}{\mathcal{N}_B^2}\int d^{N^2}\tilde{\phi}\left(d^{N^2}\psi\, d^{N^2}\tilde{\bar{\psi}}\right) e^{-N\,\text{tr}\left[\frac{1}{2}B^2+iBW'(\tilde{\phi})+\tilde{\bar{\psi}}QW'(\tilde{\phi})\right]}$$
$$\times N\,\text{tr}(B\tilde{\bar{\psi}})\,N\,\text{tr}(B\psi)\,e^{-(e^{i\alpha}-1)N\,\text{tr}(\tilde{\bar{\psi}}\psi)}, \tag{25}$$

which is valid for $B \neq 0$. It does not vanish in general by the effect of the twist $e^{i\alpha}-1$. This suggests that the localization is incomplete by the twist. Although we can proceed the computation further, it is more convenient to invoke another method based on the Nicolai mapping we will present in the next section.

4 $(e^{i\alpha}-1)$-Expansion and Nicolai Mapping

In this section, we instead compute Z_α^{M} in an expansion with respect to $(e^{i\alpha}-1)$. For the purpose of examining the spontaneous SUSY breaking, we are interested in behavior of Z_α^{M} in the $\alpha \to 0$ limit. Thus it is expected that it will be often sufficient to compute Z_α^{M} in the leading order of the $(e^{i\alpha}-1)$-expansion for our purpose.

Let us expand this with respect to $(e^{i\alpha}-1)$ as

$$Z_\alpha^{\text{M}} = \sum_{k=0}^{N^2}(e^{i\alpha}-1)^k Z_{\alpha,k}, \tag{26}$$

and derive a formula in the leading order of this expansion. In terms of the eigenvalues, the Nicolai mapping [11] for each i can be applied even in the presence of the external field:

$$\Lambda_i = (e^{i\alpha}-1)\lambda_i + W'(\lambda_i), \tag{27}$$

giving

$$Z_\alpha^{\text{M}} = \tilde{C}_N\int\left(\prod_{i=1}^{N} d\Lambda_i\right)\prod_{i>j}(\Lambda_i-\Lambda_j)^2 e^{-N\sum_i\frac{1}{2}\Lambda_i^2}e^{-N\sum_i\left(-A\Lambda_i\lambda_i+\frac{1}{2}A^2\lambda_i^2\right)}, \tag{28}$$

where $A = e^{i\alpha} - 1$. Note that λ_i has several branches as a function of Λ_i and it has a different expression according to each of the branches. In the last factor of (28) contains $\lambda_i(\Lambda_i)$, we have to take account of the branches and divide the integration region of Λ_i accordingly. Nevertheless, we can derive a rather simple formula at least in the leading order of the expansion in terms of A owing to the Nicolai mapping (27). In the following, let us concentrate on the cases where $\Lambda_i \to \infty$ as $\lambda_i \to \pm\infty$, or $\Lambda_i \to -\infty$ as $\lambda_i \to \pm\infty$. In such cases, we can expect spontaneous SUSY breaking, in which the leading nontrivial expansion coefficient is relevant since $Z^{\mathrm{M}}_{\alpha=0} = Z_{\alpha,0} = 0$. By considering the expansion of the last factor in (28), we see that in the expansion (26), $Z_{\alpha,k} = 0$ for $k = 0, \cdots, N-1$ and that the first possibly nonvanishing contribution starts from $\mathscr{O}(A^N)$. Although the integration over Λ_i should be divided into the branches, after changing the integration variables by $\Lambda_i = W'(x_i)$, we end up with

$$Z_{\alpha,N} = \tilde{C}_N N^N \int_{-\infty}^{\infty} \left(\prod_{i=1}^{N} dx_i\right) \prod_{i=1}^{N} \left(W''(x_i)W'(x_i)x_i\right) \prod_{i>j} (W'(x_i) - W'(x_j))^2 \times e^{-N\sum_{i=1}^{N} \frac{1}{2} W'(x_i)^2}, \tag{29}$$

which does not vanish in general. The nontrivial $\mathscr{O}(A^N)$ contribution can be regarded as a specific feature of SUSY matrix models.

5 Large-N Solutions of a Double-Well SUSY Matrix Model

As an application of (29), let us discuss SUSY breaking/restoration in the large-N limit of our SUSY matrix models. From (29), introducing the eigenvalue density $\rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i)$ rewrites the leading $\mathscr{O}(A^N)$ part of Z^{M}_{α}. In the large-N limit, $\rho(x)$ is given as a solution to the saddle point equation:

$$\int dy\, \rho(y) \mathrm{P}\frac{1}{x-y} + \int dy\, \rho(y) \mathrm{P}\frac{1}{x+y} = x^3 - \mu^2 x \tag{30}$$

for the case $W'(x) = x^2 - \mu^2$. Let us first consider the case $\mu^2 > 0$, where the shape of the scalar potential is a double-well $\frac{1}{2}\left(x^2 - \mu^2\right)^2$.

5.1 General Two-Cut Solutions

Let us consider configurations that $\nu_+ N$ eigenvalues are located around one minimum $x = +\sqrt{\mu^2}$ of the double-well, and the remaining $\nu_- N (= N - \nu_+ N)$ eigenvalues are around the other minimum $x = -\sqrt{\mu^2}$.

We can apply the method in [2, 4] to cases $(\nu_+,\nu_-) = (1,0),(0,1),(1/2,1/2)$. Then, it is found that the eigenvalue distribution $\rho_0(x)$ for general (ν_+,ν_-) having the support $\Omega = [-b,-a]\cup[a,b]$ given by

$$\rho_0(x) = \begin{cases} \frac{\nu_+}{\pi}\, x\sqrt{(x^2-a^2)(b^2-x^2)} & (a<x<b) \\ \frac{\nu_-}{\pi}\, |x|\sqrt{(x^2-a^2)(b^2-x^2)} & (-b<x<-a) \end{cases} \tag{31}$$

with $a^2 = -2+\mu^2$, $b^2 = 2+\mu^2$ satisfies (30). The large-N free energy and all the expectation values of $\frac{1}{N}\operatorname{tr} B^n$ $(n = 1,2,\cdots)$ are proved to vanish for the solution (31). Thus, we can conclude that the SUSY matrix model with the double-well potential restores SUSY at large N, and has an infinitely many degenerate supersymmetric saddle points parametrized by (ν_+,ν_-) at large N for the case $\mu^2 > 2$.

It is somewhat surprising that the end points of the cut a, b and the large-N free energy are the same for all (ν_+,ν_-), which is recognized as a new interesting feature of the supersymmetric models and can be never seen in the case of bosonic double-well matrix models.

5.2 Symmetric One-Cut Solution

Here we obtain a one-cut solution with a symmetric support $[-c,c]$. The same method as before leads to the solution

$$\rho_0(x) = \frac{1}{2\pi}\left(x^2-\mu^2+\frac{c^2}{2}\right)\sqrt{c^2-x^2}, \quad x\in[-c,c] \tag{32}$$

with $c = \left[\frac{2}{3}\left(\mu^2+\sqrt{\mu^4+12}\right)\right]^{1/2}$. The condition $\rho_0(x)\geq 0$ implies that this solution is valid for $\mu^2 \leq 2$, which is indeed the complement of the region of μ^2 where the general two-cut solutions exist. (32) is valid also for $\mu^2 < 0$. Given $\rho_0(x)$, the large-N free energy is positive for $\mu^2 < 2$, and the expectation value of $\frac{1}{N}\operatorname{tr} B$ is nonzero, which are strong evidence suggesting the spontaneous SUSY breaking.

The μ^2-derivatives of the free energy show that the transition between the SUSY phase ($\mu^2 \geq 2$) and the SUSY broken phase ($\mu^2 < 2$) is of the third order.

Acknowledgements The author would like to thank Tsunehide Kuroki for collaboration which gives a basis for this article. The author is grateful to the organizers of the conference for giving me an opportunity of my talk in the wonderful environment of Varna. The work is supported in part by Grant-in-Aid for Scientific Research (C), 21540290.

References

1. Banks, T., Fischler, W., Shenker, S.H., Susskind, L.: Phys. Rev. D **55**, 5112–5128 (1996)
2. Brezin, E., Itzykson, C., Parisi, G., Zuber, J.B.: Comm. Math. Phys. **59**, 35–51 (1978)
3. Dijkgraaf, R., Verlinde, E.P., Verlinde, H.L.: Nucl. Phys. B **500**, 43–61 (1997)
4. Eynard, B., Kristjansen, C.: Nucl. Phys. B **455**, 577–618 (1995)
5. Ishibashi, N., Kawai, H., Kitazawa, Y., Tsuchiya, A.: Nucl. Phys. B **498**, 467–491 (1996)
6. Kanamori, I., Suzuki, H., Sugino, F.: Phys. Rev. D **77**, 091502 (2008)
7. Kanamori, I., Suzuki, H., Sugino, F.: Prog. Theor. Phys. **119**, 797–827 (2008)
8. Kuroki, T., Sugino, F.: Nucl. Phys. B **796**, 471–499 (2008)
9. Kuroki, T., Sugino, F.: Nucl. Phys. B **830**, 434–473 (2010)
10. Kuroki, T., Sugino, F.: Nucl. Phys. B **844**, 409–449 (2011)
11. Nicolai, H.: Phys. Lett. B **89**, 341–346 (1979)
12. Witten, E.: Nucl. Phys. B **188**, 513–554 (1981)
13. Witten, E.: Nucl. Phys. B **202**, 253–316 (1982)

Mirror Map as Generating Function of Intersection Numbers

Masao Jinzenji

Abstract In this article, we review our recent results on geometrical reconstruction of the B-model data used in the mirror computation of projective hypersurfaces, which was presented in Jinzenji (Lett. Math. Phys. 86(2–3):99–114, 2008; Mirror map as generating function of intersection numbers: Toric manifolds with two Kähler forms. Preprint, arXiv:1006.0607).

1 Introduction

Gromov-Witten invariants are mathematical objects that correspond to correlation functions of topological (non-linear) sigma model. In [1, 2], we showed that the Gromov-Witten invariants of the degree k hypersurface in CP^{N-1} (we denote this hypersurface by M_N^k) are computed by using mirror symmetry . Especially, we can compute the following Gromov-Witten invariant:

$$\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d} = \int_{\overline{M}_{0,2}(CP^{N-1},d)} ev_1^*(h^a)\wedge ev_2^*(h^b)\wedge c_{top}(R^0\pi_* ev_3^*\mathcal{O}_{CP^{N-1}}(k)). \quad (1)$$

In (1), $\overline{M}_{0,n}(CP^{N-1},d)$ is the moduli space of stable maps of degree d from genus 0 semi-stable curves with n marked points to CP^{N-1}. $ev_i : \overline{M}_{0,n}(CP^{N-1},d)\to CP^{N-1}$ is the evaluation map at the i-th marked point. $\pi : \overline{M}_{0,3}(CP^{N-1},d)\to \overline{M}_{0,2}(CP^{N-1},d)$ is the forgetful map that forgets the third marked point. In [1, 2], we start from the Picard-Fuchs equation:

$$\left((\partial_x)^{N-1} - k\cdot e^x\cdot (k\partial_x + k-1)(k\partial_x+k-2)\cdots(k\partial_x+1)\right)w(x) = 0. \quad (2)$$

M. Jinzenji (✉)
Division of Mathematics, Graduate School of Science, Hokkaido University, Kita-ku, Sapporo, 060-0810, Japan
e-mail: jin@math.sci.hokudai.ac.jp

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_12,

Then we construct the virtual Gauss-Manin system associated with (2) and obtain the virtual structure constants $\tilde{L}_n^{N,k,d}$ as the matrix elements of the connection matrix of the virtual Gauss-Manin system. From the virtual structure constants given by $\tilde{L}_{1+(k-N)d}^{N,k,d}$, we can construct the mirror map. Finally, we can obtain the Gromov-Witten invariant (1) by operating the generalized mirror transformation, which is induced by the mirror map, on the virtual structure constants $\tilde{L}_n^{N,k,d}$.

Let us illustrate the process of the mirror computation more explicitly when $N=k$, i.e., M_N^k is a Calabi-Yau manifold. In this case the virtual structure constants $\tilde{L}_n^{k,k,d}$ are obtained from the following equality:

$$\left((\frac{d}{dx})^{k-1} - k\cdot e^x \cdot \left(k\frac{d}{dx}+k-1\right)\cdots\left(k\frac{d}{dx}+2\right)\cdot\left(k\frac{d}{dx}+1\right)\right)w(x)$$
$$= \frac{1}{\tilde{L}_{k-1}^{k,k}(e^x)}\left(\frac{d}{dx}\frac{1}{\tilde{L}_{k-2}^{k,k}(e^x)}\left(\frac{d}{dx}\cdots\frac{1}{\tilde{L}_1^{k,k}(e^x)}\left(\frac{d}{dx}\frac{w(x)}{\tilde{L}_0^{k,k}(e^x)}\right)\cdots\right)\right), \quad (3)$$

where $\tilde{L}_n^{k,k}(e^x)$ is the generating function of $\tilde{L}_n^{k,k,d}$:

$$\tilde{L}_n^{k,k}(e^x) := 1 + \sum_{d=1}^{\infty} \tilde{L}_n^{k,k,d} e^{dx}. \quad (4)$$

In (3), $w(x)$ is an arbitrary function with adequate differentiable property. We can construct $\tilde{L}_n^{k,k}(e^x)$ that satisfies (3) from the solutions of the differential equation:

$$\left(\left(\frac{d}{dx}\right)^{k-1} - k\cdot e^x\cdot\left(k\frac{d}{dx}+k-1\right)\cdots\left(k\frac{d}{dx}+2\right)\cdot\left(k\frac{d}{dx}+1\right)\right)w(x)=0. \quad (5)$$

Linearly independent solutions of (5) around $x=-\infty$ are explicitly given by $w_j(x)$ $(j=0,\,1,\,2,\,\cdots,\,k-2)$:

$$w(x,y) := \sum_{d=0}^{\infty}\frac{\prod_{j=1}^{kd}(j+ky)}{\prod_{j=1}^{d}(j+y)^k}e^{(d+y)x}, \;\; w_j(x) := \frac{1}{j!}\frac{\partial^j w}{\partial y^j}(x,0). \quad (6)$$

Then $\tilde{L}_n^{k,k}(e^x)$ is inductively determined by the following relation[1]:

$$\tilde{L}_0^{k,k}(x) = w_0(x),$$
$$\tilde{L}_j^{k,k}(e^x) = \frac{d}{dx}\left(\frac{1}{\tilde{L}_{j-1}^{k,k}(e^x)}\frac{d}{dx}\left(\frac{1}{\tilde{L}_{j-2}^{k,k}(e^x)}\frac{d}{dx}\left(\frac{1}{\tilde{L}_{j-3}^{k,k}(e^x)}\cdots\frac{d}{dx}\left(\frac{1}{\tilde{L}_1^{k,k}(e^x)}\frac{d}{dx}\frac{w_j(x)}{\tilde{L}_0^{k,k}(e^x)}\right)\cdots\right)\right)\right). \quad (7)$$

[1] In (7), we need to use formally $w_{k-1}(x)$ to determine $\tilde{L}_{k-1}^{k,k}(e^x)$ though it is not a solution of (5).

The most important relation derived from the above equation is the following equality:

$$x + \sum_{d=1}^{\infty} \frac{\tilde{L}_1^{k,k,d}}{d} e^{dx} = \frac{w_1(x)}{w_0(x)}, \tag{8}$$

where the r.h.s. gives us the celebrated mirror map: $t(x) = \frac{w_1(x)}{w_0(x)}$ used in the mirror computation. With this mirror map, We can compute $\frac{\langle \mathcal{O}_{h^{k-2-n}} \mathcal{O}_{h^{n-1}} \rangle_{0,d}}{k}$ from the equality:

$$t + \sum_{d=1}^{\infty} \frac{\langle \mathcal{O}_{h^{k-2-n}} \mathcal{O}_{h^{n-1}} \rangle_{0,d}}{k} e^{dt} = x(t) + \sum_{d=1}^{\infty} \frac{\tilde{L}_n^{k,k,d}}{d} e^{dx(t)}. \tag{9}$$

This is the mirror transformation induced by the mirror map in (8).

In this article, we review geometrical characterization of the virtual structure constant $\tilde{L}_n^{N,k,d}$ as the intersection number of the moduli space of polynomial maps with two marked points. This moduli space is obtained from compactifying the moduli space of holomorphic maps from CP^1 to CP^{N-1} by using rational maps. Note that this compactification is different from the stable map compactification that is used to define $\overline{M}_{0,2}(CP^{N-1}, d)$ in (1). We denote by $\widetilde{Mp}_{0,2}(N,d)$ the compactified moduli space of polynomial maps from CP^1 to CP^{N-1} of degree d with two marked points, which was introduced in [3] and was explicitly defined in [4]. We then introduce an intersection number:

$$w(\mathcal{O}_{h^\alpha} \mathcal{O}_{h^\beta})_{0,d} := \int_{\widetilde{Mp}_{0,2}(N,d)} ev_1^*(h^\alpha) \wedge ev_2^*(h^\beta) \wedge c_{top}(\mathcal{E}_d^k), \tag{10}$$

where $\mathcal{E}_d^k$ is a rank $kd+1$ orbi-bundle on $\widetilde{Mp}_{0,2}(N,d)$ that corresponds to $R^0\pi_* ev_3^* \mathcal{O}_{CP^{N-1}}(k)$ on $\overline{M}_{0,2}(CP^{N-1}, d)$. In Sect. 2, we roughly illustrate construction process of $\widetilde{Mp}_{0,2}(N,d)$ and the procedure to compute $w(\mathcal{O}_{h^\alpha} \mathcal{O}_{h^\beta})_{0,d}$ by localization techniques. Our main result presented in [4] is,

Theorem 1. *The equality:*

$$k \cdot \frac{\tilde{L}_n^{N,k,d}}{d} = w(\mathcal{O}_{h^{N-2-n}} \mathcal{O}_{h^{n-1+(N-k)d}})_{0,d}. \tag{11}$$

holds true for arbitrary M_N^k *if* $0 \le N-2-n \le N-2$ *and if* $0 \le n-1+(N-k)d \le N-2$.

2 CP^{N-1} Case

Let $\mathbf{a}_j$, $(j = 0, 1, \cdots, d)$ be vectors in $\mathbf{C}^N$ and let $\pi_N : \mathbf{C}^N \setminus \{\mathbf{0}\} \to CP^{N-1}$ be the projection map. We define a degree d polynomial map p from $\mathbf{C}^2$ to $\mathbf{C}^N$ as a map that consists of $\mathbf{C}^N$vector-valued degree d homogeneous polynomials in two coordinates s, t of $\mathbf{C}^2$:

$$\begin{aligned} &p : \mathbf{C}^2 \to \mathbf{C}^N \\ &p(s,t) = \mathbf{a}_0 s^d + \mathbf{a}_1 s^{d-1} t + \mathbf{a}_2 s^{d-2} t^2 + \cdots + \mathbf{a}_d t^d. \end{aligned} \tag{12}$$

The parameter space of polynomial maps is given by $\mathbf{C}^{N(d+1)} = \{(\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_d)\}$. We denote by $Mp_{0,2}(N,d)$ the space defined as follows:

$$Mp_{0,2}(N,d) = \{(\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_d) \in \mathbf{C}^{N(d+1)} \mid \mathbf{a}_0, \mathbf{a}_d \neq \mathbf{0}\}/(\mathbf{C}^\times)^2, \tag{13}$$

where the two $\mathbf{C}^\times$ actions are given by,

$$\begin{aligned} &(\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_d) \to (\mu\mathbf{a}_0, \mu\mathbf{a}_1, \cdots, \mu\mathbf{a}_{d-1}, \mu\mathbf{a}_d), \\ &(\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_d) \to (\mathbf{a}_0, \nu\mathbf{a}_1, \cdots, \nu^{d-1}\mathbf{a}_{d-1}, \nu^d\mathbf{a}_d). \end{aligned} \tag{14}$$

This space can be regarded as the parameter space of degree d rational maps from CP^1 to CP^{N-1} with two marked points in CP^1: $0(= (1:0))$ and $\infty(= (0:1))$. The condition $\mathbf{a}_0, \mathbf{a}_d \neq \mathbf{0}$ assures that the images of 0 and ∞ are well-defined in CP^{N-1}. The two $\mathbf{C}^\times$ actions are induced from the following two $\mathbf{C}^\times$ actions on $\mathbf{C}^2$.

$$(s,t) \to (\mu s, \mu t), \ \ (s,t) \to (s, \nu t). \tag{15}$$

At this stage, we have to note the difference between the moduli space of holomorphic maps from CP^1 to CP^{N-1} and the moduli space of polynomial maps from CP^1 to CP^{N-1}. In short, the latter includes the points that are not the actual maps from CP^1 to CP^{N-1} but the rational maps from CP^1 to CP^{N-1}. These points are called freckled instantons by physicists. More explicitly, a freckled instanton is a polynomial map $\sum_{j=0}^{d} \mathbf{a}_j s^j t^{d-j}$ which can be factorized as

$$\sum_{j=0}^{d} \mathbf{a}_j s^j t^{d-j} = p_{d-d_1}(s,t) \cdot (\sum_{j=0}^{d_1} \mathbf{b}_j s^j t^{d_1-j}), \tag{16}$$

where $p_{d-d_1}(s,t)$ is a homogeneous polynomial of degree $d - d_1 (> 0)$. If we consider $\sum_{j=0}^{d} \mathbf{a}_j s^j t^{d-j}$ as a map from CP^1 to CP^{N-1}, it should be regarded as a rational map whose images of the zero points of p_{d-d_1} is undefined. Moreover, the closure of the image of this map is a rational curve of degree $d_1 (< d)$ in CP^{N-1}. The reason why we include point instantons is that we can obtain simpler compactification of the moduli space than the moduli space of the stable maps $\overline{M}_{0,2}(CP^{N-1},d)$, the standard moduli space used to define the two-point Gromov-Witten invariants.

Now, let us turn into the problem of compactification of $Mp_{0,2}(N,d)$. If $d=1$, $Mp_{0,2}(N,1)$ is identified with $CP^{N-1}\times CP^{N-1}$ and is already compact. If $d\geq 2$, we have to use the two $\mathbf{C}^{\times}$ actions in (14) to turn $\mathbf{a}_0$ and $\mathbf{a}_d$ into the points in CP^{N-1}, $[\mathbf{a}_0]$ and $[\mathbf{a}_d]$. Therefore, we can easily see,

$$Mp_{0,2}(N,d)=\{([\mathbf{a}_0],\mathbf{a}_1,\cdots,\mathbf{a}_{d-1},[\mathbf{a}_d])\in CP^{N-1}\times \mathbf{C}^{N(d-1)}\times CP^{N-1}\,|\}/\mathbf{Z}_d.$$

In this way, we can see that $Mp_{0,2}(N,d)$ is not compact if $d\geq 2$. In order to compactify $Mp_{0,2}(N,d)$, we imitate the stable map compactification and add the following chains of polynomial maps

$$\cup_{j=1}^{l}\Big(\sum_{m_j=0}^{d_j-d_{j-1}}\mathbf{a}_{d_{j-1}+m_j}(s_j)^{m_j}(t_j)^{d_j-d_{j-1}-m_j}\Big),\ \ \big(\mathbf{a}_{d_j}\neq \mathbf{0},\ \ j=0,1,\cdots,l\big),\tag{17}$$

to the infinity locus of $Mp_{0,2}(N,d)$. In (17), d_j's are integers that satisfy,

$$0=d_0<d_1<d_2<\cdots<d_{l-1}<d_l=d.\tag{18}$$

We denote by $\widetilde{Mp}_{0,2}(N,d)$ the space obtained after this compactification. It is explicitly constructed as a toric orbifold by introducing boundary divisor coordinates $u_1,u_2,\cdots u_{d-1}$ as follows.

$$\widetilde{Mp}_{0,2}(N,d)=$$

$$\{(\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d,u_1,u_2,\cdots,u_{d-1})|\,\mathbf{a}_0,(\mathbf{a}_1,u_1),\cdots,(\mathbf{a}_{d-1},u_{d-1}),\mathbf{a}_d\neq\mathbf{0}\}/(\mathbf{C}^{\times})^{d+1},$$

where the $(d+1)$ $\mathbf{C}^{\times}$actions are given by the following $(d+1)\times 2d$ weight matrix W_d:

$$W_d:=\begin{array}{c} \\ h_0\\ h_1\\ h_2\\ \vdots\\ \vdots\\ \vdots\\ h_d\end{array}\begin{array}{c}\begin{array}{cccccccccccc}\mathbf{a}_0&\mathbf{a}_1&\mathbf{a}_2&\cdots&\cdots&\mathbf{a}_{d-1}&\mathbf{a}_d&u_1&u_2&u_3&\cdots&u_{d-1}\end{array}\\ \left(\begin{array}{cccccccccccc}1&0&0&\cdots&\cdots&0&0&-1&0&0&\cdots&0\\ 0&1&0&\cdots&\cdots&0&0&2&-1&0&\cdots&0\\ 0&0&1&\ddots&\cdots&\vdots&0&-1&2&-1&\ddots&0\\ \vdots&\vdots&\ddots&\ddots&\ddots&\vdots&\vdots&\vdots&\ddots&\ddots&\ddots&\vdots\\ \vdots&\vdots&\ddots&\ddots&\ddots&0&0&0&0&\ddots&\ddots&-1\\ \vdots&\vdots&\vdots&\ddots&\ddots&1&0&0&0&\ddots&-1&2\\ 0&0&0&\cdots&\cdots&0&1&0&0&\cdots&0&-1\end{array}\right)\end{array}$$

The chain of polynomial maps presented in (17) is contained in the locus:

$$u_{d_j}=0\ \ (j=1,2,\cdots,l-1),\ \ \ u_i\neq 0\ \ (i\neq d_j).\tag{19}$$

With this set-up, we define the following intersection number on $\widetilde{Mp}_{0,2}(N,d)$, which is an analogue of a two point Gromov-Witten invariant of the degree k hypersurface in CP^{N-1}:

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d} := \int_{\widetilde{Mp}_{0,2}(N,d)} ev_1^*(h^a)\wedge ev_2^*(h^b)\wedge c_{top}(\mathcal{E}_d^k). \tag{20}$$

In (20), h is the hyperplane class of CP^{N-1}, and $ev_1 : \widetilde{Mp}_{0,2}(N,d)\to CP^{N-1}$ (resp. $ev_2 : \widetilde{Mp}_{0,2}(N,d)\to CP^{N-1}$) is the evaluation map at the first (resp. second) marked point. These maps are easily constructed as follows:

$$\begin{aligned} ev_1([(\mathbf{a}_0,\cdots,\mathbf{a}_d,u_1,\cdots,u_{d-1})]) &:= [\mathbf{a}_0]\in CP^{N-1}, \\ ev_2([(\mathbf{a}_0,\cdots,\mathbf{a}_d,u_1,\cdots,u_{d-1})]) &:= [\mathbf{a}_d]\in CP^{N-1}. \end{aligned} \tag{21}$$

$\mathcal{E}_d^k$ is the orbi-bundle that guarantees the image of the (chain of) polynomial maps lie inside the degree k hypersurface.

We can compute the intersection number $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ by using the localization theorem. First, we introduce the following $\mathbf{C}^{\times}$ action on $\widetilde{Mp}_{0,2}(N,d)$.

$$[(e^{\lambda_0 t}\mathbf{a}_0, e^{\lambda_1 t}\mathbf{a}_1,\cdots,e^{\lambda_{d-1}t}\mathbf{a}_{d-1},e^{\lambda_d t}\mathbf{a}_d,u_1,u_2,\cdots,u_{d-1})]. \tag{22}$$

The fixed point sets of $\widetilde{Mp}_{0,2}(N,d)$ consist of connected components $F_{(d_0,d_1,\cdots,d_l)}$'s labeled by the sequence of integers given in (18). Explicitly, a point in $F_{(d_0,d_1,\cdots,d_l)}$ is represented by the following chain of polynomial maps.

$$\bigcup_{j=1}^{l}(\mathbf{a}_{d_{j-1}}(s_j)^{d_j-d_{j-1}}+\mathbf{a}_{d_j}(t_j)^{d_j-d_{j-1}}). \tag{23}$$

Note here that $(\mathbf{a}_{d_{j-1}}(s_j)^{d_j-d_{j-1}}+\mathbf{a}_{d_j}(t_j)^{d_j-d_{j-1}})$ is the $\mathbf{Z}_{d_j-d_{j-1}}$ singularity in $Mp_{0,2}(N,d_j-d_{j-1})$. We can easily see from (23) that $F_{(d_0,d_1,\cdots,d_l)}$ is set-theoretically isomorphic to $\prod_{j=0}^{l}(CP^{N-1})_{d_j}$ where $(CP^{N-1})_{d_j}$ is the CP^{N-1} whose point is given by $[\mathbf{a}_{d_j}]$. After applying the standard procedure of the localization computation, we obtain the following closed formula for $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$:

$$\begin{aligned} & w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d} = \\ & \sum_{0=d_0<d_1<\cdots<d_{l-1}<d_l=d}\frac{1}{\prod_{j=1}^{l}(d_j-d_{j-1})}\frac{1}{(2\pi\sqrt{-1})^{l+1}}\oint_{C_{(0)}}\frac{dz_{d_0}}{(z_{d_0})^N}\cdots\oint_{C_{(0)}}\frac{dz_{d_l}}{(z_{d_l})^N}\times \\ & (z_{d_0}+\lambda_{d_0})^a\frac{\prod_{j=1}^{l}\prod_{m=0}^{k(d_j-d_{j-1})}\left(\frac{(k(d_j-d_{j-1})-m)(z_{d_{j-1}}+\lambda_{d_{j-1}})+m(z_{d_j}+\lambda_{d_j})}{d_j-d_{j-1}}\right)}{\prod_{j=1}^{l}\prod_{i=1}^{d_j-d_{j-1}-1}\left(\frac{(d_j-d_{j-1}-i)(z_{d_{j-1}}+\lambda_{d_{j-1}})+i(z_{d_j}+\lambda_{d_j})}{d_j-d_{j-1}}-\lambda_{d_{j-1}+i}\right)^N}\times \end{aligned}$$

$$\frac{1}{\prod_{j=1}^{l-1}\left(\frac{z_{d_j}+\lambda_{d_j}-z_{d_{j-1}}-\lambda_{d_{j-1}}}{d_j-d_{j-1}}+\frac{z_{d_j}+\lambda_{d_j}-z_{d_{j+1}}-\lambda_{d_{j+1}}}{d_{j+1}-d_j}\right)\left(k(z_{d_j}+\lambda_{d_j})\right)}(z_{d_l}+\lambda_{d_l})^b, \quad (24)$$

where $\frac{1}{(2\pi\sqrt{-1})}\oint_{C_{(0)}} dz$ means the operation of taking a residue at $z=0$. Here, we used the equality: $\frac{1}{(2\pi\sqrt{-1})}\oint_{C_{(0)}}\frac{dz}{z^N}z^j = \int_{CP^{N-1}} h^j$. In the above formula, we can integrate the variable z_{d_j} in arbitrary order. The formula (24) has the form of residue integral and we can take non-equivariant limit $\lambda_j \to 0$. For simplicity, we introduce the following notations. We define the following two polynomials in z and w:

$$e(k,d;z,w) := \prod_{j=0}^{kd}\left(\frac{jz+(kd-j)w}{d}\right), \quad t(N,d;z,w) := \prod_{j=1}^{d-1}\left(\frac{jz+(d-j)w}{d}\right)^N. \quad (25)$$

We also introduce the ordered partition of a positive integer d:

Definition 1. Let OP_d be the set of ordered partitions of a positive integer d:

$$OP_d = \{\sigma_d = (d_1, d_2, \cdots, d_{l(\sigma_d)}) \mid \sum_{j=1}^{l(\sigma_d)} d_j = d, \ d_j \in \mathbf{N}\}. \quad (26)$$

In (26), we denoted the length of the ordered partition σ_d by $l(\sigma_d)$.

The increasing sequence of integers $(d_0, d_1, \cdots, d_l)$ $(0 = d_0 < d_1 < \cdots < d_{l-1} < d_l{=}d)$ used in (24) can be replaced by the ordered partition $\sigma_d = (\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_l) \in OP_d$ if we use the following correspondence:

$$\tilde{d}_j = d_j - d_{j-1}, \ (j = 1, 2, \cdots, l). \quad (27)$$

With this setup, we can simplify the formula for $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ after taking the non-equivariant limit, by relabeling the subscript of z_*'s as follows.

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d} = \sum_{\sigma_d \in OP_d} \frac{1}{(2\pi\sqrt{-1})^{l(\sigma_d)+1}\prod_{j=0}^{l(\sigma_d)} d_j} \oint_{C_0}\frac{dz_0}{(z_0)^N}\cdots\oint_{C_0}\frac{dz_{l(\sigma_d)}}{(z_{l(\sigma_d)})^N}(z_0)^a$$
$$\times \prod_{j=1}^{l(\sigma_d)-1}\frac{1}{\left(\frac{z_j-z_{j-1}}{d_j}+\frac{z_j-z_{j+1}}{d_{j+1}}\right)kz_j}\prod_{j=1}^{l(\sigma_d)}\frac{e(k,d_j;z_{j-1},z_j)}{t(N,d_j;z_{j-1},z_j)}(z_{l(\sigma_d)})^b. \quad (28)$$

After taking non-equivariant limit, we have to take care of the order of integration of z_j's. In (28), we have to integrate z_j's in all the summands of the formula in descending (or ascending) order of the subscript j. We can further simplify (28) into the following form:

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d} = \frac{1}{(2\pi\sqrt{-1})^{d+1}}\oint_{C_0}\frac{dz_0}{(z_0)^N}\oint_{E_1}\frac{dz_1}{(z_1)^N}\cdots\oint_{E_{d-1}}\frac{dz_{d-1}}{(z_{d-1})^N}\oint_{C_0}\frac{dz_d}{(z_d)^N}$$
$$\times\frac{(z_0)^{N-2-n}(z_d)^{n-1+(N-k)d}}{\prod_{i=1}^{d-1}((2z_i-z_{i-1}-z_{i+1})}\frac{\prod_{j=1}^{d}e(k,1;z_{j-1},z_j)}{\prod_{i=1}^{d-1}(kz_i)}, \tag{29}$$

where $\frac{1}{2\pi\sqrt{-1}}\oint_{E_j}dz_j,\ (i=1,\cdots,d-1)$ represents the operation of taking residues at $z_j=0,\frac{z_{j-1}+z_{j+1}}{2}$.

In [4], we also extended the construction so far to some toric manifolds with two dimensional Kähler cones. In these examples, we also reconstructed B-model data of mirror computation as the intersection number of the (compactified) moduli space of polynomial maps.

We end this article by presenting the numerical data for the quintic threefold ($N=k=5$). In this case, we have the following data of 2-point Gromov-Witten invariants.

$$\langle\mathcal{O}_{h^0}\mathcal{O}_{h^2}\rangle_{0,1}=0,\ \langle\mathcal{O}_{h^0}\mathcal{O}_{h^2}\rangle_{0,2}=0,\ \langle\mathcal{O}_{h^0}\mathcal{O}_{h^2}\rangle_{0,3}=0,\cdots,$$
$$\langle\mathcal{O}_{h^1}\mathcal{O}_{h^1}\rangle_{0,1}=2875,\ \langle\mathcal{O}_{h^1}\mathcal{O}_{h^1}\rangle_{0,2}=\frac{4876875}{2},\ \langle\mathcal{O}_{h^1}\mathcal{O}_{h^1}\rangle_{0,3}=\frac{8564575000}{3},\cdots.$$

The fact that $\langle\mathcal{O}_{h^0}\mathcal{O}_{h^2}\rangle_{0,d}=0$ follows from the puncture axiom of Gromov-Witten invariants. On the other hand, the corresponding $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$'s are given as follows.

$$\begin{aligned}w(\mathcal{O}_{h^0}\mathcal{O}_{h^2})_{0,1} &= 3850,\ w(\mathcal{O}_{h^0}\mathcal{O}_{h^2})_{0,2}=3589125,\ w(\mathcal{O}_{h^0}\mathcal{O}_{h^2})_{0,3}\\ &= \frac{16126540000}{3},\cdots,\\ w(\mathcal{O}_{h^1}\mathcal{O}_{h^1})_{0,1} &= 6725,\ w(\mathcal{O}_{h^1}\mathcal{O}_{h^1})_{0,2}=\frac{16482625}{2},\ w(\mathcal{O}_{h^1}\mathcal{O}_{h^1})_{0,3}\\ &= \frac{44704818125}{3},\cdots.\end{aligned}$$

In this case, $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ and $\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d}$ differ from each other. Let us consider here the generating function:

$$t(x):=x+\sum_{d=1}^{\infty}\frac{w(\mathcal{O}_{h^0}\mathcal{O}_{h^2})_{0,d}}{5}e^{dx}=x+770e^x+717825e^{2x}+\frac{3225308000}{3}e^{3x}+\cdots. \tag{30}$$

This is nothing but the mirror map used in the mirror computation of the quintic threefold! If we introduce another generating function:

$$F(x) := 5x + \sum_{d=1}^{\infty} w(\mathcal{O}_{h^1}\mathcal{O}_{h^1})_{0,d}e^{dx} = 5x + 6725e^x + \frac{16482625}{2}e^{2x} + \frac{44704818125}{3}e^{3x} + \cdots, \tag{31}$$

$F(x(t))$ gives us the generating function of $\langle\mathcal{O}_{h^1}\mathcal{O}_{h^1}\rangle_{0,d}$.

$$F(x(t)) = 5t + 2875e^t + \frac{4876875}{2}e^{2t} + \frac{8564575000}{3}e^{3t} + \cdots. \tag{32}$$

References

1. Jinzenji, M.: Gauss-Manin system and the virtual structure constants. Int. J. Math. **13**, 445–478 (2002)
2. Jinzenji, M.: Coordinate change of Gauss-Manin system and generalized mirror transformation. Int. J. Modern Phys. A **20**(10), 2131–2156 (2005)
3. Jinzenji, M.: Virtual structure constants as intersection numbers of moduli space of polynomial maps with two marked points. Lett. Math. Phys. **86**(2–3), 99–114 (2008)
4. Jinzenji, M.: Mirror map as generating function of intersection numbers: Toric manifolds with two Kähler forms. Preprint, arXiv:1006.0607

Operadic Construction of the Renormalization Group

Jean-Louis Loday[†] **and Nikolay M. Nikolov**

Abstract First, we give a functorial construction of a group associated to a symmetric operad. Applied to the endomorphism operad it gives the group of formal diffeomorphisms. Second, we associate a symmetric operad to any family of decorated graphs stable by contraction. In the case of Quantum Field Theory models it gives the renormalization group. As an example we get an operadic interpretation of the group of "diffeographisms" attached to the Connes–Kreimer Hopf algebra.

1 Introduction

The combinatorics underlying the renormalization of Quantum Field Theory (QFT) is encoded into the Feynman diagrams. The diagram technique is a powerful tool in perturbative QFT. It was discovered by Connes and Kreimer that the combinatorics in renormalization can be described by a Hopf algebra structure on the space of Feynman diagrams since the attached group is the renormalization group. In this paper our aim is to systematize this procedure by means of symmetric operads. First we show that a family of decorated graphs which is stable for the contraction of the internal edges determines a symmetric operad. Second, we show that to any symmetric operad is attached a (formal) group which takes care of the symmetric group action. Combining the two constructions we get the construction of a group attached to families of diagrams. In the case of QFT we get the renormalization group.

[†]Professor Jean-Louis Loday passed away on 6 June 2012.

J.-L. Loday
Institut de Recherche Mathématique Avancée, CNRS et Université de Strasbourg,
Zinbiel Institute of Mathematics, France

N.M. Nikolov (✉)
INRNE, Bulgarian Academy of Sciences, Tsarigradsko Chaussee 72 Blvd.,
Sofia 1784, Bulgaria

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_13,

Forthe notation and terminology on operads we follow [6] for which we refer for details.

2 Operadic Construction of the Group of Formal Diffeomorphisms

Let $V \equiv \mathbb{R}^N$ be a vector space and $\vec{\mathrm{x}} = (x_1,\ldots,x_N), \vec{\mathrm{y}}, \vec{\mathrm{z}} \in V$. Consider the formal power series[1]

$$\vec{\mathrm{y}} = \vec{\mathrm{f}}\,(\vec{\mathrm{x}}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mu_1,\ldots,\mu_n=1}^{N} \vec{\mathrm{f}}_{\mu_1,\ldots,\mu_n} x_{\mu_1} \cdots x_{\mu_n}\,, \tag{1}$$

$$\vec{\mathrm{z}} = \vec{\mathrm{g}}\,(\vec{\mathrm{y}}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mu_1,\ldots,\mu_n=1}^{N} \vec{\mathrm{g}}_{\mu_1,\ldots,\mu_n} x_{\mu_1} \cdots x_{\mu_n}\,,$$

where $\vec{\mathrm{f}}_{\mu_1,\ldots,\mu_n} = (f_{\nu;\mu_1,\ldots,\mu_n})_{\nu=1}^{N}$ and $\vec{\mathrm{g}}_{\mu_1,\ldots,\mu_n} = (g_{\nu;\mu_1,\ldots,\mu_n})_{\nu=1}^{N}$ are the series coefficients. Since these series do not have constant terms (i.e., terms with $n = 0$) it is well known that their composition

$$\vec{\mathrm{z}} = \vec{\mathrm{g}} \circ \vec{\mathrm{f}}\,(\vec{\mathrm{x}}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mu_1,\ldots,\mu_n=1}^{N} \vec{\mathrm{h}}_{\mu_1,\ldots,\mu_n} x_{\mu_1} \cdots x_{\mu_n}\,, \tag{2}$$

can be determined completely algebraically. A less popular fact is the formula for the coefficients $\vec{\mathrm{h}}_{\mu_1,\ldots,\mu_n} = (h_{\nu;\mu_1,\ldots,\mu_n})_{\nu=1}^{N}$ of the composition series[2]:

---

[1] According to the meaning of a formal power series (cf. [5, Chap. 2]) the notation $\vec{\mathrm{y}} = \vec{\mathrm{f}}\,(\vec{\mathrm{x}})$ is just an abbreviation of the formal sum in the right hand side of (1), which in turn, is nothing but just the list of coefficients $(f_{\nu;\mu_1,\ldots,\mu_n})_{\nu,\mu_1,\ldots,\mu_n=1}^{N}$. One defines the formal derivative series $\frac{\partial^m \vec{\mathrm{f}}}{\partial x_{\nu_1} \cdots \partial x_{\nu_m}}(\vec{\mathrm{x}})$ $:= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mu_1,\ldots,\mu_n=1}^{N} \vec{\mathrm{f}}_{\nu_1,\ldots,\nu_m,\mu_1,\ldots,\mu_n}\, x_{\mu_1} \cdots x_{\mu_n}$ and then the coefficient $\vec{\mathrm{f}}_{\nu_1,\ldots,\nu_m}$ coincides with the leading term of the derivative series $\frac{\partial^m \vec{\mathrm{f}}}{\partial x_{\nu_1} \cdots \partial x_{\nu_m}}(\vec{\mathrm{x}})$. In particular, the coefficients $\vec{\mathrm{f}}_{\nu_1,\ldots,\nu_m}$ must be symmetric in the indices, which is equivalent to the symmetry of the derivatives.

[2] Equation (3) follows from the formula for the nth formal derivative of the composition series $\vec{\mathrm{g}} \circ \vec{\mathrm{f}}\,(\vec{\mathrm{x}})$:

$$\frac{\partial^n h_\mu}{\partial x_{\mu_1} \cdots \partial x_{\mu_n}}(\vec{\mathrm{x}}) = \sum_{\mathfrak{P} \in \mathrm{Part}\{1,\ldots,n\}} \sum_{\rho_1,\ldots,\rho_k=1}^{N} \Big(\frac{\partial^k g_\nu}{\partial x_{\rho_1} \cdots \partial x_{\rho_k}} \circ \vec{\mathrm{f}}\Big)(\vec{\mathrm{x}})$$
$$\times \frac{\partial^{j_1} f_{\rho_1}}{\partial \mu_{i_{1,1}} \cdots \partial \mu_{i_{1,j_1}}}(\vec{\mathrm{x}}) \cdots \frac{\partial^{j_k} f_{\rho_k}}{\partial \mu_{i_{k,1}} \cdots \partial \mu_{i_{k,j_k}}}(\vec{\mathrm{x}}),$$

$$h_{\nu;\mu_1,\ldots,\mu_n} = \sum_{\mathfrak{P} \in \mathrm{Part}\{1,\ldots,n\}} \sum_{\rho_1,\ldots,\rho_k=1}^{N} g_{\nu;\rho_1,\ldots,\rho_k} f_{\rho_1;\mu_{i_{1,1}},\ldots,\mu_{i_{1,j_1}}} \cdots f_{\rho_k;\mu_{i_{k,1}},\ldots,\mu_{i_{k,j_k}}}, \quad (3)$$

which, in the case $N = 1$, is known as the *Faà di Bruno formula*. Here are the notations used in (3):

- The sum is over the set $\mathrm{Part}\{1,\ldots,n\}$ of all *unordered* partitions (cf. below)

$$\mathfrak{P} = \Big\{ \{i_{1,1},\ldots,i_{1,j_1}\},\ldots,\{i_{k,1},\ldots,i_{k,j_k}\} \Big\} \quad (4)$$

  of the set $\{1,\ldots,n\}$.
- In particular, k is the cardinality $|\mathfrak{P}|$ of the partition $\mathfrak{P}$ and $j_1, \ldots, j_k$ are the cardinalities of its pieces.
- The partitions $\mathfrak{P}$ are unordered, but we shall introduce a "canonical order" such that inside each group the elements are in increasing order and the groups are ordered according to the order of their minimal elements

$$i_{\ell,1} < \cdots < i_{\ell,j_\ell}, \quad i_{1,1} < i_{2,2} < \cdots < i_{k,j_k}. \quad (5)$$

Note that all the coefficients $\vec{\mathrm{f}}_{\mu_1,\ldots,\mu_n}$, $\vec{\mathrm{g}}_{\mu_1,\ldots,\mu_n}$ and $\vec{\mathrm{h}}_{\mu_1,\ldots,\mu_n}$ are symmetric in their indices $\mu_1,\ldots,\mu_n$ (cf. footnote 1) and hence, our convention in (3) about the order on $\mathfrak{P}$ is not essential. However, we shall see that dropping the symmetry condition on the coefficients still defines an associative product.

Let us try to simplify a little bit (3) by absorbing some summations: the coefficients $\vec{\mathrm{f}}_{\mu_1,\ldots,\mu_n}$ define a multi-linear map[3]

$$f_n = \big(\vec{\mathrm{f}}_{\mu_1,\ldots,\mu_n}\big) : V^{\otimes n} \to V \quad (6)$$

and vice versa, every multi-linear map $f_n : V^{\times n} \to V$ defines a system of coefficients $\vec{\mathrm{f}}_{\mu_1,\ldots,\mu_n}$ by its matrix elements. Furthermore, the coefficients $\vec{\mathrm{f}}_{\mu_1,\ldots,\mu_n}$ are symmetric in $\mu_1,\ldots,\mu_n$ iff the map f_n is symmetric. Similarly, we set

$$g_n = \big(\vec{\mathrm{g}}_{\mu_1,\ldots,\mu_n}\big) : V^{\otimes n} \to V, \quad h_n = \big(\vec{\mathrm{h}}_{\mu_1,\ldots,\mu_n}\big) : V^{\otimes n} \to V$$

($n = 1,2,\ldots$). Then (3) reads

$$h_n = \sum_{\mathfrak{P} \in \mathrm{Part}\{1,\ldots,n\}} g_k \circ \big(f_{j_1} \otimes \cdots \otimes f_{j_k}\big) \circ \sigma_{\mathfrak{P}}, \quad (7)$$

---

which in turn is derived by induction in n.

[3]Thus, $f_n(x_{1;1},\ldots,x_{1;N};\ldots;x_{n;1},\ldots,x_{n;N})_\nu := \sum_{\mu_1,\ldots,\mu_n=1}^{N} f_{\nu;\mu_1,\ldots,\mu_n}\, x_{1;\mu_1} \cdots x_{n;\mu_n}$.

where the numbers $k, j_1, \dots, j_k$ are defined by conventions (4) and (5) together with the permutation $\sigma_{\mathfrak{P}} \in \mathbb{S}_n$, which is

$$\sigma_{\mathfrak{P}} := \left(i_{1,1}, \dots, i_{1,j_1}, \dots, i_{k,1}, \dots, i_{k,j_k}\right).$$

Thus, the formal power series $\vec{\mathrm{y}} = \vec{\mathrm{f}}\left(\vec{\mathrm{x}}\right)$ of formula (1) is encoded by a sequence

$$\underline{f} = \left(f_1, f_2, \dots, f_n, \dots\right) \in \prod_{n=1}^{\infty} \mathrm{Hom}\left(V^{\otimes n}, V\right)^{\mathbb{S}_n}$$

($\mathrm{Hom}\left(V^{\otimes n}, V\right)^{\mathbb{S}_n}$ being the subspace of $\mathbb{S}_n$-invariant maps in $\mathrm{Hom}\left(V^{\otimes n}, V\right)$). The multiplication in $\prod\limits_{n=1}^{\infty} \mathrm{Hom}\left(V^{\otimes n}, V\right)^{\mathbb{S}_n}$,

$$\underline{h} = \underline{g} \bullet \underline{f} := \left(h_n\right)_{n=1}^{\infty},$$

that is defined by (7) is associative. It has a unit, the composition unit:

$$\underline{1} = (\mathrm{id}_V, 0, \dots)$$

Furthermore, if we assume that $f_1 = \mathrm{id}_V$ (the identity map of V), then $\underline{f}$ has a composition inverse $\underline{f}^{-1} = \left(1, (\underline{f}^{-1})_2, \dots\right)$ since for $n > 1$ we have

$$0 = \left(\underline{1}\right)_n = \left(\underline{f}^{-1} \bullet \underline{f}\right)_n = (\underline{f}^{-1})_n + f_n + \text{low order terms},$$

which inductively fixes $(\underline{f}^{-1})_n$.

The so described *group of formal diffeomorphisms* is denoted by

$$\mathrm{FDiff}(V) \cong \{\mathrm{id}_V\} \times \prod_{n=2}^{\infty} \mathrm{Hom}\left(V^{\otimes n}, V\right)^{\mathbb{S}_n}. \tag{8}$$

Note that the vector space V can be even arbitrary linear vector space: N then will be the cardinality (possibly, infinite) of the linear basis of V and the series (1) would be neither more nor less formal. We note also that $f_{v;\mu_1,\dots,\mu_n}$ for fixed $\mu_1, \dots, \mu_n$ are nonzero only for no more than a finite number of indices v since they are coordinates of the vector $\vec{\mathrm{f}}_{\mu_1,\dots,\mu_n}$. Hence, the correspondence $\vec{\mathrm{f}}\left(\vec{\mathrm{x}}\right) \leftrightarrow \underline{f}$ defined by (6) remains valid and the composition (2) is again well defined algebraically.

3 Group Associated to a Symmetric Operad

We now observe that the group multiplication (7) has a straightforward generalization in a symmetric operad (see (9) below). Indeed, it uses two basic structures which are axiomatized in the operad theory (cf. [6, Sect. 5.3]). These structures are first, the composition $g_k \circ \big(f_{j_1} \otimes \cdots \otimes f_{j_k}\big)$ that in the general case becomes an operadic composition $\gamma\big(g_k; f_{j_1}, \ldots, f_{j_k}\big)$, and the second structure is the right action of the permutation group, $f_n \mapsto (f_n)^{\sigma_{\mathfrak{P}}} := f_n \circ \sigma_{\mathfrak{P}}$.

Theorem 3.4 ([7]). *There is a functor together with a subfunctor:*

$$\left\{\begin{matrix} \textit{Category of} \\ \textit{Symmetric operads} \end{matrix}\right\} \to \left\{\begin{matrix} \textit{Category of} \\ \textit{Groups} \end{matrix}\right\}$$

$$\mathscr{P} = \{\mathscr{P}(n)\}_{n=1}^{\infty} \quad \mapsto \mathfrak{G}(\mathscr{P}) = \{\mathrm{id}\} \times \prod_{n=2}^{\infty} \mathscr{P}(n)$$

$$\cup\!\shortparallel$$

$$\mathscr{P} = \{\mathscr{P}(n)\}_{n=1}^{\infty} \quad \mapsto \mathfrak{G}(\mathscr{P})^{\mathbf{S}} = \{\mathrm{id}\} \times \prod_{n=2}^{\infty} \mathscr{P}(n)^{\mathbb{S}_n},$$

where $\mathscr{P}(n)^{\mathbb{S}_n}$ stands for the subspace of $\mathbb{S}_n$-invariant elements. The multiplication law is given by

$$(\underline{\beta} \bullet \underline{\alpha})_n = \sum_{\mathfrak{P} \in \mathrm{Part}\{1,\ldots,n\}} \gamma\big(\beta_k; \alpha_{j_1}, \ldots, \alpha_{j_k}\big)^{\sigma_{\mathfrak{P}}} \tag{9}$$

for $\underline{\alpha} = (\alpha_n)_{n=1}^{\infty}$ and $\underline{\beta} = (\beta_n)_{n=1}^{\infty}$ and the notations of (7). On operadic morphisms $\vartheta : \mathscr{P} \to \mathscr{P}'$ $(= \{\vartheta_n : \mathscr{P}(n) \to \mathscr{P}'(n)\}_{n=1}^{\infty})$ the functor gives

$$\mathfrak{G}(\vartheta) := \prod_{n=1}^{\infty} \vartheta_n.$$

In the case of End_V *we have a natural isomorphism*

$$\mathfrak{G}(\mathrm{End}_V)^{\mathbf{S}} \cong \mathrm{FDiff}(V). \tag{10}$$

The most nontrivial part of the above statement is the associativity of the operation $\bullet$ (9). It can be proven by straightforward inspection. The existence of a unit and inverse elements follows exactly by the same arguments as for the group of formal diffeomorphisms.

Remark 3.1. There is a natural group associated with a nonsymmetric operad $\mathscr{Q} = \big\{\mathscr{Q}_n\big\}_{n\geq 1}$ (cf. [6, Sect. 5.8.15]). The underlying set of this group is $\mathfrak{G}_{\mathscr{Q}} := \{\mathrm{id}\} \times \prod_{n=2}^{\infty} \mathscr{Q}_n$. Since any symmetric operad $\mathscr{P}$ can be considered as a nonsymmetric operad $\widetilde{\mathscr{P}}$ by forgetting the action of the symmetric group, $\widetilde{\mathscr{P}}_n = \mathscr{P}(n)$, then we have a second way of associating a group to a symmetric operad. Though in both ways the underlying sets are the same, $\mathfrak{G}_{\widetilde{\mathscr{P}}} = \mathfrak{G}(\mathscr{P})$, the group structures are completely different. It is shown in [2, Sect. 1] that the group $\mathfrak{G}_{\widetilde{\mathscr{P}}}$ admits a quotient by taking co-invariants, i.e., the set $\{\mathrm{id}\} \times \prod_{n=2}^{\infty} \mathscr{P}(n)_{\mathbb{S}_n}$ has a group structure coming from the group structure of $\mathfrak{G}_{\widetilde{\mathscr{P}}}$.

On the other hand, if we start with a nonsymmetric operad $\mathscr{Q}$ one can associate a symmetric operad $\mathscr{Q} \otimes As$ called a regular operad. The functor $\mathscr{Q} \mapsto \mathfrak{G}(\mathscr{Q} \otimes As)^{\mathbf{S}}$ associates a group to any nonsymmetric operad. The underlying set of this group is $\{\mathrm{id}\} \times \prod_{n=2}^{\infty} \mathscr{Q}_n$.

We will give below some facts about the structure of the groups related to symmetric operads.

Proposition 1 ([7]). *Let us set for* $m > 0$

$$\mathfrak{G}(\mathscr{P})_m = \Big\{\underline{\alpha} = (\alpha_n)_{n=1}^{\infty} \in \mathfrak{G}(\mathscr{P}) \Big| \alpha_2 = \cdots = \alpha_m = 0\Big\}$$

(for $m = 1$, $\mathfrak{G}(\mathscr{P})_1 := \mathfrak{G}(\mathscr{P})$*). Then* $\mathfrak{G}(\mathscr{P})_m$ *is a normal subgroup of* $\mathfrak{G}(\mathscr{P})$.

Note that

$$\mathfrak{G}(\mathscr{P}) = \lim_{\longleftarrow} \mathfrak{G}(\mathscr{P})/\mathfrak{G}(\mathscr{P})_m$$

and in the case when the operadic spaces $\mathscr{P}(n)$ are finite dimensional the quotient groups are (finite dimensional) Lie groups. Hence, in the latter case the group $\mathfrak{G}(\mathscr{P})$ is a *pro-Lie group*. We use this fact to derive the Lie algebra corresponding to the group $\mathfrak{G}(\mathscr{P})$ together with the exponential map.

Theorem 3.5 ([7]). *The Lie algebra corresponding to the group* $\mathfrak{G}(\mathscr{P})$ *is*

$$\mathfrak{g}(\mathscr{P}) = \{0\} \times \prod_{n=2}^{\infty} \mathscr{P}(n)$$

The Lie bracket on $\mathfrak{g}(\mathscr{P})$ *is built from a pre-Lie bracket*

$$[\underline{\mu}, \underline{\nu}] = \underline{\mu} * \underline{\nu} - \underline{\nu} * \underline{\mu}$$

$(\underline{\mu}, \underline{\nu} \in \mathfrak{g}(\mathscr{P}))$*, where*[4]

$$(\underline{\mu} * \underline{\nu})_n = \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} \big(\nu_k \circ_{\min J} \mu_j\big)^{\sigma_{\mathfrak{P}_J}} \equiv \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} \gamma\big(\nu_k; \mathrm{id}, \ldots, \mathrm{id}, \underset{\substack{\uparrow \\ \min J}}{\mu_j}, \mathrm{id}, \ldots, \mathrm{id}\big)^{\sigma_{\mathfrak{P}_J}} \qquad (11)$$

where $j = |J|$ *and the partition* $\mathfrak{P}_J$ *is the partition* $\big\{\{i\}\big| i \in \{1,\ldots,n\}\backslash J\big\} \cup \{J\}$. *(Note that the sum in (11) is the subsum in (9) corresponding to partitions* $\mathfrak{P}$ *of a form* $\mathfrak{P}_J$.*)*

The Lie algebra $\mathfrak{g}(\mathscr{P})$ is again an inverse limit of finite dimensional Lie algebras

$$\mathfrak{g}(\mathscr{P}) = \lim_{\longleftarrow} \mathfrak{g}(\mathscr{P})/\mathfrak{g}(\mathscr{P})_m$$

where $\mathfrak{g}(\mathscr{P})_m$ is the ideal

$$\mathfrak{g}(\mathscr{P})_m = \Big\{\underline{\mu} = (\mu_n)_{n=1}^{\infty} \in \mathfrak{g}(\mathscr{P}) \Big| \mu_2 = \cdots = \mu_m = 0\Big\}.$$

[4] $\circ_i$ is the ith operadic partial composition.

Note that the quotient group $\mathfrak{G}(\mathscr{P})\big/\mathfrak{G}(\mathscr{P})_m$ and Lie algebra $\mathfrak{g}(\mathscr{P})\big/\mathfrak{g}(\mathscr{P})_m$ are isomorphic as sets to the set $\prod_{n=2}^{m} \mathscr{P}(n)$ and the group and pre-Lie products on this set are just $\bullet$ (9) and $*$ (11) truncated up to order m.

4 Feynman Diagrams and Their Combinatorics

Feynman diagrams are a powerful tool in perturbation theory. They indicate the terms of perturbative expansions. Furthermore, many manipulation on the corresponding formal perturbation series have a combinatorial description by operations on diagrams.

(a) Basic definitions

A Feynman diagram is a finite graph with various decorations.

A graph Γ is a set of points, called *vertices*, with attached *flags* (or *half-edges*) to them. Some pairs of these flags are further joined to become edges connecting the corresponding vertices. All these structures are contained in the following data: two finite sets, the set of vertices vert(Γ) and the set of flags flag(Γ), and two maps

$$s : \text{flag}(\Gamma) \to \text{vert}(\Gamma)\,, \quad \sigma : \text{flag}(\Gamma) \to \text{flag}(\Gamma) \tag{12}$$

such that $\sigma^2 = \text{id}$. Thus, the map s represents the process of attaching flags to vertices, i.e., the flag f is attached to the vertex $s(f)$. The map σ represents the process of joining flags, i.e., the flag f is joined with the flag $\sigma(f)$. If $f = \sigma(f)$ then we call the flag f an *external line*; such a line is attached to only one vertex. If $f \neq \sigma(f)$ then the unordered pair $\{f, \sigma(f)\}$ form an *edge*, or an *internal line* of the graph, which is attached to the vertices $s(f)$ and $s(\sigma(f))$. When $s(f) = s(\sigma(f))$ but $f \neq \sigma(f)$ we have an internal line attached to a unique vertex. Such an internal line is called a *tadpole* and we will not deal with graphs containing tadpoles.

To every graph we assign a topological space: its *geometric realization*. To this end we assign to each edge a copy of the closed interval $[0,1]$ (without the orientation) and to each vertex a point. Then we glue all of these spaces according to the incidence between the edges and the vertices.

A *decorated graph* is a graph with some extra data. Forgetting these extra structure we obtain just a graph that is called the *body* of the decorated graph. We shall consider graphs with the following decorations:

(a) *Colors* for the vertices and for the flags. They form two sets

- A set of colors for the vertices: Colv
- A set of colors for the flags: Colf

Then we have maps assigning colors:

$$\text{c}_\text{v} : \text{vert}(\Gamma) \to \text{Colv}\,, \quad \text{c}_\text{f} : \text{flag}(\Gamma) \to \text{Colf}\,. \tag{13}$$

At the beginning we do not impose any conditions on the above maps $\mathrm{c_v}$ and $\mathrm{c_f}$. However, in Sect. 6 we shall explain how one can consider the restrictions on the decorations, which arise in the physical models.

(b) The second type of decoration we shall consider is an *enumeration*

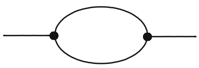
$$\nu : \mathrm{vert}(\Gamma) \cong \{1,\dots,n\} \tag{14}$$

of the set of vertices.

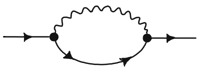
(b) Examples

These are the notion of graph and decorated graph, or also diagram. Here are some examples to illustrate them.

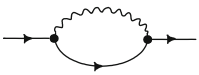
Example 1. An example of a graph is: $\mathrm{vert}(\Gamma) = \{0,1\}$, $\mathrm{flag}(\Gamma) = \{a,b,c,d,e,f\}$, $s(a) = s(b) = s(c) = 0$, $s(d) = s(e) = s(f) = 1$, $\sigma(a) = a$, $\sigma(b) = e$, $\sigma(c) = d$, $\sigma(f) = f$. The geometric realization is:



Example 2. A decoration for the graph in Example 1 is provided by $\mathrm{Colv} = \{\bullet\}$, $\mathrm{Colf} = \{\wr, \uparrow, \downarrow\}$, and coloring maps: $\mathrm{c_v}(0) = \mathrm{c_v}(1) = \bullet$, $\mathrm{c_f}(a) = \downarrow$, $\mathrm{c_f}(b) = \uparrow$, $\mathrm{c_f}(e) = \downarrow$, $\mathrm{c_f}(f) = \uparrow$, $\mathrm{c_f}(c) = \wr = \mathrm{c_f}(d)$. The result can be drawn as



So, we indicated the colors in this example by shapes, which is common in physics. Also if the colors of two joined flags coincide we indicate this as a color of the corresponding edge. In the above example we also meet situation of edges of the form $\bullet\!\!\rightarrow\!\!-\!\!\rightarrow\!\!\bullet$ and in this case it is also convenient to think of such an edge as an oriented edge $\bullet\!\!-\!\!\rightarrow\!\!-\!\!\bullet$. Then we can draw the diagram of this example as



(c) Types of graphs and diagrams

A graph is called *connected* if its geometric realization is a connected space.

Another important type of graphs are the so-called *one particle irreducible* (1PI) graphs. A graph Γ is called one particle irreducible if it is connected and after cutting any of its inner edges it remains connected. Here cutting of an inner edge determined by a pair of flags $f \neq \sigma(f)$ means to change the second structure map σ to a new map σ' such that $\sigma'(f') := \sigma(f)$ if $f' \neq f$ and $f' \neq \sigma(f)$, and $\sigma'(f') := f'$ if $f' = f$ or $f' = \sigma(f)$. We shall impose in addition the requirement that 1PI graphs have no tadpoles and have at least two vertices (or equivalently, at least one inner edge).

If the body of a decorated graph is connected, then the graph is also called connected. Similarly a decorated graph is called 1PI if its body is 1PI.

(e) Operations on graphs and diagrams

A *subgraph* of a graph Γ is a subset $J \subseteq \mathrm{vert}(\Gamma)$. It determines a graph Γ_J as follows: the set of vertices of Γ_J is $\mathrm{vert}(\Gamma_J) := J \subseteq \mathrm{vert}(\Gamma)$. The set of flags of Γ_J is $\mathrm{flag}(\Gamma_J) := s^{-1}(J) \equiv s^{-1}\big(\mathrm{vert}(\Gamma_J)\big)$ and we set the map $s_J : \mathrm{flag}(\Gamma_J) \to \mathrm{vert}(\Gamma_J)$ to be the restriction of the map s. The map $\sigma_J : \mathrm{flag}(\Gamma_J) \to \mathrm{flag}(\Gamma_J)$ coincides with σ whenever f and $\sigma(f)$ belong to $\mathrm{flag}(\Gamma_J)$: such pairs $\{f, \sigma(f)\}$ of different flags are the inner edges of the subgraph. For the remaining $f \in \mathrm{flag}(\Gamma_J)$ we set $\sigma_J(f) = f$ and they are the outer edges of the subgraph. Note that the outer edges of the graph Γ_J are either outer edges of Γ attached to a vertex in J or they are inner edges of Γ with only one end belonging to J.

If the graph Γ is colored then the graph Γ_J determined by a subgraph J has an induced coloring defined just by the restrictions of the coloring maps $\mathrm{c_v}$ and $\mathrm{c_f}$ to $\mathrm{vert}(\Gamma_J)$ and $\mathrm{flag}(\Gamma_J)$, respectively.

If the graph Γ is enumerated, then the graph Γ_J has an induced enumeration provided by the unique monotonically increasing isomorphism $\nu(J) \cong \{1, \ldots, |J|\}$.

Another important operation on graphs is the *contraction* of a subgraph.

For every graph Γ and its subgraph $J \subseteq \mathrm{vert}(\Gamma)$ we define the contracted graph Γ/J as follows. We introduce a new vertex v_J, which can be identified with the set J in order to be accurate. Then we set

$$\begin{aligned}
\mathrm{vert}(\Gamma/J) &:= \big(\mathrm{vert}(\Gamma) \backslash J\big) \cup \{v_J\}, \\
\mathrm{flag}(\Gamma/J) &:= \big\{ f \in \mathrm{flag}(\Gamma) \,\big|\, \text{if } s(f) \text{ and } s(\sigma(f)) \in J \text{ then } f = \sigma(f) \big\} \\
&\equiv \mathrm{flag}(\Gamma) \big\backslash \big\{ f \in \mathrm{flag}(\Gamma) \,\big|\, s(f), s(\sigma(f)) \in J \text{ and } f \neq \sigma(f) \big\},
\end{aligned}$$

in other words, $\mathrm{flag}(\Gamma/J)$ contains all the flags of $\mathrm{flag}(\Gamma)$ except those ones that form the inner edges of the graph Γ_J. The structure maps $s_{\Gamma/J}$ and $\sigma_{\Gamma/J}$ are defined as follows:

$$s_{\Gamma/J}(f) := s(f) \;\text{ if }\; s(f) \notin J \quad \text{and} \quad s_{\Gamma/J}(f) := v_J \;\text{ if }\; s(f) \in J,$$

$$\sigma_{\Gamma/J} := \sigma\big|_{\mathrm{vert}(\Gamma/J)},$$

where the second identity is provided by the fact that $\mathrm{flag}(\Gamma/J)$ is defined as a σ-invariant subset. To summarize, the graph Γ/J is obtained by shrinking all the vertices in J to a single vertex v_J and removing all the internal lines of Γ_J. Note that if the graph Γ is connected or 1PI, respectively, then so is Γ/J.

If the graph Γ is colored, then for every pair (J, L) consisting of a subset $J \subseteq \mathrm{vert}(\Gamma)$ and an element $L \in \mathrm{Colv}$ we can define a colored contracted graph $\Gamma/(J, L)$ constructed as the graph Γ/J endowed with the following coloring maps $\mathrm{c'_v}$ and $\mathrm{c'_f}$:

$$\mathrm{c}'_{\mathrm{v}}\big|_{\mathrm{vert}(\Gamma)\setminus J} := \mathrm{c}_{\mathrm{v}}\big|_{\mathrm{vert}(\Gamma)\setminus J}\,, \qquad \mathrm{c}'_{\mathrm{v}}(v_J) := L\,,$$
$$\mathrm{c}'_{\mathrm{f}} := \mathrm{c}_{\mathrm{f}}\big|_{\mathrm{flag}(\Gamma/J)}\,.$$

Finally, if we have an enumerated graph Γ, then the contracted graph Γ/J will be endowed with the enumeration provided by the unique monotonically increasing isomorphism

$$\nu(\mathrm{vert}(\Gamma)\setminus J)\cup\{\min\nu(J)\}\cong\{1,\dots,n-|J|+1\}\,.$$

Note that if the graph Γ has no tadpoles, then the graphs Γ_J and Γ/J have no tadpoles for every subgraph J of Γ.

(f) Isomorphic diagrams

Let us introduce the notion of an *isomorphism* of two enumerated diagrams Γ and Γ'. An isomorphism of graphs $\Gamma\cong\Gamma'$ consists of a pair of bijections $j_v:\mathrm{vert}(\Gamma)\cong\mathrm{vert}(\Gamma')$ and $j_f:\mathrm{flag}(\Gamma)\cong\mathrm{flag}(\Gamma')$, which commute with the structure maps s,s' and σ,σ', respectively. In other words, $j_v\circ s=s'\circ j_f$ and $j_f\circ\sigma=\sigma'\circ j_v$. An isomorphism of colored graphs is an isomorphism of graphs, which in addition satisfies $\mathrm{c}_{\mathrm{v}}=\mathrm{c}'_{\mathrm{v}}\circ j_v$ and $\mathrm{c}_{\mathrm{f}}=\mathrm{c}'_{\mathrm{f}}\circ j_f$ (compatibility with the coloring maps). Finally, an isomorphism of enumerated colored graphs is an isomorphism of colored graphs which preserves the enumeration.

We shall consider two isomorphic diagrams as identical. More precisely, we shall work with the set:

$$\mathrm{Dgm}(n) := \text{set of all equivalence classes of isomorphic enumerated colored graphs with } n \text{ vertices.} \tag{15}$$

(g) Combinatorial Feynman rules, or, representation of diagrams in a monoid

There is a convenient one-to-one correspondence between $\mathrm{Dgm}(n)$ and a free commutative monoid. This construction follows on an abstract algebraic (or combinatorial) level the so called "Feynman rules" that assign in QFT to every Feynman diagram an analytic expression. Let

$$\mathfrak{M}(n) := \text{the free commutative monoid with a set of generators}$$
$$\big(\{1,\dots,n\}\times\mathrm{Colv}\big)\cup\big(\{1,\dots,n\}\times\mathrm{Colf}\big)\cup\big(\{1,\dots,n\}\times\mathrm{Colf}\big)^{\times 2}. \tag{16}$$

Let us introduce "physical" names and notation for the elements in the above three disjoint sets. We call the elements of Colf the basic "fields" and denote them by ϕ, ψ, etc. The element $(i,\phi)\in\{1,\dots,n\}\times\mathrm{Colf}$ will be denoted by $\phi(i)$ and called a "field at the point i". Next, the elements $(i,\phi;j,\psi)\in\big(\{1,\dots,n\}\times\mathrm{Colf}\big)^{\times 2}$ will be denoted by $C_{\phi,\psi}(i,j)$ and will be called "propagators". Finally, the elements $L\in\mathrm{Colv}$ will be called "interactions" and a pair $(i,L)\in\{1,\dots,n\}\times\mathrm{Colv}$ will be called an interaction at the point i and will be denoted by $L(i)$.

Thus, in the above notations the set of generators (16) for the monoid $\mathfrak{M}(n)$ reads:

$$\begin{aligned}\{L(i)\,|\,L\in \mathrm{Colv}, i=1,\dots,n\}\cup\{\phi(i)\,|\,\phi\in \mathrm{Colf}, i=1,\dots,n\}\\ \cup\{C_{\phi,\psi}(i,j)\,|\,\phi,\psi\in \mathrm{Colf}, i,j=1\dots,n\}\,.\end{aligned} \tag{17}$$

Now, to each enumerated colored graph Γ we assign a monomial in $\mathfrak{M}(n)$ in the following way. To the vertex $\nu^{-1}(i)$ (i.e., to the vertex with number i) we assign $L(i)$ if its color is $L\in \mathrm{Colv}$. To each outer edge attached to the vertex $\nu^{-1}(i)$ we assign $\phi(i)$ if the color of the corresponding flag is $\phi\in\mathrm{Colf}$. To each inner edge connecting the vertices $\nu^{-1}(i)$ and $\nu^{-1}(j)$ we assign $C_{\phi,\psi}(i,j)$ if the colors of the flags attached to $\nu^{-1}(i)$ and $\nu^{-1}(j)$ are ϕ and ψ, respectively. Finally, we multiply all the above obtained generators in $\mathfrak{M}(n)$. The resulting monomial in $\mathfrak{M}(n)$ is denoted by M_Γ.

Example 3. In the case of Example 2 with vertex enumeration $\nu(0)=1$, $\nu(1)=2$ we have

$$M_\Gamma = \overline{\psi}(1)\,\psi(2)\,L(1)\,L(2)\,C_{A,A}(1,2)\,C_{\psi,\overline{\psi}}(1,2)\,,$$

where we denoted now the colors by letters: $L := \bullet \in$Colv and $A :=$ , $\psi :=$, $\overline{\psi} :=$ .

Proposition 2. *The correspondence $\Gamma\mapsto M_\Gamma$ is a bijection* $\mathrm{Dgm}(n)\cong\mathfrak{M}(n)$*, i.e., it is a one-to-one correspondence between the equivalence classes of isomorphic enumerated colored graphs with n vertices and the elements of the monoid* $\mathfrak{M}(n)$*.*

Proof. It is clear that $\Gamma\mapsto M_\Gamma$ maps injectively the equivalence classes of diagrams to elements of $\mathfrak{M}(n)$. To see that this map is surjective one constructs for every element of $\mathfrak{M}(n)$ a diagram that reproduces this monomial. □

5 The Universal Contraction Operad

Recall that $\mathrm{Dgm}(n)$ is the set of all equivalence classes of isomorphic enumerated colored graphs with n vertices. Let us define

$$\mathfrak{R}(n) := \mathrm{Hom}_{\mathbb{K}}\left(\mathbb{K}^{(\mathrm{Dgm}(n))},\mathbb{K}^{(\mathrm{Colv})}\right)\cong\mathbb{K}^{\mathrm{Dgm}(n)\times\mathrm{Colv}}, \tag{18}$$

where $\mathbb{K}^{(I)}$ stands for the vector space over the ground field (ring) $\mathbb{K}$ spanned by a basis indexed by I and the existence of the second *canonical* isomorphism follows in the case when Colv is a *finite* set, which we shall assume further. This canonical isomorphism is provided by the decomposition

$$Q(\Gamma) = \sum_{L \in \mathrm{Colv}} q(\Gamma, L)\, L\,, \tag{19}$$

where $Q \in \mathfrak{R}(n)$. We shall treat the isomorphism at the second equality in (18) as an identification, $\mathfrak{R}(n) = \mathbb{K}^{\mathrm{Dgm}(n) \times \mathrm{Colv}}$.

We call the elements of $\mathfrak{R}(n)$ *contraction maps*. This is motivated by the fact that they can be thought of as prescriptions for contracting subgraphs as we shall describe below.

Note that the action of the permutation group $\mathbb{S}_n$ on $\mathrm{Dgm}(n)$ induces an action on $\mathfrak{R}(n)$. We shall endow now the so-defined $\mathbb{S}$-module $\mathfrak{R} = \{\mathfrak{R}(n)\}_{n \geq 1}$ with a structure of a symmetric operad.

To this end we shall define the partial composition maps:

$$\circ_i : \mathfrak{R}(n) \otimes \mathfrak{R}(j) \to \mathfrak{R}(n-1+j)\,, \tag{20}$$

$i = 1, \ldots, n,\ j = 1, 2, \ldots$. Let us introduce for every enumerated diagram Γ the subsets of vertices $J := J(i,j) \subseteq \mathrm{vert}(\Gamma)$:

$$J\ (\equiv J(i,j)) := \left\{ \nu^{-1}(\ell) \,\middle|\, \ell = i+1, \ldots, i+j \right\}. \tag{21}$$

We define for $Q'' \in \mathfrak{R}(n)$, $Q' \in \mathfrak{R}(j)$ and Γ that is a representative of an isomorphism class in $\mathrm{Dgm}(n-1+j)$:

$$(Q'' \circ_i Q')(\Gamma) = \sum_{L \in \mathrm{Colv}} q'(\Gamma_J, L)\, Q''\big(\Gamma/(J,L)\big)\,, \tag{22}$$

where

$$Q'(\Gamma_J) =: \sum_{L \in \mathrm{Colv}} q'(\Gamma_J, L)\, L\,. \tag{23}$$

Note that if we set

$$\begin{aligned} Q''(\Gamma'') &= \sum_{L \in \mathrm{Colv}} q''(\Gamma'', L)\, L\,, \\ Q(\Gamma) &= (Q'' \circ_i Q')(\Gamma) = \sum_{L \in \mathrm{Colv}} q(\Gamma, L)\, L, \end{aligned} \tag{24}$$

then (22) reads

$$q(\Gamma, K) = \sum_{L \in \mathrm{Colv}} q'(\Gamma_J, L)\, q''\big(\Gamma/(J,L), K\big)\,. \tag{25}$$

Proposition 3 ([7]). $\mathfrak{R} = \{\mathfrak{R}(n)\}_{n \geq 1}$ *is a symmetric operad.*

The *proof* is straightforward checking and we omit it.

6 Suboperads in $\mathfrak{R}$ and Concrete Combinatorial Models of Quantum Field Theory

In the previous section we have defined a universal operad $\mathfrak{R}$ on decorated graphs, which can include, at the combinatorial level, any concrete model of Quantum Field Theory (QFT) provided that we have sufficiently many colors in Colv and Colf. So, the QFT models can be considered as particular suboperads of $\mathfrak{R}$. Describing these suboperads can be quite cumbersome in general and we shall do this in several steps. At each step we shall impose certain restrictions on the contraction maps $Q \in \mathfrak{R}(n)$. These restrictions include, in particular, requirements that Q should vanish on certain classes of diagrams that are "not admissible for contraction".

For instance, excluding tadpoles was a first example of such a restriction on diagrams. It was "stable with respect to contractions and subdiagrams" and hence, it defined a suboperad in $\mathfrak{R}$. More precisely, the statement is that the subspaces in $\mathfrak{R}(n)$ for every $n = 1, 2, \ldots$, which consist of those contraction maps that vanish on diagrams with tadpoles, form a suboperad.

Let us formulate the argument in a more general principle:

Proposition 4 ([7]). *Let* $\Phi = \{\Phi(n)\}_{n\geq 1}$ *be a system of subsets* $\Phi(n) \subseteq \mathrm{Dgm}(n) \times \mathrm{Colv}$ *for* $n = 1, 2, \ldots$ *and let us define*

$$\mathfrak{R}_{\Phi}(n) = \mathbb{K}^{\Phi(n)} \subseteq \mathbb{K}^{\mathrm{Dgm}(n)\times\mathrm{Colv}} \equiv \mathfrak{R}(n),$$
$$\mathfrak{R}_{\Phi}(n) \equiv \left\{ Q = \sum qL \in \mathfrak{R}(n) \,\middle|\, q\big|_{(\mathrm{Dgm}(n)\times\mathrm{Colv})\setminus\Phi(n)} = 0 \right\}, \qquad (26)$$

where we use the expansion (19) and embeddings of type $\mathbb{K}^A \hookrightarrow \mathbb{K}^B$ *for* $A \subseteq B$*, which are defined by* $(x_a)_{a\in A} \mapsto (y_b)_{b\in B}$ *such that* $y_a = x_a$ *for* $a \in A$ *and* $y_b = 0$ *for* $b \in B\setminus A$.

Then the following conditions are equivalent:

(i) The system $\mathfrak{R}_{\Phi} = \{\mathfrak{R}_{\Phi}(n)\}_{n\geq 1}$ *is a suboperad of* $\mathfrak{R}$.
(ii) Each subset $\Phi(n)$ *is* $\mathbb{S}_n$*-invariant and the system* $\{\Phi(n)\}_{n\geq 1}$ *has the property*

$$(\Gamma_J, L) \in \Phi(|J|) \ \text{ and } \ (\Gamma/(J,L), K) \in \Phi(n - |J| + 1) \Rightarrow (\Gamma, K) \in \Phi(n) \qquad (27)$$

for every $\Gamma \in \mathrm{Dgm}(n)$*,* $J \subseteq \mathrm{vert}(\Gamma)$ *and* $K, L \in \mathrm{Colv}$.

Corollary 1. *The following systems form a suboperad in* $\mathfrak{R}$*:*

$$\mathfrak{R}_{\mathrm{1PI}}(n) := \left\{ Q \in \mathfrak{R}(n) \,\middle|\, Q(\Gamma) = 0 \text{ if } \Gamma \text{ is not 1PI} \right\}.$$

Let us give another example for a restriction on diagrams that induces a suboperad. Let us consider a *nonempty* subset

$$\mathscr{E} \subset \mathrm{Colf}^{\times 2}$$

and call it a set of *admissible connections*. A colored graph Γ is called $\mathscr{E}$-*admissible* if for all flags $f \in \mathrm{flag}(\Gamma)$ such that $f \neq \sigma(f)$ we have $(\mathrm{c_f}(f), \mathrm{c_f}(\sigma(f))) \in \mathscr{E}$. Or in other words, if the pairs of colors of the flags corresponding to the inner edges are contained in $\mathscr{E}$. As an application of Proposition 4 we get:

Corollary 2. *Let $\mathscr{E}$ be any symmetric subset in* $\mathrm{Colf}^{\times 2}$ *and let $\mathfrak{R}_{\mathscr{E}}(n)$ be the space that consists of all contraction maps $Q \in \mathfrak{R}(n)$, which vanish on all diagrams that either are not $\mathscr{E}$-admissible, or have tadpoles. Then $\{\mathfrak{R}_{\mathscr{E}}(n)\}_{n\geq 1}$ is a suboperad of $\mathfrak{R}$.*

Note that in Corollaries 1 and 2 the sets $\boldsymbol{\Phi}(n)$ are of the form

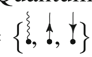
$$\boldsymbol{\Phi}(n) = \mathrm{Dgm}'(n) \times \mathrm{Colv}$$

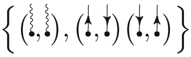
for some subsets $\mathrm{Dgm}'(n) \subseteq \mathrm{Dgm}(n)$. In this case condition (27) reads

$$\Gamma_J \in \mathrm{Dgm}'(|J|) \ \text{ and } \ \Gamma/J \in \mathrm{Dgm}'(n-|J|+1) \ \Rightarrow \ \Gamma \in \mathrm{Dgm}'(n).$$

and $\mathfrak{R}_{\boldsymbol{\Phi}}$ is

$$\begin{aligned}\mathfrak{R}_{\boldsymbol{\Phi}}(n) = \ & \left\{Q \in \mathfrak{R}(n) \,\middle|\, Q\big|_{\mathrm{Dgm}(n)\setminus\mathrm{Dgm}'(n)} = 0\right\} \equiv \mathrm{Hom}_{\mathbb{K}}\left(\mathbb{K}^{(\mathrm{Dgm}'(n))}, \mathbb{K}^{\mathrm{Colv}}\right) \\ = \ & \mathbb{K}^{\mathrm{Dgm}'(n)\times\mathrm{Colv}}.\end{aligned}$$

Example 4. Let us introduce an example of the set $\mathscr{E}$ for the case of Quantum Electrodynamics (QED). In this case we use three colors for flags Colf = 
The set of admissible connections is:

$\mathscr{E}$ =  .

The diagram of Example 2 was thus $\mathscr{E}$-admissible for QED and as there we can use for edges single colors, one with no orientation and one with orientation. The nonoriented lines are called "photon lines" and the oriented lines are called "electron lines".

Our next "selection rule" for contraction maps is by the type of vertices. A vertex is a colored graph with one vertex and no tadpoles. So, it contains only outer edges which are called *corolla* of the vertex. The number of the external edges of the vertex is called its *valency*.

Let $\mathscr{V} \subseteq \mathrm{Dgm}(1)$ be a set of vertices. We call the set $\mathscr{V}$ *types of vertices* in the theory. Let us define then the system $\boldsymbol{\Phi}_{\mathscr{V}} = \{\boldsymbol{\Phi}_{\mathscr{V}}(n)\}_{n\geq 1}$

$$\begin{aligned}\boldsymbol{\Phi}_{\mathscr{V}}(n) \ = \ & \Big\{(\Gamma, L) \in \mathrm{Dgm}(n) \times \mathrm{Colv} \Big| \forall J \subseteq \Gamma \ \big(\text{if } |J| = 1 \text{ then } \Gamma_J \in \mathscr{V}\big) \text{ and} \\ & \Gamma/(\mathrm{vert}(\Gamma), L) \in \mathscr{V}\Big\}.\end{aligned}$$

It follows that $\Phi_{\mathscr{V}}$ satisfies condition (ii) of Proposition 4 and hence,

$$\mathfrak{R}_{\mathscr{V}} := \mathfrak{R}_{\Phi_{\mathscr{V}}} ,$$

is a suboperad of $\mathfrak{R}$.

Thus, a physical theory can be defined as intersection of the operads

$$\mathfrak{R}_{\mathscr{E},\mathscr{V}} := \mathfrak{R}_{1\mathrm{PI}} \cap \mathfrak{R}_{\mathscr{E}} \cap \mathfrak{R}_{\mathscr{V}} . \tag{28}$$

In the next section we shall consider the main examples of physical theories.

Remark 1. If $\{\Phi_i\}_{i\in I}$ is a collection of systems $\Phi_i = \{\Phi_i(n)\}_{n\geq 1}$ each satisfying condition (ii) of Proposition 4 then

$$\bigcap_{i\in I} \mathfrak{R}_{\Phi_i} = \mathfrak{R}_{\Phi} \quad \text{where} \quad \Phi = \{\Phi(n)\}_{n\geq 1} \quad \text{with} \quad \Phi(n) = \bigcap_{i\in I} \Phi_i(n) ,$$

and Φ also satisfies condition (ii) of Proposition 4.

7 The Group Related to the Contraction Operad and Its Representation in the Group of Formal Diffeomorphisms on the Space of Interactions

Having defined a symmetric operad $\mathfrak{R}$ for each particular QFT model we have automatically a group $\mathfrak{G}(\mathfrak{R})^{\mathbf{S}}$ associated to it. This group is precisely the operadic construction of the renormalization group.

7.1 Notions of Renormalization Group

There are several widespread notions of renormalization group in physics and they do not lead to equal objects although they are closely related to each other. We shall review below some of them. For recent related works we refer the reader to [1, 9].

In renormalization theory a physical quantity U (an observable for instance, or a correlation function in QFT) is derived as a function $U = U(\kappa_1,\ldots,\kappa_N;\varepsilon)$ ($\equiv U(\vec{\kappa};\varepsilon)$) of various parameters including:

- Physical constants $\kappa_1,\ldots,\kappa_N$. In QFT these are called coupling constants.
- An additional subsidiary parameter $\varepsilon > 0$ called a regularization parameter. It makes meaningful the value of $U(\kappa_1,\ldots,\kappa_N;\varepsilon)$ that is usually ill-defined for $\varepsilon \to 0$. The latter limit corresponds exactly to the actual physical value of U and the purpose of the renormalization is to understand how to do it.

- There might be further variables but we consider them as a "part" of U (so that U is then valued in some vector or function space).

Furthermore, in perturbation theory, one has defined U only as a formal power series in the coupling constants

$$U(\vec{\kappa};\varepsilon) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^{N} U_{i_1,\dots,i_n}(\varepsilon)\, \kappa_{i_1} \cdots \kappa_{i_n}\,, \tag{29}$$

with coefficients $U_{i_1,\dots,i_n}(\varepsilon)$ that are functions in $\varepsilon > 0$. The renormalization issue now is to find such a change of the physical parameters:

$$\vec{\kappa}\,' = \vec{\mathrm{K}}(\vec{\kappa};\varepsilon), \qquad \kappa_i' = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^{N} \mathrm{K}_{i;i_1,\dots,i_n}(\varepsilon)\, \kappa_{i_1} \cdots \kappa_{i_n}\,, \tag{30}$$

again as a formal power series, so that after the substitution[5]

$$U^{\mathrm{ren}}(\vec{\kappa};\varepsilon) := U\big(\vec{\mathrm{K}}(\vec{\kappa};\varepsilon);\varepsilon\big) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^{N} U^{\mathrm{ren}}_{i_1,\dots,i_n}(\varepsilon)\, \kappa_{i_1} \cdots \kappa_{i_n}\,, \tag{31}$$

the resulting coefficients $U^{\mathrm{ren}}_{i_1,\dots,i_n}(\varepsilon)$ would have a finite limit for $\varepsilon \to 0$. We set the final renormalized physical quantity U^{ren} to be

$$U^{\mathrm{ren}}(\vec{\kappa}) := \lim_{\varepsilon \to 0} U^{\mathrm{ren}}(\vec{\kappa};\varepsilon)\,. \tag{32}$$

The existence of such a formal diffeomorphism $\vec{\kappa}\,' = \vec{\mathrm{K}}(\vec{\kappa};\varepsilon)$ (30) for a given in advance series $U(\vec{\kappa};\varepsilon)$ (29) so that the limit (32) exists is far from being a trivial statement. This phenomena is called *renormalizability* of U. The physical interpretation of this procedure is that we pass by the change (30) to a new set of coupling constants called "renormalized couplings" so that the initial "bare couplings" become infinite (meaningless) for $\varepsilon \to 0$.

Still, the above renormalization procedure has a built in ambiguity. Namely, if we have one solution $\vec{\mathrm{K}}(\vec{\kappa};\varepsilon)$ (30) of this problem then any composition

$$\vec{\mathrm{K}}_1(\vec{\kappa};\varepsilon) = \vec{\mathrm{K}}\big(\vec{\mathrm{X}}(\vec{\kappa});\varepsilon\big)$$

with a formal diffeomorphism $\vec{\mathrm{X}}(\vec{\kappa})$ will also be a solution. Thus, the group of formal diffeomorphisms of the couplings $\vec{\kappa}$ appears naturally as acting on the

[5]In terms of formal power series; note that the series $\vec{\mathrm{K}}(\vec{\kappa};\varepsilon)$ starts from $n=1$ but for $U(\vec{\kappa};\varepsilon)$ we do not have such a restriction.

renormalization schemes. This is the first notion of a renormalization group. It is simply the group of formal diffeomorphism.

We see that the above concept of renormalization is rather general. It leads also to the most primary concept of a renormalization group and so, it should be related to any other such notion. More precisely, any other notion of a renormalization group should have a representation (a homomorphism) in the group of formal diffeomorphisms of the coupling constants. In this case we speak about "renormalization group action", i.e., it is an action of the corresponding group by formal diffeomorphisms of the couplings.

We pass now to a second notion of the renormalization group that is specific for QFT and it is finer than the above one. In QFT there are additional technical features of the renormalization procedure. Namely, each of the terms $U_{i_1,\dots,i_n}(\varepsilon)$ in series (29) is additionally expanded in a finite sum labeled by a Feynman graph with n vertices. The renormalization adds to every diagram contribution a counter-term together with recursively determined counter-terms for subdiagrams. Without going more into the details we will only mention that the ambiguity in the renormalization in QFT is described exactly by contraction maps introduced in Sect. 5. So, we obtain now a finer notion of renormalization group that is formed by sequences of contraction maps. One further shows that the composition in this group is exactly given by the rule following from the operadic structure on contraction maps. The latter is shown in [8, Sect. 2.6] in a more general context of renormalization than the graph-combinatorial one.

Thus, from this second perspective the renormalization group appears exactly as a group related to the contraction operad on Feynman diagrams. Then, as explained above, there should be related a "renormalization group action", i.e., a homomorphism from this group to the group of formal diffeomorphisms of the couplings. The existence and the derivation of this homomorphism follow also from the general renormalization theory and are not a part of the present work. However, our result is that the resulting homomorphism corresponds to an operadic morphism via the functor established in Theorem 3.4. Let us summarize all this:

There is an operadic morphism, $\Xi : \mathfrak{R}_{\mathscr{E},\mathscr{V}} \to \mathrm{End}_{\mathbb{R}^{\mathscr{V}}}$, from the contraction operad to the operad $\mathrm{End}_{\mathbb{R}^{\mathscr{V}}}$ over the vector space spanned by the set of type of vertices $\mathscr{V}$. The latter set indexes the set of coupling constants in the QFT model that is determined by the combinatorial data $(\mathscr{E},\mathscr{V})$. The induced map between the related groups

$$\mathfrak{G}(\Xi)^{\mathbf{S}} : \mathfrak{G}(\mathfrak{R}_{\mathscr{E},\mathscr{V}})^{\mathbf{S}} \to \mathfrak{G}(\mathrm{End}_{\mathbb{R}^{\mathscr{V}}})^{\mathbf{S}} \cong \mathrm{FDiff}\big(\mathbb{R}^{\mathscr{V}}\big) \tag{33}$$

coincides with the renormalization group action determined from the renormalization theory.

In the subsequent subsections we will construct the morphism $\Xi : \mathfrak{R}_{\mathscr{E},\mathscr{V}} \to \mathrm{End}_{\mathbb{R}^{\mathscr{V}}}$. We shall continue our considerations on a general ground field (ring) $\mathbb{K}$ but the above application uses the case $\mathbb{K} = \mathbb{R}$.

7.2 Bosons and Fermions

We introduce a subdivision of the set of fields, i.e. the set Colf of flags' colors, into two disjoint subsets called bosons and fermions. According to this we assign $(\mathbb{Z}/2\mathbb{Z})$-parities to the set of generators (17) of the monoid $\mathfrak{M}(n)$. For a bosonic ϕ the element $\phi(i)$ is even and for fermionic ϕ, $\phi(i)$ is odd. The parity of the propagator $C_{\phi,\psi}(i,j)$ is the sum of the parities of the coupled fields ϕ and ψ. Usually bosons are coupled only to bosons and fermions—to fermions, so that the propagators are then always even. Finally, the interactions $L(i)$ are even as well.

Recall that we introduced in Sect. 4 g a canonical isomorphism $\mathrm{Dgm}(n) \cong \mathfrak{M}(n)$ between the set $\mathrm{Dgm}(n)$ of all classes of isomorphic enumerated colored diagrams with n vertices and the elements in the free monoid $\mathfrak{M}(n)$ generated by the set (17). Let us introduce the linear envelope of the monoid $\mathfrak{M}(n)$:

$$\mathscr{M}(n) := \mathbb{K}^{(\mathfrak{M}(n))} \cong \mathbb{K}^{(\mathrm{Dgm}(n))}, \tag{34}$$

which is thus an algebra.[6] In the more general case of presence of fermions we redefine the algebra structure on $\mathscr{M}(n)$ (34) and set

$$\mathscr{M}(n) := \text{the graded commutative algebra generated by the set (17).} \tag{35}$$

Note that in all the constructions up to now the division of the fields (i.e., the set Colf) into bosons and fermions is inessential.

7.3 The Wick Generating Operator of Diagrams

Let us assume first that we have a theory only with bosons so that the algebras $\mathscr{M}(n)$ are commutative.

Let us have n vertices $I_1,\dots,I_n \in \mathscr{V}$ and consider them as one enumerated colored graph that is completely disconnected (i.e., it has no inner lines). The monomial in $\mathfrak{M}(n)$ corresponding to this diagram is thus $I_1(1)\cdots I_n(n) \equiv I_1 \otimes \cdots \otimes I_n$, where the number in bracket "(j)" indicates the number assigned to the corresponding vertex. Denote

$$\begin{aligned}\mathrm{Wick}_n^{\mathscr{E}}(I_1,\dots,I_n) :=& \sum \text{all possible ways of connecting the vertices} \\ & I_1(1),\dots,I_n(n) \text{ into } \mathscr{E}\text{-admissible enumerated colored} \\ & \text{graphs with no tadpoles} \\ =& \; I_1(1)\cdots I_n(n) + \cdots, \end{aligned} \tag{36}$$

[6] However, we remark that the algebra structure induced by the monoid structure of $\mathfrak{M}(n)$ is quite different from the algebra structure on the space of diagrams that is usually used in the Connes–Kreimer approach.

where $\mathscr{E} \subseteq \mathrm{Colf}^{\times 2}$ is a set of admissible connections as defined in Sect. 6. This defines us a multilinear map

$$\mathrm{Wick}_n^{\mathscr{E}} \,:\, \big(\mathbb{K}^{\mathscr{V}}\big)^{\times n} \to \mathscr{M}(n)\,.$$

Proposition 5 ([7]). *Under the isomorphism* $\mathrm{Dgm}(n) \cong \mathfrak{M}(n)$ *(Proposition 2) the following equation holds*

$$\mathrm{Wick}_n^{\mathscr{E}}(I_1,\ldots,I_n) = \left[\prod_{1\leqslant i<j\leqslant n} \exp\left(\sum_{(\phi,\psi)\in\mathscr{E}} C_{\phi,\psi}(i,j)\frac{\partial^2}{\partial\phi(i)\partial\psi(j)}\right)\right] I_1(1)\cdots I_n(n)\,. \quad (37)$$

In the presence of fermions (37) continues to generate the terms in the right hand side of (36) but with some signs that depend on the order of writing of the remaining generators of $\mathscr{M}(n)$. The derivatives $\frac{\partial}{\partial\phi(i)}$ are understood as left Grassmann derivatives for odd $\phi(i)$.

7.4 Construction of Operadic Morphism $\Xi : \mathfrak{R}_{\mathscr{E},\mathscr{V}} \to \mathrm{End}_{\mathbb{K}^{\mathscr{V}}}$

The operadic morphism $\Xi : \mathfrak{R}_{\mathscr{E},\mathscr{V}} \to \mathrm{End}_{\mathbb{K}^{\mathscr{V}}}$ consists of a sequence of linear maps

$$\Xi_n \,:\, \mathfrak{R}_{\mathscr{E},\mathscr{V}}(n) \to \mathrm{End}_{\mathbb{K}^{\mathscr{V}}}(n) \equiv \mathrm{Hom}\big(\big(\mathbb{K}^{\mathscr{V}}\big)^{\otimes n},\mathbb{K}^{\mathscr{V}}\big)\,. \quad (38)$$

The Ansatz for Ξ_n is

$$\Xi_n(Q)(I_1\otimes\cdots\otimes I_n) = \widehat{Q}\Big(\mathrm{Wick}_n^{\mathscr{E}}(I_1,\ldots,I_n)\Big) \in \mathbb{K}^{\mathscr{V}}\,, \quad (39)$$

where $Q \in \mathfrak{R}_{\mathscr{E},\mathscr{V}}(n) \subseteq \mathfrak{R}(n)$ is generally given by (19) and $\widehat{Q}$ is then set to be

$$\widehat{Q}(\Gamma) = \sum_{L\in\mathrm{Colv}} q(\Gamma,L)\,\big[\Gamma\big/(\mathrm{vert}(\Gamma),L)\big]\,, \qquad \widehat{Q} : \mathbb{K}^{(\mathrm{Dgm}(n))} \to \mathbb{K}^{\mathrm{Dgm}(1)}\,, \quad (40)$$

i.e., $\widehat{Q}(\Gamma)$ contracts the diagram Γ to a sum of single vertices according to the color prescription of $Q : \mathbb{K}^{(\mathrm{Dgm}(n))} \to \mathbb{K}^{(\mathrm{Colv})}$.

Let us explain by words the meaning of (38). The value of $\Xi_n(Q)(I_1\otimes\cdots\otimes I_n)$ is a sum of single vertices obtained by making first a sum over all possible ways of connecting the vertices $I_1(1),\ldots,I_n(n)$ into enumerated diagrams; then we contract each of the terms in the latter sum to a sum of single vertices via Q. Shortly speaking, $\Xi_n(Q)(I_1\otimes\cdots\otimes I_n)$ is the Q-contraction of all possible connections of $I_1,\ldots,I_n$ into diagrams.

Proposition 6 ([7]). *Equation (40) determines an operadic morphism.*

8 Outlook

We make here a connection with the Connes–Kreimer Hopf algebra of “formal diffeographisms” [3], which in details will appear in a forthcoming work.

The first step towards the comparison with the Connes–Kreimer approach is to study the dual (commutative) Hopf algebra to the Lie algebra associated with a symmetric operad. In fact, it can be associated directly to a symmetric *co-operad*. When this construction is applied to the contraction operads on diagrams we obtain a Hopf algebra that is very close to the Connes–Kreimer Hopf algebra. However, there is an important difference. On a technical level, in our approach a subdiagram is always contracted to a vertex, while in the Connes–Kreimer theory some subdiagrams that have two external lines can be contracted also to an edge with no intermediate vertex.

The origin for this difference comes from physics. The Connes–Kreimer Hopf algebra incorporates an additional step in the renormalization called a “field renormalization”. Let us briefly explain this. Our set of vertices $\mathscr{V}$ corresponds to all the monomials in the Lagrangian of a given QFT model. Some of these vertices of valence two correspond to quadratic terms in the Lagrangian, which are called “kinetic terms” since they basically determine the propagators. For this reason in physics there are no physical parameters related to these terms: we always normalize them with some standard normalization coefficients like

$$\frac{1}{2}(\partial\phi)\cdot(\partial\phi), \qquad \overline{\psi}(\gamma\cdot\partial)\psi,$$

for a scalar and a spinor field, respectively ($\gamma\cdot\partial$ being the Dirac operator). On the other hand, as a result of the renormalization the coefficients in front of these kinetic terms are changed (renormalized). Then we absorb this change by a redefinition of the field strengths. For instance, in the above examples we pass to new fields $\phi' = Z_\phi\,\phi$, $\psi' = Z_\psi\,\psi$ and $\overline{\psi}' = \overline{Z}_\psi\overline{\psi}$ so that the kinetic terms are changed by Z_ϕ^2 and $\overline{Z}_\psi Z_\psi$, respectively, in such a way that compensate the renormalization change.

Acknowledgements We thank Dorothea Bahns, Kurusch Ebrahimi-Fard, Alessandra Frabetti, Klaus Fredenhagen and Raymond Stora for fruitful discussions. The work was partially supported by the French-Bulgarian Project Rila under the contract Egide-Rila N112 and by grant DO 02-257 of the Bulgarian National Science Foundation. N.N. thanks the Courant Research Center “Higher order structures in mathematics” (Göttingen) and the II. Institute for Theoretical Physics at the University of Hamburg for support and hospitality.”

References

1. Brunetti, R., Duetsch, M., Fredenhagen, K.: Adv. Theor. Math. Phys. **13**, 1–56 (2009)
2. Chapoton, F.: Rooted trees and an exponential-like series [arXiv:math/0209104]
3. Connes, A., Kreimer, D.: Comm. Math. Phys. **199**, 203–242 (1998)

4. Frabetti, A.: J. Algebra**319**(1), 377–413 (2008)
5. Lang, S.: Complex analysis, 4th edn. In: Graduate Texts in Mathematics, vol. 103. Springer, New York, (1999)
6. Loday, J.-L., Vallette, B.: Algebraic operads. In: Grundlehren Math. Wiss. vol. 346. Springer, Heidelberg (2012)
7. Loday, J.-L., Nikolov, N.M.: Renormalizationfrom the operadic point of view (in preparation)
8. Nikolov, N.M.: Anomalies in quantum field theory and cohomologies of configuration spaces [arXiv:0903.0187]
9. Stora, R.: Causalité et Groupesde Renormalisation Perturbatifs. In: Boudjedaa, T., Makhlouf, A. (eds.) Théorie Quantique des Champs Méthode et Applications, p. 67 (2007)

Part III
String and Gravity Theories

Lightlike Braneworlds in Anti-de Sitter Bulk Space-Times

Eduardo Guendelman, Alexander Kaganovich, Emil Nissimov, and Svetlana Pacheva

Abstract We consider five-dimensional Einstein–Maxwell–Kalb–Ramond system self-consistently coupled to a *lightlike* 3-brane, where the latter acts as material, charge and variable cosmological constant source. We find wormhole-like solutions whose total space-time manifold consists of either (a) two "universes", which are identical copies of the exterior space-time region (beyond the horizon) of five-dimensional Schwarzschild–anti-de Sitter black hole, or (b) a "right" "universe" comprising the exterior space-time region of Reissner–Nordström–anti-de Sitter black hole and a "left" "universe" being the Rindler "wedge" of five-dimensional flat Minkowski space. The wormhole "throat" connecting these "universes", which is located on their common horizons, is self-consistently occupied by the lightlike 3-brane as a direct result of its dynamics given by an explicit reparametrization-invariant world-volume Lagrangian action. The intrinsic world-volume metric on the 3-brane turns out to be flat, which allows its interpretation as a *lightlike* braneworld.

1 Introduction

Lightlike branes (*"LL-branes"* for short) play an important role in modern general relativity. *LL-branes* are singular null (lightlike) hypersurfaces in Riemannian space-time which provide dynamical description of various physically important phenomena in cosmology and astrophysics such as: (a) impulsive lightlike signals

E. Guendelman • A. Kaganovich
Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva, Israel
e-mail: guendel@bgu.ac.il; alexk@bgu.ac.il

E. Nissimov (✉) • S. Pacheva
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria
e-mail: nissimov@inrne.bas.bg; svetlana@inrne.bas.bg

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_14,

arising in cataclysmic astrophysical events (supernovae, neutron star collisions) [1]; (b) dynamics of horizons in black hole physics—the so called "membrane paradigm" [2]; (c) the thin-wall approach to domain walls coupled to gravity [3–6]. More recently, *LL-branes* became significant also in the context of modern non-perturbative string theory [7–10].

In our previous papers [11–20] we have provided an explicit reparametrization invariant world-volume Lagrangian formulation of lightlike p-branes (a brief review is given in Sect. 2) and we have used them to construct various types of wormhole, regular black hole and lightlike braneworld solutions in $D=4$ or higher-dimensional asymptotically flat or asymptotically anti-de Sitter bulk space-times (for a detailed account of the general theory of wormholes see the book [21] and also [22–28]). In particular, in [18–20] we have shown that lightlike branes can trigger a series of spontaneous compactification–decompactification transitions of space-time regions, e.g., from ordinary compactified ("tube-like") Levi–Civita–Bertotti–Robinson [29–31] space to non-compact Reissner–Nordström or Reissner–Nordström–de-Sitter region or *vice versa*. Let us note that wormholes with "tube-like" structure (and regular black holes with "tube-like" core) have been previously obtained within different contexts in [32–40].

Let us emphasize the following characteristic features of *LL-branes* which drastically distinguish them from ordinary Nambu–Goto branes:

(a) They describe intrinsically lightlike modes, whereas Nambu–Goto branes describe massive ones.
(b) The tension of the *LL-brane* arises as an *additional dynamical degree of freedom*, whereas Nambu–Goto brane tension is a given *ad hoc* constant. The latter characteristic feature significantly distinguishes our *LL-brane* models from the previously proposed *tensionless* p-branes (for a review of the latter, see [41]) which rather resemble a p-dimensional continuous distribution of massless point-particles.
(c) Consistency of *LL-brane* dynamics in a spherically or axially symmetric gravitational background of codimension one requires the presence of a horizon which is automatically occupied by the *LL-brane* ("horizon straddling" according to the terminology of [4]).
(d) When the *LL-brane* moves as a *test* brane in spherically or axially symmetric gravitational backgrounds its dynamical tension exhibits exponential "inflation/deflation" time behavior [42]—an effect similar to the "mass inflation" effect around black hole horizons [43, 44].

Here we will focus on studying four-dimensional lightlike braneworlds in 5-dimensional bulk anti-de Sitter spaces—an alternative to the standard Randall–Sundrum scenario [45, 46] (for a systematic overview to braneworld theory, see [47–49]). Namely, we will present explicit solutions of five-dimensional Einstein–Maxwell–Kalb–Ramond system self-consistently interacting with codimension-one *LL-branes*, which are special kinds of *"wormhole"-like* space-times of either one of the following structures:

(A) Two "universes" which are identical copies of the exterior space-time region (beyond the horizon) of five-dimensional Schwarzschild–anti-de Sitter black hole.
(B) "Right" "universe" comprising the exterior space-time region of Reissner–Nordström–anti-de Sitter black hole and "left" "universe" being the Rindler "wedge" of five-dimensional flat Minkowski space.

Both "right" and "left" "universes" in (A)–(B) are glued together along their common horizons occupied by the *LL-brane* with *flat* four-dimensional intrinsic world-volume metric, in other words, a flat lightlike braneworld (*LL-braneworld*) at the wormhole "throat". In case (A) the *LL-brane* is electrically neutral whereas in case (B) it is both electrically charged as well as it couples also to a bulk Kalb–Ramond tensor gauge field.

2 Lagrangian Formulation of Lightlike Brane Dynamics

In what follows we will consider gravity/gauge-field system self-consistently interacting with a lightlike p-brane of codimension one ($D=(p+1)+1$). In a series of previous papers [11–20] we have proposed manifestly reparametrization invariant world-volume Lagrangian formulation in several dynamically equivalent forms of *LL-branes* coupled to bulk gravity $G_{\mu\nu}$ and bulk gauge fields, in particular, Maxwell A_μ and Kalb–Ramond $A_{\mu_1\ldots\mu_{D-1}}$. Here we will use our Polyakov-type formulation given by the world-volume action:

$$S_{\rm LL}[q,\beta] = -\frac{1}{2}\int d^{p+1}\sigma\, T b_0^{\frac{p-1}{2}}\sqrt{-\gamma}\left[\gamma^{ab}\bar{g}_{ab} - b_0(p-1)\right], \tag{1}$$

$$-\frac{\beta}{(p+1)!}\int d^{p+1}\sigma\, \varepsilon^{a_1\ldots a_{p+1}}\partial_{a_1}X^{\mu_1}\ldots\partial_{a_{p+1}}X^{\mu_{p+1}}A_{\mu_1\ldots\mu_{p+1}} \tag{2}$$

where:

$$\bar{g}_{ab} \equiv \partial_a X^\mu G_{\mu\nu}\partial_b X^\nu - \frac{1}{T^2}(\partial_a u + qA_a)(\partial_b u + qA_b), \quad A_a \equiv \partial_a X^\mu A_\mu \,. \tag{3}$$

Here and below the following notations are used:

- $X^\mu(\sigma)$ are the p-brane embedding coordinates in the bulk D-dimensional space-time with Riemannian metric $G_{\mu\nu}(x)$ ($\mu,\nu = 0,1,\ldots,D-1$); $(\sigma)\equiv\left(\sigma^0\equiv\tau,\sigma^i\right)$ with $i=1,\ldots,p$; $\partial_a \equiv \frac{\partial}{\partial\sigma^a}$.
- γ_{ab} is the *intrinsic* Riemannian metric on the world-volume with $\gamma = \det\|\gamma_{ab}\|$; g_{ab} is the *induced* metric on the world-volume:

$$g_{ab} \equiv \partial_a X^\mu G_{\mu\nu}(X)\partial_b X^\nu \,, \tag{4}$$

which becomes *singular* on-shell (manifestation of the lightlike nature), cf. (9) below; b_0 is a positive constant measuring the world-volume "cosmological constant".

- u is auxiliary world-volume scalar field defining the lightlike direction of the induced metric (see (9) below) and it is a non-propagating degree of freedom (cf. [20]).
- T is *dynamical (variable)* brane tension (also a non-propagating degree of freedom).
- The coupling parameters q and β are the electric surface charge density and the Kalb–Rammond charge of the *LL-brane*, respectively.

The corresponding equations of motion w.r.t. X^μ, u, γ_{ab} and T read accordingly (using short-hand notation (3)):

$$\partial_a\left(T\sqrt{|\bar{g}|}\bar{g}^{ab}\partial_b X^\mu\right)+T\sqrt{|\bar{g}|}\bar{g}^{ab}\partial_a X^\lambda\partial_b X^\nu\Gamma^\mu_{\lambda\nu}$$
$$+\frac{q}{T}\sqrt{|\bar{g}|}\bar{g}^{ab}\partial_a X^\nu(\partial_b u+qA_b)F_{\lambda\nu}G^{\mu\lambda}$$
$$-\frac{\beta}{(p+1)!}\varepsilon^{a_1\ldots a_{p+1}}\partial_{a_1}X^{\mu_1}\ldots\partial_{a_{p+1}}X^{\mu_{p+1}}F_{\lambda\mu_1\ldots\mu_{p+1}}G^{\lambda\mu}=0, \tag{5}$$

$$\partial_a\left(\frac{1}{T}\sqrt{|\bar{g}|}\bar{g}^{ab}(\partial_b u+qA_b)\right)=0,\quad \gamma_{ab}=\frac{1}{b_0}\bar{g}_{ab}, \tag{6}$$

$$T^2+\bar{g}^{ab}(\partial_a u+qA_a)(\partial_b u+qA_b)=0. \tag{7}$$

Here $\bar{g}=\det\|\bar{g}_{ab}\|$, $\Gamma^\mu_{\lambda\nu}$ denotes the Christoffel connection for the bulk metric $G_{\mu\nu}$ and:

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\quad F_{\mu_1\ldots\mu_D}=D\partial_{[\mu_1}A_{\mu_2\ldots\mu_D]}=\mathscr{F}\sqrt{-G}\varepsilon_{\mu_1\ldots\mu_D} \tag{8}$$

are the corresponding gauge field strengths.

The on-shell singularity of the induced metric g_{ab} (4), i.e., the lightlike property, directly follows from (7) and the definition of $\bar{g}_{ab}$ (3):

$$g_{ab}\left(\bar{g}^{bc}(\partial_c u+qA_c)\right)=0. \tag{9}$$

Explicit world-volume reparametrization invariance of the *LL-brane* action (1) allows to introduce the standard synchronous gauge-fixing conditions for the intrinsic world-volume metric

$$\gamma^{00}=-1,\quad \gamma^{0i}=0\ \ (i=1,\ldots,p). \tag{10}$$

which reduces (6)–(7) to the following relations:

$$\frac{(\partial_0 u + qA_0)^2}{T^2} = b_0 + g_{00}, \quad \partial_i u + qA_i = (\partial_0 u + qA_0) g_{0i} (b_0 + g_{00})^{-1} \ ,$$

$$g_{00} = g^{ij} g_{0i} g_{0j}, \quad \partial_0 \left(\sqrt{g^{(p)}}\right) - \partial_i \left(\sqrt{g^{(p)}} g^{ij} g_{0j}\right) = 0 \ , \ g^{(p)} \equiv \det\|g_{ij}\| \ , \quad (11)$$

(recall that g_{00}, g_{0i}, g_{ij} are the components of the induced metric (4); g^{ij} is the inverse matrix of g_{ij}). Then, as shown in [11–20], consistency of *LL-brane* dynamics in static "spherically-symmetric"-type backgrounds (in what follows we will use Eddington–Finkelstein coordinates, $dt = dv - \frac{d\eta}{A(\eta)}$):

$$ds^2 = -A(\eta)dv^2 + 2dvd\eta + C(\eta)h_{ij}(\theta)d\theta^i d\theta^j \ ,$$
$$F_{v\eta} = F_{v\eta}(\eta), \ \text{rest} = 0, \quad \mathscr{F} = \mathscr{F}(\eta), \quad (12)$$

with the standard embedding Ansatz:

$$X^0 \equiv v = \tau, \quad X^1 \equiv \eta = \eta(\tau), \quad X^i \equiv \theta^i = \sigma^i \ \ (i = 1, \ldots, p). \quad (13)$$

requires the corresponding background (12) to possess a horizon at some $\eta = \eta_0$, which is automatically occupied by the *LL-brane*.

Indeed, in the case of (12)–(13) (11) reduce to:

$$g_{00} = 0, \ \ \partial_0 C\big(\eta(\tau)\big) \equiv \dot{\eta} \ \partial_\eta C \,\big|_{\eta=\eta(\tau)} = 0, \ \ \frac{(\partial_0 u + qA_0)^2}{T^2} = b_0, \ \ \partial_i u = 0 \quad (14)$$

($\dot{\eta} \equiv \partial_0 \eta \equiv \partial_\tau \eta(\tau)$). Thus, in the generic case of non-trivial dependence of $C(\eta)$ on the "radial-like" coordinate η, the first two relations in (14) yield:

$$\dot{\eta} = \frac{1}{2} A\big(\eta(\tau)\big), \quad \dot{\eta} = 0 \quad \rightarrow \quad \eta(\tau) = \eta_0 = \text{const}, \quad A(\eta_0) = 0 \ . \quad (15)$$

The latter property is called "horizon straddling" according to the terminology of [4]. Similar "horizon straddling" has been found also for *LL-branes* moving in rotating axially symmetric (Kerr or Kerr-Newman) and rotating cylindrically symmetric black hole backgrounds [16, 17].

3 Gravity/Gauge-Field System Interacting with Lightlike Brane

The generally covariant and manifestly world-volume reparametrization-invariant Lagrangian action describing a bulk Einstein–Maxwell–Kalb–Ramond system (with bulk cosmological constant Λ) self-consistently interacting with a codimension-one *LL-brane* is given by:

$$S = \int d^D x \sqrt{-G} \left[\frac{R(G) - 2\Lambda}{16\pi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{D!2} F_{\mu_1 \ldots \mu_D} F^{\mu_1 \ldots \mu_D} \right] + S_{\rm LL}[q, \beta] \, , \tag{16}$$

where again $F_{\mu\nu}$ and $F_{\mu_1 \ldots \mu_D}$ are the Maxwell and Kalb–Ramond field-strengths (8) and $S_{\rm LL}[q,\beta]$ indicates the world-volume action of the *LL-brane* of the form (1). It is now the *LL-brane* which will be the material and charge source for gravity and electromagnetism, as well as it will generate dynamically an additional space-varying bulk cosmological constant (see (20) and second relation (28) below).

The equations of motion resulting from (16) read:

(a) Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R + \Lambda G_{\mu\nu} = 8\pi \left(T^{(EM)}_{\mu\nu} + T^{(KR)}_{\mu\nu} + T^{(\rm brane)}_{\mu\nu} \right) \, . \tag{17}$$

(b) Maxwell equations:

$$\partial_\nu \left[\sqrt{-G} F_{\kappa\lambda} G^{\mu\kappa} G^{\nu\lambda} \right] + j^\mu_{(\rm brane)} = 0 \, . \tag{18}$$

(c) Kalb–Ramond equations (recall definition of $\mathscr{F}$ in (8)):

$$\varepsilon^{\nu\mu_1 \ldots \mu_{p+1}} \partial_\nu \mathscr{F} - J^{\mu_1 \ldots \mu_{p+1}}_{(\rm brane)} = 0 \, . \tag{19}$$

(d) The *LL-brane* equations of motion have already been written down in (5)–(7) above.

The energy-momentum tensors of bulk gauge fields are given by:

$$T^{(EM)}_{\mu\nu} = F_{\mu\kappa} F^{\mu\nu} - G_{\mu\nu} \frac{1}{4} F_{\kappa\lambda} F^{\kappa\lambda}, \quad T^{(KR)}_{\mu\nu} = -\frac{1}{2} \mathscr{F}^2 G_{\mu\nu}, \tag{20}$$

where the last relation indicates that $\Lambda \equiv 4\pi \mathscr{F}^2$ can be interpreted as dynamically generated cosmological "constant".

The energy-momentum (stress-energy) tensor $T^{(\rm brane)}_{\mu\nu}$ and the electromagnetic $j^\mu_{(\rm brane)}$ and Kalb–Ramond $J^{\mu_1 \ldots \mu_{p+1}}_{(\rm brane)}$ charge current densities of the *LL-brane* are straightforwardly derived from the pertinent *LL-brane* action (1):

$$T^{\mu\nu}_{(\rm brane)} = -\int d^{p+1}\sigma \, \frac{\delta^{(D)}\Big(x - X(\sigma)\Big)}{\sqrt{-G}} T \sqrt{|\bar{g}|} \bar{g}^{ab} \partial_a X^\mu \partial_b X^\nu \, , \tag{21}$$

$$j^\mu_{(\rm brane)} = -q \int d^{p+1}\sigma \, \delta^{(D)}\Big(x - X(\sigma)\Big) \sqrt{|\bar{g}|} \bar{g}^{ab} \partial_a X^\mu \left(\partial_b u + q A_b \right) T^{-1} \, , \tag{22}$$

$$J^{\mu_1 \ldots \mu_{p+1}}_{(\rm brane)} = \beta \int d^{p+1}\sigma \, \delta^{(D)}\Big(x - X(\sigma)\Big) \varepsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} X^{\mu_1} \ldots \partial_{a_{p+1}} X^{\mu_{p+1}} \, . \tag{23}$$

Construction of "wormhole"-like solutions of static "spherically-symmetric"-type (12) for the coupled gravity-gauge-field-*LL-brane* system (16) proceeds along the following simple steps:

(a) Choose "vacuum" static "spherically-symmetric"-type solutions (12) of (17)–(19) (i.e., without the delta-function terms due to the *LL-branes*) in each region $-\infty < \eta < \eta_0$ and $\eta_0 < \eta < \infty$ with a common horizon at $\eta = \eta_0$.
(b) The *LL-brane* automatically locates itself on the horizon according to "horizon straddling" property (15).
(c) Match the discontinuities of the derivatives of the metric and the gauge field strength (12) across the horizon at $\eta = \eta_0$ using the explicit expressions for the *LL-brane* stress-energy tensor, electromagnetic and Kalb–Ramond charge current densities (21)–(23).

Using (11)–(13) we find:

$$T^{\mu\nu}_{(\text{brane})} = S^{\mu\nu}\,\delta(\eta-\eta_0), \quad j^{\mu}_{(\text{brane})} = \delta^{\mu}_{0}\, q\sqrt{\det\|G_{ij}\|}\,\delta(\eta-\eta_0),$$
$$\frac{1}{(p+1)!}\varepsilon_{\mu\nu_1\ldots\nu_{p+1}} J^{\nu_1\ldots\nu_{p+1}}_{(\text{brane})} = \beta\,\delta^{\eta}_{\mu}\,\delta(\eta-\eta_0) \tag{24}$$

where $G_{ij} = C(\eta)h_{ij}(\theta)$ (cf. (12)) and the surface energy-momentum tensor reads:

$$S^{\mu\nu} \equiv \frac{T}{b_0^{1/2}}\left(\partial_\tau X^\mu \partial_\tau X^\nu - b_0 G^{ij}\partial_i X^\mu \partial_j X^\nu\right)_{v=\tau,\,\eta=\eta_0,\,\theta^i=\sigma^i}\ . \tag{25}$$

The non-zero components of $S_{\mu\nu}$ (with lower indices) and its trace are:

$$S_{\eta\eta} = \frac{T}{b_0^{1/2}}, \quad S_{ij} = -Tb_0^{1/2}G_{ij}, \quad S^{\lambda}_{\lambda} = -pTb_0^{1/2}. \tag{26}$$

Taking into account (24)–(26) together with (12)–(15), the matching relations at the horizon $\eta = \eta_0$ become [18–20] (for a systematic introduction to the formalism of matching different bulk space-time geometries on codimension-one hypersurfaces ("thin shells") see the textbook [50]):

(a) Matching relations from Einstein equations (17):

$$\left[\partial_\eta A\right]_{\eta_0} = -16\pi T\sqrt{b_0}, \quad \left[\partial_\eta \ln C\right]_{\eta_0} = -\frac{16\pi}{p\sqrt{b_0}}T \tag{27}$$

with notation $\left[Y\right]_{\eta_0} \equiv Y\big|_{\eta\to\eta_0+0} - Y\big|_{\eta\to\eta_0-0}$ for any quantity Y.
(b) Matching relation from gauge field equations (18)–(19):

$$\left[F_{v\eta}\right]_{\eta_0} = q, \quad \left[\mathscr{F}\right]_{\eta_0} = -\beta. \tag{28}$$

(c) X^0-equation of motion of the *LL-brane* (the only non-trivial contribution of second-order *LL-brane* (5) in the case of embedding (13)):

$$\frac{T}{2}\left(\left\langle\partial_\eta A\right\rangle_{\eta_0}+pb_0\left\langle\partial_\eta \ln C\right\rangle_{\eta_0}\right)-\sqrt{b_0}\left(q\left\langle F_{v\eta}\right\rangle_{\eta_0}-\beta\left\langle\mathscr{F}\right\rangle_{\eta=\eta_0}\right)=0 \quad (29)$$

with notation $\langle Y\rangle_{\eta_0}\equiv\frac{1}{2}\left(Y\,|_{\eta\to\eta_0+0}+Y\,|_{\eta\to\eta_0-0}\right)$.

4 Explicit Solutions: Braneworlds via Lightlike Brane

Consider five-dimensional AdS–Schwarzschild black hole in Eddington–Finkelstein coordinates $(v,r,\mathbf{x})$ (with $\mathbf{x}\equiv(x^1,x^2,x^3)$):

$$ds^2=-A(r)dv^2+2dv\,dr+Kr^2d\mathbf{x}^2,\quad A(r)=Kr^2-m/r^2\,, \quad (30)$$

where $\Lambda=-6K$ is the bare negative five-dimensional cosmological constant and m is the mass parameter of the black hole. The pertinent horizon is located at:

$$A(r_0)=0\to r_0=(m/K)^{1/4},\ \text{where}\ \partial_r A(r_0)>0\,. \quad (31)$$

First, let us consider self-consistent Einstein-*LL-brane* system (16) with a neutral *LL-brane* source (i.e. no *LL-brane* couplings to bulk Maxwell and Kalb–Ramond gauge fields: $q,\beta=0$ in $S_{\rm LL}[q,\beta]$). A simple trick to obtain *"wormhole"*-like solution to this coupled system is to change variables in (30):

$$r\to r(\eta)=r_0+|\eta| \quad (32)$$

with r_0 being the AdS–Schwarzschild horizon (31), where now $\eta\in(-\infty,+\infty)$, i.e., consider:

$$ds^2=-A(\eta)dv^2+2dv\,d\eta+C(\eta)d\mathbf{x}^2, \quad (33)$$

$$A(\eta)=K(r_0+|\eta|)^2-\frac{m}{(r_0+|\eta|)^2},\quad C(\eta)=K(r_0+|\eta|)^2, \quad (34)$$

$$A(0)=0,\quad A(\eta)>0\ \text{for}\ \eta\neq 0\,.$$

Obviously, (32) is *not* a smooth local coordinate transformation due to $|\eta|$. The coefficients of the new metric (33)–(34) are continuous at the horizon $\eta_0=0$ with discontinuous first derivatives across the horizon. The *LL-brane* automatically locates itself on the horizon according to the "horizon-straddling" property of its world-volume dynamics (15).

Substituting (33)–(34) into the matching relations (27)–(29) we find the following relation between bulk space-time parameters $(K = |\Lambda|/6, m)$ and the *LL-brane* parameters (T, b_0) :

$$T^2 = \frac{3}{8\pi^2}K, \quad T < 0, \quad b_0 = \frac{2}{3}\sqrt{Km}. \tag{35}$$

Taking into account second equation (6) and (10) the intrinsic metric γ_{ab} on the *LL-braneworld* becomes *flat*:

$$\gamma_{00} = -1, \quad \gamma_{0i} = 0, \quad \gamma_{ij} = \frac{3}{2}\delta_{ij} \,. \tag{36}$$

The solution (34)–(36) describes a “wormhole”-like $D = 5$ bulk space-time consisting of two “universes” being identical copies of the exterior region beyond the horizon $(r > r_0)$ of the five-dimensional AdS–Schwarzschild black hole glued together along their common horizon (at $r = r_0$) by the *LL-brane*, i.e., the latter serving as a wormhole “throat”, which in turn can be viewed as a *LL-braneworld* with flat intrinsic geometry (36).

Let us now consider the five-dimensional AdS–Reissner–Nordström black hole (in Eddington–Finkelstein coordinates $(v, r, \mathbf{x})$):

$$ds^2 = -A(r)dv^2 + 2dv\,dr + Kr^2 d\mathbf{x}^2, \quad \Lambda = -6K \,,$$
$$A(r) = Kr^2 - \frac{m}{r^2} + \frac{Q}{r^4}, \quad F_{vr} = \sqrt{\frac{3}{4\pi}}\frac{Q}{r^3}. \tag{37}$$

We can construct, following the same procedure, another *non-symmetric* “wormhole”-like solution with a flat *LL-braneworld* occupying its “throat” provided the *LL-brane* is electrically charged and couples to bulk Kalb–Ramond gauge field, i.e., $q, \beta \neq 0$ in (16), (1). This solution describes:

(a) “Left” universe being a five-dimensional flat Rindler space-time—the Rindler “wedge” of $D = 5$ Minkowski space [51, 52] (here $|\eta| = X^2$, where X is the standard Rindler coordinate):

$$ds^2 = \eta dv^2 + 2dv\,d\eta + d\mathbf{x}^2, \quad \text{for } \eta < 0. \tag{38}$$

(b) “Right” universe comprising the exterior $D = 5$ space-time region of the AdS–Reissner–Nordström black hole beyond the *outer* AdS–Reissner–Nordström horizon r_0 ($A(r_0) = 0$ with $A(r)$ as in (37) and where again we apply the non-smooth coordinate change (32)):

$$ds^2 = -A(\eta)dv^2 + 2dv\,d\eta + K(r_0 + \eta)^2 d\mathbf{x}^2 \tag{39}$$

$$A(\eta) = K(r_0 + \eta)^2 - \frac{m}{(r_0 + \eta)^2} + \frac{Q^2}{(r_0 + \eta)^4} \tag{40}$$

$$F_{v\eta} = \sqrt{\frac{3}{4\pi}}\frac{Q}{(r_0 + \eta)^3}, \quad A(0) = 0, \;\; \partial_\eta A(0) > 0, \quad \text{for } \eta > 0. \tag{41}$$

All physical parameters of the "wormhole"-like solution (38)–(41) are determined in terms of (q,β)—the electric and Kalb–Ramond *LL-braneworld* charges:

$$m = \frac{3}{2\pi\beta^2}\left(1+\frac{2q^2}{\beta^2}\right), \quad Q^2 = \frac{9q^2}{2\pi\beta^6}, \quad |\Lambda| \equiv 6K = 4\pi\beta^2 \tag{42}$$

$$|T| = \frac{1}{8\pi}\sqrt{\frac{3}{2}\sqrt{K}+4\pi(\beta^2-q^2)},\; b_0 = \frac{1}{6\sqrt{K}}\left[1+\frac{8\pi}{3}\sqrt{K}(\beta^2-q^2)\right] \tag{43}$$

Here again $T < 0$. Let is stress the importance of the third relation in (42). Namely, the dynamically generated space-varying effective cosmological constant (cf. second (20)) through the Kalb–Ramond coupling of the *LL-brane* (cf. second matching relation in (28)) has zero value in the "right" AdS–Reissner–Nordström "universe" and has positive value $4\pi\beta^2$ in the "left" flat Rindler "universe" (38) compensating the negative bare cosmological constant Λ.

The intrinsic metric γ_{ab} on the *LL-braneworld* is again *flat*:

$$\gamma_{00} = -1, \quad \gamma_{0i} = 0, \quad \gamma_{ij} = \frac{1}{b_0}\delta_{ij} \tag{44}$$

5 Traversability and Trapping Near the Lightlike Braneworld

The "wormhole"-like solutions presented in the previous section share the following important properties:

(a) The *LL-braneworlds* at the wormhole "throats" represent "exotic" matter with $T < 0$, i.e., negative brane tension implying violation of the null-energy conditions as predicted by general wormhole arguments [21] (although the latter could be remedied via quantum fluctuations).

(b) The wormhole space-times constructed via *LL-branes* at their "throats" are *not* traversable w.r.t. the "laboratory" time of a static observer in either of the different "universes" comprising the pertinent wormhole space-time manifold since the *LL-branes* sitting at the "throats" look as black hole horizons to the static observer. On the other hand, these wormholes *are traversable* w.r.t. the *proper time* of a traveling observer.

Indeed, proper-time traversability can be easily seen by considering dynamics of test particle of mass m_0 ("traveling observer") in a wormhole background, which is described by the reparametrization-invariant world-line action:

$$S_{\text{particle}} = \frac{1}{2}\int d\lambda \left[\frac{1}{e}\,\dot{x}^\mu \dot{x}^\nu\, G_{\mu\nu} - e m_0^2\right]. \tag{45}$$

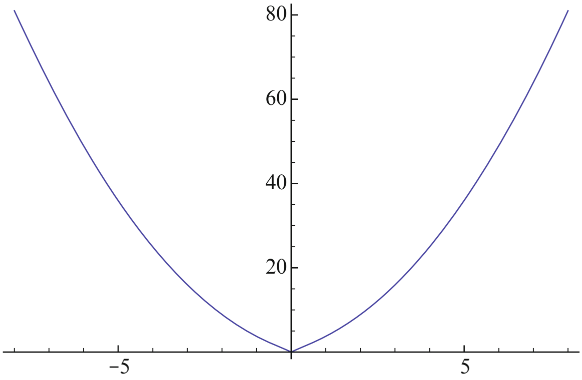


Fig. 1 Shape of the "effective potential" $\mathcal{V}_{\text{eff}}(\eta) = A(\eta)$ with $A(\eta)$ as in (34). Travelling observer along the extra fifth dimension will "shuttle" between the two five-dimensional AdS "universes" crossing in either direction the four-dimensional flat braneworld within *finite* proper-time intervals

Using energy $\mathcal{E}$ and orbital momentum $\mathcal{J}$ conservation and introducing the *proper* world-line time s ($\frac{ds}{d\lambda} = em_0$), the "mass-shell" constraint equation (the equation w.r.t. the "einbein" e) produced by the action (45) yields:

$$\left(\frac{d\eta}{ds}\right)^2 + \mathcal{V}_{\text{eff}}(\eta) = \frac{\mathcal{E}^2}{m_0^2}, \quad \mathcal{V}_{\text{eff}}(\eta) \equiv A(\eta)\left(1 + \frac{\mathcal{J}^2}{m_0^2 C(\eta)}\right) \tag{46}$$

where the metric coefficients $A(\eta), C(\eta)$ are those in (12).

Since the "effective potential" $\mathcal{V}_{\text{eff}}(\eta)$ in (46) is everywhere non-negative and vanishes only at the wormhole throat(s) ($\eta = \eta_0$, where $A(\eta_0) = 0$), "radially" moving test matter (e.g. a traveling observer) with zero "impact" parameter $\mathcal{J} = 0$ and with sufficiently large energy $\mathcal{E}$) will always cross from one "universe" to another within *finite* amount of its proper-time (see Fig. 1). Moreover, this test matter (travelling observer) will "shuttle" between the turning points $\eta_\pm$:

$$\mathcal{V}_{\text{eff}}(\eta_\pm) = \frac{\mathcal{E}^2}{m_0^2}, \;\; \eta_+ > 0, \;\; \eta_- < 0, \tag{47}$$

so that in fact it will be trapped in the vicinity of the *LL-braneworld*. This effect is analogous to the gravitational trapping of matter near domain wall of a stable false vacuum bubble in cosmology [53].

6 Discussion

Let us recapitulate the crucial properties of the dynamics of *LL-branes* interacting with gravity and bulk space-time gauge fields which enabled us to construct the *LL-braneworld* solutions presented above:

1. "Horizon straddling"—automatic positioning of *LL-branes* on (one of) the horizon(s) of the bulk space-time geometry.
2. Intrinsic nature of the *LL-brane* tension as an additional *degree of freedom* unlike the case of standard Nambu–Goto p-branes; (where it is a given *ad hoc* constant), and which might in particular acquire negative values. Moreover, the variable tension feature significantly distinguishes *LL-brane* models from the previously proposed *tensionless* p-branes—the latter rather resemble p-dimensional continuous distributions of independent massless point-particles without cohesion among them.
3. The stress-energy tensors of the *LL-branes* are systematically derived from the underlying *LL-brane* Lagrangian actions and provide the appropriate source terms on the r.h.s. of Einstein equations to enable the existence of consistent non-trivial wormhole-like solutions.
4. Electrically charged *LL-branes* naturally produce *asymmetric* wormholes with the *LL-branes* themselves materializing the wormhole "throats" and uniquely determining the pertinent wormhole parameters.
5. *LL-branes* naturally couple to Kalb–Ramond bulk space-time gauge fields which results in *dynamical* generation of space-time varying cosmological constant.
6. *LL-branes* naturally produce *lightlike* braneworlds (extra dimensions are undetectable for observers confined on the *LL-brane* universe).

In our previous works we have also shown that:

7. *LL-branes* trigger sequences of spontaneous compactification/decompactification transitions of space-time [18–20].
8. *LL-branes* remove physical singularities of black holes [15].

The crucial importance of *LL-branes* in wormhole physics is underscored by the role they are playing in the self-consistent construction of the famous Einstein–Rosen "bridge" wormhole in its *original* formulation [54]—historically the first explicit wormhole solution. To this end let us make the following important remark. In several standard textbooks, e.g. [52, 55], the formulation of the Einstein–Rosen "bridge" uses the Kruskal–Szekeres manifold, where the Einstein–Rosen "bridge" geometry becomes *dynamical* (see [52], p. 839, Fig. 31.6, and [55], p. 228, Fig. 5.15). The latter notion of the Einstein–Rosen "bridge" is *not* equivalent to the original Einstein–Rosen's formulation in the classic paper [54], where the space-time manifold is *static* spherically symmetric consisting of two identical copies of the outer Schwarzschild space-time region ($r > 2m$) glued together along the horizon at $r = 2m$. Namely, the two regions in Kruskal–Szekeres space-time corresponding to the outer Schwarzschild space-time region ($r > 2m$) and labeled (I) and (III) in [52] are generally *disconnected* and share only a two-sphere (the angular part) as a common border ($U = 0, V = 0$ in Kruskal–Szekeres coordinates), whereas in the original Einstein–Rosen "bridge" construction [54] the boundary between the two identical copies of the outer Schwarzschild space-time region ($r > 2m$) is their common horizon ($r = 2m$)—a three-dimensional *lightlike* hypersurface. In [14, 17] it has been shown that the Einstein–Rosen "bridge" in its original formulation [54] naturally arises as the simplest particular case of static spherically symmetric wormhole solutions produced by *LL-branes* as gravitational sources, where the two identical "universes" with Schwarzschild outer-region geometry are glued together by a *LL-brane* occupying their common horizon—the wormhole "throat". An understanding of this picture within the framework of Kruskal–Szekeres manifold was subsequently given in [56], which uses Rindler's identification of antipodal future event horizons.

One of the most interesting physical phenomena in wormhole physics is the well-known Misner–Wheeler "charge without charge" effect [57]. Namely, Misner and Wheeler have shown that wormholes connecting two asymptotically flat space-times provide the possibility of existence of electromagnetically non-trivial solutions, where *without being produced by any charge source* the flux of the electric field flows from one universe to the other, thus giving the impression of being positively charged in one universe and negatively charged in the other universe.

In our recent paper [58] we found an opposite "charge-hiding" effect in wormhole physics, namely, that a genuinely charged matter source of gravity and electromagnetism may appear *electrically neutral* to an external observer. This phenomenon takes place when coupling self-consistently an electrically charged *LL-brane* to gravity and a *non-standard* form of nonlinear electrodynamics, whose Lagrangian contains a square-root of the ordinary Maxwell term:

$$L(F^2) = -\frac{1}{4}F^2 - \frac{f}{2}\sqrt{-F^2}, \quad F^2 \equiv F_{\mu\nu}F^{\mu\nu}, \tag{48}$$

f being a positive coupling constant. In flat space-time the theory (48) is known to produce a QCD-like effective potential between charged fermions [59–64]. When coupled to gravity it generates an effective global cosmological constant $\Lambda_{\text{eff}} = 2\pi f^2$ as well a nontrivial constant radial vacuum electric field $f/\sqrt{2}$ [65]. When in addition gravity and nonlinear electrodynamics (48) also interact self-consistently with a charged *LL-brane* we found in [58] a new type of wormhole solution which connects a non-compact "universe", comprising the exterior region of Schwarzschild-de Sitter black hole beyond the internal (Schwarzschild-type horizon), to a Levi–Civita–Bertotti–Robinson-type "tube-like" "universe" with two compactified dimensions (cf. [29–31]) via a wormhole "throat" occupied by the charged *LL-brane*. In this solution the whole electric flux produced by the charged *LL-brane* is pushed into the "tube-like" Levi–Civita–Bertotti–Robinson-type "universe" and thus the brane is detected as neutral by an observer in the Schwarzschild–de-Sitter "universe".

In the subsequent recent paper [66] we succeeded to find a truly "charge-confining" wormhole solution when the coupled system of gravity and non-standard nonlinear electrodynamics (48) are self-consistently interacting with *two* separate oppositely charged *LL-branes*. Namely, we found a self-consistent "two-throat" wormhole solution where the "left-most" and the "right-most" "universes" are two identical copies of the exterior region of the electrically neutral Schwarzschild–de-Sitter black hole beyond the Schwarzschild horizon, whereas the "middle" "universe" is of generalized Levi–Civita–Bertotti–Robinson "tube-like" form with geometry $dS_2 \times S^2$ (dS_2 is the two-dimensional de Sitter space). It comprises the finite-size intermediate region of dS_2 between its two horizons. Both "throats" are occupied by the two oppositely charged *LL-branes* and the whole electric flux produced by the latter is confined entirely within the middle finite-size "tube-like" "universe".

One of the most important issues to be studied is the problem of stability of the wormhole(-like) solutions with *LL-branes* at their "throats", in particular, the above presented *LL-braneworld* solutions in anti-de Sitter bulk space-times. The "horizon-straddling" property (15) of *LL-brane* dynamics will impose severe restrictions on the impact of the perturbations of the bulk space-time geometry.

Acknowledgements E.N. and S.P. are grateful to the organizers for hospitality during the IX-th International Workshop "Lie Theory and Its Applications in Physics". E.N. and S.P. are supported by Bulgarian NSF grant *DO 02-257*. Also, all of us acknowledge support of our collaboration through the exchange agreement between the Ben-Gurion University and the Bulgarian Academy of Sciences.

References

1. Barrabés, C., Hogan, P.: Singular Null-Hypersurfaces in General Relativity. WorldScientific, Singapore (2004)
2. Thorne, K., Price, R., Macdonald, D. (eds.): Black Holes: The Membrane Paradigm. Yale University Press, New Haven (1986)
3. Israel, W.: Nuovo Cimento **B44**, 1 (1966) [Erratum, Nuovo Cimento **B48**, 463 (1967)]
4. Barrabés, C., Israel, W.: Phys. Rev. **D43**, 1129 (1991)
5. Dray, T., 't Hooft, G.: Class. Quant. Gravit. **3**, 825 (1986)
6. Berezin, V., Kuzmin, A., Tkachev, I.: Phys. Rev. **D36**, 2919 (1987)
7. Kogan, I., Reis, N.: Int. J. Mod. Phys. **A16**, 4567 (2001) (arxiv:hep-th/0107163)
8. Mateos, D., Ng, S.: J. High Energy Phys. **0208**, 005 (2002) (arxiv:hep-th/0205291)
9. Mateos, D., Mateos, T., Townsend, P.K.: J. High Energy Phys. **0312**, 017 (2003) (arxiv:hep-th/0309114)
10. Bredthauer, A., Lindström, U., Persson, J., Wulff, L.: J. High Energy Phys. **0402**, 051 (2004) (arxiv:hep-th/0401159)
11. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Phys. Rev. **D72**, 0806011 (2005) (arxiv:hep-th/0507193)
12. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Fortschr. Phys. **55**, 579 (2007) (arxiv:hep-th/0612091)
13. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Fortschr. Phys. **57**, 566 (2009) (arxiv:0901.4443[hep-th])
14. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Phys. Lett. **681B**, 457 (2009) (arxiv:0904.3198[hep-th])
15. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Int. J. Mod. Phys. **A25**, 1571–1596 (2010) (arxiv:0908.4195[hep-th])
16. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Phys. Lett. **673B**, 288–292 (2009) (arxiv:0811.2882[hep-th])
17. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Int. J. Mod. Phys. **A25**, 1405 (2010) (arxiv:0904.0401[hep-th])

18. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Gen. Relativity Gravit. **43**, 1487–1513 (2011) (arxiv:1007.4893[hep-th])
19. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Invertis Journal of Science & Technology **3**, 91–100 (2010) (arxiv:0912.3712[hep-th])
20. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: In: Dragovic, B., Rakic, Z. (eds.) Sixth Meeting in Modern Mathematical Physics, vol. 22. Belgrade Institute of Physics Press, Belgrade (2011) (arxiv:1011.6241[hep-th])
21. Visser, M.: Lorentzian Wormholes. From Einstein to Hawking. Springer, Berlin (1996)
22. Hochberg, D., Visser, M.: Phys. Rev. **D56**, 4745 (1997) (arxiv:9704082[gr-qc])
23. Hochberg, D., Visser, M.: Phys. Rev. **D58**, 044021 (1998) (arxiv:9802046[gr-qc])
24. Lemos, J., Lobo, F., de Oliveira, S.: Phys. Rev. **D68**, 064004 (2003) (arxiv:0302049[gr-qc])
25. Sushkov, S.: Phys. Rev. **D71**, 043520 (2005) (arxiv:0502084[gr-qc])
26. Lobo, F.: Exotic Solutions in General Relativity-Traversable wormholes and warp drive spacetimes (arxiv:0710.4474[gr-qc])
27. Bronnikov, B., Lemos, J.: Phys. Rev. **D79**, 104019 (2009) (arxiv:0902.2360[gr-qc])
28. Bronnikov, K., Skvortsova, M., Starobinsky, A.: Gravit. Cosmol. **16**, 216 (2010) (arxiv:1005.3262[gr-qc])
29. Levi-Civita, T.: Rend. R. Acad. Naz. Lincei **26**, 519 (1917)
30. Bertotti, B.: Phys. Rev. **D116**, 1331 (1959)
31. Robinson, I.: Bull. Akad. Pol. **7**, 351 (1959)
32. Guendelman, E.: Gen. Relativity Gravit. **23**, 1415 (1991)
33. Dzhunushaliev, V., Singleton, D.: Class. Quant. Gravit. **16**, 973 (1999) (arxiv:gr-qc/9805104)
34. Dzhunushaliev, V., Singleton, D.: Phys. Rev. **D59**, 064018 (1999) (arxiv:gr-qc/9807086)
35. Dzhunushaliev, V., Kasper, U., Singleton, D.: Phys. Lett. **479B**, 249 (2000) (arxiv:gr-qc/9910092)
36. Dzhunushaliev, V.: Gen. Relativity Gravit. **35**, 1481 (2003) (arxiv:gr-qc/0301046)
37. Matyjasek, J., Zaslavsky, O.: Phys. Rev. **D64**, 044005 (2001) (arxiv:gr-qc/0006014)
38. Zaslavskii, O.: Phys. Rev. **D70**, 104017 (2004) (arxiv:gr-qc/0410101)
39. Zaslavskii, O.: Phys. Lett. **634B**, 111 (2006) (arxiv:gr-qc/0601017)
40. Zaslavskii, O.: Phys. Rev. **D80**, 064034 (2009) (arxiv:0909.2270[gr-qc])
41. Lindström, U., Svendsen, H.: Int. J. Mod. Phys. **A16**, 1347 (2001) (arxiv:hep-th/0007101)
42. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Cent. Eur. J. Phys. **7**, 668 (2009) (arxiv:0711.2877[hep-th])
43. Israel, W., Poisson, E.: Phys. Rev. Lett. **63**, 1663 (1989)
44. Israel, W., Poisson, E.: Phys. Rev. **D41**, 1796 (1990)
45. Randall, L., Sundrum, R.: Phys. Rev. Lett. **83**, 3370 (1999) (arxiv:hep-ph/9905221)
46. Randall, L., Sundrum, R.: Phys. Rev. Lett. **83**, 4690 (1999) (arxiv:hep-ph/9906064)
47. Mannheim, P.: Brane-Localized Gravity. World Scientific, Singapore (2005)
48. Gregory, R.: Braneworld black holes, chap. 7. In: Papantonopoulos, E. (ed.) Physics of Black Holes: A Guided Tour. Lecture Notes in Physics, vol. 769, pp. 259–298. Springer, Berlin (2009)
49. Maartens, R., Koyama, K.: Living Rev. Relativity **13**, 5 (2010) (arxiv:1004.3962[hep-th])
50. Poisson, E.: A Relativist's Toolkit. The Mathematics of Black-Hole Mechanics. Cambridge University Press, Cambridge (2004)
51. Rindler, W.: Relativity: Special, General and Cosmological, Sect. 3.8, 2nd edn. Oxford University Press, Oxford (2006)
52. Misner, Ch., Thorne, K., Wheeler, J.: Gravitation, Sect. 6.6. W.H. Freeman and Co, San Francisco (1973)
53. Guendelman, E., Shilon, I.: Phys. Rev. **D76**, 025021 (2007) (arxiv:gr-qc/0612057)
54. Einstein, A., Rosen, N.: Phys. Rev. **43**, 73 (1935)
55. Carroll, S.: Spacetime and Geometry. An Introduction to General Relativity. Addison Wesley, San Francisco (2003)
56. Poplawski, N.: Phys. Lett. **687B**, 110 (2010) (arxiv:0902.1994[gr-qc])
57. Misner, Ch., Wheeler, J.: Ann. Phys. **2**, 525–603 (1957)

58. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Open Nucl. Part. Phys. J. **4**, 27–34 (2011) (arxiv:1108.3735[hep-th])
59. Guendelman, E.: Int. J. Mod. Phys. **A19**, 3255 (2004) (arxiv:0306162[hep-th])
60. Gaete, P., Guendelman, E.: Phys. Lett. **640B**, 201–204 (2006) (arxiv:0607113[hep-th])
61. Gaete, P., Guendelman, E., Spalluci, E.: Phys. Lett. **649B**, 217 (2007) (arxiv:0702067[hep-th])
62. Guendelman, E.: Mod. Phys. Lett. **A22**, 1209–1215 (2007) (arxiv:0703139[hep-th])
63. Korover, I., Guendelman, E.: Int. J. Mod. Phys. **A24**, 1443–1456 (2009)
64. Guendelman, E.: Int. J. Mod. Phys. **A25**, 4195–4220 (2010) (arxiv:1005.1421[hep-th])
65. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Phys. Lett. **704B**, 230 (2011) [Erratum Phys. Lett. **705B** 545 (2011)] (arxiv:1108.0160[hep-th])
66. Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S.: Int. J. Mod. Phys. **A26**, 5211–5239 (2011) (arxiv: 1109.0453[hep-th])

Generalized Bernoulli Polynomials and the Casimir Effect in the Einstein Universe

Patrick Moylan

Abstract We consider various regularization schemes for calculating the renormalized vacuum energy of a massless scalar field in the n-dimensional Einstein universe. We also study a related problem, namely, the Casimir energy for a massless scalar field in the n-dimensional Einstein universe subject to Dirichlet boundary conditions on a sphere of maximal radius. In a recent work the author used the representation theory of $SO(2,n)$ to obtain exact results but not in closed form for the second problem with n arbitrary. Here we make use of generating functions for generalized Bernoulli polynomials and an extension of a result of Srivastava and Todorov about generalized Bernoulli numbers (Srivastava, Todorov, J. Math. Anal. Appl. 130:509–513, 1988) to obtain new results involving exact expressions in closed form for both problems. We also consider expansions of the generalized Bernoulli polynomials into Hurwitz zeta functions which enables us to explicitly demonstrate the equivalence of the cutoff function technique with the zeta regularization technique. Our method of approach confirms the results of Herdeiro et al. (Class. Quant. Gravit. 25:165010, 2008) and Özcan (Class. Quant. Gravit. 23:5531–5546, 2006). We conclude the paper by showing that useful information about the analogous problem in n-dimensional Minkowski space can also be obtained out of our analysis.

P. Moylan (✉)
Department of Physics, The Pennsylvania State University, Abington College,
Abington, PA 19001 USA
e-mail: pjm11@psu.edu

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_15,

1 Preliminaries on the Einstein Universe and Minkowski Space in *n* Dimensions

Let $Q(x)$ be the quadratic form associated with the line element $ds^2 = dx_1^2 + dx_0^2 - dx_1^2 - \ldots - dx_n^2$. Consider $G = SO_0(2,n)$, the connected component of the group of linear transformations of $\mathbb{R}^{n+2}$ preserving the symmetric bilinear form which is associated to $Q(x)$ by polarization. We call G the n-dimensional conformal group and we denote by $G^\sim = SO_0(2,n)^\sim$ the universal cover of G. Let $\mathscr{G}$ be the Lie algebra of G. Consider the $n+1$ dimensional isotropic cone in $\mathbb{R}^{n+2}$ defined by

$$C = \{x \in \mathbb{R}^{n+2} | Q(x) = 0\}. \tag{1}$$

Let $\mathbb{R}^{n+2^*}$ and C^* be the sets of nonzero elements in $\mathbb{R}^{n+2}$ and C, respectively. Let $P \subset G$ be the stabilizer subgroup of $e = e_{-1} + e_n$. ($e_{-1} = (1,0,\ldots,0) \in \mathbb{R}^{n+2}$ and $e_n = (0,0,\ldots,1) \in \mathbb{R}^{n+2}$.) P is the n dimensional Poincaré group [1]. The orbit of e under G is C^* [1]. Hence $C^* \cong G/P$.

Let S^{n-1} denote the $(n-1)$-sphere:

$$S^{n-1} = \left\{u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n | u_1^2 + \ldots u_n^2 = 1\right\} \tag{2}$$

Define the upper and lower hemispheres of S^{n-1} and the equator Σ^{n-1} as

$$S_{\pm}^{n-1} = \{s \in S^{n-1} \mid u_n \gtrless 0\},\ \Sigma^{n-1} = \partial\, S_{+}^{n-1} = \partial\, S_{-}^{n-1}, \tag{3}$$

respectively. Spherical coordinates on S^{n-1} are as follows:

$$u = (sin(\rho)\ \omega, cos(\rho)) \in S^{n-1} \text{ with } \omega \in S^{n-2}. \tag{4}$$

Let $Proj(\mathbb{R}^{n+2})$ be the real projective variety of all one dimensional subspaces in $\mathbb{R}^{n+2}$. We have the map $\pi : X^* \longrightarrow Proj(\mathbb{R}^{n+2})$, $\pi(x) = \bar{x}$ where $\bar{x}$ is the equivalence class of x with the equivalence relation $x \sim \lambda x$ ($\lambda \in \mathbb{R}$). Let $M \subset Proj(\mathbb{R}^{n+2})$ be the image of C^* under π. The above defined action of $SO_0(2,n)^\sim$ on $\mathbb{R}^{n+2}$ induces an action of $SO_0(2,n)^\sim$ on $Proj(\mathbb{R}^{n+2})$. Since C is stable under the $SO_0(2,n)^\sim$ action, M is stable under the action of $SO_0(2,n)^\sim$ on $Proj(\mathbb{R}^{n+2})$. $SO_0(2,n)$ and therefore $SO_0(2,n)^\sim$ are transitive on C^*, and hence $SO_0(2,n)$ and $SO_0(2,n)^\sim$ are transitive on M. M is naturally diffeomorphic to $(S^1 \times S^{n-1})/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ action is the product of antipodal maps on S^1 and S^{n-1}. Denote $S^1 \times S^{n-1}$ by $\bar{M}$. $K = SO(2) \times SO(n)$ acts transitively on $\bar{M}$, and $\bar{M}$ is the homogeneous space $\bar{M} \cong K/K_0$ where $K_0 = SO(1) \times SO(n-1)$. The universal cover $\bar{M}^\sim$ of $\bar{M}$ is the (n dimensional) Einstein universe. (See [2] for the definition in four dimensions.) Since M is the conformal compactification of M_0 [1], the Einstein universe is the universal cover of the conformal compactification of Minkowski space.

2 Quantization of a Real Massless Scalar Field on $\tilde{M}$

The line bundle $L^s(M)$ over M associated with the character $\lambda \to |\lambda|^{-s}$ of $\mathbb{R}^*$ is the bundle whose fibre over $\bar{x}$ is the set of all pairs $(\lambda x, |\lambda|^s) \in C \times \mathbb{C}$, $(s \in \mathbb{C})$. Denote by $\Gamma^s(M)$ the space of smooth sections of $L^s(M)$. There is a unique isomorphism between $\Gamma^s(M)$ and the space of smooth functions $f : C^* \longrightarrow \mathbb{C}$ which satisfy the homogeneity condition $f(\lambda x) = |\lambda|^{-s} f(x)$. $\Gamma^s(M)$ is an $SO_0(2,n)^\sim$ module with respect to the representation π_s defined by $(\pi_s(g)\, f)(x) = f(g^{-1}x)$ $(f \in \Gamma^s(M)$, $g \in SO_0(2,n)$, $x \in C^*$, and $g^{-1}\, x$ denotes the action of g^{-1} on $x \in C^*)$. We denote the associated representation of the Lie algebra $so(2,n)$ by $d\pi_s$.

Let $C^s_K(\bar{M})$ be the space of all K finite elements of $C^\infty(\bar{M})$ for which $\phi(w) = (-1)^s\phi(-w)$ $(w \in \bar{M})$. For $s = -2 + \frac{n+2}{2}$ the representation π_s is reducible and the space of $\phi \in C^s_K(\bar{M})$ for which

$$\left\{\frac{\partial^2}{\partial\tau^2} - \Delta_{S^{n-1}} + \left(\frac{n-2}{2}\right)^2\right\}\phi = 0. \tag{5}$$

defines a subrepresentation [3]. This subrepresentation splits into two irreducible components, which are spaces of positive and negative energy massless, spin zero fields on $\bar{M}$ [4].

Let

$$\mathscr{H}^\sigma = \left\{f \in C^\infty(S^{n-1}) | \Delta_{S^{n-1}} f = -\sigma(\sigma+n-2)f\right\}. \tag{6}$$

The spherical harmonics are

$$Y_{\sigma\ell\{m\}}(u) = N(k,\ell,\{m\}) \sin^\ell\rho\, C^{\ell+\frac{n-2}{2}}_{\sigma-\ell}(\cos\rho) Y^{\{m\}}_\ell(\theta_1,\theta_2,\,.\,.\,,\theta_{n-2}) \tag{7}$$

where $\theta_1, \theta_2, \ldots, \theta_{n-2}, \rho$ are spherical coordinates of the point $u \in S^{n-1}$, $\{m\}$ is used for the other labels and $N(k,\ell,\{m\})$ is the normalization factor for the spherical harmonics. For ϕ a K-finite function of the form

$$\phi_{\nu\sigma\ell\{m\}}(\tau,u) = e^{i\nu\tau} Y_{\sigma\ \ell\ \{m\}}(u) \tag{8}$$

which are in the massless subspace (i.e. the space of K-finite solutions of (5) we have the spectral equation

$$\nu^2 - \sigma(\sigma+n-2) - \left(\frac{n-2}{2}\right)^2 = 0. \tag{9}$$

We denote by Δ_j the covariant derivative determined by the semi-Riemannian metric g on $\tilde{M}$, then the energy operator for a massless, scalar field on $\tilde{M}$ is [5]:

$$H = \int_{S^{n-1}} \frac{1}{2}\left\{\sum_{j=1}^{n} (\Delta_j\phi)^*(\Delta_j\phi) + \left(\frac{n-2}{2R}\right)^2 |\phi|^2\right\} du \tag{10}$$

where $du = R^{n-1}\sin^{n-2}(\rho)d\rho \wedge d\omega$ is the volume form on S^{n-1}, with $d\omega$ denoting the volume form on S^{n-2} and ϕ in the massless subspace. R is the radius of the Einstein universe which in the above was set equal to one. The standard quantization of a real, massless scalar field on $\tilde{M}$ is carried out in a similar way to the quantization of a real, massless scalar field on M_0 [5]. Let c be the speed of light, $\hbar$ Planck's constant and $c_\sigma = \left(\frac{\hbar}{E_\sigma}\right)^{1/2}$ then, for a point $x \in \tilde{M}$ with coordinates $(t = \frac{R}{c}\tau, \omega, \rho)$, the quantum field $\phi(x)$ is given by:

$$\phi(t,\omega,\rho) = \sum_{\sigma,\ell,\{m\}} c_\sigma \left\{ a_{\sigma\ell\{m\}} e^{-iE_\sigma t} Y_{\sigma\ell\{m\}}(\rho,\omega) + a^\dagger_{\sigma\ell\{m\}} e^{iE_\sigma t} Y^*_{\sigma\ell\{m\}}(\rho,\omega) \right\} \quad (11)$$

where $E_\sigma = \frac{cv}{R}$ is specified by (9), and the $a_{\sigma\ell\{m\}}$ and $a^\dagger_{\sigma\ell\{m\}}$ satisfy the following relations on the Fock space:

$$\left[a_{\sigma\ell\{m\}}\ ,\ a^\dagger_{\sigma'\ell'\{m'\}}\right] = \delta_{\sigma,\sigma'}$$

$$\delta_{\ell,\ell'}\delta_{\{m\},\{m'\}}, \left[a_{\sigma\ell\{m\}}\ ,\ a_{\sigma'\ell'\{m'\}}\right] = \left[a^\dagger_{\sigma\ell\{m\}}, a^\dagger_{\sigma'\ell'\{m'\}}\right] = 0. \quad (12)$$

By a calculation, which uses (5), we may rewrite the Hamiltonian as

$$H = \int_{S^{n-1}} \frac{1}{2}\left\{(\partial_t\phi)^*(\partial_t\phi) - \phi^*\partial_t^2\phi\right\} du. \quad (13)$$

3 The Renormalized Vacuum Energy of a Massless Scalar Field on $\tilde{M}$ and the Casimir Energy of a Massless Scalar Field on $\tilde{M}$ with Dirichlet Boundary Conditions on Σ^{n-1}

We have

$$S^{n-1}_{\pm} = \left\{ v \in S^{n-1} \mid \rho \gtrless \frac{\pi}{2} \right\} \quad (14)$$

and

$$\Sigma^{n-1} = \left\{ v \in S^{n-1} | \rho = \frac{\pi}{2} \right\}. \quad (15)$$

We now determine the Casimir energy of a quantized massless, scalar field on $\tilde{M}$ subject to vanishing of the field on the equator Σ^{n-1}. Using (11)–(13) and some results on spherical harmonics, we find for the vacuum expectation value of the energy operator H given in (13) the following [6]:

$$E = \frac{\hbar c}{R} \sum_{\substack{\sigma = 0 \\ \sigma - \ell \text{ odd} \\ \ell, \{m\}}}^{\infty} \sqrt{\sigma(\sigma+n-2)+(\frac{n-2}{2})^2}, \tag{16}$$

where the sum includes all multiplicities. Using $SO(n-1)$ representation theory to account for multiplicity we may perform the finite sums over ℓ to get for $n > 3$ (cf. [6]):

$$E = \frac{\hbar c}{4R} \sum_{\sigma=0}^{\infty} (2\sigma+n-2) \frac{\Gamma(\sigma+n-2)}{\Gamma(\sigma)\Gamma(n-1)}. \tag{17}$$

For the much simpler $n = 3$ case the analysis is slightly different due to the difference in multiplicities in this case and we leave it as an exercise to the reader to work out the details in this case (cf. [6]). For the even simpler $n = 2$ case there are no multiplicities and it is easy to see that (17) is replaced by $E = \frac{\hbar c}{4R} \sum_{\sigma=0}^{\infty} \sigma$. For computing the vacuum energy of a massless scalar field on $\tilde{M}$ the only difference from the above analysis is that in (16) the restriction that $\sigma - \ell$ is odd is removed, and we leave it to the reader to work out the details for this very similar problem.

Equation (17) is infinite for all n and we need to regularize it. Standard regularization techniques to extract the finite part of the energy, which is the Casimir energy, are the point splitting method (coincidence limit) [7], cutoff function method [8], zeta function regularization [9] and dimensional regularization [10]. Additionally, we have considered in [11,12] a q-regularization using representations at roots of unity of a q deformation of the symmetry algebra of the problem. For the cutoff function method we introduce an exponential damping function inside the summation in (17) to get:

$$\begin{aligned} E_{1/2}^{reg}(\alpha,n) &= \frac{\hbar c}{4R} \sum_{\sigma=0}^{\infty} (2\sigma+n-2) \frac{\Gamma(\sigma+n-2)}{\Gamma(\sigma)\Gamma(n-1)} e^{-\frac{\alpha}{2R}(2\sigma+n-2)} = \\ &= \frac{\hbar c}{2^n} \frac{d}{d\alpha} \left(e^{-\frac{\alpha}{2R}} \left\{ \operatorname{csch} \frac{\alpha}{2R} \right\}^{(n-1)} \right). \end{aligned} \tag{18}$$

For the vacuum energy of a massless scalar field on $\tilde{M}$ a similar analysis yields:

$$E^{reg}(\alpha,n) = \frac{R}{(n-2)} \frac{\hbar c}{2^{n-2}} \frac{d^2}{d\alpha^2} \left\{ \operatorname{csch} \frac{\alpha}{2R} \right\}^{(n-2)}. \tag{19}$$

4 Generalized Bernoulli Polynomials and the Casimir Energy

It is clear from (18) that the Casimir interaction energy $E^{c}_{1/2}(R,n)$ is the α independent part of $E^{reg}_{1/2}(\alpha,n)$ and similarly the renormalized vacuum energy $E^{c}(R,n)$ of the Einstein universe is the α independent part of $E^{reg}(\alpha,n)$ (cf. [13]). We obtain as an immediate consequence of (19) that $E^{c}(R,n)$ vanishes for n odd due to the fact that $\mathrm{csch}(x)$ is an odd function of x, which confirms a result in [14] obtained in a much more complicated way. To extract the α independent part for the other cases we relate the expressions in (18) and (19) to generating functions for generalized Bernoulli polynomials. Generalized Bernoulli polynomials are defined as

$$e^{xz}\left(\frac{z}{e^{z}-1}\right)^{\sigma}=\sum_{n=0}^{\infty}B_{n}^{\{\sigma\}}(x)\frac{z^{n}}{n!}\ (|z|<2\pi;1^{\sigma}=1) \tag{20}$$

We have the following explicit formula for the generalized Bernoulli polynomials [15]:

$$B_{n}^{\{\sigma\}}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{\sigma+k-1}{k}\frac{k!}{(2k)!}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}j^{2k}(x+j)^{n-k} \times {}_{2}F_{1}[k-n,k-\sigma;2k+1;j/(x+j)] \tag{21}$$

where ${}_{2}F_{1}[k-n,k-\sigma;2k+1;j/(x+j)]$ is the Gaussian hypergeometric function (cf. [16]). Using some trigonometry and (20) we can rewrite (18) as

$$E^{reg}_{1/2}(\alpha,n)=\frac{\hbar c}{2}\frac{d}{d\alpha}\left(\sum_{m=0}^{\infty}B_{m}^{\{n-1\}}\left(\frac{n-2}{2}\right)\frac{\alpha^{m-n+1}}{R^{m-n+1}m!}\right). \tag{22}$$

and, in a similar way, we may rewrite (19) as follows:

$$E^{reg}(\alpha,n)=\frac{\hbar c}{n-2}\frac{d^{2}}{d\alpha^{2}}\left(\sum_{m=0}^{\infty}B_{m}^{\{n-2\}}\left(\frac{n-2}{2}\right)\frac{\alpha^{m-n+2}}{R^{m-n+1}m!}\right). \tag{23}$$

Using these equations we can easily extract the α independent parts of $E^{reg}_{1/2}(\alpha,n)$ and $E^{reg}(\alpha,n)$ to get:

$$E^{c}_{1/2}(R,n)=-\frac{\hbar c}{2n!R}B_{n}^{\{n-1\}}\left(\frac{n-2}{2}\right) \tag{24}$$

and

$$E^{c}(R,n)=\frac{2\hbar c}{(n-2)n!R}B_{n}^{\{n-2\}}\left(\frac{n-2}{2}\right). \tag{25}$$

By using (21) we can express (24) and (25) as finite sums of terms involving hypergeometric functions. It is straightforward to show that (24) and (25) agree with the results in [14] and [17], which are only obtained for specific n values ranging from $n = 2$ to $n = 11$. With a little more work we may even compare our (25) with the nonzero cases of (44) in [18], which again is only given for specific n values, namely for $n = 2p+3,\ p = 0,\ 1,\ 2,\ 3,\ \ldots$.

For the zeta function regularization consider [16]

$$\zeta(s,q) = \sum_{\sigma=0}^{\infty} \frac{1}{(\sigma+q)^s}\ , \ \mathrm{Re}\, s\ >\ 1,\ q \neq\ 0,\ -1,\ -2,\ \ldots \tag{26}$$

By analytic continuation we can extend $\zeta(s,q)$ to all s and desired q except for $s = 1$. In [14], (17), for special values ranging from n = 2 to 12, is rewritten as finite linear combinations of series of powers of σ or $\sigma + \frac{1}{2}$ for n even or n odd, respectively. This, according to [14], is the zeta regularization technique. In [19] it is shown how to write a generalized Bernoulli polynomial as a finite sum of Hurwitz zeta functions, which result enables us to explicitly demonstrate the equivalence of the exponential cutoff method and the zeta regularization method. For the $n = 4$ case we follow [19] to obtain

$$B_4^{\{2\}}(c) = \frac{4!}{2!}\{\zeta(-3,c) + (1-c)\zeta(-2,c)\}. \tag{27}$$

Using this equation with $c = \frac{n-2}{2}$ together with the value of $E^c(R,n)$ for $n = 4$ in 25 we find

$$E^c(R, n=4) = \frac{2\hbar c}{2\cdot 4!R} B_4^{\{2\}}(1) = \frac{\hbar c}{2R}\zeta(-3,1). \tag{28}$$

We leave it to the reader to follow Example 2.10 and Theorem 2.13 in [19] in order to write $B_n^{\{n-\delta\}}(x)$ $(\delta = 1,2)$ as finite linear combinations of Hurwitz zeta functions and establish the equivalence for the other cases.

We conclude with showing how our results on the Casimir energy for Dirichlet boundary conditions on Σ^{n-1} give useful information about the Casimir energy for a massless, scalar field in n-dimensional Minkowski space with Dirichlet boundary conditions on a sphere of radius R. Using (24) together with the requirement that the Casimir energy be continuous as a function of the sphere's radius, we conclude that the Casimir energy with Dirichlet boundary conditions on spheres of slightly smaller radii must also be positive. Apart from certain global differences, which, in our case, do not matter, fields in $\bar{M}$ approximate fields in M_0 as $R \to \infty$. Since the Einstein energy differs from the Minkowski energy by an amount that goes as $\frac{1}{R}$, it clear that the Casimir energy for a sphere of radius of cosmic extent in $\mathbb{R}^{n-1}$ with the field vanishing on the boundary of the sphere is approximately given by (24) for its Casimir energy. Thus our method can be used to settle the important question about the sign of the Casimir energy for massless fields in n dimensional Minkowski space with Dirichet boundary conditions on spheres. In particular, for the physically important case of $n = 4$, the Casimir energy is positive.

References

1. Angelopolous, E., Laoues, M.: Rev. Math. Phys. **10**(3), 371–299 (1998)
2. Birrell, N., Davies, P.: Quantum Fields in Curved Space. Cambridge University Press, Cambridge (1982)
3. Kostant, B.: Progress of Mathematics, pp. 67–86. Birkhäuser, Boston (1992)
4. Kobayashi, T., Orsted, B.: Adv. Math. **180**, 486–595 (2003)
5. Baez, J.: Lett. Math. Phys. **21**, 117–121 (1991)
6. Moylan, P.: J. Phys. Conf. Ser. **128**, 012010 (2008)
7. Dowker, J., Critchley, R.: J. Phys. A Math. Gen. **9**, 535 (1976)
8. Ford, L.H.: Phys. Rev. D **11**, 3370 (1975)
9. Elizalde, E., Odintsov, S.D., Romeo, A., Bytsenko, A.A., Zerbini, S.: Zeta Regularization Techniques with Applications. World Scientific, Singapore (1994)
10. Reuter, M., Dittrich, W.: Eur. J. Phys. **6**, 33–40 (1985)
11. Moylan, P.: Bulg. J. Phys. **34**(3), 687–706 (2007)
12. Moylan, P.: The Casimir Effect in the Einstein Universe, R.O.M.P (tentatively accepted for publication)
13. Verschelde, H., Wille, L., Phariseau, P.: Phys. Lett. **149B**(4, 5), 396–398 (1984)
14. Özcan, M.: Class. Quant. Gravit. **23**, 5531–5546 (2006)
15. Sirvastava, H.M., Todorov, P.G.: J. Math. Anal. Appl. **130**, 509–513 (1988)
16. Gradshteyn, I.C., Rhyshik, I.M.: Tables of Integrals, Sums, Series and Derivatives. Academic, New York (2000)
17. Bayin, S., Özcan, M.: Class. Quant. Gravit. **10**, L115–L121 (1993)
18. Herdiero, C., Bibeiro, R., Sampaio, M.: Class. Quant. Gravit. **28**, 165010 (2008)
19. Kirsten, K.: Basic zeta functions and some applications in physics. In: Kirsten, K., Williams, F.L. (eds.) Zeta and Modular Physics, vol. 57, pp. 101–143. MSRI Publ./Cambridge University Press, Cambridge (2010)

From Singularities to Algebras to Pure Yang–Mills with Matter

Tamar Friedmann

Abstract Since the advent of dualities in string theory, it has been well-known that codimension 4 orbifold singularities that appear in extra-dimensional spaces, such as Calabi–Yau or G_2 spaces, may be interpreted as ADE gauge theories. As to orbifold singularities of higher codimension, there has not been an analog of this interpretation. Here we show how the search for such an analog led us from the singularities to the creation of Lie Algebras of the Third Kind ("LATKes"). We introduce an example of a LATKe that arises from the singularity $\mathbf{C}^3/\mathbf{Z}_3$, and prove it to be simple and unique. We explain that the uniqueness of the LATKe serves as a vacuum selection mechanism. We also show how the LATKe leads to a new kind of gauge theory in which the matter field arises naturally and which is tantalizingly close to the Standard Model of particle physics.

1 Introduction and Motivation

One of the outcomes of the "string revolution" of the mid-1990s was an interpretation of ADE singularities in Calabi–Yau spaces as gauge theories with ADE gauge groups [1, 2]. This interpretation arose via string dualities, and later on was applied to the same singularities within manifolds of G_2 holonomy in the context of M-theory compactifications [3–5]. The usefulness of this interpretation lies in the fact that it enhances our understanding of the four-dimensional theory that is obtained when string/M theory is compactified on Calabi–Yau or G_2 spaces which have those ADE singularities. A particularly encouraging result of this interpretation was the first manifestation from M-theory [6–8] of Georgi–Glashow grand unification [9], where the $SU(5)$ grand unified group is obtained from an A_4

T. Friedmann (✉)
University of Rochester, 500 Wilson Boulevard, Rochester, NY, USA
e-mail: tamarf@pas.rochester.edu

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_16, © Springer Japan 2013

singularity in a G_2 manifold, and is then naturally broken by Wilson lines precisely to $SU(3) \times SU(2) \times U(1)$, the gauge group of the standard model of particle physics.

Singularities other than ADE have arisen in the same context [6], but an analogous interpretation of those other singularities in terms of gauge groups and gauge theories was not available. For example, take orbifold singularities of codimension 6, of the form $\mathbf{C}^3/\Gamma$, where Γ is a discrete finite subgroup of $SU(3)$; these are direct generalizations of ADE singularities which are codimension 4 orbifold singularities of the form $\mathbf{C}^2/\Gamma$, where Γ is a discrete finite subgroup of $SU(2)$. For the codimension 6 singularities we ask: what is the four-dimensional physical theory that arises from string/M theory compactifications on CY or G_2 manifolds that have these codimension 6 singularities?

As it turns out, string dualities do not provide for a generalization of the interpretation of codimension 4 orbifold singularities to one for codimension 6 orbifold singularities. Instead, we address this question by turning to the mathematical roots of these dualities. Our approach is to analyze the mathematical aspects of codimension 4 singularities in a way that will allow us to generalize to codimension 6, and then obtain an interpretation of the results for the physical theory.

On the mathematical side, we introduce a new set of relations, which we call the Commutator-Intersection Relations, that illuminate the connection between codimension 4 singularities and Lie algebras. These relations pave the way to construct Lie Algebras of the Third Kind, or LATKes, a kind of algebras that arise from codimension 6 orbifold singularities. We also learn and prove the existence and uniqueness of a simple LATKe.

On the physics side, we discover a new kind of Yang–Mills theory, called "LATKe Yang–Mills," which arises from the LATKe. Unlike any known Yang–Mills theory, the LATKe Yang–Mills theory in its purest form automatically contains matter. We also propose that the uniqueness of the simple LATKe is a vacuum selection mechanism. The selected vacuum theory is an $SU(2) \times SU(2)$ gauge theory with matter in the $(2,2)$ representation, and the corresponding singularity is $\mathbf{C}^3/\mathbf{Z}_3$. The algebra $\mathfrak{su}(2) \times \mathfrak{su}(2)$ is protected by the LATKe from being broken. The selected singularity $\mathbf{C}^3/\mathbf{Z}_3$ is one of those which arose in the G_2 spaces of [6], and which at the time we put on hold in anticipation of the outcome of this investigation.

This paper is based on [10]; due to space constraints, we leave out many details and references which may be found there.

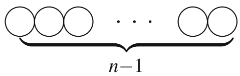
2 The Commutator-Intersection Relations (CIRs)

Let $\mathbf{C}^2/\Gamma$ be an orbifold singularity of codimension 4, with Γ a discrete, finite subgroup of $SU(2)$; the groups Γ were studied by Klein [11], who found them to have an ADE classification. Work of DuVal and of Artin [12–14] then provided a correspondence between these singularities and Lie algebras. We now present the correspondence in a way that will lead us to a new relation between intersection numbers of the blow-ups of the singularities and commutators of the Lie algebras. These relations, which we name the Commutator-Intersection Relations, will then be generalized to the $\mathbf{C}^3/\Gamma$ case.

We proceed via an example. Let $\Gamma = \mathbf{Z}_n \subset SU(2)$, which corresponds to A_{n-1} in the ADE classification, be generated by the $SU(2)$ matrix

$$\begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}. \tag{1}$$

Its action on $(x,y) \in \mathbf{C}^2$ is given by $(x,y) \longmapsto (e^{2\pi i/n}x, e^{-2\pi i/n}y)$. This action is free except at the origin where $\mathbf{C}^2/\mathbf{Z}_n$ has a singularity. The blow-up of this singularity has an exceptional divisor made up of $n-1$ spheres S^2 that intersect as follows:



When the spheres are replaced by nodes and their intersections are replaced by edges, we obtain the Dynkin diagram of the Lie algebra $\mathfrak{sl}_{n-1}$:

$$\bullet - \bullet - \bullet \quad \cdots \quad \bullet - \bullet \tag{2}$$

Furthermore, the intersection numbers between pairs of spheres of the exceptional divisor are exactly minus of the entries of the Cartan matrix of the Lie algebra:

$$I_{ij} = -C_{ij} \qquad i,j = 1,\ldots,n-1. \tag{3}$$

As shown in the work of Duval and of Artin, this relation between the $\mathbf{Z}_n$ singularity and the $\mathfrak{sl}_{n-1}$ Lie algebra generalizes to a correspondence between all the ADE singularities and ADE Lie algebras: in all cases, the blow-up of the ADE singularity corresponds to the Dynkin diagram of the ADE Lie algebra, and (3) holds.

Using the above correspondence, we now show how to obtain a direct relation between the intersection numbers of the blow-up of the ADE singularity and the commutators of the ADE Lie algebras.

Recall that a complex simple Lie algebra is generated by k triples $\{X_i, Y_i, H_i\}_{i=1}^k$ with their commutators determined by the following relations:

$$\begin{aligned} &[H_i, H_j] = 0\,; && [X_i, Y_j] = \delta_{ij} H_j; \\ &[H_i, X_j] = C_{ij} X_j\,; && [H_i, Y_j] = -C_{ij} Y_j; \\ &\mathrm{ad}(X_i)^{1-C_{ij}}(X_j) = 0\,; && \mathrm{ad}(Y_i)^{1-C_{ij}}(Y_j) = 0. \end{aligned} \tag{4}$$

Here, the H_i form the Cartan subalgebra, the X_i are simple positive roots, the Y_i are simple negative roots, k is the rank of the Lie algebra, C_{ij} is the Cartan matrix, and $\mathrm{ad}(X_i)(A) = [X_i, A]$. These equations are the familiar Chevalley–Serre relations.

Using (3), we can replace C_{ij} in (4) by $-I_{ij}$, giving a new set of relations:

$$[H_i,H_j]=0\,; \qquad [X_i,Y_j]=\delta_{ij}H_j;$$
$$[H_i,X_j]=-I_{ij}X_j\,; \qquad [H_i,Y_j]=I_{ij}Y_j;$$
$$\mathrm{ad}(X_i)^{1+I_{ij}}(X_j)=0\,; \qquad \mathrm{ad}(Y_i)^{1+I_{ij}}(Y_j)=0. \tag{5}$$

These are the CIRs relations, which are central in what follows. They demonstrate that *the intersection numbers of the exceptional divisor completely determine the commutators of the corresponding Lie algebra*.

3 Lie Algebras of the Third Kind

Here we generalize the CIRs relations to the case of codimension 6 orbifold singularities.

For codimension $2n$ singularities, $n \geq 2$, the components of the exceptional divisor are $(2n-2)$-cycles, and the intersection of a pair of those has dimension

$$\dim(C_1 \cap C_2) = \dim C_1 + \dim C_2 - 2n = 2n-4. \tag{6}$$

Therefore, when $n = 2$ (the codimension 4 case), $\dim C_1 = \dim C_2 = 2$ and a pair of cycles intersect in a zero-dimensional space, yielding a number. But for codimension 6 orbifolds, the components C_i of the exceptional divisor are 4-cycles, and the intersection of any pair C_1, C_2 of 4-cycles does not yield a number but a two-dimensional space:

$$\dim(C_1 \cap C_2) = 4+4-6 = 2. \tag{7}$$

To obtain intersection numbers, we consider instead intersections of *triples* of 4-cycles. By iterating (6), we see that such intersections are zero-dimensional. They yield intersection numbers I_{ijk} with three indices.

The triple intersection numbers enable us to generalize the CIRs to the codimension 6 case. Take the second line of (5)

$$[H_i,X_j]=-I_{ij}X_j\,; \qquad [H_i,Y_j]=I_{ij}Y_j\,. \tag{8}$$

Using I_{ijk}, we may generalize this to

$$[A_i,B_j,X_k]=-I_{ijk}X_k\,; \qquad [A_i,B_j,Y_k]=I_{ijk}Y_k\,. \tag{9}$$

The A_i, B_j, and X_k are as yet not defined. However, it is now clear how to generalize the original correspondence of Duval and of Artin to the codimension 6 case: there is a new algebraic object that takes the place of Lie algebras, and it involves a commutator of three objects.

Definition 1. A ***Lie Algebra of the Third Kind (a "LATKe")*** $\mathfrak{L}$ is a vector space equipped with a commutator of the third kind, which is a trilinear anti-symmetric map

$$[\,\cdot\,,\,\cdot\,,\,\cdot\,] : \Lambda^3 \mathfrak{L} \to \mathfrak{L} \tag{10}$$

that satisfies the Jacobi identity of the third kind (or the LATKe Jacobi identity):

$$[X,Y,[Z_1,Z_2,Z_3]] = [[X,Y,Z_1],Z_2,Z_3] + [Z_1,[X,Y,Z_2],Z_3] + [Z_1,Z_2,[X,Y,Z_3]] \tag{11}$$

for $X,Y,Z_i \in \mathfrak{L}$.

We can now easily generalize this definition to an algebra that would correspond to codimension $2n$ orbifold singularities for any $n \geq 2$:

Definition 2. A ***Lie Algebra of the n-th Kind (a "LAnKe")*** $\mathfrak{L}$ is a vector space equipped with a commutator of the n-th kind, which is an n-linear, totally antisymmetric map

$$[\cdot,\cdot, \quad ,\cdot] : \wedge^n \mathfrak{L} \to \mathfrak{L}\,, \tag{12}$$

that satisfies the Jacobi identity of the n-th kind:

$$[X_1,\ldots X_{n-1},[Z_1,\ldots Z_n]] = \sum_{i=1}^{n} [Z_1,\ldots,[X_1,\ldots X_{n-1},Z_i],\ldots Z_n]\,, \tag{13}$$

for $X_i, Z_j \in \mathfrak{L}$.

Since our original physical motivation involved singularities in the extra-dimensional manifolds of string and M-theory, and those are at most seven-dimensional, we will concentrate on codimension 6 orbifolds rather than higher dimensional ones.

4 Example of a LATKe

Before we go any further, we construct an explicit example of a LATKe arising from a singularity. We construct it directly from the singularity $\mathbf{C}^3/\mathbf{Z}_3$, where the $\mathbf{Z}_3$ action on $\mathbf{C}^3$ is given by

$$\varepsilon : (x,y,z) \longmapsto (\varepsilon x, \varepsilon y, \varepsilon z)\,, \text{ where } \varepsilon^3 = 1, (x,y,z) \in \mathbf{C}^3. \tag{14}$$

The blow-up at the origin of this singularity is the 4-cycle $\mathbf{P}^2$. Recall that in the codimension-4 case, each component of the exceptional divisor corresponds to a node in the Dynkin diagram of the corresponding Lie algebra, and therefore to a simple root. Here too, the $\mathbf{P}^2$ corresponds to a "root" of the LATKe, which we now define.

Recall that for a Lie algebra $\mathfrak{g}$, a root is an element of the dual space of the Cartan subalgebra $\mathfrak{h}$, where $H \in \mathfrak{h}$ acts as an operator on $\mathfrak{g}$ via

$$H : X_\alpha \longmapsto [H, X_\alpha] = \alpha(H) X_\alpha, \tag{15}$$

where $X_\alpha \in \mathfrak{g}$ is a root vector. For a LATKe, there is no natural action of a subalgebra. However, given a subalgebra $\mathfrak{h}_{\mathfrak{L}} \subset \mathfrak{L}$, there *is* a natural action on $\mathfrak{L}$ of a pair $H_1, H_2 \in \mathfrak{h}_{\mathfrak{L}}$ given by

$$H_1 \wedge H_2 : X \longmapsto [H_1, H_2, X] \,, \tag{16}$$

where $X \in \mathfrak{L}$. Therefore, if we define a Cartan subalgebra $\mathfrak{h}_{\mathfrak{L}}$ to be a maximal commuting subalgebra of $\mathfrak{L}$ such that $\Lambda^2 \mathfrak{h}_{\mathfrak{L}}$ acts diagonally on $\mathfrak{L}$, then we can define a root as follows:

Definition 3. Let $\mathfrak{L}$ be a LATKe and let $\mathfrak{h}_{\mathfrak{L}}$ be a Cartan subalgebra of $\mathfrak{L}$. A ***root*** α of $\mathfrak{L}$ is a map in the dual space of $\Lambda^2 \mathfrak{h}_{\mathfrak{L}}$:

$$\alpha : \Lambda^2 \mathfrak{h}_{\mathfrak{L}} \longrightarrow \mathbf{C} \,. \tag{17}$$

Since we have a single cycle in our exceptional divisor (i.e. the $\mathbf{P}^2$), our root space is one-dimensional. From the definition of a root, we see this means that the Cartan subalgebra is two-dimensional. So we have, so far, four elements in the LATKe: H_1 and H_2 (making up the Cartan subalgebra), a positive root X, and a negative root Y. We also have

$$[H_1, H_2, X] = \alpha(H_1 \wedge H_2) X \,, \tag{18}$$

$$[H_1, H_2, Y] = -\alpha(H_1 \wedge H_2) Y \,, \tag{19}$$

where α is a simple root. Note that $\alpha(H_1 \wedge H_2) = -I_{111}$, the triple intersection of the exceptional divisor of our singularity, but we can normalize H_1 and H_2 so that

$$[H_1, H_2, X] = X \,;$$

$$[H_1, H_2, Y] = -Y \,. \tag{20}$$

We have but two commutators left to determine: $[H_i, X, Y]$, $i = 1, 2$. To do so, we use the LATKe Jacobi identity, which can be shown to require, among other things, that $[H_i, X, Y] \in \mathfrak{h}_{\mathfrak{L}}$. We also restrict our attention to "simple" LATKes, which we now define.

Definition 4. An ***ideal*** of $\mathfrak{L}$ is a subalgebra $\mathscr{I}$ that satisfies

$$[\mathfrak{L}, \mathfrak{L}, \mathscr{I}] \subset \mathscr{I} \,. \tag{21}$$

Definition 5. A LATKe is ***simple*** if it is non-Abelian and has no non-trivial ideals.

Now our example of a simple LATKe, which we name $\mathfrak{L}_3$, is fully determined as follows:

Example 1. The simple LATKe $\mathfrak{L}_3$ corresponding to the singularity $\mathbf{C}^3/\mathbf{Z}_3$, with $\mathbf{Z}_3$ action given by (14), is four dimensional, with commutators

$$[H_1, H_2, X] = X, \qquad [H_1, H_2, Y] = -Y,$$
$$[H_1, X, Y] = H_2, \qquad [H_2, X, Y] = H_1,$$

where H_1, H_2 form the Cartan subalgebra $\mathfrak{h}_{\mathfrak{L}_3}$ and X, Y are positive and negative root vectors, respectively.

One may easily check that the LATKe Jacobi identity is satisfied. Note that with an appropriate change of basis [10], one can see that this algebra is given by the cross product in four dimensions, or equivalently by the algebra of differential forms in four dimensions with the triple commutator given by the Hodge dual of the exterior product of three 1-forms.

5 Classification of Simple LATKes

Having constructed a LATKe from a codimension 6 orbifold singularity, we turn to the task of classifying all simple finite dimensional LATKes. We present here only a brief sketch of the proof of the classification; the complete proof and any omitted details can be found in [10].

Let $\mathrm{Der}(\mathfrak{L}) = \mathfrak{g}_{\mathfrak{L}}$ be the Lie algebra of derivations of $\mathfrak{L}$, consisting of operators D satisfying

$$D[X, Y, Z] = [DX, Y, Z] + [X, DY, Z] + [X, Y, DZ]\,, \tag{22}$$

with the Lie bracket

$$[D_1, D_2] = D_1 D_2 - D_2 D_1\ . \tag{23}$$

Then $\mathfrak{L}$ itself is a representation space for $\mathfrak{g}_{\mathfrak{L}}$. In fact, it can be shown that if $\mathfrak{L}$ is simple, it is irreducible and faithful as a representation of $\mathfrak{g}_{\mathfrak{L}}$, so $g_{\mathfrak{L}}$ is reductive. Further, it can be shown that the center of $\mathfrak{g}_{\mathfrak{L}}$ is trivial, leading to:

Lemma 1. *If $\mathfrak{L}$ is simple then $\mathfrak{g}_{\mathfrak{L}}$ is semi-simple.*

The surjective morphism of representations of $\mathfrak{g}_{\mathfrak{L}}$

$$\mathrm{ad} : \Lambda^2 \mathfrak{L} \longrightarrow \mathfrak{g}_{\mathfrak{L}} \tag{24}$$

indicates a close relation between weights of $\Lambda^2 \mathfrak{L}$ and roots of $\mathfrak{g}_{\mathfrak{L}}$. By studying this morphism and its kernel we obtain an equation relating highest roots of $\mathfrak{g}_{\mathfrak{L}}$ to the highest weight of $\mathfrak{L}$:

Lemma 2. *Let θ be a highest root of $\mathfrak{g}_{\mathfrak{L}}$, and let Λ be the highest weight of $\mathfrak{L}$ as a representation of $\mathfrak{g}_{\mathfrak{L}}$. Then*

$$\theta = 2\Lambda - \alpha \tag{25}$$

for some simple positive root α of $\mathfrak{g}_{\mathfrak{L}}$.

The two lemmas put together mean that for a Lie algebra $\mathfrak{g}$ to serve as $\mathfrak{g}_{\mathfrak{L}}$ for some LATKe $\mathfrak{L}$, it must be semisimple and it must admit a faithful, irreducible representation whose highest weight Λ satisfies (25). Interestingly, the condition in (25) was studied in an entirely different context by Kac [15].

For our purposes, the condition in (25) is necessary but not sufficient; an additional requirement is that the map $\omega : \Lambda^2 V \to \mathfrak{g}$ of representations of $\mathfrak{g}$ must yield a LATKe commutator via

$$[v_1, v_2, v_3] = (\omega(v_1 \wedge v_2)) \cdot v_3 \ , \qquad v_i \in V, \tag{26}$$

where the expression on the right hand side must be antisymmetric in all three variables.

As it happens, rather surprisingly, there is only one Lie algebra that satisfies all these conditions. It is $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, the Lie algebra of derivations of our example $\mathfrak{L}_3$. So we have

Theorem 1. *There is precisely one simple LATKe, namely $\mathfrak{L}_3$ of Example 1.*

6 The Physics of LATKes

Having constructed a LATKe directly from a codimension 6 singularity, and having discovered its uniqueness, we now turn to two physical applications: first, we describe LATKe gauge theory, which is a new kind of gauge theory that arises from codimension 6 orbifold singularities; and second, we interpret the very uniqueness of the LATKe as a new kind of vacuum selection mechanism for the string landscape.

6.1 LATKe Gauge Theory

In analogy with the traditional treatment of Lie algebras and their applications in particle physics, we define a representation for LATKes. We begin with an example: the adjoint representation. This is a map that utilizes the commutator in a natural way:

$$\mathrm{ad} : \mathfrak{L} \wedge \mathfrak{L} \longrightarrow \mathrm{End}(\mathfrak{L}) \tag{27}$$

$$\mathrm{ad}(X \wedge Y) : Z \longmapsto [X, Y, Z] \ . \tag{28}$$

The map ad satisfies the condition

$$[\mathrm{ad}(X_1 \wedge X_2), \mathrm{ad}(X_3 \wedge X_4)] = \mathrm{ad}([X_1, X_2, X_3] \wedge X_4) + \mathrm{ad}(X_3 \wedge [X_1, X_2, X_4]), \quad (29)$$

which is equivalent to the LATKe Jacobi identity. If we generalize (27) and (29), we have

Definition 6. A ***representation*** of a LATKe $\mathfrak{L}$ is a map

$$\rho : \Lambda^2 \mathfrak{L} \longrightarrow \mathrm{End}(V) \quad (30)$$

for some vector space V, subject to the condition

$$[\rho(X_1 \wedge X_2), \rho(X_3 \wedge X_4)] = \rho([X_1, X_2, X_3] \wedge X_4) + \rho(X_3 \wedge [X_1, X_2, X_4]). \quad (31)$$

In traditional Yang–Mills theory, one studies matter fields ψ in certain representations ρ of the gauge group or Lie algebra, and the Yang–Mills Lagrangian contains terms in which the fields are transformed according to those representations. For the LATKe, we are able [10] to construct an analogous system, using the definition of representations of a LATKe rather than representations of an ordinary Lie algebra.

We end up with more than we could have hoped for: in conventional Yang Mills theory, we have what is known as "pure Yang–Mills theory," where the gauge fields, which live in the adjoint representation of the gauge group, are the only fields. There are no matter fields—that is, no field ψ appears—and the Lagrangian consists only of the kinetic term of the gauge field. In general, for physical theories to include matter fields they typically have to be put in by hand.

But in the LATKe Yang–Mills theory, this is not the case. Built into the theory is not just the adjoint representation $\Lambda^2\mathfrak{L}$ of $\mathfrak{g}_{\mathfrak{L}}$, but also the adjoint representation of the LATKe itself, i.e. $\mathfrak{L}$. This representation is in fact a matter representation of $\mathfrak{g}_{\mathfrak{L}}$ and an inseparable part of *pure* LATKe Yang–Mills theory.

Therefore, unlike pure Yang–Mills theory, pure *LATKe* Yang–Mills theory *automatically includes matter*, without the need to put it in by hand. The fact that matter, which must of course be included in any physical theory, is intrinsic to LATKe gauge theory makes it all the more compelling.

6.2 Uniqueness of the LATKe as a Vacuum Selection Mechanism

One of the central outcomes of the "string revolution" of the mid-1980s [16–18] was that string theory came along with gauge theories. At the time, the gauge theories that arose were far larger than the Standard Model gauge group: anomaly cancellation dictated they may be only $E_8 \times E_8$ or $SO(32)$. However, the fact that gauge theories appeared at all was a triumph for string theory, as it gave hope for applications of string theory to the real world. It led physicists to believe for many years that upon searching further, the Calabi–Yau or G_2 manifold that results precisely in the Standard Model of Particle Physics would be found.

After a while it became apparent [19] that there is a staggering number of possible CY or G_2 manifolds, forming what is now known as the "string landscape." Therefore, the idea of a "vacuum selection mechanism," which is some principle that would single out one vacuum or at least narrow down the choices considerably, has been sought after.

The uniqueness of the LATKe *is* a vacuum selection mechanism. The selected compactification space is a Calabi–Yau or G_2 space with a $\mathbf{C}^3/\mathbf{Z}_3$ singularity, and the selected vacuum theory is a supersymmetric $\mathfrak{su}(2) \times \mathfrak{su}(2)$ gauge theory with matter in the $(2,2)$ representation.

While it has been accepted that no vacuum selection mechanisms have as yet been proposed [19], in retrospect we claim that before the present work, there did exist a vacuum selection mechanism: anomaly cancellation. It selected a string theory with gauge group either $E_8 \times E_8$ or $SO(32)$.

While neither the uniqueness of the LATKe nor anomaly cancellation actually selects the standard model itself, our unique, simple LATKe Yang–Mills is tantalizingly close to the standard model.

Acknowledgements The author is grateful to the organizers, and especially to V. Dobrev, for putting together such a vibrant, stimulating, and enjoyable workshop.

References

1. Hull, C.M., Townsend, P.K.: Enhanced gauge symmetries in superstring theories. Nucl. Phys. B **451**, 525 (1995) [arXiv:hep-th/9505073]
2. Witten, E.: String theory dynamics in various dimensions. Nucl. Phys. B **443**, 85 (1995) [arXiv:hep-th/9503124]
3. Acharya, B.S.: M theory, Joyce orbifolds and super Yang-Mills. Adv. Theor. Math. Phys. **3**, 227 (1999) [arXiv:hep-th/9812205]
4. Acharya, B.S.: On realising N = 1 super Yang-Mills in M theory [arXiv:hep-th/0011089]
5. Atiyah, M., Witten, E.: M-theory dynamics on a manifold of G_2 holonomy. Adv. Theor. Math. Phys. **6**, 1 (2003) [arXiv:hep-th/0107177]
6. Friedmann, T.: On the quantum moduli space of M theory compactifications. Nucl. Phys. B **635**, 384 (2002) [arXiv:hep-th/0203256]
7. Friedmann, T.: Physics through extra dimensions: on dualities, unification, and pair production. Ph.D. thesis, Princeton University, 2003, UMI-31-03026; http://physics.princeton.edu/www/jh/thesis_friedmann_tamar.pdf
8. Friedmann, T., Witten, E.: Unification scale, proton decay, and manifolds of G_2 holonomy. Adv. Theor. Math. Phys. **7**, 577 (2003) [arXiv:hep-th/0211269]
9. Georgi, H., Glashow, S.L.: Unity of all elementary particle forces. Phys. Rev. Lett. **32**, 438 (1974)
10. T. Friedmann, Orbifold singularities, Lie algebras of the third kind (LATKes), and pure Yang-Mills with matter. J. Math. Phys. **52**, 022304 (2011)

11. Klein, F.: Lectures on the Ikosahedron and the Solution of Equations of the Fifth Degree, Translated by G.G. Morrice. Trubner & Co., London (1888)
12. Du Val, P.: On isolated singularities of surfaces which do not affect the conditions of adjunctions. Proc. Camb. Philos. Soc. **30**, 453–465, 483–491 (1933/1934)
13. Artin, M.: On isolated rational singularities of surfaces. Am. J. Math. **88**, 129–136 (1966)
14. Steinberg, R.: Kleinian singularities and unipotent elements. Proc. Symp. Pure Math. **37**, 265 (1980)
15. Kac, V.: Lie superalgebras. Adv. Math. **26**, 8 (1977)
16. Green, M.B., Schwarz, J.H.: Anomaly cancellation in supersymmetric D=10 gauge theory and superstring theory. Phys. Lett. B **149**, 117 (1984)
17. Gross, D.J., Harvey, J.A., Martinec, E.J., Rohm, R.: The heterotic string. Phys. Rev. Lett. **54**, 502 (1985)
18. Candelas, P., Horowitz, G.T., Strominger, A., Witten, E.: Vacuum configurations for superstrings. Nucl. Phys. B **258**, 46 (1985)
19. Douglas, M.R.: The statistics of string/M theory vacua. J. High Energy Phys. **0305**, 046 (2003) [arXiv:hep-th/0303194]

On Modified Gravity

Ivan Dimitrijevic, Branko Dragovich, Jelena Grujic, and Zoran Rakic

Abstract We consider some aspects of nonlocal modified gravity, where nonlocality is of the type $R\mathscr{F}(\Box)R$. In particular, using Ansatz of the form $\Box R = cR^{\gamma}$, we find a few special cosmological solutions for the spatially flat FLRW metric. There are singular and nonsingular bounce solutions. For late cosmic time, scalar curvature $R(t)$ is in low regime and scale factor $a(t)$ is decelerated.

1 Introduction

General theory of relativity was founded by Einstein at the end of 1915 and has been successfully verified as modern theory of gravity for the Solar System. It is done by the Einstein equations of motion for gravitational field: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$, which can be derived from the Einstein–Hilbert action $S = \frac{1}{16\pi G}\int\sqrt{-g}Rd^4x + \int\sqrt{-g}\mathscr{L}_{mat}d^4x$.

Attempts to modify general relativity started already at its early times and it was mainly motivated by research of possible mathematical generalizations. Recently there has been an intensive activity in gravity modification, motivated by discovery of accelerating expansion of the Universe, which has not yet generally accepted theoretical explanation. If general relativity is theory of gravity for the Universe as

I. Dimitrijevic (✉) • Z. Rakic
Faculty of Mathematics, University of Belgrade, Studentski trg 16, Belgrade, Serbia
e-mail: ivand@matf.bg.ac.rs; zrakic@matf.bg.ac.rs

B. Dragovich
Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Zemun, Belgrade, Serbia
e-mail: dragovich@ipb.ac.rs

J. Grujic
Teachers Training Faculty, University of Belgrade, Kraljice Natalije 43, Belgrade, Serbia
e-mail: jelenagg@gmail.com

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_17,

a whole then it has to be some new kind of matter with negative pressure, dubbed *dark energy*, which is responsible for acceleration. However, general relativity has not been verified at the cosmic scale (low curvature regime) and dark energy has not been directly detected. This situation has motivated a new interest in modification of general relativity, which should be some kind of its generalization (for a recent review of various approaches, see [1], and for renormalizability [2]). However there is not a unique way how to modify general relativity. Among many approaches there are two of them, which have been much investigated: (1) $f(R)$ theories of gravity (for a review, see [3]) and (2) nonlocal gravities (see, e.g. [4, 5] and references therein).

In the case of $f(R)$ gravity, the Ricci scalar R in the action is replaced by a function $f(R)$. This is extensively investigated for the various forms of function $f(R)$. We have had some investigation when $f(R) = R\cosh\frac{\alpha R+\beta}{\gamma R+\delta}$ and, after completion of research, the results will be presented elsewhere.

In the sequel we shall consider some aspects of nonlocal gravity. Nonlocality means that Lagrangian contains an infinite number of space-time derivatives, i.e. derivatives up to an infinitive order in the form of d'Alembert operator $\Box$. In string theory nonlocality emerges as a consequence of extendedness of strings. Since string theory contains gravity as well as other kinds of interaction and matter, it is natural to expect nonlocality not only in the matter sector but also in geometrical sector of gravity. On some developments in cosmology with nonlocality in the matter sector one can see, e.g., [6–9] and references therein. In the next section we shall discuss a nonlocal modification of only geometry sector of gravity and its corresponding cosmological solutions (on nonlocality in both sectors, see [10]).

2 On a Nonlocal Modification of Gravity

Under nonlocal modification of gravity we understand replacement of the Ricci curvature R in the action by a suitable function $F(R,\Box)$, where $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$.

Inspired by [5] (for recent developments, see [11, 12]), we consider nonlocal Lagrangian without matter in the form

$$S = \int d^4x\sqrt{-g}\Big(\frac{R}{16\pi G} + \frac{c}{2}R\mathscr{F}(\Box)R\Big), \tag{1}$$

which was proposed in [13], where $\mathscr{F}(\Box) = \sum\limits_{n=0}^{\infty} f_n\Box^n$ and c is a constant. By variation of the Lagrangian (1) with respect to metric $g^{\mu\nu}$ one obtains the equation of motion for $g_{\mu\nu}$

$$(1+16\pi Gc\mathscr{F}(\Box)R)G_{\mu\nu} = 4\pi Gc\sum_{n=1}^{+\infty} f_n \sum_{l=0}^{n-1}(\partial_\mu \Box^l R \partial_\nu \Box^{n-1-l}R$$
$$+\partial_\nu \Box^l R \partial_\mu \Box^{n-1-l}R - g_{\mu\nu}(g^{\rho\sigma}\partial_\rho \Box^l R \partial_\sigma \Box^{n-1-l}R + \Box^l R \Box^{n-l}R))$$
$$-4\pi G g_{\mu\nu} cR\mathscr{F}(\Box)R + 16\pi Gc(D_\mu \partial_\nu - g_{\mu\nu}\Box)\mathscr{F}(\Box)R. \tag{2}$$

The trace of (2) is also a useful formula and it is

$$\sum_{n=1}^{+\infty} f_n \sum_{l=0}^{n-1}(\partial_\mu \Box^l R \partial^\mu \Box^{n-1-l}R + 2\Box^l R \Box^{n-l}R) + 6\Box\mathscr{F}(\Box)R = \frac{R}{8\pi Gc}. \tag{3}$$

We mainly use the spatially flat (homogeneous and isotropic) Friedmann–Lemaître–Robertson–Walker (FLRW) metric $ds^2 = -dt^2 + a^2(t)\big(dx^2+dy^2+dz^2\big)$. Investigation of (2) and finding its general solution is a very difficult task. Hence it is important to find some special solutions. To this end some Ansätze of the form $\Box R = cR^\gamma$ seem to be useful. In the sequel we construct a few such Ansätze.

2.1 Case $\Box R = rR$

At the beginning, to illustrate method, we investigate Ansatz of the simplest form: $\Box R = rR$. For this Ansatz, where r is a constant, we have

$$\Box^n R = r^n R, \qquad \mathscr{F}(\Box)R = \mathscr{F}(r)R. \tag{4}$$

In the FLRW metric $\Box = -\partial_t^2 - 3H\partial_t$ and Ansatz $\Box R = rR$ becomes

$$\ddot{R} + 3H\dot{R} + rR = 0, \tag{5}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Replacing

$$R = 6(\dot{H} + 2H^2) \tag{6}$$

in (5) we get

$$\dddot{H} + 4\dot{H}^2 + 7H\ddot{H} + 12H^2\dot{H} + r(\dot{H} + 2H^2) = 0. \tag{7}$$

A solution of this equation is

$$H(t) = \frac{1}{2t+C_1}. \tag{8}$$

This implies scale factor $a(t) = C_2\sqrt{|2t+C_1|}$ and acceleration $\ddot{a} = -\frac{C_2}{|2t+C_1|\sqrt{|2t+C_1|}}$, where $C_2 > 0, C_1 \in \mathbb{R}$. Calculation of R by expression (6) gives $R = 0$ and it is consistent with other formula containing R, including (3).

It is natural to take constant $C_1 = 0$, because it yields symmetrical solutions with respect to $t = 0$. Result $a(t) = C_2\sqrt{|2t|}$ is an example of the symmetric singular bounce solution.

Note that the above solutions hold also when $r = 0$ in the Ansatz, i.e. $\Box R = 0$. More general Ansatz $\Box R = rR + s$ was considered in [5].

2.2 Case $\Box R = q R^2$

The corresponding differential equation for the Hubble parameter is

$$\dddot{H} + 4\dot{H}^2 + 7H\ddot{H} + 12H^2\dot{H} + 6q(\dot{H}^2 + 4H^2\dot{H} + 4H^4) = 0 \tag{9}$$

with solution

$$H_\eta(t) = \frac{2\eta+1}{3}\frac{1}{t+C_1}, \quad q_\eta = \frac{6(\eta-1)}{(2\eta+1)(4\eta-1)}, \ \eta \in \mathbb{R}. \tag{10}$$

Another solution is $H = \frac{1}{2}\frac{1}{t+C_1}$ with arbitrary coefficient q, what is equivalent to the Ansatz $\Box R = rR$ with $R = 0$.

The corresponding scalar curvature is given by

$$R_\eta = \frac{2}{3}\frac{(2\eta+1)(4\eta-1)}{(t+C_1)^2}, \ \eta \in \mathbb{R}. \tag{11}$$

It is interesting that $\Box^n R_n = 0$ when $n \in \mathbb{N}$. This can be shown by mathematical induction by the following way. It is evident that $\Box R_1 = 0$. Suppose that $\Box^n R_n = 0$, then $\Box^{n+1} R_{n+1} = \Box\Box^n R_n + \frac{16n+10}{9}\Box^n\Box R_1 = 0$.

This $\Box^n R_n = 0$ property simplifies the equations considerably. For this special case of solutions trace equation (3) effectively becomes

$$\sum_{k=1}^{n+1} f_k \sum_{l=0}^{k-1} (\partial_\mu \Box^l R \partial^\mu \Box^{k-1-l} R + 2\Box^l R \Box^{k-l} R) + 6\Box\mathscr{F}(\Box)R = \frac{R}{8\pi Gc}, \tag{12}$$

where

$$\mathscr{F}(\Box)R = \sum_{k=0}^{n-1} f_k \Box^k R. \tag{13}$$

In particular case $n = 2$ the trace formula becomes

$$\frac{36}{35}f_0R^2 + f_1(-\dot{R}^2 + \frac{12}{35}R^3) + f_2(-\frac{24}{35}R\dot{R}^2 + \frac{72}{1225}R^4) + f_3(-\frac{144}{1225}R^2\dot{R}^2) = \frac{R}{8\pi Gc}. \quad (14)$$

2.3 Case $\Box^n R = c_n R^{\alpha n+\beta}$

We consider[1] another Ansatz of the form $\Box^n R = c_n R^{\alpha n+\beta}$, where α and β are constants, and $n \in \mathbb{N}$. From the equalities

$$\begin{aligned}\Box^{n+1}R &= \Box c_n R^{\alpha n+\beta}\\ &= c_n((\alpha n+\beta)R^{\alpha n+\beta-1}\Box R - (\alpha n+\beta)(\alpha n+\beta-1)R^{\alpha n+\beta-2}\dot{R}^2)\\ &= c_n(\alpha n+\beta)(c_1R^{\alpha n+\alpha+2\beta-1} - (\alpha n+\beta-1)R^{\alpha n+\beta-2}\dot{R}^2) = c_{n+1}R^{\alpha n+\alpha+\beta}\end{aligned} \quad (15)$$

we get the following conditions:

$$\alpha n+\alpha+2\beta-1 = \alpha n+\alpha+\beta, \quad (16)$$

$$\dot{R}^2 = R^{\alpha+\beta+1}, \quad (17)$$

$$c_{n+1} = c_n(\alpha n+\beta)(c_1-\alpha n-\beta+1). \quad (18)$$

Equation (16) implies that β is equal to 1. Sequence c_n is defined by (18) and can be explicitly written ($\beta = 1$) as

$$c_n = c_1\prod_{k=1}^{n-1}(\alpha k+1)(c_1-\alpha k), \quad (19)$$

where c_1 is an arbitrary constant. General solution of (17) is of the form

$$R(t) = 2^{2/\alpha}\left(\alpha\left(\pm t-d_1\right)\right)^{-2/\alpha}, \quad d_1 \in \mathbb{R} \quad (20)$$

with arbitrary constant d_1.

Case $\alpha = 1$. In the case $\alpha = 1$ the coefficients c_n are given by $c_n = (n!)^2\binom{c_1}{n}$, where c_1 is the first element. Putting $\alpha = 1$ into (20) one obtains

$$R(t) = \frac{4}{(t-d_1)^2}. \quad (21)$$

The corresponding expressions for $H(t)$ and $a(t)$ are:

[1] We thank A.S. Koshelev for suggestion of Ansatz $\Box^n R \sim R^{n+1}$.

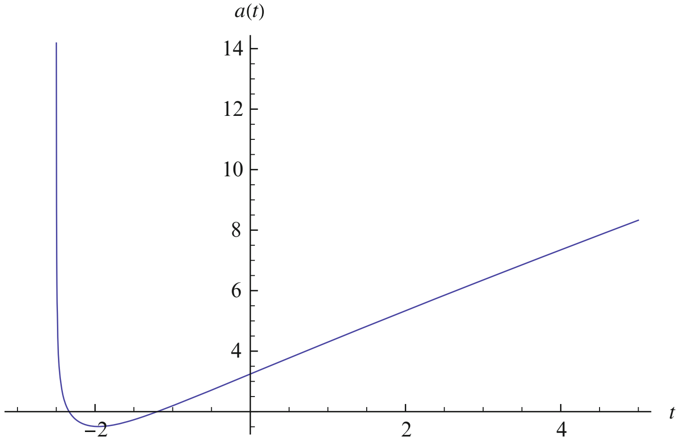


Fig. 1 Scale factor $a(t)$ given by (23) for $d_1 = -2.5, \quad d_2 = 2$ and $d_3 = 1$

$$H(t) = \frac{\left(3+\sqrt{57}\right) d_2 (t-d_1)^{\sqrt{\frac{19}{3}}} - \sqrt{57}+3}{12(t-d_1)\left(d_2(t-d_1)^{\sqrt{\frac{19}{3}}}+1\right)}, \tag{22}$$

$$a(t) = d_3 (t-d_1)^{\frac{3-\sqrt{57}}{12}} \sqrt{d_2 (t-d_1)^{\sqrt{\frac{19}{3}}}+1}, \tag{23}$$

where d_1, d_2, d_3 are arbitrary real constants (Fig. 1).

The function $a(t)$ has a vertical asymptote at the point $t = d_1$. If $d_1 < 0$ then $a(t) > 0$ for all $t > 0$. For large values of t, $a(t)$ is asymptotically equivalent to $t^{\frac{1}{2}\sqrt{\frac{19}{3}}+\frac{1}{12}\left(3-\sqrt{57}\right)} \approx t^{0.879}$.

$$\ddot{a}(T) = -\frac{d_3 T^{\frac{1}{12}\left(-21-\sqrt{57}\right)}\left(\left(\sqrt{57}-5\right) d_2^2 T^{2\sqrt{\frac{19}{3}}} - 48 d_2 T^{\sqrt{\frac{19}{3}}} - \sqrt{57}-5\right)}{24\left(d_2 T^{\sqrt{\frac{19}{3}}}+1\right)^{3/2}}, \tag{24}$$

where $T = t - d_1$. The expansion is accelerated for $d_2(t-d_1)^{\sqrt{\frac{19}{3}}} < \frac{24}{\sqrt{57}-5} + \frac{4\sqrt{38}}{\sqrt{57}-5}$ and it is decelerated otherwise.

Note that this Ansatz $\Box^n R = c_n R^{n+1}$ for $n = 1$ coincides with Ansatz $\Box R = q_\eta R^2$, when $\eta = \frac{-1\pm\sqrt{57}}{8}$, because then one can take $c_1 = q_\eta = \frac{-9\pm\sqrt{57}}{8}$. In this particular case they have the same scalar curvature R and the same Hubble parameter for $d_2 = 0$. However, apart from this special case $\eta = \frac{-1\pm\sqrt{57}}{8}$, constant q_η is different of $c_1 = \frac{-9\pm\sqrt{57}}{8}$.

Case $\alpha = \frac{1}{2}$. Putting $\alpha = \frac{1}{2}$ into (20) we obtain

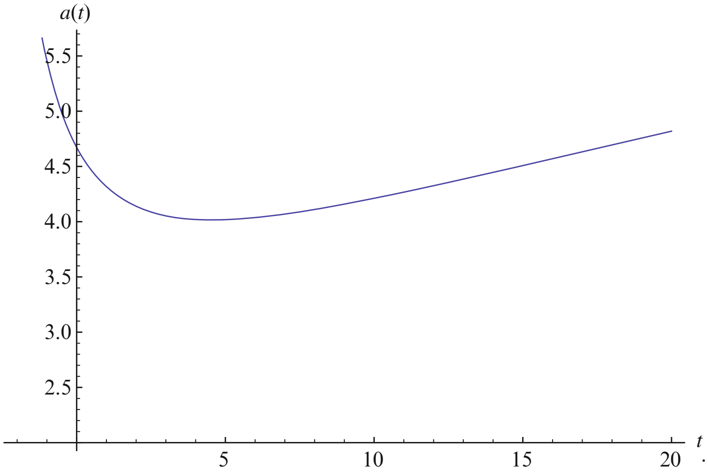


Fig. 2 Scale factor $a(t)$ given by (27) for $d_1 = \frac{8}{\sqrt{3}}$, $d_2 = 2$ and $d_3 = \frac{1}{10}$

$$R(t) = \frac{256}{(t+d_1)^4}. \tag{25}$$

From (6) we obtain

$$H(t) = \frac{-512\sqrt{3}d_2 e^{\frac{32}{\sqrt{3}(d_1+t)}} + 3(t+d_1)\left(32d_2 e^{\frac{32}{\sqrt{3}(d_1+t)}} + \sqrt{3}\right) + 48}{6\,(d_1+t)^2\left(32d_2 e^{\frac{32}{\sqrt{3}(d_1+t)}} + \sqrt{3}\right)} \tag{26}$$

and then

$$a(t) = d_3 e^{-\frac{8}{\sqrt{3}(d_1+t)}}\sqrt{d_1+t}\sqrt{32d_2 e^{\frac{32}{\sqrt{3}(d_1+t)}} + \sqrt{3}}, \tag{27}$$

where d_1, d_2, d_3 are some real constants (Fig. 2).

The corresponding acceleration is

$$\begin{aligned} \ddot{a}(t) = {} & \frac{d_3 e^{-\frac{8}{\sqrt{3}(d_1+t)}}}{12\,(d_1+t)^{7/2}\left(32d_2 e^{\frac{32}{\sqrt{3}(d_1+t)}} + \sqrt{3}\right)^{3/2}}\Bigg(1024\,d_2^2\,e^{\frac{64}{\sqrt{3}(d_1+t)}} \\ & \times\left(-6d_1 t - 3d_1^2 + 32\sqrt{3}d_1 - 3t^2 + 32\sqrt{3}t + 256\right) \\ & - 3\left(6d_1 t + 3d_1^2 + 32\sqrt{3}d_1 + 3t^2 + 32\sqrt{3}t - 256\right) \\ & - 192\sqrt{3}d_2 e^{\frac{32}{\sqrt{3}(d_1+t)}}\,(d_1+t-16)\,(d_1+t+16)\Bigg). \end{aligned} \tag{28}$$

The acceleration is positive for $t < t_0$ and negative for $t > t_0$, where t_0 is the zero of $\ddot{a}(t)$. For large values of t, $\ddot{a}(t)$ converges to 0.

Case $\alpha = 2$. In the case $\alpha = 2$ only one integration can be performed and it gives

$$H(t) = \frac{-\sqrt{3}d_2 I_1\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right) - d_2\sqrt{t-d_1}I_0\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right) - d_2\sqrt{t-d_1}I_2\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right)}{4\sqrt{3}(t-d_1)\left(K_1\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right) - d_2 I_1\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right)\right)} + \frac{-\sqrt{t-d_1}K_0\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right) + \sqrt{3}K_1\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right) - \sqrt{t-d_1}K_2\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right)}{4\sqrt{3}(t-d_1)\left(K_1\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right) - d_2 I_1\left(\frac{2\sqrt{t-d_1}}{\sqrt{3}}\right)\right)}. \tag{29}$$

I_i and K_i are modified Bessel functions of the first and the second kind, respectively, and d_1, d_2 are real constants.

Case $\alpha = -2$. For $\alpha = -2$ we obtain expression for $H(t)$ involving Airy functions

$$H(t) = \frac{\sqrt[3]{-\frac{1}{3}}\left(d_2\mathrm{Ai}'\left(\sqrt[3]{-\frac{1}{3}}(t-d_1)\right) + \mathrm{Bi}'\left(\sqrt[3]{-\frac{1}{3}}(t-d_1)\right)\right)}{2\left(d_2\mathrm{Ai}\left(\sqrt[3]{-\frac{1}{3}}(t-d_1)\right) + \mathrm{Bi}\left(\sqrt[3]{-\frac{1}{3}}(t-d_1)\right)\right)}. \tag{30}$$

3 Concluding Remarks

In this article we presented three Ansätze, two of them are quite new and can be adjusted so that $\Box^n R = 0$. These two Ansätze have solutions for scalar curvature of the form $R = \frac{C_2}{(t+C_1)^2}$, which satisfy all but extended Einstein equations (2) and related trace formula (3). It is a consequence of the quadratic form in R of the Lagrangian (1). However these Ansätze are promising for some new nonlocal Lagrangians, which investigation is in progress.

It is worth mentioning that all the above Ansätze contain solution $R = 0$, which satisfies all (including (2) and (3)) equations with curvature constant $k = -1$. Namely, for $R = 0$, (2) reduces to $G_{\mu\nu} = 0$ and it gives

$$\frac{\ddot{a}}{a} = 0, \qquad \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = 0. \tag{31}$$

If $k = 0$ one has only static solution $a = constant$. However, when $k = -1$ then $a(t) = |t|$ and it contains a crunch preceding to a big bang.

Above considered Ansätze may be also useful in analysis of some other nonlocal gravity and cosmology models. Further investigation of nonlocality governed by the Riemann zeta function in p-adic strings dynamics [14] extends interesting cases and can give new insights.

Acknowledgements This investigation is supported by Ministry of Education and Science of the Republic of Serbia, grant No 174012. The authors are grateful to Alexey Koshelev for useful discussions and they thank the anonymous referee for constructive comments helping to improve the presentation and clarify some assertions. This is an extended version of the talk presented by I.D. at the IX International Workshop "Lie Theory and its Applications in Physics", 20–26 June 2011, Varna, Bulgaria. I.D. thanks the organizers for the hospitality and ICTP for financial support.

References

1. Clifton, T., Ferreira, P.G., Padilla, A., Skordis, C.: Modified gravity and cosmology [arXiv:1106.2476v2 [astro-ph.CO]]
2. Modesto, L.: Super-renormalizable quantum gravity [arXiv:1107.2403v1 [hep-th]]
3. Sotiriou, T.P., Faraoni, V.: $f(R)$ theories of gravity. Rev. Mod. Phys. **82**, 451–497 (2010) [arXiv:0805.1726v4 [gr-qc]]
4. Nojiri, S., Odintsov, S.D.: Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models. Phys. Rep. **505**, 59–144 (2011) [arXiv:1011.0544v4 [gr-qc]]
5. Biswas, T., Koivisto, T., Mazumdar, A.: Towards a resolution of the cosmological singularity in non-local higher derivative theories of gravity. J. Cosmology Astropart. Phys. **1011**, 008 (2010) [arXiv:1005.0590v2 [hep-th]]
6. Aref'eva, I.Ya., Joukovskaya, L.V., Vernov, S.Yu.: Bouncing and accelerating solutions in nonlocal stringy models. J. High Energy Phys. **0707**, 087 (2007) [hep-th/0701184]
7. Calcagni, G., Montobbio, M., Nardelli, G.: A route to nonlocal cosmology. Phys. Rev. D **76**, 126001 (2007) [arXiv:0705.3043v3 [hep-th]]
8. Barnaby, N., Biswas, T., Cline, J.M.: p-Adic inflation. J. High Energy Phys. **0704**, 056 (2007) [hep-th/0612230]
9. Koshelev, A.S., Vernov, S.Yu.: Analysis of scalar perturbations in cosmological models with a non-local scalar field. Class. Quant. Gravit. **28**, 085019 (2011) [arXiv:1009.0746v2 [hep-th]]
10. Calcagni, G., Nardelli, G.: Nonlocal gravity and the diffusion equation. Phys. Rev. D **82**, 123518 (2010) [arXiv:1004.5144v2 [hep-th]]
11. Koshelev, A.S.: Modified non-local gravity [arXiv:1112.6410v1 [hep-th]]
12. Koshelev, A.S., Vernov, S.Yu.: On bouncing solutions in non-local gravity [arXiv:1202.1289v1 [hep-th]]
13. Biswas, T., Mazumdar, A., Siegel, W.: Bouncing universes in string-inspired gravity. J. Cosmology Astropart. Phys. **0603**, 009 (2006) [arXiv:hep-th/0508194]
14. Dragovich, B.: Nonlocal dynamics of p-adic strings. Theor. Math. Phys. **164**(3), 1151–115 (2010) [arXiv:1011.0912v1 [hep-th]]

Part IV
Quantum Groups and Related Objects

The q-Wakimoto Realization of the Superalgebras $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$

Takeo Kojima

Abstract We give bosonizations of the superalgebras $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbf{C}$. We introduce the submodule by the ξ-η system, that we call the q-Wakimoto realization.

1 Introduction

Bosonizations are known to be a powerful method to construct correlation functions in not only conformal field theory [1], but also exactly solvable lattice models [2]. The quantum algebra $U_q(g)$ and the elliptic algebra $U_{q,p}(g)$ play an important role in exactly solvable lattice models. The level parameter k plays an important role in representation theory for $U_q(g)$ and $U_{q,p}(g)$. Bosonizations for an arbitrary level k are completely different from those of level $k=1$. In the case for level $k=1$, bosonizations have been constructed for quantum algebra $U_q(g)$ in many cases $g=(ADE)^{(r)}$, $(BC)^{(1)}$, $G_2^{(1)}$, $\widehat{sl}(M|N)$, $osp(2|2)^{(2)}$ [3–10]. Using the dressing method developed in non-twisted algebra [11] and twisted algebra $A_2^{(2)}$ [12], we have bosonizations of the elliptic algebra $U_{q,p}(g)$ for $g=(ADE)^{(1)}, (BC)^{(1)}, G_2^{(1)}$ and $A_2^{(2)}$. In the case of an arbitrary level k, bosonizations have been constructed only for $U_q(\widehat{sl}(N))$ [13–15], $U_q(\widehat{sl}(2|1))$ [16], $U_{q,p}(\widehat{sl}(N))$ [11], and $U_{q,p}(\widehat{sl}(2|1))$ [17]. In this paper we give a bosonization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level k [18]. Using the dressing method developed in [17], we give a bosonization of the quantum superalgebra $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k. The level k bosonizations on the boson Fock space of $U_q(\widehat{sl}(N))$ and

T. Kojima (✉)
Graduate School of Science and Engineering, Yamagata University,
Jonan 4-3-16, Yonezawa 992-8510, Japan
e-mail: kojima@yz.yamagata-u.ac.jp

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_18,

$U_q(\widehat{sl}(N|1))$ [15, 16, 18] are not irreducible realizations. The construction of the irreducible highest weight module $V(\lambda)$ is nontrivial problem. We recall the non-quantum algebra $\widehat{sl}(2)$ case [19]. The irreducible highest weight module $V(\lambda)$ for the affine algebra $\widehat{sl}(2)$ was constructed from the Wakimoto realization on the boson Fock space [20] by the Felder complex. We recall the quantum algebra $U_q(\widehat{sl}(2))$ case [13, 14, 21]. The irreducible highest weight module $V(\lambda)$ for $U_q(\widehat{sl}(2))$ was constructed from the level k bosonizations on the boson Fock space [13, 14] by two steps; the first step is the resolution by the ξ-η system, and the second step is the resolution by the Felder complex [19, 21]. The submodule of the quantum algebra $U_q(\widehat{sl}(2))$, induced by the ξ-η system, plays the same role as the Wakimoto realization of the non-quantum algebra $\widehat{sl}(2)$. We would like to call this submodule induced by the ξ-η system "the q-Wakimoto realization". Constructions of the irreducible highest weight module $V(\lambda)$ for $U_q(\widehat{sl}(N))$ $(N \geq 3)$ and $U_q(\widehat{sl}(N|1))$ $(N \geq 2)$ are still an open problem. In this paper we study the ξ-η system and introduce the q-Wakimoto realization for the superalgebra $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$.

This paper is organized as follows. In Sect. 2, after preparing notations, we give the definition of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ and the elliptic superalgebra $U_{q,p}(\widehat{sl}(N|1))$. In Sect. 3 we give bosonizations of the superalgebras $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k. In Sect. 4 we introduce the q-Wakimoto realization of by the ξ-η system.

2 Superalgebra $U_q(\widehat{sl}(\mathbf{N}|\mathbf{1}))$ and $U_{q,p}(\widehat{sl}(\mathbf{N}|\mathbf{1}))$

In this section we recall the definitions of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ [22] and the elliptic deformed superalgebra $U_{q,p}(\widehat{sl}(N|1))$ [17] for $N \geq 2$. We fix a complex number $q \neq 0, |q| < 1$. We set

$$[x,y] = xy - yx,\ \{x,y\} = xy + yx,\ [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}. \tag{1}$$

Let us fix complex numbers $r, k \in \mathbf{C}$, $\mathrm{Re}(r) > 0, \mathrm{Re}(r-k) > 0$. We use the abbreviation $r^* = r - k$. We set $p = q^{2r}$. We set the Jacobi theta functions

$$[u] = q^{\frac{u^2}{r} - u} \frac{\Theta_{q^{2r}}(q^{2u})}{(q^{2r}; q^{2r})_\infty^3},\ [u]^* = q^{\frac{u^2}{r^*} - u} \frac{\Theta_{q^{2r^*}}(q^{2u})}{(q^{2r^*}; q^{2r^*})_\infty^3}, \tag{2}$$

where we have used

$$\Theta_p(z) = (z;p)_\infty (pz^{-1};p)_\infty (p;p)_\infty,\ (z;p)_\infty = \prod_{m=0}^{\infty} (1 - p^m z). \tag{3}$$

The Cartan matrix $(A_{i,j})_{0\le i,j\le N}$ of the affine Lie algebra $\widehat{sl}(N|1)$ is given by

$$A_{i,j}=(\nu_i+\nu_{i+1})\delta_{i,j}-\nu_i\delta_{i,j+1}-\nu_{i+1}\delta_{i+1,j}. \tag{4}$$

Here we set $\nu_1=\cdots=\nu_N=+, \nu_{N+1}=\nu_0-$.

2.1 Quantum Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we recall the definition of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$.

Definition 1 ([22]). The Drinfeld generators of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ are

$$x_{i,m}^{\pm},\ h_{i,m},\ c\ (1\le i\le N, m\in\mathbf{Z}). \tag{5}$$

Defining relations are

$$c:\text{central},\ [h_i,h_{j,m}]=0, \tag{6}$$

$$[a_{i,m},h_{j,n}]=\frac{[A_{i,j}m]_q[cm]_q}{m}q^{-c|m|}\delta_{m+n,0}\ (m,n\neq 0), \tag{7}$$

$$[h_i,x_j^{\pm}(z)]=\pm A_{i,j}x_j^{\pm}(z), \tag{8}$$

$$[h_{i,m},x_j^{+}(z)]=\frac{[A_{i,j}m]_q}{m}q^{-c|m|}z^m x_j^{+}(z)\ (m\neq 0), \tag{9}$$

$$[h_{i,m},x_j^{-}(z)]=-\frac{[A_{i,j}m]_q}{m}z^m x_j^{-}(z)\ (m\neq 0), \tag{10}$$

$$(z_1-q^{\pm A_{i,j}}z_2)x_i^{\pm}(z_1)x_j^{\pm}(z_2)=(q^{\pm A_{j,i}}z_1-z_2)x_j^{\pm}(z_2)x_i^{\pm}(z_1)\ \text{for}\ |A_{i,j}|\neq 0, \tag{11}$$

$$x_i^{\pm}(z_1)x_j^{\pm}(z_2)=x_j^{\pm}(z_2)x_i^{\pm}(z_1)\ \text{for}\ |A_{i,j}|=0, (i,j)\neq(N,N), \tag{12}$$

$$\{x_N^{\pm}(z_1),x_N^{\pm}(z_2)\}=0, \tag{13}$$

$$[x_i^{+}(z_1),x_j^{-}(z_2)]=\frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\Big(\delta(q^{-c}z_1/z_2)\psi_i^{+}(q^{\frac{c}{2}}z_2) \\ -\delta(q^{c}z_1/z_2)\psi_i^{-}(q^{-\frac{c}{2}}z_2)\Big),\quad \text{for}\ (i,j)\neq(N,N), \tag{14}$$

$$\{x_N^{+}(z_1),x_N^{-}(z_2)\}=\frac{1}{(q-q^{-1})z_1z_2}\Big(\delta(q^{-c}z_1/z_2)\psi_N^{+}(q^{\frac{c}{2}}z_2) \\ -\delta(q^{c}z_1/z_2)\psi_N^{-}(q^{-\frac{c}{2}}z_2)\Big), \tag{15}$$

$$\Big(x_i^\pm(z_1)x_i^\pm(z_2)x_j^\pm(z)-(q+q^{-1})x_i^\pm(z_1)x_j^\pm(z)x_i^\pm(z_2)+x_j^\pm(z)x_i^\pm(z_1)x_i^\pm(z_2)\Big)$$
$$+(z_1\leftrightarrow z_2)=0 \text{ for } |A_{i,j}|=1,\ i\neq N, \tag{16}$$

where we have used $\delta(z)=\sum_{m\in\mathbf{Z}}z^m$. Here we have used the abbreviation $h_i=h_{i,0}$. We have set the generating function

$$x_j^\pm(z)=\sum_{m\in\mathbf{Z}}x_{j,m}^\pm z^{-m-1}, \tag{17}$$

$$\psi_i^+(q^{\frac{c}{2}}z)=q^{h_i}\exp\left((q-q^{-1})\sum_{m>0}h_{i,m}z^{-m}\right), \tag{18}$$

$$\psi_i^-(q^{-\frac{c}{2}}z)=q^{-h_i}\exp\left(-(q-q^{-1})\sum_{m>0}h_{i,-m}z^{m}\right). \tag{19}$$

2.2 Elliptic Superalgebra $U_{q,p}(\widehat{sl}(N|1))$

In this section we recall the definition of the elliptic superalgebra $U_{q,p}(\widehat{sl}(N|1))$.

Definition 2 ([17]). The elliptic superalgebra $U_{q,p}(\widehat{sl}(N|1))$ is the associative algebra generated by the currents $E_j(z),F_j(z),H_j^\pm(z)$ $(1\leq j\leq N)$ and $B_{j,m}(1\leq j\leq N, m\in\mathbf{Z}_{\neq 0})$, h_j $(1\leq j\leq N)$ that satisfy the following relations.

$$[h_i,B_{j,m}]=0,\ [B_{i,m},B_{j,n}]=\frac{[A_{i,j}m]_q[km]_q}{m}\frac{[r^*m]_q}{[rm]_q}\delta_{m+n,0}, \tag{20}$$

$$[h_i,E_j(z)]=A_{i,j}E_j(z),\ [h_i,F_j(z)]=-A_{i,j}F_j(z), \tag{21}$$

$$[B_{i,m},E_j(z)]=\frac{[A_{i,j}m]_q}{m}z^mE_j(z), \tag{22}$$

$$[B_{i,m},F_j(z)]=-\frac{[A_{i,j}m]_q}{m}\frac{[r^*m]_q}{[rm]_q}z^mF_j(z). \tag{23}$$

For $1\leq i,j\leq N$ such that $(i,j)\neq(N,N)$ they satisfy

$$\left[u_1-u_2-\frac{A_{i,j}}{2}\right]^*E_i(z_1)E_j(z_2)=\left[u_1-u_2+\frac{A_{i,j}}{2}\right]^*E_j(z_2)E_i(z_1), \tag{24}$$

$$\left[u_1-u_2+\frac{A_{i,j}}{2}\right]F_i(z_1)F_j(z_2)=\left[u_1-u_2-\frac{A_{i,j}}{2}\right]F_j(z_2)F_i(z_1), \tag{25}$$

$$[E_i(z_1),F_j(z_2)]=\frac{\delta_{i,j}}{(q-q^{-1})z_1z_2}\Big(\delta(q^{-k}z_1/z_2)H_i(q^rz_2)-\delta(q^kz_1/z_2)H_i(q^{-r}z_2)\Big), \tag{26}$$

$$\{E_N(z_1),E_N(z_2)\}=0,\qquad \{F_N(z_1),F_N(z_2)\}=0, \tag{27}$$

$$\{E_N(z_1),F_N(z_2)\}=\frac{1}{(q-q^{-1})z_1z_2}\Big(\delta(q^{-k}z_1/z_2)H_N(q^rz_2)-\delta(q^kz_1/z_2)H_N(q^{-r}z_2)\Big). \tag{28}$$

For $1\le i,j\le N$ they satisfy

$$H_i(z_1)H_j(z_2)=\frac{[u_2-u_1-\frac{A_{i,j}}{2}]^*[u_2-u_1+\frac{A_{i,j}}{2}]}{[u_2-u_1+\frac{A_{i,j}}{2}]^*[u_2-u_1-\frac{A_{i,j}}{2}]}H_j(z_2)H_i(z_1), \tag{29}$$

$$H_i(z_1)E_j(z_2)=\frac{[u_1-u_2+\frac{r^*}{2}+\frac{A_{i,j}}{2}]^*}{[u_1-u_2+\frac{r^*}{2}-\frac{A_{i,j}}{2}]^*}E_j(z_2)H_i(z_1), \tag{30}$$

$$H_i(z_1)F_j(z_2)=\frac{[u_1-u_2+\frac{r}{2}+\frac{A_{i,j}}{2}]}{[u_1-u_2+\frac{r}{2}-\frac{A_{i,j}}{2}]}F_j(z_2)H_i(z_1). \tag{31}$$

For $1\le i,j\le N,(i\ne N)$ such that $|A_{i,j}|=1$, they satisfy the Serre relations.

$$\begin{aligned}
&\Bigg\{E_i(z_1)E_i(z_2)E_j(z)\frac{\left(q^{2r^*+A_{i,j}}\frac{z}{z_1};q^{2r^*}\right)_\infty\left(q^{2r^*+A_{i,j}}\frac{z}{z_2};q^{2r^*}\right)_\infty}{\left(q^{2r^*-A_{i,j}}\frac{z}{z_1};q^{2r^*}\right)_\infty\left(q^{2r^*-A_{i,j}}\frac{z}{z_2};q^{2r^*}\right)_\infty}\left(\frac{z}{z_2}\right)^{\frac{1}{r^*}A_{i,j}}\\
&-(q+q^{-1})E_i(z_1)E_j(z)E_i(z_2)\frac{\left(q^{2r^*+A_{i,j}}\frac{z}{z_1};q^{2r^*}\right)_\infty\left(q^{2r^*+A_{i,j}}\frac{z_2}{z};q^{2r^*}\right)_\infty}{\left(q^{2r^*-A_{i,j}}\frac{z}{z_1};q^{2r^*}\right)_\infty\left(q^{2r^*-A_{i,j}}\frac{z_2}{z};q^{2r^*}\right)_\infty}\\
&+E_j(z)E_i(z_1)E_i(z_2)\frac{\left(q^{2r^*+A_{i,j}}\frac{z_1}{z};q^{2r^*}\right)_\infty\left(q^{2r^*+A_{i,j}}\frac{z_2}{z};q^{2r^*}\right)_\infty}{\left(q^{2r^*-A_{i,j}}\frac{z_1}{z};q^{2r^*}\right)_\infty\left(q^{2r^*-A_{i,j}}\frac{z_2}{z};q^{2r^*}\right)_\infty}\left(\frac{z_1}{z}\right)^{\frac{1}{r^*}A_{i,j}}\Bigg\}\\
&\times\frac{\left(q^{2r^*+A_{i,i}}\frac{z_2}{z_1};q^{2r^*}\right)_\infty}{\left(q^{2r^*-A_{i,i}}\frac{z_2}{z_1};q^{2r^*}\right)_\infty}z_1^{-\frac{1}{r^*}(A_{i,i}+A_{i,j})}+(z_1\leftrightarrow z_2)=0,
\end{aligned} \tag{32}$$

$$\left\{ F_i(z_1)F_i(z_2)F_j(z) \frac{\left(q^{2r-A_{i,j}}\frac{z}{z_1};q^{2r}\right)_\infty \left(q^{2r-A_{i,j}}\frac{z}{z_2};q^{2r}\right)_\infty}{\left(q^{2r+A_{i,j}}\frac{z}{z_1};q^{2r}\right)_\infty \left(q^{2r+A_{i,j}}\frac{z}{z_2};q^{2r}\right)_\infty} \left(\frac{z_2}{z}\right)^{\frac{1}{r}A_{i,j}} \right.$$

$$-(q+q^{-1})F_i(z_1)F_j(z)F_i(z_2) \frac{\left(q^{2r-A_{i,j}}\frac{z}{z_1};q^{2r}\right)_\infty \left(q^{2r-A_{i,j}}\frac{z_2}{z};q^{2r}\right)_\infty}{\left(q^{2r+A_{i,j}}\frac{z}{z_1};q^{2r}\right)_\infty \left(q^{2r+A_{i,j}}\frac{z_2}{z};q^{2r}\right)_\infty}$$

$$\left. +F_j(z)F_i(z_1)F_i(z_2) \frac{\left(q^{2r-A_{i,j}}\frac{z_1}{z};q^{2r}\right)_\infty \left(q^{2r-A_{i,j}}\frac{z_2}{z};q^{2r}\right)_\infty}{\left(q^{2r+A_{i,j}}\frac{z_1}{z};q^{2r}\right)_\infty \left(q^{2r+A_{i,j}}\frac{z_2}{z};q^{2r}\right)_\infty} \left(\frac{z}{z_1}\right)^{\frac{1}{r}A_{i,j}} \right\}$$

$$\times \frac{\left(q^{2r-A_{i,i}}\frac{z_2}{z_1};q^{2r}\right)_\infty}{\left(q^{2r+A_{i,i}}\frac{z_2}{z_1};q^{2r}\right)_\infty} z_1^{\frac{1}{r}(A_{i,i}+A_{i,j})} + (z_1 \leftrightarrow z_2) = 0. \tag{33}$$

Here we have used $z_j = q^{2u_j}$.

3 Bosonization

In this section we give bosonizations of the superalgebras $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k [16–18].

3.1 Boson

We fix the level $c = k \in \mathbf{C}$. We introduce the bosons and the zero-mode operators a_m^j, Q_a^j $(m \in \mathbf{Z}, 1 \le j \le N)$, $b_m^{i,j}, Q_b^{i,j}$ $(m \in \mathbf{Z}, 1 \le i < j \le N+1)$, $c_m^{i,j}, Q_c^{i,j}$ $(m \in \mathbf{Z}, 1 \le i < j \le N)$. The bosons $a_m^i, b_m^{i,j}, c_m^{i,j}$, $(m \in \mathbf{Z}_{\neq 0})$ and the zero-mode operators a_0^i, Q_a^i, $b_0^{i,j}, Q_b^{i,j}$, $c_0^{i,j}, Q_c^{i,j}$ satisfy

$$[a_m^i, a_n^j] = \frac{[(k+N-1)m]_q [A_{i,j}m]_q}{m} \delta_{m+n,0}, \ [a_0^i, Q_a^j] = (k+N-1)A_{i,j}, \tag{34}$$

$$[b_m^{i,j}, b_n^{i',j'}] = -\nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \ [b_0^{i,j}, Q_b^{i',j'}] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'}, \tag{35}$$

$$[c_m^{i,j}, c_n^{i',j'}] = \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \ [c_0^{i,j}, Q_c^{i',j'}] = \delta_{i,i'} \delta_{j,j'}. \tag{36}$$

We impose the cocycle condition on the zero-mode operator $Q_b^{i,j}$, $(1 \le i < j \le N+1)$ by

$$[Q_b^{i,j}, Q_b^{i',j'}] = \delta_{j,N+1} \delta_{j',N+1} \pi \sqrt{-1} \text{ for } (i,j) \neq (i',j'). \tag{37}$$

We have the following (anti) commutation relations

$$\left[e^{Q_b^{i,j}}, e^{Q_b^{i',j'}}\right] = 0 \ (1 \le i < j \le N, 1 \le i' < j' \le N), \tag{38}$$

$$\left\{e^{Q_b^{i,N+1}}, e^{Q_b^{j,N+1}}\right\} = 0 \ (1 \le i \ne j \le N). \tag{39}$$

We use the standard symbol of the normal orderings ::. In what follows we use the abbreviations $b^{i,j}(z), c^{i,j}(z), b_\pm^{i,j}(z), a_\pm^j(z)$ given by

$$\begin{aligned} b^{i,j}(z) &= -\sum_{m\neq 0} \frac{b_m^{i,j}}{[m]_q} z^{-m} + Q_b^{i,j} + b_0^{i,j} \log z, \ c^{i,j}(z) \\ &= -\sum_{m\neq 0} \frac{c_m^{i,j}}{[m]_q} z^{-m} + Q_c^{i,j} + c_0^{i,j} \log z, \end{aligned} \tag{40}$$

$$\begin{aligned} b_\pm^{i,j}(z) &= \pm(q - q^{-1}) \sum_{\pm m > 0} b_m^{i,j} z^{-m} \pm b_0^{i,j} \log q, \ a_\pm^j(z) \\ &= \pm(q - q^{-1}) \sum_{\pm m > 0} a_m^j z^{-m} \pm a_0^j \log q. \end{aligned} \tag{41}$$

3.2 Quantum Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we give a bosonization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level k.

Theorem 1 ([18]). *The Drinfeld currents $x_i^\pm(z)$, $\psi_i^\pm(z)$, $(1 \le i \le N)$ of $U_q(\widehat{sl}(N|1))$ for an arbitrary level k are realized by the bosonic operators as follows.*

$$\begin{aligned} x_i^+(z) = \frac{1}{(q-q^{-1})z} \sum_{j=1}^{i} &: \exp\left((b+c)^{j,i}(q^{j-1}z) + \sum_{l=1}^{j-1} (b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^l z))\right) \\ &\times \left\{\exp\left(b_+^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^j z)\right) - \right. \\ &\left. - \exp\left(b_-^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^{j-2}z)\right)\right\} :, \end{aligned} \tag{42}$$

$$\begin{aligned} x_N^+(z) = \sum_{j=1}^{N} &: \exp\left((b+c)^{j,N}(q^{j-1}z) + b^{j,N+1}(q^{j-1}z) \right. \\ &\left. - \sum_{l=1}^{j-1} (b_+^{l,N+1}(q^l z) + b_+^{l,N}(q^l z))\right) :, \end{aligned} \tag{43}$$

$$
\begin{aligned}
x_i^-(z) = q^{k+N-1} &: \exp\Big(a_+^i(q^{\frac{k+N-1}{2}}z) - b^{i,N+1}(q^{k+N-1}z) \\
&- b_+^{i+1,N+1}(q^{k+N-1}z) + b^{i+1,N+1}(q^{k+N}z)\Big) : \\
&+ \frac{1}{(q-q^{-1})z} \sum_{j=1}^{i-1} : \exp\Big(a_-^i(q^{-\frac{k+N-1}{2}}z)\Big) \times \\
&\times\ \exp\Big((b+c)^{j,i+1}(q^{-k-j}z) + b_-^{i,n+1}(q^{-k-n}z) - b_-^{i+1,n+1}(q^{-k-n+1}z)\Big) \\
&\times \exp\Bigg(\sum_{l=j+1}^{i}(b_-^{l,i+1}(q^{-k-l+1}z) - b_-^{l,i}(q^{-k-l}z)) + \\
&+ \sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z))\Bigg) \\
&\times \Big\{\exp\Big(-b_-^{j,i}(q^{-k-j}z) - (b+c)^{j,i}(q^{-k-j+1}z)\Big) \\
&-\ \exp\Big(-b_+^{j,i}(q^{-k-j}z) - (b+c)^{j,i}(q^{-k-j-1}z)\Big)\Big\} : \\
&+ \frac{1}{(q-q^{-1})z} : \Big\{\exp\Big(a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{i,i+1}(q^{-k-i}z) \\
&+ \sum_{l=i+1}^{N}(b_-^{i,l}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z)) \\
&+ b_-^{i,N+1}(q^{-k-N}z) - b_-^{i+1,N+1}(q^{-k-N+1}z)\Big) \\
&- \exp\Big(a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,i+1}(q^{k+i}z) \\
&+ \sum_{l=i+1}^{N}(b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)) \\
&+ b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z)\Big)\Big\} : \\
&- \frac{1}{(q-q^{-1})z} \sum_{j=i+1}^{N-1} : \exp\Big(a_+^i(q^{\frac{k+N-1}{2}}z)\Big) \\
&\times \exp\Big((b+c)^{i,j+1}(q^{k+j}z) + b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z)\Big) \\
&\times \exp\Bigg(\sum_{l=j+1}^{N}(b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z))\Bigg) \\
&\times\ \Big\{\exp\Big(b_+^{i+1,j+1}(q^{k+j}z) - (b+c)^{i+1,j+1}(q^{k+j+1}z)\Big) \\
&-\ \exp\Big(b_-^{i+1,j+1}(q^{k+j}z) - (b+c)^{i+1,j+1}(q^{k+j-1}z)\Big)\Big\} : . \qquad (44)
\end{aligned}
$$

$$\begin{aligned}
x_N^-(z) = & \frac{1}{(q-q^{-1})z}\left\{\sum_{j=1}^{N-1} q^{j-1} : \exp\left(a_-^N(q^{-\frac{k+N-1}{2}}z)\right)\right. \\
& \times \exp\left(-b_+^{j,N+1}(q^{-k-j}z) - b^{j,N+1}(q^{-k-j-1}z)\right. \\
& \left. - \sum_{l=j+1}^{N-1}(b_-^{l,N}(q^{-k-l}z) + b_-^{l,N+1}(q^{-k-l}z))\right) \\
& \times \left\{\exp\left(-b_+^{j,N}(q^{-k-j}z) - (b+c)^{j,N}(q^{-k-j-1}z)\right)\right. \\
& \left. - \exp\left(-b_-^{j,N}(q^{-k-j}z) - (b+c)^{j,N}(q^{-k-j+1}z)\right)\right\} : \\
& + q^{N-1} : \left\{\exp\left(a_+^N(q^{\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{k+N-1}z)\right)\right. \\
& \left.\left. - \exp\left(a_-^N(q^{-\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{-k-N+1}z)\right)\right\} : \right\}. \qquad (45)
\end{aligned}$$

$$\begin{aligned}
\psi_i^{\pm}(q^{\pm\frac{k}{2}}z) = & \exp\left(a_\pm^i(q^{\pm\frac{k+N-1}{2}}z) + \sum_{l=1}^{i}(b_\pm^{l,i+1}(q^{\pm(l+k-1)}z) - b_\pm^{l,i}(q^{\pm(l+k)}z)\right) \\
& \times \exp\left(\sum_{l=i+1}^{N}(b_\pm^{i,l}(q^{\pm(k+l)}z) - b_\pm^{i-1,l}(q^{\pm(k+l-1)}z)\right. \qquad (46) \\
& \left. + b_\pm^{i,N+1}(q^{\pm(k+N)}z) - b_\pm^{i+1,N+1}(q^{\pm(k+N-1)}z)\right),
\end{aligned}$$

$$\psi_N^{\pm}(q^{\pm\frac{k}{2}}z) = \exp\left(a_\pm^N(q^{\pm\frac{k+N-1}{2}}z) - \sum_{l=1}^{N-1}(b_\pm^{l,N}(q^{\pm(k+l)}z) + b_\pm^{l,N+1}(q^{\pm(k+l)}z))\right) \qquad (47)$$

3.3 Elliptic Superalgebra $U_{q,p}(\widehat{sl}(N|1))$

In this section we give a bosonization of the elliptic superalgebra $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k, using the dressing deformation [17]. Let us introduce the zero-mode operators P_i, Q_i, $(1 \le i \le N)$ by

$$[P_i, Q_j] = -\frac{A_{i,j}}{2}\ (1 \le i, j \le N), \qquad (48)$$

where $(A_{i,j})_{1\le i,j\le N}$ is the Cartan matrix of the classical $sl(N|1)$. In [18] the bosonization of the Drinfeld generator $h_{i,m}$ $(1\le i\le N, m\in\mathbf{Z})$ is given by

$$\begin{aligned} h_{i,m} = q^{-\frac{k+N-1}{2}|m|}a_m^i + \sum_{l=1}^{i}(q^{-(k+l-1)|m|}b_m^{l,i+1} - q^{-(k+l)|m|}b_m^{l,i}) \\ + \sum_{l=i+1}^{N}(q^{-(k+l)|m|}b_m^{i,l} - q^{-(k+l-1)|m|}b_m^{i+1,l}) \\ + q^{-(k+N)|m|}b_m^{i,N+1} - q^{-(k+N-1)|m|}b_m^{i+1,N+1}, \end{aligned} \tag{49}$$

$$h_{N,m} = q^{-\frac{k+N-1}{2}|m|}a_m^N - \sum_{l=1}^{N-1}(q^{-(k+l)|m|}b_m^{l,N} + q^{-(k+l)|m|}b_m^{l,N+1}). \tag{50}$$

Let us set the boson $B_{j,m}$ $(1\le j\le N, m\in\mathbf{Z}_{\neq 0})$ by

$$B_{j,m} = \begin{cases} \frac{[r^*m]_q}{[rm]_q}h_{j,m} & (m>0), \\ q^{k|m|}h_{j,m} & (m<0). \end{cases} \tag{51}$$

Theorem 2 ([17, 18]). *The currents $E_j(z), F_j(z), H_j^{\pm}(z)$ $(1\le j\le N)$ of the elliptic superalgebra $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level k are realized by the bosonic operators as follows.*

$$E_j(z) = U_j^+(z)x_j^+(z)e^{2Q_j}z^{-\frac{1}{r^*}P_j}, \tag{52}$$

$$F_j(z) = x_j^-(z)U_j^-(z)z^{\frac{1}{r}(P_j+h_j)}, \tag{53}$$

$$H_j^{\pm}(z) = H_j(q^{\pm(r-\frac{k}{2})}z), \tag{54}$$

$$H_j(z) = :\exp\left(-\sum_{m\neq 0}\frac{B_{j,m}}{[r^*m]_q}z^{-m}\right): e^{2Q_j}z^{-\frac{k}{rr^*}P_j+\frac{1}{r}h_j}. \tag{55}$$

Here we have used the dressing operators $U_j^+(z), U_j^-(z)$ $(1\le j\le N)$ given by

$$U_j^+(z) = \exp\left(\sum_{m>0}\frac{q^{rm}}{[rm]_q}B_{j,-m}z^m\right),\ U_j^-(z) = \exp\left(-\sum_{m>0}\frac{q^{r^*m}}{[rm]_q}B_{j,m}z^{-m}\right). \tag{56}$$

4 q-Wakimoto Realization

In this section we introduce the q-Wakimoto realization by the ξ-η system, following $U_q(\widehat{sl}(2|1))$ case [23]. We introduce the vacuum state $|0\rangle$ of the boson Fock space by

$$a_m^i|0\rangle = b_m^{i,j}|0\rangle = c_m^{i,j}|0\rangle = 0 \ (m \geq 0). \tag{57}$$

For complex numbers $p_a^i \in \mathbf{C}$ $(1 \leq i \leq N)$, $p_b^{i,j} \in \mathbf{C}$ $(1 \leq i < j \leq N+1)$, $p_c^{i,j} \in \mathbf{C}$ $(1 \leq i < j \leq N)$, we set

$$\begin{aligned}|p_a, p_b, p_c\rangle = \\ &= \exp\left(\sum_{i,j=1}^{N} \frac{\mathrm{Min}(i,j)(N-1-\mathrm{Max}(i,j))}{(N-1)(k+N-1)} p_a^i Q_a^j\right. \\ &\left. - \sum_{1\leq i<j\leq N+1} p_b^{i,j} Q_b^{i,j} + \sum_{1\leq i<j\leq N} p_c^{i,j} Q_c^{i,j}\right)|0\rangle.\end{aligned} \tag{58}$$

It satisfies

$$\begin{aligned}&a_0^i|p_a, p_b, p_c\rangle = p_a^i|p_a, p_b, p_c\rangle,\ b_0^{i,j}|p_a, p_b, p_c\rangle = p_b^{i,j}|p_a, p_b, p_c\rangle, \\ &c_0^{i,j}|p_a, p_b, p_c\rangle = p_c^{i,j}|p_a, p_b, p_c\rangle.\end{aligned} \tag{59}$$

The boson Fock space $F(p_a, p_b, p_c)$ is generated by the bosons $a_m^i, b_m^{i,j}, c_m^{i,j}$ on the vector $|p_a, p_b, p_c\rangle$. We set $U_q(\widehat{sl}(N|1))$-module $F(p_a)$ by

$$F(p_a) = \bigoplus_{\substack{p_b^{i,j} = -p_c^{i,j} \in \mathbf{Z}\ (1\leq i<j\leq N) \\ p_b^{i,N+1} \in \mathbf{Z}\ (1\leq i\leq N)}} F(p_a, p_b, p_c). \tag{60}$$

We have imposed the restriction $p_b^{i,j} = -p_c^{i,j} \in \mathbf{Z}$ $(1 \leq i < j \leq N)$, because the $x_{i,m}^{\pm}$ change $Q_b^{i,j} + Q_c^{i,j}$. The module $F(p_a)$ is not irreducible representation. For instance, the irreducible highest weight module $V(\lambda)$ for $U_q(\widehat{sl}(2))$ was constructed from the similar space as $F(p_a)$ by two steps; the first step is the construction of the q-Wakimoto realization by the ξ-η system, and the second step is the resolution by the Felder complex [21]. In this paper we study the ξ-η system and introduce the q-Wakimoto realization for $U_q(\widehat{sl}(N|1))$. For $1 \leq i < j \leq N$ we introduce

$$\eta^{i,j}(z) = \sum_{m\in\mathbf{Z}} \eta_m^{i,j} z^{-m-1} =: e^{c^{i,j}(z)} :,\ \xi^{i,j}(z) = \sum_{m\in\mathbf{Z}} \xi_m^{i,j} z^{-m} =: e^{-c^{i,j}(z)} :. \tag{61}$$

The Fourier components $\eta_m^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^m \eta^{i,j}(z)$, $\xi_m^{i,j} = \oint \frac{dz}{2\pi\sqrt{-1}} z^{m-1} \xi^{i,j}(z)$ $(m \in \mathbf{Z})$ are well defined on the space $F(p_a)$. They satisfy

$$\{\eta_m^{i,j}, \xi_n^{i,j}\} = \delta_{m+n,0},\ \{\eta_m^{i,j}, \eta_n^{i,j}\} = \{\xi_m^{i,j}, \xi_n^{i,j}\} = 0 \ (1 \leq i < j \leq N), \tag{62}$$

$$[\eta_m^{i,j}, \xi_n^{i',j'}] = [\eta_m^{i,j}, \eta_n^{i',j'}] = [\xi_m^{i,j}, \xi_n^{i',j'}] = 0 \ (i,j) \neq (i',j'). \tag{63}$$

We focus our attention on the operators $\eta_0^{i,j}, \xi_0^{i,j}$ satisfying $(\eta_0^{i,j})^2=0$, $(\xi_0^{i,j})^2=0$ and $\mathrm{Im}(\eta_0^{i,j})=\mathrm{Ker}(\eta_0^{i,j})$, $\mathrm{Im}(\xi_0^{i,j})=\mathrm{Ker}(\xi_0^{i,j})$. The products $\eta_0^{i,j}\xi_0^{i,j}$ and $\xi_0^{i,j}\eta_0^{i,j}$ are the projection operators

$$\eta_0^{i,j}\xi_0^{i,j}+\xi_0^{i,j}\eta_0^{i,j}=1. \tag{64}$$

We have a direct sum decomposition.

$$F(p_a)=\eta_0^{i,j}\xi_0^{i,j}F(p_a)\oplus\xi_0^{i,j}\eta_0^{i,j}F(p_a), \tag{65}$$

$$\mathrm{Ker}(\eta_0^{i,j})=\eta_0^{i,j}\xi_0^{i,j}F(p_a),\ \mathrm{Coker}(\eta_0^{i,j})=\xi_0^{i,j}\eta_0^{i,j}F(p_a). \tag{66}$$

Definition 3. We introduce the subspace $\mathscr{F}(p_a)$ that we call the q-Wakimoto realization.

$$\mathscr{F}(p_a)=\left(\prod_{1\leq i<j\leq N}\eta_0^{i,j}\xi_0^{i,j}\right)F(p_a)=\bigcap_{1\leq i<j\leq N}\mathrm{Ker}(\eta_0^{i,j}), \tag{67}$$

The dressing operators $U_i^{\pm}(z)$ and the zero-mode operators P_i, Q_i commute with $\eta_0^{i',j'}$. The bosonizations commute with the operators $\eta_0^{i',j'}$, $\xi_0^{i',j'}$ up to sign $\pm$.

Proposition 1. *The subspace $\mathscr{F}(p_a)$ is both $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$ module.*

Let $\bar{\alpha}_i$, $\bar{\Lambda}_i$, $(1\leq i\leq N)$ and $(\cdot|\cdot)$ be the simple roots, the fundamental weights, and the symmetric bilinear norm ; $(\bar{\alpha}_i,\bar{\alpha}_j)=A_{i,j}$, $(\bar{\alpha}_i,\bar{\Lambda}_j)=\delta_{i,j}$. It is expected that we have the irreducible highest weight module $V(\lambda)$ with the highest weight λ, whose classical part $\bar{\lambda}=\sum_{j=1}^N p_a^i\bar{\Lambda}_i$, by the Felder complex of the q-Wakimoto realization. We would like to report this problem for $U_q(\widehat{sl}(N))$ and $U_q(\widehat{sl}(N|1))$ in the future publication.

Acknowledgements This work is supported by the Grant-in-Aid for Scientific Research **C** (21540228) from Japan Society for Promotion of Science. The author would like to thank the organizing committee of the 9-th International Workshop "Lie Theory and Its Applications in Physics"s for an invitation to Bulgaria.

References

1. Bouwknegt, P., McCarthy, J., Pilch, K.: Prog. Theor. Phys. **102**, 67–135 (1990)
2. Jimbo, M., Miwa, T.: Algebraic analysis of solvable lattice models. In: CBMS Regional Conference Series in Mathematics, vol. 85. Amer. Math. Soc., Providence, Rhode Island (1994)
3. Frenkel, I.B., Jing, N.: Proc. Natl. Acad. Sci. **85**, 9373–9377 (1988)
4. Bernard, D.: Lett. Math. Phys. **17**, 239–245 (1989)
5. Jing, N., Koyama, Y., Misra, K.: Selecta Math. **5**(2), 243–255 (1999)

6. Jing, N.: Invent. Math. **102**, 663–690 (1990)
7. Jing, N.: Proc. Am. Math. Soc. **127**(1), 21–27 (1999)
8. Kimura, K., Shiraishi, J., Uchiyama, J.: Comm. Math. Phys. **188**(2), 367–378 (1997)
9. Zhang, Y.-Z.: J. Math. Phys. **40**(11), 6110–6124 (1999)
10. Yang, W.-L., Zhang, Y.-Z.: Phys. Lett. **A261**, 252–258 (1999)
11. Jimbo, M., Konno, H., Odake, S., Shiraishi, J.: Comm. Math. Phys. **199**, 605–647 (1999)
12. Kojima, T., Konno, H.: J. Math. Phys. **45**, 3146–3179 (2004)
13. Matsuo, A.: Comm. Math. Phys. **160**, 33–48 (1994)
14. Shiraishi, J.: Phys. Lett. **A171**, 243–248 (1992)
15. Awata, H., Odake, S., Shiraishi, J.: Comm. Math. Phys. **162** (1), 61–83 (1994)
16. Awata, H., Odake, S., Shiraishi, J.: Lett. Math. Phys. **42**(3), 271–279 (1997)
17. Kojima, T.: J. Phys.: Math. Theor. **A44**, 485205 (23pp) (2011)
18. Kojima, T.: J. Math. Phys. **53**, 013515(1–15) (2012)
19. Bernard, D., Felder, G.: Comm. Math. Phys. **127**, 145–168 (1990)
20. Wakimoto, M.: Comm. Math. Phys. **104**(4), 605–609 (1986)
21. Konno, H.: Mod. Phys. Lett. **A9**, 1253–1265 (1994)
22. Yamane, H.: Publ. Res. Inst. Math. Sci. **35**, 321–390 (1990)
23. Zhang, Y.-Z., Gould, M.D.: J. Math. Phys. **41**, 5577–5291 (2000)

Quantum Phases in Noncommutative Space

Ö.F. Dayi and B. Yapışkan

Abstract Instead of the common procedure of using star product we present an alternative method of constructing quantum mechanics in noncommutative coordinates. Within this approach we study quantum phases in noncommutative coordinates.

1 Introduction

The star product

$$\star_\theta \equiv \exp\left[\frac{i}{2}\theta_{IJ}\frac{\overleftarrow{\partial}}{\partial Q^I}\frac{\overrightarrow{\partial}}{\partial Q^J}\right], \tag{1}$$

where θ_{IJ} is an antisymmetric, constant deformation parameter is commonly used to imply noncommutativity of coordinates. This is equivalent to the shift of coordinates in terms of $P_I^{op} = -i\hbar\frac{\partial}{\partial Q^I}$ as

$$Q_I \to Q_I - \frac{1}{2\hbar}\theta_{IJ}P_J^{op}. \tag{2}$$

In this talk we would like to present an alternative approach and study its applications to diverse quantum systems as reported in [1]. The alternative method is

Ö.F. Dayi (✉)
Faculty of Science and Letters, Physics Department, Istanbul Technical University, 34469, Maslak–Istanbul, Turkey
e-mail: dayi@itu.edu.tr

B. Yapışkan
Faculty of Science and Letters, Physics Department, Mimar Sinan Fine Arts University, Çırağan Cad. Çiğdem Sok. No:1, 34349, Beşiktaş–Istanbul, Turkey
e-mail: yapiska1@gmail.com

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_19,

first introduced in [2]. It can be employed as far as in the starting Hamiltonian there exist terms which can be interpreted as minimally coupled gauge fields. We apply the new deformation procedure to obtain velocity independent formulations of Aharonov–Bohm (AB) [3], Aharonov–Casher (AC) [4], He–McKellar–Wilkens (HMW) [5, 6] and Anandan [7, 8] phases in noncommutative coordinates. Most of the earlier formulations yielded velocity dependent quantum phases in noncommutative spaces, in spite of the fact that the distinguished property of the original phases is their independence from the velocity of scattered particles. We discuss how to select the suitable realization.

2 The Alternative θ-Deformation of Quantum Mechanics

If the non-deformed system possess the gauge field A_α which may be matrix valued we introduced the field strength:

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial r^\alpha} - \frac{\partial A_\alpha}{\partial r^\beta} - \frac{i\rho}{\hbar}[A_\alpha, A_\beta]. \tag{3}$$

It worths noting that commutators appearing in this semiclassical formulation are the ordinary matrix commutators. The stating point of the alternative method can be taken as the θ-deformed quantum commutators denoted by the subscript q to distinguish them from matrix commutators:

$$[\hat{r}^\alpha, \hat{r}^\beta]_q = i\theta^{\alpha\beta}, \tag{4}$$

$$[\hat{p}^\alpha, \hat{p}^\beta]_q = i\hbar\rho F^{\alpha\beta} - i\rho^2 (F\theta F)^{\alpha\beta}, \tag{5}$$

$$[\hat{r}^\alpha, \hat{p}^\beta]_q = i\hbar\delta^{\alpha\beta} - i\rho(\theta F)^{\alpha\beta}, \tag{6}$$

$$[\hat{p}^\alpha, \hat{r}^\beta]_q = -i\hbar\delta^{\alpha\beta} + i\rho(F\theta)^{\alpha\beta}. \tag{7}$$

Because of being first order in θ, the right hand sides of (4)–(7) may only possess $\hat{r}_\alpha|_{\theta=0} = r_\alpha$ dependence. Hence $F_{\alpha\beta}$ is still as in (3).

The covariant derivative defined as

$$D_\alpha = -i\hbar\frac{\partial}{\partial r^\alpha} - \rho A_\alpha$$

can be employed to write the operators

$$\hat{p}_\alpha = D_\alpha - \frac{\rho}{2\hbar} F_{\alpha\beta}\theta^{\beta\gamma} D_\gamma, \tag{8}$$

$$\hat{r}_\alpha = r_\alpha - \frac{1}{2\hbar}\theta_{\alpha\beta} D^\beta, \tag{9}$$

which leads to a realization of the generalized algebra (4)–(7) satisfying the Jacobi identities, as far as the conditions

$$-i\hbar\nabla_\alpha F_{\beta\gamma} - \rho[A_\alpha, F_{\beta\gamma}] = 0, \quad [F_{\alpha\beta}, F_{\gamma\delta}] = 0 \tag{10}$$

are fulfilled. To illustrate the method let the initial Hamiltonian be $H_0(p) = p^2/2m$. Substituting p with the quantum operator (8) one obtains the θ-deformed Hamiltonian

$$H_0(\hat{p}) \equiv \hat{H}_{nc} = \frac{1}{2m}\left(D_\alpha - \frac{\rho}{2\hbar}F_{\alpha\beta}\theta^{\beta\gamma}D_\gamma\right)^2. \tag{11}$$

Setting $\theta = 0$ yields the Hamiltonian operator

$$\hat{H} = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial r_\alpha} - \rho A_\alpha\right)^2. \tag{12}$$

Therefore, (11) gives the noncommutative dynamics corresponding to the Hamiltonian (12).

When (10) are valid, another representation of the algebra (4)–(7) is given by

$$\hat{p}_\alpha = -i\hbar\nabla_\alpha + \frac{\rho}{2}F_{\alpha\beta}(r^\beta + 2i\theta^{\beta\gamma}\nabla_\gamma), \tag{13}$$

$$\hat{r}_\alpha = r_\alpha - \frac{1}{2\hbar}\theta_{\alpha\beta}(-i\hbar\nabla^\beta - \frac{\rho}{2}F^{\beta\gamma}r_\gamma) \tag{14}$$

In this representation only the gauge invariant field strength $F_{\alpha\beta}$ appears, in contrast to (8)–(9) where the gauge field A_α explicitly appears.

3 Quantum Phases in Noncommutative Space

Let us first present a unified formulation of the different phases which were considered in [9–13]. We start with the Hamiltonian operator

$$H = \frac{1}{2m}\left(p_\alpha - \rho A_\alpha(r)\right)^2, \tag{15}$$

where ρ is a constant. One implements noncommutativity by the shift

$$r_\alpha \to r_\alpha - \frac{1}{2\hbar}\theta_{\alpha\beta}p^\beta = r_\alpha - \frac{1}{2\hbar}\theta_{\alpha\beta}\left(\hbar k^\beta + \rho A^\beta(r)\right), \tag{16}$$

where k_α is the eigenvalue of the kinetic momentum operator:

$$\left(p_\alpha - \rho A_\alpha(r)\right)\psi(r) = \hbar k_\alpha \psi(r). \tag{17}$$

Hence, at the first order in θ the Hamiltonian (15) becomes

$$H=\frac{1}{2m}\left[p_\alpha-\rho A_\alpha(r)+\frac{\rho}{2\hbar}\theta^{\beta\sigma}\left(\hbar k_\sigma+\rho A_\sigma(r)\right)\partial_\beta A_\alpha(r)\right]^2. \quad (18)$$

Identifying,

$$\tilde{A}_\alpha(r,\theta)=A_\alpha(r)-\frac{1}{2\hbar}\theta^{\beta\sigma}\left(\hbar k_\sigma+\rho A_\sigma(r)\right)\partial_\beta A_\alpha(r) \quad (19)$$

as the gauge field in noncommutative space, one defines the θ-deformed quantum phase by

$$\Phi(\theta)=\frac{i\rho}{\hbar}\oint\tilde{A}_\alpha(r,\theta)dr^\alpha. \quad (20)$$

Different phases can be considered by choosing the original field A_α appropriately. To study the AB phase on the noncommutative plane let the nonvanishing components of the deformation parameter be

$$\theta_{ij}=\theta\varepsilon_{ij},$$

where $i,j=1,2$. Moreover, choose $\rho=-e/c$ and an appropriate three-vector potential $\boldsymbol{A}$, whose third component vanishes $A_3=0$. Hence, (20) leads to

$$\Phi^I_{AB}(\theta)=-\frac{ie}{\hbar c}\oint\boldsymbol{A}(r)\cdot d\boldsymbol{r}-\frac{ime\theta}{2\hbar c}\oint\left[(\boldsymbol{v}\times\nabla A_i)_3-\frac{e}{\hbar mc}(\boldsymbol{A}\times\nabla A_i)_3\right]dr_i,$$

where $\boldsymbol{k}=m\boldsymbol{v}$. This is the deformed phase obtained in [9, 10].

To formulate the AC, HMW and Anandan phases in noncommutative coordinates we set

$$c\rho\boldsymbol{A}=\boldsymbol{\mu}\times\boldsymbol{E}-\boldsymbol{d}\times\boldsymbol{B} \quad (21)$$

where $\boldsymbol{\mu}$ and $\boldsymbol{d}$ are the magnetic and the electric dipole moments which are proportional to the Pauli spin matrices $\boldsymbol{\sigma}$. We deal with the standard configuration where dipole moments are in z-direction and the external electric and magnetic fields are in the polar radial direction, so that $\boldsymbol{\mu}\cdot\boldsymbol{B}=0$, $\boldsymbol{d}\cdot\boldsymbol{E}=0$. Moreover, let there be no change in the dipoles along the external fields: $\boldsymbol{E}\cdot\nabla\mu=0,\boldsymbol{B}\cdot\nabla d=0$. After implying these conditions, insert (19) into (20) to obtain

$$\begin{aligned}\Phi^A(\theta)=&\frac{i}{\hbar c}\oint(\boldsymbol{\mu}\times\boldsymbol{E}-\boldsymbol{d}\times\boldsymbol{B})\cdot d\boldsymbol{r}\\&+\frac{i}{2\hbar^2c^2}\theta_{ab}\oint(\boldsymbol{k}+\boldsymbol{\mu}\times\boldsymbol{E}-\boldsymbol{d}\times\boldsymbol{B})_a\partial_b(\boldsymbol{\mu}\times\boldsymbol{E}-\boldsymbol{d}\times\boldsymbol{B})\cdot d\boldsymbol{r},\end{aligned} \quad (22)$$

where $a,b=1,2,3$. For $\boldsymbol{d}=0$ the θ-deformation of the AC phase obtained in [11] follows

$$\Phi^{AC}(\theta) = \frac{i}{\hbar c}\oint(\boldsymbol{\mu}\times\boldsymbol{E})\cdot d\boldsymbol{r} + \frac{i}{2\hbar^2c^2}\theta_{ab}\oint(\boldsymbol{k}+\boldsymbol{\mu}\times\boldsymbol{E})_a\partial_b(\boldsymbol{\mu}\times\boldsymbol{E})\cdot d\boldsymbol{r}. \quad (23)$$

For $\boldsymbol{\mu} = 0$, the HMW phase in noncommuting coordinates is obtained in accord with [12] as

$$\Phi^{HMW}(\theta) = -\frac{i}{\hbar c}\oint(\boldsymbol{d}\times\boldsymbol{B})\cdot d\boldsymbol{r} - \frac{i}{2\hbar^2c^2}\theta_{ab}\oint(\boldsymbol{k}-\boldsymbol{d}\times\boldsymbol{B})_a\partial_b(\boldsymbol{d}\times\boldsymbol{B})\cdot d\boldsymbol{r}. \quad (24)$$

By putting (23) and (24) together

$$\Phi^{A1}(\theta) = \Phi^{AC}(\theta) + \Phi^{HMW}(\theta),$$

which means ignoring the terms behaving as μd in (22), the deformation of [13] follows. Although we used 3-dimensional vectors the formalism is effectively 2-dimensional because of the selected configurations leading to the AC and HMW phases.

The approach of [14] differs from the above formulation. In [14] one considers the θ-deformed Hamiltonian defined as the generalization of the one obtained in the uniform transverse magnetic field B. In terms of the related path integral one identifies

$$\tilde{A}_i(\theta, r) = \left(1 - \frac{e\theta B}{4\hbar c}\right)^{-1} A_i(r).$$

Then, one employs it in (20) with $\rho = -e/c$ to get the AB phase in noncommutative coordinates as

$$\Phi^{II}_{AB}(\theta) = -\frac{ie}{\hbar c}\left(1 + \frac{e\theta B}{4\hbar c}\right)\oint A_i(r)dr_i.$$

Now, let us present our approach following in part the receipt given in [14]. We deal with the configurations leading to vanishing scalar potentials, so that in general the Hamiltonian in noncommutative coordinates is written in terms of $\hat{p}$ which is a realization of the algebra (4)–(7) as

$$H^{nc} = \frac{\hat{p}^2}{2m}. \quad (25)$$

Obviously, different realizations will lead to different Hamiltonians. Let (r_α, p_α) define the classical phase space variables corresponding to the operators $(r^{op}_\alpha,\ p^{op}_\alpha = -i\hbar\partial_\alpha)$. The classical Hamiltonian $H_{\text{eff}}(r,p)$ will be obtained from the related Hamiltonian operator in noncommutative space by substituting $p^{op}_\alpha, r^{op}_\alpha$ with the c-number variables p, r. To keep the discussion general let us define the classical θ-deformed Hamiltonian corresponding to (25) as

$$H^{nc}_{\text{eff}} = a_{\alpha\beta}(r,\theta)p_\alpha p_\beta + b_\alpha(r,\theta)p_\alpha + c(r,\theta), \quad (26)$$

without specifying the coefficients $a_{\alpha\beta}(r,\theta)$, $b_\alpha(r,\theta)$ and $c(r,\theta)$. Plugging (26) into the path integral

$$Z = N\int d^d p\, d^d r\, \exp\left\{\frac{i}{\hbar}\int dt[p^\alpha \dot{r}_\alpha - H_{\rm eff}(p,r)]\right\}, \tag{27}$$

where N is the normalization factor, yields the partition function in the d-dimensional phase space:

$$Z = N\int d^d p\, d^d r\, \exp\left\{\frac{i}{\hbar}\int dt\, \left[p^\alpha\left(\dot{r}_\alpha - b_\alpha(r,\alpha)\right) - a_{\alpha\beta}(r,\theta)p_\alpha p_\beta - c(r,\theta)\right]\right\}.$$

Integration over the momenta gives the partition function in configuration space with the normalization factor N' as

$$Z = N'\int d^d r\, \exp\left\{\frac{i}{\hbar}\int dt\, \left[\frac{1}{4}a^{-1}_{\alpha\beta}(r,\theta)\left(\dot{r}_\alpha - b_\alpha(r,\theta)\right)\left(\dot{r}_\beta - b_\beta(r,\theta)\right) - c(r,\theta)\right]\right\}.$$

This can be written as

$$Z = N'\int d^d r\, \exp\left\{\frac{i}{\hbar}S + \frac{i}{\hbar}\int dr_\alpha \mathscr{A}^\alpha(r,\theta)\right\},$$

in terms of

$$S = \int dt\,[\frac{1}{4}a^{-1}_{\alpha\beta}(r,\theta)(\dot{r}^\alpha\dot{r}^\beta + b^\alpha(r,\theta)b^\beta(r,\theta)) - c(r,\theta)]$$

and the θ-deformed gauge field defined as

$$\mathscr{A}_\alpha(r,\theta) \equiv -\frac{1}{2}a^{-1}_{\alpha\beta}(r,\theta)b^\beta(r,\theta). \tag{28}$$

Hence, in general we can introduce the quantum phase as follows

$$\Phi = \frac{i}{\hbar}\oint \mathscr{A}^\alpha(r,\theta)dr_\alpha = -\frac{i}{2\hbar}\oint a^{-1}_{\alpha\beta}(r,\theta)b^\beta(r,\theta)dr_\alpha. \tag{29}$$

As the first specific example we would like to discuss the AB phase in noncommutative space adopting some different realizations. Hence, let the particles be confined to move on the $r_i = (x,y)$ plane, in the presence of an infinitely long, tiny solenoid placed along the z-axis. Obviously we set $\rho = -e/c$, moreover the nonvanishing components of θ and F are

$$\theta_{ij} = \varepsilon_{ij}\theta,\; F_{ij} = \varepsilon_{ij}F_{12} = \left\{\begin{array}{ll}\varepsilon_{ij}B & \text{in,}\\ 0 & \text{out.}\end{array}\right\}$$

Except on the solenoid, the conditions (10) are fulfilled, due to the fact that F_{12} is constant inside the solenoid and vanishes outside the solenoid. Thus, we are equipped with the realizations (8)–(9) and (13)–(14) in a consistent manner. We first deal with the realization given in (8) but ignore the e^2/c^2 terms, so that the related coefficients are

$$a_{ij}^{(1)}(r,\theta) = \frac{1}{2m}\left(1 - \frac{eF_{12}\theta}{2\hbar c}\right)^2 \delta_{ij}, \quad b_i^{(1)}(r,\theta) = \frac{e}{mc}\left(1 - \frac{eF_{12}\theta}{2\hbar c}\right) A_i. \tag{30}$$

The trajectory in (29) is chosen to enclose the origin, thus it yields

$$\Phi_{AB}^{nc(1)} = -\frac{ie}{\hbar c}\oint\left(1+\frac{eF_{12}\theta}{2\hbar c}\right)A_i dr_i = -\frac{ie}{\hbar c}\left(1+\frac{e\theta B}{2\hbar c}\right)\int \varepsilon_{ij}\nabla_i A_j ds = \left(1+\frac{e\theta B}{2\hbar c}\right)\Phi_{AB}, \tag{31}$$

where the AB phase is given in terms of $\Phi_0 = hc/e$ and the cross-sectional area of the solenoid S as

$$\Phi_{AB} = -2\pi i \frac{BS}{\Phi_0}.$$

When we consider (8) keeping the e^2/c^2 terms the coefficients become

$$a_{ij}^{(2)}(r,\theta) = \frac{1}{2m}\left(1 - \frac{eF_{12}\theta}{2\hbar c}\right)^2 \delta_{ij}, \quad b_i^{(2)}(r,\theta) = \frac{e}{mc}\left(1 - \frac{eF_{12}\theta}{2\hbar c}\right)^2 A_i. \tag{32}$$

Observe that (28) does not acquire any θ-deformation. As a result of this the phase is not deformed:

$$\Phi_{AB}^{nc(2)} = \Phi_{AB}. \tag{33}$$

For the realization (13) one can read the coefficients as follows

$$a_{ij}^{(3)}(r,\theta) = \frac{1}{2m}\left(1 - \frac{eF_{12}\theta}{\hbar c}\right)^2 \delta_{ij}, \quad b_i^{(3)}(r,\theta) = -\frac{eB}{2mc}\left(1 - \frac{eF_{12}\theta}{\hbar c}\right)\varepsilon_{ij} r_j. \tag{34}$$

Hence, the θ-deformed AB phase is deduced as

$$\Phi_{AB}^{nc(3)} = \frac{ie}{2\hbar c}\oint\left(1 + \frac{eF_{12}\theta}{\hbar c}\right)F_{12}\varepsilon_{ij} r_j dr_i = \left(1 + \frac{e\theta B}{\hbar c}\right)\Phi_{AB}, \tag{35}$$

where we used $S = \oint \varepsilon_{ij} r_i dr_j/2$.

When the terms at the order of e^2/c^2 are kept, i.e. (32), does not procure any θ-deformation of the AB phase (33). However, the other realizations (30) and (34) led to (31) and (35) with different θ-dependent factors. An approach to determine which formulation should be preferred is presented in the last section.

To discuss the AC, HMW and Anandan phases in noncommutative coordinates we will consider the realization (13) in three dimensions: $a,b = 1,2,3$. In general it leads to the θ-deformed gauge field (21), where

$$a_a^b = \frac{1}{2m}\left(\delta_a^b - \frac{2\rho}{\hbar}F_{ac}\theta^{cb}\right)$$

and

$$b_a = \frac{\rho}{2m}\left(F_{ab} - \frac{\rho}{\hbar}\theta_{ac}F^{cd}F_{db}\right)r^b.$$

As far as the conditions (10) are satisfied this construction is valid also for non-Abelian gauge fields. The θ-deformed phase factor is

$$\Phi^{nc} = -\frac{i\rho}{2\hbar}\oint\left(F^{ab} + \frac{\rho}{\hbar}F^{ac}F_{cd}\theta^{db}\right)r_a dr_b \tag{36}$$

Now we specify the gauge field as in (21) which is appropriate to discuss the AC, HMW and Anandan phases and consider the configuration: $\boldsymbol{\mu} = \mu\hat{z}, \boldsymbol{d} = d\hat{z}$; $\boldsymbol{\mu}\cdot\boldsymbol{B} = 0$, $\boldsymbol{d}\cdot\boldsymbol{E} = 0$ and $\boldsymbol{E}\cdot\nabla\mu = 0, \boldsymbol{B}\cdot\nabla d = 0$. Hence the problem is effectively 2-dimensional. The gauge field (21) is now Abelian and the nonvanishing components of the field strength are

$$F_{ij} = \varepsilon_{ij}(-\mu\nabla\cdot\boldsymbol{E} + d\nabla\cdot\boldsymbol{B}). \tag{37}$$

Moreover, we consider the noncommutative plane by setting $\theta_{ij} = \varepsilon_{ij}\theta$. As usual the electromagnetic fields are taken in the radial direction and their divergence vanish except in the infinitesimal regions around the origin where they satisfy [15]

$$\nabla\cdot\boldsymbol{E} = \frac{\lambda_e}{s'}, \quad \nabla\cdot\boldsymbol{B} = \frac{\lambda_m}{s''}.$$

We introduced s' and s'' which are, respectively, the areas of the infinitesimal regions where $\nabla\cdot\boldsymbol{E}$ and $\nabla\cdot\boldsymbol{B}$ are nonvanishing. Obviously, s' and s'' do not play any role in the original definition of the Anandan phase:

$$\Phi_A = \frac{\rho}{2}\oint F_{ij}r^i dr^j = -\frac{1}{\hbar c}(\mu\lambda_e - d\lambda_m). \tag{38}$$

The field strength (37) satisfies the conditions (10) so that we can use the realization leading to the phase (36). The Anandan phase in noncommutative space can be calculated as

$$\Phi_A^{nc} = \Phi_A\left[1 + \theta\left(\frac{\mu\lambda_e}{\hbar c s'} - \frac{d\lambda_m}{\hbar c s''}\right)\right], \tag{39}$$

where (38) is employed.

Imposing, respectively, $\lambda_e = 0$ and $\lambda_m = 0$ in (39), the noncommutative AC and HMW phases can be deduced as

$$\Phi_{AC}^{nc} = \frac{d\lambda_m}{\hbar c}\left(1 - \theta\frac{d\lambda_m}{\hbar c s''}\right),$$
$$\Phi_{HMW}^{nc} = -\frac{\mu\lambda_e}{\hbar c}\left(1 + \theta\frac{\mu\lambda_e}{\hbar c s\prime}\right).$$

Hence, the deformed phases which we obtained are independent of the velocity of the scattered particles which is one of the main features of the original quantum phases.

4 Discussions

The alternative procedure itself leads to different deformed dynamical systems which is equivalent to identify the θ-deformed quantum phase space variables. Depending on to the realization adopted the resulting AB phase acquire diverse deformation factors in noncommutative coordinates. Although at first sight this may seem to be a pathological fact, as we will explain it is an embarrassment of riches permitting us to choose the realization adequate to the problem considered. One of the interpretations of the noncommutativity of coordinates is to consider it as an effective method of introducing interactions whose dynamical origins can be complicated [14, 16]. Once we determine which realization leads to the desired effective theory we can select to work within that representation.

References

1. Dayi, Ö.F., Yapışkan, B.: Phys. Lett. A **374**, 3810 (2010)
2. Dayi, Ö.F., Jellal, A.: J. Math. Phys. **51**, 063522 (2010)
3. Aharonov, Y., Bohm, D.: Phys. Rev. **115**, 485 (1959)
4. Aharonov, Y., Casher, A.: Phys. Rev. Lett. **53**, 319 (1984)
5. He, X.-G., McKellar, B.H.J.: Phys. Rev. A **47**, 3424 (1993)
6. Wilkens, M.: Phys. Rev. Lett. **72**, 5 (1994)
7. Anandan, J.: Phys. Rev. Lett. **85**, 1354 (2000)
8. Furtado, C., de Lima Ribeiro, C.A.: Phys. Rev. A **69**, 064104 (2004)
9. Gamboa, J., Loewe, M., Rojas, J.C.: Noncommutativity and the Aharonov–Bohm effect (hep-th/0101081)
10. Chaichian, M., Presnajder, P., Sheikh–Jabbari, M.M., Tureanu, A.: Phys. Lett. B **527**, 149 (2002)
11. Mirza, B., Zarei, M.: Eur. Phys. J. C **32**, 583 (2004)
12. Wang, J., Li, K.: J. Phys. A Math. Theor. **40**, 2197 (2007)
13. Passos, E., Ribeiro, L.R., Furtado, C., Nascimento, J.R.: Phys. Rev. A **76**, 012113 (2007)
14. Dayi, Ö.F., Jellal, A.: J. Math. Phys. **43**, 4592 (2002) [Erratum: **45**, 827 (2004)]

15. Dayi, Ö.F.: Europhys. Lett. **85**, 41002 (2009)
16. Scholtz, F.G., Chakraborty, B., Gangopadhyay, S., Hazra, A.G.: Phys. Rev. D **71**, 085005 (2005); J. Phys. A Math. Gen. **38**, 9849 (2005)

On Quantum WZNW Monodromy Matrix: Factorization, Diagonalization, and Determinant

Ludmil Hadjiivanov and Paolo Furlan

Abstract We review the basic algebraic properties of the quantum monodromy matrix M in the canonically quantized chiral $SU(n)_k$ Wess–Zumino–Novikov–Witten model with a quantum group symmetry.

1 Introduction

The Wess–Zumino–Novikov–Witten (WZNW) model [17] on a $2D$ cylindric space-time (with periodic space coordinate) describes the conformal invariant free motion of a closed string on a Lie group manifold [13]. We will only consider here the case of a compact semisimple Lie group G and positive integer level k, and the explicit calculations will apply exclusively to $G = SU(n)$. Canonical quantization prescribes replacing the classical Poisson brackets (PB) by commutators or, in the case of quadratic PB, by *exchange relations* such that the classical symmetries are recovered in the quasiclassical limit. Here is a short list of references on the subject covered below: [1, 2, 5, 7, 9, 10, 12, 15].

L. Hadjiivanov (✉)
Theoretical and Mathematical Physics Division, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria

INFN, Sezione di Trieste, Trieste, Italy
e-mail: lhadji@inrne.bas.bg

P. Furlan
Dipartimento di Fisica dell' Università degli Studi di Trieste, Strada Costiera 11, 34014 Trieste, Italy

INFN, Sezione di Trieste, Trieste, Italy
e-mail: furlan@trieste.infn.it

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_20,

The $2D$ WZNW field admits a chiral splitting in a product of left and right movers. The chiral field $g(z)$ (where $z = e^{ix}$ and x is a light cone variable) is only twisted-periodic,

$$g(e^{2\pi i}z) = g(z)M \,, \tag{1}$$

where M is the *monodromy matrix*.[1] The corresponding exchange relations with a constant statistics matrix $\hat{R}$ read

$$g^A{}_\alpha(z_1)\,g^B{}_\beta(z_2) = g^B{}_\rho(z_2)\overset{\curvearrowright}{\,}g^A{}_\sigma(z_1)\,\hat{R}^{\rho\sigma}{}_{\alpha\beta} \quad (\,|z_1| > |z_2|\,,\ \pi > \arg(z_1) > \arg(z_2) > -\pi\,) \tag{2}$$

where $z_{12} \overset{\curvearrowright}{\to} z_{21} = e^{-i\pi}z_{12}$ [10]. It is assumed that $\hat{R}_{12} = P_{12}R_{12}$ (we are using the common tensor product notation) where P_{12} is the permutation matrix, $P^{\alpha\beta}_{\rho\sigma} = \delta^\alpha_\sigma\delta^\beta_\rho$, and R_{12} is a solution of the quantum Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \;\Leftrightarrow\; \hat{R}_1\hat{R}_2\hat{R}_1 = \hat{R}_2\hat{R}_1\hat{R}_2\,, \quad \hat{R}_i := \hat{R}_{ii+1}$$
$$\text{and, trivially,} \quad \hat{R}_i\hat{R}_j = \hat{R}_j\hat{R}_i \quad \text{for} \quad |i-j| > 1\,. \tag{3}$$

The virtue of the exchange relations (2) is that they reveal, along with the left G-symmetry (acting on the capital Latin indices of $g^A{}_\alpha(z)$), also right *quantum group* [4] invariance with respect to transformations satisfying the *RTT relations*

$$R_{12}T_1T_2 = T_2T_1R_{12} \;\Leftrightarrow\; \hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12} \tag{4}$$

which is the quantum counterpart of the Lie–Poisson symmetry of the corresponding classical Poisson brackets. The relations (3) identify $\hat{R}_i$ as generators of the (non-Abelian) braid group statistics of the model.

The first sign that the WZNW model is somehow related to quantum groups appeared in [16]. Although it became soon clear that the quantum group symmetry does not hold in the unitary version of the model (in particular, the quantum group representation ring does not close on the "physical" representations), it seems to be the appropriate internal ("gauge") symmetry for a logarithmic extension of it (see e.g. [8, 11, 14]).

The monodromy matrix M obeys the *reflection equation*

$$M_1\,R_{12}\,M_2\,R_{21} = R_{12}\,M_2\,R_{21}\,M_1 \;\Leftrightarrow\; \hat{R}_{12}\,M_2\,\hat{R}_{12}\,M_2 = M_2\,\hat{R}_{12}\,M_2\,\hat{R}_{12}\,, \tag{5}$$

while its exchange relations with $g(z)$ read

---

[1] We start with a general monodromy matrix (classically, $M \in G$). The case when M belongs to the maximal torus will be considered later as a diagonalization problem. The possibility of analytic continuation in z (in correlation functions) due to energy positivity is implicitly assumed.

$$g_1(z)\,R^-_{12}M_2 = M_2\,g_1(z)\,R^+_{12} \quad (R^-_{12} := R_{12}\,,\; R^+_{12} := R^{-1}_{21}\,) \Leftrightarrow$$
$$M_1\,g_2(z) = g_2(z)\,\hat{R}_{12}\,M_2\,\hat{R}_{12}\;. \tag{6}$$

The quantum group properties of the chiral field $g(z)$ become transparent by taking as R_{12} the $U_q(\mathscr{G}_{\mathbb{C}})$ Drinfeld–Jimbo quantum R-matrix (where $\mathscr{G}_{\mathbb{C}}$ is the complexification of the Lie algebra $\mathscr{G}$ of G) and performing the *factorization* of M into a product $M_+M_-^{-1}$ of two upper, resp. lower triangular matrices such that

$$\mathrm{diag}\,M_+ = \mathrm{diag}\,M_-^{-1},\;\; R_{12}M_{\pm 2}M_{\pm 1} = M_{\pm 1}M_{\pm 2}R_{12},\;\; R_{12}M_{+2}M_{-1} = M_{-1}M_{+2}R_{12}. \tag{7}$$

According to a deep result of Faddeev et al. [6], a quotient of the Hopf algebra generated by the entries of $M_\pm$ and endowed with a coalgebra structure in which the coproduct, counit and antipode are defined as

$$\Delta((M_\pm)^\alpha_{\;\beta}) = (M_\pm)^\alpha_{\;\sigma} \otimes (M_\pm)^\sigma_{\;\beta}\;, \quad \varepsilon((M_\pm)^\alpha_{\;\beta}) = \delta^\alpha_\beta\;, \quad S((M_\pm)^\alpha_{\;\beta}) = (M^{-1}_\pm)^\alpha_{\;\beta}, \tag{8}$$

respectively, is equivalent to a certain cover U_q of $U_q(\mathscr{G}_{\mathbb{C}})$. The exchange relation

$$M_{\pm 2}\,g_1(z)\,M^{-1}_{\pm 2} \;\; (= M_{\pm 2}\,g_1(z)\,S(M_\pm)_2 = Ad_{M_{\pm 2}}g_1(z)\,) \;= g_1(z)\,R^{\mp}_{12} \tag{9}$$

(leading to (6)) implies that each row of $g(z) = (g^A_{\;\alpha}(z))$ is a U_q *vector operator*. The factorization of M actually involves a "quantum prefactor" [10]; in particular, for $G = SU(n)$ when the deformation parameter is $q = e^{-i\frac{\pi}{h}}\,,\; h = k+n\,,$

$$M = q^{\frac{1}{n}-n}\,M_+M_-^{-1} \qquad (\mathscr{G}_{\mathbb{C}} = s\ell(n))\;. \tag{10}$$

The quantum $SU(n)$ WZNW monodromy matrix M and its components $M_\pm$, as matrices with *non-commutative* entries, are the main objects of interest for us in this paper. In Sect. 2 we remind the FRT construction and provide some important technical details of it. Section 3 is devoted to the diagonalization of M. In the last Sect. 4 we introduce the quantum determinant $\det_q(M)$ [9] and discuss some of its properties. The results are illustrated by explicit formulae for small n.

2 U_q in Disguise: The FRT Construction

One of the amazing results in [6] is that a quotient of the RTT algebra (4), regarded as a deformation of the algebra of functions on a matrix Lie group G, is Hopf dual to a certain cover of the QUEA $U_q(\mathscr{G})$. The "classical" ($q = 1$) counterpart of this fact is the realization, due to L. Schwartz, of the universal enveloping algebra $U(\mathscr{G})$ as the non-commutative algebra of distributions on G supported by its unit element,

$U(\mathscr{G}) \simeq C_e^{-\infty}(G)$ (see Theorem 3.7.1 in [3]). The details below concern the case $\mathscr{G} = s\ell(n)$. As shown in [6], the Hopf algebra (7), (8) is dual to $Fun(SL_q(n))$, the $\det_q(T) = 1$ quotient of the RTT algebra (4) (for an appropriate definition of the quantum determinant) with coalgebra relations written in matrix form as

$$\Delta(1) = 1 \otimes 1 \,, \quad \Delta(T) = T \otimes T \,, \quad \varepsilon(T) = \boldsymbol{I} \,, \quad S(T) = T^{-1} \,. \tag{11}$$

The Chevalley generators of $U_q(s\ell(n))$ obey the commutation relations

$$\begin{aligned} &K_i K_j = K_j K_i \,, \quad K_i E_j K_i^{-1} = q^{c_{ij}} E_j \,, \quad K_i F_j K_i^{-1} = q^{-c_{ij}} F_j \,, \\ &[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \,, \quad i,j = 1,\dots,n-1 \end{aligned} \tag{12}$$

and, for $n > 2$, also the *q-Serre relations*

$$\begin{aligned} &E_i^2 E_j + E_j E_i^2 = [2] E_i E_j E_i \,, \qquad F_i^2 F_j + F_j F_i^2 = [2] F_i F_j F_i \\ &\text{for} \quad |i-j| = 1 \,, \qquad [E_i, E_j] = 0 = [F_i, F_j] \quad \text{for} \quad |i-j| > 1 \,. \end{aligned} \tag{13}$$

Here (c_{ij}) is the $s\ell(n)$ Cartan matrix, $c_{ii} = 2$, $c_{i\,i\pm1} = -1$, $c_{ij} = 0$ for $|i-j| > 1$. The coalgebra structure is defined on the generators as follows:

$$\begin{aligned} &\Delta(K_i) = K_i \otimes K_i \,, \; \Delta(E_i) = E_i \otimes K_i + \boldsymbol{I} \otimes E_i \,, \; \Delta(F_i) = F_i \otimes \boldsymbol{I} + K_i^{-1} \otimes F_i \,, \\ &\varepsilon(K_i) = 1 \,, \; \varepsilon(E_i) = \varepsilon(F_i) = 0 \,, \quad S(K_i) = K_i^{-1} \,, \; S(E_i) = -E_i K_i^{-1} \,, \; S(F_i) = -K_i F_i \,. \end{aligned} \tag{14}$$

On the other hand, using the explicit form of the Drinfeld–Jimbo $U_q(s\ell(n))$ R-matrix,

$$R_{12} = (R^{\alpha\beta}_{\rho\sigma}) \,, \; R^{\alpha\beta}_{\rho\sigma} = q^{\frac{1}{n}} \left(\delta^\alpha_\rho \delta^\beta_\sigma + (q^{-1} - q^{\varepsilon_{\alpha\beta}}) \delta^\alpha_\sigma \delta^\beta_\rho \right) \,, \; \varepsilon_{\alpha\beta} = \begin{cases} 1 \,, & \alpha > \beta \\ 0 \,, & \alpha = \beta \\ -1 \,, & \alpha < \beta \end{cases} \,, \tag{15}$$

Equation (7) give rise to the following relations for the components of $M_\pm$:

$$\begin{aligned} &[(M_\pm)^\alpha_{\ \rho}, (M_\pm)^\beta_{\ \sigma}] = (q^{\varepsilon_{\sigma\rho}} - q^{\varepsilon_{\alpha\beta}}) (M_\pm)^\alpha_{\ \sigma} (M_\pm)^\beta_{\ \rho} \,, \\ &[(M_-)^\alpha_{\ \rho}, (M_+)^\beta_{\ \sigma}] = (q^{-1} - q^{\varepsilon_{\alpha\beta}}) (M_+)^\alpha_{\ \sigma} (M_-)^\beta_{\ \rho} - (q^{-1} - q^{\varepsilon_{\sigma\rho}}) (M_-)^\alpha_{\ \sigma} (M_+)^\beta_{\ \rho} \,. \end{aligned} \tag{16}$$

We will denote

$$\operatorname{diag} M_+ = \operatorname{diag} M_-^{-1} \;=: D = (d_\alpha \, \delta^\alpha_\beta) \,, \quad \det D := \prod_{\alpha=1}^n d_\alpha = 1, \tag{17}$$

thus introducing a quotient of the algebra (7). From (16) we obtain, in particular,

$$
\begin{aligned}
& d_\alpha\, d_\beta = d_\beta\, d_\alpha \ , \\
& d_\alpha\, (M_+)^\beta{}_\alpha = q^{-1}\, (M_+)^\beta{}_\alpha\, d_\alpha \ , \quad d_\beta (M_+)^\beta{}_\alpha = q\, (M_+)^\beta{}_\alpha\, d_\beta \ , \qquad \alpha > \beta \ , \\
& d_\alpha\, (M_-)^\alpha_{\ \beta} = q\, (M_-)^\alpha_{\ \beta}\, d_\alpha \ , \quad d_\beta (M_-)^\alpha_{\ \beta} = q^{-1}\, (M_-)^\alpha_{\ \beta}\, d_\beta \ , \quad \alpha > \beta \ , \\
& [(M_-)^\alpha_{\ \beta}, (M_+)^\beta{}_\alpha] = \lambda\, (d_\alpha^{-1} d_\beta - d_\alpha d_\beta^{-1}) \ , \quad \alpha > \beta \quad (\lambda = q - q^{-1}) \ . \qquad (18)
\end{aligned}
$$

As d_α commute, their order in the product defining $\det D$ in (17) is not important. Using the triangularity of M_+ and M_- in deriving (18) is crucial. Moreover, due to it, the coproduct (8) of a matrix element of M_+ or M_- belonging to the corresponding "m-th diagonal" (for $m = 1, \ldots, n$) contains exactly m summands. Thus, the diagonal elements $d_\alpha\,,\ \alpha = 1, 2, \ldots, n\ (m = 1)$ are *group-like* ($\Delta(d_\alpha) = d_\alpha \otimes d_\alpha\,,\ \varepsilon(d_\alpha) = 1\,,$ $S(d_\alpha) = d_\alpha^{-1}$), while

$$
\begin{aligned}
\Delta((M_+)^i{}_{i+1}) &= d_i \otimes (M_+)^i{}_{i+1} + (M_+)^i{}_{i+1} \otimes d_{i+1} \ , \\
\Delta((M_-)^{i+1}_{\ i}) &= (M_-)^{i+1}_{\ i} \otimes d_i^{-1} + d_{i+1}^{-1} \otimes (M_-)^{i+1}_{\ i} \qquad (19)
\end{aligned}
$$

for $1 \le i \le n-1$ (here $m = 2$). The comparison with (14) suggests that

$$
(M_+)^i{}_{i+1} = x_i\, F_i\, d_{i+1} \ , \quad (M_-)^{i+1}_{\ i} = y_i\, d_{i+1}^{-1}\, E_i \ , \quad d_i^{-1} d_{i+1} = K_i \qquad (20)
$$

where x_i and y_i are some yet unknown q-dependent coefficients. For $\alpha = i+1, \beta = i$, the second and third relation in (18) as well as the condition (17) are satisfied if

$$
d_\alpha = k_{\alpha-1} k_\alpha^{-1} \quad (k_0 = k_n = 1\,), \qquad (21)
$$

the new set of independent Cartan generators $k_1, \ldots, k_{n-1}$ obeying

$$
\begin{aligned}
& k_i = \prod_{\ell=1}^{i} d_\ell^{-1} \ , \quad K_i = k_{i-1}^{-1} k_i^2 k_{i+1}^{-1} \ , \quad i = 1, 2, \ldots, n-1 \ , \\
& k_i k_j = k_j k_i \ , \quad k_i\, E_j = q^{\delta_{ij}} E_j\, k_i \ , \quad k_i\, F_j = q^{-\delta_{ij}} F_j\, k_i \ , \\
& \Delta(k_i) = k_i \otimes k_i \ , \quad \varepsilon(k_i) = 1 \ , \quad S(k_i) = k_i^{-1} \ . \qquad (22)
\end{aligned}
$$

Inserting (20) into the last (18) and using the second and third relation (18) from which it follows that $[d_{i+1}, (M_-)^{i+1}_{\ i} (M_+)^i{}_{i+1}] = 0$, we obtain

$$
x_i\, y_i = -\lambda^2 \ , \quad i = 1, \ldots, n-1 \ . \qquad (23)
$$

The commutation relation (16) of $(M_+)^i{}_{i+2}$ with d_α (21) suggests that $(M_+)^i{}_{i+2}$ contains the step operators F_i and F_{i+1} only. Assuming that it is proportional to

$(F_{i+1}F_i - zF_iF_{i+1})D_{i+2}$ where D_{i+2} is some group-like element and z is another unknown q-dependent coefficient, taking the corresponding coproduct (8) and using (20), (14), we obtain

$$(M_+)^i{}_{i+2} = -\frac{x_i x_{i+1}}{\lambda}\,[F_{i+1},F_i]_q\, d_{i+2}\,, \quad ([A,B]_q := AB - qBA)\,. \tag{24}$$

A similar calculation shows that $(M_-)^{i+2}_i = \frac{y_i y_{i+1}}{\lambda}\, d^{-1}_{i+2}\,[E_i,E_{i+1}]_{q^{-1}}$. We will fix the coefficients x_i and y_i satisfying (23) in a symmetric way: $x_i = -\lambda\,,\ y_i = \lambda$. The commutators

$$\begin{aligned} &[(M_+)^i{}_{i+1},(M_+)^i{}_{i+2}]_q = 0\,, \quad [(M_+)^i{}_{i+2},(M_+)^{i+1}_{i+2}]_q = 0\,,\\ &[(M_-)^{i+1}_i,(M_-)^{i+2}_i]_q = 0\,, \quad [(M_-)^{i+2}_i,(M_-)^{i+2}_{i+1}]_q = 0 \end{aligned} \tag{25}$$

are in fact the non-trivial q-Serre relations (13) written as

$$\begin{aligned} &[F_i,[F_i,F_{i+1}]_{q^{-1}}]_q = 0 = [F_{i+1},[F_{i+1},F_i]_q]_{q^{-1}}\,,\\ &[E_i,[E_i,E_{i+1}]_{q^{-1}}]_q = 0 = [E_{i+1},[E_{i+1},E_i]_q]_{q^{-1}}\,. \end{aligned} \tag{26}$$

One can obtain in a similar way the higher off-diagonal terms of the matrices $M_\pm$ (for example, $(M_+)^1{}_4 = -\lambda\,[F_3,[F_2,F_1]_q]_q\, d_4$). The result can be summarized in

$$M_+ = (\boldsymbol{I} - \lambda N_+)D\,, \qquad M_- = D^{-1}(\boldsymbol{I} + \lambda N_-) \tag{27}$$

where the *nilpotent* matrices N_+ and N_- are upper and lower triangular, respectively, with matrix elements given by the corresponding (lowering and raising) *Cartan–Weyl* generators, while the non-trivial entries $d_\alpha\,,\ \alpha = 1,\dots,n$ of the diagonal matrix D are expressed in terms of k_i (21). Writing $K_i = q^{H_i}\,,\ i = 1,\dots,n-1$ and using (22) allows to present k_i as $k_i = q^{h^i}$ where h^i are dual to the fundamental weights,

$$H_i = \sum_{j=1}^{n-1} c_{ij}\, h^j = 2h^i - h^{i-1} - h^{i+1}\,. \tag{28}$$

As $\det c^{(n)} = n$ for $c^{(n)} := (c_{ij})^{s\ell(n)}$, (28) infers that an inverse formula expressing k_i in terms of K_i would involve "n-th roots" of the latter[2]; indeed,

$$h^i = \sum_{j=1}^{n-1} (c^{-1})^{ij} H_j = \sum_{j=1}^{i} j\,(1-\frac{i}{n})H_j + \sum_{j=i+1}^{n-1} i\,(1-\frac{j}{n})H_j\,. \tag{29}$$

[2]The determinant of the $s\ell(n)$ Cartan matrix obeys

$$\det c^{(n)} = 2\det c^{(n-1)} - \det c^{(n-2)}\,, \quad \det c^{(2)} = 2\,, \quad \det c^{(3)} = 3 \quad \Rightarrow \quad \det c^{(n)} = n\,.$$

Thus the Hopf algebra U_q generated by E_i, F_i, k_i is an *n-fold cover* of $U_q(s\ell(n))$.

Note that the U_q invariance of the vacuum vector can be written as

$$X \mid 0\rangle = \varepsilon(X) \mid 0\rangle \quad \forall X \in U_q \,, \tag{30}$$

where $\varepsilon(X)$ is the counit (see (8) or, equivalently, (27), (14), (22)).

We display below the matrices D and $N_\pm$ (27) in the cases $n=2$ and $n=3$.

$\mathbf{n=2}$:

$$D = \begin{pmatrix} k^{-1} & 0 \\ 0 & k \end{pmatrix} \quad (K = k^2) \,, \quad N_+ = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \,, \quad N_- = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix} , \tag{31}$$

$\mathbf{n=3}$:

$$D = \begin{pmatrix} k_1^{-1} & 0 & 0 \\ 0 & k_1 k_2^{-1} & 0 \\ 0 & 0 & k_2 \end{pmatrix} \qquad (K_1 = k_1^2 k_2^{-1} \,, \;\; K_2 = k_1^{-1} k_2^2) \,,$$

$$N_+ = \begin{pmatrix} 0 & F_1 & [F_2, F_1]_q \\ 0 & 0 & F_2 \\ 0 & 0 & 0 \end{pmatrix} \,, \quad N_- = \begin{pmatrix} 0 & 0 & 0 \\ E_1 & 0 & 0 \\ [E_1, E_2]_{q^{-1}} & E_2 & 0 \end{pmatrix} \,, \tag{32}$$

$$(\boldsymbol{I} + \lambda N_-)^{-1} = \boldsymbol{I} - \lambda \begin{pmatrix} 0 & 0 & 0 \\ E_1 & 0 & 0 \\ [E_1, E_2]_q & E_2 & 0 \end{pmatrix} \,. \tag{33}$$

3 The Diagonal Monodromy Matrix M_p

The natural solution of the diagonalization problem for the chiral $SU(n)$ WZNW monodromy matrix M appears to be the diagonal matrix M_p defined as

$$M_p \, a = a M \,, \quad M_p = q^{1-\frac{1}{n}} \operatorname{diag}(q^{-2p_1}, \ldots, q^{-2p_n}) \tag{34}$$

(see e.g. [10]). Here q^{p_i} form a commutative set of operators ($q^{p_i} q^{p_j} = q^{p_j} q^{p_i}$) satisfying $\prod_{i=1}^n q^{p_i} = 1$, the *zero modes'* matrix (with non-commutative entries) a obeys the relations

$$q^{p_j} \, a^i_\alpha = a^i_\alpha \, q^{p_j + \delta^i_j - \frac{1}{n}} \,, \quad \hat{R}_{12}(p) \, a_1 \, a_2 = a_1 \, a_2 \hat{R}_{12} \tag{35}$$

as well as an appropriate (n-linear) determinant condition, and $\hat{R}_{12}(p)$ in (35) is a solution of the quantum *dynamical* Yang–Baxter equation [15].

The $q^{1-\frac{1}{n}}$ prefactor of M_p (34) has a quantum origin [9, 10]. Applying both sides of the first relation (34) to the vacuum and using (10), (30) and the first equation (35), we deduce that the equality

$$a^i_\alpha \, q^{-2p_i} \mid 0\rangle = q^{1-n} \, a^i_\alpha \mid 0\rangle \tag{36}$$

should hold for any i (and α). The natural way to satisfy (36) is to set

$$q^{p_i}\,|\,0\rangle = q^{\frac{n+1}{2}-i}\,|\,0\rangle\ , \quad i=1,\dots,n\ , \quad a^i_\alpha\,|\,0\rangle = 0 \quad \text{for} \quad i\geq 2\ . \tag{37}$$

Here $p_i^{(0)} = \frac{n+1}{2} - i$ are the "barycentric coordinates" ($\sum_{i=1}^n p_i^{(0)} = 0$) of the Weyl vector ρ in the orthogonal basis of the $s\ell(n)$ weights.

These two relations give rise to a Fock representation of the zero modes' matrix algebra generated by polynomials $\mathscr{P}(a)$ applied to the vacuum vector. For homogeneous polynomials, the action of a^i_α on the vector $\mathscr{P}(a)\,|\,0\rangle$ can be depicted as adding a box to the i-th row of a Young-type diagram. In the case of admissible $s\ell(n)$ diagrams (associated to irreducible representations (IR) with highest weight Λ) the eigenvalues of q^{p_i} on $\mathscr{P}^\Lambda(a)\,|\,0\rangle$ are expressed in terms of the barycentric coordinates of the *shifted* weight $\Lambda+\rho$. For q generic, the Fock space is in fact a *model space* (a direct sum of all IR with multiplicity one) of U_q [10]. In the case at hand q is an (even) root of unity, and a more complicated structure including indecomposable U_q representations occurs (see [11] where the simplest, $n=2$ case has been studied).

The first equation (35) implies the following exchange relation of M_p and a:

$$M_{p1}a_2 = q^{-2\sigma_{12}}a_2M_{p1} \ \Leftrightarrow\ a_1M_{p2}a_1^{-1} = q^{2\sigma_{12}}M_{p2}\ , \quad (q^{2\sigma_{12}})^{ij}_{\ell m} = q^{2(\delta_{ij}-\frac{1}{n})}\,\delta^i_\ell\,\delta^j_m\ . \tag{38}$$

On the other hand, the exchange relation between M and a is similar to (6):

$$a_1\,R^-_{12}M_2 \,=\, M_2\,a_1\,R^+_{12} \ \Leftrightarrow\ M_1\,a_2 = a_2\,\hat{R}_{12}\,M_2\,\hat{R}_{12}\ . \tag{39}$$

The compatibility of (38) and (39) requires the relation

$$\hat{R}^{-1}_{12}(p) \,=\, q^{2\sigma_{12}}\,M_{p2}\,\hat{R}_{12}(p)\,M^{-1}_{p1} \tag{40}$$

to hold (it takes place indeed, being equivalent to (6.17) of [15] with $\hat{R}_{12}(p)\leftrightarrow\hat{R}^{-1}_{12}(p)$). To prove this, we start with (39) and then use $M = a^{-1}M_p\,a$ (34), the second equation (35) rewritten as $a_2\,\hat{R}_{12}\,a_2^{-1} = a_1^{-1}\,\hat{R}_{12}(p)\,a_1$, and (38):

$$\begin{aligned}
&M_1\,a_2 = a_2\,\hat{R}_{12}\,M_2\,\hat{R}_{12} \quad\Rightarrow\quad (a_1^{-1}\,M_{p1}\,a_1)\,a_2 = a_2\,\hat{R}_{12}\,(a_2^{-1}\,M_{p2}\,a_2)\,\hat{R}_{12} \quad\Rightarrow\\
&a_1^{-1}\,M_{p1}\,a_1 \,=\, (a_2\,\hat{R}_{12}\,a_2^{-1})\,M_{p2}\,(a_2\,\hat{R}_{12}\,a_2^{-1}) \quad\Rightarrow\\
&a_1^{-1}\,M_{p1}\,a_1 \,=\, (a_1^{-1}\,(\hat{R}_{12}(p)\,a_1)\,M_{p2}\,(a_1^{-1}\,\hat{R}_{12}(p))\,a_1) \quad\Rightarrow\\
&M_{p1} = \hat{R}_{12}(p)\,(a_1\,M_{p2}\,a_1^{-1})\,\hat{R}_{12}(p) \quad\Rightarrow\quad \hat{R}^{-1}_{12}(p) \,=\, q^{2\sigma_{12}}M_{p2}\,\hat{R}_{12}(p)\,M^{-1}_{p1}\,.
\end{aligned} \tag{41}$$

It is easy to verify (40) for $n=2$ when

$$\hat{R}_{12}^{\pm 1}(p) = q^{\pm\frac{1}{2}} \begin{pmatrix} q^{\mp 1} & 0 & 0 & 0 \\ 0 & \frac{q^{\mp p}}{[p]} & q^{-\alpha}\frac{[p-1]}{[p]} & 0 \\ 0 & q^{\alpha}\frac{[p+1]}{[p]} & -\frac{q^{\pm p}}{[p]} & 0 \\ 0 & 0 & 0 & q^{\mp 1} \end{pmatrix} , \quad M_p = q^{\frac{1}{2}} \begin{pmatrix} q^{-p} & 0 \\ 0 & q^p \end{pmatrix} \tag{42}$$

(here $p := p_{12}$ and $\alpha = \alpha(p)$), so that

$$q^{-\frac{1}{2}} M_{p2} = \mathrm{diag}\,(q^{-p}\,, q^{p}\,, q^{-p}\,, q^{p})\,, \quad q^{\frac{1}{2}} M_{p1}^{-1} = \mathrm{diag}\,(q^{p}\,, q^{p}\,, q^{-p}\,, q^{-p})\,,$$
$$q^{2\sigma_{12}} = \mathrm{diag}\,(q\,, q^{-1}\,, q^{-1}\,, q). \tag{43}$$

4 The Quantum Determinant $\det_q(M)$

As shown in [9], the appropriate definition of the quantum determinant of M is

$$\det_q(M) := \frac{1}{[n]!}\, \varepsilon_{\alpha_1\ldots\alpha_n} \left[(\hat{R}_{12}\hat{R}_{23}\ldots\hat{R}_{n-1\,n}M_n)^n \right]^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n} \varepsilon^{\beta_1\ldots\beta_n}. \tag{44}$$

Here $[n]! = [n][n-1]\ldots[1]$ and the *quantum* antisymmetric tensors vanish whenever some of their indices coincide, while their non-zero components are given by

$$\varepsilon^{\alpha_1\ldots\alpha_n} = \varepsilon_{\alpha_1\ldots\alpha_n} = q^{-\frac{n(n-1)}{4}} (-q)^{\ell(\alpha)} \quad \Rightarrow \quad \varepsilon_{\alpha_1\ldots\alpha_n}\varepsilon^{\alpha_1\ldots\alpha_n} = [n]! \tag{45}$$

for $(\alpha_1,\ldots,\alpha_n)$ a permutation of $(n,\ldots,1)$ of length $\ell(\alpha)$.

The corresponding independent definition of $\det_q(M_\pm)$ does not involve the R-matrix and is thus simpler; due to the triangularity of the matrices, only the $n!$ products of (commuting) diagonal entries survive in the sum so that, by (45), the end result complies with (17):

$$\det_q(M_\pm) := \frac{1}{[n]!}\, \varepsilon_{\alpha_1\ldots\alpha_n} (M_\pm)^{\alpha_n}_{\ \beta_n} \ldots (M_\pm)^{\alpha_1}_{\ \beta_1}\, \varepsilon^{\beta_1\ldots\beta_n} = \prod_{\alpha=1}^{n} (M_\pm)^{\alpha}_{\ \alpha} = \prod_{\alpha=1}^{n} d_\alpha^{\pm 1} = 1 \tag{46}$$

One can prove that the formula (44) possesses the following factorization property. Substituting M by (10) (including the prefactor!), one obtains just the product of the quantum determinants of M_+ and M_-^{-1} (both equal to 1), and hence

$$\det_q(M) = \det_q(M_+)\,.\det_q(M_-^{-1}) = 1\,. \tag{47}$$

Of course, this is a highly desirable result, as it appears as a quantum counterpart of the similar classical property.

We will end up by calculating $\det_q(M)$ for $n=2$ directly from (44). In this case $\varepsilon_{12}=\varepsilon^{12}=-q^{\frac{1}{2}}\,,\ \varepsilon_{21}=\varepsilon^{21}=q^{-\frac{1}{2}}$, and with

$$\hat{R}_{12}=q^{\frac{1}{2}}\begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & -\lambda & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix}\,,\quad M:=\begin{pmatrix} m^1{}_1 & m^1{}_2\\ m^2{}_1 & m^2{}_2\end{pmatrix} \tag{48}$$

we obtain the expression

$$\begin{aligned}\det_q(M) &= \frac{1}{[2]}\,\varepsilon_{\alpha\beta}\left(\hat{R}_{12}M_2\hat{R}_{12}M_2\right)^{\alpha\beta}_{\ \ \rho\sigma}\varepsilon^{\rho\sigma} = \\ &= \frac{q^2}{[2]}\,(m^1{}_1m^2{}_2+m^2{}_2m^1{}_1+q\,\lambda\,(m^2{}_2)^2-q^{-2}m^1{}_2m^2{}_1-m^2{}_1m^1{}_2)\end{aligned} \tag{49}$$

which reproduces the classical one, $m^1{}_1m^2{}_2-m^1{}_2m^2{}_1\,$, for $q=1$ and commuting $m^\alpha{}_\beta$. Through (27) and (31), the entries of $M=q^{-\frac{3}{2}}M_+M_-^{-1}$ are expressed in terms of the U_q generators:

$$m^1{}_1=q^{-\frac{1}{2}}(\lambda^2FE+q^{-1}K^{-1})\,,\ \ m^1{}_2=-q^{-\frac{3}{2}}\lambda FK\,,\ \ m^2{}_1=-q^{-\frac{1}{2}}\lambda E\,,\ \ m^2{}_2=q^{-\frac{3}{2}}K. \tag{50}$$

(Note that only $k^2=K\in U_q(s\ell(2))$ appears in (50) and not $k\in U_q$ alone [8, 11].) Now using $KE=q^2EK\,,\ [E,F]=\frac{K-K^{-1}}{\lambda}\,,\ [2]=q+q^{-1}$ we obtain

$$\det_q(M)=\frac{1}{[2]}\,(2\,q^{-1}-\lambda^2\,[E,F]K+\lambda\,K^2)=1\,, \tag{51}$$

as prescribed by (47).

Acknowledgements L.H. thanks the organizers of the 9th International Workshop “Lie Theory and Its Applications in Physics” (LT-9), 20–26 June 2011, held in Varna, Bulgaria. The work of L.H. has been supported in part by the Bulgarian National Science Fund (grant DO 02-257) and INFN, Sezione di Trieste. P.F. acknowledges the support of the Italian Ministry of University and Research (MIUR).

References

1. Alekseev, A., Shatashvili, S.: Comm. Math. Phys. **133**, 353–368 (1990)
2. Balog, J., Fehér, L., Palla, L.: Nucl. Phys. B **568**, 503–542 (2000) (hep-th/9910046)
3. Cartier, P.: A primer of Hopf algebras. IHES Bures-sur-Yvette preprint IHES/M/06/40 (2006), p. 80 (Available at http://inc.web.ihes.fr/prepub/PREPRINTS/2006/M/M-06-40.pdf)

4. Drinfeld, V.G.: Quantum groups. In: Proceedings of the International Congress of Mathematicians, vol. 1, pp. 798–820, Berkeley, 1986. Academic, New York (1986)
5. Faddeev, L.D.: Comm. Math. Phys. **132**, 131–138 (1990)
6. Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Algebra I Anal. **1**, 178–206 (1989) [English translation: Leningrad Math. J. **1**, 193–225 (1990)]
7. Falceto, F., Gawędzki, K.: On quantum group symmetries in conformal field theories. In: Talk given at the XXth International Congress on Differential Geometric Methods in Theoretical Physics, New York, 3–7 June 1991 (hep-th/9109023)
8. Feigin, B.L., Gainutdinov, A.M., Semikhatov, A.M., Tipunin, I.Yu.: Comm. Math. Phys. **265**, 47–93 (2006) (hep-th/0504093)
9. Furlan, P., Hadjiivanov, L.: Quantum $su(n)_k$ monodromy matrices (arXiv:1111.2037[math-ph])
10. Furlan, P., Hadjiivanov, L., Isaev, A.P., Ogievetsky, O.V., Pyatov, P.N., Todorov, I.: J. Phys. A **36**, 5497–5530 (2003) (hep-th/0003210)
11. Furlan, P., Hadjiivanov, L., Todorov, I.: Lett. Math. Phys. **82**, 117–151 (2007) (arXiv:0710.1063 [hep-th])
12. Gawędzki, K.: Comm. Math. Phys. **139**, 201–213 (1991)
13. Gepner, D., Witten, E.: Nucl. Phys. B **278**, 493–549 (1986)
14. Hadjiivanov, L., Popov, T.: Eur. Phys. J. B **29**, 183–187 (2002) (hep-th/0109219)
15. Hadjiivanov, L., Isaev, A.P., Ogievetsky, O.V., Pyatov, P.N., Todorov, I.: J. Math. Phys. **40**, 427–448 (1999) (q-alg/9712026)
16. Tsuchia, A., Kanie, Y.: Lett. Math. Phys. **13**, 303–312 (1987)
17. Witten, E.: Comm. Math. Phys. **92**, 455–472 (1984)

Part V
Representation Theory

Some Properties of Planar Galilean Conformal Algebras

Naruhiko Aizawa

Abstract An infinite dimensional extension of the spin 1 Galilean conformal algebra in the plane is investigated. We present the coadjoint representation and a classification of all possible central extensions. Furthermore, we study representations of the algebra with central extensions. Kac determinant for the highest weight Verma modules is given explicitly which shows that the Verma modules are irreducible for nonvanishing highest weights. A boson realization corresponding to unit central charge is also presented.

1 Introduction

Importance of infinite dimensional Lie algebras in physics and mathematics has been recognized for a long time. Many of them are related to semisimple Lie algebra of finite dimension. The celebrated Virasoro algebra contains infinitely many $sl(2)$ as its subalgebras and each classical Lie algebra is associated with Kac–Moody algebra. Recent studies of nonrelativistic AdS/CFT correspondence have introduced variety of infinite dimensional Lie algebras [1–9]. Some of them were introduced as a formal extension of finite dimensional Lie algebras and some others are derived by contraction form some copies of Virasoro algebra. Physical implication of some of such infinite dimensional algebras is not clear at this moment. However, they have two interesting features: They have Virasoro algebra as a subalgebra, and they are associated with non-semisimple Lie algebras which are regarded as a nonrelativistic analogue of conformal algebras. This would suggest that such infinite dimensional algebras are of physical importance and their representation theory is different from semisimple counterparts.

N. Aizawa (✉)
Department of Mathematics and Information Sciences, Graduate School of Science,
Osaka Prefecture University, Nakamozu Campus, Sakai, Osaka 599-8531, Japan
e-mail: aizawa@mi.s.osakafu-u.ac.jp

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_21,

In the present work, we pick up one of such infinite dimensional algebras and study its central extensions and representations. The algebra we pick up was introduced by Martelli and Tachikawa [4] which is an infinite dimensional extension of the so-called *spin 1* Galilean conformal algebra (GCA) [10–12]. We focus on the algebra defined in $(2+1)$ dimensional spacetime because the GCA of finite dimension has an *exotic* central extension which is allowed only in this dimension of spacetime [4, 13–15].

To make clear the difference between finite and infinite dimensional GCA, we start with a brief review of GCA in arbitrary dimensional spacetime. It is followed by a study of central extensions of infinite dimensional GCA of spin 1 in the plane where it is shown that the algebra does not have the exotic central extension. We then consider the Verma modules over the algebra with the central extensions. The explicit formula of the Kac determinant is given which shows that the Verma modules are irreducible for nonvanishing highest weights. It is also shown that the algebra has a boson realization similar to that of Virasoro algebra. In Sect. 4 we return to the centerless algebra and derive the coadjoint representation. Section 5 is devoted to concluding remarks.

2 Review of Galilean Conformal Algebras in $(d+1)$D Spacetime

We present the GCA of finite dimension in d spatial dimension according to [11, 12] (see also [4]). By a given d, GCA is labelled by a half-integer ℓ (sometimes called spin). Let us consider the following generators of spacetime transformations:

$$\begin{aligned} &H = \partial_t, && P_i^n = (-t)^n \partial_i, && J_{ij} = -x_i\partial_j + x_j\partial_i, \\ &D = -t\partial_t - \ell x_i \partial_i, && C = t^2\partial_t + 2\ell t x_i \partial_i, && \end{aligned} \tag{1}$$

where $i,j = 1,2,\cdots,d$ and $n = 0,1,\cdots,2\ell$. Commutation relations are easily deduced from (1) and one see that $\langle\, D,H,C\,\rangle$ span the subalgebra $sl(2,\mathbb{R})$ and the rotation $\langle\, J_{ij}\,\rangle$ dose $so(n)$. The subalgebra $\langle\, P_i^n\,\rangle$ forms an Abelian ideal so that the structure of the algebra is summarised as

$$(\, sl(2,\mathbb{R}) \,\oplus\, so(n)\,) \;\ni\!\!\!+ \langle\, P_i^n\,\rangle.$$

It is known that GCA has two types of central extensions. One exists for any d and half-integer ℓ :

$$[P_i^m, P_j^n] = I^{mn}\delta_{ij}M, \tag{2}$$

and another exists only for $d = 2$ and integer ℓ which is called exotic extension:

$$[P_i^m, P_j^n] = I^{mn}\varepsilon_{ij}\Theta, \quad \varepsilon_{12} = -\varepsilon_{21} = 1, \tag{3}$$

where I^{mn} is a symmetric tensor. The simplest case $\ell = 1/2$ corresponds to the Schrödinger algebra with mass central extension. The $\ell = 1$ algebra with the exotic central extension is of some physical interest (see for example, [4, 13–16]). The representation theory of the algebra for $d = 2,\ \ell = 1$ has been investigated in [17].

The vector field realization (1) is formally extended to infinite dimension [4]:

$$
\begin{aligned}
&L^m = -t^{m+1}\partial_t - \ell(m+1)t^m x_i \partial_i, \\
&J_{ij}^m = -t^m(x_i\partial_j - x_j\partial_i), \qquad P_i^n = -t^{n+\ell}\partial_i,
\end{aligned} \tag{4}
$$

where $m \in \mathbb{Z},\ n \in \mathbb{Z}+\ell$. The subalgebra $sl(2,\mathbb{R})$ is recast into single expression L^m with $m = 0, \pm 1$. J_{ij}^0 and P_i^n with $-\ell \le n \le \ell$ are corresponds to those of finite dimensional GCA. One may verify that (4) defines an Lie algebra with the structure

$$
(\ \langle L^m\ \rangle \ni \langle\ J_{ij}^m\ \rangle\ ) \ni \langle\ P_i^n\ \rangle,
$$

where $\langle\ L^m\ \rangle$, $\langle\ J_{ij}^m\ \rangle$ span the Virasoro and $\widehat{so(n)}$ subalgebras, respectively. The subalgebra $\langle\ P_i^m\ \rangle$ remains Abelian ideal.

In this work we focus on $d = 2,\ \ell = 1$ algebra and denote it by $\mathfrak{g}$. We present the defining relation of $\mathfrak{g}$ with a slight change of notations.

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n}, \ \ [J_m, J_n] = [P_m^i, P_n^j] = 0, \\
&[L_m, J_n] = -nJ_{m+n}, \ \ [L_m, P_n^i] = (m-n)P_{m+n}^i, \\
&[J_m, P_n^i] = \varepsilon_{ij} P_{m+n}^j,
\end{aligned} \tag{5}
$$

where $i, j = 1, 2$ and $m, n \in \mathbb{Z}$.

3 Centrally Extended Algebra and Its Verma Modules

Probably, the first question one may be interested in is the central extension of the algebra $\mathfrak{g}$. Does $\mathfrak{g}$ have the central extension of exotic type? One has the negative answer to this question from the following theorem proved in [18]:

Theorem 1. *The algebra $\mathfrak{g}$ has the following central extensions:*

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{\alpha}{12} m(m^2-1)\delta_{m+n,0}, \\
&[J_m, J_n] = \beta m \delta_{m+n,0},
\end{aligned}
$$

where α and β are independent central charges. However, the exotic type extension is impossible.

We denote the algebra with the central extensions by $\tilde{\mathfrak{g}}$.

Let us turn to representations of $\tilde{\mathfrak{g}}$. We study the Verma modules of highest weight type. Define the degree of $X_n \in \tilde{\mathfrak{g}}$ by $\deg(X_n) = -n$ where $X = L, J, P^i$. This allows us to define the triangular decomposition of $\tilde{\mathfrak{g}}$:

$$\begin{aligned}\tilde{\mathfrak{g}} &= \tilde{\mathfrak{g}}^- \oplus \tilde{\mathfrak{g}}^0 \oplus \tilde{\mathfrak{g}}^+ \\ &= \langle\, L_{-n}, J_{-n}, P^i_{-n} \,\rangle \oplus \langle\, L_0, J_0, P^i_0 \,\rangle \oplus \langle\, L_n, J_n, P^i_n \,\rangle, \qquad n \in \mathbb{Z}_+\end{aligned}$$

Let $|0\rangle$ be the highest weight vector:

$$\begin{aligned}& L_n\,|0\rangle = J_n\,|0\rangle = P^i_n\,|0\rangle = 0, \quad n \in \mathbb{Z}_+ \\ & L_0\,|0\rangle = h\,|0\rangle\,, \quad J_0\,|0\rangle = \mu\,|0\rangle\,, \quad P^i_0\,|0\rangle = \rho_i\,|0\rangle\,.\end{aligned}$$

Following the usual definition of Verma modules, we define the Verma modules for $\tilde{\mathfrak{g}}$:

$$V^{\mathscr{I}} = U(\tilde{\mathfrak{g}}^-)\,|0\rangle\,,$$

where $\mathscr{I} = \{\, h, \mu, \rho_i \,\}$. The Verma module $V^{\mathscr{I}}$ is a graded-modules through a natural extension of the degree from $\tilde{\mathfrak{g}}$ to $U(\tilde{\mathfrak{g}})$ by $\deg(XY) = \deg(X) + \deg(Y)$, $X, Y \in U(\tilde{\mathfrak{g}})$,

$$V^{\mathscr{I}} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V^{\mathscr{I}}_n\,, \quad V^{\mathscr{I}}_n = \{X\,|0\rangle \mid X \in U(\tilde{\mathfrak{g}}^-),\, \deg(X) = n\,\}.$$

There exists an algebraic anti-automorphism $\omega : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ defined by

$$\omega(L_m) = L_{-m}, \quad \omega(J_m) = J_{-m}, \quad \omega(P^i_m) = P^i_{-m}. \tag{6}$$

One can introduce an inner product in $V^{\mathscr{I}}$ by extending the anti-automorphism ω to $U(\tilde{\mathfrak{g}})$. We define the inner product of $X\,|0\rangle\,, Y\,|0\rangle \in V^{\mathscr{I}}$ by

$$\langle 0|\,\omega(X)Y\,|0\rangle\,, \quad \langle 0|0\rangle = 1.$$

The reducibility of $V^{\mathscr{I}}$ may be investigated by the Kac determinant. The Kac determinant is defined as usual [19]. Let $|i\rangle\,(i = 1, \cdots \dim V^{\mathscr{I}}_n)$ be a basis of $V^{\mathscr{I}}_n$, then the Kac determinant at level (degree) n is given by

$$\Delta_n = \det(\,\langle i|j\rangle\,).$$

We want to calculate Δ_n. The main obstacles of this calculation are rapid increase of $\dim V^{\mathscr{I}}_n$ as a function of n and Δ_n is never reduced to the determinant of diagonal matrix. For illustration we list $\dim V^{\mathscr{I}}_n$ for some small n:

n	0	1	2	3	4	5
$\dim V_n^{\mathscr{I}}$	1	4	14	40	105	252

However, one can carry out the computation by the method similar to that for Schrödinger–Virasoro algebra used in [20]. We here merely mention the result and do not go into the detail. The computational details are found in [18]. To mention the result we need some preparation.

A partition $A = (a_1 a_2 \cdots a_\ell)$ of a positive integer a is the sequence of positive integers such that

$$a = a_1 + a_2 + \cdots + a_\ell,$$
$$a_1 \geq a_2 \geq \cdots \geq a_\ell > 0.$$

The integers ℓ is called length of the partition A and denoted by $\ell(A)$. For a given partition A of a, we decompose a set of integers $a_1, a_2, \cdots, a_\ell$ to two subsets by selecting s integers from them $(0 \leq s \leq \ell)$:

$$A_1 = \{\, a_{\sigma_1} \geq a_{\sigma_2} \geq \cdots \geq a_{\sigma_s} \,\}, \quad A_2 = \{\, a_{\rho_1} \geq a_{\rho_2} \geq \cdots \geq a_{\rho_{\ell-s}} \,\}. \qquad (7)$$

A_1 consists of the selected s integers and the members of A_2 are the rest integers so that the partition A is decomposed into a pair of partitions $(A_1 A_2)$. We denote the number of all possible pairs $(A_1 A_2)$ by $s(A)$. For instance, let $A = (21)$ then the possible pairs are $(A_1 A_2) = ((21)\phi),\ ((2)(1)),\ ((1)(2)),\ (\phi(21))$ so that $s((21))$=4.Now we mention our result of Δ_n :

Theorem 2. *Level n Kac determinant is given by*

$$\Delta_n = c_n \prod_{a,b} \prod_{A,B} (\rho_1^2 + \rho_2^2)^{\frac{1}{2} s(A) s(B) (\ell(A) + \ell(B))}$$

where the pair (a,b) runs all possible non-negative integers satisfying $n = a + b$ and the pair (A,B) runs all possible partitions of fixed a and b. The coefficient c_n is a numerical constant.

Explicit values of c_n (up to sign) and the power of $\rho_1^2 + \rho_2^2$ for $n = 1,2,3$ are listed below:

n	1	2	3
(c_n, power)	$(2,2)$	$(2^{18}, 12)$	$(2^{72} 3^6, 48)$

Remark 1. Δ_n is independent of the central charges α, β. Thus the formula of Δ_n is common for the algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$.

Remark 2. For $\rho_1^2 + \rho_2^2 \neq 0$ the Kac determinant Δ_n never vanish so that there exist no singular vectors in $V_n^{\mathscr{I}}$ for any n. Thus the Verma modules $V^{\mathscr{I}}$ are irreducible. This is a sharp contrast to spin 1 GCA of finite dimension with the exotic central extension where some Verma modules for certain nonvanishing highest weights are reducible [17].

Remark 3. The facts mentioned in the above remarks are also observed for the Schrödinger–Virasoro algebra in $(1+1)$ dimensional space-time [20].

We close this section with the boson realization of $\tilde{\mathfrak{g}}$. We introduce the set of bosons used to realize the Virasoro algebra [19]:

$$[a_m, a_n] = m\delta_{m+n,0}, \quad m,n \in \mathbb{Z}. \tag{8}$$

Observe that the commutation relation is identical to that of J_m if $\beta = 1$. Let us introduce three more bosons:

$$[b_i, \bar{b}_j] = \delta_{i,j}, \quad [b_i, b_j] = [\bar{b}_i, \bar{b}_j] = 0, \quad [c, \bar{c}] = 1,$$

where $i, j = 1, 2$. We suppose that these three types of bosons (a, b, c) commute one another. Then the algebra $\tilde{\mathfrak{g}}$ with $\alpha = \beta = 1$ is realized as follows:

$$\begin{aligned} L_m &= \frac{1}{2}\sum_{k\in\mathbb{Z}} : a_k a_{m-k} : -\bar{c}^{m+1}c - (m+1)c^m \sum_i \bar{b}_i b_i, \\ J_m &= a_m - \bar{c}^m \sum_{ij} \varepsilon_{ij}\bar{b}_i b_j, \quad P^i_m = -\bar{c}^{m+1} b_i, \end{aligned} \tag{9}$$

where the normal ordering is defined as usual:

$$: a_m a_k := \begin{cases} a_m a_k & m \le k \\ a_k a_m & m > k \end{cases}$$

One may prove (9) by direct computation.

4 Coadjoint Representation of $\mathfrak{g}$

In this section, we return to the algebra $\mathfrak{g}$ without the central extensions and study its coadjoint representation by employing the procedure of [20]. We first determine the algebra $\mathfrak{g}^*$ dual to $\mathfrak{g}$. This is done by regarding $\mathfrak{g}$ as a sum of modules of centerless Virasoro algebra. The centerless Virasoro algebra has a one-parameter family of representation $\mathscr{F}_\lambda,\ \lambda \in \mathbb{R}$ on $\mathbb{C}[z, z^{-1}]$ which is defined by

$$L_n z^m = (\lambda n - m) z^{n+m}, \quad n,m \in \mathbb{Z}. \tag{10}$$

By introducing the current $L_f = \sum_n f_n L_n$, an element of $\mathscr{F}_\lambda$ is understood as $(-\lambda)$-density $\phi(z)dz^{-\lambda}$ acted by L_f :

$$f(z)\partial_z \phi(z) dz^{-\lambda} = (f\phi' - \lambda f'\phi) dz^{-\lambda}. \tag{11}$$

If $L_n = ie^{in\theta}\partial_\theta$, $\phi_m = ie^{im\theta}$ in terms of an angular variable $0 \leq \theta \leq 2\pi$, one gets $L_n\phi_m = (\lambda n - m)\phi_{n+m}$. The dual module $\mathscr{F}_\lambda^*$ may be identified with $\mathscr{F}_{-1-\lambda}$ through the paring:

$$\langle\, u(z)dz^{1+\lambda},\, f(z)zdz^{-\lambda}\,\rangle = \int_{S^1} u(z)f(z)dz. \tag{12}$$

The current algebra on $\mathfrak{g}$ may reads as follows:

$$L_f = f(\theta)\partial_\theta + f'(\theta)x_i\partial_i, \quad P_g^i = g(\theta)\partial_i, \quad J_h = h(\theta)(x_1\partial_2 - x_2\partial_1). \tag{13}$$

It follows that

$$[L_f, L_g] = L_{\{f,g\}}, \quad [L_f, P_g^i] = P^i_{\{f,g\}}, \quad [L_f, J_h] = J_{fh'},$$

with $\{f,g\} = fg' - f'g$. Thus, by the adjoint action of L_f, L_g and P_g^i behave like $\mathscr{F}_1$, J_h like $\mathscr{F}_0$, so that $\mathfrak{g}$ is regarded as a sum of L_f modules:

$$\mathfrak{g} = \mathscr{F}_1 \oplus \mathscr{F}_1 \oplus \mathscr{F}_1 \oplus \mathscr{F}_0.$$

The dual algebra is given by

$$\mathfrak{g}^* = \mathscr{F}_{-2} \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-2} \oplus \mathscr{F}_{-1}.$$

We shall denote the element $\gamma_0 dz^2 + \gamma_1 dz^2 + \gamma_2 dz^2 + \gamma_3 dz \in \mathfrak{g}^*$ by the vector form $\overrightarrow{\gamma} = {}^t(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, that is,

$$\langle\, \overrightarrow{\gamma},\, L_{f_0} + P^1_{f_1} + P^2_{f_2} + J_{f_3}\,\rangle = \sum_{i=0}^{3}\int_{S^1} \gamma_i f_i dz. \tag{14}$$

The coadjoint action ad^* of $\mathfrak{g}$ on $\mathfrak{g}^*$ is defined by

$$\langle\, ad^*(X)\gamma,\, Y\,\rangle = \langle\, \gamma,\, [Y,X]\,\rangle, \qquad X, Y \in \mathfrak{g}, \quad \gamma \in \mathfrak{g}^* \tag{15}$$

Theorem 3. *The coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is given as follows*:

$$ad^*(L_{f_0})\overrightarrow{\gamma} = \begin{pmatrix} 2\gamma_0 f_0' + \gamma_0' f_0 \\ 2\gamma_1 f_0' + \gamma_1' f_0 \\ 2\gamma_2 f_0' + \gamma_2' f_0 \\ \gamma_3 f_0' + \gamma_3' f_0 \end{pmatrix}, \qquad ad^*(J_{f_3})\overrightarrow{\gamma} = \begin{pmatrix} \gamma_3 f_3' \\ \gamma_2 f_3 \\ -\gamma_1 f_3 \\ 0 \end{pmatrix},$$

$$ad^*(P^1_{f_1})\overrightarrow{\gamma} = \begin{pmatrix} 2\gamma_1 f_1' + \gamma_1' f_1 \\ 0 \\ 0 \\ -\gamma_2 f_1 \end{pmatrix}, \qquad ad^*(P^2_{f_2})\overrightarrow{\gamma} = \begin{pmatrix} 2\gamma_2 f_2' + \gamma_2' f_2 \\ 0 \\ 0 \\ \gamma_1 f_2 \end{pmatrix}. \tag{16}$$

Proof. We show $ad^*(P^1_{f_1})\overrightarrow{\gamma}$ for illustration. Others are obtained in a similar way. By definitions (14) and (15) we obtain

$$\begin{aligned}
\langle\, ad^*(P^1_{f_1})\overrightarrow{\gamma},\, L_{h_0}\,\rangle &= \langle\, \overrightarrow{\gamma},\, [L_{h_0}, P^1_{f_1}]\,\rangle = \langle\, \overrightarrow{\gamma},\, P^1_{\{h_0,f_1\}}\,\rangle \\
&= \int_{S^1} \gamma_1 (h_0 f_1' - h_0' f_1) dz = \int_{S^1} h_0 (2\gamma_1 f_1' + \gamma_1' f_1) dz, \\
\langle\, ad^*(P^1_{f_1})\overrightarrow{\gamma},\, P^j_{h_j}\,\rangle &= \langle\, \overrightarrow{\gamma},\, [P^j_{h_j}, P^1_{f_1}]\,\rangle = 0, \qquad (j = 1,2), \\
\langle\, ad^*(P^1_{f_1})\overrightarrow{\gamma},\, J_{h_3}\,\rangle &= \langle\, \overrightarrow{\gamma},\, [J_{h_3}, P^1_{f_1}]\,\rangle = \langle\, \overrightarrow{\gamma},\, -P^2_{h_3 f_1}\,\rangle = -\int_{S^1} h_3 \gamma_2 f_1 dz.
\end{aligned}$$

It follows the result for $ad^*(P^1_{f_1})\overrightarrow{\gamma}$. □

5 Concluding Remarks

We have investigated the infinite dimensional extension of the spin 1 Galilean conformal algebra in the plane. It was shown that the algebra $\mathfrak{g}$ does not have the central extension of exotic type. This is a sharp contrast to the algebra of finite dimension. We then studied the representations of both $\mathfrak{g}$ and its central extension $\tilde{\mathfrak{g}}$. The coadjoint representation of $\mathfrak{g}$ was derived using the *regular dual* of $\mathfrak{g}$. This will be followed by the study of coadjoint representation and coadjoint orbit for the infinite dimensional group which is an integration of $\mathfrak{g}$. In such studies it might be useful to have a more general vector field realization of $\mathfrak{g}$. Here we give a two parameter (λ, μ) realization which is reduced to (4) for $\lambda = \mu = 0$:

$$\begin{aligned}
L_m &= -t^{m+1}(\partial_t + \lambda) - (m+1)t^m(x_i\partial_i + \mu), \\
P^i_m &= -t^{m+1}\partial_i, \qquad J_m = -t^m(x_1\partial_2 - x_2\partial_1).
\end{aligned}$$

Turn to the algebra $\tilde{\mathfrak{g}}$ with the central extension, we have studied reducibility of the Verma modules by computing the Kac determinant. It was shown that if $\rho_1^2 + \rho_2^2 \neq 0$ then the Verma modules are irreducible and this is independent of the central charges. On the other hand, if $\rho_1^2 + \rho_2^2 = 0$ then the Verma modules will be reducible. This may be verified by seeking singular vectors. It will be a future work.

In the present work, only mathematical aspects of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ have been investigated. It has its own interest, however, it would be more exciting if one elucidate physical implication of the algebras. Unfortunately, this is an open question at this stage. We know that physical application of algebraic structure usually appear through representations. In this sense it may be worth to carry out further studies of representations of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$.

Acknowledgements The author is supported by a grants-in-aid from JSPS (Contract No.23540154).

References

1. Henkel, M.: J. Stat. Phys. **75**, 1023–1061 (1994) [arXiv:hep-th/9310081]
2. Bagchi, A., Gopakumar, R.: J. High Energy Phys. **07**, 037 (2009) (arXiv:0902.1385 [hep-th])
3. Alishahiha, M., Davody, A., Vahedi, A.: J. High Energy Phys. **08**, 022 (2009) (arXiv:0903.3953 [hep-th])
4. Martelli, D., Tachikawa, Y.: J. High Energy Phys. **05**, 091 (2010) (arXiv:0903.5184 [hep-th])
5. Bagchi, A., Gopakumar, R., Mandal, I., Miwa, A.: J. High Energy Phys. **08**, 004 (2010) (arXiv:0912.1090 [hep-th])
6. Hosseiny, A., Rouhani, S.: J. Math. Phys. **51**, 102303 (2010) (arXiv:1001.1036 [hep-th])
7. Hosseiny, A., Rouhani, S.: J. Math. Phys. **51**, 052307 (2010) (arXiv:0909.1203 [hep-th])
8. Henkel, M., Schott, R., Stoimenov, S., Unterberger, J.: arXiv:math-ph/0601028.
9. Galajinsky, A.V., Masterov, I.V.: Phys. Lett. **B 702**, 265–267 (2011) (arXiv:1104.5115 [hep-th])
10. Havas, P., Plebański, J.: J. Math. Phys. **19**, 482–488 (1978)
11. Negro, J., del Olmo, M.A., Rodríguez-Marco, A.: J. Math. Phys. **38**, 3786–3809 (1997)
12. Negro, J., del Olmo, M.A., Rodríguez-Marco, A.: J. Math. Phys. **38**, 3810–3831 (1997)
13. Stichel, P.C., Zakrzewski, W.J.: Ann. Phys. **310**, 158–180 (2004) [arXiv:hep-th/0309038]
14. Lukierski, J., Stichel, P.C., Zakrzewski, W.J.: Phys. Lett. **A357**, 1–5 (2006) [hep-th/0511259]
15. Lukierski, J., Stichel, P.C., Zakrzewski, W.J.: Phys. Lett. **B650**, 203–207 (2007) [arXiv:hep-th/0702179]
16. Gomis, J., Kamimura, K.: Phys. Rev. **D85**, 045023 (2012) (arXiv:1109.3773v1 [hep-th])
17. Aizawa, N., Isaac, P.S.: J. Phys. A Math. Theor. **44**, 035401 (2011) (arXiv:1010.4075 [math-ph])
18. Aizawa, N.: arXiv:1112.0634 [math-ph]
19. Kac, V.G., Raina, A.K.: Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras. World Scientific, Singapore (1987)
20. Roger, C., Unterberger, J.: The Schrödinger-Virasoro Algebra: Mathematical Structure and Dynamical Schrodinger Symmetries. Springer (2012)

Invariant Differential Operators for Non-compact Lie Groups: The $Sp(n,\mathbb{R})$ Case

V.K. Dobrev

Abstract In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact algebras $sp(n,\mathbb{R})$, in detail for $n = 6$. Our choice of these algebras is motivated by the fact that they belong to a narrow class of algebras, which we call "conformal Lie algebras", which have very similar properties to the conformal algebras of Minkowski space-time. We give the main multiplets and the main reduced multiplets of indecomposable elementary representations for $n = 6$, including the necessary data for all relevant invariant differential operators. In fact, this gives by reduction also the cases for $n < 6$, since the main multiplet for fixed n coincides with one reduced case for $n+1$.

1 Introduction

Consider a Lie group G, e.g., the Lorentz, Poincaré, conformal groups, and differential equations

$$\mathscr{I} f = j$$

which are G-invariant. These play a very important role in the description of physical symmetries—recall, e.g., the early examples of Dirac, Maxwell, d'Allembert, equations and nowadays the latest applications of (super-)differential operators in conformal field theory, supergravity, string theory, (for a recent review, cf. e.g., [1]). Naturally, it is important to construct systematically such invariant equations and operators.

V.K. Dobrev (✉)
Theory Division, Department of Physics, CERN, CH-1211 Geneva 23, Switzerland

Institute for Nuclear Research and Nuclear Energy, Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria
e-mail: Vladimir.Dobrev@cern.ch; dobrev@inrne.bas.bg

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_22, © Springer Japan 2013

In a recent paper [2] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on the groups $Sp(n,\mathbb{R})$, which are very interesting for several reasons. First of all, they belong to the class of Hermitian symmetric spaces, i.e., the pair (G,K) is a Hermitian symmetric pair (K is the maximal compact subgroup of the noncompact semisimple group G). Further, $Sp(n,\mathbb{R})$ belong to a narrower class of groups/algebras, which we call 'conformal Lie groups or algebras' since they have very similar properties to the canonical conformal algebras $so(n,2)$ of n-dimensional Minkowski space-time. This class was identified from our point of view in [3]. Besides $so(n,2)$ it includes the algebras $su(n,n)$, $sp(n,\mathbb{R})$, $so^*(4n)$, $E_{7(-25)}$, (omitting to mention coincidences between the low-dimensional cases, cf. [3]). The corresponding groups are also called Hermitian symmetric spaces of tube type [4]. The same class was identified from different considerations in [5], where these groups/algebras were called 'conformal groups of simple Jordan algebras'. It was identified from still different considerations also in [6], where the objects of the class were called simple space-time symmetries generalizing conformal symmetry.

In our further plans it shall be very useful that (as in [2]) we follow a procedure in representation theory in which intertwining differential operators appear canonically [7] and which procedure has been generalized to the supersymmetry setting and to quantum groups.

The present paper is organized a follows. In Sect. 2 we give the preliminaries, actually recalling and adapting facts from [2]. In Sect. 3 we specialize to the $sp(n,\mathbb{R})$ case. In Sect. 4 we present some results on the multiplet classification of the representations and intertwining differential operators between them. Sect. 5 is an outlook. There are eight figures that are referred to throughout the text and for typographical reasons (due to their size) are collected in Sect. 6.

2 Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an Iwasawa decomposition $G = KA_0N_0$, where A_0 is abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of G. A parabolic subgroup $P = M'A'N'$ is any subgroup of G (including G itself) which contains a minimal parabolic subgroup.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [8]. For the classification of all irreducible representations it is enough to use only the so-called *cuspidal* parabolic subgroups $P = M'A'N'$, singled out by the condition that $\mathrm{rank}\, M' = \mathrm{rank}\, M' \cap K$ [9, 10], so that M' has discrete series representations [11]. However, often induction from non-cuspidal parabolics is also convenient, cf. [2, 12–14].

Let ν be a (non-unitary) character of A', $\nu \in \mathscr{A}'^*$, let μ fix an irreducible representation D^μ of M' on a vector space V_μ.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [15–17]. (These are called *generalized principal series representations* (or *limits thereof*) in [18].) Their spaces of functions are:

$$\mathscr{C}_\chi \ = \ \{\mathscr{F} \in C^\infty(G, V_\mu) \mid \mathscr{F}(gman) \ = \ e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathscr{F}(g)\} \tag{1}$$

where $a = \exp(H) \in A'$, $H \in \mathscr{A}'$, $m \in M'$, $n \in N'$. The representation action is the *left* regular action:

$$(\mathscr{T}^\chi(g)\mathscr{F})(g') \ = \ \mathscr{F}(g^{-1}g')\,, \quad g, g' \in G. \tag{2}$$

For our purposes we need to restrict to *maximal* parabolic subgroups P, (so that $\mathrm{rank}\, A' = 1$), that may not be cuspidal. For the representations that we consider the character ν is parameterized by a real number d, called the conformal weight or energy.

Further, let μ fix a discrete series representation D^μ of M' on the Hilbert space V_μ, or the so-called limit of a discrete series representation (cf. [18]). Actually, instead of the discrete series we can use the finite-dimensional (non-unitary) representation of M' with the same Casimirs.

An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathscr{G}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathscr{G}^{\mathbb{C}}$, where $\Lambda \in (\mathscr{H}^{\mathbb{C}})^*$, $\mathscr{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathscr{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [7]. In this setting we can consider also unitarity, which here means positivity w.r.t. the Shapovalov form in which the conjugation is the one singling out $\mathscr{G}$ from $\mathscr{G}^{\mathbb{C}}$.

Actually, since our ERs may be induced from finite-dimensional representations of $\mathscr{M}'$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu\, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [7, 19]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the BGG [20] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) \;=\; m\,, \quad \beta^\vee \equiv 2\beta/(\beta, \beta). \tag{3}$$

When (3) holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\; v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [21]. More explicitly, [7], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs).[1] Then there exists [7] an intertwining differential operator

$$\mathcal{D}_{m,\beta} \;:\; \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{4}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} \;=\; \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}) \tag{5}$$

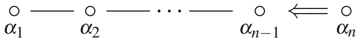
where $\widehat{\mathcal{G}^-}$ denotes the *right* action on the functions $\mathcal{F}$, cf. (1).

3 The Non-compact Lie Algebras $sp(n, \mathbb{R})$

Let $n \geq 2$. Let $\mathcal{G} \;=\; sp(n, \mathbb{R})$, the split real form of $sp(n, \mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subgroup of $\mathcal{G}$ is $\mathcal{K} \cong u(1) \oplus su(n)$, $\dim_{\mathbb{R}} \mathcal{P} = n(n+1)$, $\dim_{\mathbb{R}} \mathcal{N} = n^2$. This algebra has discrete series representations and highest/lowest weight representations.

The split rank is equal to n, while $\mathcal{M} = 0$.

The Satake diagram [24] of $sp(n, \mathbb{R})$ is the same as the Dynkin diagram of $sp(n, \mathbb{C})$:



Also the root systems coincide.

We choose a *maximal* parabolic $\mathcal{P} = \mathcal{M}'\mathcal{A}'\mathcal{N}'$ such that $\mathcal{A}' \cong so(1,1)$, while the factor $\mathcal{M}'$ has the same finite-dimensional (nonunitary) representations as the finite-dimensional (unitary) representations of the semi-simple subalgebra of $\mathcal{K}$, i.e., $\mathcal{M}' = \; sl(n, \mathbb{R})$, cf. [2]. Thus, these induced representations are representations of finite $\mathcal{K}$-type [25]. Relatedly, the number of ERs in the corresponding multiplets

[1] For explicit expressions for singular vectors we refer to [22, 23].

is equal to $|W(\mathscr{G}^{\mathbb{C}},\mathscr{H}^{\mathbb{C}})|\,/\,|W(\mathscr{K}^{\mathbb{C}},\mathscr{H}^{\mathbb{C}})| \;=\; 2^n$, cf. [26], where $\mathscr{H}$ is a Cartan subalgebra of both $\mathscr{G}$ and $\mathscr{K}$. Note also that $\mathscr{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(n,\mathbb{C}) \cong \mathscr{M}'^{\mathbb{C}} \oplus \mathscr{A}'^{\mathbb{C}}$. Finally, note that $\dim_{\mathbb{R}} \mathscr{N}' = n(n+1)/2$.

We label the signature of the ERs of $\mathscr{G}$ as follows:

$$\chi \;=\; \{\, n_1\,, \ldots, \, n_{n-1}\,; c\,\}\,, \quad n_j \in \mathbb{N}\,, \quad c = d - (n+1)/2 \tag{6}$$

where the last entry of χ labels the characters of $\mathscr{A}'$, and the first $n-1$ entries are labels of the finite-dimensional nonunitary irreps of $\mathscr{M}'$, (or of the finite-dimensional unitary irreps of $su(n)$).

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simpler way, as we shall see.

Below we shall use the following conjugation on the finite-dimensional entries of the signature:

$$(n_1, \ldots, n_{n-1})^* \;\doteq\; (n_{n-1}, \ldots, n_1) \tag{7}$$

The ERs in the multiplet are related also by intertwining integral operators.

The integral operators were introduced by Knapp and Stein [27]. In fact, these operators are defined for any ER, not only for the reducible ones, the general action being:

$$\begin{aligned} G_{KS} \,:\, \mathscr{C}_\chi \;&\longrightarrow\; \mathscr{C}_{\chi'}\,, \\ \chi \;&=\; \{\, n_1, \ldots, n_{n-1}\,; c\,\}\,, \\ \chi' \;&=\; \{\, (n_1, \ldots, n_{n-1})^*\,; -c\,\} \end{aligned} \tag{8}$$

The above action on the signatures is also called restricted Weyl reflection, since it represents the nontrivial element of the two-element restricted Weyl group which arises canonically with every maximal parabolic subalgebra.[2]

Further, we need more explicitly the root system of the algebra $sp(n,F)$.

In terms of the orthonormal basis ε_i, $i = 1, \ldots, n$, the positive roots are given by

$$\Delta^+ = \{\varepsilon_i \pm \varepsilon_j,\; 1 \le i < j \le n;\; 2\varepsilon_i, 1 \le i \le n\}, \tag{9}$$

while the simple roots are:

$$\pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1},\; 1 \le i \le n-1;\; \alpha_n = 2\varepsilon_n\} \tag{10}$$

With our choice of normalization of the long roots $2\varepsilon_k$ have length 4, while the short roots $\varepsilon_i \pm \varepsilon_j$ have length 2.

From these the compact roots are those that form (by restriction) the root system of the semisimple part of $\mathscr{K}^{\mathbb{C}}$, the rest are noncompact, i.e.,

$$\begin{aligned} \text{compact}: \;\; \alpha_{ij} \;&\equiv\; \varepsilon_i - \varepsilon_j, \quad 1 \le i < j \le n, \\ \text{noncompact}: \;\; \beta_{ij} \;&\equiv\; \varepsilon_i + \varepsilon_j, \quad 1 \le i \le j \le n \end{aligned} \tag{11}$$

Thus, the only non-compact simple root is $\alpha_n = \beta_{nn}$.

---

[2] Generically, the Knapp–Stein operators can be normalized so that indeed $G_{KS} \circ G_{KS} = \mathrm{Id}_{\mathscr{C}_\chi}$. However, this usually fails exactly for the reducible ERs that form the multiplets, cf., e.g., [15–17].

We adopt the following ordering of the roots:

$$\begin{array}{ccccccccc}
\beta_{11} & & & & & & & & \\
\vee & & & & & & & & \\
\beta_{12} & > & \beta_{22} & & & & & & \\
\vee & & \vee & & & & & & \\
\dots & \dots & \dots & \dots\dots\dots & & & & & \\
\vee & & \vee & & \dots & & \vee & & \\
\beta_{1n} & > & \beta_{2n} & > & \dots > & & \beta_{n-1,n} & > & \beta_{nn} = \alpha_n \\
\vee & & \vee & & \dots & & \vee & & \\
\alpha_{1n} & > & \alpha_{2n} & > & \dots > & \alpha_{n-1,n} = \alpha_{n-1} & & & \\
\vee & & \vee & & \dots & & & & \\
\dots & \dots & \dots & \dots\dots\dots & & & & & \\
\vee & & \vee & & & & & & \\
\alpha_{13} & > & \alpha_{23} = \alpha_2 & & & & & & \\
\vee & & & & & & & & \\
\alpha_{12} = \alpha_1 & & & & & & & &
\end{array} \tag{12}$$

This ordering is lexicographical adopting the ordering of the ε:

$$\varepsilon_1 > \cdots > \varepsilon_n \tag{13}$$

Further, we shall use the so-called Dynkin labels:

$$m_i \equiv \ (\Lambda + \rho, \alpha_i^\vee) \ , \quad i = 1, \dots, n, \tag{14}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$.

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta) \ , \tag{15}$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms. In particular, in the case of the noncompact roots we have:

$$m_{\beta_{ij}} \ = \ \Big(\sum_{s=i}^{n} + \sum_{s=j}^{n} \Big) m_s \ , \ i < j \ ; \qquad m_{\beta_{ii}} \ = \ \sum_{s=i}^{n} m_s \tag{16}$$

Now we can give the correspondence between the signatures χ and the highest weight Λ. The explicit connection is:

$$n_i = m_i \ , \quad c \ = \ -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_n) \ = \ -\tfrac{1}{2}(m_1 + \cdots + m_{n-1} + 2m_n) \tag{17}$$

where $\tilde{\alpha} \ = \ \beta_{11}$ is the highest root.

There are several types of multiplets: the main type, (which contains maximal number of ERs/GVMs, the finite-dimensional and the discrete series representations), and some reduced types of multiplets.

In the next section we give the main type of multiplets and the main reduced types for $sp(n,\mathbb{R})$ for $n \leq 6$.

4 Multiplets

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $sp(n,\mathbb{R})$, i.e., they will be labelled by the n positive Dynkin labels $m_i \in \mathbb{N}$. As we mentioned, each such multiplet contains 2^n ERs/GVMs. It is difficult to give explicitly the multiplets for general n. Thus, we shall give explicitly the case $n = 6$ which can still be represented and comprehended, and then show how to obtain the cases $n < 6$.

4.1 *sp(6,*$\mathbb{R}$*)*

4.1.1 Main Multiplets

The main multiplets R^6 contain $64 (= 2^6)$ ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{\,(m_1,m_2,m_3,m_4,m_5)^\pm\,;\,\pm\tfrac{1}{2}(m_{\tilde{\alpha}}+m_6)\,\} \\
\chi_a^\pm &= \{\,(m_1,m_2,m_3,m_4,m_5+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{15}\,\} \\
\chi_b^\pm &= \{\,(m_1,m_2,m_3,m_{45},m_5+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{14}\,\} \\
\chi_c^\pm &= \{\,(m_1,m_2,m_{34},m_5,m_{45}+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{13}\,\} \\
\chi_{c'}^\pm &= \{\,(m_1,m_2,m_3,m_{45}+2m_6,m_5)^\pm\,;\,\pm\tfrac{1}{2}m_{14}\,\} \\
\chi_d^\pm &= \{\,(m_1,m_{23},m_4,m_5,m_{35}+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{d'}^\pm &= \{\,(m_1,m_2,m_{34},m_5+2m_6,m_{45})^\pm\,;\,\pm\tfrac{1}{2}m_{13}\,\} \\
\chi_e^\pm &= \{\,(m_{12},m_3,m_4,m_5,m_{25}+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{e'}^\pm &= \{\,(m_1,m_{23},m_4,m_5+2m_6,m_{35})^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{e''}^\pm &= \{\,(m_1,m_2,m_{35},m_5+2m_6,m_4)^\pm\,;\,\pm\tfrac{1}{2}m_{13}\,\} \\
\chi_f^\pm &= \{\,(m_2,m_3,m_4,m_5,m_{15}+2m_6)^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_{f'}^\pm &= \{\,(m_{12},m_3,m_4,m_5+2m_6,m_{25})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{f''}^\pm &= \{\,(m_1,m_{23},m_{45},m_5+2m_6,m_{34})^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{f'''}^\pm &= \{\,(m_1,m_2,m_{35}+2m_6,m_5,m_4)^\pm\,;\,\pm\tfrac{1}{2}m_{13}\,\} \\
\chi_g^\pm &= \{\,(m_2,m_3,m_4,m_5+2m_6,m_{15})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_{g'}^\pm &= \{\,(m_{12},m_3,m_{45},m_5+2m_6,m_{24})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}
\end{aligned} \tag{18}$$

$$\begin{aligned}
\chi_{g''}^{\pm} &= \{\,(m_1, m_{23}, m_{45}+2m_6, m_5, m_{34})^{\pm}\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi_{h}^{\pm} &= \{\,(m_2, m_3, m_{45}, m_5+2m_6, m_{14})^{\pm}\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi_{h'}^{\pm} &= \{\,(m_{12}, m_3, m_{45}+2m_6, m_5, m_{24})^{\pm}\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi_{h''}^{\pm} &= \{\,(m_2, m_3, m_{45}+2m_6, m_5, m_{14})^{\pm}\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi_{j}^{\pm} &= \{\,(m_2, m_{34}, m_5, m_{45}+2m_6, m_{13})^{\pm}\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi_{j'}^{\pm} &= \{\,(m_{12}, m_{34}, m_5, m_{45}+2m_6, m_{23})^{\pm}\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi_{j''}^{\pm} &= \{\,(m_1, m_{24}, m_5, m_{45}+2m_6, m_3)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi_{k}^{\pm} &= \{\,(m_2, m_{34}, m_5+2m_6, m_{45}, m_{13})^{\pm}\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi_{k'}^{\pm} &= \{\,(m_{12}, m_{34}, m_5+2m_6, m_{45}, m_{23})^{\pm}\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi_{k''}^{\pm} &= \{\,(m_1, m_{24}, m_5+2m_6, m_{45}, m_3)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi_{\ell}^{\pm} &= \{\,(m_2, m_{35}, m_5+2m_6, m_4, m_{13})^{\pm}\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi_{\ell'}^{\pm} &= \{\,(m_{12}, m_{35}, m_5+2m_6, m_4, m_{23})^{\pm}\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi_{\ell''}^{\pm} &= \{\,(m_1, m_{25}, m_5+2m_6, m_4, m_3)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi_{m}^{\pm} &= \{\,(m_2, m_{35}+2m_6, m_5, m_4, m_{13})^{\pm}\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi_{m'}^{\pm} &= \{\,(m_{12}, m_{35}+2m_6, m_5, m_4, m_{23})^{\pm}\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi_{m''}^{\pm} &= \{\,(m_1, m_{25}+2m_6, m_5, m_4, m_3)^{\pm}\,;\,\pm\tfrac{1}{2}m_{12}\,\}
\end{aligned}$$

where the notation $(\dots)^{\pm}$ employs the conjugation (7) :

$$(n_1,\dots,n_5)^- = (n_1,\dots,n_5)\,, \quad (n_1,\dots,n_5)^+ = (n_1,\dots,n_5)^* = (n_5,\dots,n_1) \tag{19}$$

Obviously, the pairs in (18) are related by Knapp–Stein integral operators, i.e.,

$$G_{KS}\;:\;\mathscr{C}_{\chi^{\mp}} \longleftrightarrow \mathscr{C}_{\chi^{\pm}} \tag{20}$$

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathscr{E}$. The latter corresponds to the finite-dimensional irrep of $sp(6)$ with signature $\{m_1,\dots,m_6\}$. The subspace $\mathscr{E}$ is annihilated by the operator G^+, and is the image of the operator G^-. The subspace $\mathscr{E}$ is annihilated also by the intertwining differential operator acting from χ^- to χ'^- (more about this operator below). When all $m_i = 1$ then $\dim\mathscr{E} = 1$, and in that case $\mathscr{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathscr{G}$. Furthermore in that case the conformal weight is zero: $d = \frac{7}{2} + c = \frac{7}{2} - \frac{1}{2}(m_1 + \dots + m_5 + 2m_6)_{|_{m_i=1}} = 0$.

Analogously, in every multiplet only the ER with signature χ_0^+ contains holomorphic discrete series representation. This is guaranteed by the criterion [11] that for such an ER all Harish-Chandra parameters for non-compact roots must be negative, i.e., in our situation, $m_\alpha < 0$, for α from the second row of (11). [That this holds for our χ^+ can be easily checked using the signatures (18).]

In fact, the Harish-Chandra parameters are reflected in the division of the ERs into χ^- and χ^+: for the χ^- less than half of the 21 non-compact Harish-Chandra parameters are negative, (none for χ_0^-), while for the χ^+ more than half of the 21 non-compact Harish-Chandra parameters are negative, (all for χ_0^+),

Note that the ER χ_0^+ contains also the conjugate anti-holomorphic discrete series. The direct sum of the holomorphic and the antiholomorphic representations are realized in an invariant subspace $\mathscr{D}$ of the ER χ_0^+. That subspace is annihilated by the operator G^-, and is the image of the operator G^+.

Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.
The conformal weight of the ER χ_0^+ has the restriction $d = \frac{7}{2} + c = \frac{7}{2} + \frac{1}{2}(m_1 + \cdots + m_5 + 2m_6) \geq 7$.

The multiplets are given explicitly in Fig. 1, where we use the notation: $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j...k}$ encoding the root $\beta_{j...k}$ and the number $m_{\beta_{j...k}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data β, m_β, which is involved in the embedding $V^\Lambda \longleftrightarrow V^{\Lambda - m_\beta, \beta}$ turns out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 2n-1\}$, such that $m_\beta = m_i$. Hence the data $\beta_{j...k}, m_{\beta_{j...k}}$ is represented by $i_{j...k}$ on the arrows.

The pairs $\Lambda^\pm$ are symmetric w.r.t. to the bullet in the middle of the figure—this represents the Weyl symmetry realized by the Knapp–Stein operators.

4.1.2 Reduced Multiplets R_1^6

The reduced multiplets of type R_1^6 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{(0, m_2, m_3, m_4, m_5)^\pm; \pm\tfrac{1}{2}(m_{25} + 2m_6)\} \qquad (21)\\
\chi_a^\pm &= \{(0, m_2, m_3, m_4, m_5 + 2m_6)^\pm; \pm\tfrac{1}{2}m_{25}\}\\
\chi_b^\pm &= \{(0, m_2, m_3, m_{45}, m_5 + 2m_6)^\pm; \pm\tfrac{1}{2}m_{24}\}\\
\chi_c^\pm &= \{(0, m_2, m_{34}, m_5, m_{45} + 2m_6)^\pm; \pm\tfrac{1}{2}m_{23}\}\\
\chi_{c'}^\pm &= \{(0, m_2, m_3, m_{45} + 2m_6, m_5)^\pm; \pm\tfrac{1}{2}m_{24}\}\\
\chi_d^\pm &= \{(0, m_{23}, m_4, m_5, m_{35} + 2m_6)^\pm; \pm\tfrac{1}{2}m_2\}\\
\chi_{d'}^\pm &= \{(0, m_2, m_{34}, m_5 + 2m_6, m_{45})^\pm; \pm\tfrac{1}{2}m_{23}\}\\
\chi_e^\pm &= \{(m_2, m_3, m_4, m_5, m_{25} + 2m_6)^\pm; 0\}\\
\chi_{e'}^\pm &= \{(0, m_{23}, m_4, m_5 + 2m_6, m_{35})^\pm; \pm\tfrac{1}{2}m_2\}\\
\chi_{e''}^\pm &= \{(0, m_2, m_{35}, m_5 + 2m_6, m_4)^\pm; \pm\tfrac{1}{2}m_{23}\}\\
\chi_{f'}^\pm &= \{(m_2, m_3, m_4, m_5 + 2m_6, m_{25})^\pm; 0\}\\
\chi_{f''}^\pm &= \{(0, m_{23}, m_{45}, m_5 + 2m_6, m_{34})^\pm; \pm\tfrac{1}{2}m_2\}
\end{aligned}$$

$$
\begin{aligned}
\chi^\pm_{f'''} &= \{\,(0,m_2,m_{35}+2m_6,m_5,m_4)^\pm\,;\pm\tfrac{1}{2}m_{23}\,\}\\
\chi^\pm_{g'} &= \{\,(m_2,m_3,m_{45},m_5+2m_6,m_{24})^\pm\,;0\,\}\\
\chi^\pm_{g''} &= \{\,(0,m_{23},m_{45}+2m_6,m_5,m_{34})^\pm\,;\pm\tfrac{1}{2}m_2\,\}\\
\chi^\pm_{h'} &= \{\,(m_2,m_3,m_{45}+2m_6,m_5,m_{24})^\pm\,;0\,\}\\
\chi^\pm_{j'} &= \{\,(m_2,m_{34},m_5,m_{45}+2m_6,m_{23})^\pm\,;0\,\}\\
\chi^\pm_{j''} &= \{\,(0,m_{24},m_5,m_{45}+2m_6,m_3)^\pm\,;\pm\tfrac{1}{2}m_2\,\}\\
\chi^\pm_{k'} &= \{\,(m_2,m_{34},m_5+2m_6,m_{45},m_{23})^\pm\,;0\,\}\\
\chi^\pm_{k''} &= \{\,(0,m_{24},m_5+2m_6,m_{45},m_3)^\pm\,;\pm\tfrac{1}{2}m_2\,\}\\
\chi^\pm_{\ell'} &= \{\,(m_2,m_{35},m_5+2m_6,m_4,m_{23})^\pm\,;0\,\}\\
\chi^\pm_{\ell''} &= \{\,(0,m_{25},m_5+2m_6,m_4,m_3)^\pm\,;\pm\tfrac{1}{2}m_2\,\}\\
\chi^\pm_{m'} &= \{\,(m_2,m_{35}+2m_6,m_5,m_4,m_{23})^\pm\,;0\,\}\\
\chi^\pm_{m''} &= \{\,(0,m_{25}+2m_6,m_5,m_4,m_3)^\pm\,;\pm\tfrac{1}{2}m_2\,\}
\end{aligned}
$$

The multiplets are given explicitly in Fig. 2.

4.1.3 Reduced Multiplets R^6_2

The reduced multiplets of type R^6_2 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi^\pm_0 &= \{\,(m_1,0,m_3,m_4,m_5)^\pm\,;\pm\tfrac{1}{2}(m_{1,35}+2m_6)\,\} \qquad (22)\\
\chi^\pm_a &= \{\,(m_1,0,m_3,m_4,m_5+2m_6)^\pm\,;\pm\tfrac{1}{2}m_{1,35}\,\}\\
\chi^\pm_b &= \{\,(m_1,0,m_3,m_{45},m_5+2m_6)^\pm\,;\pm\tfrac{1}{2}m_{1,34}\,\}\\
\chi^\pm_c &= \{\,(m_1,0,m_{34},m_5,m_{45}+2m_6)^\pm\,;\pm\tfrac{1}{2}m_{1,3}\,\}\\
\chi^\pm_{c'} &= \{\,(m_1,0,m_3,m_{45}+2m_6,m_5)^\pm\,;\pm\tfrac{1}{2}m_{1,34}\,\}\\
\chi^\pm_d &= \{\,(m_1,m_3,m_4,m_5,m_{35}+2m_6)^\pm\,;\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{d'} &= \{\,(m_1,0,m_{34},m_5+2m_6,m_{45})^\pm\,;\pm\tfrac{1}{2}m_{1,3}\,\}\\
\chi^\pm_{e'} &= \{\,(m_1,m_3,m_4,m_5+2m_6,m_{35})^\pm\,;\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{e''} &= \{\,(m_1,0,m_{35},m_5+2m_6,m_4)^\pm\,;\pm\tfrac{1}{2}m_{1,3}\,\}\\
\chi^\pm_f &= \{\,(0,m_3,m_4,m_5,m_{1,35}+2m_6)^\pm\,;\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{f''} &= \{\,(m_1,m_3,m_{45},m_5+2m_6,m_{34})^\pm\,;\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{f'''} &= \{\,(m_1,0,m_{35}+2m_6,m_5,m_4)^\pm\,;\pm\tfrac{1}{2}m_{1,3}\,\}\\
\chi^\pm_g &= \{\,(0,m_3,m_4,m_5+2m_6,m_{1,35})^\pm\,;\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{g''} &= \{\,(m_1,m_3,m_{45}+2m_6,m_5,m_{34})^\pm\,;\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_h &= \{\,(0,m_3,m_{45},m_5+2m_6,m_{1,34})^\pm\,;\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{h''} &= \{\,(0,m_3,m_{45}+2m_6,m_5,m_{1,34})^\pm\,;\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_j &= \{\,(0,m_{34},m_5,m_{45}+2m_6,m_{1,3})^\pm\,;\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{j''} &= \{\,(m_1,m_{34},m_5,m_{45}+2m_6,m_3)^\pm\,;\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_k &= \{\,(0,m_{34},m_5+2m_6,m_{45},m_{1,3})^\pm\,;\mp\tfrac{1}{2}m_1\,\}
\end{aligned}
$$

$$\begin{aligned}
\chi^\pm_{k''} &= \{\,(m_1,m_{34},m_5+2m_6,m_{45},m_3)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{\ell} &= \{\,(0,m_{35},m_5+2m_6,m_4,m_{1,3})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{\ell''} &= \{\,(m_1,m_{35},m_5+2m_6,m_4,m_3)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{m} &= \{\,(0,m_{35}+2m_6,m_5,m_4,m_{1,3})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{m''} &= \{\,(m_1,m_{35}+2m_6,m_5,m_4,m_3)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}
\end{aligned}$$

The multiplets are given explicitly in Fig. 3.

4.1.4 Reduced Multiplets R^6_3

The reduced multiplets of type R^6_3 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^\pm_{0} &= \{\,(m_1,m_2,0,m_4,m_5)^\pm\,;\,\pm\tfrac{1}{2}(m_{12,45}+2m_6)\,\} \qquad (23)\\
\chi^\pm_{a} &= \{\,(m_1,m_2,0,m_4,m_5+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{12,45}\,\}\\
\chi^\pm_{b} &= \{\,(m_1,m_2,0,m_{45},m_5+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{12,4}\,\}\\
\chi^\pm_{c'} &= \{\,(m_1,m_2,0,m_{45}+2m_6,m_5)^\pm\,;\,\pm\tfrac{1}{2}m_{12,4}\,\}\\
\chi^\pm_{d} &= \{\,(m_1,m_2,m_4,m_5,m_{45}+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi^\pm_{e} &= \{\,(m_{12},0,m_4,m_5,m_{2,45}+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{e'} &= \{\,(m_1,m_2,m_4,m_5+2m_6,m_{45})^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi^\pm_{f} &= \{\,(m_2,0,m_4,m_5,m_{12,45}+2m_6)^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{f'} &= \{\,(m_{12},0,m_4,m_5+2m_6,m_{2,45})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{f''} &= \{\,(m_1,m_2,m_{45},m_5+2m_6,m_4)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi^\pm_{g} &= \{\,(m_2,0,m_4,m_5+2m_6,m_{12,45})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{g'} &= \{\,(m_{12},0,m_{45},m_5+2m_6,m_{2,4})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{g''} &= \{\,(m_1,m_2,m_{45}+2m_6,m_5,m_4)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi^\pm_{h} &= \{\,(m_2,0,m_{45},m_5+2m_6,m_{12,4})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{h'} &= \{\,(m_{12},0,m_{45}+2m_6,m_5,m_{2,4})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{h''} &= \{\,(m_2,0,m_{45}+2m_6,m_5,m_{12,4})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{j} &= \{\,(m_2,m_4,m_5,m_{45}+2m_6,m_{12})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{j'} &= \{\,(m_{12},m_4,m_5,m_{45}+2m_6,m_2)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{j''} &= \{\,(m_1,m_{2,4},m_5,m_{45}+2m_6,0)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi^\pm_{k} &= \{\,(m_2,m_4,m_5+2m_6,m_{45},m_{12})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{k'} &= \{\,(m_{12},m_4,m_5+2m_6,m_{45},m_2)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{k''} &= \{\,(m_1,m_{2,4},m_5+2m_6,m_{45},0)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi^\pm_{\ell} &= \{\,(m_2,m_{45},m_5+2m_6,m_4,m_{12})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{\ell'} &= \{\,(m_{12},m_{45},m_5+2m_6,m_4,m_2)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\}\\
\chi^\pm_{\ell''} &= \{\,(m_1,m_{2,45},m_5+2m_6,m_4,0)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\}\\
\chi^\pm_{m} &= \{\,(m_2,m_{45}+2m_6,m_5,m_4,m_{12})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\}
\end{aligned}$$

$$\begin{aligned}
\chi^\pm_{m'} &= \{\,(m_{12}, m_{45} + 2m_6, m_5, m_4, m_2)^\pm\,;\,\pm\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{m''} &= \{\,(m_1, m_{2,45} + 2m_6, m_5, m_4, 0)^\pm\,;\,\pm\tfrac{1}{2} m_{12}\,\}
\end{aligned}$$

The multiplets are given explicitly in Fig. 4.

4.1.5 Reduced Multiplets R^6_4

The reduced multiplets of type R^6_4 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi^\pm_0 &= \{\,(m_1, m_2, m_3, 0, m_5)^\pm\,;\,\pm\tfrac{1}{2}(m_{13,5} + 2m_6)\,\} \qquad (24) \\
\chi^\pm_a &= \{\,(m_1, m_2, m_3, 0, m_5 + 2m_6)^\pm\,;\,\pm\tfrac{1}{2} m_{13,5}\,\} \\
\chi^\pm_c &= \{\,(m_1, m_2, m_3, m_5, m_5 + 2m_6)^\pm\,;\,\pm\tfrac{1}{2} m_{13}\,\} \\
\chi^\pm_d &= \{\,(m_1, m_{23}, 0, m_5, m_{3,5} + 2m_6)^\pm\,;\,\pm\tfrac{1}{2} m_{12}\,\} \\
\chi^\pm_{d'} &= \{\,(m_1, m_2, m_3, m_5 + 2m_6, m_5)^\pm\,;\,\pm\tfrac{1}{2} m_{13}\,\} \\
\chi^\pm_e &= \{\,(m_{12}, m_3, 0, m_5, m_{23,5} + 2m_6)^\pm\,;\,\pm\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{e'} &= \{\,(m_1, m_{23}, 0, m_5 + 2m_6, m_{3,5})^\pm\,;\,\pm\tfrac{1}{2} m_{12}\,\} \\
\chi^\pm_{e''} &= \{\,(m_1, m_2, m_{3,5}, m_5 + 2m_6, 0)^\pm\,;\,\pm\tfrac{1}{2} m_{13}\,\} \\
\chi^\pm_f &= \{\,(m_2, m_3, 0, m_5, m_{13,5} + 2m_6)^\pm\,;\,\mp\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{f'} &= \{\,(m_{12}, m_3, 0, m_5 + 2m_6, m_{23,5})^\pm\,;\,\pm\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{f'''} &= \{\,(m_1, m_2, m_{3,5} + 2m_6, m_5, 0)^\pm\,;\,\pm\tfrac{1}{2} m_{13}\,\} \\
\chi^\pm_g &= \{\,(m_2, m_3, 0, m_5 + 2m_6, m_{13,5})^\pm\,;\,\mp\tfrac{1}{2} m_1\,\} \\
\chi^\pm_j &= \{\,(m_2, m_3, m_5, m_5 + 2m_6, m_{13})^\pm\,;\,\mp\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{j'} &= \{\,(m_{12}, m_3, m_5, m_5 + 2m_6, m_{23})^\pm\,;\,\pm\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{j''} &= \{\,(m_1, m_{23}, m_5, m_5 + 2m_6, m_3)^\pm\,;\,\pm\tfrac{1}{2} m_{12}\,\} \\
\chi^\pm_k &= \{\,(m_2, m_3, m_5 + 2m_6, m_5, m_{13})^\pm\,;\,\mp\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{k'} &= \{\,(m_{12}, m_3, m_5 + 2m_6, m_5, m_{23})^\pm\,;\,\pm\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{k''} &= \{\,(m_1, m_{23}, m_5 + 2m_6, m_5, m_3)^\pm\,;\,\pm\tfrac{1}{2} m_{12}\,\} \\
\chi^\pm_\ell &= \{\,(m_2, m_{3,5}, m_5 + 2m_6, 0, m_{13})^\pm\,;\,\mp\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{\ell'} &= \{\,(m_{12}, m_{3,5}, m_5 + 2m_6, 0, m_{23})^\pm\,;\,\pm\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{\ell''} &= \{\,(m_1, m_{23,5}, m_5 + 2m_6, 0, m_3)^\pm\,;\,\pm\tfrac{1}{2} m_{12}\,\} \\
\chi^\pm_m &= \{\,(m_2, m_{3,5} + 2m_6, m_5, 0, m_{13})^\pm\,;\,\mp\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{m'} &= \{\,(m_{12}, m_{3,5} + 2m_6, m_5, 0, m_{23})^\pm\,;\,\pm\tfrac{1}{2} m_1\,\} \\
\chi^\pm_{m''} &= \{\,(m_1, m_{23,5} + 2m_6, m_5, 0, m_3)^\pm\,;\,\pm\tfrac{1}{2} m_{12}\,\}
\end{aligned}$$

The multiplets are given explicitly in Fig. 5.

4.1.6 Reduced Multiplets R_5^6

The reduced multiplets of type R_5^6 contain 48 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{\,(m_1,m_2,m_3,m_4,0)^\pm\,;\,\pm\tfrac{1}{2}(m_{14}+2m_6)\,\} \\
\chi_a^\pm &= \{\,(m_1,m_2,m_3,m_4,2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{14}\,\} \\
\chi_b^\pm &= \{\,(m_1,m_2,m_3,m_4,2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{14}\,\} \\
\chi_c^\pm &= \{\,(m_1,m_2,m_{34},0,m_4+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{13}\,\} \\
\chi_{c'}^\pm &= \{\,(m_1,m_2,m_3,m_4+2m_6,0)^\pm\,;\,\pm\tfrac{1}{2}m_{14}\,\} \\
\chi_d^\pm &= \{\,(m_1,m_{23},m_4,0,m_{34}+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{d'}^\pm &= \{\,(m_1,m_2,m_{34},2m_6,m_4)^\pm\,;\,\pm\tfrac{1}{2}m_{13}\,\} \\
\chi_e^\pm &= \{\,(m_{12},m_3,m_4,0,m_{24}+2m_6)^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{e'}^\pm &= \{\,(m_1,m_{23},m_4,2m_6,m_{34})^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{e''}^\pm &= \{\,(m_1,m_2,m_{34},2m_6,m_4)^\pm\,;\,\pm\tfrac{1}{2}m_{13}\,\} \\
\chi_f^\pm &= \{\,(m_2,m_3,m_4,0,m_{14}+2m_6)^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_{f'}^\pm &= \{\,(m_{12},m_3,m_4,2m_6,m_{24})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{f''}^\pm &= \{\,(m_1,m_{23},m_4,2m_6,m_{34})^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{f'''}^\pm &= \{\,(m_1,m_2,m_{34}+2m_6,0,m_4)^\pm\,;\,\pm\tfrac{1}{2}m_{13}\,\} \\
\chi_g^\pm &= \{\,(m_2,m_3,m_4,2m_6,m_{14})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_{g'}^\pm &= \{\,(m_{12},m_3,m_4,2m_6,m_{24})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{g''}^\pm &= \{\,(m_1,m_{23},m_4+2m_6,0,m_{34})^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_h^\pm &= \{\,(m_2,m_3,m_4,2m_6,m_{14})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_{h'}^\pm &= \{\,(m_{12},m_3,m_4+2m_6,0,m_{24})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{h''}^\pm &= \{\,(m_2,m_3,m_4+2m_6,0,m_{14})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_j^\pm &= \{\,(m_2,m_{34},0,m_4+2m_6,m_{13})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_{j'}^\pm &= \{\,(m_{12},m_{34},0,m_4+2m_6,m_{23})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{j''}^\pm &= \{\,(m_1,m_{24},0,m_4+2m_6,m_3)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_k^\pm &= \{\,(m_2,m_{34},2m_6,m_4,m_{13})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_{k'}^\pm &= \{\,(m_{12},m_{34},2m_6,m_4,m_{23})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{k''}^\pm &= \{\,(m_1,m_{24},2m_6,m_4,m_3)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\} \\
\chi_m^\pm &= \{\,(m_2,m_{34}+2m_6,0,m_4,m_{13})^\pm\,;\,\mp\tfrac{1}{2}m_1\,\} \\
\chi_{m'}^\pm &= \{\,(m_{12},m_{34}+2m_6,0,m_4,m_{23})^\pm\,;\,\pm\tfrac{1}{2}m_1\,\} \\
\chi_{m''}^\pm &= \{\,(m_1,m_{24}+2m_6,0,m_4,m_3)^\pm\,;\,\pm\tfrac{1}{2}m_{12}\,\}
\end{aligned} \tag{25}$$

The multiplets are given explicitly in Fig. 6.

4.1.7 Reduced Multiplets R_6^6

The reduced multiplets of type R_6^6 contain 32 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{(m_1,m_2,m_3,m_4,m_5)^\pm;\pm\tfrac{1}{2}m_{15}\} \\
\chi_{c'}^\pm &= \{(m_1,m_2,m_3,m_{45},m_5)^\pm;\pm\tfrac{1}{2}m_{14}\} \\
\chi_{d'}^\pm &= \{(m_1,m_2,m_{34},m_5,m_{45})^\pm;\pm\tfrac{1}{2}m_{13}\} \\
\chi_{e'}^\pm &= \{(m_1,m_{23},m_4,m_5,m_{35})^\pm;\pm\tfrac{1}{2}m_{12}\} \\
\chi_{f'}^\pm &= \{(m_{12},m_3,m_4,m_5,m_{25})^\pm;\pm\tfrac{1}{2}m_1\} \\
\chi_{f'''}^\pm &= \{(m_1,m_2,m_{35},m_5,m_4)^\pm;\pm\tfrac{1}{2}m_{13}\} \\
\chi_g^\pm &= \{(m_2,m_3,m_4,m_5,m_{15})^\pm;\mp\tfrac{1}{2}m_1\} \\
\chi_{g''}^\pm &= \{(m_1,m_{23},m_{45},m_5,m_{34})^\pm;\pm\tfrac{1}{2}m_{12}\} \\
\chi_{h'}^\pm &= \{(m_{12},m_3,m_{45},m_5,m_{24})^\pm;\pm\tfrac{1}{2}m_1\} \\
\chi_{h''}^\pm &= \{(m_2,m_3,m_{45},m_5,m_{14})^\pm;\mp\tfrac{1}{2}m_1\} \\
\chi_k^\pm &= \{(m_2,m_{34},m_5,m_{45},m_{13})^\pm;\mp\tfrac{1}{2}m_1\} \\
\chi_{k'}^\pm &= \{(m_{12},m_{34},m_5,m_{45},m_{23})^\pm;\pm\tfrac{1}{2}m_1\} \\
\chi_{k''}^\pm &= \{(m_1,m_{24},m_5,m_{45},m_3)^\pm;\pm\tfrac{1}{2}m_{12}\} \\
\chi_\ell^\pm &= \{(m_2,m_{35},m_5,m_4,m_{13})^\pm;\mp\tfrac{1}{2}m_1\} \\
\chi_{\ell'}^\pm &= \{(m_{12},m_{35},m_5,m_4,m_{23})^\pm;\pm\tfrac{1}{2}m_1\} \\
\chi_{\ell''}^\pm &= \{(m_1,m_{25},m_5,m_4,m_3)^\pm;\pm\tfrac{1}{2}m_{12}\}
\end{aligned} \tag{26}$$

The multiplets are given explicitly in Fig. 7.

Here the ER χ_0^+ contains the limits of the (anti)holomorphic discrete series representations. This is guaranteed by the fact that for this ER all Harish-Chandra parameters for non-compact roots are non-positive, i.e., $m_\alpha \leq 0$, for α from (16). (Actually, we have: $m_{11}=0$, $m_\alpha<0$ for the rest of the non-compact α.) Its conformal weight has the restriction $d=\frac{7}{2}+\frac{1}{2}(m_1+\cdots+m_5)\geq 6$.

4.2 The Cases $sp(n,\mathbb{R})$ for $n \leq 5$

We start with $sp(5,\mathbb{R})$. The main multiplets R^5 contain $32(=2^5)$ ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= \{(m_1,m_2,m_3,m_4)^\pm;\pm\tfrac{1}{2}(m_{14}+2m_5)\} \\
\chi_a^\pm &= \{(m_1,m_2,m_3,m_4+2m_5)^\pm;\pm\tfrac{1}{2}m_{14}\} \\
\chi_b^\pm &= \{(m_1,m_2,m_{34},m_4+2m_5)^\pm;\pm\tfrac{1}{2}m_{13}\} \\
\chi_c^\pm &= \{(m_1,m_{23},m_4,m_{34}+2m_5)^\pm;\pm\tfrac{1}{2}m_{12}\} \\
\chi_{c'}^\pm &= \{(m_1,m_2,m_{34}+2m_5,m_4)^\pm;\pm\tfrac{1}{2}m_{13}\} \\
\chi_d^\pm &= \{(m_{12},m_3,m_4,m_{24}+2m_5)^\pm;\pm\tfrac{1}{2}m_1\}
\end{aligned} \tag{27}$$

$$
\begin{aligned}
\chi_{d'}^{\pm} &= \{\,(m_1, m_{23}, m_4 + 2m_5, m_{34})^{\pm}\,;\, \pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{e}^{\pm} &= \{\,(m_2, m_3, m_4, m_{14} + 2m_5)^{\pm}\,;\, \mp\tfrac{1}{2}m_1\,\} \\
\chi_{e'}^{\pm} &= \{\,(m_{12}, m_3, m_4 + 2m_5, m_{24})^{\pm}\,;\, \pm\tfrac{1}{2}m_1\,\} \\
\chi_{e''}^{\pm} &= \{\,(m_1, m_{24}, m_4 + 2m_5, m_3)^{\pm}\,;\, \pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{f}^{\pm} &= \{\,(m_2, m_3, m_4 + 2m_5, m_{14})^{\pm}\,;\, \mp\tfrac{1}{2}m_1\,\} \\
\chi_{f'}^{\pm} &= \{\,(m_{12}, m_{34}, m_4 + 2m_5, m_{23})^{\pm}\,;\, \pm\tfrac{1}{2}m_1\,\} \\
\chi_{f''}^{\pm} &= \{\,(m_1, m_{24} + 2m_5, m_4, m_3)^{\pm}\,;\, \pm\tfrac{1}{2}m_{12}\,\} \\
\chi_{g}^{\pm} &= \{\,(m_2, m_{34}, m_4 + 2m_5, m_{13})^{\pm}\,;\, \mp\tfrac{1}{2}m_1\,\} \\
\chi_{g'}^{\pm} &= \{\,(m_{12}, m_{34} + 2m_5, m_4, m_{23})^{\pm}\,;\, \pm\tfrac{1}{2}m_1\,\} \\
\chi_{h}^{\pm} &= \{\,(m_2, m_{34} + 2m_5, m_4, m_{13})^{\pm}\,;\, \mp\tfrac{1}{2}m_1\,\}
\end{aligned}
$$

Recalling that the $Sp(6,\mathbb{R})$ reduced multiplets of type R_6^6 also have 32 members we check whether they may be coinciding. Indeed, that turns out to be the case and this is obvious from the corresponding figures, Figs. 7 and 8 (though our graphical representations are a little distorted!). To make it explicit via the signatures we do the following manipulations of (26): in each signature we just drop the entry m_5 (there is exactly one such entry in each signature). Then we replace each entry of the kind: m_{k5}, $(k = 1,2,3,4)$, by $m_{k4} + 2m_5$ (identifying $m_{44} \equiv m_4$). Thus (26) becomes exactly (27). (Of course, this does not mean that the contents is the same. For instance, the ER χ_0^+ from (27) contains the (anti)holomorphic discrete series representations of $sp(5,\mathbb{R})$, while the ER χ_0^+ from (26) contains the limits of the (anti)holomorphic discrete series representations of $sp(6,\mathbb{R})$.)

Thus, it is clear how to obtain from the case $sp(6,\mathbb{R})$ all the cases $sp(n,\mathbb{R})$ for $n \leq 5$. We shall not do it here due to the lack of space.

5 Outlook

In the present paper we continued the programme outlined in [2] on the example of the non-compact group $Sp(n,\mathbb{R})$. Similar explicit descriptions are planned for the other non-compact groups, in particular, those with highest/lowest weight representations. From the latter we have considered so far the cases of $E_{7(-25)}$ [3],[3] $E_{6(-14)}$ [29], $SU(n,n)$ ($n \leq 4$) [30]. We plan also to extend these considerations to the supersymmetric cases and also to the quantum group setting. Such considerations are expected to be very useful for applications to string theory and integrable models, cf., e.g., [31]. In our further plans it shall be very useful that (as in [2]) we follow a procedure in representation theory in which intertwining differential operators appear canonically [7] and which procedure has been generalized to the supersymmetry setting [32–34] and to quantum groups [35, 36].

[3] For a different use of $E_{7(-25)}$, see, e.g., [28].

6 Figures

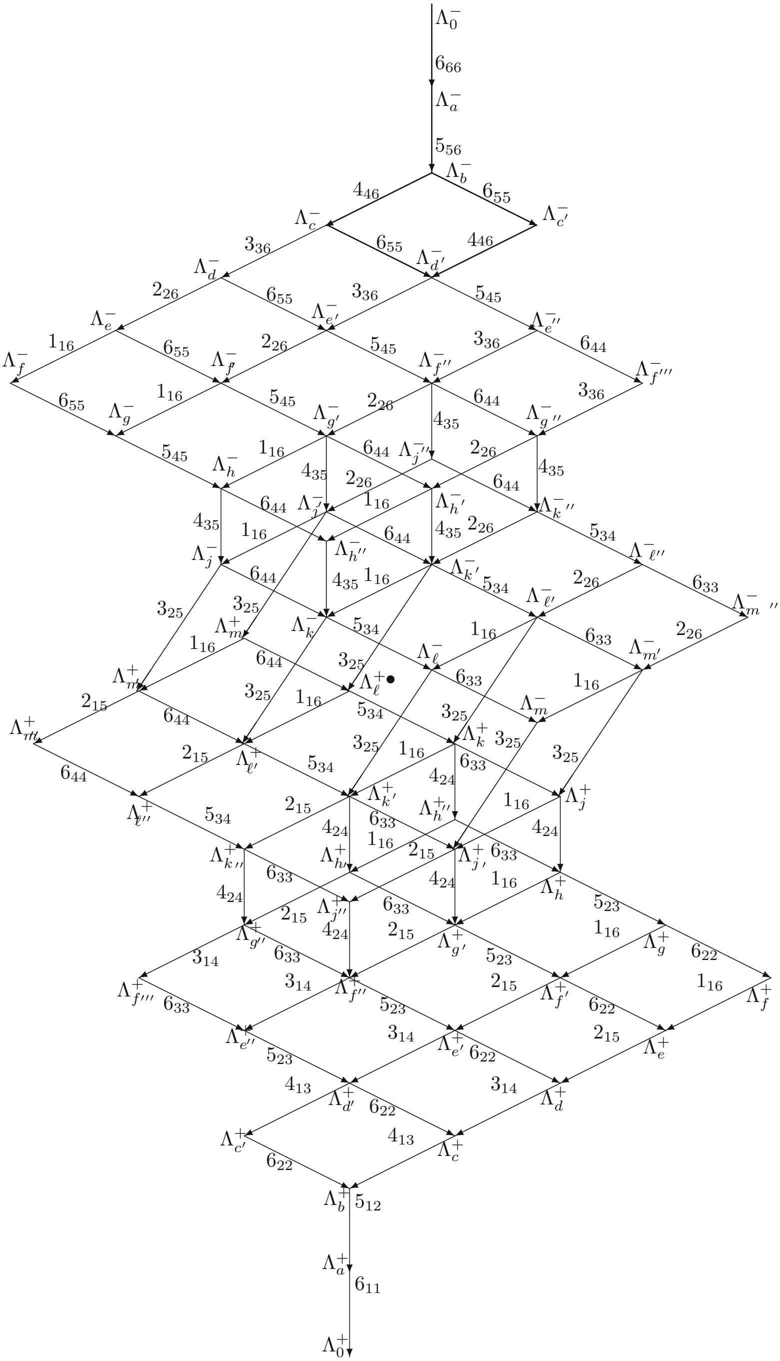


Fig. 1 Main multiplets for $Sp(6,\mathbb{R})$

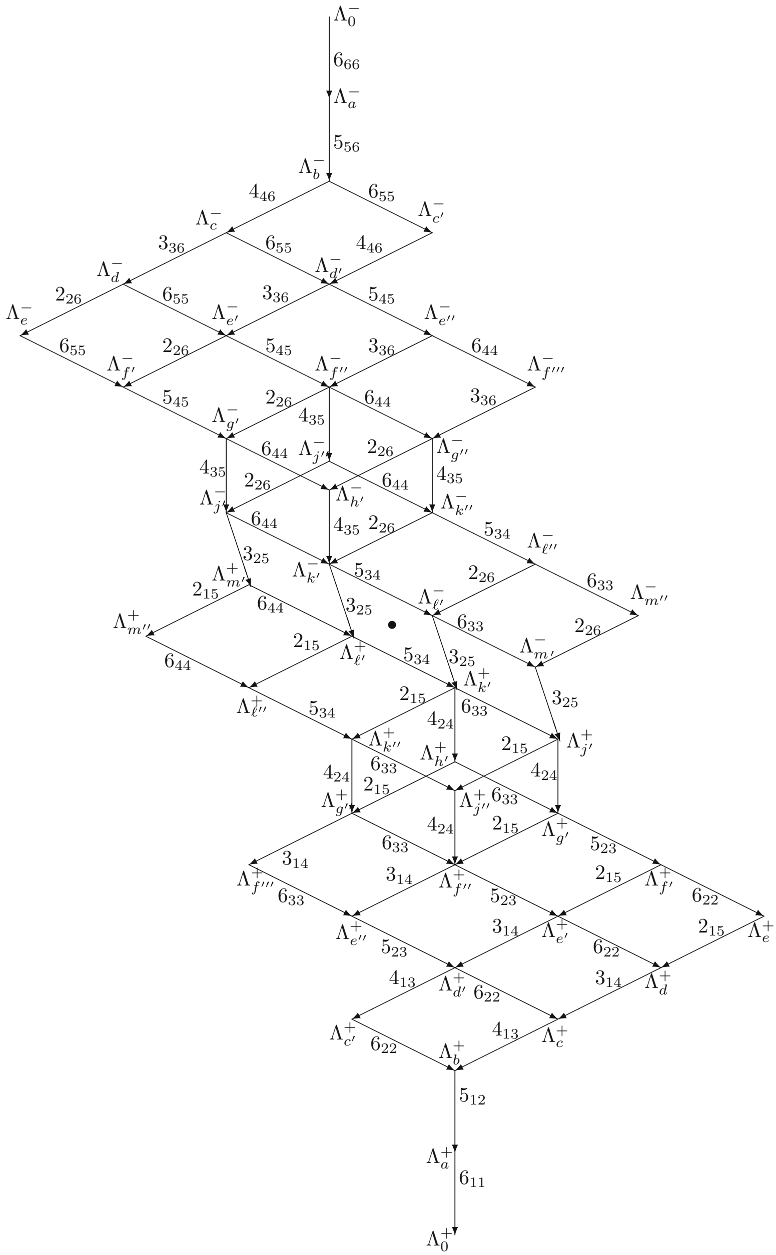


Fig. 2 Reduced multiplets for R_1^6 for $Sp(6, \mathbb{R})$

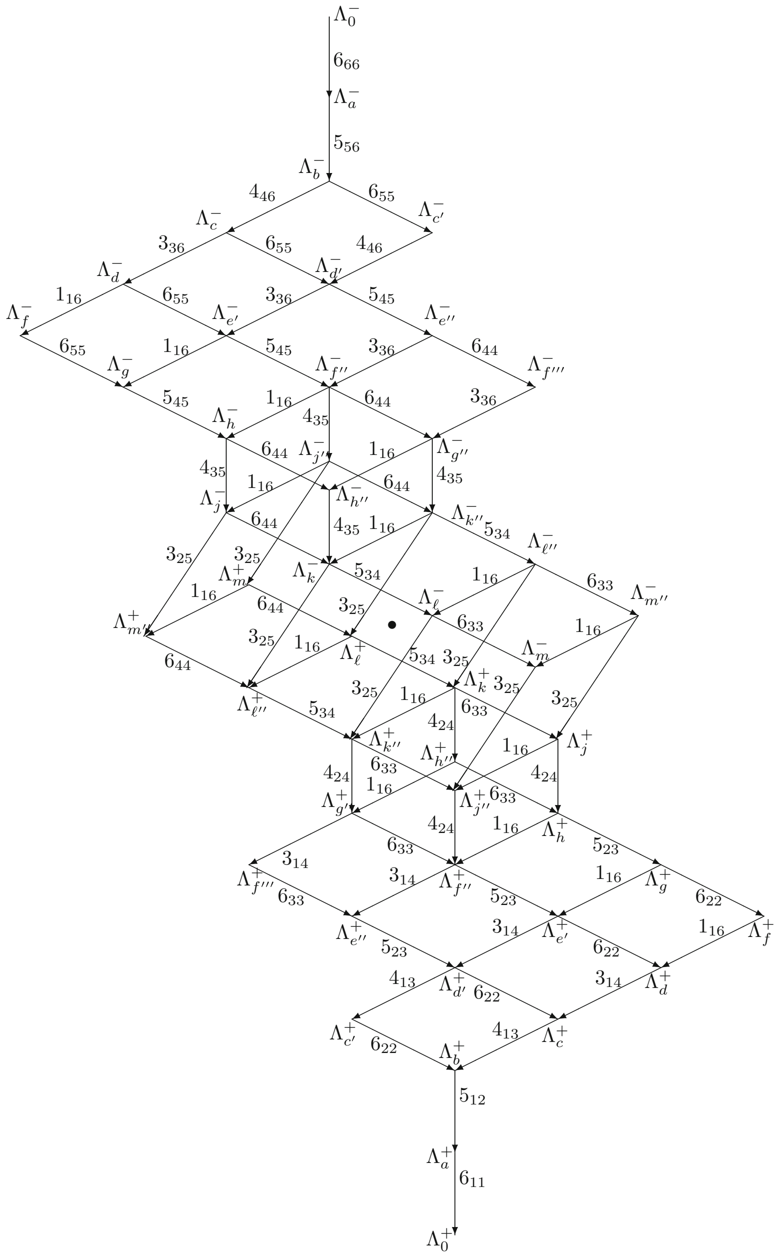


Fig. 3 Reduced multiplets for R_2^6 for $Sp(6,\mathbb{R})$

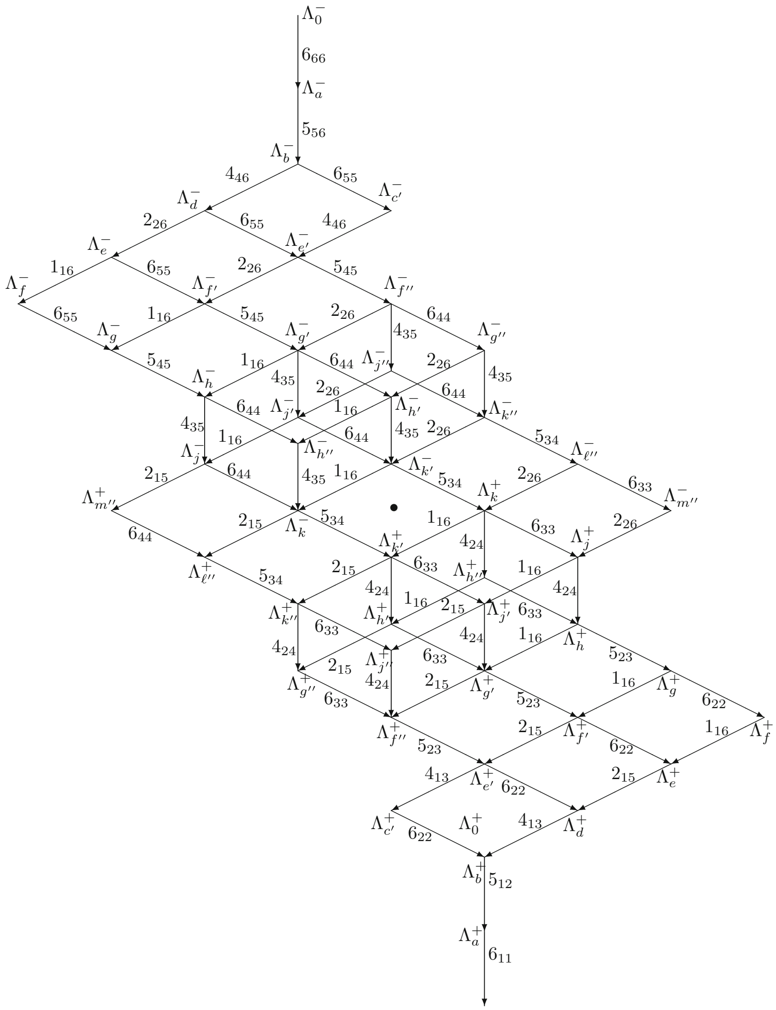


Fig. 4 Reduced multiplets for R_3^6 for $Sp(6,\mathbb{R})$

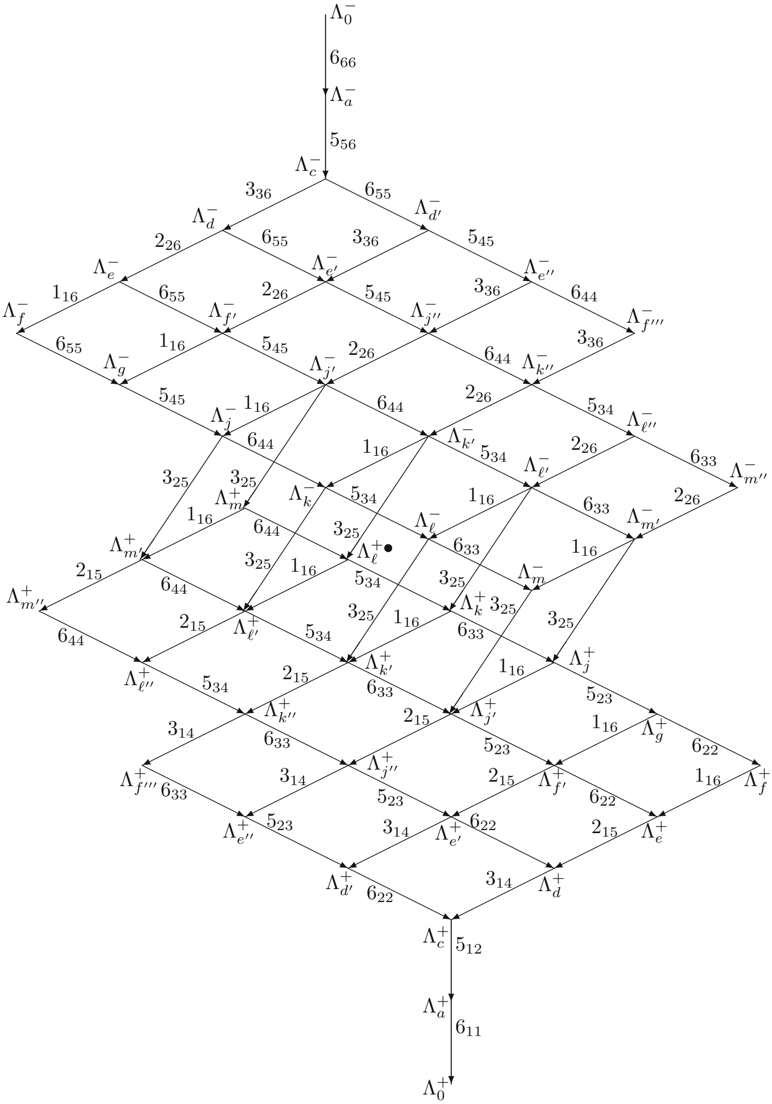


Fig. 5 Reduced multiplets for R_4^6 for $Sp(6,\mathbb{R})$

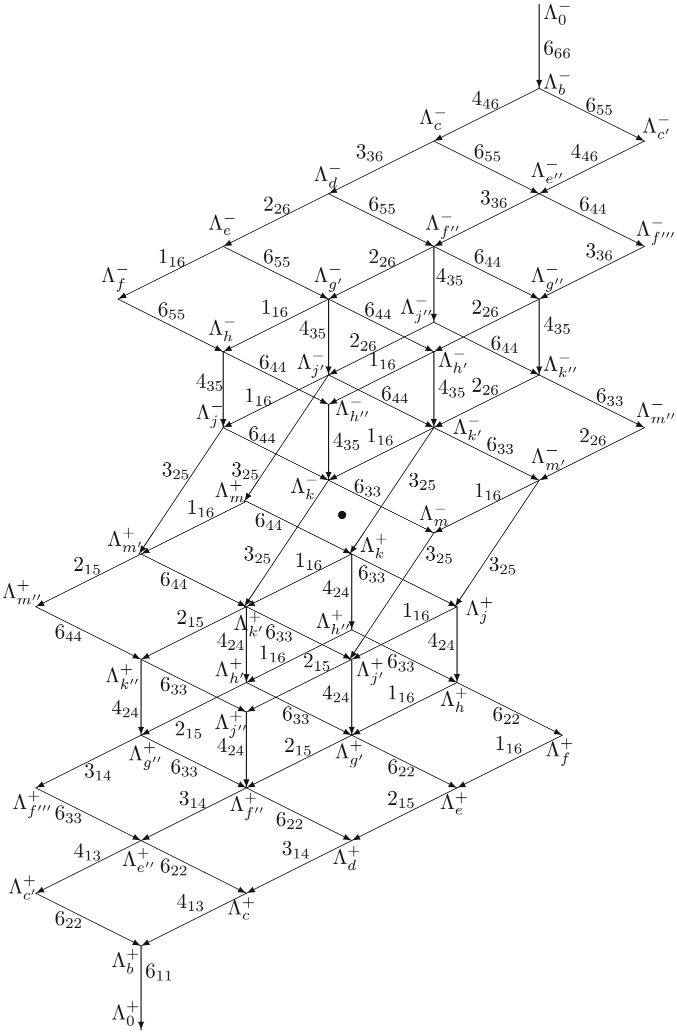


Fig. 6 Reduced multiplets for R_5^6 for $Sp(6,\mathbb{R})$

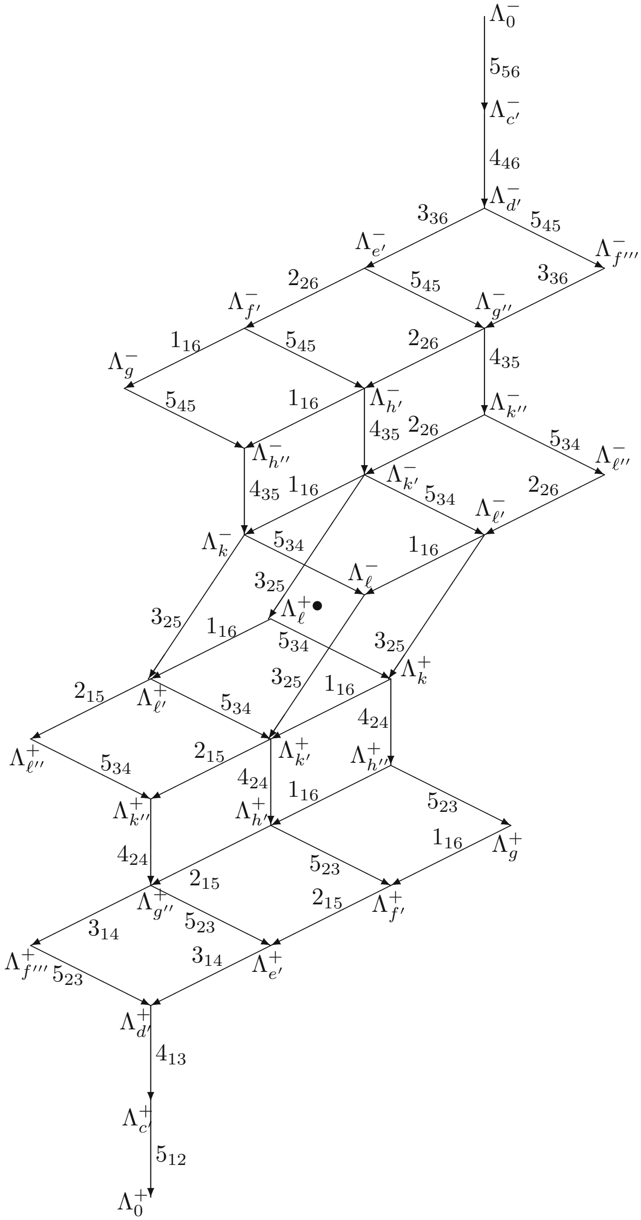


Fig. 7 Reduced multiplets for R_6^6 for $Sp(6,\mathbb{R})$

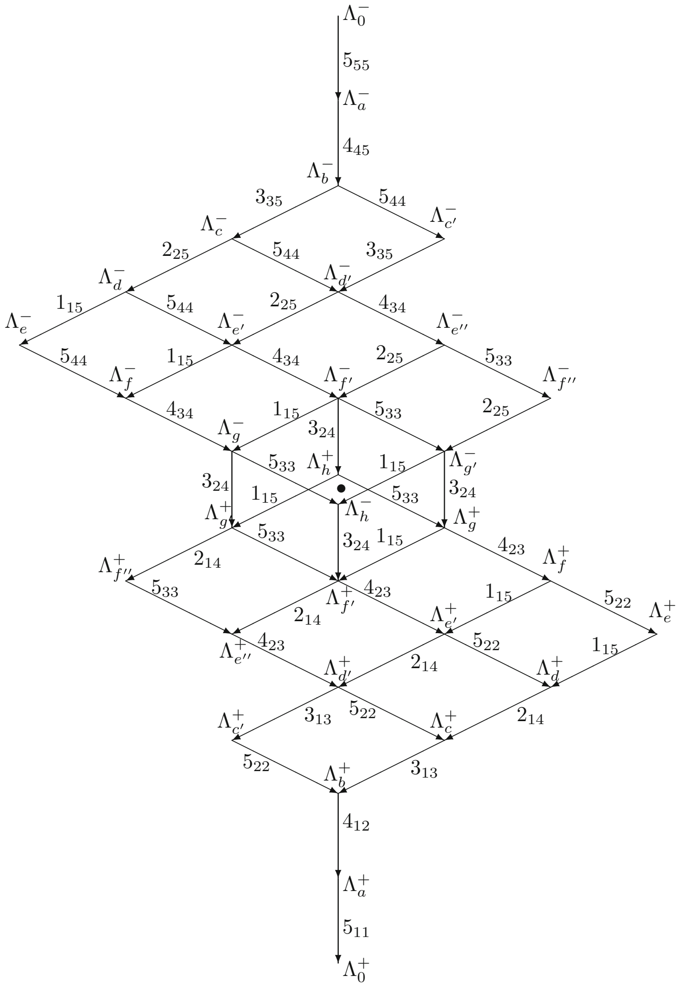


Fig. 8 Main multiplets for R^5 for $Sp(5,\mathbb{R})$

Acknowledgements This work was supported in part by the Bulgarian National Science Fund, grant DO 02-257. The author thanks the Theory Division of CERN for hospitality during the course of this work.

References

1. Terning, J.: Modern supersymmetry: Dynamics and duality. In: International Series of Monographs on Physics, vol. 132, Oxford University Press, Oxford (2005)
2. Dobrev, V.K.: Rev. Math. Phys. **20**, 407–449 (2008) [hep-th/0702152]
3. Dobrev, V.K.: J. Phys. A Math. Theor. **42**, 285203 (2009) (arXiv:0812.2690 [hep-th])
4. Faraut, J., Korányi, A.: Analysis on symmetric cones. In: Oxford Mathematical Monographs. Clarendon Press, Oxford (1994)
5. Gunaydin, M.: Mod. Phys. Lett. **A8**, 1407–1416 (1993)
6. Mack, G., de Riese, M.: J. Math. Phys. **48**, 052304 (2007) [hep-th/0410277v2]
7. Dobrev, V.K.: Rept. Math. Phys. **25**, 159–181 (1988) [first as ICTP Trieste preprint IC/86/393 (1986)]
8. Langlands, R.P.: On the classification of irreducible representations of real algebraic groups. In: Mathematical Surveys and Monographs, vol. 31. American Mathematical Society (1988) [first as IAS Princeton preprint (1973)]
9. Zhelobenko, D.P.: Harmonic Analysis on Semisimple Complex Lie Groups. Nauka, Moscow (1974) (in Russian)
10. Knapp, A.W., Zuckerman, G.J.: Classification theorems for representations of semisimple groups. In: Lecture Notes in Mathematics, vol. 587. Springer, Berlin (1977); Classification of irreducible tempered representations of semisimple groups. Ann. Math. **116**, 389–501 (1982)
11. Harish-Chandra, Discrete series for semisimple Lie groups: II. Ann. Math. **116**, 1–111 (1966)
12. Enright, T., Howe, R., Wallach, W.: A classification of unitary highest weight modules. In: Trombi, P. (ed.) Representations of Reductive Groups, pp. 97–143. Birkhäuser, Boston (1983)
13. Dobrev, V.K.: J. Phys. A Math. Theor. **41**, 425206 (2008) [arXiv:0712.4375]
14. Grosse, H., Prešnajder, P., Wang, Zh.: J. Math. Phys. **53**, 013508 (2012)
15. Dobrev, V.K., Mack, G., Petkova, V.B., Petrova, S.G., Todorov, I.T.: Harmonic analysis on the n-dimensional Lorentz group and its applications to conformal quantum field theory. In: Lecture Notes in Physics, vol. 63. Springer, Berlin (1977)
16. Dobrev, V.K., Mack, G., Petkova, V.B., Petrova, S.G., Todorov, I.T.: Rep. Math. Phys. **9**, 219–246 (1976)
17. Dobrev, V.K., Petkova, V.B.: Rep. Math. Phys. **13**, 233–277 (1978)
18. Knapp, A.W.: Representation Theory of Semisimple Groups (An Overview Based on Examples). Princeton University Press, Princeton (1986)
19. Dobrev, V.K.: Lett. Math. Phys. **9**, 205–211 (1985); J. Math. Phys. **26**, 235–251 (1985)
20. Bernstein, I.N., Gel'fand, I.M., Gel'fand, S.I.: Structure of representations generated by highest weight vectors. Funkts. Anal. Prilozh. **5**(1), 1–9 (1971) [English translation: Funct. Anal. Appl. **5**, 1–8 (1971)]
21. Dixmier, J.: Enveloping Algebras. North Holland, New York (1977)
22. Dobrev, V.K.: Lett. Math. Phys. **22**, 251–266 (1991)
23. Chair, N., Dobrev, V.K., Kanno, H.: Phys. Lett. **283B**, 194–202 (1992)
24. Satake, I.: Ann. Math. **71**, 77–110 (1960)
25. Harish-Chandra, Representations of semisimple Lie groups: IV, V. Am. J. Math. **77**, 743–777 (1955); **78**, 1–41 (1956)
26. Bourbaki, N.: Groupes at algèbres de Lie, Chapitres 4, 5 et 6. Hermann, Paris (1968)
27. Knapp, A.W., Stein, E.M.: Intertwining operators for semisimple groups. Ann. Math. **93**, 489–578 (1971); II: Inv. Math. **60**, 9–84 (1980)

28. Cacciatori, S.L., Cerchiai, B.L., Marrani, A.: Magic Coset Decompositions [arXiv:1201.6314], CERN-PH-TH/2012-020
29. Dobrev, V.K.: Invited lectures at 5th meeting on modern mathematical physics, Belgrade, 6–17 July 2008. In: Dragovich, B., Rakic, Z. (eds.) Proceedings, pp. 95–124. Institute of Physics, Belgrade (2009) (arXiv:0812.2655 [math-ph])
30. Dobrev, V.K.: Invariant Differential Operators for Non-Compact Lie Groups: the Main SU(n,n) Cases, Plenary talk at SYMPHYS XV (Dubna, 12–16.7.2011). In: Pogosyan, G., et al. (eds.) Proceedings, Physics of Atomic Nuclei.
31. Witten, E.: Conformal Field Theory in Four and Six Dimensions [arXiv:0712.0157]
32. Dobrev, V.K., Petkova, V.B.: Phys. Lett. **B162**, 127–132 (1985); Lett. Math. Phys. **9**, 287–298 (1985); Fortsch. Phys. **35**, 537–572 (1987)
33. Dobrev, V.K.: Phys. Lett. **B186**, 43–51 (1987); J. Phys. **A35**, 7079–7100 (2002) [hep-th/0201076]; Phys. Part. Nucl. **38**, 564–609 (2007) [hep-th/0406154]
34. Dobrev, V.K., Ganchev, A.Ch.: Mod. Phys. Lett. **A3**, 127–137 (1988)
35. Dobrev, V.K.: J. Math. Phys. **33**, 3419–3430 (1992); J. Phys. **A26**, 1317–1334 (1993) [first as Göttingen University preprint (July 1991)]; J. Phys. **A27**, 4841–4857 (1994) [Erratum-ibid. **A27**, 6633–6634 (1994) (hep-th/9405150)]; Phys. Lett. **B341**, 133–138 (1994) [Erratum-ibid. **B346**, 427 (1995)]
36. Dobrev, V.K., Moylan, P.J.: Phys. Lett. **B315**, 292–298 (1993)

Generalization of the Gell–Mann Decontraction Formula for $sl(n,\mathbb{R})$ and Its Applications in Affine Gravity

Igor Salom and Djordje Šijački

Abstract The Gell–Mann Lie algebra decontraction formula was proposed as an inverse to the Inonu–Wigner contraction formula. We considered recently this formula in the content of the special linear algebras $sl(n)$, of an arbitrary dimension. In the case of these algebras, the Gell–Mann formula is not valid generally, and holds only for some particular algebra representations. We constructed a generalization of the formula that is valid for an arbitrary irreducible representation of the $sl(n)$ algebra. The generalization allows us to explicitly write down, in a closed form, all matrix elements of the algebra operators for an arbitrary irreducible representation, irrespectively whether it is tensorial or spinorial, finite or infinite dimensional, with or without multiplicity, unitary or nonunitary. The matrix elements are given in the basis of the $Spin(n)$ subgroup of the corresponding $SL(n,R)$ covering group, thus covering the most often cases of physical interest. The generalized Gell–Mann formula is presented, and as an illustration some details of its applications in the Gauge Affine theory of gravity with spinorial and tensorial matter manifields are given.

1 Introduction

The Inönü–Wigner contraction [7] is a well known transformation of algebras (groups) with numerous applications in various fields of physics. Just to mention a few: contractions from the Poincaré algebra to the Galilean one; from the Heisenberg algebras to the Abelian ones of the same dimensions (a symmetry background of a transition processes from relativistic and quantum mechanics to classical mechanics); contractions in the Kaluza–Klein gauge theories framework;

I. Salom (✉) • D. Šijački
Institute of Physics, Belgrade, Serbia
e-mail: isalom@ipb.ac.rs; sijacki@ipb.ac.rs

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_23,

from (Anti-)de Sitter to the Poincaré algebra; various cases involving the Virasoro and Kac–Moody algebras; relation of strong to weak coupling regimes of the corresponding theories; relation of geometrically curved to "less curved" and/or flat spaces....

However, existence of a transformation (i.e. algebra deformation) inverse to the Inönü–Wigner contraction, so called the "Gell–Mann formula" [1,3,5,6], is far less known. The aim of the formula is to express the elements of the starting algebra as explicitly given expressions containing elements of the contracted algebra. In this way, a relation between certain representations of the two algebras is also established. This, in turn, can be very useful since, by a rule, various properties of the contracted algebras are much easier to explore (e.g. construction of representations [8], decompositions of a direct product of representations [5], etc.).

Before we write down the Gell–Mann formula in the general case, some notation is in order. Let $\mathscr{A}$ be a symmetric Lie algebra $\mathscr{A} = M + T$ with a subalgebra $\mathscr{M}$ such that:

$$[\mathscr{M},\mathscr{M}] \subset \mathscr{M}, \quad [\mathscr{M},\mathscr{T}] \subset \mathscr{T}, \quad [\mathscr{T},\mathscr{T}] \subset \mathscr{M}. \tag{1}$$

Further, let $\mathscr{A}'$ be its Inönü–Wigner contraction algebra w.r.t its subalgebra $\mathscr{M}$, i.e. $\mathscr{A}' = \mathscr{M} + \mathscr{U}$, where

$$[\mathscr{M},\mathscr{M}] \subset \mathscr{M}, \quad [\mathscr{M},\mathscr{U}] \subset \mathscr{U}, \quad [\mathscr{U},\mathscr{U}] = \{0\}. \tag{2}$$

The Gell–Mann formula states that the elements $T \in \mathscr{T}$ can be in certain cases expressed in terms of the contracted algebra elements $M \in \mathscr{M}$ and $U \in \mathscr{U}$ by the following rather simple expression:

$$T = i\frac{\alpha}{\sqrt{U \cdot U}}[C_2(\mathscr{M}), U] + \sigma U. \tag{3}$$

Here, $C_2(\mathscr{M})$ and $U \cdot U$ denote the second order Casimir operators of the $\mathscr{M}$ and $\mathscr{A}'$ algebras respectively, while α is a normalization constant and σ is an arbitrary parameter. For a mathematically more strict definition, cf. [3].

Probably the main reason why this formula is not widely known—in spite of its potential versatility—is the lack of its general validity. Namely, there is a number of references dealing with the question when this formula is applicable [1, 5, 6, 14]. Apart form the case of (pseudo) orthogonal algebras where, loosely speaking, the Gell–Mann formula works very well [17], there are only some subclasses of representations when the formula can be applied [5, 6]. To make the things worse, the question of its applicability is not completely resolved.

Recently, we studied the $\overline{SL}(n,\mathbb{R})$ group cases, contracted w.r.t the maximal compact $Spin(n)$ subgroups. By $\overline{SL}(n,\mathbb{R})$ we denote the double cover of $SL(n,\mathbb{R})$. Note that there faithful spinorial representations are always infinite dimensional and physically correspond to fermionic matter. In these cases the Gel–Mann formula does not hold as a general operator expression and its validity depends heavily on the

$sl(n,\mathbb{R})$ algebra representation space. An exhaustive list of the cases for which the Gell–Mann formula for $sl(n,\mathbb{R})$ algebras hold was obtained [14]. In particular, we have shown that the Gell–Mann formula is not valid for any spinorial representation, nor for any representation with nontrivial $Spin(n)$ multiplicity, rendering the Gell–Mann formula here useless for most of physical applications.

There were some attempts to generalize the Gell–Mann formula for the "decontracted" algebra operators of the complex simple Lie algebras g with respect to decomposition $g = k + ik = k_c$ [9, 19], that resulted in a form of relatively complicated polynomial expressions. Recently we have managed to obtain a generalized form of this formula, first in the concrete case of $sl(5,\mathbb{R})$ algebra, and then also in the case of $sl(n,\mathbb{R})$ algebra, for any n.

In this paper we shall consider the obtained generalized expressions and illustrate applicability of the formula in the context of affine theory of gravity. In particular, we analyze the five dimensional affine gravity models.

2 Generalized Formula

The $sl(n,\mathbb{R})$ algebra operators, i.e. the $SL(n,\mathbb{R})$, $\overline{SL}(n,\mathbb{R})$ group generators, can be split into two subsets: M_{ab}, $a,b = 1,2,\ldots,n$ operators of the maximal compact subalgebra $so(n)$ (corresponding to the antisymmetric real $n\times n$ matrices, $M_{ab} = -M_{ba}$), and the, so called, sheer operators T_{ab}, $a,b = 1,2,\ldots,n$ (corresponding to the symmetric traceless real $n\times n$ matrices, $T_{ab} = T_{ba}$). The $sl(n,\mathbb{R})$ commutation relations, in this basis, read:

$$[M_{ab},M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}), \tag{4}$$

$$[M_{ab},T_{cd}] = i(\delta_{ac}T_{bd} + \delta_{ad}T_{cb} - \delta_{bc}T_{ad} - \delta_{bd}T_{ca}), \tag{5}$$

$$[T_{ab},T_{cd}] = i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). \tag{6}$$

The Inönü–Wigner contraction of $sl(n,\mathbb{R})$ with respect to its maximal compact subalgebra $so(n)$ is given by the limiting procedure:

$$U_{ab} \equiv \lim_{\varepsilon\to 0}(\varepsilon T_{ab}), \tag{7}$$

which leads to the following commutation relations:

$$[M_{ab},M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}) \tag{8}$$

$$[M_{ab},U_{cd}] = i(\delta_{ac}U_{bd} + \delta_{ad}U_{cb} - \delta_{bc}U_{ad} - \delta_{bd}U_{ca}) \tag{9}$$

$$[U_{ab},U_{cd}] = 0. \tag{10}$$

Therefore, the Inönü–Wigner contraction of $sl(n,\mathbb{R})$ gives a semidirect sum $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra, where $r_{\frac{n(n+1)}{2}-1}$ is an Abelian subalgebra (ideal) of "translations" in $\frac{n(n+1)}{2}-1$ dimensions.

The generalized Gell–Mann formula for $sl(n,\mathbb{R})$, obtained in [15], reads:

$$T_{ab}^{\sigma_2\ldots\sigma_n} = i\sum_{c=2}^{n}\frac{1}{2}[C_2(so(c)_K), U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)}. \tag{11}$$

Operators T_{ab} live in the space $\mathscr{L}^2(Spin(n))$ of square integrable functions over the $Spin(n)$ manifold and it is known that this space is rich enough to contain all representatives from equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra representations [2]. A natural discrete orthonormal basis in this space is given by properly normalized functions of the $Spin(n)$ representation matrix elements:

$$\left\{\left|\begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix}\right\rangle \equiv \int\sqrt{dim(\{J\})}D^{\{J\}}_{\{k\}\{m\}}(g^{-1})dg\,|g\rangle\right\},$$

$$\left\langle\begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix}\middle|\begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix}\right\rangle = \delta_{\{J'\}\{J\}}\delta_{\{k'\}\{k\}}\delta_{\{m'\}\{m\}}, \tag{12}$$

where dg is an (normalized) invariant Haar measure and $D^{\{J\}}_{\{k\}\{m\}}$ are the $Spin(n)$ irreducible representation matrix elements:

$$D^{\{J\}}_{\{k\}\{m\}}(g) \equiv \left\langle\begin{matrix}\{J\}\\ \{k\}\end{matrix}\middle|R(g)\middle|\begin{matrix}\{J\}\\ \{m\}\end{matrix}\right\rangle. \tag{13}$$

Here, $\{J\}$ stands for a set of the $Spin(n)$ irreducible representation labels, while $\{k\}$ and $\{m\}$ labels enumerate the $dim(D^{\{J\}})$ representation basis vectors.

In the basis (12) sets of labels $\{J\}$ and $\{m\}$ determine transformation properties of a basis vector under the $Spin(n)$ subgroup: $\{J\}$ label irreducible representation of $Spin(n)$, while numbers $\{m\}$ label particular vector within that representation. The set of parameters $\{k\}$ serve to enumerate $Spin(n)$ multiplicity of representation $\{J\}$ within the given representation of $\overline{SL}(n,\mathbb{R})$. These parameters $\{k\}$ are mathematically related to the left action of $Spin(n)$ subgroup in representation space $\mathscr{L}^2(Spin(n))$.

Operators $U_{ab}^{(cc)}$ appearing in (11) are concrete (normalized) representations (in $\mathscr{L}^2(Spin(n))$ space) of the Inönü–Wigner contractions of shear generators T_{ab}. In basis (12) these operators act in the following way:

$$\left\langle\begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix}\middle|U_{ab}^{(cd)}\middle|\begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix}\right\rangle = \sqrt{\frac{dim(\{J\})}{dim(\{J'\})}}C^{\{J\}\square\square\{J'\}}_{\{k\}(cd)\{k'\}}C^{\{J\}\square\square\{J'\}}_{\{m\}(ab)\{m'\}}, \tag{14}$$

where $\square\square$ denotes $Spin(n)$ representation that corresponds to second order symmetric tensors (shear generators, as well as their Inönü–Wigner contractions, transform in this way w.r.t. $Spin(n)$ subgroup) and C stands for Clebsch–Gordan coefficients of $Spin(n)$.

In (11) we also used notation $C_2(so(c)_K) \equiv \frac{1}{2}\sum_{a,b=1}^{c}(K_{ab})^2$, where K_{ab} are generators of $Spin(n)$ group left action in basis (12). These operators behave exactly as the rotation generators M_{ab}, but, instead of acting on right-hand $\{m\}$ indices, they act on the lower left-hand side indices $\{k\}$ that label multiplicity:

$$\left\langle \begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix} \middle| K_{ab} \middle| \begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix} \right\rangle = \delta_{\{J'\}\{J\}}\delta_{\{m'\}\{m\}}\sqrt{C_2(\{J\})}\, C^{\{J\}\ \square\ \{J'\}}_{\{k\}(ab)\{k'\}}. \tag{15}$$

Finally, the set of $n-1$ indices $\sigma_2,\sigma_3,\ldots\sigma_n$ in (11) label the particular representation of the $\overline{SL}(n,\mathbb{R})$. The formula (11) covers all cases: infinite and finite dimensional representations, spinorial and tensorial, with and without multiplicity, unitary and non unitary.

We note that the term $c = n$ in (11) is, essentially, the original Gell–Mann formula, since $C_2(so(n)_K) = C_2(so(n)_M)$. The rest of the terms can be seen as necessary corrections securing the formula validity in the entire representation space. The additional terms vanish for some particular representations thus yielding the original formula.

An immediate mathematical benefit of the generalized formula is the expression for matrix elements of shear generators in basis (12) [15]:

$$\begin{aligned}\left\langle \begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix} \middle| T_{ab} \middle| \begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix} \right\rangle &= \frac{i}{2}\sqrt{\frac{dim(\{J\})}{dim(\{J'\})}}\, C^{\{J\}\ \square\square\ \{J'\}}_{\{m\}\ ab\ \{m'\}} \\ \times \sum_{c=2}^{n}\sqrt{\frac{c-1}{c}}\Big(C_2(so(c)_{\{k'\}}) &- C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c\Big)\, C^{\{J\}(\square\square)^{n-c+1}\{J'\}}_{\{k\}\ \ (0)^{c-2}\ \ \{k'\}}.\end{aligned} \tag{16}$$

In order to demonstrate application of this result in the context of five dimensional affine gravity models, we introduce a concrete $n = 5$ adapted notation (for all $n = 5$ notation we adhere to that of our paper [13]). As a basis for $Spin(5)$ representations we pick vectors:

$$\left\{ \left| \begin{matrix}\bar{J}_1 & \bar{J}_2\\ J_1 & J_2\\ m_1 & m_2\end{matrix} \right\rangle, \bar{J}_i = 0, \frac{1}{2}, \ldots; \bar{J}_1 \geq \bar{J}_2; m_i = -J_i, \ldots J_i \right\}. \tag{17}$$

with respect to decomposition $so(5) \supset so(4) = so(3) \oplus so(3)$. Basis of $\overline{SL}(5,\mathbb{R})$ representation space, corresponding to (12) is then given by vectors:

$$\left\{ \left| \begin{matrix}\bar{J}_1 & \bar{J}_2 & & \\ K_1 & K_2 & J_1 & J_2\\ k_1 & k_2 & m_1 & m_2\end{matrix} \right\rangle \right\}. \tag{18}$$

The reduced matrix elements of the $sl(5,\mathbb{R})$ shear (noncompact) operators, derived from an alternative form of Gell–Mann formula that we have given in the paper [13], read:

$$
\left\langle \begin{matrix} \bar J'_1\, \bar J'_2 \\ K'_1 K'_2 \\ k'_1\, k'_2 \end{matrix} \right\| \left| T \right| \left\| \begin{matrix} \bar J_1\, \bar J_2 \\ K_1 K_2 \\ k_1\, k_2 \end{matrix} \right\rangle = \sqrt{\frac{dim(\bar J_1,\bar J_2)}{dim(\bar J'_1,\bar J'_2)}}
$$
$$
\times \Bigg(\Big(\sigma_1 + i\sqrt{\tfrac{4}{5}}(\bar J'_1(\bar J'_1+2)+\bar J'_2(\bar J'_2+1)-\bar J_1(\bar J_1+2)-\bar J_2(\bar J_2+1)) \Big) C^{\bar J_1 \bar J_2\; \overline{1}\overline{1}\; \bar J'_1 \bar J'_2}_{\substack{K_1K_2\; 00\; K'_1K'_2 \\ k_1\,k_2\; 00\; k'_1\,k'_2}}
$$
$$
+\, i\big(\sigma_2+K'_1(K'_1+1)+K'_2(K'_2+1)-K_1(K_1+1)-K_2(K_2+1)\big) C^{\bar J_1 \bar J_2\; \overline{1}\overline{1}\; \bar J'_1 \bar J'_2}_{\substack{K_1K_2\; 11\; K'_1K'_2 \\ k_1\,k_2\; 00\; k'_1\,k'_2}} \tag{19}
$$
$$
-\, i(\delta_1+k_1-k_2) C^{\bar J_1 \bar J_2\; \overline{1}\overline{1}\; \bar J'_1 \bar J'_2}_{\substack{K_1K_2\; 11\; K'_1K'_2 \\ k_1\,k_2\; 1-1\; k'_1\,k'_2}} - i(\delta_1-k_1+k_2) C^{\bar J_1 \bar J_2\; \overline{1}\;\overline{1}\; \bar J'_1 \bar J'_2}_{\substack{K_1K_2\; 1\;1\; K'_1K'_2 \\ k_1\,k_2\; -11\; k'_1\,k'_2}}
$$
$$
+\, i(\delta_2+k_1+k_2) C^{\bar J_1 \bar J_2\; \overline{1}\overline{1}\; \bar J'_1 \bar J'_2}_{\substack{K_1K_2\; 11\; K'_1K'_2 \\ k_1\,k_2\; 11\; k'_1\,k'_2}} + i(\delta_2-k_1-k_2) C^{\bar J_1 \bar J_2\; \overline{1}\;\overline{1}\; \bar J'_1 \bar J'_2}_{\substack{K_1K_2\; 1\;1\; K'_1K'_2 \\ k_1\,k_2\; -1-1\; k'_1\,k'_2}} \Bigg),
$$

where $dim(\bar J_1,\bar J_2) = (2\bar J_1 - 2\bar J_2 + 1)(2\bar J_1 + 2\bar J_2 + 3)(2\bar J_1 + 2)(2\bar J_2 + 1)/6$ is the dimension of the $so(5)$ irreducible representation characterized by $(\bar J_1,\bar J_2)$. In this notation, $\overline{SL}(5,\mathbb{R})$ irreducible representations are labelled by parameters $\sigma_1,\sigma_2,\delta_1$ and δ_2, that appear in the formula (19).

3 Gauge Affine Action

The space-time symmetry of the affine models of gravity (prior to any symmetry breaking) is given by the General Affine Group $GA(n,\mathbb{R}) = T^n \wedge GL(n,\mathbb{R})$ (or, sometimes, by the Special Affine Group $SA(n,\mathbb{R}) = T^n \wedge SL(n,\mathbb{R})$). In the quantum case, the General Affine Group is replaced by its double cover counterpart $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$, which contains double cover of $\overline{GL}(n,\mathbb{R})$ as a subgroup. This subgroup here plays the role that Lorentz group has in the Poincaré symmetry case. Thus it is clear that knowledge of $\overline{GL}(n,\mathbb{R})$ representations is a must-know for any serious analysis of affine gravity models. On the other hand, the essential nontrivial representation determining part of the $\overline{GL}(n,\mathbb{R}) = R_+ \otimes \overline{SL}(n,\mathbb{R})$ group is its $\overline{SL}(n,\mathbb{R})$ subgroup (R_+ is subgroup of dilatations). We will make use of the $\overline{SL}(n,\mathbb{R})$ matrix elements expression in order to obtain coefficients for some of the gauge field–matter interaction vertices.

A standard way to introduce interactions into affine gravity models is by localization of the global affine symmetry $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$. Thus, quite generally, affine Lagrangian consists of a gravitational part (i.e. kinetic terms for gauge potentials) and Lagrangian of the matter fields: $L = L_g + L_m$. Gravitational part L_g is a function of gravitational gauge potentials and their derivatives, and also of the dilaton field φ (that ensures action invariance under local dilatations). In the case of the standard Metric Affine [4], i.e. Gauge Affine Gravity [10], the gravitational potentials are tetrads $e^a{}_\mu$, metrics g_{ab} and affine connection $\Gamma^a_{b\mu}$, so that we can write: $L_g = L_g(e,\partial e,g,\partial g,\Gamma,\partial\Gamma,\varphi)$. More precisely, due to action invariance under local affine transformations, gravitational part of Lagrangian must be a function of the form $L_g = L_g(e,g,T,R,N,\varphi)$, where $T^a_{\mu\nu} = \partial_\mu e^a{}_\nu + \Gamma^a_{b\mu} e^b{}_\nu - (\mu \leftrightarrow \nu)$, $R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a_{b\nu} + \Gamma^c_{b\mu}\Gamma^a_{c\nu} - (\mu \leftrightarrow \nu)$, $N_{\mu ab} = D_\mu g_{ab}$ are, respectively, torsion, curvature and nonmetricity. Assuming, as usual, that equations of motion are linear in second derivatives of gauge fields, we are confined to no higher than quadratic powers of the torsion, curvature and nonmetricity. Covariant derivative is of the form $D_\mu = \partial_\mu - i\Gamma_a{}^b{}_\mu Q_b^a$, where Q_b^a denote generators of $\overline{GL}(n,\mathbb{R})$ group. The matter Lagrangian (assuming minimal coupling for all fields except the dilaton one) is a function of some number of affine fields ϕ^I and their covariant derivatives, together with metrics and tetrads (affine connection enters only through covariant derivative): $L_m = L_m(\phi^I, D\phi^I, e, g)$.

With all these general remarks, we will consider a class of affine Lagrangians, in arbitrary number of dimensions n, of the form:

$$\begin{aligned} &L(e_\mu{}^a, \partial_\nu e_\mu{}^a, \Gamma_{b\mu}{}^a, \partial_\nu \Gamma_{b\mu}{}^a, g_{ab}, \Psi_A, \partial_\nu \Psi_A, \Phi_A, \partial_\nu \Phi_A, \varphi, \partial_\nu \varphi) = \\ &e\Big[\varphi^2 R - \varphi^2 T^2 - \varphi^2 N^2 + \\ &\bar{\Psi} i g^{ab}\gamma_a e_b{}^\mu D_\mu \Psi + \tfrac{1}{2} g^{ab} e_a{}^\mu e_b{}^\nu (D_\mu \Phi)^+ (D_\nu \Phi) + \tfrac{1}{2} g^{ab} e_a{}^\mu e_b{}^\nu D_\mu \varphi D_\nu \varphi\Big]. \end{aligned} \tag{20}$$

The terms in the first row represent general gravitational part of the Lagrangian, that is invariant w.r.t. affine transformations (dilatational invariance is obtained with the aid of field φ, of mass dimension $n/2-1$). Here T^2 and N^2 stand for linear combination of terms quadratic in torsion and nonmetricity, respectively, formed by irreducible components of these fields. For the scope of this paper, we need not fix these terms any further. This is a general form of gravitational kinetic terms, invariant for an arbitrary space-time dimension $n \geq 3$.

The Lagrangian matter terms, invariant w.r.t. the local $\overline{GA}(n,\mathbb{R})$, $n \geq 3$, transformations, are written in the second row. The field Ψ denotes a spinorial $\overline{GL}(n,\mathbb{R})$ field—components of that field transform under some appropriate spinorial $\overline{GL}(n,\mathbb{R})$ irreducible representations. All spinorial $\overline{GL}(n,\mathbb{R})$ representations are necessarily infinite dimensional [11], and thus the field Ψ will have infinite number of components. The concrete spinorial irreducible representation of field Ψ is given by a set of $n-1$ $\overline{SL}(n,\mathbb{R})$ labels $\{\sigma_c^\Psi\}$ together with the dilatation charge d_Ψ. The field Φ is a representative of a tensorial $\overline{GL}(n,\mathbb{R})$ field, transforming under a tensorial $\overline{GL}(n,\mathbb{R})$ representation (i.e. one transforming w.r.t. single-valued representation of the $SO(n)$ subgroup) labelled by parameters $\{\sigma_c^\Phi\}$ and d_Φ. Since, as it is briefly argued later, the noncompact $\overline{SL}(n-1,\mathbb{R})$ affine subgroup is to be represented unitarily, the tensorial field Φ is also to transform under an infinite-dimensional representation and to have an infinite number of components. The remaining dilaton field φ is scalar with respect to $\overline{SL}(n,\mathbb{R})$ subgroup, and thus has only one component.

Interaction of affine connection with matter fields is determined by terms containing covariant derivatives. We write these terms in a component notation, where the component labelling is done with respect to the physically important Lorenz $Spin(1,n-1)$ subgroup of $\overline{GL}(n,\mathbb{R})$. Such a labelling allows, in principle, to identify affine field components with Lorentz fields of models based on the Poincaré symmetry. Namely, the affine models of gravity necessarily imply existence of some symmetry breaking mechanism that reduces the global symmetry to the Poincaré one, reflecting the subgroup structure $T^n \wedge \overline{SO}(1,n-1) \subset T^n \wedge \overline{GL}(n,\mathbb{R})$. Therefore, we consider the field Ψ (and similarly for Φ field) as a sum of its Lorentz components:

$$\sum_{\substack{\{J\}\\\{k\}\{m\}}} \Psi^{\{J\}}_{\{k\}\{m\}} \big|{}^{\{J\}}_{\{k\}\{m\}}\big\rangle.$$

The interaction term connecting fields g^{cd}, $e_d^{\ \mu}$, Γ^{ab}_{μ}, $\bar{\Psi}^{\{J\}}_{\{k\}\{m\}}$, $\Psi^{\{J'\}}_{\{k'\}\{m'\}}$ is now:

$$g^{cd} e_d^{\ \mu} \Gamma^{ab}_{\mu} \bar{\Psi}^{\{J\}}_{\{k\}\{m\}} \Psi^{\{J'\}}_{\{k'\}\{m'\}} \sum_{\substack{\{J''\}\\\{k''\}\{m''\}}} \big\langle{}^{\{J\}}_{\{k\}\{m\}}\big|\gamma_c\big|{}^{\{J''\}}_{\{k''\}\{m''\}}\big\rangle \big\langle{}^{\{J''\}}_{\{k''\}\{m''\}}\big|Q_{ab}\big|{}^{\{J'\}}_{\{k'\}\{m'\}}\big\rangle, \quad (21)$$

while the interaction of tensorial field with connection is given by:

$$-\tfrac{i}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_{\nu} \partial_\mu \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \Phi^{\{J'\}}_{\{k'\}\{m'\}} \big\langle{}^{\{J\}}_{\{k\}\{m\}}\big|Q_{ab}\big|{}^{\{J'\}}_{\{k'\}\{m'\}}\big\rangle + \quad (22)$$

$$\tfrac{i}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_{\nu} \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \partial_\mu \Phi^{\{J'\}}_{\{k'\}\{m'\}} \big\langle{}^{\{J'\}}_{\{k'\}\{m'\}}\big|Q_{ab}\big|{}^{\{J\}}_{\{k\}\{m\}}\big\rangle^* + \quad (23)$$

$$\tfrac{1}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_{\mu} \Gamma^{a'b'}_{\nu} \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \partial_\mu \Phi^{\{J'\}}_{\{k'\}\{m'\}} \cdot$$

$$\sum_{\substack{\{J''\}\\\{k''\}\{m''\}}} \big\langle{}^{\{J\}}_{\{k\}\{m\}}\big|Q_{ab}\big|{}^{\{J''\}}_{\{k''\}\{m''\}}\big\rangle \big\langle{}^{\{J''\}}_{\{k''\}\{m''\}}\big|Q_{a'b'}\big|{}^{\{J'\}}_{\{k'\}\{m'\}}\big\rangle. \quad (24)$$

The scalar dilaton field interact only with the trace of affine connection:

$$\tfrac{1}{2} g^{ab} e_a^{\ \mu} e_b^{\ \nu} (\partial_\mu - i\Gamma_{a\ \mu}^{\ a} d_\varphi)\varphi(\partial_\nu - i\Gamma_{a\ \nu}^{\ a} d_\varphi)\varphi, \quad (25)$$

where d_φ denotes dilatation charge of φ field.

In the above interaction terms we note an appearance of matrix elements of $\overline{GL}(n,\mathbb{R})$ generators, written in a basis of the Lorenz subgroup $Spin(1,n-1)$. The dilatation generator (that is, the trace Q^a_a) acts merely as multiplication by dilatation charge, so it is really the $\overline{SL}(n,\mathbb{R})$ matrix elements that should be calculated. (An infinite dimensional generalization of Dirac's gamma matrices also appear in the term (21); more on these matrices can be found in papers of Šijački [18].) However, before presenting examples of the matrix elements evaluations, and thus calculations of the vertex coefficients, it is due to note that the correct physical interpretation of the $\overline{SL}(n,\mathbb{R})$ representations requires these representations to be unitary w.r.t. its $\overline{SL}(n-1,\mathbb{R})$ subgroup and to be nonunitary w.r.t. its lorentz-like $Spin(1,n-1)$ subgroup. It turns out that these requirements can be properly satisfied by making use of the so called deunitarizing automorphism [11].

4 Gauge Affine Symmetry Vertex Coefficients Evaluation

Now we return to evaluation of vertex coefficients for interaction between various Lorentz components of the $\overline{GL}(n,\mathbb{R})$ fields. The nontrivial part is to find matrix elements of $\overline{SL}(n,\mathbb{R})$ shear generators in expressions (21)–(24), and, to do that in $n=5$ case we will use expression (19). However, this formula is given in the basis of the compact $Spin(n)$ subgroup, and not in the basis of the physically important Lorentz group $Spin(1,n-1)$. On the other hand, it turns out that taking into account deunitarizing automorphism exactly amounts to keeping reduced matrix element from (16) and replacing the remaining Clebsch–Gordan coefficient of the $Spin(n)$ group by the corresponding coefficient of the Lorenz group $Spin(1,n-1)$ [12].

As the first example, let the field Φ correspond to an unitary multiplicity free $\overline{SL}(5,\mathbb{R})$ representation, defined by labels $\sigma_2=-4, \delta_1=\delta_2=0$, with σ_1 arbitrary real. The representation space is spanned by vectors (18) satisfying $\overline{J}_1=\overline{J}_2=\overline{J}\in \mathbb{N}_0+\frac{1}{2}; K_1=K_2=0; J_1=J_2=J\leq \overline{J}$. This is a simplest class of multiplicity free representations that is unitary assuming usual scalar product. If we denote $\Phi^a, a=1\dots 5$ the five Φ components with $\overline{J}_1=\overline{J}_2=\frac{1}{2}$ (in this sense Φ^a corresponds to a Lorenz 5-vector) then the interaction vertex (22) connecting fields $\Phi^{a\dagger}$, $\partial_\mu\Phi^d$ and affine shear connection Γ_ν^{bc} is:

$$\frac{i}{2}g^{ef}e_e^{\ \mu}e_f^{\ \nu}\Phi^{a\dagger}\Gamma_\nu^{bc}\partial_\mu\Phi^d\,\frac{\sqrt{5}}{14}\,\sigma_1(\eta_{ab}\eta_{dc}+\eta_{ac}\eta_{db}-\frac{2}{n}\eta_{ad}\eta_{bc}). \tag{26}$$

To obtain this result we used an easily derivable formula for Clebsch–Gordan coefficient connecting Lorentz vector and symmetric second order Lorenz tensor representations:

$$C^{L\square\,\square\square\,\square}_{\ a\ (bc)\ d}=\sqrt{\tfrac{n}{2(n+2)(n-1)}}(\eta_{ab}\eta_{dc}+\eta_{ac}\eta_{db}-\frac{2}{n}\eta_{ad}\eta_{bc}), \tag{27}$$

where we labelled $Spin(1,n-1)$ irreducible representations by Young diagrams, as in [15]. More importantly, we also used value of the reduced matrix element:

$$\left\langle \begin{matrix}\frac{1}{2}\,\frac{1}{2}\\ 0\,0\\ 0\,0\end{matrix}\right\| Q \left\| \begin{matrix}\frac{1}{2}\,\frac{1}{2}\\ 0\,0\\ 0\,0\end{matrix}\right\rangle=\sqrt{\frac{2}{7}}\sigma_1, \tag{28}$$

that we obtained by using formula (19) (based on this formula, a Mathematica program was generated that directly calculates $sl(5,\mathbb{R})$ matrix elements [12], taking into account $Spin(5)$ Clebsch–Gordan coefficients found in [16]).

It is no more difficult to obtain coefficients of the vertices of the form (24). Lagrangian term (24) connecting Lorenz 5-vector Φ components Φ_5, $\Phi_5^\dagger$ and affine connection component $\Gamma_{(55)\mu}$ is:

$$\frac{1}{15}\left(\sigma_1^2-25\right)g^{cd}e_c^{\ \mu}e_d^{\ \nu}\Gamma_\mu^{55}\Gamma_\nu^{55}\Phi_5^\dagger\partial_\mu\Phi_5. \tag{29}$$

Next we will consider an example where Φ field corresponds to a representation with multiplicity. Let us, again, consider 5-vector component $\overline{J}_1 = \overline{J}_2 = \frac{1}{2}$ of Φ, only this time without any restriction to the values of $\sigma_1, \sigma_2, \delta_1, \delta_2$. In general, this will correspond to a representation with non trivial multiplicity. Quantum numbers $\{k\} = (K_1, K_2, k_1, k_2)$, that label multiplicity, now can take values: $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and $(0,0,0,0)$. Therefore, this a priori corresponds to 5 observable 5-vector fields, differentiated by the $\{k\}$ values, and these five vector fields mutually interact by gravitational interaction. Part of the Lagrangian term (22), responsible for this interaction, has the form:

$$\frac{i}{2} g^{ef} e_e^{\ \mu} e_f^{\ \nu} \Phi^{a\dagger}_{\{k'\}} \Gamma_\nu^{bc} \partial_\mu \Phi^d_{\{k\}} \left\langle \begin{matrix} \overline{\tfrac{1}{2}}\ \overline{\tfrac{1}{2}} \\ K_1' K_2' \\ k_1'\ k_2' \end{matrix} \left\| Q \right\| \begin{matrix} \overline{\tfrac{1}{2}}\ \overline{\tfrac{1}{2}} \\ K_1 K_2 \\ k_1\ k_2 \end{matrix} \right\rangle \frac{\sqrt{5}}{\sqrt{56}} (\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{5}\eta_{ad}\eta_{bc}). \tag{30}$$

The reduced matrix element is obtained from the generalized Gell–Mann formula:

$$\left\langle \begin{matrix} \overline{\tfrac{1}{2}}\ \overline{\tfrac{1}{2}} \\ \tfrac{1}{2}\ \tfrac{1}{2} \\ k_1' k_2' \end{matrix} \left\| Q \right\| \begin{matrix} \overline{\tfrac{1}{2}}\ \overline{\tfrac{1}{2}} \\ \tfrac{1}{2}\ \tfrac{1}{2} \\ k_1 k_2 \end{matrix} \right\rangle =$$

$$\frac{1}{4\sqrt{14}} \Bigg(-2\sigma_1 C_3{}^{\frac{1}{2}\,0\,\frac{1}{2}}_{k_1\,0\,k_1'} C_3{}^{\frac{1}{2}\,0\,\frac{1}{2}}_{k_2\,0\,k_2'} + 15\sigma_2 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,0\,k_1'} C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,0\,k_2'} -$$

$$-15 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,-1\,k_1'} \left((k_1 + k_2 - \delta_2) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,-1\,k_2'} + (-k_1 + k_2 + \delta_1) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,1\,k_2'} \right)$$

$$-15 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,1\,k_1'} \left((k_1 - k_2 + \delta_1) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,-1\,k_2'} - (k_1 + k_2 + \delta_2) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,1\,k_2'} \right) \Bigg),$$

$$\left\langle \begin{matrix} \overline{\tfrac{1}{2}\,\tfrac{1}{2}} \\ 0\,0 \\ 0\,0 \end{matrix} \left\| Q \right\| \begin{matrix} \overline{\tfrac{1}{2}}\ \overline{\tfrac{1}{2}} \\ K_1 K_2 \\ k_1\ k_2 \end{matrix} \right\rangle = 0, \qquad \left\langle \begin{matrix} \overline{\tfrac{1}{2}\,\tfrac{1}{2}} \\ 0\,0 \\ 0\,0 \end{matrix} \left\| Q \right\| \begin{matrix} \overline{\tfrac{1}{2}\,\tfrac{1}{2}} \\ 0\,0 \\ 0\,0 \end{matrix} \right\rangle = \sqrt{\frac{2}{7}}\,\sigma_1, \tag{31}$$

where C_3 denotes an usual $Spin(3)$ Clebsch–Gordan coefficient.

References

1. Berendt, G.: Acta Phys. Austriaca **25**, 207 (1967)
2. Harish-Chandra, Proc. Natl. Acad. Sci. **37**, 170, 362, 366, 691 (1951)
3. Hazewinkel, M. (ed.): Encyclopaedia of Mathematics, Supplement I, p. 269. Springer, Berlin (1997)
4. Hehl, F.W., McCrea, J.D., Mielke, E.W., Ne'eman, Y.: Phys. Rep. **258**, 1 (1995)
5. Hermann, R.: Lie Groups for Physicists. W. A. Benjamin Inc, New York (1965)
6. Hermann, R.: Comm. Math. Phys. **2**, 78 (1966)
7. Inönü, E., Wigner, E.P.: Proc. Natl. Acad. Sci. **39**, 510 (1953)
8. Mackey, G.: Induced Representations of Groups and Quantum Mechanics. Benjamin, New York (1968)
9. Mukunda, N.: J. Math. Phys. **10**, 897 (1969)

10. Ne'eman, Y., Šijački, Dj.: Ann. Phys. (N.Y.) **120**, 292 (1979)
11. Ne'eman, Y., Šijački, Dj.: Int. J. Mod. Phys. **A2**, 1655 (1987)
12. Salom, I.: Decontraction formula for $sl(n,\mathbb{R})$ algebras and applications in theory of gravity. Ph.D. Thesis, Physics Department, University of Belgrade, 2011 (in Serbian)
13. Salom, I., Šijački, Dj.: Int. J. Geom. Met. Mod. Phys. **7**, 455 (2010)
14. Salom, I., Šijački, Dj.: Lie theory and its applications in physics. AIPCP **1243**, 191 (2010)
15. Salom, I., Šijački, Dj.: Int. J. Geom. Met. Mod. Phys. **8**, 395 (2011)
16. Salom, I., Šijački, Dj.: [arXiv:math-ph/0904.4200v1]
17. Sankaranarayanan, A.: Nuovo Cimento **38**, 1441 (1965)
18. Šijački, Dj.: Class. Quant. Gravit. **21**, 4575 (2004)
19. Štoviček, P.: J. Math. Phys. **29**, 1300 (1988)

W-Algebras Extending $\widehat{\mathfrak{gl}}(1|1)$

Thomas Creutzig and David Ridout

Abstract We have recently shown that $\widehat{\mathfrak{gl}}(1|1)$ admits an infinite family of simple current extensions. Here, we review these findings and add explicit free field realizations of the extended algebras. We use them for the computation of leading contributions of the operator product algebra. Amongst others, we find extensions that contain the Feigin–Semikhatov $W_N^{(2)}$ algebra at levels $k = N(3-N)/(N-2)$ and $k = -N+1+N^{-1}$ as subalgebras.

1 Introduction

The affine Kac–Moody superalgebra $\widehat{\mathfrak{gl}}(1|1)$ is an attractive candidate for study. On the one hand, its highest weight theory is particularly easy to analyse. On the other, one is naturally led to study indecomposable modules of the type that arise in logarithmic conformal field theory. In [1], we reviewed and consolidated what was known about this superalgebra, drawing in particular upon the previous works [2–8].

One motivation for undertaking this work was to understand how one could reconcile the observation that conformal field theories with $\widehat{\mathfrak{gl}}(1|1)$ symmetry appeared to admit only continuous spectra, whereas one might expect that the Wess–Zumino–Witten model on the real form $\mathsf{U}(1|1)$ would have the same sym-

T. Creutzig (✉)
Fachbereich Mathematik, Technische Universität Darmstadt,
Schloßgartenstraße 7 64289 Darmstadt, Germany
e-mail: tcreutzig@mathematik.tu-darmstadt.de

D. Ridout
Department of Theoretical Physics, Research School of Physics and Engineering and
Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200 Australia
e-mail: david.ridout@anu.edu.au

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_24,

metry, but a discrete spectrum. Another was to understand whether $\widehat{\mathfrak{gl}}(1|1)$ could be related to other infinite-dimensional algebras, thus providing relationships between certain (logarithmic) conformal field theories. For the first question, we were able to show that certain discrete spectra seem to be consistent provided one *extends* the chiral algebra appropriately. For the second, we identified a certain $\widehat{\mathfrak{u}}(1)$-coset of $\widehat{\mathfrak{gl}}(1|1)$ as the chiral algebra of the well-known $\beta\gamma$ ghost system. Previous work [9] then links $\widehat{\mathfrak{gl}}(1|1)$ to the affine Kac–Moody algebra $\widehat{\mathfrak{sl}}(2)_{-1/2}$ [10, 11], the triplet algebra $\mathfrak{W}(1,2)$ of Gaberdiel and Kausch [12] and the symplectic fermions algebra [13] ($\widehat{\mathfrak{psl}}(1|1)$).

This article describes a certain family of *extended algebras* of $\widehat{\mathfrak{gl}}(1|1)$. In [1], we noted that the fusion rules give rise to an infinite family of simple currents labelled by $n \in \mathbb{R}$ and $\ell \in \mathbb{Z}$. It follows that these algebra extensions may be computed algorithmically [14, 15]. Here, we perform the computations up to a certain order, using a well-known free field realisation [16]. More precisely, we study the resulting W-algebras and show that for certain infinite families of n and ℓ, there is a bosonic subalgebra which we conjecture to be the $W_N^{(2)}$ algebra of Feigin and Semikhatov [17].

2 $\mathfrak{gl}(1|1)$ and Its Representations

2.1 Algebraic Structure

In this section, we review our notation [1] for $\mathfrak{gl}(1|1)$. The Lie superalgebra $\mathfrak{gl}(1|1)$ consists of the endomorphisms of the super vector space $\mathbb{C}^{1|1}$ equipped with the standard graded commutator. It is convenient to choose the following basis

$$N = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

in which N and E are parity-preserving (bosonic) whereas ψ^+ and ψ^- are parity-reversing (fermionic). The non-vanishing brackets are then

$$\left[N, \psi^{\pm}\right] = \pm\psi^{\pm}, \qquad \left\{\psi^+, \psi^-\right\} = E. \tag{2}$$

We note that E is central, so this superalgebra is not simple. In fact, $\mathfrak{gl}(1|1)$ does not decompose as a direct sum of ideals. Equivalently, the adjoint representation of $\mathfrak{gl}(1|1)$ is reducible, but indecomposable.

The standard non-degenerate bilinear form $\kappa(\cdot,\cdot)$ on $\mathfrak{gl}(1|1)$ is given by the supertrace of the product in the defining representation (1). With respect to the basis elements (1), this form is

$$\kappa(N,E) = \kappa(E,N) = 1, \qquad \kappa(\psi^+,\psi^-) = -\kappa(\psi^-,\psi^+) = 1, \tag{3}$$

with all other combinations vanishing. From this, we compute the quadratic Casimir $Q \in \mathfrak{U}(\mathfrak{gl}(1|1))$ (up to an arbitrary polynomial in the central element E). We find it convenient to take

$$Q = NE + \psi^- \psi^+. \tag{4}$$

2.2 Representation Theory

The obvious triangular decomposition of $\mathfrak{gl}(1|1)$ regards ψ^+ as a raising (annihilation) operator, ψ^- as a lowering (creation) operator, and N and E as Cartan elements. A highest weight state of a $\mathfrak{gl}(1|1)$-representation is then defined to be an eigenstate of N and E which is annihilated by ψ^+. Such states generate Verma modules in the usual way and as ψ^- squares to zero in any representation, every Verma module has dimension 2. If (n,e) denotes the weight (the N- and E-eigenvalues) of a highest weight state generating a Verma module, then its unique descendant will have weight $(n-1,e)$. We will denote this Verma module by $\mathcal{V}_{n-1/2,e}$, remarking that the convention of characterising a highest weight module by the *average* N-eigenvalue of its states, rather than that of the highest weight state itself, turns out to symmetrise many of the formulae to follow.

Suppose now that $|v\rangle$ is a (generating) highest weight state of $\mathcal{V}_{n,e}$. It satisfies

$$\psi^+ \psi^- |v\rangle = \{\psi^+, \psi^-\} |v\rangle = E|v\rangle = e|v\rangle, \tag{5}$$

so the descendant $\psi^- |v\rangle \neq 0$ is a singular vector if and only if $e = 0$. Verma modules are therefore irreducible for $e \neq 0$, and have irreducible quotients of dimension 1 when $e = 0$. Modules with $e \neq 0$ are called *typical* while those with $e = 0$ are *atypical*. We will denote a typical irreducible by $\mathcal{T}_{n,e} \cong \mathcal{V}_{n,e}$ and an atypical irreducible by $\mathcal{A}_n$. Our convention of labelling modules by their average N-eigenvalue leads us to define the latter to be the irreducible quotient of $\mathcal{V}_{n-1/2,0}$. This is summarised in the short exact sequence

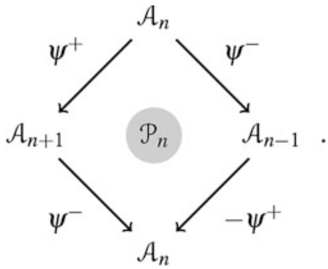
$$0 \longrightarrow \mathcal{A}_{n-1/2} \longrightarrow \mathcal{V}_{n,0} \longrightarrow \mathcal{A}_{n+1/2} \longrightarrow 0 \tag{6}$$

and structure diagram

$$\mathcal{V}_{n,0} \quad : \qquad \mathcal{A}_{n+1/2} \xrightarrow{\quad \psi^- \quad} \mathcal{A}_{n-1/2} \, .$$

Such diagrams illustrate the irreducible composition factors of an indecomposable module, with arrows indicating (schematically) the action of the algebra. In the above diagram, the composition factor $\mathcal{A}_{n-1/2}$ is seen to be a submodule as the arrow points towards it. The factor $\mathcal{A}_{n+1/2}$ is not a submodule, but rather is the quotient $\mathcal{V}_{n,0}/\mathcal{A}_{n-1/2}$: Its preimage in $\mathcal{V}_{n,0}$ is sent to the submodule $\mathcal{A}_{n-1/2}$ by the action of ψ^-.

Atypical modules also appear as submodules of larger indecomposable modules. Of particular importance are the four-dimensional projectives[1] $\mathcal{P}_n$ whose structure diagrams take the form



We remark that these modules may be viewed as particularly simple examples of staggered modules [18]. Indeed, they may be regarded as extensions of highest weight modules via the exact sequence

$$0 \longrightarrow \mathcal{V}_{n+1/2,0} \longrightarrow \mathcal{P}_n \longrightarrow \mathcal{V}_{n-1/2,0} \longrightarrow 0, \tag{7}$$

and one can verify that the Casimir Q acts non-diagonalisably on $\mathcal{P}_n$, taking the generator associated with the top $\mathcal{A}_n$ factor to the generator of the bottom $\mathcal{A}_n$ factor, while annihilating the other states.

2.3 The Representation Ring

The relevance of the projectives $\mathcal{P}_n$ is that they appear in the representation ring generated by the irreducibles.[2] The tensor product rules governing this ring are [2]

$$\begin{gathered}
\mathcal{A}_n \otimes \mathcal{A}_{n'} = \mathcal{A}_{n+n'}, \qquad \mathcal{A}_n \otimes \mathcal{T}_{n',e'} = \mathcal{T}_{n+n',e'}, \qquad \mathcal{A}_n \otimes \mathcal{P}_{n'} = \mathcal{P}_{n+n'}, \\
\mathcal{T}_{n,e} \otimes \mathcal{T}_{n',e'} = \begin{cases} \mathcal{P}_{n+n'} & \text{if } e+e'=0, \\ \mathcal{T}_{n+n'+1/2,e+e'} \oplus \mathcal{T}_{n+n'-1/2,e+e'} & \text{otherwise,} \end{cases} \\
\mathcal{T}_{n,e} \otimes \mathcal{P}_{n'} = \mathcal{T}_{n+n'+1,e} \oplus 2\,\mathcal{T}_{n+n',e} \oplus \mathcal{T}_{n+n'-1,e}, \\
\mathcal{P}_n \otimes \mathcal{P}_{n'} = \mathcal{P}_{n+n'+1} \oplus 2\,\mathcal{P}_{n+n'} \oplus \mathcal{P}_{n+n'-1}.
\end{gathered} \tag{8}$$

[1] We mention that the typical irreducibles are also projective in the category of finite-dimensional $\mathfrak{gl}(1|1)$-modules.

[2] It is perhaps also worth pointing out that the adjoint representation of $\mathfrak{gl}(1|1)$ is isomorphic to $\mathcal{P}_0$.

There are other indecomposables which may be constructed from submodules and quotients of the $\mathcal{P}_n$ by taking tensor products. We will not need them and refer to [19] for further discussion.

3 $\widehat{\mathfrak{gl}}(1|1)$ and Its Representations

3.1 Algebraic Structure

Here, we summarise the important results of [1] that we need for what follows. Our conventions for $\mathfrak{gl}(1|1)$ carry over to its affinisation $\widehat{\mathfrak{gl}}(1|1)$ in the usual way. Explicitly, the non-vanishing brackets are

$$\left[N_r, E_s\right] = rk\delta_{r+s,0}, \quad \left[N_r, \psi_s^{\pm}\right] = \pm\psi_{r+s}^{\pm}, \quad \left\{\psi_r^+, \psi_s^-\right\} = E_{r+s} + rk\delta_{r+s,0}, \tag{9}$$

where $k \in \mathbb{R}$ is called the level and $r, s \in \mathbb{Z}$. We emphasise that when $k \neq 0$, the generators can be rescaled so as to normalise k to 1:

$$N_r \longrightarrow N_r, \qquad E_r \longrightarrow \frac{E_r}{k}, \qquad \psi_r^{\pm} \longrightarrow \frac{\psi_r^{\pm}}{\sqrt{k}}. \tag{10}$$

As in the more familiar case of $\widehat{\mathfrak{u}}(1)$, we see that the actual value of $k \neq 0$ is not physical.

The Virasoro generators are constructed using (a modification of) the Sugawara construction. Because the quadratic Casimir of $\mathfrak{gl}(1|1)$ is only defined modulo polynomials in E, one tries the Ansatz [2]

$$T(z) = \mu : NE + EN - \psi^+\psi^- + \psi^-\psi^+ : (z) + \nu : EE : (z), \tag{11}$$

finding that this defines an energy-momentum tensor if and only if $\mu = 1/2k$ and $\nu = 1/2k^2$. Moreover, the $\widehat{\mathfrak{gl}}(1|1)$ currents $N(z)$, $E(z)$ and $\psi^{\pm}(z)$ are found to be Virasoro primaries of conformal dimension 1 and the central charge is zero.

The structure theory of highest weight modules for $\widehat{\mathfrak{gl}}(1|1)$ turns out to be particularly accessible because of certain automorphisms. These consist of the automorphism w which defines the notion of conjugation and the family [4] of spectral flow automorphisms σ^{ℓ}, $\ell \in \mathbb{Z}$. Explicitly,

$$\begin{aligned} &\mathsf{w}(N_r) = -N_r, \quad \mathsf{w}(E_r) = -E_r, \qquad \mathsf{w}\left(\psi_r^{\pm}\right) = \pm\psi_r^{\mp}, \quad \mathsf{w}(L_0) = L_0. \\ &\sigma^{\ell}(N_r) = N_r, \quad \sigma^{\ell}(E_r) = E_r - \ell k\delta_{r,0}, \ \sigma^{\ell}\left(\psi_r^{\pm}\right) = \psi_{r\mp\ell}^{\pm}, \ \sigma^{\ell}(L_0) = L_0 - \ell N_0. \end{aligned} \tag{12}$$

These automorphisms may be used to construct new modules $\mathsf{w}^*(\mathcal{M})$ and $\sigma^*(\mathcal{M})$ by twisting the action of the algebra on a module $\mathcal{M}$:

$$J \cdot \mathsf{w}^*\left(\left|v\right\rangle\right) = \mathsf{w}^*\left(\mathsf{w}^{-1}(J)\left|v\right\rangle\right), \quad J \cdot \sigma^*\left(\left|v\right\rangle\right) = \sigma^*\left(\sigma^{-1}(J)\left|v\right\rangle\right) \quad (J \in \widehat{\mathfrak{gl}}(1|1)). \tag{13}$$

Note that $\mathsf{w}^*(\mathcal{M})$ is precisely the module conjugate to $\mathcal{M}$.

3.2 Representation Theory

We can now define affine highest weight states, affine Verma modules $\widehat{\mathcal{V}}_{n,\ell}$, and their irreducible quotients as before. We remark only that (10) suggests that we characterise modules by the invariant ratio $\ell = e/k$ rather than by the E_0-eigenvalue e. The affine highest weight state $\left|v_{n,\ell}\right\rangle$ of $\widehat{\mathcal{V}}_{n,\ell}$, whose weight (its N_0- and E_0/k-eigenvalues) is $\left(n+\frac{1}{2},\ell\right)$, has conformal dimension

$$\Delta_{n,\ell} = n\ell + \frac{1}{2}\ell^2. \tag{14}$$

Of course, this formula also applies to singular vectors. Again, the label n refers to the average N_0-eigenvalue of the zero-grade subspace of $\widehat{\mathcal{V}}_{n,\ell}$, generalising the labelling convention of Sect. 2.2.

Verma modules for $\widehat{\mathfrak{gl}}(1|1)$ are infinite-dimensional and their characters have the form

$$\chi_{\widehat{\mathcal{V}}_{n,\ell}}(z;q) = \operatorname{tr}_{\widehat{\mathcal{V}}_{n,\ell}} z^{N_0} q^{L_0} = z^{n+1/2} q^{\Delta_{n,\ell}} \prod_{i=1}^{\infty} \frac{\left(1+zq^i\right)\left(1+z^{-1}q^{i-1}\right)}{\left(1-q^i\right)^2}. \tag{15}$$

For the irreducible quotients, the case with $\ell = 0$ is particularly easy. As in Sect. 2.2, we regard (n,ℓ) (and modules so-labelled) as being *typical* if $\widehat{\mathcal{V}}_{n,\ell}$ is irreducible and *atypical* otherwise.

Proposition 1. *The affine Verma module $\widehat{\mathcal{V}}_{n,0}$ has an exact sequence*

$$0 \longrightarrow \widehat{\mathcal{A}}_{n-1/2,0} \longrightarrow \widehat{\mathcal{V}}_{n,0} \longrightarrow \widehat{\mathcal{A}}_{n+1/2,0} \longrightarrow 0 \tag{16}$$

in which the $\widehat{\mathcal{A}}_{n,0}$ are (atypical) irreducibles whose characters are given by

$$\chi_{\widehat{\mathcal{A}}_{n,0}}(z;q) = z^n \prod_{i=1}^{\infty} \frac{\left(1+zq^i\right)\left(1+z^{-1}q^i\right)}{\left(1-q^i\right)^2}. \tag{17}$$

Proof. Since $\ell = 0$, every singular vector of $\widehat{\mathcal{V}}_{n,0}$ has dimension 0 by (14). The space of singular vectors is thus spanned by $\left|v_{n,0}\right\rangle$ and $\psi_0^-\left|v_{n,0}\right\rangle$. Taking the quotient by the module generated by $\psi_0^-\left|v_{n,0}\right\rangle$ gives a module with a one-dimensional zero-grade subspace. The only singular vector is then the highest weight state, so this quotient is irreducible. We denote it by $\widehat{\mathcal{A}}_{n+1/2,0}$ as its zero-grade subspace has N_0-eigenvalue $n+\frac{1}{2}$. Its character follows trivially. The submodule of $\widehat{\mathcal{V}}_{n,0}$ generated by $\psi_0^-\left|v_{n,0}\right\rangle$ is not a Verma module because $\left(\psi_0^-\right)^2\left|v_{n,0}\right\rangle = 0$. It must therefore be a proper quotient of $\widehat{\mathcal{V}}_{n-1,0}$ and, by the above argument, the only such quotient is the irreducible $\widehat{\mathcal{A}}_{n-1/2,0}$. The exact sequence follows. □

For $\ell \neq 0$, one proves by direct calculation [1] that for $0 < |\ell| < 1$, $\widehat{\mathcal{V}}_{n,\ell}$ is irreducible. In other words, the corresponding irreducibles are typical, hence we denote them by $\widehat{\mathcal{T}}_{n,\ell}$. For $|\ell| \geq 1$, the structure of the Verma modules now follows from considering the induced action of the spectral flow automorphisms. More precisely, one proves [1] that any Verma module is isomorphic to a twisted version of a Verma module with $-1 < |\ell| < 1$ (or the conjugate of such a Verma module). We summarise the result as follows.

Proposition 2. *When $\ell \notin \mathbb{Z}$, the affine Verma module $\widehat{\mathcal{V}}_{n,\ell}$ is irreducible, $\widehat{\mathcal{V}}_{n,\ell} \cong \widehat{\mathcal{T}}_{n,\ell}$, so its character is given by (15). When $\ell \in \mathbb{Z}$, the affine Verma module $\widehat{\mathcal{V}}_{n,\ell}$ has an exact sequence*

$$\begin{aligned} &0 \longrightarrow \widehat{\mathcal{A}}_{n+1,\ell} \longrightarrow \widehat{\mathcal{V}}_{n,\ell} \longrightarrow \widehat{\mathcal{A}}_{n,\ell} \longrightarrow 0 \qquad (\ell = +1,+2,+3,\ldots), \\ &0 \longrightarrow \widehat{\mathcal{A}}_{n-1,\ell} \longrightarrow \widehat{\mathcal{V}}_{n,\ell} \longrightarrow \widehat{\mathcal{A}}_{n,\ell} \longrightarrow 0 \qquad (\ell = -1,-2,-3,\ldots), \end{aligned} \tag{18}$$

in which the $\widehat{\mathcal{A}}_{n,\ell}$ are (atypical) irreducibles whose characters are given by

$$\chi_{\widehat{\mathcal{A}}_{n,\ell}}(z;q) = \begin{cases} \dfrac{z^{n+1/2} q^{\Delta_{n,\ell}}}{1+zq^{\ell}} \displaystyle\prod_{i=1}^{\infty} \dfrac{\left(1+zq^i\right)\left(1+z^{-1}q^{i-1}\right)}{(1-q^i)^2} & (\ell = +1,+2,+3,\ldots), \\ \dfrac{z^{n+1/2} q^{\Delta_{n,\ell}}}{1+z^{-1}q^{-\ell}} \displaystyle\prod_{i=1}^{\infty} \dfrac{\left(1+zq^i\right)\left(1+z^{-1}q^{i-1}\right)}{(1-q^i)^2} & (\ell = -1,-2,-3,\ldots). \end{cases} \tag{19}$$

(The exact sequence and character for $\ell = 0$ was given in Proposition 1.)

Note that the $\widehat{\mathcal{V}}_{n,\ell}$ with $\ell \in \mathbb{Z}$ have a non-trivial singular vector at grade $|\ell|$. We emphasise that the $\widehat{\mathcal{A}}_{n,\ell}$ with $\ell \neq 0$ therefore possess a two-dimensional zero-grade subspace.

This description of the Verma modules, their irreducible quotients and characters relies upon being able to identify the result of applying the spectral flow automorphisms to modules. For irreducibles, we have

$$\left(\sigma^{\ell'}\right)^* \left(\widehat{\mathcal{T}}_{n,\ell}\right) = \widehat{\mathcal{T}}_{n-\ell',\ell+\ell'}, \qquad \left(\sigma^{\ell'}\right)^* \left(\widehat{\mathcal{A}}_{n,\ell}\right) = \widehat{\mathcal{A}}_{n-\ell'+\varepsilon(\ell+\ell')-\varepsilon(\ell),\ell+\ell'}, \tag{20}$$

where we introduce a convenient variant ε of the sign function on $\mathbb{Z}$, defined by taking $\varepsilon(\ell)$ to be $\frac{1}{2}$, 0 or $-\frac{1}{2}$ according as to whether $\ell \in \mathbb{Z}$ is positive, zero or negative, respectively.

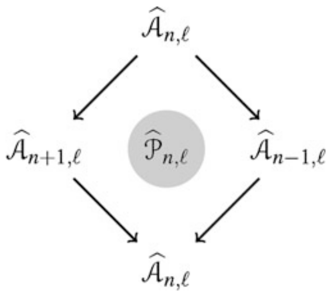
3.3 Fusion

The fusion rules of the irreducible $\widehat{\mathfrak{gl}}(1|1)$-modules (among others) were first deduced in [5] using three-point functions computed in a free field realisation and a conjectured completeness of the spectrum. These rules and the spectrum conjecture were confirmed in [1] through a direct argument involving the Nahm–Gaberdiel–Kausch fusion algorithm [20, 21] and spectral flow. The fusion ring generated by the irreducibles may be understood [22] as a "constrained lift" of the representation ring (8) of $\mathfrak{gl}(1|1)$ where the constraints are effectively implemented by spectral flow. Explicitly, the rules are

$$\begin{aligned}
\widehat{\mathcal{A}}_{n,\ell} \times \widehat{\mathcal{A}}_{n',\ell'} &= \widehat{\mathcal{A}}_{n+n'-\varepsilon(\ell,\ell'),\ell+\ell'}, \quad \widehat{\mathcal{A}}_{n,\ell} \times \widehat{\mathcal{T}}_{n',\ell'} = \widehat{\mathcal{T}}_{n+n'-\varepsilon(\ell),\ell+\ell'}, \\
\widehat{\mathcal{A}}_{n,\ell} \times \widehat{\mathcal{P}}_{n',\ell'} &= \widehat{\mathcal{P}}_{n+n'-\varepsilon(\ell,\ell'),\ell+\ell'}, \\
\widehat{\mathcal{T}}_{n,\ell} \times \widehat{\mathcal{T}}_{n',\ell'} &= \begin{cases} \widehat{\mathcal{P}}_{n+n'+\varepsilon(\ell+\ell'),\ell+\ell'} & \text{if } \ell+\ell'=0, \\ \widehat{\mathcal{T}}_{n+n'+1/2,\ell+\ell'} \oplus \widehat{\mathcal{T}}_{n+n'-1/2,\ell+\ell'} & \text{otherwise,} \end{cases} \\
\widehat{\mathcal{T}}_{n,\ell} \times \widehat{\mathcal{P}}_{n',\ell'} &= \widehat{\mathcal{T}}_{n+n'+1-\varepsilon(\ell'),\ell+\ell'} \oplus 2\,\widehat{\mathcal{T}}_{n+n'-\varepsilon(\ell'),\ell+\ell'} \oplus \widehat{\mathcal{T}}_{n+n'-1-\varepsilon(\ell'),\ell+\ell'}, \\
\widehat{\mathcal{P}}_{n,\ell} \times \widehat{\mathcal{P}}_{n',\ell'} &= \widehat{\mathcal{P}}_{n+n'+1-\varepsilon(\ell,\ell'),\ell+\ell'} \oplus 2\,\widehat{\mathcal{P}}_{n+n'-\varepsilon(\ell,\ell'),\ell+\ell'} \oplus \widehat{\mathcal{P}}_{n+n'-1-\varepsilon(\ell,\ell'),\ell+\ell'}.
\end{aligned} \tag{21}$$

Here, we have defined $\varepsilon(\ell,\ell') = \varepsilon(\ell) + \varepsilon(\ell') - \varepsilon(\ell+\ell')$ for convenience.

These fusion rules also introduce the indecomposable modules $\widehat{\mathcal{P}}_{n,\ell}$ which are the counterparts of the projective $\mathfrak{gl}(1|1)$-modules $\mathcal{P}_n$ discussed in Sect. 2.2.[3] The $\widehat{\mathcal{P}}_{n,\ell}$ are staggered with structure diagram



and a non-diagonalisable action of the Virasoro mode L_0. It follows that conformal field theories whose spectra contain typical modules will also contain such $\widehat{\mathcal{P}}_{n,\ell}$ (by fusion), and so will be *logarithmic*.

[3]More precisely, $\widehat{\mathcal{P}}_{n,0}$ is the affine counterpart to $\mathcal{P}_n$ and the remaining $\widehat{\mathcal{P}}_{n,\ell}$ are obtained by spectral flow.

4 W-Algebras Extending $\widehat{\mathfrak{gl}}(1|1)$

4.1 Chiral Algebra Extensions

Our search for extended algebras is guided by the following considerations: First, note that if we choose to extend by a zero-grade field associated to any irreducible $\widehat{\mathfrak{gl}}(1|1)$-module, then we must include the rest of its zero-grade fields in the extension. Second, the fields we extend by should be closed under conjugation. Third, extending by fields from typical irreducibles will lead to logarithmic behaviour in the extended chiral algebra because fusing typicals with their conjugates yields the staggered indecomposable $\widehat{\mathcal{P}}_{0,0}$.

It seems then that the most tractable extensions will involve zero-grade fields from atypical modules $\widehat{\mathcal{A}}_{n,\ell}$ and their conjugates $\widehat{\mathcal{A}}_{-n,-\ell}$. The simplest extension we could hope for would involve a single atypical and its conjugate and have the further property that these extension fields generate no new fields at the level of the commutation relations. This may be achieved for extension fields of integer or half-integer conformal dimension by requiring that the operator product expansions of the zero-grade fields of $\widehat{\mathcal{A}}_{n,\ell}$ are regular. From the fusion rules (21), we obtain

$$\widehat{\mathcal{A}}_{n,\ell} \times \widehat{\mathcal{A}}_{n,\ell} = \widehat{\mathcal{A}}_{2n-\varepsilon(\ell),2\ell}, \tag{22}$$

from which it follows that the zero-grade fields of $\widehat{\mathcal{A}}_{n,\ell}$ will have regular operator product expansions with one another if $2\,\Delta_{n,\ell} \leq \Delta_{2n-\varepsilon(\ell),2\ell}$, that is, if

$$|\ell| \leq 2\,\Delta_{n,\ell}. \tag{23}$$

We may take ℓ positive without loss of generality. Further, we require that the conformal dimension of the extension fields be a positive half-integer (so $2n\ell \in \mathbb{Z}$). Equation (23) then implies that there are m distinct possibilities to extend by fields of dimension $m/2$. We denote by $\mathfrak{W}_{n,\ell}$ the algebra obtained upon extending $\widehat{\mathfrak{gl}}(1|1)$ by the atypical module $\widehat{\mathcal{A}}_{n,\ell}$ and its conjugate $\widehat{\mathcal{A}}_{-n,-\ell}$.

4.2 Characters of Extended Algebras

The complete extended algebra also contains normally-ordered products of the extension fields and their descendants. Indeed, the extended algebra $\mathfrak{W}_{n,\ell}$ may be identified, at least at the level of graded vector spaces, with the orbit of the $\widehat{\mathfrak{gl}}(1|1)$ vacuum module under fusion by the simple current modules $\widehat{\mathcal{A}}_{n,\ell}$ and $\widehat{\mathcal{A}}_{-n,-\ell}$. In other words,

$$\mathfrak{W}_{n+1/2,\ell} = \widehat{\mathcal{A}}_{0,0} \oplus \bigoplus_{m=1}^{\infty} \left(\widehat{\mathcal{A}}_{mn+1/2,m\ell} \oplus \widehat{\mathcal{A}}_{-mn-1/2,-m\ell}\right). \tag{24}$$

The character of the extended vacuum module is therefore

$$\chi_{\mathfrak{W}_{n+1/2,\ell}}(y;z;q) = \chi_{\widehat{\mathcal{A}}_{0,0}}(y;z;q) + \sum_{m=1}^{\infty}\Big[\chi_{\widehat{\mathcal{A}}_{mn+1/2,m\ell}}(y;z;q) + \chi_{\widehat{\mathcal{A}}_{-mn-1/2,-m\ell}}(y;z;q)\Big]$$
$$= z\sum_{m\in\mathbb{Z}} \frac{y^{m\ell}z^{mn}q^{(mn+1/2)m\ell+m^2\ell^2/2}}{1+zq^{m\ell}} \cdot \prod_{i=1}^{\infty} \frac{\left(1+zq^i\right)\left(1+z^{-1}q^{i-1}\right)}{\left(1-q^i\right)^2}. \tag{25}$$

Here, we have introduced an additional formal variable y in order to keep track of the eigenvalues of E_0/k. One can likewise identify the irreducible modules of the extended algebra with the other orbits of the extension modules. We will not consider these modules, their characters, nor their interesting modular properties here, but will return to this in a future publication.

4.3 Free Field Realisations

The affine Kac–Moody superalgebra $\widehat{\mathfrak{gl}}(1|1)$ has two well-known free field realizations, the standard Wakimoto realization [4] and one constructed from a pair of symplectic fermions, a Euclidean boson, and a Lorentzian boson [16]. An explicit equivalence between the two realisations was established in [23]. Here, we review the latter one.

We take the symplectic fermions $\chi^{\pm}$ and bosons Y, Z to have the following operator product expansions:

$$\chi^{+}(z)\chi^{-}(w) = \frac{1}{(z-w)^2} + \text{regular terms}, \quad \partial Y(z)\,\partial Z(w) = \frac{1}{(z-w)^2} + \text{regular terms} \tag{26}$$

(the others are regular). The $\widehat{\mathfrak{gl}}(1|1)$ current fields are then given by

$$E(z) = k\partial Y(z), \qquad N(z) = \partial Z(z), \qquad \psi^{\pm}(z) = \sqrt{k} : \mathsf{e}^{\pm Y(z)} : \chi^{\pm}(z), \tag{27}$$

and a moderately tedious computation shows that the $\widehat{\mathfrak{gl}}(1|1)$ energy momentum tensor (11) indeed corresponds to the sum of those of the bosonic and symplectic fermion systems.

It remains to construct the $\widehat{\mathfrak{gl}}(1|1)$ primaries that generate our extended algebras. As these correspond to atypical modules, this is relatively straight-forward. First, we introduce some convenient notation: Let $X_{n,\ell}$ be the bosonic linear combination $nY + \ell Z$ and define composite fields $F_r^{\pm}$, with $r \in \mathbb{N}$, by $F_0^{\pm} = 1$ and $F_r^{\pm} = \,: F_{r-1}^{\pm}\partial^{r-1}\chi^{\pm}:$ for $r \geq 1$. The conformal dimension of $F_r^{\pm}$ is then $\frac{1}{2}r(r+1)$. The zero-grade fields of the atypicals $\widehat{\mathcal{A}}_{n,\ell}$ for $\ell > 0$ have conformal dimension $\Delta_{n,\ell} = \ell(n+\ell/2)$ and are realised by

$$V_{n,\ell}^{+} = \,: \mathsf{e}^{X_{n+1/2,\ell}} : F_{\ell-1}^{-}, \qquad V_{n,\ell}^{-} = \,: \mathsf{e}^{X_{n-1/2,\ell}} : F_{\ell}^{-}. \tag{28}$$

This follows from their operator product expansions with the $\widehat{\mathfrak{gl}}(1|1)$ currents:

$$N(z)\,V^{\pm}_{n,\ell}(w) = \frac{(n\pm 1/2)\,V^{\pm}_{n,\ell}(w)}{z-w} + \dots, \qquad \psi^{+}(z)\,V^{-}_{n,\ell}(w) = (-1)^{\ell-1}\,\ell!\,\frac{\sqrt{k}V^{+}_{n,\ell}(w)}{z-w} + \dots,$$
$$E(z)\,V^{\pm}_{n,\ell}(w) = \frac{\ell k V^{\pm}_{n,\ell}(w)}{z-w} + \dots, \qquad \psi^{-}(z)\,V^{+}_{n,\ell}(w) = \frac{(-1)^{\ell-1}}{(\ell-1)!}\,\frac{\sqrt{k}V^{-}_{n,\ell}(w)}{z-w} + \dots, \tag{29}$$

the others being regular. The zero-grade fields of the conjugate module $\widehat{\mathcal{A}}_{-n,-\ell}$ are realised as

$$V^{+}_{-n,-\ell} = \,:\mathsf{e}^{X_{-n+1/2,-\ell}} : F^{+}_{\ell}, \quad V^{-}_{-n,-\ell} = \,:\mathsf{e}^{X_{-n-1/2,-\ell}} : F^{+}_{\ell-1}. \tag{30}$$

Their operator product expansions with the current fields are similar.

4.4 The Extended Operator Product Algebra

In order to compute the leading contributions to the extended algebra operator product expansions, we need the expansion of the bosonic vertex operators. To second order, this is

$$:\mathsf{e}^{X_{n,\ell}(z)}: \; :\mathsf{e}^{X_{n',\ell'}(w)}: = (z-w)^{n\ell'+n'\ell}\Big[:\mathsf{e}^{X_{n+n',\ell+\ell'}(w)}: + :\partial X_{n,\ell}(w)\,\mathsf{e}^{X_{n+n',\ell+\ell'}(w)}:\,(z-w)$$
$$+\frac{1}{2} : \Big(\partial X_{n,\ell}(w)\,\partial X_{n,\ell}(w) + \partial^2 X_{n,\ell}(w)\Big)\mathsf{e}^{X_{n+n',\ell+\ell'}(w)} : (z-w)^2 + \dots\Big]. \tag{31}$$

Note that it follows that $:\mathsf{e}^{X_{n,\ell}(w)}:$ and $:\mathsf{e}^{X_{n',\ell'}(w)}:$ will be mutually bosonic when $n\ell'+n'\ell$ is an even integer and mutually fermionic when $n\ell'+n'\ell$ is odd. The implication of this for the statistics of the extended algebra generators $V^{\pm}_{n,\ell}$ and $V^{\pm}_{-n,-\ell}$ is a little subtle. It turns out that when $2n\ell$ is even, these generators may be consistently assigned a bosonic or fermionic parity—$\mathfrak{W}_{n,\ell}$ is a superalgebra. In fact, $V^{+}_{n,\ell}$ and $V^{-}_{-n,-\ell}$ will be fermions and $V^{-}_{n,\ell}$ and $V^{+}_{-n,-\ell}$ will be bosons in this case. However, when $2n\ell$ is odd, such an assignment is impossible—$\mathfrak{W}_{n,\ell}$ is *not* a superalgebra. In this case, separately taking $V^{+}_{n,\ell}$ and $V^{-}_{-n,-\ell}$ to be bosons and $V^{-}_{n,\ell}$ and $V^{+}_{-n,-\ell}$ to be fermions is consistent, but the mutual locality of a boson and a fermion will now be -1 instead of $+1$. We will remark further on this subtlety in Sect. 4.5.

We moreover need the leading terms of certain operator product expansions of the $F^{\pm}_{r}$. In particular,

$$F_r^+(z)F_r^-(w) = (z-w)^{-r(r+1)}\left[\mu_r^{(0)}+\mu_{r-1}^{(2)} : \chi^+(w)\chi^-(w) : (z-w)^2+\dots\right],$$

$$F_{r-1}^-(z)F_r^+(w) = (z-w)^{-(r-1)(r+1)}\left[\mu_{r-1}^{(1)}\chi^+(w)+\dots\right],$$

$$F_r^-(z)F_{r-1}^+(w) = (z-w)^{-(r-1)(r+1)}\left[\mu_{r-1}^{(1)}\chi^-(w)+\dots\right], \tag{32}$$

where the coefficients $\mu_r^{(a)}$, for $a = 0, 1, 2$, are given by

$$\mu_r^{(a)} = \sum_{\sigma\in\mathsf{S}_r}(-1)^{|\sigma|}\prod_{i=1}^{r}(i+\sigma(i)+a-1)! = \prod_{i=1}^{r}(i-1)!(i+a)! \tag{33}$$

This last equality follows from recognising the $\mu_r^{(a)}$ as determinants of Hankel matrices for which LU-decompositions are easily found. In detail, consider the $r\times r$ matrix $A_r(a)$, for a non-negative integer a, with entries $(A_r(a))_{ij} = (i+j+a-1)!$ Defining $r\times r$ matrices $L_r(a)$ and $U_r(a)$ by

$$(L_r(a))_{ij} = \frac{(i+a)!}{(j+a)!}\binom{i-1}{j-1}, \qquad (U_r(a))_{ij} = (i-1)!(j+a)!\binom{j-1}{i-1}, \tag{34}$$

and noting that $L_r(a)$ is lower-triangular with diagonal entries equal to 1 and $U_r(a)$ is upper-triangular, we see that $L_r(a)U_r(a)$ is an LU-decomposition of $A_r(a)$:

$$\begin{aligned}(L_r(a)U_r(a))_{ij} &= \sum_{k=1}^{r}\frac{(i+a)!(i-1)!(j+a)!(j-1)!}{(k+a)!(k-1)!(i-k)!(j-k)!}\\ &= (j+a)!(i-1)!\sum_{k=1}^{r}\binom{i+a}{k+a}\binom{j-1}{k-1}\\ &= (j+a)!(i-1)!\binom{i+j+a-1}{i-1} = (A_r(a))_{ij}.\end{aligned} \tag{35}$$

Since $\det L_r(a) = 1$, we obtain $\det A_r(a) = \det U_r(a) = \prod_{i=1}^{r}(i-1)!(i+a)!$ and hence (33).

We are now in a position to obtain the leading contributions to the operator product expansions of the extension fields $V_{n,\ell}^{\pm}$ and their conjugates $V_{-n,-\ell}^{\mp}$. Since we assume (23), there are only four non-regular expansions and these take the form

$$V_{n,\ell}^+(z)V_{-n,-\ell}^+(w) = \frac{\mu_{\ell-1}^{(1)}\psi^+(w)/\sqrt{k}}{(z-w)^{2\Delta_{n,\ell}-1}}+\dots,$$

$$\begin{aligned}V_{-n,-\ell}^-(z)V_{n,\ell}^+(w) = \mu_{\ell-1}^{(0)}\Bigg[&\frac{1}{(z-w)^{2\Delta_{n,\ell}}}-\frac{\partial X_{n+1/2,\ell}(w)}{(z-w)^{2\Delta_{n,\ell}-1}}+\frac{\ell(\ell-1)}{2}\frac{:\chi^+(w)\chi^-(w):}{(z-w)^{2\Delta_{n,\ell}-2}}\\ &+\frac{1}{2}\frac{:\partial X_{n+1/2,\ell}(w)\partial X_{n+1/2,\ell}(w): -\partial^2 X_{n+1/2,\ell}(w)}{(z-w)^{2\Delta_{n,\ell}-2}}+\dots\Bigg],\end{aligned}$$

$$V^+_{-n,-\ell}(z)V^-_{n,\ell}(w) = \mu^{(0)}_\ell \left[\frac{1}{(z-w)^{2\Delta_{n,\ell}}} - \frac{\partial X_{n-1/2,\ell}(w)}{(z-w)^{2\Delta_{n,\ell}-1}} + \frac{\ell(\ell+1)}{2} \frac{:\chi^+(w)\chi^-(w):}{(z-w)^{2\Delta_{n,\ell}-2}} + \frac{1}{2} \frac{:\partial X_{n-1/2,\ell}(w)\partial X_{n-1/2,\ell}(w): - \partial^2 X_{n-1/2,\ell}(w)}{(z-w)^{2\Delta_{n,\ell}-2}} + \dots \right],$$

$$V^-_{n,\ell}(z)V^-_{-n,-\ell}(w) = \frac{\mu^{(1)}_{\ell-1}\psi^-(w)/\sqrt{k}}{(z-w)^{2\Delta_{n,\ell}-1}} + \dots \tag{36}$$

Here, we have used (33) to evaluate the ratios $\mu^{(2)}_{r-1}/\mu^{(0)}_r = \frac{1}{2}r(r+1)$ appearing in these expansions.

4.5 Examples

Let us now illustrate the results of the above calculations with a few simple examples. First, (23) tells us that the extended algebra $\mathfrak{W}_{n,\ell}$ will be unique if we insist that the extension fields have conformal dimension $\frac{1}{2}$. Indeed, this requires $\ell = 1$ and $n = 0$. We are therefore extending $\widehat{\mathfrak{gl}}(1|1)$ by the fields associated with the atypical modules $\widehat{\mathcal{A}}_{0,1}$ and $\widehat{\mathcal{A}}_{0,-1}$. Since $2n\ell = 0$ is even, the generators of the resulting extended algebra, $\mathfrak{W}_{0,1}$, may be assigned a definite parity: $\varkappa = V^+_{0,1}$ and $\bar{\varkappa} = V^-_{0,-1}$ are odd, $\beta = V^-_{0,1}$ and $\gamma = -V^+_{0,-1}$ are even. The expansions (36) become

$$\varkappa(z)\bar{\varkappa}(w) = \frac{1}{z-w} + N(w) + \frac{1}{2k}E(w) + \dots, \qquad \beta(z)\varkappa(w) = +\frac{\psi^+(w)}{\sqrt{k}} + \dots,$$
$$\beta(z)\gamma(w) = \frac{1}{z-w} + N(w) - \frac{1}{2k}E(w) + \dots, \qquad \gamma(z)\bar{\varkappa}(w) = -\frac{\psi^-(w)}{\sqrt{k}} + \dots, \tag{37}$$

which we recognise as a free complex fermion $(\varkappa, \bar{\varkappa})$ and a $\beta\gamma$ ghost system. Because the mixed operator product expansions are regular, $\mathfrak{W}_{0,1}$ decomposes into the direct sum of the chiral algebras of these theories.

If we choose to extend by dimension 1 fields, then there are two distinct choices: $n = \frac{1}{2}$ and $\ell = 1$ or $n = \frac{1}{2}$ and $\ell = -2$. We expect a current algebra symmetry in both cases. Indeed, if we set $\mathbf{H} = N + E/\ell k$ and $\mathbf{Z} = N - E/\ell k$, then we discover that the $(\mathbf{H}, \mathbf{Z})$-weights of the $\widehat{\mathfrak{gl}}(1|1)$ currents and the extension fields $V^\pm_{n,\ell}$, $V^\pm_{-n,-\ell}$ precisely match the $(\mathbf{H}, \mathbf{Z})$-weights of the adjoint representation of $\mathfrak{sl}(2|1)$.[4] Moreover, we have

[4]Here, $\mathbf{H}$ and $\mathbf{Z}$ should be associated with the matrices diag$\{1,-1,0\}$ and diag$\{1,1,2\}$ in the defining representation of $\mathfrak{sl}(2|1)$.

$$\mathbf{H}(z)\,\mathbf{H}(w) = \frac{2/\ell}{(z-w)^2} + \dots, \quad \mathbf{Z}(z)\,\mathbf{Z}(w) = \frac{-2/\ell}{(z-w)^2} + \dots, \tag{38}$$

and $\mathbf{H}(z)\,\mathbf{Z}(w)$ regular, which suggests that the extended algebra will be $\widehat{\mathfrak{sl}}(2|1)$ at level $1/\ell$.

Checking this for the choice $\ell = -2$ is easy. As $2n\ell = -2$ is even, $\mathfrak{W}_{1/2,-2}$ admits a superalgebra structure. Moreover, the fusion rules

$$\widehat{\mathcal{A}}_{0,1} \times \widehat{\mathcal{A}}_{0,1} = \widehat{\mathcal{A}}_{-1/2,2}, \quad \widehat{\mathcal{A}}_{0,-1} \times \widehat{\mathcal{A}}_{0,-1} = \widehat{\mathcal{A}}_{1/2,-2} \tag{39}$$

imply that $\mathfrak{W}_{1/2,-2}$ is a subalgebra of the extended algebra $\mathfrak{W}_{0,1}$ considered above. One readily checks that by taking normally-ordered products, the $\beta\gamma$ ghost fields of $\mathfrak{W}_{0,1}$ generate the bosonic subalgebra $\widehat{\mathfrak{sl}}(2)_{-1/2} \subset \widehat{\mathfrak{sl}}(2|1)_{-1/2}$, the complex fermion gives the $\widehat{\mathfrak{u}}(1)$-subalgebra, and the mixed products yield the remaining fermionic currents. This establishes the superalgebra isomorphism $\mathfrak{W}_{1/2,-2} \cong \widehat{\mathfrak{sl}}(2|1)_{-1/2}$.

The computation when $\ell = 1$ is, however, more subtle because $2n\ell = 1$ is odd, so $\mathfrak{W}_{1/2,1}$ does not admit the structure of a superalgebra. To impose the correct parities on the extended algebra currents, we must adjoin an operator-valued function μ which is required to satisfy

$$\mu_{a,b}\mu_{c,d} = (-1)^{ad}\mu_{a+b,c+d}, \quad (a,b,c,d \in \mathbb{Z}). \tag{40}$$

Note that the algebra generated by these operators has unit $\mu_{0,0}$. The currents are then given by

$$\begin{aligned} \mathbf{E} &= +\mu_{1,1}V^{+}_{1/2,1}, & \mathbf{H} &= N + E/k, & \mathbf{e}^{+} &= -\mu_{1,0}\psi^{+}/\sqrt{k}, & \mathbf{f}^{+} &= \mu_{0,-1}V^{+}_{-1/2,-1}, \\ \mathbf{F} &= -\mu_{-1,-1}V^{-}_{-1/2,-1}, & \mathbf{Z} &= N - E/k, & \mathbf{f}^{-} &= +\mu_{-1,0}\psi^{-}/\sqrt{k}, & \mathbf{e}^{-} &= \mu_{0,1}V^{-}_{1/2,1}, \end{aligned} \tag{41}$$

and routine computation now verifies that these currents indeed generate $\widehat{\mathfrak{sl}}(2|1)_1$.

As our final example, we briefly consider the case of extensions of conformal dimension $\frac{3}{2}$. There are now three distinct choices, corresponding to $n = 1$, $\ell = 1$, or $n = -\frac{1}{4}$, $\ell = 2$, or $n = -1$, $\ell = 3$. The latter choice again results in an extended algebra which is a subalgebra of $\mathfrak{W}_{0,1}$ because

$$\widehat{\mathcal{A}}_{0,1} \times \widehat{\mathcal{A}}_{0,1} \times \widehat{\mathcal{A}}_{0,1} = \widehat{\mathcal{A}}_{-1,3}. \tag{42}$$

Both $\mathfrak{W}_{1,1}$ and $\mathfrak{W}_{-1,3}$ are superalgebras, while $\mathfrak{W}_{-1/4,2}$ is not. We expect, however, that a modification similar to (40) will restore the superalgebra parity requirements. We will not analyse this in any detail as our interest in $\Delta_{n,\ell} = \frac{3}{2}$ lies not with the full extended algebra, but rather with one of its subalgebras.

We start with the superalgebras $\mathfrak{W}_{1,1}$ and $\mathfrak{W}_{-1,3}$. Both $V^{+}_{-n,-\ell}$ and $V^{-}_{n,\ell}$ are bosonic and upon defining

$$\mathsf{g}^{+} = \sqrt{\frac{3\alpha\left(3\alpha-1\right)}{2\mu_{\ell}^{(0)}}}\,V^{+}_{-n,-\ell}, \qquad \mathsf{g}^{-} = \sqrt{\frac{3\alpha\left(3\alpha-1\right)}{2\mu_{\ell}^{(0)}}}\,V^{-}_{n,\ell},$$
$$\mathsf{j} = -\alpha\partial X_{n-1/2,\ell}, \quad \mathsf{t} = \frac{\alpha}{2} :\partial X_{n-1/2,\ell}\partial X_{n-1/2,\ell}: - \frac{\ell\left(\ell+1\right)}{2}\frac{\alpha\left(3\alpha-1\right)}{\alpha+1}\frac{:\psi^{+}\psi^{-}:}{k} \tag{43}$$

where

$$\alpha = \frac{1}{\left(2n-1\right)\ell}, \tag{44}$$

we obtain the defining relations of the *Bershadsky–Polyakov algebra* $W_3^{(2)}$ [24, 25]:

$$\begin{aligned}
\mathsf{g}^{+}\left(z\right)\mathsf{g}^{-}\left(w\right) &= \frac{\left(K+1\right)\left(2K+3\right)}{\left(z-w\right)^{3}} + \frac{3\left(K+1\right)\mathsf{j}\left(w\right)}{\left(z-w\right)^{2}} \\
&\quad + \frac{3:\mathsf{jj}:\left(w\right) + \frac{3}{2}\left(K+1\right)\partial\mathsf{j}\left(w\right) - \left(K+3\right)\mathsf{t}\left(w\right)}{z-w} + \ldots, \\
\mathsf{j}\left(z\right)\mathsf{g}^{\pm}\left(w\right) &= \frac{\pm\mathsf{g}^{\pm}\left(w\right)}{z-w} + \ldots, \qquad \mathsf{j}\left(z\right)\mathsf{j}\left(w\right) = \frac{\left(2K+3\right)/3}{\left(z-w\right)^{2}} + \ldots, \\
\mathsf{t}\left(z\right)\mathsf{g}^{\pm}\left(w\right) &= \frac{3}{2}\frac{\mathsf{g}^{\pm}\left(w\right)}{\left(z-w\right)^{2}} + \frac{\partial\mathsf{g}^{\pm}\left(w\right)}{z-w} + \ldots, \qquad \mathsf{t}\left(z\right)\mathsf{j}\left(w\right) = \frac{\mathsf{j}\left(w\right)}{\left(z-w\right)^{2}} + \frac{\partial\mathsf{j}\left(w\right)}{z-w} + \ldots, \\
\mathsf{t}\left(z\right)\mathsf{t}\left(w\right) &= \frac{-\left(2K+3\right)\left(3K+1\right)/2\left(K+3\right)}{\left(z-w\right)^{4}} + \frac{2\mathsf{t}\left(w\right)}{\left(z-w\right)^{2}} + \frac{\partial\mathsf{t}\left(w\right)}{z-w} + \ldots
\end{aligned} \tag{45}$$

Here, the $\widehat{\mathfrak{sl}}(3)$-level $K = \frac{3}{2}\left(\alpha-1\right)$ is 0 for $\mathfrak{W}_{1,1}$ and $-\frac{5}{3}$ for $\mathfrak{W}_{-1,3}$. The central charge of the $W_3^{(2)}$-subalgebra is in both cases -1.

For $\mathfrak{W}_{-1/4,2}$, this procedure does not yield a Bershadsky–Polyakov algebra because $V^{+}_{-n,-\ell}$ and $V^{-}_{n,\ell}$ are, in this case, mutually fermionic. Rather, these fields generate a copy of the $\mathcal{N}=2$ superconformal algebra of central charge -1. Instead, we must consider the mutually bosonic fields $V^{+}_{n,\ell}$ and $V^{-}_{-n,-\ell}$. Taking

$$\mathsf{g}^{+} = \sqrt{3}\,V^{-}_{1/4,-2}, \quad \mathsf{g}^{-} = \sqrt{3}\,V^{+}_{-1/4,2}, \quad \mathsf{j} = -\partial X_{1/4,2},$$
$$\mathsf{t} = \frac{1}{2}:\partial X_{1/4,2}\partial X_{1/4,2}: - \frac{1}{k}:\psi^{+}\psi^{-}: \tag{46}$$

in particular, now leads to the Bershadsky–Polyakov algebra of level 0 and central charge -1. (In contrast, $V^{+}_{n,\ell}$ and $V^{-}_{-n,-\ell}$ are fermionic in both $\mathfrak{W}_{1,1}$ and $\mathfrak{W}_{-1,3}$, generating copies of the $\mathcal{N}=2$ superconformal algebra with central charges 1 and -1, respectively.)

4.6 $W_N^{(2)}$-Subalgebras

In the previous section, we found the Bershadsky–Polyakov algebra $W_3^{(2)}$, at certain levels, appearing as a subalgebra of the extended algebras $\mathfrak{W}_{1,1}$, $\mathfrak{W}_{-1/4,2}$ and $\mathfrak{W}_{-1,3}$. We now generalise this observation. The algebra $W_3^{(2)}$ is defined [24, 25] as the Drinfel'd–Sokolov reduction of $\widehat{\mathfrak{sl}}(3)$ corresponding to the non-principal embedding of $\mathfrak{sl}(2)$ in $\mathfrak{sl}(3)$. Feigin and Semikhatov [17] found that it could also be realised as a subalgebra of $\widehat{\mathfrak{sl}}(3|1) \oplus \widehat{\mathfrak{u}}(1)$ commuting with an $\widehat{\mathfrak{sl}}(3)$-subalgebra. They then studied a generalisation $W_N^{(2)} \subset \widehat{\mathfrak{sl}}(N|1) \oplus \widehat{\mathfrak{u}}(1)$ which commutes with the obvious $\widehat{\mathfrak{sl}}(N)$-subalgebra.

When $N = 1$, these generalisations reduce to the chiral algebra of the $\beta\gamma$ ghost system. For $N = 2$, one gets $\widehat{\mathfrak{sl}}(2)$, and as mentioned above, $N = 3$ recovers the Bershadsky–Polyakov algebra. The examples studied in Sect. 4.5 therefore lead us to the plausible conjecture that the $W_N^{(2)}$ algebras of Feigin and Semikhatov may be realised, at least for certain levels, as subalgebras of certain of our extended algebras $\mathfrak{W}_{n,\ell}$. We mention that there is a second construction of these $W_N^{(2)}$ algebras, but restricted to the critical level $K = -N$ (see (48)), starting from the affine superalgebra $\widehat{\mathfrak{psl}}(N|N)$ at (critical) level 0 [26].

Feigin and Semikhatov only computed the first few terms of the defining operator product expansions of $W_N^{(2)}$. We will compare these terms with those obtained from our extended algebras, finding decidedly non-trivial agreement. Our findings will, however, be stated as conjectures because the full operator product expansion of $W_N^{(2)}$ is not currently known. $W_N^{(2)}$ is generated by two fields $\mathcal{E}_N^{\pm}$ of dimension $\frac{1}{2}N$, a $\widehat{\mathfrak{u}}(1)$-current $\mathcal{H}_N$ and an energy-momentum tensor $\mathcal{T}_N$. The defining expansions are:

$$\mathcal{H}_N(z)\,\mathcal{H}_N(w) = \frac{(N-1)K/N+N-2}{(z-w)^2}+\dots, \qquad \mathcal{H}_N(z)\,\mathcal{E}_N^{\pm}(w) = \pm\frac{\mathcal{E}_N^{\pm}(w)}{z-w}+\dots,$$

$$\begin{aligned}\mathcal{E}_N^{+}(z)\,\mathcal{E}_N^{-}(w) = {} & \frac{\lambda_{N-1}}{(z-w)^N}+\frac{N\lambda_{N-2}\mathcal{H}_N(w)}{(z-w)^{N-1}}-\frac{(K+N)\,\lambda_{N-3}\mathcal{T}_N(w)}{(z-w)^{N-2}} \\ & +\frac{\lambda_{N-3}}{(z-w)^{N-2}}\left[\frac{N(N-1)}{2}:\mathcal{H}_N\mathcal{H}_N:(w)\right. \\ & \left.+\frac{N\big((N-2)(K+N-1)-1\big)}{2}\partial\mathcal{H}_N(w)\right]+\dots \end{aligned} \tag{47}$$

Here, $\lambda_m = \prod_{i=1}^{m}\big(i(K+N-1)-1\big)$, K is the level of the $W_N^{(2)}$ algebra, and the central charge is given by

$$C = -\frac{\big((K+N)(N-1)-N\big)\big((K+N)(N-2)N-N^2+1\big)}{K+N}. \tag{48}$$

Suppose first that $2n\ell$ is even, so we can consider the bosonic subalgebra generated by the fields

$$\mathcal{E}_N^+ = \sqrt{\frac{\lambda_{N-1}}{\mu_\ell^{(0)}}} V_{-n,-\ell}^+, \qquad \mathcal{E}_N^- = \sqrt{\frac{\lambda_{N-1}}{\mu_\ell^{(0)}}} V_{n,\ell}^-. \tag{49}$$

Evaluating the operator product expansion of these fields using (36) and comparing with (47), we find that the first two singular terms agree provided that $N = 2\Delta_{n,\ell}$ and $\mathcal{H}_N = -\partial X_{n-1/2,\ell}/(2n-1)\ell$. This also fixes the $W_N^{(2)}$ level K. Comparing the third terms fixes the form of the $W_N^{(2)}$ energy-momentum tensor $\mathcal{T}_N$ and $\mathcal{H}_N$ is then verified to have dimension 1. However, the $\mathcal{E}_N^\pm$ only have the required dimension $\frac{1}{2}N = \Delta_{n,\ell}$ if $n = 1$ or $2n+\ell = 1$.[5] These constraints also let us check that $\mathcal{T}_N$ is an energy-momentum tensor and the central charge turns out to be $C = -1$. When $2n\ell$ is odd, we instead consider the bosonic subalgebra generated by

$$\mathcal{E}_N^+ = \sqrt{\frac{\lambda_{N-1}}{\mu_{\ell-1}^{(0)}}} V_{-n,-\ell}^-, \qquad \mathcal{E}_N^- = \sqrt{\frac{\lambda_{N-1}}{\mu_{\ell-1}^{(0)}}} V_{n,\ell}^+. \tag{50}$$

A similar analysis reveals that this subalgebra agrees with $W_N^{(2)}$ up to the first three terms in the operator product expansions provided that $N = 2\Delta_{n,\ell}$ and either $\ell = 1$ or $\ell = 2$.[6] In the first case, $C = 1$; in the second, $C = -1$.

We summarise our findings as follows:

Conjecture 1. The extended algebra $\mathfrak{W}_{n,\ell}$ has a subalgebra isomorphic to $W_N^{(2)}$ of level K when:

- $\ell = 1$ and $n = 0, 1, 2, \ldots$ Then, $N = 2n+1$ and $K = -2(n-1)(2n+1)/(2n-1)$.
- $\ell = 1$ and $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ Then, $N = 2n+1$ and $K = -\left(2n^2-1\right)/n$.
- $\ell = 2$ and $n = -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \ldots$ Then, $N = 4(n+1)$ and $K = -2(n+1)(4n+1)/(2n+1)$.
- $n = -\frac{1}{2}(\ell-1)$ and $\ell = 1, 2, 3, \ldots$ Then, $N = \ell$ and $K = -\left(\ell^2-\ell-1\right)/\ell$.

Note that the examples considered in Sect. 4.5 exhaust the $W_N^{(2)}$-subalgebras with $N \leq 3$ except for $\ell = 2$ and $n = -\frac{3}{4}$. This latter case is excluded if one insists, as we did with (23), that the operator product expansion of $\mathcal{E}^\pm$ with itself is regular. We mention that Feigin and Semikhatov actually computed the first *four* terms of the $W_N^{(2)}$ operator product expansions, finding in the fourth term a Virasoro primary field $\mathcal{W}_N$ of dimension 3 and $\mathcal{H}_N$-weight 0. We have extended (31), (32) and (36)

[5] There is a third solution, $\Delta_{n,\ell} + \ell + 1 = 0$, but this is invalid as we require $\ell, \Delta_{n,\ell} > 0$.

[6] Taking $n = -\frac{1}{2}(\ell+1)$ also satisfies these requirements, but then $2n\ell$ is necessarily even. Moreover, there is again a solution of the form $\Delta_{n,\ell} - \ell + 1 = 0$, but it is easy to check that it leads to the wrong operator product expansion of $\mathcal{T}_N$ with itself.

to compute $\mathcal{W}_N$ in our extended algebras and have checked that for each ℓ and n appearing in our conjecture, this field indeed has the required properties. It follows that our conjecture has been verified for all $N \leq 4$.

References

1. Creutzig, T., Ridout, D.: Relating the Archetypes of Logarithmic Conformal Field Theory. (arXiv:1107.2135 [hep-th])
2. Rozansky, L., Saleur, H.: Quantum field theory for the multivariable Alexander-Conway polynomial. Nucl. Phys. **B376**, 461–509 (1992)
3. Rozansky, L., Saleur, H.: S and T matrices for the super $U(1,1)$ WZW model: application to surgery and three manifolds invariants based on the Alexander-Conway polynomial. Nucl. Phys. **B389**, 365–423 (1993) (arXiv:hep-th/9203069)
4. Saleur, H., Schomerus, V.: The $GL(1 \mid 1)$ WZW model: from supergeometry to logarithmic CFT. Nucl. Phys. **B734**, 221–245 (2006) (arXiv:hep-th/0510032)
5. Creutzig, T., Quella, T., Schomerus, V.: Branes in the $GL(1|1)$ WZNW-Model. Nucl. Phys. **B792**, 257–283 (2008) (arXiv:0708.0583 [hep-th])
6. Creutzig, T., Schomerus, V.: Boundary correlators in supergroup WZNW models. Nucl. Phys. **B807**, 471–494 (2009) (arXiv:0804.3469 [hep-th])
7. Creutzig, T.: Branes in Supergroups. Ph.D. thesis, DESY Theory Group, 2009 (arXiv:0908.1816 [hep-th])
8. Creutzig, T., Rønne, P.: From world-sheet supersymmetry to super target spaces. J. High Energy Phys. **1011**, 021 (2010) (arXiv:1006.5874 [hep-th])
9. Ridout, D.: $\widehat{\mathfrak{sl}}(2)_{-1/2}$ and the Triplet Model. Nucl. Phys. **B835**, 314–342 (2010) (arXiv:1001.3960 [hep-th])
10. Ridout, D.: $\widehat{\mathfrak{sl}}(2)_{-1/2}$: a Case Study. Nucl. Phys. **B814**, 485–521 (2009) (arXiv:0810.3532 [hep-th])
11. Ridout, D.: Fusion in fractional level $\widehat{\mathfrak{sl}}(2)$-theories with $k = -\frac{1}{2}$. Nucl. Phys. **B848**, 216–250 (2011) (arXiv:1012.2905 [hep-th])
12. Gaberdiel, M., Kausch, H.: A rational logarithmic conformal field theory. Phys. Lett. **B386**, 131–137 (1996) (arXiv:hep-th/9606050)
13. Kausch, H.: Symplectic fermions. Nucl. Phys. **B583**, 513–541 (2000) (arXiv:hep-th/0003029)
14. Mathieu, P., Ridout, D.: The extended algebra of the $SU(2)$ Wess-Zumino-Witten models. Nucl. Phys. **B765**, 201–239 (2007) (arXiv:hep-th/0609226)
15. Mathieu, P., Ridout, D.: The extended algebra of the minimal models. Nucl. Phys. **B776**, 365–404 (2007) (arXiv:hep-th/0701250)
16. Guruswamy, S., LeClair, A., Ludwig, A.: $gl(N|N)$ super current algebras for disordered Dirac fermions in two-dimensions. Nucl. Phys. **B583**, 475–512 (2000) (arXiv:cond-mat/9909143)
17. Feigin, B., Semikhatov, A.: $W_n^{(2)}$ algebras. Nucl. Phys. **B698**, 409–449 (2004) (arXiv:math/0401164)
18. Kytölä, K., Ridout, D.: On staggered indecomposable Virasoro modules. J. Math. Phys. **50**, 123503 (2009) (arXiv:0905.0108 [math-ph])
19. Gotz, G., Quella, T., Schomerus, V.: Representation theory of $\mathfrak{sl}(2|1)$. J. Algebra **312**, 829–848 (2007) (arXiv:hep-th/0504234)
20. Nahm, W.: Quasirational fusion products. Int. J. Mod. Phys. **B8**, 3693–3702 (1994) (arXiv:hep-th/9402039)
21. Gaberdiel, M., Kausch, H.: Indecomposable fusion products. Nucl. Phys. **B477**, 293–318 (1996) (arXiv:hep-th/9604026)
22. Quella, T., Schomerus, V.: Free fermion resolution of supergroup WZNW models. J. High Energy Phys. **0709**, 085 (2007) (arXiv:0706.0744 [hep-th])

23. Creutzig, T., Rønne, P: The $GL(1|1)$-symplectic fermion correspondence. Nucl. Phys. **B815**, 95–124 (2009) (arXiv:0812.2835 [hep-th])
24. Polyakov, A.: Gauge transformations and diffeomorphisms. Int. J. Mod. Phys. **A5**, 833–842 (1990)
25. Bershadsky, M.: Conformal field theories via Hamiltonian reduction. Commun. Math. Phys. **139**, 71–82 (1991)
26. Creutzig, T., Gao, P., Linshaw, A.: A commutant realization of $W_n^{(2)}$ at critical level. Int. Math. Res. Not. (arXiv:1109.4065[math]) doi:10.1093/imrn/rns229

Non-Local Space-Time Transformations Generated from the Ageing Algebra

Stoimen Stoimenov and Malte Henkel

Abstract The ageing algebra is a local dynamical symmetry of many ageing systems, far from equilibrium, and with a dynamical exponent $z = 2$. Here, new representations for an integer dynamical exponent $z = n$ are constructed, which act non-locally on the physical scaling operators. The new mathematical mechanism which makes the infinitesimal generators of the ageing algebra dynamical symmetries, is explicitly discussed for a n-dependent family of linear equations of motion for the order-parameter. Finite transformations are derived through the exponentiation of the infinitesimal generators and it is proposed to interpret them in terms of the transformation of distributions of spatio-temporal coordinates. The two-point functions which transform co-variantly under the new representations are computed, which quite distinct forms for n even and n odd. Depending on the sign of the dimensionful mass parameter, the two-point scaling functions either decay monotonously or in an oscillatory way towards zero.

1 Introduction

Non-relativistic space-time transformations have recently met with a lot of interest. In addition to fields such as hydrodynamics [27, 33], they have been playing an increasing rôle in the analysis of the long-time behaviour of strongly interacting many-body systems far from equilibrium [9, 21] and even more recently have

S. Stoimenov (✉)
Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, BG–1784 Sofia, Bulgaria
e-mail: spetrov@inrne.bas.bg

M. Henkel
Groupe de Physique Statistique, Département de Physique de la Matière et des Matériaux, Institut Jean Lamour, Nancy Université, B.P. 70239, F–54506 Vandœuvre lès Nancy Cedex, France
e-mail: henkel@lpm.u-nancy.fr

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_25,

discussed in non-relativistic limits of the AdS/CFT correspondence [1, 2, 23, 25]. Physically interesting sets of space-time transformations are these which define some kind of conformal invariance and recently, a classification of non-relativistic conformal space-time transformations was presented [11]. Indeed, the list of sets of admissible generators (with $0 < z < \infty$), which close into a Lie algebra is a rather short one: the *conformal algebra* itself, in $d+1$ dimensions, with $z=1$; the *conformal galilean algebra* $\mathrm{CGA}(d)$, first identified in [14], poses representations with $z=1$ and $z=2$ [20]; the *Schrödinger algebra* with dynamical exponent $z=2$ [13, 22, 24, 26] and its subalgebra (with time-translation left out) known as *ageing algebra* [7, 9, 15, 21] (for the full list see [11, 30]). This short list illustrates the difficulty of constructing sets of "conformal" space-time transformations for a generic dynamical exponent $z \neq 1,2$. It is at present not fully understood how to construct a dynamical symmetry (beyond the obvious translation, dilatation and rotation symmetries) even for a simple linear equation of the form (where $z \neq 1,2$)

$$S\psi(t,r) := \left(z\mu\partial_t - \partial_r^z\right)\psi(t,r) = 0 \tag{1}$$

which arises as one of the most simple equations of motion of the order-parameter in studies of ageing far from equilibrium [6, 8]. Indeed, current attempts to find further dynamical symmetries of (1) beyond the obvious translation, dilatation and rotations (if $d > 1$) only succeed at the price that the further generators (build by a fractional derivatives) must be required to vanish on certain states [16, 17, 21]. In the context of statistical physics, the order-parameter does not really satisfy a deterministic equation, but rather the r.h.s of (1) is replaced by a random noise term, which leads to a Langevin equation. However, since the non-relativistic algebras mentioned above are all non-semi-simple and their representations are projective, it is possible to study first the symmetries of the deterministic equation (1) and then use the resulting Bargman super-selection rules [3] in order to reduce the calculation of any average to the calculation of averages within the deterministic part of the theory as defined by (1) [28]. This procedure works not only for thermal noises and a simple diffusion equation with $z=2$, but can be generalised to generic values of z and fairly general noises, such as they may arise in reaction-diffusion systems [4–6, 10, 29], see [21] for a systematic presentation.

In this paper,[1] we shall explore properties of a new kind of representations of the common sub-algebra $age(d)$ of the Schrödinger algebra. The standard representation in $d=1$ space dimensions, on sufficiently differentiable space-time functions $f(t,r)$, of the Lie algebra $age(1) := \left\langle X_{0,1}, Y_{\pm\frac{1}{2}}, M_0\right\rangle$ is given by

$$\begin{aligned} X_0 &= -t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2}, \quad X_1 = -t^2\partial_t - tr\partial_r - \frac{\mathscr{M}}{2}r^2 - (x+\xi)t \\ Y_{-\frac{1}{2}} &= -\partial_r, \quad Y_{\frac{1}{2}} = -t\partial_r - \mathscr{M}r, \quad M_0 = -\mathscr{M} \end{aligned} \tag{2}$$

[1] This paper contains the main results from the original one [30] which the first author presented on LT-9 conference

with the non-vanishing commutators given by

$$\left[X_0, Y_{\pm\frac{1}{2}}\right] = \mp\frac{1}{2}Y_{\pm\frac{1}{2}} \ , \ \left[X_0, X_1\right] = -X_1 \ , \ \left[Y_{\frac{1}{2}}, Y_{-\frac{1}{2}}\right] = M_0. \tag{3}$$

This representation is characterised by the "mass" $\mathscr{M}$ and the pair of scaling dimensions (x, ξ) whose values depend on the scaling operator on which these generators act. If one defines the Schrödinger operator

$$S := 2\mathscr{M}\partial_t - \partial_r^2 + 2\mathscr{M}\left(x + \xi - \frac{1}{2}\right)\frac{1}{t} \tag{4}$$

then the equation $S\psi(t,r) = 0$ has $age(1)$ as dynamical symmetry (see [30]). A physical example for (4) is given by the relaxation kinetics of the spherical model, or equivalently the $N \to \infty$ limit of the O(N) model, after a quench to a temperature $T \leq T_c$ at or below its critical temperature $T_c > 0$ [12]. The representation (2) has a dynamical exponent $z = 2$ and acts *locally* on the space-time coordinates. While the time-translations are not included, a system with an *age*-symmetry is not at a stationary state. The scaling dimension ξ arises as a further universal characteristics of the relaxation process [19]. When trying to extend (2) to a representation of $\text{CGA}(1) \supset age(1)$, the extra generators are not necessarily first-order differential operators [18]. For this reason, and in order to find further representations of $age(1)$ with different values of z, we shall give in Sect. 2 *non-local* representations of $age(1)$, which admit any integer value $z = n \in N$. In Sect. 3, we address the question how to interpret geometrically such infinitesimal generators. Next, in Sect. 4, we derive the co-variant two-point functions which depend strongly on the parity of n.

2 Non-Local Representation of the Ageing Algebra *age*(1)

Consider a dynamical exponent with integer values $2 \leq z = n \in N$. The generators of $age(1)$ we are interested in take the form

$$X_0 = -\frac{n}{2}t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2}, \quad X_1 = -\frac{n}{2}t^2\partial_t\partial_r^{n-2} - tr\partial_r^{n-1} - \frac{1}{2}\mu r^2 - (x+\xi)t\partial_r^{n-2}$$
$$Y_{-\frac{1}{2}} = -\partial_r, \quad Y_{\frac{1}{2}} = -t\partial_r^{n-1} - \mu r, \quad M_0 = -\mu \tag{5}$$

and satisfy the commutators of $age(1)$. However, there is a notable exception, namely $\left[X_1, Y_{\frac{1}{2}}\right] = \frac{n-2}{2}t^2\partial_r^{n-3}S$ with the Schrödinger operator

$$S := n\mu\frac{\partial}{\partial t} - \frac{\partial^n}{\partial r^n} + 2\mu\left(x + \xi - \frac{n-1}{2}\right)\frac{1}{t}. \tag{6}$$

The generators (5) form a dynamical symmetry of the Schrödinger equation $S\psi(t,r) = 0$, as can be seen from the commutators

$$\left[S, Y_{\pm\frac{1}{2}}\right] = \left[S, M_0\right] = 0 \ , \ \left[S, X_0\right] = -\frac{n}{2}S \ , \ \left[S, X_1\right] = -nt\partial_r^{n-2}S. \qquad (7)$$

In order to close the representation (5), we must restrict the function space *modulo* solutions of $S\psi = 0$. Then a natural function space for our purposes is $\mathscr{F} := C^1\left(R_+, C^n(R)\right) / \sim = C^1\left(R_+, H^n(R)\right) / \sim$ [30]. Restricted to the space $\mathscr{F}$, the generators (5) give for each integer $n > 2$ a non-local representation of $age(1)$ which is a dynamical symmetry of the Schrödinger equation $S\psi = 0$.

3 Finite Transformations

Besides the usual local generators of dilatations X_0, of spatial translations $Y_{-\frac{1}{2}}$ and of phase shifts M_0, the representation (5) also contains the non-local generators $Y_{\frac{1}{2}}, X_1$ whose effect cannot be interpreted as a simple space-time coordinate transformation $t \mapsto t'(t,r)$, $r \mapsto r'(t,r)$. On the other hand, we can still write the formal Lie series $F(\varepsilon,t,r) = e^{-\varepsilon Y_{1/2}}F(0,t,r)$ and $F(\varepsilon,t,r) = e^{-\varepsilon X_1}F(0,t,r)$. They are given as the solutions of the two initial-value problems

$$\left(\partial_\varepsilon - t\partial_r^{n-1} - \mu r\right)0\varepsilon,t,r) = 0, \ \ F(0,t,r) = \phi(t,r) \qquad (8)$$

$$\left(\partial_\varepsilon - \frac{n}{2}t^2\partial_t\partial_r^{n-2} - tr\partial_r^{n-1} - xt\partial_r^{n-2} - \frac{1}{2}\mu r^2\right)F(\varepsilon,t,r) = 0, \ \ F(0,t,r) = \phi(t,r) \qquad (9)$$

such that the initial function $\phi \in \mathscr{F}$.

In Tables 1 and 2, we illustrate these Lie series for the choices $\phi(t,r) = t^m$ and $\phi(t,r) = r^k$ with $m \in N$ and $1 \le k \le n-1$, which for $\mu = 0$ solve the Schrödinger equation $S\phi(t,r) = 0$. Comparison with the local Galilei- and special Schrödinger transformation shows important differences. For example, although the spatial coordinate r is left invariant by both generators when $n > 2$, this does not imply that

Table 1 Comparison of the finite transformations $e^{-\varepsilon Y_{1/2}}\phi(t,r)$ for the generalised, non-local Galilei-transformation when $z = n > 2$ with the standard local Galilei-transformation for $z = n = 2$. The initial distribution $\phi \in \mathscr{F}$ and $\mu = 0$

$\phi(t,r)$	Non-local, $n > 2$	Local, $n = 2$	
t^m	t^m	t^m	$m \in N$
r^k	r^k	$(r + t\varepsilon)^k$	$1 \le k \le n-2$
r^{n-1}	$r^{n-1} + (n-1)!t\varepsilon$	$\left(r + t\varepsilon\right)^{n-1}$	

Table 2 Comparison of the finite transformations $e^{-\varepsilon X_1}\phi(t,r)$ for the generalised, non-local special Schrödinger-transformation when $z=n=3$ or 4 with the standard local special Schrödinger-transformation for $z=n=2$. The initial distribution $\phi \in \mathscr{F}$ and $\mu=0$

	Non-local		Local	
$\phi(t,r)$	$n=3$	$n=4$	$n=2$	
t^m	t^m	t^m	$t^m/(1-t\varepsilon)^{m+x+\xi}$	$m\in N$
r	$r+(x+\xi)t\varepsilon$	r	$r/(1-t\varepsilon)^{1+x+\xi}$	
r^2	$r^2+2(x+\xi+1)tr\varepsilon$ $+\frac{1}{2}(x+\xi+1)(2x+2\xi+3)t^2\varepsilon^2$	$r^2+2(x+\xi)t\varepsilon$	$r/(1-t\varepsilon)^{2+x+\xi}$	
r^3	——	$r^3+6(x+\xi+1)tr\varepsilon$	$r/(1-t\varepsilon)^{3+x+\xi}$	

these generators would not generate any spatial transformation, as we see from the transformation behaviour of the higher powers of r. While in the local case $n=2$, the transformation of the powers r^k is simply given by taking the corresponding power of the transformation law of r itself, this is no longer true in the non-local cases $n>2$. The action of the generators $Y_{\frac{1}{2}}$ and X_1, in our example, look reminiscent to a transformation of a statistical distribution, where the first moment happens to be invariant, but the higher ones change. Therefore, these examples suggest that a better interpretation might be to consider a transformation of an initial distribution of spatial (or temporal) coordinates, where $\phi(t,r)$ would then take the rôle of a distribution function. Next we give further results on the transformation of $\phi(t,r)$ and discuss possible consequences for an interpretation on two cases $z=3$ and $z=4$. These are the values of z in the Bray–Rutenberg theory of the growth of the relevant time-dependent length scale $L(t)\sim t^{1/z}$ in O(n)-symmetric systems with a conserved order parameter and quenched to $T<T_c$ [7].

(i). $\underline{z=n=3}$

We now give the full transformation laws of the distribution $\phi(t,r)$. We begin with the generalised Galilei transformation (8):

$$F(\varepsilon,t,r)=\frac{1}{\sqrt{4\pi t\varepsilon}}\int_R \mathrm{d}r'\,\phi(t,r')e^{-\frac{1}{4t\varepsilon}\left(\left(r-r'-t\mu\varepsilon^2\right)^2-4\mu t r'\varepsilon^2-\frac{4}{3}\mu^2t^2\varepsilon^4\right)}. \tag{10}$$

Setting $\mu=0$, we obtain the entries in Table 1. Up to the μ-dependent terms, (10) is a convolution of the initial distribution with a gaussian. It can be checked that the group property holds true.

In particular, if one tentatively interprets $\phi(r)$ as a probability distribution such that $\int_R \mathrm{d}r\,\phi(t,r)=1$, this normalisation condition remains unchanged for $\mu=0$, viz. $\int_R \mathrm{d}r\,F(\varepsilon,t,r)|_{\mu=0}=1$. Furthermore, one may consider

$$\widehat{\phi}(t,k)=\left\langle e^{-\mathrm{i}kr}\right\rangle=\frac{1}{\sqrt{2\pi}}\int_R \mathrm{d}r\,e^{-\mathrm{i}kr}\phi(t,r) \tag{11}$$

as the associated characteristic function. For example, if we consider a shifted gaussian with characteristic function $\widehat{\phi}(t,k) = \exp\left(-\lambda k^2 + \mathrm{i}\gamma k\right)$, this transforms into

$$\widehat{\phi}(t,k) \mapsto \widehat{F}(\varepsilon,t,k) = e^{-(\lambda+t\varepsilon)k^2+\mathrm{i}\gamma k}\, e^{-\mathrm{i}k(2\mu\lambda\varepsilon+\mu t\varepsilon^2)}\, e^{\mu^2(\lambda+t\varepsilon/3)\varepsilon^2-\mu\gamma\varepsilon} \tag{12}$$

For $\mu = 0$ the center stays unchanged at γ, while the width becomes $\lambda \mapsto \lambda + t\varepsilon$. Gaussian distributions are therefore co-variant under the generalised Galilei generator $Y_{\frac{1}{2}}$ with $\mu = 0$. However, since the gaussian distribution is not a solution of the Schrödinger equation with $n \neq 2$, one can realise a gaussian distribution at best as an initial condition which has to be evolved in time.

The integration of the generalised special transformation (9) gives the result (first we set $\mu = 0$):

$$\begin{aligned} F(\varepsilon,t,r) = \frac{1}{2\pi}\int_{R^2} \mathrm{d}k\mathrm{d}r'\, e^{\mathrm{i}k(r-r')} \left(1+\frac{tk\varepsilon}{2\mathrm{i}}\right)^{2(1-x-\xi)} \times \\ \times\, \phi\left[t\left(1+\frac{tk\varepsilon}{2\mathrm{i}}\right)^{-3}, r'\left(1+\frac{tk\varepsilon}{2\mathrm{i}}\right)^{-2}\right] \end{aligned} \tag{13}$$

In particular, the entries in Table 2 are recovered.

When we consider a gaussian distribution, we find the formal transformation

$$\widehat{\phi}(t,k) = \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda k^2} \mapsto \widehat{F}(\varepsilon,t,k) = \left(1+\frac{tk\varepsilon}{2\mathrm{i}}\right)^{2(1-x-\xi)} \sqrt{\frac{\lambda_{\rm eff}(tk)}{\pi}}\, e^{-\lambda_{\rm eff}(tk)k^2} \tag{14}$$

but now with a k-dependent effective width $\lambda_{\rm eff}(tk) = \lambda\left(1+tk\varepsilon/(2\mathrm{i})\right)^4$. Again, a gaussian distribution can at best be realised as an initial distribution.

Alternatively, one may implement the constraint of resting in the reduced function space of solutions of the Schrödinger equation directly, in order to treat the case $\mu \neq 0$ (see [30] for details).

(ii). $\underline{z = n = 4}$

The finite form of the generalised Galilei transformation is found by solving (8). We find

$$F(\varepsilon,t,r) = \frac{\exp(t\mu^3\varepsilon^4/4)}{2\pi}\int_R \mathrm{d}r'\; \phi(t,r')\, e^{\mu\varepsilon r'} \int_R \mathrm{d}k\; e^{\mathrm{i}k(r-r'+t\mu^2\varepsilon^3)+\frac{3}{2}t\mu\varepsilon^2k^2-\mathrm{i}t\varepsilon k^3}. \tag{15}$$

Setting $\mu = 0$, the results in Table 1 can be recovered and we also have the same conservation of the normalisation, when $\mu = 0$. Next, we integrate the special generator X_1 by solving (9). The result is:

$$F(\varepsilon,t,r) = \frac{1}{2\pi}\int_{R^2} \mathrm{d}k\mathrm{d}r'\; e^{\mathrm{i}k(r-r')-(x+\xi-2)\varepsilon tk^2}\; \phi\left(e^{-2\varepsilon tk^2}t, e^{-\varepsilon tk^2}r'\right) \tag{16}$$

from which the corresponding entries in Table 2 follow.[2] The main difference with respect to (13) is the exponential rescaling of time and space.

4 Covariant Two-Point Functions

We derive the form of the co-variant two-point function $F = F(t_1,t_2;r_1,r_2) = \langle\phi_1(t_1,r_1)\phi_2(t_2,r_2)\rangle$ ((x_i,ξ_i) and μ_i are the scaling dimensions and the "mass" of the scaling operators ϕ_i) from the co-variance conditions $X^{(2)}F = 0$, where $X^{(2)}$ is the two-body extension of the generators $X \in age(1)$ constructed in Sect. 2. It is turned out (here we give the final results [30]) that one must distinguish between the cases (i) n even and (ii) n odd.

(i). n even. We rewrite the two-point function as

$$F = F(u,v,r)\ ,\ u := t_1 - t_2\ ,\ v := t_1/t_2\ ,\ r := r_1 - r_2$$
$$F(u,v,r) = t_2^{-(x_1+x_2)/n}\,(v-1)^{-\frac{2}{n}[(x_1+x_2)/2+\xi_1+\xi_2-n+2]}\,v^{-\frac{1}{n}[x_2-x_1+2\xi_2-n+2]}\, f\left(ru^{-1/n}\right) \tag{17}$$

where the form of the last scaling function $f = f(y)$ satisfy the equation

$$\frac{\mathrm{d}^{n-1}f(y)}{\mathrm{d}y^{n-1}} + \mu_1 y f(y) = 0. \tag{18}$$

(ii). n odd. In this case the two-point function is

$$F = F(u,v,r)\ ,\ u := t_1 + t_2\ ,\ v := t_1/t_2\ ,\ r := r_1 - r_2$$
$$F(u,v,r) = t_2^{-(x_1+x_2)/n}\,(v+1)^{-\frac{2}{n}[(x_1+x_2)/2+\xi_1+\xi_2-n+2]}\,v^{-\frac{2}{n}[x_2-x_1+\xi_1-\xi_2]}\, f\left(ru^{-1/n}\right) \tag{19}$$

and where the scaling function $f(y)$ is again given by (18).

It remains to discuss the remaining scaling function $f(y)$. The general solution of (18) is

$$f(y) = \sum_{\ell=0}^{n-2} f_\ell\, y^\ell\, {}_1F_{n-1}\left(1; \frac{2+\ell}{n}, \frac{3+\ell}{n}, \ldots, \frac{n+\ell}{n}; -\frac{\mu_1 y^n}{n^{n-1}}\right) \tag{20}$$

[2] All entries in Tables 1 and 2 can be checked by direct substitution.

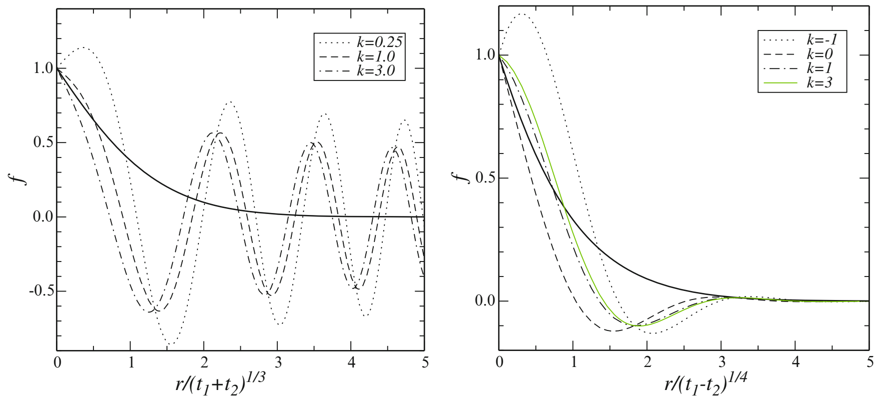


Fig. 1 Scaling function $f(y)$ in the case $z = n = 3$ (on the *left hand side*), normalised to $f(0) = 1$. The *solid line* gives the behaviour if $\mu_1 = 1 > 0$, while the *broken lines* indicate the behaviour, for $\mu_1 = -8 < 0$ and several values of k, of the function $f(y) = (\mathrm{Ai}(|\mu_1|^{1/3}y) + k\mathrm{Bi}(|\mu_1|^{1/3}y))/((\mathrm{Ai}(0)) + k\mathrm{Bi}(0))$. On the *right hand side* is the scaling function $f(y)$ in the case $z = n = 4$, normalised to $f(0) = 1$. The *thick solid line* gives the behaviour if $\mu_1 = 1 > 0$. The *broken lines* and the *grey line* indicate the behaviour, for $\mu_1 = -8 < 0$ and several values of k, of the function $f(y) = (\mathscr{F}_1(y) + k\mathscr{F}_2(y))/(\mathscr{F}_1(0) + k\mathscr{F}_2(0))$, with the $\mathscr{F}_i(y)$ defined in (24)

where ${}_1F_{n-1}$ are generalised hypergeometric functions and the f_ℓ are normalisation constants. On this, physically reasonable boundary conditions must be imposed, especially $\lim_{y\to\infty} f(y) = 0$. It may be more instructive, however, to look at explicit examples.

1. $\underline{n=3}$. In this case, (18) reduces essentially to Airy's equation and the solutions can be compactly expressed in terms of Airy's functions and the normalisation constants $f_{1,2}$

$$
\begin{aligned}
f(y) &= f_1 \mathrm{Ai}\left(-\mu_1^{1/3} y\right);\ \mu_1 > 0 \\
f(y) &= f_1 \mathrm{Ai}\left(|\mu_1|^{1/3} y\right) + f_2 \mathrm{Bi}\left(|\mu_1|^{1/3} y\right);\ \mu_1 < 0
\end{aligned}
\tag{21}
$$

   For $\mu_1 > 0$, the second independent solution of (18) was suppressed, since it diverges for $y \to \infty$. Figure 1(left hand side) illustrates the behaviour of the scaling function, for positive and negative values of μ_1.
2. $\underline{n=4}$. The solution of (18) now takes the more simple form

$$
\begin{aligned}
f(y) = {} & f_0\, {}_0F_2\left(\frac{1}{2}, \frac{3}{4}; -\frac{\mu_1 y^4}{64}\right) + f_1\, y\, {}_0F_2\left(\frac{3}{4}, \frac{5}{4}; -\frac{\mu_1 y^4}{64}\right) \\
& + f_2\, y^2\, {}_0F_2\left(\frac{5}{4}, \frac{3}{2}; -\frac{\mu_1 y^4}{64}\right)
\end{aligned}
\tag{22}
$$

This may be analysed using the leading asymptotic behaviour of the hypergeometric function ${}_0F_2$, which may be read off from Wright's formulæ [31]. Its turns out that for both $\mu_1 > 0$ and $\mu_1 < 0$, this implies that the function $f(y)$ diverges exponentially fast as $y \to \infty$. We absorb this divergence by choosing the constants $f_{0,1,2}$ accordingly and then find

$$
\begin{aligned}
f(y) &= f_0 \left[{}_0F_2\left(\frac{1}{2}, \frac{3}{4}; -\frac{\mu_1 y^4}{64}\right) - \frac{\sqrt{2}\,\Gamma(3/4)}{\Gamma(1/2)} \mu_1^{1/4} y\, {}_0F_2\left(\frac{3}{4}, \frac{5}{4}; -\frac{\mu_1 y^4}{64}\right) \right. \\
&\quad \left. + \frac{\Gamma(3/4)}{\Gamma(1/4)} \mu_1^{1/2} y^2\, {}_0F_2\left(\frac{5}{4}, \frac{3}{2}; -\frac{\mu_1 y^4}{64}\right) \right]; \qquad \mu_1 > 0 \\
f(y) &= f_{(0)} \left[\mathscr{F}_1(y) + k \mathscr{F}_2(y) \right] \qquad \mu_1 < 0
\end{aligned} \tag{23}
$$

where f_0 and $f_{(0)}$ are normalisation constants, k is a free parameter and

$$
\begin{aligned}
\mathscr{F}_1(y) &:= |\mu_1|^{1/4} y\, {}_0F_2\left(\frac{3}{4}, \frac{5}{4}; \frac{|\mu_1| y^4}{64}\right) - \frac{\Gamma(1/2)}{\Gamma(3/4)}\, {}_0F_2\left(\frac{1}{2}, \frac{3}{4}; \frac{|\mu_1| y^4}{64}\right) \\
\mathscr{F}_2(y) &:= |\mu_1|^{1/2} y^2\, {}_0F_2\left(\frac{5}{4}, \frac{3}{2}; \frac{|\mu_1| y^4}{64}\right) - \frac{\Gamma(1/4)}{\Gamma(3/4)}\, {}_0F_2\left(\frac{1}{2}, \frac{3}{4}; \frac{|\mu_1| y^4}{64}\right)
\end{aligned} \tag{24}
$$

The behaviour of these scaling functions is illustrated in Fig. 1(right hand side). We notice that although the scaling function satisfies for both n odd and n even the same differential equation (18), the interpretation of the scaling variable $|\mu_1|^{1/4} y$ is different. Furthermore, we see that for $\mu_1 > 0$, only a single independent solution remains, which decreases from $f(0) = 1$ monotonously and very rapidly towards zero when y is increased. On the other hand, for $\mu_1 < 0$, we find two independent admissible solutions whose decay towards zero is an oscillatory function of y. This feature may allow to distinguish at least qualitatively between two physically distinct situations with $z > 2$:

- Non-equilibrium relaxation kinetics with a *conserved* order-parameter (model B dynamics). Below the critical point, viz. $T < T_c$, in systems with a global O(n)-symmetry it is known that $z = 3$ for a scalar order-parameter ($n = 1$), and $z = 4$ for vector order-parameters ($n \geq 2$) [7]. At criticality $z = 4 - \eta = 4 - \frac{1}{2}\frac{n+2}{(n+8)^2}\varepsilon^2 + \mathrm{O}(\varepsilon^3)$ in $d = 4 - \varepsilon$ dimensions [32]. In these cases, scaling functions are generically seen to be oscillating.[3]
- In critical dynamics, viz. $T = T_c$, and without any conservation law on the order-parameter (model A dynamics), the dynamical exponent is $z \geq 2$ [32]. Here, the decay of the scaling functions is in general monotonous.

Our results suggest that these physically distinct cases, even with the same value of z,might be distinguished through the *sign* of the dimensionful parameter μ_1.

[3] For example, the scaling function $\mathscr{F}_2(y)$ in (24) reproduces the exactly known two-time response in the 3D Mullins–Herring model of surface growth with a conserved order-parameter [29].

5 Conclusions

We have constructed new representations of the ageing algebra $age(1)$, corresponding to an integer dynamical exponent $z = n \geq 2$ to explore the mathematical structure of dynamical symmetries whose infinitesimal generators are no longer described by the usual vector fields, provided that we restrict the admissible function space to the solution space of the Schrödinger equation $S\psi = 0$. We have given an explicit n-dependent family of linear partial differential equations which are indeed $age(1)$-invariant in the sense introduced here. The non-local infinitesimal generators of $age(1)$ contain higher-order differential operators. Their exponentiation does not lead to local spatio-temporal coordinate transformations and we have considered the possibility that a better interpretation might be formulated in terms of transformation rules for distributions of spatio-temporal coordinates. Several examples of such transformation rules have been derived.

Finally, we also studied the scaling form of co-variant two-point functions. Surprisingly, for $z = n$ *even* the scaling forms are compatible with the expectations of a two-time *response* function (as it is usually the case in present theories of local scale-invariance in ageing systems) since they depend on the time difference $t_1 - t_2$. On the other hand, this is not so for $z = n$ *odd*, where the arguments of the scaling functions are much more reminiscent of co-variant two-time *correlators*, since they contain the sum $t_1 + t_2$. We have also seen that the shape of the space-dependent part of the scaling functions can at least qualitatively account for the different forms found for non-conserved (model A) dynamics, where one expects a monotonous decay, and for conserved (model B) dynamics, where scaling functions are oscillatory. This is achieved through a simple change in the sign of the dimensionful "mass parameter" μ. Although we think it unlikely that our non-local representations of $age(1)$ should be directly applicable to physical models, we consider this qualitative feature encouraging.

Acknowledgements Most of the work on this paper was done during the visits of S.S. at the Université Henri Poincaré Nancy I. S.S. is supported in part by the Bulgarian NSF grant *DO 02-257*.

References

1. Bagchi, A., Gopakumar, R.: J. High Energ Phys. **0907**, 307 (2009)
2. Balasubramanian, K., McGrevy, J.: Phys. Rev. Lett. **101**, 061601 (2008)
3. Bargman, V.: Ann. Math. **56**, 1 (1954)
4. Baumann, F., Stoimenov, S., Henkel, M.: J. Phys. **A39**, 4095 (2006)
5. Baumann, F., Henkel, M.: J. Stat. Mech. P01012 (2007)
6. Baumann, F., Dutta, S.B., Henkel, M.: J. Phys. A Math. Gen. **40**, 7389 (2007)
7. Bray, A.J.: Adv. Phys. **43**, 357 (1994)
8. Cannas, S.A., Stariolo, D.A., Tamarit, F.A.: Physica **A294**, 362 (2001)

9. Cugliandolo, L.F.: In: Barrat, J.-L., Dalibard, J., Kurchan, J., Feigel'man, M.V. (eds.) Slow relaxation and non equilibrium dynamics in condensed matter, Les Houches Session 77 July 2002. Springer, Heidelberg (2003) (cond-mat/0210312)
10. Durang, X., Henkel, M.: J. Phys. A Math. Theor. **42**, 395004 (2009)
11. Duval, C., Horváthy, P.A.: J. Phys. A Math. Theor. **42**, 465206 (2009)
12. Godrèche, C., Luck, J.M.: J. Phys. A. Math. Gen. **33**, 9141 (2000)
13. Hagen, C.R.: Phys. Rev. **D5**, 377 (1972); Phys. Lett. **A279**, 215 (2001)
14. Havas, P., Plebanski, J.: J. Math. Phys. **19**, 482 (1978)
15. Henkel, M.: J. Stat. Phys. **75**, 1023 (1994)
16. Henkel, M.: Phys. Rev. Lett. **78**, 1940 (1997)
17. Henkel, M.: Nucl. Phys. **B641**, 405 (2002)
18. Henkel, M., Unterberger, J.: Nucl. Phys. **B660**, 407 (2003)
19. Henkel, M., Enss, T., Pleimling, M.: J. Phys. A Math. Gen. **39**, L589 (2006)
20. Henkel, M., Schott, R., Stoimenov, S., Unterberger, J.: On the dynamical symmetry algebra of ageing: Lie structure, representations and Appell systems, to appear in *Confluents Mathematici* 2013, preprint math-ph/0601028.
21. Henkel, M., Pleimling, M.: Non-equilibrium Phase Transitions, vol. 2: Ageing and Dynamical Scaling Far from Equilibrium. Springer, Heidelberg (2010)
22. Jackiw, R.: Phys. Today **25**, 23 (1972)
23. Jottar, J.I., Leigh, R.G., Minic, D., Pando Zayas, L.A.: J. High-energy Phys. 1011:034 (2010), preprint arxiv:1004.3752
24. Kastrup, H.A.: Nucl. Phys. **B7**, 545 (1968)
25. Minic, D., Pleimling, M.: Phys. Rev. **E78**, 061108 (2008)
26. Niederer, U.: Helv. Phys. Acta **45**, 802 (1972)
27. Ovsiannikov, L.V.: Group Analysis of Differential Equations. Academic Press, New York (1982)
28. Picone, A., Henkel, M.: Nucl. Phys. **B688** 217 (2004)
29. Röthlein, A., Baumann, F., Pleimling, M.: Phys. Rev. E **74**, 061604 (2006)
30. Stoimenov, S., Henkel, M.: Nucl. Phys, **B847**, 612–627 (2011)
31. Wright, E.M.: J. Lond. Math. Soc. **10**, 287 (1935); Proc. Lond. Math. Soc. **46**, 389 (1940); erratum J. Lond. Math. Soc. **27**, 256 (1952)
32. Zinn-Justin, J.: Quantum Field-Theory and Critical Phenomena, 4th edn. Oxford University Press, Oxford (2002)
33. Zhang, P.-M., Horváthy, P.A.: Eur. Phys. J. **C65**, 607 (2010)

Construction of the Noncommutative Rank I Bergman Domain

Zhituo Wang

Abstract In this paper we present a harmonic oscillator realization of the most degenerate discrete series representations of the $SU(2,1)$ group and the deformation quantization of the coset space $D = SU(2,1)/U(2)$ with the method of coherent state quantization. This short article is based on a talk given at the 9-th International Workshop, Varna "Lie Theory and Its Applications in Physics" (LT-9).

1 Introduction

It is believed that ordinary differential geometry should be replaced by noncommutative geometry [3] when we are approaching the Planck scale and quantum field theories defined on noncommutative space time (NCQFT) [4, 13, 15] are considered as the right way to explore the effects of quantum gravity.

The simplest noncommutative space is the Moyal space, which is a symplectic manifold generated by the noncommutative coordinates x_μ, such that $[x_\mu, x_\mu] = i\theta_{\mu\nu}$, where $\theta_{\mu\nu}$ is a constant. The first well defined quantum field theory on 4 dimensional Moyal space is the Grosse–Wulkenhaar model [6]. It is not only perturbative renormalisable to all orders but also asymptotically safe, namely the beta function for the coupling constant is zero at the fixed point of this model. Hence this model is a candidate to be constructed nonperturbatively, namely it's possible to obtain the exact Green's function which is unique and analytic in the coupling constant, by resumming the perturbation series [12]. Recently the two dimensional Grosse–Wulkenhaar model has been constructed in [16].

Z. Wang (✉)
Laboratoire de Physique Théorique, CNRS UMR 8627, Université Paris XI,
F-91405 Orsay Cedex, France
e-mail: zhituo.wang@th.u-psud.fr

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_26, © Springer Japan 2013

Since the noncommutative quantum fields theories are better behaved than their commutative counterparts, it is very natural to construct other noncommutative manifolds and physics models over them.

In this paper we construct the noncommutative coset space $D = SU(2,1)/S(U(2) \times U(1))$, with the method of coherent state quantization. For doing this we also introduce a harmonic oscillator realization of the most degenerate discrete series representation of the group $SU(2,1)$ which is a generalization for the $SU(1,1)$ case introduced by Grosse and Presnajder [5]. The interested reader could look at [5, 11] for more details about the coherent state quantization and [9, 10, 14] for more details about the representation theory of noncompact Lie group. In [7] and [17] we have studied the harmonic oscillator realization of the maximal degenerate discrete series representations for an arbitrary $SU(m,n)$ group.

The construction of the noncommutative coset space $SU(2,1)/U(2)$ has been also studied by [2, 8] with the method of Berezin–Toeplitz quantization and by [1] with the method of "WKB quantization". The interested reader could go to the references for details.

2 The $SU(m,1)$ Group and Its Lie Algebra

The group $G = SU(m,1)$ is defined as a subgroup of the matrix group $SL(m+n,C)$:

$$g = \begin{pmatrix} a_{m\times m} & b_{m\times 1} \\ c_{1\times m} & d \end{pmatrix} \in G \tag{1}$$

satisfies the constraint

$$g^\dagger \Gamma g = g \Gamma g^\dagger = \Gamma, \quad \Gamma = \begin{pmatrix} I_{m\times m} & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Here I's represents unit matrices and 0's the blocks of zeros.

The maximal compact subgroup is defined by matrices

$$K = S(U(m) \times U(1)) = \left\{ \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \ \det(K_1 K_2) = 1 \right\}. \tag{3}$$

The Bergman domain is defined as the coset space $D = G/K$:

$$D = \{Z \,|\, 1 - |Z|^2 > 0\} \ = \ \{z \,|\, 1 - |z_1|^2 - |z_2| - \cdots - |z_m|^2 > 0\}, \tag{4}$$

where $Z = (z_1, \cdots, z_m)$ are the coordinates of the coset space D. It is a pseudo-convex domain over which we could define a holomorphic Hilbert spaces [2] with the reproducing Bergman kernel:

$$K(W^\dagger, Z) = (1 - W^\dagger Z)^{-N}, \tag{5}$$

where Z and W are complex m-columns, and $N = m+1, m+2, \cdots$ is a natural number characterizing the representation.

D is also a Kähler manifold with the Kähler metric defined by the derivations of the Bergman kernel:

$$g_{i\bar{j}} = \frac{1}{N}\partial_{\bar{z}^i}\partial_{z^j} \log K(Z^\dagger, Z). \tag{6}$$

More explicitly we have:

$$g_{i\bar{j}} = [\frac{\delta_{ij}}{1-|Z|^2} + \frac{z_i\bar{z}_j}{(1-|Z|^2)^2}],\ \ g^{i\bar{j}} = (1-|Z|^2)(\delta_{ij} - \bar{z}_i z_j). \tag{7}$$

We could easily calculate that the Ricci tensor: $R_{i\bar{j}} = -(m+1)g_{i\bar{j}}$ and the curvature $R = -(m+1)$ and verify that the metric $g_{i\bar{j}}$ is a solution to the Einstein's equation in the vacuum:

$$R_{i\bar{j}} - \frac{1}{2}g_{i\bar{j}}R + \Lambda g_{i\bar{j}} = 0 \tag{8}$$

with the cosmological constant $\Lambda = \frac{m+1}{2}$.

The Lie algebra $\mathbf{g} = Lie(G) = su(m,1)$ is defined by $M = \left\{ \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix} \right\} \in \mathbf{g}$, $M^\dagger \Gamma = -\Gamma M$. where $A^\dagger = -A, D^\dagger = -D, \mathrm{tr}(A+D) = 0$.

Consider the Cartan decomposition of the Lie algebra $\mathbf{g} = \mathbf{l} + \mathbf{p}$ and let $\mathbf{a} \in \mathbf{p}$ be a maximal Abelian subalgebra. We could choose for $\mathbf{a}$ the set of all matrices of the form

$$H_t = \begin{pmatrix} O_{(m-1)\times(m-1)} & O_{(m-1)\times 1} & O_{(m-1)\times 1} \\ O_{1\times(m-1)} & 0 & t \\ O_{1\times(m-1)} & t & 0 \end{pmatrix} \tag{9}$$

where t is a real number.

Define the linear functional over H_t by $\alpha(H_t) = t$, the roots of $(\mathbf{g}, \mathbf{a})$ are given by

$$\pm\alpha,\ \pm 2\alpha, \tag{10}$$

with multiplicities $m_\alpha = 2$ and $m_{2\alpha} = 1$.

Define

$$\delta := \{\mathbf{a}_t |\ \mathbf{a}_t = \exp H_t,\ H_t \in \mathbf{a}\}. \tag{11}$$

so we have

$$\mathbf{a}_t = \begin{pmatrix} I & O & 0 \\ O & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}, \tag{12}$$

where the symbol I stands for the identity matrix and O is the matrix with entries 0.

3 The Holomorphic Discrete Series of Representations of the $SU(m,1)$ Group

The unitary irreducible representations for $G = SU(m,1)$ are the principal series, the discrete series and the supplementary series. We consider only the discrete series of representations, which are realized in the Hilbert space $\mathscr{L}^2_N(D)$ of holomorphic functions with the inner product defined by:

$$(f,g)_N = \int d\mu_N(Z,\bar{Z})\ \bar{f}(\bar{Z})g(Z), \tag{13}$$

where $d\mu_N(Z,\bar{Z}) = c_N[\det(E-Z^\dagger Z)]^{N-(m+1)}|dZ|$ is the normalised measure and $c_N = \pi^{-2}(N-2)(N-1)$.

The discrete series of representations T_N is defined by

$$T_N f(Z) = [\det(CZ+d)]^{-N} f(Z'),\ N = m+1, m+2, \cdots \tag{14}$$

where

$$Z' = (AZ+B)(CZ+d)^{-1} \tag{15}$$

In the following we consider only the $m = 2$ case and construct the harmonic oscillator realization of the most degenerate discrete series representation (14). We introduce a 3×1 matrix $\hat{Z} = (\hat{z}_a)$, $a = 1,2,3$, of bosonic oscillators acting in Fock space and satisfying commutation relations

$$[\hat{z}_a, \hat{z}_b^\dagger] = \Gamma_{ab},\ a,b = 1,2,3 \tag{16}$$

$$[\hat{z}_a, \hat{z}_b] = [\hat{z}_a^\dagger, \hat{z}_b^\dagger] = 0, \tag{17}$$

where Γ is a 3×3 matrix defined in (2). It can be easily seen that for all $g \in SU(2,1)$ these commutation relations are invariant under transformations:

$$\hat{Z} \mapsto g\hat{Z},\quad \hat{Z}^\dagger \mapsto \hat{Z}^\dagger g^\dagger. \tag{18}$$

We could define the creation and anhilation operators $\hat{a}_\alpha$ and $\hat{b}$ as:

$$\hat{Z} = \begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \end{pmatrix}:\ [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}, \alpha,\beta = 1,2.\ [\hat{b},\hat{b}^\dagger] = 1, \tag{19}$$

and all other commutation relations among oscillator operators vanish.

The Fock space $\mathscr{F}$ in question is generated from a normalized vacuum state $|0\rangle$, satisfying $\hat{a}_\alpha|0\rangle = \hat{b}|0\rangle = 0$, by repeated actions of creation operators:

$$|m_\alpha, n\rangle = \prod_\alpha \frac{(\hat{a}_\alpha^\dagger)^{m_\alpha}(\hat{b}^\dagger)^n}{\sqrt{m_\alpha!n!}}|0\rangle. \tag{20}$$

We shall use the terminology that the state $|m_\alpha, n\rangle$ contains $m = \sum m_\alpha$ particles a and n particles b.

Consider a basis of $su(2,1)$ Lie algebra $X = X^A_{ab}$ $A = 1,\cdots 8$, $a,b = 1,2,3$, we assign the operator

$$\hat{X} = -\hat{Z}^\dagger \Gamma X \hat{Z} = -\hat{z}^\dagger_a \Gamma^A_{ab} X^A_{ab} \hat{z}_b\,. \tag{21}$$

Their anti-hermiticity follows directly:

$$\hat{X}^\dagger = -\mathrm{tr}\,(\hat{Z}^\dagger X^\dagger \Gamma \hat{Z}) = +\mathrm{tr}\,(\hat{Z}^\dagger \Gamma X \hat{Z}) = -\hat{X}.$$

Using commutation relations for annihilation and creation operators we have:

$$[\hat{X},\hat{Y}] = [\hat{Z}^\dagger \Gamma X \hat{Z}, \hat{Z}^\dagger \Gamma Y \hat{Z}] = -\hat{Z}^\dagger \Gamma\,[X,\,Y]\,\hat{Z}. \tag{22}$$

So that the operators $\hat{X}_a$ satisfy in Fock space the $su(2,1)$ commutation relations. The assignment

$$g = e^{\xi^A X_A} \in SU(2,1) \Rightarrow \hat{T}(g) = e^{\xi^A \hat{X}_A} \tag{23}$$

then defines a unitary $SU(2,1)$ representation in Fock space.

4 The Coherent States Quantization of $D = SU(2,1)/U(2)$ and the Star Product

We briefly describe the construction of coherent states on coset space of a Lie group following [11]. Let T_g be an unitary irreducible representation of an arbitrary Lie group G in a Hilbert space $\mathscr{H}$, $|z_0\rangle \in \mathscr{H}$ is a normalized state in the Garding space of T_g. Let K be the stability group of the $|z_0\rangle$, for which $T_k|z_0\rangle = e^{i\alpha(k)}|z_0\rangle$, for $k \in K$. Then for each $z = g_z z_0 \in D = G/K$ we could assign a coherent states : $|z\rangle = \psi_z = T(g_z)|z_0\rangle$. Define the functions $\omega_0(g) =< z_0|T(g)|z_0 >$ and $\omega(g,z) =< z|T_g|z >= \omega_0(g_z^{-1} g g_z)$. As $|z_0\rangle$ is in the Garding space, $\omega(g)$ is a smooth function in g.

For $G = SU(2,1)$, the state $|z_0\rangle$ is defined in the Fock space as:

$$|z_0\rangle = \frac{(\hat{b}^\dagger)^N}{\sqrt{(N)!}}|0>= \frac{1}{\sqrt{N}}\,|0,0;N\rangle\,. \tag{24}$$

Here N is a natural number that specifies the representation: $\hat{N}|z_0\rangle = N|z_0\rangle$. All other states in the representation space are obtained by the action of rising operators given in (20). The stability group for $|z_0\rangle$ is $K = S(U(2)\times U(1))$.

Using the $K\delta K$ decomposition of $g = k^\dagger\,\delta\,q$ [10], for which $k,q \in K$, we obtain:

$$\omega_0(g) = \langle z_0|\,\hat{T}(g)\,|z_0\rangle = \frac{1}{\cosh t}\left[(1+\ln\cosh t)e^{i(\alpha(q)-\alpha(k))}\right]^N \tag{25}$$

Consider an operator acting on $\mathscr{H}$:

$$\hat{F} = \int dg\tilde{F}(g)T(g) = \int dg\tilde{F}(g)\omega(g,z) \tag{26}$$

where $\tilde{F}(g)$ ia a distribution on a group G with compact support. We also define for each $\hat{F}$ a biholomorphic function:

$$F(z,\bar{z}) = \langle \Psi_z|\hat{F}|\Psi_z\rangle. \tag{27}$$

The star product of two such bi-holomorphic functions F and G is then defined by [5]:

$$\begin{aligned}(F\star G)(z,\bar{z}) &= \langle \Psi_z|\hat{F}\hat{G}|\Psi_z\rangle = (F\star G)(z,\bar{z}) \\ &= \int dg_1 dg_2 \tilde{F}(g_1)\tilde{F}(g_2)\omega(g_1 g_2,z).\end{aligned} \tag{28}$$

Obviously the star product defined above is noncommutative, associative and is invariant under the action of the group G. The noncommutative algebra of functions $\{F(z)\}$ induces a noncommutative structure on the coset space D. That's how we construct the noncommutative version of the Bergman domain, which is noted as $\hat{D}$.

Now we shall study the explicit form of the star product for $G = SU(2,1)$. Using the explicit form of the group element $g = e^{\xi^A X_A}$ and integration by parts we have:

$$F_{A_1\ldots A_n}(z) \;=\; (-1)^n(\partial_{\xi_{A_1}}\ldots\partial_{\xi_{A_n}}\omega)(e^{\xi^A\hat{X}_A}z)|_{\xi=0} = (-1)^n\langle z|\hat{X}_{A_1}\ldots\hat{X}_{A_n}|z\rangle. \tag{29}$$

Here $\hat{X}_A$, $(A = 1\cdots,8)$ are the left-invariant vector field on group G corresponding to the Lie algebra basis X_A whose explicit form is given in [17].

From the definition of the star product (28) it follows that:

$$(F_{A_1\ldots A_n}\star F_{B_1\ldots B_m})(z) \quad = (-1)^{n+m}(\hat{X}_{A_n}\ldots\hat{X}_{A_1}\hat{X}_{B_m}\ldots\hat{X}_{B_1}\omega)(g,z)|_{g=e}. \tag{30}$$

We define the function ξ_A as the expectation value of the operator $\hat{X}_A$ between the coherent states as:

$$\xi_A(z) = \frac{1}{N}\langle z|\hat{X}_A|z\rangle = \frac{1}{N}\langle z_0|\hat{T}^\dagger(g_z)\hat{X}_A\hat{T}(g_z)|z_0\rangle. \tag{31}$$

The star product between these coordinates functions reads:

$$(\xi_A\star\xi_B)(z) \;=\frac{1}{N^2}\langle z|\hat{X}_A\hat{X}_B|z\rangle = \;(1+A_N)\,\xi_A(z)\,\xi_B(z) + \frac{1}{2N}f^C_{A,B}\,\xi_C(z) + B_N\,\delta_{A,B}, \tag{32}$$

where A_N and B_N depend on the Bernoulli numbers coming from the Baker–Campbell–Hausdorff formula and are of order $1/N$. We see that the parameter of the non-commutativity is $\lambda_N = 1/N$. For $N\to\infty$ we recover the commutative product.

According to the Harish–Chandra imbedding theorem, we could always imbed the commutative maximal Hermitian symmetric space into the noncompact part of the Cartan subalgebra. So the coordinates of the noncommutative Bergman domain $\hat{D}$ can be identified as the coordinate functions corresponding to the noncompact Cartan subalgebra.

5 Conclusions and Prospectives

In this paper we have constructed the noncommutative Bergman domain $\hat{D}$ whose commutative counterpart is the coset space $D = G/K$, where $G = SU(2,1)$ and $K = S(U(2) \times U(1))$. This result could be generalized to an arbitrary type one rank one Cartan domain $D = G = SU(m,1)/S(U(m) \times U(1))$ straightforwardly.

In [17] we have build a model of quantum theory of real scalar fields on the noncommutative manifold $\hat{D}$ and find that the one loop quantum correction to the 2 point function is finite. This is a hint of the finiteness of quantum field theory on $\hat{D}$ and this deserves further studies.

Acknowledgements The author is very grateful to Harald Grosse and Peter Presnajder for useful discussions.

References

1. Bieliavsky, P., Gurau, R., Rivasseau, V.: Non commutative field theory on rank one symmetric spaces. J. Noncommut. Geom. **3**, 99–123 (2009). [arXiv:0806.4255 [hep-th]]
2. Bortwick, D., Lesniewski, A., Upmeier, H.: Non-perturbative deformation quantization of Cartan domains. J. Funct. Anal. **113**, 153–176 (1993); Upmeier, H.: Toeplitz operators and index theory in several complex variavbles, Harmonic Analysis 1996. Birkhäuser, Basel (1996)
3. Connes, A.: Noncommutative Geometry. Academic Press, Boston (1994)
4. Douglas, M.R., Nekrasov, N.A.: Noncommutative field theory. Rev. Mod. Phys. **73**, 977 (2001) [hep-th/0106048]
5. Grosse, H., Prešnajder, P.: The construction of noncommutative manifolds using coherent states. Let. Math. Phys. **28**, 239–250 (1993)
6. Grosse, H., Wulkenhaar, R.: Renormalisation of phi**4 theory on noncommutative R**4 in the matrix base. Commun. Math. Phys. **256**, 305 (2005) [arXiv:hep-th/0401128]
7. Grosse, H., Presnajder, P., Wang, Z.: Quantum field theory on quantized bergman domain. J. Math. Phys. **53**, 013508 (2012) [arXiv:1005.5723 [math-ph]]
8. Jakimowicz, G., Odziewicz, A.: Quantum complex Minkowski space. J. Geom. Phys. **56**, 1576–1599 (2006)
9. Kirillov, A.A.: Elements of Theory of Representations. Springer, Berlin (1976); Representation theory and noncommutative harmonic analysis II. Encyklopedia Math. Sci. **59**, Springer, Berlin (1995)
10. Knapp, Anthony K.: Lie Group: Beyond An Introduction, 2nd edn. Birkhauser Boston, Basel, Berlin (2002)
11. Perelomov, A.M.: Generalized Coherent States and Their Applications. Springer, Berlin (1986)

12. Rivasseau, V.: From Perturbative to Constructive Renormalization. Princeton University Press, Princeton (1991)
13. Rivasseau, V.: Quantum Spaces, Progress in Mathematical Physics, Birkhäuser Basel, **53**, pp 19–107, (2007)
14. Rudin, W.: Quantum Spaces, Progress in Mathematical Physics, Birkhäuser Basel, **53**, pp 19–107 (2007)
15. Szabo, R.J.: Quantum field theory on noncommutative spaces. Phys. Rept. **378**, 207 (2003) [hep-th/0109162]
16. Wang, Z.: Construction of 2-dimensional Grosse–Wulkenhaar Model. arXiv:1104.3750 [math-ph]
17. Wang, Z.: Deformation quantization of rank I Bergman domain. arXiv:1110.2632 [math-ph]

Part VI
Vertex Algebras

Singular Vectors and Zhu's Poisson Algebra of Parafermion Vertex Operator Algebras

Tomoyuki Arakawa, Ching Hung Lam, and Hiromichi Yamada

Abstract We study Zhu's Poisson algebra of parafermion vertex operator algebras associated with integrable highest weight modules for the affine Kac-Moody Lie algebra $\widehat{sl}_2$. Using singular vectors, we show that the parafermion vertex operator algebras are C_2-cofinite and rational.

1 Introduction

Fix a positive integer k. Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody Lie algebra associated with a finite dimensional simple Lie algebra $\mathfrak{g}$ and $V_{\widehat{\mathfrak{g}}}(k,0)$ a Weyl module for $\widehat{\mathfrak{g}}$ with level k. Let $L_{\widehat{\mathfrak{g}}}(k,0)$ be its simple quotient. Then $L_{\widehat{\mathfrak{g}}}(k,0)$ is a simple vertex operator algebra and it contains a Heisenberg vertex operator algebra generated by a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The commutant $K(\mathfrak{g},k)$ of the Heisenberg vertex operator algebra in $L_{\widehat{\mathfrak{g}}}(k,0)$ is called a parafermion vertex operator algebra. The most basic case, namely, the case $\mathfrak{g} = sl_2$ was studied in [1, 2] and the structure of the parafermion vertex operator algebra for a general $\mathfrak{g}$ was discussed in [3, 4]. In [3, 4], C. Dong and Q. Wang emphasize the importance of the sl_2 case, since $K(\mathfrak{g},k)$ for a general $\mathfrak{g}$ is generated by subalgebras isomorphic to $K(sl_2,k_\alpha)$ which correspond to positive roots α. In this article we continue the study of the parafermion vertex operator algebra for the case $\mathfrak{g} = sl_2$.

T. Arakawa
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
e-mail: arakawa@kurims.kyoto-u.ac.jp

C.H. Lam
Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan, Republic of China
e-mail: chlam@math.sinica.edu.tw

H. Yamada (✉)
Department of Mathematics, Hitotsubashi University, Kunitachi, Tokyo 186-8601, Japan
e-mail: yamada@econ.hit-u.ac.jp

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_27,

For a vertex operator algebra V, Y. Zhu [5] introduced two intrinsic associative algebras, one is Zhu's algebra $A(V)$ and the other is Zhu's Poisson algebra $V/C_2(V)$, where $C_2(V)$ is the subspace spanned by the vectors $a_{-2}b$ for $a,b \in V$. The vertex operator algebra V is said to be C_2-cofinite if $V/C_2(V)$ is finite dimensional. Also, V is said to be rational if every V-module is semisimple. Our purpose of this article is to establish the C_2-cofiniteness and the rationality of $K(sl_2,k)$. Combining the results with a theorem of [4], we obtain the C_2-cofiniteness of $K(\mathfrak{g},k)$ for a general $\mathfrak{g}$ also. Since the results are known for $k \le 4$, we will discuss the case $k \ge 5$.

Our main tool is a detailed analysis of two kinds of singular vectors. Actually, we use the null field $\mathbf{v}^0$ of weight 8 and its image under the operator W_1^3 to establish an embedding of $N_0/C_2(N_0)$ into $V(k,0)/C_2(V(k,0))$. The singular vector $\mathbf{u}^0$ of weight $k+1$ and its image under the operator W_1^3 are necessary to show that the dimension of $K_0/C_2(K_0)$ is at most $k(k+1)/2$. This upper bound is sufficient to obtain the C_2-cofiniteness and the rationality of K_0.

We will follow the notations and use the results in [1, 2]. Thus, $K_0 = K(sl_2,k)$ and N_0 is the commutant of the Heisenberg vertex operator algebra in the Weyl module $V(k,0) = V_{\widehat{sl}_2}(k,0)$. Also, we denote by W^2 the Virasoro element ω of [1, 2]. Then $\{W^2,W^3,W^4,W^5\}$ is a set of strong generators for N_0 [2, Lemma 2.4]. Our arguments depend on the operator product expansions among those generators [2, Appendix B]. The unique maximal ideal $\mathscr{I}$ of N_0 is generated by the singular vector $\mathbf{u}^0 = f(0)^{k+1}e(-1)^{k+1}\mathbf{1}$ [1, Theorem 4.2] and $K_0 = N_0/\mathscr{I}$.

In this article we present an outline of the proof. Details will appear elsewhere.

2 N_0 Modulo $C_2(N_0)$

Since $K_0 = N_0/\mathscr{I}$, we have $C_2(K_0) = (C_2(N_0)+\mathscr{I})/\mathscr{I}$ and $K_0/C_2(K_0) \cong N_0/(C_2(N_0)+\mathscr{I})$. First, we study $\widetilde{N}_0 = N_0/C_2(N_0)$. It is infinite dimensional. Since N_0 is strongly generated by W^s, $s = 2,3,4,5$, $\widetilde{N}_0$ is a commutative associative algebra generated by $\widetilde{W}^s = W^s + C_2(N_0)$, $s = 2,3,4,5$. That is, $\widetilde{N}_0$ is the image of a homomorphism

$$\varphi : \mathbb{C}[x_2,x_3,x_4,x_5] \to \widetilde{N}_0; \quad x_s \mapsto \widetilde{W}^s$$

of a polynomial algebra with four variables x_2,x_3,x_4,x_5. (φ is denoted by $\tilde{\rho}$ in [2].)

We consider the action of the weight 1 operator $W_1^3 = (W^3)_1$, which is a component operator of the vertex operator associated with the vector W^3. Note that $W_0^3 W^s \in C_2(N_0)$, $s = 2,3,4,5$ [2, Appendix B]. Then the formula $[W_1^3, a_{-2}]b = (W_0^3 a)_{-1}b + (W_1^3 a)_{-2}b$ implies that $C_2(N_0)$ is invariant under W_1^3.

Lemma 1. $W_1^3 C_2(N_0) \subset C_2(N_0)$.

Thus we can define an action of W_1^3 on $\widetilde{N}_0$ by $W_1^3 \cdot \widetilde{u} = W_1^3 u + C_2(N_0)$ for $u \in N_0$. In fact, W_1^3 acts as a derivation on the commutative associative algebra $\widetilde{N}_0$ generated by $\widetilde{W}^s$, $s = 2,3,4,5$. The next lemma follows from Appendix B of [2].

Lemma 2. *The action of* W^3_1 *on* $\widetilde{W}^s$, $s = 2,3,4,5$ *is as follows.*

$$\begin{aligned}
W^3_1 \cdot \widetilde{W}^2 &= 3\widetilde{W}^3,\\
W^3_1 \cdot \widetilde{W}^3 &= \frac{1}{16k+17}\big(288k^3(k-2)(k+2)^2(3k+4)(\widetilde{W}^2)^2 + 36k(2k+3)\widetilde{W}^4\big),\\
W^3_1 \cdot \widetilde{W}^4 &= \frac{1}{64k+107}\big(1248k^2(k-3)(k+2)(2k+1)(2k+3)\widetilde{W}^2\widetilde{W}^3\\
&\quad -12k(3k+4)(16k+17)\widetilde{W}^5\big),\\
W^3_1 \cdot \widetilde{W}^5 &= \frac{240k^4(k+2)^3(2k+3)(3k+4)(202k-169)}{16k+17}(\widetilde{W}^2)^3\\
&\quad -15k(2k+3)(41k+61)(\widetilde{W}^3)^2\\
&\quad +\frac{60k^2(k+2)(404k^2+1170k+835)}{16k+17}\widetilde{W}^2\widetilde{W}^4.
\end{aligned}$$

3 N_0 Modulo $C_2((V(k,0))$

In this section we study N_0 modulo $C_2((V(k,0))$. Let $\overline{V(k,0)} = V(k,0)/C_2(V(k,0))$, which is isomorphic to a polynomial algebra $\mathbb{C}[y_0,y_1,y_2]$ with three variables $y_0 = h(-1)\mathbf{1}$, $y_1 = e(-1)\mathbf{1}$, $y_2 = f(-1)\mathbf{1}$. Since $C_2(N_0) \subset N_0 \cap C_2(V(k,0))$, there is a natural homomorphism ψ from $\widetilde{N_0}$ onto the image $\overline{N_0}$ of N_0 in $\overline{V(k,0)}$. Let $y = y_0$, $z = y_1y_2$. Then $\overline{N_0} \subset \mathbb{C}[y,z]$. In fact, $\overline{W^s} = W^s + C_2(V(k,0)) \in \overline{V(k,0)}$, $s = 2,3,4,5$ are as follows.

$$\begin{aligned}
\overline{W^2} &= -\frac{1}{2k(k+2)}(y^2-2kz),\\
\overline{W^3} &= 2(y^3-3kyz),\\
\overline{W^4} &= -(11k+6)y^4+4k(11k+6)y^2z-2k^2(6k-5)z^2,\\
\overline{W^5} &= -2(19k+12)y^5+10k(19k+12)y^3z-10k^2(10k-7)yz^2,
\end{aligned}$$

and $\overline{N_0}$ is generated by $\overline{W^s}$, $s = 2,3,4,5$ as a commutative associative algebra.

We assign the weight to each term of $\mathbb{C}[y,z]$ by $\mathrm{wt}\, y = 1$ and $\mathrm{wt}\, z = 2$. Since the weight of the elements $h(-1)\mathbf{1}$ and $e(-1)f(-1)\mathbf{1}$ of $V(k,0)$ is 1 and 2, respectively, the weight on $\mathbb{C}[y,z]$ is compatible with that on $V(k,0)$.

Now, consider the composition $\psi \circ \varphi$ of the homomorphism $\varphi : \mathbb{C}[x_2,x_3,x_4,x_5] \to \widetilde{N_0}$ and the natural homomorphism $\psi : \widetilde{N_0} \to \overline{N_0}$.

$$\psi \circ \varphi : \mathbb{C}[x_2,x_3,x_4,x_5] \to \widetilde{N_0} \to \overline{N_0}; \quad x_s \mapsto \widetilde{W}^s \mapsto \overline{W^s}.$$

The kernel Ker $\psi \circ \varphi$ of the composition $\psi \circ \varphi$ is the algebraic relations among $\overline{W^s}$, $s = 2,3,4,5$ in the polynomial algebra $\mathbb{C}[y,z]$.

In [2], three polynomials $B_0, B_1, B_2 \in \mathbb{C}[x_2,x_3,x_4,x_5]$ corresponding to null fields $\mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2$ are introduced. In fact, $\varphi(B_i) = \widetilde{\mathbf{v}}^i = 0$, $i = 0,1,2$, since $\mathbf{v}^i = 0$ in N_0.

At this stage, using a computer algebra system Risa/Asir, we can verify that the kernel Ker$\psi \circ \varphi$ coincides with the ideal $\langle B_0, B_1, B_2 \rangle$ of $\mathbb{C}[x_2,x_3,x_4,x_5]$ generated by B_0, B_1, B_2. This implies that the kernel of φ is also $\langle B_0, B_1, B_2 \rangle$ and ψ is an isomorphism.

Theorem 1. (1) $\widetilde{N_0} \cong \mathbb{C}[x_2,x_3,x_4,x_5]/\langle B_0, B_1, B_2 \rangle$.
(2) $C_2(N_0) = N_0 \cap C_2(V(k,0))$ *and* $\widetilde{N_0} \cong \overline{N_0}$.

We replace $\overline{W^s}$ with simpler polynomials in $\mathbb{C}[y,z]$, $s = 2,3,4,5$, that is, we set

$$g_2 = y^2 - 2kz, \quad g_3 = y^3 - 3kyz, \quad g_4 = z^2, \quad g_5 = yz^2.$$

Indeed, we can express $\overline{W^s}$ as follows.

$$\begin{aligned}
\overline{W^2} &= -\frac{1}{2k(k+2)} g_2, \\
\overline{W^3} &= 2g_3, \\
\overline{W^4} &= -(11k+6)g_2^2 + 2k^2(16k+17)g_4, \\
\overline{W^5} &= -2(19k+12)g_2g_3 + 2k^2(64k+107)g_5.
\end{aligned}$$

Conversely,

$$\begin{aligned}
g_2 &= -2k(k+2)\overline{W^2}, \\
g_3 &= \frac{1}{2}\overline{W^3}, \\
g_4 &= \frac{2(k+2)^2(11k+6)}{16k+17}(\overline{W^2})^2 + \frac{1}{2k^2(16k+17)}\overline{W^4}, \\
g_5 &= -\frac{(k+2)(19k+12)}{k(64k+107)}\overline{W^2}\cdot\overline{W^3} + \frac{1}{2k^2(64k+107)}\overline{W^5}.
\end{aligned}$$

Therefore, $\overline{N_0}$ is generated by g_s, $s = 2,3,4,5$ as a commutative associative algebra. We note that

$$\begin{aligned}
g_2^4 - g_2g_3^2 - 5k^2g_2^2g_4 + 4k^4g_4^2 + 2k^2g_3g_5 &= 0, \\
g_2^3g_3 - g_3^3 - 5k^2g_2g_3g_4 + 2k^2g_2^2g_5 - 2k^4g_4g_5 &= 0, \\
g_2^3g_4 - g_3^2g_4 - 4k^2g_2g_4^2 + k^2g_5^2 &= 0
\end{aligned}$$

are the algebraic relations among g_2,g_3,g_4,g_5 in $\mathbb{C}[y,z]$. The terms in g_2, g_3, g_4 and g_5 of weight at most 7 are linearly independent in $\mathbb{C}[y,z]$. We also have

$$\begin{aligned} z^2 &= g_4, \\ yz^2 &= g_5, \\ z^3 &= \frac{3}{2k}g_2g_4 - \frac{1}{2k^3}(g_2^3 - g_3^2), \\ yz^3 &= \frac{1}{k}(g_2g_5 - g_3g_4). \end{aligned}$$

From now on, we set $\mathscr{A} = \overline{N}_0$ for simplicity of notation. Denote by $\mathbb{C}[y,z]_{(n)}$ and $\mathscr{A}_{(n)}$ the weight n subspaces of $\mathbb{C}[y,z]$ and $\mathscr{A}$ with respect to the weight defined by $\mathrm{wt}\, y = 1$ and $\mathrm{wt}\, z = 2$. The terms $y^{n-2j}z^j$, $0 \le j \le [n/2]$ form a basis of $\mathbb{C}[y,z]_{(n)}$ for $n \ge 0$, where $[n/2]$ is the largest integer which does not exceed $n/2$. In particular, $\dim \mathbb{C}[y,z]_{(n)} = [n/2]+1$. As to $\mathscr{A}_{(n)}$, we have $\dim \mathscr{A}_{(0)} = 1$, $\mathscr{A}_{(1)} = 0$ and the terms $y^n - nky^{n-2}z$, $y^{n-2j}z^j$, $2 \le j \le [n/2]$ form a basis of $\mathscr{A}_{(n)}$ for $n \ge 2$. Thus, $\dim \mathscr{A}_{(n)} = [n/2]$ if $n \ge 1$.

Next, we study the action $W_1^3 \cdot f(y,z)$ of W_1^3 on $f(y,z) \in \mathscr{A}$. Note that $C_2(V(k,0))$ is not invariant under the operator W_1^3. For instance, $W_1^3 e(-2)\mathbf{1}$ is not contained in $C_2(V(k,0))$. Thus, we first consider the action of W_1^3 on $\widetilde{N}_0$ as in Sect. 2 and then transform it to $\mathscr{A} = \overline{N}_0$ by the isomorphism ψ. In fact, we have the following lemma by Lemma 2 and the relations between $\overline{W^2}$, $\overline{W^3}$, $\overline{W^4}$, $\overline{W^5}$ and g_2, g_3, g_4, g_5.

Lemma 3. *W_1^3 acts on g_s, $s = 2,3,4,5$ as follows.*

$$\begin{aligned} W_1^3 \cdot g_2 &= -12k(k+2)g_3, \\ W_1^3 \cdot g_3 &= -18k(k+2)g_2^2 + 36k^3(2k+3)g_4, \\ W_1^3 \cdot g_4 &= -12k(3k+4)g_5, \\ W_1^3 \cdot g_5 &= \frac{6(7k+9)}{k}(g_2^3 - g_3^2) - 6k(28k+37)g_2g_4. \end{aligned}$$

We also note that W_1^3 acts on a polynomial in g_s, $s = 2,3,4,5$ as a derivation, since it acts on a polynomial in $\widetilde{W}^s$, $s = 2,3,4,5$ similarly.

Now, we define a differential operator D on $\mathbb{C}[y,z]$ by

$$D = \big((k+2)y^2 - 2kz\big)\frac{\partial}{\partial y} + (3k+4)yz\frac{\partial}{\partial z}.$$

Theorem 2. *The restriction of the action of $-6kD$ to $\mathscr{A}$ coincides with the action of W_1^3 on $\mathscr{A}$.*

4 Singular Vectors and Ideals of $\mathscr{A}$

In this section we study the image $\overline{\mathbf{u}^0}$ of the singular vector $\mathbf{u}^0 = f(0)^{k+1}e(-1)^{k+1}\mathbf{1}$ of the parafermion vertex operator algebra in $\overline{V(k,0)} = V(k,0)/C_2(V(k,0))$.

Let $f_0(y,z) = ((-1)^{k+1}/(k+1)!)\overline{\mathbf{u}^0} \in \mathbb{C}[y,z]$. We can calculate $f_0(y,z)$ explicitly.

Lemma 4. *We have*

$$f_0(y,z) = \sum_{j=0}^{[(k+1)/2]} c_j y^{k+1-2j} z^j$$

with

$$c_j = (-1)^j \frac{(k+1)!}{(k+1-2j)!\,(j!)^2}.$$

Here, $[(k+1)/2]$ denotes the largest integer which does not exceed $(k+1)/2$.

We consider the action of the differential operator D. For a homogeneous polynomial

$$f(y,z) = \sum_{j=0}^{[n/2]} a_j y^{n-2j} z^j$$

of weight n, we have

$$D \cdot f(y,z) = \sum_{j=0}^{[(n+1)/2]} \big(-2k(n+2-2j)a_{j-1} + (n(k+2)+jk)a_j\big) y^{n+1-2j} z^j,$$

where a_{-1} and $a_{[n/2]+1}$ are understood to be 0.

Let $f_r(y,z) = D^r \cdot f_0(y,z)$ be the image of $f_0(y,z)$ under the operator D^r, $r = 1,2,\ldots$. Then $f_r(y,z)$ is a homogeneous polynomial of weight $k+r+1$. We can verify that

$$\begin{vmatrix} \partial f_0/\partial y & \partial f_0/\partial z \\ \partial f_1/\partial y & \partial f_1/\partial z \end{vmatrix} \neq 0.$$

Thus $f_0(y,z)$ and $f_1(y,z)$ are algebraically independent.

Let $p(y,z) = -(k+1)(k+2)^2((k+1)y^2 + kz)$ and $q(y) = (k+2)(2k+3)y$. Then, we have

$$f_2(y,z) = p(y,z)f_0(y,z) + q(y)f_1(y,z).$$

Let $J = \mathbb{C}[y,z]f_0(y,z) + \mathbb{C}[y,z]f_1(y,z)$ be the ideal of $\mathbb{C}[y,z]$ generated by $f_0(y,z)$ and $f_1(y,z)$. The above equation implies that $f_r(y,z) \in J$ for $r \geq 0$.

The weight 0 operator $L_{\mathrm{aff}}(0) = (\omega_{\mathrm{aff}})_1$ associated with the Virasoro element ω_{aff} of $V(k,0)$ can be written in the form

$$L_{\mathrm{aff}}(0) = y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$$

as an operator on $\mathbb{C}[y,z]$.

The determinant of the 2×2 matrix consisting of the coefficients of $\partial/\partial y$ and $\partial/\partial z$ in the differential operator D and the Virasoro operator $L_{\mathrm{aff}}(0)$ is

$$\begin{vmatrix} (k+2)y^2 - 2kz & (3k+4)yz \\ y & 2z \end{vmatrix} = -k(y^2+4z)z.$$

On the other hand, it is easy to see that $f_0(y, -y^2/4) \neq 0$. Thus the above determinant and $f_0(y,z)$ can not be 0 simultaneously. This implies that the codimension of J in $\mathbb{C}[y,z]$ is finite.

Lemma 5. $\dim \mathbb{C}[y,z]/J < \infty$.

Now, the weight n subspace of J is

$$J_{(n)} = \mathbb{C}[y,z]_{(n-k-1)} f_0(y,z) + \mathbb{C}[y,z]_{(n-k-2)} f_1(y,z).$$

and $\dim \mathbb{C}[y,z]_{(n)} = [n/2]+1$. Using the equation

$$[n/2] = [(n-k-1)/2] + [(n-k-2)/2] - [(n-2k-3)/2]$$

for $n \geq 2k+3$, we obtain the following lemma.

Lemma 6. *Let F be a free $\mathbb{C}[y,z]$-module with basis $\{P_0,P_1\}$, where P_0 and P_1 are elements with weight $k+1$ and $k+2$, respectively. Then the kernel of the $\mathbb{C}[y,z]$-module homomorphism from F onto J which maps P_r to $f_r(y,z)$, $r=0,1$ is generated by a single element*

$$f_1(y,z)P_0 - f_0(y,z)P_1$$

of weight $2k+3$.

Clearly, $J_{(n)} = 0$ if $n \leq k$. The weight n subspace $J_{(n)}$ for $n \geq k+1$ is as follows.

Lemma 7. (1) $\dim J_{(k+1)} = 1$.
(2) $\dim J_{(n)} = [(n-k-1)/2] + [(n-k-2)/2] + 2$ *if* $k+2 \leq n \leq 2k$.
(3) $J_{(n)} = \mathbb{C}[y,z]_{(n)}$ *if* $n \geq 2k+1$.

We also have $\dim(J_{(k+1)} \cap \mathscr{A}) = 1$, $\dim(J_{(n)} \cap \mathscr{A}) = \dim J_{(n)} - 1$ for $k+2 \leq n \leq 2k$ and $J_{(n)} \cap \mathscr{A} = \mathscr{A}_{(n)}$ for $n \geq 2k+1$.

Next, we consider some ideals of $\mathscr{A}$. Let I_s be the ideal of $\mathscr{A}$ generated by $f_r(y,z)$, $0 \leq r \leq s-1$ and I the ideal of $\mathscr{A}$ generated by $f_r(y,z)$, $r \geq 0$. The dimension of weight n subspaces $I_{s(n)}$ and $I_{(n)}$ of these ideals is as follows.

Lemma 8. (1) $\dim I_{2(n)} = 1$ *if* $n = k+1, k+2$.
(2) $\dim I_{2(n)} = [(n-k-1)/2] + [(n-k-2)/2]$ *if* $k+3 \leq n \leq 2k+2$.
(3) $\dim I_{2(2k+3)} = k$.
(4) $I_{2(n)} = \mathscr{A}_{(n)}$ *if* $n \geq 2k+4$.

Lemma 9. (1) $\dim I_{3(n)} = \dim I_{2(n)} + 1$ *if* $n = k+3$ *or* $k+5 \leq n \leq 2k+3$.
(2) $I_{3(n)} = I_{2(n)}$ *if* $n \leq k+2$, $n = k+4$, *or* $n \geq 2k+4$.

Lemma 10. (1) $\dim I_{4(k+4)} = \dim I_{3(k+4)} + 1$.
(2) $I_{4(n)} = I_{3(n)}$ *if* $n \neq k+4$.

These calculations are sufficient to obtain the following theorem.

Theorem 3. (1) $\dim \mathscr{A}/I = k(k+1)/2$.
(2) $J \cap \mathscr{A} = I$.
(3) $I = I_4$.

Recall that $\mathscr{I}$ is the unique maximal ideal of N_0 generated by the singular vector $\mathbf{u}^0$ and $K_0 = N_0/\mathscr{I}$. Hence, $\dim K_0/C_2(K_0) \leq \dim \mathscr{A}/I$. On the other hand, it is shown in [2] that there are at least $k(k+1)/2$ inequivalent irreducible modules for the parafermion vertex operator algebra K_0. Also, Zhu's algebra $A(K_0)$ of K_0 is known to be commutative. In particular, $\dim A(K_0) \geq k(k+1)/2$. Now, $\dim A(V) \leq \dim V/C_2(V)$ for any vertex operator algebra V. Therefore, the following theorem holds.

Theorem 4. (1) $\dim K_0/C_2(K_0) = \dim A(K_0) = k(k+1)/2$ *and* $K_0/C_2(K_0) \cong \mathscr{A}/I$.
(2) K_0 *is* C_2*-cofinite and rational.*

Remark 1. Combining the above theorem with C. Dong and Q. Wang [3, Theorem 4.1], we obtain the C_2-cofiniteness of the parafermion vertex operator algebra $K(\mathfrak{g},k)$ for an arbitrary finite dimensional simple Lie algebra $\mathfrak{g}$.

Acknowledgements Part of our computation was done by a computer algebra system Risa/Asir. The authors are indebted to Kazuhiro Yokoyama for helpful advice concerning the system. T. A. was partially supported by JSPS Grant-in-Aid for Scientific Research (B) No. 20340007 and JSPS Grant-in-Aid for Challenging Exploratory Research No. 23654006. C. L. was partially supported by NSC grant 97-2115-M-006-015-MY3 of Taiwan, R.O.C., H. Y. was partially supported by JSPS Grant-in-Aid for Scientific Research (C) No. 23540009.

References

1. Dong, C., Lam, C.H., Wang, Q., Yamada, H.: J. Algebra **323**, 371–381 (2010)
2. Dong, C., Lam, C.H., Yamada, H.: J. Algebra **322**, 2366–2403 (2009)
3. Dong, C., Wang, Q.: Commun. Math. Phys. **299**, 783–792 (2010)
4. Dong, C., Wang, Q.: J. Algebra **328**, 420–431 (2011)
5. Zhu, Y.: J. Am. Math. Soc. **9**, 237–302 (1996)

Boson-Fermion Correspondence of Type B and Twisted Vertex Algebras

Iana I. Anguelova

Abstract The boson-fermion correspondence of type A is an isomorphism between two super vertex algebras (and so has singularities in the operator product expansions only at $z = w$). The boson-fermion correspondence of type B plays similarly important role in many areas, including representation theory, integrable systems, random matrix theory and random processes. But the vertex operators describing it have singularities in their operator product expansions at both $z = w$ and $z = -w$, and thus need a more general notion than that of a super vertex algebra. In this paper we present such a notion: the concept of a twisted vertex algebra, which generalizes the concept of super vertex algebra. The two sides of the correspondence of type B constitute two examples of twisted vertex algebras. The boson-fermion correspondence of type B is thus an isomorphism between two twisted vertex algebras.

1 Introduction

In 1+1 dimensions (1 time and 1 space dimension) the bosons and fermions are related by the boson-fermion correspondences. The simplest, and best known, case of a boson-fermion correspondence is that of type A, but there are other examples of boson-fermion correspondences, for instance the boson-fermion correspondence of type B, the super boson-fermion correspondences of type A and B, and others. They are extensively studied in many physics and mathematics papers, some of the first and most influential being the papers by Date-Jimbo-Kashiwara-Miwa, Igor Frenkel, Sato and Segal-Wilson, which make the connection between the representation theory of Lie algebras and soliton theory. (Exposition of some of

---

I.I. Anguelova (✉)
College of Charleston, Math Department, 66 George Street,
Charleston SC 29414, USA
e-mail: anguelovai@cofc.edu

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_28,

the mathematical results concerning the boson-fermion correspondence of type A are given in [12, 16].) As with any mathematical concept, there are at least two distinct directions of inquiry. One is: what types of applications and structures we can get as a result of such a boson-fermion correspondence. And the second direction addresses the fundamental questions: What *is* a boson-fermion correspondence? A correspondence of what mathematical structures? For the simplest boson-fermion correspondence, that of type A (often called the charged free boson-fermion correspondence), both these directions of inquiry have been addressed, and applications thereof continue to be found. The first of these directions was also studied first, and the structures, properties, and applications of this boson-fermion correspondence turned out to be very rich and many varied. As was mentioned above, Date-Jimbo-Kashiwara-Miwa and Igor Frenkel discovered its connection to the theory of integrable systems, namely to the KP and KdV hierarchies, to the theory of symmetric polynomials and representation theory of infinite-dimensional Lie algebras, namely the a_∞ algebra, (whence the name "type A" derives), as well as to the $\hat{sl}_n$ and other affine Lie algebras. Their work sparked further interest in it, and there are now connections to many other areas, including number theory and geometry, as well as random matrix theory and random processes (see for example papers by Harnad, Orlov and van de Leur). As this boson-fermion correspondence turned out to have so many applications and connections to various mathematical areas, the natural question needed to be addressed: what *is* a boson-fermion correspondence—a correspondence of what mathematical structures? A partial answer early on was given by Igor Frenkel in [8], but the full answer had to wait for the development of the theory of vertex algebras. Vertex operators were introduced in the earliest days of string theory and now play an important role in many areas such as quantum field theory, integrable models, representation theory, random matrix theory, and many others. The theory of super vertex algebras axiomatizes the properties of some, simplest, "algebras" of vertex operators (see for instance [4, 9–11, 14]). Thus, the answer to the question "what *is* the boson-fermion correspondence of type A" is: the boson-fermion correspondence of type A is an isomorphism between two super vertex algebras [11].

For other well-known boson-fermion correspondences, e.g. the type B, the super correspondence of type B, and others, the question of applications and connections to other mathematical structures already has many answers. For example, Date, Jimbo, Kashiwara and Miwa, who introduced the correspondence of type B, discovered its connection to the theory of integrable systems, namely to the BKP hierarchy [5], to the representation theory of the b_∞ algebra (whence the name "type B" derives), to symmetric polynomials and the symmetric group (some further developments were provided by You in [18]). There are currently studies of its connection to random matrices and random processes by J. Harnad, Van de Leur, Orlov, and others (see for example [17]).

On the other hand the question of "what *is* the boson-fermion correspondence of type B" has not been answered. We know it *is not* an isomorphism between two *super* vertex algebras anymore, as the correspondence of type A was. In this paper we answer this question. To do that we need to introduce the concept of a *twisted* *vertex algebra*, which generalizes the concept of super vertex algebra. The boson-fermion correspondence of type B is then an isomorphism between two *twisted* vertex algebras.

The overview of the paper: first we briefly describe the examples of the two super vertex algebras that constitute the boson-fermion correspondence of type A. We list only one property which is the "imprint" of the boson-fermion correspondence of type A: namely the Cauchy determinant identity that follows from the equality between the vacuum expectation values of the two sides of the correspondence. (As discussed above, there are many, many properties and applications of any boson-fermion correspondence, which can, and do, occupy many papers).

Next we proceed with the definition of a twisted vertex algebra and the examples of the two twisted vertex algebras that constitute the boson-fermion correspondence of type B. Then again we list only one property which is the "imprint" of the boson-fermion correspondence of type B: namely the Schur Pfaffian identity that follows from the equality between the vacuum expectation values of the two sides of the correspondence.

2 Super Vertex Algebras and Boson-Fermion Correspondence of Type A

The following definition is well known, it can be found for instance in [9–11, 14] and others. We recall it for completeness, as "algebras" of fields are the subject of this paper. (Roughly speaking, all vertex algebras, be they super or twisted, are "singular algebras" of fields).

Definition 2.1 (Field). A field $a(z)$ on a vector space V is a series of the form

$$a(z)=\sum_{n\in\mathbf{Z}}a_{(n)}z^{-n-1},\quad a_{(n)}\in \mathrm{End}(V),\ \text{ such that }\ a_{(n)}v=0\ \text{ for any } v\in V,\ n\gg 0.$$

We also recall the following notation: For a rational function $f(z,w)$ we denote by $i_{z,w}f(z,w)$ the expansion of $f(z,w)$ in the region $|z|\gg|w|$, and correspondingly for $i_{w,z}f(z,w)$. Similarly, we will denote by $i_{z_1,z_2,\dots,z_n}$ the expansion in the region $|z_1|\gg\cdots\gg|z_n|$. And lastly, we work with the category of super vector spaces, i.e., $\mathbb{Z}_2$ graded vector spaces. The flip map τ is defined by

$$\tau(a\otimes b)=(-1)^{\tilde{a}\cdot\tilde{b}}(b\otimes a) \tag{1}$$

for any homogeneous elements a,b in the super vector space, where $\tilde{a}$, $\tilde{b}$ denote correspondingly the parity of a, b.

The definition of a super vertex algebra is well known, we refer the reader for example to [9–11, 14], as well as for notations, details and theorems. We only remark that (classical) vertex algebras have two important properties which we would like to carry over to the case of twisted vertex algebras. These are the analytic continuation and completeness with respect to Operator Product Expansions (OPEs). In fact our definition of a twisted vertex algebra is based on enforcing these two properties. Recall we have for the OPE of two fields

$$a(z)b(w) = \sum_{j=0}^{N-1} i_{z,w}\frac{c^j(w)}{(z-w)^{j+1}} + :a(z)b(w):, \tag{2}$$

where $:a(z)b(w):$ denotes the nonsingular part of the expansion of $a(z)b(w)$ as a Laurent series in $(z-w)$. We call $:a(z)b(z):$ a *normal ordered product* of the fields $a(z)$ and $b(z)$. Moreover, $Res_{z=w}a(z)b(w)(z-w)^j = c^j(w) = (a_{(j)}b)(w)$, i.e., the coefficients of the OPEs are fields in the *same* super vertex algebra. Since for the commutation relations only the singular part of the OPEs matters, we abbreviate the OPE above as:

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}. \tag{3}$$

For many examples, super vertex algebras are generated by a much smaller number of generating fields, with imposing the condition that the resulting space of fields of the vertex algebra has to be closed under certain operations: For any field $a(z)$ the field $Da(z) = \partial_z a(z)$ has to be a field in the vertex algebra. Also, the OPEs coefficients ($c^j(w)$ from (3)) and normal ordered products $:a(z)b(z):$ of any two fields $a(z)$ and $b(w)$ have to be fields in the vertex algebra. Note that the identity operator on V is always a trivial field in the vertex algebra, corresponding to the vacuum vector $|0\rangle \in V$. The OPEs are a good indicator of the restrictions placed by the definition of the super vertex algebra: for example, the only functions allowed in the OPEs when the identity field is the coefficient are the $\frac{1}{(z-w)^j}$ with $j \in \mathbb{N}$ (this is clearly seen in (3)). We will use this information later as a way to compare the different generalizations of vertex algebras existing in the math literature, see Remark 3.4 later.

Let us thus turn our attention to the boson-fermion correspondence of type A. The fermion side of the boson-fermion correspondence of type A is a super vertex algebra generated by two nontrivial odd fields—two charged fermions: the fields $\phi(z)$ and $\psi(z)$ with only nontrivial operator product expansion (OPE) (see e.g. [12, 16] and [11]):

$$\phi(z)\psi(w) \sim \frac{1}{z-w} \sim \psi(z)\phi(w), \tag{4}$$

where the 1 above denotes the identity map Id. The modes ϕ_n and ψ_n, $n \in \mathbf{Z}$ of the fields $\phi(z)$ and $\psi(z)$, which we index as follows:

$$\phi(z) = \sum_{n\in\mathbf{Z}} \phi_n z^n, \quad \psi(z) = \sum_{n\in\mathbf{Z}} \psi_n z^n, \tag{5}$$

form a Clifford algebra Cl_A with relations

$$[\phi_m,\psi_n]_\dagger=\delta_{m+n,-1}1,\quad [\phi_m,\phi_n]_\dagger=[\psi_m,\psi_n]_\dagger=0. \tag{6}$$

The indexing of the generating fields vary depending on the point of view; our indexing here corresponds to $\phi_n=\hat{v}_{n+1},\quad \psi_n=\check{v}^*_{-n}$ in [12]. This indexing and the properties of the vertex algebra dictate that the underlying space of states of this super vertex algebra—the fermionic Fock space—is the highest weight representation of Cl_A generated by the vacuum vector $|0\rangle$, so that $\phi_n|0\rangle=\psi_n|0\rangle=0$ for $n<0$.

We denote both the space of states and the resulting vertex algebra generated by the fields $\phi(z)$ and $\psi(z)$ by F_A. It is often called the charged free fermion vertex algebra.

We can calculate vacuum expectation values if we have a symmetric bilinear form $\langle\ |\ \rangle: V\otimes V\to\mathbb{C}$ on the space of states of the vertex algebra V. Recall[1] it is required that the bilinear form is normalized on the vacuum vector $|0\rangle$: by abuse of notation we just write $\langle 0\mid 0\rangle=1$ instead of $\langle\langle 0|\mid|0\rangle\rangle=1$. Also, the vacuum has to be orthogonal to the generating states (the states $\phi=\phi_0|0\rangle$, $\psi=\psi_0|0\rangle$) and their descendants (the states $\phi_n|0\rangle=\frac{D^n}{n!}\phi(z)|0\rangle|_{z=0}$ and $\psi_n|0\rangle=\frac{D^n}{n!}\psi(z)|0\rangle|_{z=0},\ n>0$).

Lemma 2.2. *The following determinant formula for the vacuum expectation values on the fermionic side F_A holds [16]:*

$$\langle 0|\phi(z_1)\phi(z_2)\dots\phi(z_n)\psi(w_1)\psi(w_2)\dots\psi(w_n)|0\rangle=(-1)^{n(n-1)/2}i_{z;w}det\Big(\frac{1}{z_i-w_j}\Big)_{i,j=1}^n.$$

Here $i_{z;w}$ stands for the expansion $i_{z_1,z_2,\dots,z_n,w_1,\dots,w_n}$.

The proof is usually given using Wick's formula, see [16], although in [1] we give a proof depending entirely on the underlying Hopf algebra structure.

The boson-fermion correspondence of type A is determined once we write the images of generating fields $\phi(z)$ and $\psi(z)$ under the correspondence. In order to do that, an *essential* ingredient is the so-called Heisenberg field $h(z)$ given by

$$h(z)=:\phi(z)\psi(z): \tag{7}$$

It follows that the Heisenberg field $h(z)=\sum_{n\in\mathbb{Z}}h_nz^{-n-1}$ has OPEs with itself given by:

$$h(z)h(w)\sim\frac{1}{(z-w)^2},\quad \text{in modes: } [h_m,h_n]=m\delta_{m+n,0}1. \tag{8}$$

i.e., its modes $h_n,\ n\in\mathbb{Z}$, generate a Heisenberg algebra $\mathscr{H}_{\mathbb{Z}}$. It is well known that any irreducible highest weight module of this Heisenberg algebra is isomorphic to the polynomial algebra with infinitely many variables $B_m\cong\mathbb{C}[x_1,x_2,\dots,x_n,\dots]$.

[1]There is a very important concept of an invariant bilinear form on a vertex algebra, for details see for example [10, 13].

The fermionic Fock space decomposes (via the charge decomposition, for details see for example [12]) as $F_A = \oplus_{i\in\mathbb{Z}} B_i$, which we can write as

$$F_A = \oplus_{i\in\mathbb{Z}} B_i \cong \mathbb{C}[e^{\alpha}, e^{-\alpha}] \otimes \mathbb{C}[x_1, x_2, \dots, x_n, \dots], \tag{9}$$

where by $\mathbb{C}[e^{\alpha}, e^{-\alpha}]$ we mean the Laurent polynomials with one variable e^{α}.[2] The isomorphism is as Heisenberg modules, where $e^{n\alpha}$ is identified as the highest weight vector for the irreducible Heisenberg module B_n. We denote the vector space on the right-hand-side of this $\mathscr{H}_{\mathbb{Z}}$-module isomorphism by B_A. B_A is then the underlying vector space of the bosonic side of the boson-fermion correspondence of type A.

Now we can write the images of generating fields $\phi(z)$ and $\psi(z)$ under the correspondence:

$$\phi(z) \mapsto e^{\alpha}(z), \quad \psi(z) \mapsto e^{-\alpha}(z), \tag{10}$$

where the generating fields $e^{\alpha}(z)$, $e^{-\alpha}(z)$ for the bosonic part of the correspondence are given by

$$e^{\alpha}(z) = \exp\Big(\sum_{n\geq 1} \frac{h_{-n}}{n} z^n\Big) \exp\Big(-\sum_{n\geq 1} \frac{h_n}{n} z^{-n}\Big) e^{\alpha} z^{\partial_\alpha},$$

$$e^{-\alpha}(z) = \exp\Big(-\sum_{n\geq 1} \frac{h_{-n}}{n} z^n\Big) \exp\Big(\sum_{n\geq 1} \frac{h_n}{n} z^{-n}\Big) e^{-\alpha} z^{-\partial_\alpha},$$

the operators e^{α}, $e^{-\alpha}$, z^{∂_α} and $z^{-\partial_\alpha}$ act in an obvious way on the space B_A.

The resulting super vertex algebra generated by the fields $e^{\alpha}(z)$ and $e^{-\alpha}(z)$ with underlying vector space B_A we denote also by B_A.

Lemma 2.3. *The following product formula for the vacuum expectation values on the bosonic side B_A holds:*

$$\langle 0| e^{\alpha}(z_1) e^{\alpha}(z_2) \dots e^{\alpha}(z_n) e^{-\alpha}(w_1) e^{-\alpha}(w_2) \dots e^{-\alpha}(w_n) |0\rangle = i_{z;w} \frac{\prod_{i<j}^n ((z_i - z_j)(w_i - w_j))}{\prod_{i,j=1}^n (z_i - w_j)}$$

Here $i_{z;w}$ stands for the expansion $i_{z_1, z_2, \dots, z_n, w_1, \dots, w_n}$.

Theorem 2.4 ([11]). *The boson-fermion correspondence of type A is the isomorphism between the charged free fermion super vertex algebra F_A and the bosonic super vertex algebra B_A.*

Lemma 2.5. *The Cauchy's determinant identity follows from the equality of the vacuum expectation values:*

$$(-1)^{n(n-1)/2} det\Big(\frac{1}{z_i - w_j}\Big)_{i,j=1}^n = AC\langle 0|\phi(z_1) \dots \phi(z_n) \psi(w_1) \dots \psi(w_n)|0\rangle =$$

[2] The reason for this notation is that the resulting vertex algebra is a lattice vertex algebra.

$$= AC\langle 0|e^{\alpha}(z_1)\dots e^{\alpha}(z_n)e^{-\alpha}(w_1)\dots e^{-\alpha}(w_n)|0\rangle = \frac{\prod_{i<j}(z_i - z_j)\prod_{i<j}(w_i - w_j)}{\prod_{i,j=1}^{n}(z_i - w_j)}$$

AC stands for Analytic Continuation.

The Cauchy determinant identity (the equality between the first and the fourth expressions) is a historic identity and is well known, one of the oldest references being [15]. The proof of it is usually given using factorization, remarkably even in [16], see Remark 5.1 there. The point here is that the Cauchy identity follows immediately from the equality of the vacuum expectation values of both sides of the boson-fermion correspondence, and is a quintessential "imprint" of the correspondence (although this identity is absent in some standard mathematical references of the boson-fermion correspondence of type A like [12] and [11]).

3 Twisted Vertex Algebras and Boson-Fermion Correspondence of Type B

Here we will only give the definition for a twisted vertex algebra of order 2, for more general definition and details see [1]. We begin with some preliminaries.

Definition 3.1. The Hopf algebra $H_{T_{-1}}$ is the Hopf algebra with a primitive generator D and a grouplike generator T_{-1} subject to the relations:

$$DT_{-1} = -T_{-1}D, \quad \text{and } (T_{-1})^2 = 1 \tag{11}$$

Denote by $\mathbf{F}^2_{\pm}(z,w)$ the space of rational functions in the variables $z,w \in \mathbb{C}$ with only poles at $z = 0,\ z = \pm w$. Note that we do not allow poles at $w = 0$, i.e., if $f(z,w) \in \mathbf{F}^2_{\pm}(z,w)$, then $f(z,0)$ is well defined. Similarly, $\mathbf{F}^2_{\pm}(z_1,z_2,\dots,z_l)$ is the space of rational functions in variables $z_1,z_2,\dots,z_l$ with only poles at $z_1 = 0$, or $z_j = \pm z_k$. $\mathbf{F}^2_{\pm}(z,w)$ is a $H_{T_{-1}} \otimes H_{T_{-1}}$ Hopf algebra module by

$$D_z f(z,w) = \partial_z f(z,w), \quad (T_{-1})_z f(z,w) = f(-z,w) \tag{12}$$

$$D_w f(z,w) = \partial_w f(z,w), \quad (T_{-1})_w f(z,w) = f(z,-w) \tag{13}$$

We will denote the action of elements $h \otimes 1 \in H_{T_{-1}} \otimes H_{T_{-1}}$ on $\mathbf{F}_{\pm}(z,w)$ by $h_z\cdot$, and similarly $h_w\cdot$ will denote the action of elements $1 \otimes h \in H_{T_{-1}} \otimes H_{T_{-1}}$.

Definition 3.2 (Twisted vertex algebra of order 2). Twisted vertex algebra of order 2 is a collection of the following data:

- the space of states: a vector space W;
- the space of fields: a vector super space V—an $H_{T_{-1}}$ module, such that $V \supset W$;
- a projection: a linear map $\pi_f : V \to W$;
- a field-state correspondence: a linear map from V to the space of fields on W;
- a vacuum vector: a vector $1 = |0\rangle \in W \subset V$.

This data should satisfy the following set of axioms:

- Vacuum axiom: $Y(1,z) = Id_W$;
- Modified creation axiom: $Y(a,z)1|_{z=0} = \pi_f(a)$, for any $a \in V$;
- Transfer of action: $Y(ha,z) = h_z \cdot Y(a,z)$ for any $h \in H_{T_{-1}}$;
- Analytic continuation: For any $a_i \in V, i = 1, \dots, k$, the composition $Y(a_1,z_1)Y(a_2,z_2)\dots Y(a_k,z_k)1$ converges in the domain $|z_1| \gg \cdots \gg |z_k|$ and can be continued to a rational vector valued function

$$X_{z_1,z_2,\dots,z_k}(a_1 \otimes a_2 \otimes \cdots \otimes a_k) : V^{\otimes k} \to W \otimes \mathbf{F}^2_{\pm}(z_1,z_2,\dots,z_k),$$

  so that $Y(a_1,z_1)Y(a_2,z_2)\dots Y(a_k,z_k)1 = i_{z_1,z_2,\dots,z_k} X_{z_1,z_2,\dots,z_k}(a_1 \otimes a_2 \otimes \cdots \otimes a_k)$
- Supercommutativity: $X_{z,w}(a \otimes b) = X_{w,z}(\tau(a \otimes b))$, with τ defined in (1).
- Completeness with respect to OPEs (modified): For each $n \in \mathbb{N}$ there exists $l_n \in \mathbf{Z}$ such that $Res_{z=\pm w} X_{z,w,0}(a \otimes b \otimes v)(z \mp w)^n = Y(c_n,w)\pi_f(v)w^{l_n}$ for some $c_n \in V$.

Remark 3.1. The axiom/property requiring completeness with respect to the OPEs is a weaker one than in the classical vertex algebra case. We can express this weaker axiom by saying that the modes of the OPE coefficients, the residues $Res_{z=\pm w}(z \mp w)^n Y(a,z)Y(b,w)$, are the modes of a field that belongs to the twisted algebra, modulo a shift (a shift by w^{l_n} is allowed, but no more). The stronger property is violated in the interesting examples, see for instance Remark 3.4 below. □

Remark 3.2. The axiom of analytic continuation expresses two requirements: one, that the products of fields are expansions of *rational* operator valued functions in appropriate regions; and two, that the only poles of these rational functions are at $z = \pm w$. Thus, if we think of the variable z as being a square root of another variable $\tilde{z}$ (similarly $w = \sqrt{\tilde{w}}$), then the only allowed singularities are at $z^2 = w^2$, i.e., at $\tilde{z} = \tilde{w}$, which is a prerequisite for the usual locality axiom (in the variable $\tilde{z}$). The usual locality axiom requires not only that the singularities are located only at $\tilde{z} = \tilde{w}$, but also that the supercommutativity axiom $X_{z,w}(a \otimes b) = X_{w,z}(\tau(a \otimes b))$ holds. These two axioms combined produce the commutation or anticommutation relations obeyed by the bosons or fermions correspondingly. If we remove the supercommutativity axiom, then we can have a more general braided locality, instead of the usual locality, and there are examples (e.g., the quantum affine Lie algebras at roots of unity) which do not obey the supercommutativity axiom. But the examples of boson-fermion correspondences do indeed obey the supercommutativity axiom, which is why we have required it as part of our definition of a twisted vertex algebra. □

Remark 3.3. There are other generalizations of the notion of super vertex algebra, for instance there is the notion(s) of a twisted module for a vertex algebra (see [2,9]); there are also the notions of a generalized vertex algebra (see e.g. [3,6]). The notion of a twisted vertex algebra as outlined above is different from any of those notions in two main respects: the first difference is in the functions allowed in the OPEs (see Remark 3.4 below); the second is the fact that the space of fields is larger than the space of states (the space of states in the examples here is a proper projection of the space of fields). In this, the notion of twisted vertex algebra resembles the notion of a Deformed Chiral Algebra of [7], and although there are differences, one can think of a twisted vertex algebra as being the root of unity symmetric version of the Deformed Chiral Algebra concept. □

Similarly to super vertex algebras, twisted vertex algebras are often generated by a smaller number of fields (for a rigorous theorem regarding that see [1]). The space of fields is then determined by requiring, as before, that it is closed under OPEs (see modification above). Also, as before, for any field $a(z)$ the field $Da(z) = \partial_z a(z)$ again has to be a field in the twisted vertex algebra. But now we also require that the field $T_{-1}a(z) = a(-z)$ is a field in the twisted vertex algebra of order 2 as well. Note that this immediately violates the stronger creation axiom for a classical vertex algebra, hence any such field cannot belong to a classical vertex algebra. This is the reason we require the modified field-state correspondence with the *modified creation axiom* for a twisted vertex algebra.

We now proceed with the two examples of a twisted vertex algebra of order 2 which give the two sides of the boson-fermion correspondence of type B. The fermionic side is generated by a single field $\phi(z) = \sum_{n\in\mathbf{Z}} \phi_n z^n$, with OPEs with itself given by ([5, 18]—modulo a factor of 2; [1]):

$$\phi(z)\phi(w) \sim \frac{z-w}{z+w}, \quad \text{in modes: } [\phi_m, \phi_n]_\dagger = 2(-1)^m \delta_{m,-n} 1. \tag{14}$$

Thus the modes generate a Clifford algebra Cl_B, and the underlying space of states, denoted by F_B, of the twisted vertex algebra is a highest weight representation of Cl_B with the vacuum vector $|0\rangle$, such that $\phi_n|0\rangle = 0$ for $n < 0$. The space of fields, which is larger than the space of states, is generated by the field $\phi(z)$ together with its descendent $T_{-1}\phi(z) = \phi(-z)$. We call the resulting twisted vertex algebra the *free neutral fermion of type B*[3], and denote also by F_B.

Remark 3.4. If we look at the defining OPE, (14), we can see that if we just write the singular part, we have the residue $Res_{z=-w}\phi(z)\phi(w) = -2w\cdot 1 = -2wId_W$, which can not be a field in any vertex algebra as it is. But a shift by w^{-1} will produce the field $-2Id_W$, which is the field corresponding to the $-2|0\rangle$. This exemplifies that in the OPEs of twisted vertex algebras when the identity field is the coefficient we do allow any function of the type $\frac{w^k}{(z\pm w)^l}$, where $k \in \mathbb{Z}_{\geq 0},\ l \in \mathbb{N}$. For instance, besides the classical vertex algebra singularity $\frac{1}{z-w}$, we allow additionally, and on its own, $\frac{1}{z+w}$, which is not allowed in twisted modules for vertex algebras, nor in the notions of generalized vertex algebras. In contrast, if the identity field is the coefficient,

[3] The reason for the name is that there is a free neutral fermion of type D, which is commonly referred to as just "the free neutral fermion". In fact, there is a boson-fermion correspondence of type D-C, see [1].

twisted modules for vertex algebras would only allow singularities in its OPEs of the type

$$\partial_{\tilde{w}}^{l}\frac{\sqrt{\tilde{z}}}{\sqrt{\tilde{w}}}\frac{1}{\tilde{z}-\tilde{w}}=\partial_{w^2}^{l}\frac{z}{w(z^2-w^2)},\quad l\in\mathbb{Z}_{\geq 0},$$

i.e., even though the singularities for twisted modules are indeed at $z=\pm w$, only particular combinations of $\frac{1}{z-w}$, $\frac{1}{z+w}$ and derivatives are allowed (see e.g. [2]). In the case of a generalized vertex algebra the singularities allowed in the OPEs when the vacuum field is the coefficient are of the type $\frac{1}{(\tilde{z}-\tilde{w})^{1/N}}=\frac{1}{(z^2-w^2)^{1/N}}$, $N\in\mathbb{N}$. □

The boson-fermion correspondence of type B is again determined once we write the image of the generating fields $\phi(z)$ (and thus of $T_{-1}\phi(z)=\phi(-z)$) under the correspondence. In order to do that, an essential ingredient is once again the *twisted* Heisenberg field $h(z)$ given by[4]

$$h(z)=\frac{1}{4}:\phi(z)T_{-1}\phi(z):=\frac{1}{4}:\phi(z)\phi(-z): \tag{15}$$

It follows that the twisted Heisenberg field, which due to the symmetry above has only odd-indexed modes, $h(z)=\sum_{n\in\mathbb{Z}}h_{2n+1}z^{-2n-1}$, has OPEs with itself given by:

$$h(z)h(w)\sim\frac{zw(z^2+w^2)}{2(z^2-w^2)^2}, \tag{16}$$

Its modes, h_n, $n\in 2\mathbb{Z}+1$, generate a *twisted* Heisenberg algebra $\mathscr{H}_{\mathbb{Z}+1/2}$ with relations $[h_m,h_n]=\frac{m}{2}\delta_{m+n,0}1$, m,n are now odd integers. It has (up-to isomorphism) only one irreducible module $B_{1/2}\cong\mathbb{C}[x_1,x_3,\dots,x_{2n+1},\dots]$. The fermionic space of states F_B decomposes as $F_B=B_{1/2}\oplus B_{1/2}$ (for details, see [1, 5, 18]). We can write this as an isomorphism of twisted Heisenberg modules for $\mathscr{H}_{\mathbb{Z}+1/2}$ in a similar way to the type A correspondence:

$$F_B=B_{1/2}\oplus B_{1/2}\cong\mathbb{C}[e^{\alpha},e^{-\alpha}]\otimes\mathbb{C}[x_1,x_3,\dots,x_{2n+1},\dots], \tag{17}$$

but now we have the extra relation $e^{2\alpha}\equiv 1$, i.e., $e^{\alpha}\equiv e^{-\alpha}$. The right-hand-side, which we denote by B_B, is the underlying vector space of *states* of the bosonic side of the boson-fermion correspondence of type B.

Now we can write the image of the generating field $\phi(z)\mapsto e^{\alpha}(z)$, which will determine the correspondence of type B (for proof see [1]):

$$e^{\alpha}(z)=\exp\Big(\sum_{k\geq 0}\frac{h_{-2k-1}}{k+1/2}z^{2k+1}\Big)\exp\Big(-\sum_{k\geq 0}\frac{h_{2k+1}}{k+1/2}z^{-2k-1}\Big)e^{\alpha}, \tag{18}$$

[4] For details on normal ordered products in this more general case see [1], the construction uses an additional Hopf algebra structure, similar to Laplace pairing.

The fields $e^{\alpha}(z)$ and $e^{\alpha}(-z)=e^{-\alpha}(z)$ (observe the symmetry) generate the resulting *twisted* vertex algebra, which we denote also by B_B.

Note that one Heisenberg $\mathscr{H}_{\mathbb{Z}+1/2}$-module $B_{1/2}$ on its own can be realized as a *twisted module* for an ordinary super vertex algebra (see [9] for details), but the point is that we need *two* of them glued together for the bosonic side of the correspondence. The two of them glued together as above no longer constitute a twisted module for an ordinary super vertex algebra.

Theorem 3.3. *The boson-fermion correspondence of type B is the isomorphism between the fermionic twisted vertex algebra F_B and the bosonic twisted vertex algebra B_B.*

Lemma 3.4. *The Schur Pfaffian identity follows from the equality between the vacuum expectation values:*

$$AC\langle 0|\phi(z_1)\ldots\phi(z_{2n})|0\rangle = Pf\left(\frac{z_i-z_j}{z_i+z_j}\right)_{i,j=1}^{2n} = \prod_{i<j}^{2n}\frac{z_i-z_j}{z_i+z_j} = AC\langle 0|e^{\alpha}(z_1)\ldots e^{\alpha}(z_{2n})|0\rangle$$

Here Pf denotes the Pfaffian of an antisymmetric matrix, AC stands for Analytic Continuation.

Remark. The general definition of a twisted vertex algebra of order N, details and proofs can be found in [1].

In conclusion, we would like to thank the organizers of the International Workshop “Lie Theory and its Applications in Physics” for a most enjoyable and productive workshop, and may it continue for many years to come!

References

1. Anguelova, Iana I.: Twisted vertex algebras, bicharacter construction and boson-fermion correspondences. (arxiv:1206.4026[math-ph]), preprint
2. Bakalov, B., Kac, V.: Twisted modules over lattice vertex algebras. In: Lie Theory and Its Applications in Physics V, pp. 3–26. World Scietific Publishing, River Edge, NJ (2004)
3. Bakalov, B., Kac, V.: Generalized vertex algebras. In: Doebner, H.-D., Dobrev, V.K. (eds.) Proceedings of the 6-th International Workshop “Lie Theory and Its Applications in Physics” (LT-6), Varna, Bulgaria, pp. 3–25. Heron Press (2006)
4. Borcherds, Richard E.: Vertex algebras, Kac-Moody algebras, and the Monster. Proc. Natl. Acad. Sci. USA **83**(10), 3068–3071 (1986)
5. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. IV. A new hierarchy of soliton equations of KP-type. Phys. D **4**(3), 343–365 (1981/82)
6. Dong, C., Lepowsky, J.: Generalized Vertex Algebras and Relative Vertex Operators, vol. 112 of Progress in Mathematics. Birkhäuser Boston, Boston, MA (1993)
7. Frenkel, E., Reshetikhin, N.: Towards deformed chiral algebras. In: Proceedings of the Quantum Group Symposium at the XXIth International Colloquium on Group Theoretical Methods in Physics, Goslar 1996, pp. 6023–+ (1997)

8. Frenkel, I.B.: Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory. J. Funct. Anal. **44**(3), 259–327 (1981)
9. Frenkel, I.B., Lepowsky, J., Meurman, A.: Vertex Operator Algebras and the Monster, vol. 134 of Pure and Applied Mathematics. Academic Press, Boston, MA (1988)
10. Frenkel, I.B., Huang, Y-Z., Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules. Mem. Am. Math. Soc. **104**(494), viii+64 (1993)
11. Kac, V.: Vertex algebras for beginners, vol. 10 of University Lecture Series, 2nd edn. American Mathematical Society, Providence, RI (1998)
12. Kac, V., Raina, A.K.: Bombay lectures on highest weight representations of infinite-dimensional Lie algebras, vol. 2 of Advanced Series in Mathematical Physics. World Scientific Publishing, Teaneck, NJ (1987)
13. Li, H.S.: Symmetric invariant bilinear forms on vertex operator algebras. J. Pure Appl. Algebra **96**(3), 279–297 (1994)
14. Lepowsky, J., Li, H.: Introduction to vertex operator algebras and their representations, vol. 227 of Progress in Mathematics. Birkhäuser Boston, Boston, MA (2004)
15. Littlewood, D.E.: A University Algebra: An Introduction to Classic and Modern Algebra. Dover, New York (1970)
16. Miwa, T., Jimbo, M., Date, E.: Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (2000)
17. van de Leur, J.W., Orlov, A. Yu.: Random turn walk on a half line with creation of particles at the origin. Phys. Lett. A **373**(31), 2675–2681 (2009)
18. You, Y.: Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups. In: Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), vol. 7 of Adv. Ser. Math. Phys., pp. 449–464. World Scientific Publishing, Teaneck, NJ (1989)

On Twisted Modules for N=2 Supersymmetric Vertex Operator Superalgebras

Katrina Barron

Abstract The classification of twisted modules for N=2 supersymmetric vertex operator superalgebras with twisting given by vertex operator superalgebra automorphisms which are lifts of a finite automorphism of the N=2 Neveu–Schwarz Lie superalgebra representation is presented. These twisted modules include the Ramond-twisted sectors and mirror-twisted sectors for N=2 vertex operator super algebras, as well as twisted modules related to more general "spectral flow" representations of the N=2 Neveu–Schwarz algebra.

1 Introduction

We give an expository presentation following [6] of the classification of twisted modules for N=2 superconformal vertex operator superalgebras for the case of vertex operator superalgebra automorphisms that arise from finite Virasoro-preserving automorphisms of the underlying N=2 Neveu–Schwarz algebra.

For g an automorphism of a vertex operator superalgebra (VOSA), V, we have the notion of "g-twisted V-module". Twisted vertex operators were discovered and used in [24], and twisted modules arose in [19] in the course of the construction of the moonshine module vertex operator algebra. This structure came to be understood as an "orbifold model" in the sense of conformal field theory and string theory. Twisted modules are the mathematical counterpart of "twisted sectors", which are the basic building blocks of orbifold models in conformal field theory and string theory. The notion of twisted module for VOSAs was developed in [25]. In general, it is an open problem as to how to construct a g-twisted V-module.

K. Barron (✉)
University of Notre Dame, Department of Mathematics, Notre Dame, IN 46556 USA
e-mail: kbarron@nd.edu

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_29,

An automorphism g of a VOSA, in particular, fixes the Virasoro vector, and thus also fixes the corresponding endomorphisms giving the representation of the Virasoro algebra. A VOSA is said to be "N=2 supersymmetric", if in addition to being a positive energy representation for the Virasoro algebra, it is a representation of the N=2 Neveu–Schwarz Lie superalgebra, an extension of the Virasoro algebra; see, for instance, [5, 11]. The group of automorphisms of the N=2 Neveu–Schwarz algebra over $\mathbb{C}$, which preserve the Virasoro algebra, is isomorphic to $\mathbb{C}^\times \times \mathbb{Z}_2$. It is generated by a continuous family of automorphisms, denoted by σ_ξ for $\xi \in \mathbb{C}^\times$, and an order two automorphism κ called the "mirror map". If ξ is a root of unity, then σ_ξ is of finite order.

Given an N=2 supersymmetric VOSA, V, some questions naturally arise: When does κ or σ_ξ lift to an automorphism of V, and when is this lift unique? When such an automorphism of the N=2 Neveu–Schwarz algebra does lift to an automorphism g of V, what is the structure of a g-twisted V module? In this paper, we present the answer to the second question, and in [6] we provide the details of this study and answer the first question for free and lattice N=2 VOSAs.

If the mirror automorphism κ of the N=2 Neveu–Schwarz algebra lifts to a VOSA automorphism of an N=2 VOSA, V, then a "mirror-twisted V-module" is naturally a representation of what we call the "mirror-twisted N=2 superconformal algebra", which is also referred to as the "twisted N=2 superconformal algebra" [16,26,29], or the "topological N=2 superconformal algebra" [21]. If the automorphism σ_ξ of the N=2 Neveu–Schwarz algebra, for ξ a root of unity, lifts to a VOSA automorphism of V, then we show that a "σ_ξ-twisted V-module" is naturally a representation of one of the algebras in the one-parameter family of Lie superalgebras we call "shifted N=2 superconformal algebras". If $\xi = -1$, then σ_ξ is the parity map, and such a shifted N=2 superconformal algebra is the N=2 Ramond algebra. The N=2 Ramond algebra and the other shifted N=2 algebras are isomorphic, as Lie superalgebras, to the N=2 Neveu–Schwarz algebra via the "spectral flow" operators, as was first realized in [29]. The mirror-twisted N=2 algebra is not isomorphic to the N=2 Neveu–Schwarz algebra.

The representation theory of the N=2 Neveu–Schwarz algebra has been studied in, for instance, [9–11, 13–15, 17, 18, 23, 27, 28, 30, 31] and from a VOSA theoretic point of view in [1]. The representation theory of the N=2 Ramond algebra has been studied in, e.g, [13, 18, 20, 21, 30], and of the mirror-twisted N=2 superconformal algebra in, e.g, [13, 16, 21, 22, 26].

The realization of the N=2 Ramond algebra (i.e. the $\frac{1}{2}$-shifted N=2 superconformal algebra) and the mirror-twisted N=2 superconformal algebra as arising from twisting an N=2 VOSA (or comparable structure) has long been known, e.g. [9, 11, 29]. However to our knowledge, the other algebras related to the N=2 Neveu–Schwarz algebra—the shifted N=2 superconformal algebras other than the N=2 Ramond algebra—have only been studied through the spectral flow operators (which do not preserve the Virasoro algebra). We believe that the realization of these algebras as arising naturally as twisted modules for an N=2 VOSA is new.

Thus this complete classification of the twisted modules for an N=2 VOSA for finite automorphisms arising from Virasoro-preserving automorphisms of the N=2 Neveu–Schwarz algebra provides a uniform way of understanding and studying all of the N=2 superconformal algebras—the continuous one-parameter family of shifted N=2 Neveu–Schwarz algebras and the mirror-twisted N=2 superconformal algebra—in the context of the theory of VOSAs and their twisted modules.

2 The N=1 and N=2 Superconformal Algebras and Their Virasoro-Preserving Automorphisms

The *N=1 Neveu–Schwarz algebra* or *N=1 superconformal algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n for $n \in \mathbb{Z}$, and odd elements G_r for $r \in \mathbb{Z}+\frac{1}{2}$, and supercommutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\, d, \tag{1}$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \quad [G_r, G_s] = 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\, d, \tag{2}$$

for $m,n \in \mathbb{Z}$, and $r,s \in \mathbb{Z}+\frac{1}{2}$. The *N=1 Ramond algebra* is the Lie superalgebra with basis the central element d, even elements L_n for $n \in \mathbb{Z}$, and odd elements G_r for $r \in \mathbb{Z}$, and supercommutation relations given by (1)–(2), where now $r,s \in \mathbb{Z}$.

Note that the only nontrivial Lie superalgebra automorphism of the N=1 Neveu–Schwarz algebra which preserves the Virasoro algebra is the parity automorphism which is the identity on the even subspace (the Virasoro Lie algebra) and acts as -1 on the odd subspace (the subspace spanned by G_r for $r \in \mathbb{Z}+\frac{1}{2}$).

The *N=2 Neveu–Schwarz Lie superalgebra* or *N=2 superconformal algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n and J_n for $n \in \mathbb{Z}$, and odd elements $G_r^{(j)}$ for $j = 1,2$ and $r \in \mathbb{Z}+\frac{1}{2}$, and such that the supercommutation relations are given as follows: L_n, d and $G_r^{(j)}$ satisfy the supercommutation relations for the N=1 Neveu–Schwarz algebra (1)–(2) for both $G_r = G_r^{(1)}$ and for $G_r = G_r^{(2)}$; the remaining relations are given by

$$[L_m, J_n] = -nJ_{m+n}, \quad [J_m, J_n] = \frac{1}{3}m\delta_{m+n,0}d, \tag{3}$$

$$\left[J_m, G_r^{(1)}\right] = -iG_{m+r}^{(2)}, \left[J_m, G_r^{(2)}\right] = iG_{m+r}^{(1)}, \left[G_r^{(1)}, G_s^{(2)}\right] = -i(r-s)J_{r+s}. \tag{4}$$

The *N=2 Ramond algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n and J_n for $n \in \mathbb{Z}$, and odd elements $G_r^{(j)}$ for $r \in \mathbb{Z}$ and $j = 1,2$, and supercommutation relations given by those of the N=2 Neveu–Schwarz algebra but with $r,s \in \mathbb{Z}$, instead of $r,s \in \mathbb{Z}+\frac{1}{2}$.

More generally, there is an infinite family of algebras which includes the N=2 Neveu–Schwarz and Ramond algebras. However these are easier to express if we make a change of basis which is ubiquitous in superconformal field theory. So consider the substitutions $G_r^{(1)} = \frac{1}{\sqrt{2}}(G_r^+ + G_r^-)$, and $G_r^{(2)} = \frac{i}{\sqrt{2}}(G_r^+ - G_r^-)$, or equivalently $G_r^\pm = \frac{1}{\sqrt{2}}(G_r^{(1)} \mp iG_r^{(2)})$. This substitution is equivalent to the change of variables $\varphi^\pm = \frac{1}{\sqrt{2}}(\varphi^{(1)} \pm i\varphi^{(2)})$ in the variables $(x, \varphi^{(1)}, \varphi^{(2)})$ representing the one even and two odd local coordinates on an N=2 superconformal worldsheet representing superstrings propagating in space-time in N=2 superconformal field theory, cf. [4]. In terms of this basis (called the *homogeneous basis*), the N=2 Neveu–Schwarz (or Ramond) algebra supercommutation relations are given by (1), (3) and

$$\left[L_m, G_r^\pm\right] = \left(\frac{m}{2} - r\right) G_{m+r}^\pm, \quad \left[J_m, G_r^\pm\right] = \pm G_{m+r}^\pm, \quad \left[G_r^\pm, G_s^\pm\right] = 0, \quad (5)$$

$$\left[G_r^+, G_s^-\right] = 2L_{r+s} + (r-s)J_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\, d, \quad (6)$$

for $m,n \in \mathbb{Z}$, and $r,s \in \mathbb{Z} + \frac{1}{2}$, or $r,s \in \mathbb{Z}$, respectively.

Observe then that there is also the notion of a Lie superalgebra generated by even elements L_n and J_n for $n \in \mathbb{Z}$ and by odd elements $G_{r\pm t}^\pm$, for $r \in \mathbb{Z} + \frac{1}{2}$ and for $t \in \mathbb{C}$. We shall call this algebra the *t-shifted N=2 superconformal algebra* or *t-shifted N=2 Neveu–Schwarz algebra*. Thus the t-shifted N=2 Neveu–Schwarz algebra is the N=2 Neveu–Schwarz algebra if $t \in \mathbb{Z}$, and is the N=2 Ramond algebra if $t \in \mathbb{Z} + \frac{1}{2}$. As was first shown in [29], the t-shifted N=2 Neveu–Schwarz algebras are all isomorphic under the continuous family of *spectral flow* maps, denoted $\mathscr{D}(t)$, for $t \in \mathbb{C}$, but which fix the Virasoro algebra only for $t = 0$. These are given by

$$\mathscr{D}(t): \quad \begin{array}{lll} L_n \mapsto L_n + tJ_n + \frac{t^2}{6}\delta_{n,o}d, & d \mapsto d, \\ J_n \mapsto J_n + \frac{t}{3}\delta_{n,0}d, & G_r^\pm \mapsto G_{r\pm t}^\pm. \end{array} \quad (7)$$

The group of automorphisms of the N=2 Neveu–Schwarz algebra (or more generally the t-shifted N=2 superconformal algebras) which preserve the Lie subalgebra generated by L_n and J_n for $n \in \mathbb{Z}$ are given by:

$$\sigma_\xi: \quad G_r^\pm \mapsto \xi^{\pm 1} G_r^\pm, \quad J_n \mapsto J_n, \quad L_n \mapsto L_n, \quad d \mapsto d, \quad (8)$$

for $\xi \in \mathbb{C}^\times$. In addition, we have the *mirror map*, given by:

$$\kappa: \quad G_r^\pm \mapsto G_r^\mp, \quad J_n \mapsto -J_n, \quad L_n \mapsto L_n, \quad d \mapsto d. \quad (9)$$

The family σ_ξ along with κ generate the Virasoro-preserving automorphisms of the N=2 Neveu–Schwarz algebra, and thus this group is isomorphic to $\mathbb{Z}_2 \times \mathbb{C}^\times$, cf. [5].

The *mirror-twisted N=2 Neveu–Schwarz algebra* is defined to be the Lie superalgebra with basis consisting of even elements L_n, and J_r and central element d, odd elements $G_r^{(1)}$ and $G_n^{(2)}$, for $n \in \mathbb{Z}$ and $r \in \mathbb{Z}+\frac{1}{2}$, and supercommutation relations given as follows: The L_n and $G_r^{(1)}$ satisfy the supercommutation relations for the N=1 Neveu–Schwarz algebra with central charge d; the L_n and $G_n^{(2)}$ satisfy the supercommutation relations for the N=1 Ramond algebra with central charge d; and the remaining supercommutation relations are

$$[L_n, J_r] = -rJ_{n+r}, \quad [J_r, J_s] = \frac{1}{3} r\delta_{r+s,0} d \tag{10}$$

$$\left[J_r, G_s^{(1)}\right] = -iG_{r+s}^{(2)}, \quad \left[J_r, G_n^{(2)}\right] = iG_{r+n}^{(1)}, \quad \left[G_r^{(1)}, G_n^{(2)}\right] = -i(r-n)J_{r+n}. \tag{11}$$

Note that this mirror-twisted N=2 Neveu–Schwarz algebra is not isomorphic to the ordinary N=2 Neveu–Schwarz algebra [29].

3 The Notions of VOSA, Supersymmetric VOSA and Twisted Module

In this section, we recall the notions of VOSA, and N=1 or N=2 Neveu–Schwarz VOSA, following the notation and terminology of [2, 3] and [5]. We also recall the notion of g-twisted V-module for a VOSA, V, and an automorphism g of V of finite order following the notation of, e.g. [7, 8, 12, 25].

Let x, x_0, x_1, x_2, denote commuting independent formal variables. Let $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$. Expressions such as $(x_1 - x_2)^n$ for $n \in \mathbb{C}$ are to be understood as formal power series expansions in nonnegative integral powers of the second variable.

Definition 3.1. A *vertex operator superalgebra* is a $\frac{1}{2}\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n$, satisfying $\dim V < \infty$ and $V_n = 0$ for n sufficiently negative, that is also $\mathbb{Z}_2$-graded by *sign*, $V = V^{(0)} \oplus V^{(1)}$, and equipped with a linear map

$$V \longrightarrow (\mathrm{End}\, V)[[x, x^{-1}]], \quad v \mapsto Y(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}, \tag{12}$$

and with two distinguished vectors $\mathbf{1} \in V_0$, (the *vacuum vector*) and $\omega \in V_2$ (the *conformal element*) satisfying the following conditions for $u, v \in V$: $u_n v = 0$ for n sufficiently large; $Y(\mathbf{1}, x)v = v$; $Y(v,x)\mathbf{1} \in V[[x]]$, and $\lim_{x\to 0} Y(v,x)\mathbf{1} = v$;

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) - (-1)^{|u||v|} x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v,x_2)Y(u,x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u,x_0)v, x_2) \tag{13}$$

(the *Jacobi identity*), where $|v| = j$ if $v \in V^{(j)}$ for $j \in \mathbb{Z}_2$; writing $Y(\omega,x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$, i.e. $L(n) = \omega_{n+1}$, for $n \in \mathbb{Z}$, then the $L(n)$ give a representation of the Virasoro algebra with central charge $c \in \mathbb{C}$ (the *central charge* of V); for $n \in \frac{1}{2}\mathbb{Z}$ and $v \in V_n$, $L(0)v = nv = (\mathrm{wt}\, v)v$; and the $L(-1)$-*derivative property* holds: $\frac{d}{dx}Y(v,x) = Y(L(-1)v,x)$.

If a VOSA, $(V,Y,\mathbf{1},\omega)$, contains an element $\tau \in V_{3/2}$ such that writing $Y(\tau,z) = \sum_{n\in\mathbb{Z}} \tau_n x^{-n-1} = \sum_{n\in\mathbb{Z}} G(n+1/2)x^{-n-2}$, the $G(n+1/2) = \tau_{n+1} \in (\mathrm{End}\, V)^{(1)}$ generate a representation of the N=1 Neveu–Schwarz Lie superalgebra, then we call $(V,Y,\mathbf{1},\tau)$ an *N=1 Neveu–Schwarz VOSA*, or an *N=1 supersymmetric VOSA*, or just an N=1 VOSA for short.

If a VOSA $(V,Y,\mathbf{1},\omega)$ has two vectors $\tau^{(1)}$ and $\tau^{(2)}$ such that $(V,Y,\mathbf{1},\tau^{(j)})$ is an N=1 VOSA for both $j = 1$ and $j = 2$, and the $\tau^{(j)}_{n+1} = G^{(j)}(n+1/2)$ generate a representation of the N=2 Neveu–Schwarz Lie superalgebra, then we call such a VOSA an *N=2 Neveu–Schwarz VOSA* or an *N=2 supersymmetric VOSA*, or for short, an N=2 VOSA. If V is an N=2 VOSA, then there exists a vector $\mu = \frac{i}{2}G^{(1)}(1/2)\tau^{(2)} = -\frac{i}{2}G^{(2)}(1/2)\tau^{(1)} \in V_{(1)}$ such that writing $Y(\mu,x) = \sum_{n\in\mathbb{Z}} \mu_n x^{-n-1} = \sum_{n\in\mathbb{Z}} J(n)x^{-n-1}$, we have that the $J(n) \in (\mathrm{End}\, V)^0$ along with the $G^{(j)}(n+1/2)$ and $L(n) = \omega_{n+1}$ for $\omega = \frac{1}{2}G^{(j)}(-1/2)\tau^{(j)}$ satisfy the supercommutation relations for the N=2 Neveu–Schwarz Lie superalgebra.

For an N=2 VOSA, it follows from the definition that $\omega = L(-2)\mathbf{1}$, $\tau^{(j)} = G^{(j)}(-3/2)\mathbf{1}$, for $j = 1,2$, $\mu = J(-1)\mathbf{1}$, and

$$L(n)\mathbf{1} = G^{(j)}(n+1/2)\mathbf{1} = J(n+1)\mathbf{1} = 0, \quad \text{for } n \geq -1,\ j = 1,2. \tag{14}$$

If V is an N=2 VOSA such that V is not only $\frac{1}{2}\mathbb{Z}$ graded by $L(0)$ but also $\mathbb{Z}$-graded by $J(0)$ such that $J(0)v = nv$ with $n \equiv j \bmod 2$, for $v \in V^{(j)}$ for $j = 0,1$, then we say that V is $J(0)$-*graded* or *graded by charge*.

An *automorphism* of a VOSA, V, is a linear map g from V to itself, preserving $\mathbf{1}$ and ω such that the actions of g and $Y(v,x)$ on V are compatible in the sense that $gY(v,x)g^{-1} = Y(gv,x)$, for $v \in V$. Then $gV_n \subset V_n$ for $n \in \frac{1}{2}\mathbb{Z}$.

If g has finite order, V is a direct sum of the eigenspaces V^j of g, i.e., $V = \bigoplus_{j\in\mathbb{Z}/k\mathbb{Z}} V^j$, where $k \in \mathbb{Z}_+$ is a period of g (i.e., $g^k = 1$ but k is not necessarily the order of g) and $V^j = \{v \in V \mid gv = \eta^j v\}$, for η a fixed primitive k-th root of unity.

Definition 3.2. Let $(V,Y,\mathbf{1},\omega)$ be a VOSA and g an automorphism of V of period $k \in \mathbb{Z}_+$. A *weak g-twisted V-module* is a vector space M equipped with a linear map

$$V \longrightarrow (\mathrm{End}\, M)[[x^{1/k},x^{-1/k}]], \quad v \mapsto Y^g(v,x) = \sum_{n\in\frac{1}{k}\mathbb{Z}} v^g_n x^{-n-1}, \tag{15}$$

satisfying the following conditions for $u,v \in V$ and $w \in M$: $v^g_n w = 0$ for n sufficiently large; $Y^g(\mathbf{1},x)w = w$; $Y^g(v,x) = \sum_{n\in\mathbb{Z}+\frac{j}{k}} v^g_n x^{-n-1}$ for $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$;

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y^g(u,x_1)Y^g(v,x_2)-(-1)^{|u||v|}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y^g(v,x_2)Y^g(u,x_1)$$
$$=x_2^{-1}\frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}}\delta\left(\eta^j\frac{(x_1-x_0)^{1/k}}{x_2^{1/k}}\right)Y^g(Y(g^ju,x_0)v,x_2) \tag{16}$$

(the *twisted Jacobi identity*) where η is a fixed primitive k-th root of unity.

If we take $g=1$, then we obtain the notion of weak V-module. The term "weak" means we are making no assumptions about a grading on M.

As a consequence of the definition, we have the following supercommutator relation on M for $u\in V^j$:

$$[Y^g(u,x_1),Y^g(v,x_2)]$$
$$=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\left(\frac{x_1-x_0}{x_2}\right)^{-j/k}Y^g(Y(u,x_0)v,x_2). \tag{17}$$

It also follows that writing $Y^g(\omega,x)=\sum_{n\in\mathbb{Z}}L^g(n)x^{-n-2}$, i.e., setting $L^g(n)=\omega^g_{n+1}$, for $n\in\mathbb{Z}$, then the $L^g(n)$ satisfy the relations for the Virasoro algebra with central charge c the central charge of V. And the $L(-1)$-derivative property for the twisted vertex operators holds:

$$\frac{d}{dx}Y^g(u,x)=Y^g(L(-1)u,x). \tag{18}$$

4 Twisting by Automorphisms Arising from Virasoro-Preserving Automorphisms of the N=2 Neveu–Schwarz Algebra

In this section, we give the structure of weak g-twisted V-modules for V an N=2 VOSA and g any automorphism of V which is a lift of a Virasoro-preserving automorphism of the N=2 Neveu–Schwarz algebra of finite order.

We first consider the mirror map κ. If an N=2 VOSA, V, with central charge c, has a VOSA automorphism κ such that $\kappa(\mu)=-\mu$, $\kappa(\tau^{(1)})=\tau^{(1)}$ and $\kappa(\tau^{(2)})=-\tau^{(2)}$, then such a VOSA automorphism of V is called an *N=2 VOSA mirror map*. If such a map exists for V, and M is a weak κ-twisted module for V, then write Y^κ for the κ-twisted operators, and

$$\begin{aligned}Y^\kappa(\omega,x)&=\sum_{n\in\mathbb{Z}}L^\kappa(n)x^{-n-2}, & Y^\kappa(\tau^{(1)},x)&=\sum_{r\in\mathbb{Z}+\frac{1}{2}}G^{(1),\kappa}(r)x^{-r-\frac{3}{2}}\\ Y^\kappa(\mu,x)&=\sum_{r\in\mathbb{Z}+\frac{1}{2}}J^\kappa(r)x^{-r-1}, & Y^\kappa(\tau^{(2)},x)&=\sum_{n\in\mathbb{Z}}G^{(2),\kappa}(n)x^{-n-\frac{3}{2}}.\end{aligned} \tag{19}$$

That is, define $J^{\kappa}(n) = \mu_n^{\kappa}$ and $G^{(2),\kappa}(n-1/2) = \tau_n^{(2),\kappa}$, for $n \in \mathbb{Z} + \frac{1}{2}$. Then, using the supercommutator relation (17) for the κ-twisted vertex operators acting on M, using the $L(-1)$-derivative property (18), using the N=2 Neveu–Schwarz supercommutation relations on V, and using (14), we have that the supercommutation relations for the κ-twisted modes of ω, μ, $\tau^{(1)}$ and $\tau^{(2)}$, given by $L^{\kappa}(n), G^{(2),\kappa}(n)$, for $n \in \mathbb{Z}$, and $J^{\kappa}(r), G^{(1),\kappa}(r)$, for $r \in \mathbb{Z} + \frac{1}{2}$, satisfy the relations of the mirror-twisted N=2 Neveu–Schwarz algebra given by (10)–(11) with central charge c.

In particular, a weak κ-twisted module, M, for an N=2 VOSA reduces the N=2 Neveu–Schwarz algebra representation to an N=1 Neveu–Schwarz algebra representation coupled with an N=1 Ramond algebra representation.

Next we consider the automorphisms σ_ξ which are of finite order. Let $\eta = e^{2\pi i/k}$, for $k \in \mathbb{Z}_+$, and let $\xi = \eta^j$, for $j = 1, \ldots, k-1$. Let σ_ξ be a VOSA automorphism of an N=2 VOSA, V, such that $\sigma_\xi(\mu) = \mu$ and $\sigma_\xi(\tau^{(\pm)}) = \xi^{\pm 1}\tau^{(\pm)}$, for $\tau^{(\pm)} = \frac{1}{\sqrt{2}}(\tau^{(1)} \mp i\tau^{(2)})$. Then $\omega, \mu \in V^0$ and $\tau^{(\pm)} \in V^{\pm j}$. If such a map exists for V, and M is a weak σ_ξ-twisted module for V, then write Y^{σ_ξ} for the σ_ξ-twisted operators, and

$$Y^{\sigma_\xi}(\omega, x) = \sum_{n \in \mathbb{Z}} L^{\sigma_\xi}(n) x^{-n-2}, \quad Y^{\sigma_\xi}(\mu, x) = \sum_{n \in \mathbb{Z}} J^{\sigma_\xi}(n) x^{-n-1}$$
$$Y^{\sigma_\xi}(\tau^{(\pm)}, x) = \sum_{r \in \mathbb{Z} - \frac{1}{2} \pm \frac{j}{k}} G^{\pm,\sigma_\xi}(r) x^{-r-\frac{3}{2}}. \tag{20}$$

Then using the supercommutator relation (17) for the σ_ξ-twisted vertex operators acting on M, using the $L(-1)$-derivative property (18), using the N=2 Neveu–Schwarz supercommutation relations on V, and using (14), we have that the supercommutation relations for the σ_ξ-twisted modes of ω, μ, and $\tau^{(\pm)}$, that is the $L^{\sigma_\xi}(n)$ and $J^{\sigma_\xi}(n)$ for $n \in \mathbb{Z}$, and $G^{\pm}(r)$ for $r \in \mathbb{Z} + \frac{1}{2} \pm \frac{j}{k}$, respectively, satisfy the relations for the $\frac{j}{k}$-shifted N=2 Neveu–Schwarz algebra (1), (3), (5)–(6) with central charge c.

That is the sectors for N=2 supersymmetric VOSAs that arise under spectral flow $\mathscr{D}(t)$, for $t = j/k$, $k \in \mathbb{Z}_+$, $j = 1, \ldots, k-1$ are twisted sectors under the Virasoro-preserving automorphisms σ_ξ of the N=2 Neveu–Schwarz algebra.

If $\xi = -1$, then the map σ_ξ always extends to V, via the parity map $\sigma_{-1}(v) = (-1)^{|v|}v$, for $v \in V$. In this case, a weak σ_{-1}-twisted V-module is a representation of the N=2 Ramond algebra. If V is an N=2 VOSA which is also $J(0)$-graded such that the $J(0)$ eigenvalues are integral with $J(0)\omega = J(0)\mu = 0$, and $J(\tau^{(\pm)}) = \pm\tau^{(\pm)}$, then setting $\sigma_\xi(v) = \xi^n v$ if $J(0)v = nv$ gives a VOSA automorphism.

We summarize these results as follows:

Theorem 4.6. *If the Virasoro-preserving automorphisms of the N=2 Neveu–Schwarz algebra, κ, and σ_ξ, for ξ a root of unity, extend to VOSA automorphisms for an N=2 VOSA, V, then:*

(i) A weak κ-twisted V-module is a representation of the mirror-twisted N=2 superconformal algebra.

(ii) A weak σ_ξ-twisted V-module, for $\xi = e^{2j\pi i/k}$, is a representation of the $\frac{j}{k}$-shifted N=2 superconformal algebra. If $\xi = -1$, such a VOSA automorphism always exists (the parity map), and in this case, a weak σ_{-1}-twisted V-module is a representation of the N=2 Ramond algebra.

In [6], we also show, in particular, that if V is a free or lattice N=2 VOSA, then each Virasoro-preserving automorphism of the N=2 Neveu–Schwarz algebra extends to a VOSA automorphism of V, but not uniquely in the case of the mirror map. In fact, we show that there are two distinct mirror maps for free and lattice N=2 VOSAs and these mirror maps give nonisomorphic mirror-twisted V-modules. We also construct examples of g-twisted V-modules for $g = \kappa$ and $g = \sigma_\xi$ of finite order.

Acknowledgements The author gratefully thanks the NSA for the grant MSPF-07G-169 that partially supported this work. In addition, the author thanks the Max Planck Institute for Mathematics in Bonn, Germany for a research grant that also partially supported this work, and thanks them for their hospitality during the 2010–2011 academic year.

References

1. Adamović, D.: Representations of the N=2 superconformal vertex algebra. Int. Math. Res. Notice **2**, 61–79 (1999). Vertex algebra approach to fusion rules for N=2 superconformal minimal models. J. Algebra **239**(2), 549–572 (2001)
2. Barron, K.: $N = 1$ Neveu–Schwarz vertex operator superalgebras over Grassmann algebras and with odd formal variables. In: Representations and Quantizations: Proceedings of the International Conference on Representation Theory, pp. 9–39. Shanghai China, Springer (2000)
3. Barron, K.: The notion of $N = 1$ supergeometric vertex operator superalgebra and the isomorphism theorem. Commun. Contemp. Math. **5**(4), 481–567 (2003)
4. Barron, K.: The moduli space of N=2 super-Riemann spheres with tubes. Commun. Contemp. Math. **9**, 857–940 (2007)
5. Barron, K.: Axiomatic aspects of N=2 vertex superalgebras with odd formal variables. Commun. Alg. **38**, 1199–1268 (2010)
6. Barron, K.: Twisted modules for N=2 supersymmetric vertex operator superalgebras arising from finite automorphisms of the N=2 Neveu–Schwarz algebra. arXiv: 1110.0229
7. Barron, K., Dong, C., Mason, G.: Twisted sectors for tensor products vertex operator algebras associated to permutation groups. Comm. Math. Phys. **227**, 349–384 (2002)
8. Barron, K., Huang, Y.-Z., Lepowsky, J.: An equivalence of two constructions of permutation-twisted modules for lattice vertex operator algebras. J. Pure Appl. Algebra **210**, 797–826 (2007)
9. Boucher, W., Friedan, D., Kent, A.: Determinant formulae and unitarity for the N=2 superconformal algebras in two dimensions or exact results on string compactification. Phys. Lett. B **172**(3–4), 316–322 (1986)
10. Di Vecchia, P., Petersen, J., Yu, M.: On the unitary representations of N=2 superconformal theory. Phys. Lett. B **172**(2), 211–215 (1986)
11. Di Vecchia, P., Petersen, J., Zheng, H.: N=2 extended superconformal theories in two dimensions. Phys. Lett. B **162**(4–6), 327–332 (1985)
12. Dong, C., Li, H., Mason, G.: Twisted representations of vertex operator algebras. Math. Ann. **310**, 571–600 (1998)

13. Dobrev, V.: Characters of the unitarizable highest weight modules over the N=2 superconformal algebras. Phys. Lett. B **186**, 43–51 (1987)
14. Dörrzapf, M.: Singular vectors of the N=2 superconformal algebra. Int. J. Modern Phys. A **10**(15), 2143–2180 (1995)
15. Dörrzapf, M., Gato-Rivera, B.: Singular dimensions of the N=2 superconformal algebras. I. Comm. Math. Phys. **206**(3), 493–531 (1999)
16. Dörrzapf, M., Gato-Rivera, B.: Singular dimensions of the N=2 superconformal algebras. II. The twisted N=2 algebra. Comm. Math. Phys. **220**(2), 263–292 (2001)
17. Feigin, B., Semikhatov, A., Sirota, V., Tipunin, I.: Resolutions and characters of irreducible representations of the N=2 superconformal algebra. Nuclear Phys. B **536**(3), 617–656 (1999)
18. Feigin, B., Semikhatov, A., Tipunin, I.: Equivalence between chain categories of representations of affine sl(2) and N=2 superconformal algebras. J. Math. Phys. **39**(7), 3865–3905 (1998)
19. Frenkel, I., Lepowsky, J., Meurman, A.: A natural representation of the Fischer-Griess Monster with the modular function J as character. Proc. Natl. Acad. Sci. USA **81**, 3256–3260 (1984). Vertex operator calculus. In: Yau, S.-T. (ed.) Mathematical Aspects of String Theory, Proc. 1986 Conference, San Diego, pp. 150–188. World Scientific, Singapore (1987). Vertex operator algebras and the monster. Pure Appl. Math., vol. 134. Academic Press, Boston, MA (1988)
20. Fu, J., Jiang, Q., Su, Y.: Classificiation of modules of the intermediate series over Ramond N=2 superconformal algebras. J. Math. Phys. **48**(043508), 17 (2007)
21. Gato-Rivera, B.: Construction formulae for singular vectors of the topological and of the Ramond N=2 superconformal algebras. Int. J. Modern Phys. A **17**, 4515–4541 (2002)
22. Iohara, K., Koga, Y.: Representation theory of $N = 2$ super Virasoro algebra: twisted sector. J. Funct. Anal. **214**, 450–518 (2004)
23. Kiritsis, E.: Character formulae and the structure of the representations of the N=1,N=2 superconformal algebras. Int. J. Modern Phys. A **3**, 1871–1906 (1988)
24. Lepowsky, J., Wilson, R.: Construction of the affine Lie algebra $A_1^{(1)}$. Comm. Math. Phys. **62**, 43–53 (1978)
25. Li, H.: Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules, in: "Moonshine, the Monster, and related topics (South Hadley, MA, 1994). Contemp. Math **193**, 203–236 (1996)
26. Li, J., Su, Y., Zhu, L.: Classification of indecomposable modules of the intermediate series over the twisted N=2 superconformal algebra. J. Math. Phys. **51**(8), 083515, 17 (2010)
27. Matsuo, Y.: Character formula of c<1 unitary representation of N=2 superconformal algebra. Prog. Theor. Phys. **77**, 793–797 (1987)
28. Nam, S.: The Kac formula for the N=1 and the N=2 super-conformal algebras. Phys. Lett. B **172**(3–4), 323–327 (1986)
29. Schwimmer, A., Seiberg, N.: Comments on the $N = 2, 3, 4$ superconformal algebras in two dimensions. Phys. Lett. B, **184**, 191–196 (1986)
30. Semikhatov, A., Tipunin, I.: The structure of Verma modules over the N=2 superconformal algebra. Comm. Math. Phys. **195**(1), 129–173 (1998)
31. Semikhatov, A., Tupunin, I., Feigin, B.: Semi-infinite realization of unitary representations of the N=2 algebra and related constructions. Theor. Math. Phys. **126**, 1–47 (2001)

Part VII
Integrability and Other Applications

The Ruijsenaars Self-Duality Map as a Mapping Class Symplectomorphism

L. Fehér and C. Klimčík

Abstract This is a brief review of the main results of our paper [Nucl. Phys. B 860, 464–515 (2012)] that contains a complete global treatment of the compactified trigonometric Ruijsenaars–Schneider system by quasi-Hamiltonian reduction. Confirming previous conjectures of Gorsky and collaborators, we have rigorously established the interpretation of the system in terms of flat $SU(n)$ connections on the one-holed torus and demonstrated that its self-duality symplectomorphism represents the natural action of the standard mapping class generator S on the phase space. The pertinent quasi-Hamiltonian reduced phase space turned out to be symplectomorphic to the complex projective space equipped with a multiple of the Fubini-Study symplectic form and two toric moment maps playing the roles of particle-positions and action-variables that are exchanged by the duality map. Open problems and possible directions for future work are also discussed.

1 Introduction

In his study of action-angle maps, Ruijsenaars [12] discovered an intriguing duality relation for both non-relativistic and relativistic Calogero type classical many-body systems associated to A_n root systems and rational, hyperbolic or trigonometric

L. Fehér (✉)
Wigner Research Centre for Physics, H-1525 Budapest, P.O.B. 49, Hungary

Department of Theoretical Physics, University of Szeged,
Tisza Lajos krt 84–86, H-6720 Szeged, Hungary
e-mail: lfeher@rmki.kfki.hu

C. Klimčík
Institut de mathématiques de Luminy, 163, Avenue de Luminy,
F-13288 Marseille, France
e-mail: klimcik@univmed.fr

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_30, © Springer Japan 2013

interaction potentials. In this paper our concern is a particular system of that kind, locally given by the trigonometric Hamiltonian (25) later on, which was invented and proved to be self-dual in [13]. Our principal goal is to give a self-contained but concise presentation of the main results of our detailed work [4], where we showed that the global variant of this system (called compactified trigonometric Ruijsenaars–Schneider $\mathrm{III_b}$ system) and its self-duality can be naturally understood by means of quasi-Hamiltonian reduction. This connects the system to the $SU(n)$ Chern–Simons theory on the one-holed torus, with a special boundary condition, and traces back its self-duality symplectomorphism to the standard duality generator $S \in SL(2,\mathbb{Z})$ of the mapping class group of the one-holed torus. Our results thus provide rigorous justification of conjectures put forward over a decade ago by Gorsky and his collaborators [6, 8] about the $\mathrm{III_b}$ system.

The plan of this contribution is as follows. In Sect. 2 we start with the definition of the concept of "Ruijsenaars duality". In particular, we shall discuss two alternative, equivalent definitions of self-duality. Necessary background information from quasi-Hamiltonian geometry is summarized next in Sect. 3.1, focusing on the example of the internally fused double that will be used subsequently. Then in Sect. 3.2 we explain how the mapping class group $SL(2,\mathbb{Z})$ acts on every reduced phase space arising from the double. Section 4 is devoted to expounding the definition of the compactified $\mathrm{III_b}$ system. The main results of [4] are presented in Sect. 5. The content of Sect. 5 and related further results are discussed in Sect. 6 together with an exposition of open problems.

2 The Concept of Ruijsenaars Duality

This concept is relevant for classical integrable many-body systems of "particles" moving in one-dimension. Due to their physical interpretation and Liouville integrability, these systems possess "particle-positions" and "action-variables" that span two Abelian subalgebras in the Poisson algebra of observables. By definition, two such systems are in duality if there exists a symplectomorphism between their phase spaces that converts the particle-positions of system (a) into the action-variables of system (b) and converts the action-variables of system (a) into the particle-positions of system (b). In particular, one speaks of self-duality if the leading Hamiltonians of both systems (which underlie the many-body interpretation) have the same form. An alternative second definition of self-duality is to consider a single integrable many-body Hamiltonian system (M,Ω,H), and call it self-dual if there exists a symplectomorphism $\mathfrak{S}$ of the phase space (M,Ω) that converts the particle-positions into the action-variables and the action-variables into the particle-positions. Notice that the second definition is the special case of the first definition where the two systems in duality are two copies of the same system and their duality relation is provided by $\mathfrak{S}$.

If not clear from the context, we propose the full name of the above duality be "Ruijsenaars duality" or "duality in the sense of Ruijsenaars" (also known as action-angle duality).

Let us further discuss the relation between the above two definitions of (Ruijsenaars) self-duality. To do this, denote by $\mathscr{J}_k$ and $\mathscr{I}_k$ $(k=1,\ldots,N)$ the particle-positions and action-variables for the system (M,Ω,H). It is required that there exists a dense open submanifold $M^{\text{loc}} \subseteq M$ where the symplectic form Ω is equal to $\Omega^{\text{loc}} = \sum_{k=1}^{N} d\theta_k \wedge d\mathscr{J}_k$, with conjugates θ_k of the $\mathscr{J}_k$. We can view $(\mathscr{J},\theta)$ and $\mathscr{I}$ as maps from M^{loc} into $\mathbb{R}^{2N}$ and $\mathbb{R}^N$, and then have

$$H^{\text{loc}} = \mathscr{H} \circ (\mathscr{J},\theta) = h \circ \mathscr{I} \tag{1}$$

with some functions $\mathscr{H}$ and h, where the form of $\mathscr{H}$ underlies the many-body interpretation. Any global symplectomorphism $\mathfrak{S}$ takes H into the integrable Hamiltonian $\tilde{H} := H \circ \mathfrak{S}$. One has the relations $\Omega^{\text{loc}} = \sum_{k=1}^{N} d\tilde{\theta}_k \wedge d\tilde{\mathscr{J}}_k$ and

$$\tilde{H}^{\text{loc}} = H^{\text{loc}} \circ \mathfrak{S} = \mathscr{H} \circ (\tilde{\mathscr{J}},\tilde{\theta}) = h \circ \tilde{\mathscr{I}} \tag{2}$$

with $(\tilde{\mathscr{J}},\tilde{\theta}) := (\mathscr{J},\theta) \circ \mathfrak{S}$ and $\tilde{\mathscr{I}} := \mathscr{I} \circ \mathfrak{S}$. Thus $\tilde{H}^{\text{loc}}$ has the same form in terms of the tilded-variables as H^{loc} in terms of the tilde-free variables. Now observe that the system (M,Ω,H) is in duality with $(M,\Omega,\tilde{H})$ if $\tilde{\mathscr{J}}$ is the same as $\mathscr{I}$ and $\tilde{\mathscr{I}}$ is the same as $\mathscr{J}$. Spelling this out in more detail: if (M,Ω,H) is self-dual in the sense of the second definition, then its dual pair $(M,\Omega,\tilde{H})$ is automatically manufactured and these two systems are in duality with respect to the identity map[1] on M. The full equivalence of our alternative definitions of self-duality is also not difficult to prove. In this paper we adopt the second definition.

To be precise, we note that in the statement "is the same as" above one must admit some sign change or re-labeling of the indices of the variables. In fact, the *self-duality symplectomorphism* $\mathfrak{S}$ is usually not an involution but has order 4. As an illustration, consider the free system with Hamiltonian $H = p^2$ on the phase space $\mathbb{R}^2 = \{(q,p)\}$, whose particle-position and action-variable are q and p, respectively. The free system is trivially self-dual with self-duality symplectomorphism $\mathfrak{S}$: $(q,p) \mapsto (p,-q)$ and dual Hamiltonian $\tilde{H} = q^2$.

Ruijsenaars [12, 13] actually found three distinct dual pairs of systems and three self-dual systems. For example, the dual of the hyperbolic Sutherland system is the rational Ruijsenaars–Schneider system, and the rational Calogero system is self-dual. See the review [14] for the other cases. Incidentally, at the quantum mechanical level, all these systems are known to enjoy the related bispectral property [2], too.

[1] In general, identifying the phase spaces of any dual pair by the symplectomorphism that appears in the definition of the duality relation given at the beginning, one may always turn this symplectomorphism into the identity map. Thus the phase spaces of the systems in duality become models of a single phase space, (not accidentally) similar to two gauge slices serving as models of the single space of gauge orbits in a gauge theory.

As was already mentioned, in this paper our concern will be the self-dual $\mathrm{III_b}$ system. For a detailed geometric treatment of a very different, not self-dual, case of the trigonometric Ruijsenaars duality, the reader may consult [3].

3 Generalities about the Internally Fused Double *D*

The basic reference for Sect. 3.1 is [1]. The mapping class group action presented in Sect. 3.2 is also well-known to experts [1, 7, 9]; in its explicit description we follow [4].

3.1 Quasi-Hamiltonian Systems on D and Their Reductions

Let G be a (connected and simply connected) compact Lie group and fix a positive definite invariant scalar product $\langle\ ,\ \rangle$ on its Lie algebra $\mathcal{G}$. Equip the Cartesian product

$$D := G \times G = \{(A,B)\,|\,A,B \in G\} \tag{3}$$

with the 2-form ω,

$$2\omega := \langle A^{-1}dA \overset{\wedge}{,} dBB^{-1}\rangle + \langle dAA^{-1} \overset{\wedge}{,} B^{-1}dB\rangle - \langle (AB)^{-1}d(AB) \overset{\wedge}{,} (BA)^{-1}d(BA)\rangle, \tag{4}$$

which is invariant under the G-action Ψ on D defined by

$$\Psi_g : (A,B) \mapsto (gAg^{-1}, gBg^{-1}), \quad \forall g \in G. \tag{5}$$

Introduce the G-equivariant map $\mu : D \to G$ by the group commutator

$$\mu(A,B) := ABA^{-1}B^{-1}. \tag{6}$$

These data satisfy

$$d\omega = -\frac{1}{12}\mu^*\langle\vartheta, [\vartheta,\vartheta]\rangle, \quad \omega(\zeta_D,\cdot) = \frac{1}{2}\mu^*\langle\vartheta + \bar{\vartheta}, \zeta\rangle, \quad \forall\zeta \in \mathcal{G}, \tag{7}$$

$$\mathrm{Ker}(\omega_x) = \{\zeta_D(x)\,|\,\zeta \in \mathrm{Ker}(\mathrm{Ad}_{\mu(x)} + \mathrm{Id}_{\mathcal{G}})\}, \qquad \forall x \in D, \tag{8}$$

where ϑ and $\bar{\vartheta}$ denote, respectively, the $\mathcal{G}$-valued left- and right-invariant Maurer–Cartan forms on G and ζ_D generates the infinitesimal action of $\zeta \in \mathcal{G}$ on D. All this means [1] that (D,ω,μ) is a so-called quasi-Hamiltonian G-space with moment map μ. This quasi-Hamiltonian G-space is nicknamed the internally fused double of G.

According to the general theory [1], every G-invariant function $h \in C^\infty(D)^G$ induces a unique vector field v_h on D by requiring that $\omega(v_h, \cdot) = dh$ and $\mathscr{L}_{v_h}\mu = 0$. The vector field v_h is G-invariant and its flow preserves ω. In this way, (D, ω, μ, h) yields a quasi-Hamiltonian dynamical system. Although (D, ω) is not a symplectic manifold, one can also introduce an honest Poisson bracket on $C^\infty(D)^G$. Naturally, for G-invariant functions f and h the Poisson bracket is furnished by

$$\{f, h\} := \omega(v_f, v_h). \tag{9}$$

Generally speaking, quasi-Hamiltonian systems are of interest since they can be reduced to true Hamiltonian systems by a generalization of the Marsden–Weinstein symplectic reduction, and this can give convenient realizations of important Hamiltonian systems. To specialize to our case, let us choose a moment map value $\mu_0 \in G$ and denote its stabilizer with respect to the adjoint action by G_0. Then consider the space of G_0-orbits

$$P(\mu_0) := \mu^{-1}(\mu_0)/G_0, \tag{10}$$

where $\mu^{-1}(\mu_0) := \{x \in D \,|\, \mu(x) = \mu_0\}$. Denote by $\iota : \mu^{-1}(\mu_0) \to D$ the tautological injection and $p : \mu^{-1}(\mu_0) \to P(\mu_0)$ the obvious projection. Under favourable circumstances (where the meaning of "favourable" is the same as for usual symplectic reduction), there exists a standard Hamiltonian system $(P(\mu_0), \hat{\omega}, \hat{h})$ such that the *symplectic* form $\hat{\omega}$ and the reduced Hamiltonian $\hat{h}$ satisfy the relations

$$p^*\hat{\omega} = \iota^*\omega, \quad p^*\hat{h} = \iota^*h. \tag{11}$$

The Hamiltonian vector field and the flow defined by $\hat{h}$ on $P(\mu_0)$ can be obtained by first restricting the quasi-Hamiltonian vector field v_h and its flow to $\mu^{-1}(\mu_0)$ and then applying the projection p. The Poisson brackets on $(P(\mu_0), \hat{\omega})$ are inherited from the Poisson brackets (9) of the G-invariant functions like in usual symplectic reduction.

Of course, the space of orbits $P(\mu_0)$ is not a smooth manifold in general. However, it always turns out to be a stratified symplectic space [9], which means that it is a disjoint union of symplectic manifolds of various dimensions glued together (in a specific manner).

The symplectic spaces obtained from quasi-Hamiltonian reduction always arise also from usual symplectic reduction of certain infinite-dimensional manifolds with respect to infinite-dimensional symmetry groups [1]. In particular, let Σ denote the torus with a hole (that is, with an open disc removed); often called the "one-holed torus". It is known that the moduli space (space of gauge equivalence classes) of flat principal G-connections on Σ whose holonomy along the boundary of the hole is constrained to the conjugacy class of μ_0 is a stratified symplectic space, which can be canonically identified with the quasi-Hamiltonian reduced phase space $P(\mu_0)$ in (10). It is also worth noting that this space supports two natural *Abelian Poisson algebras*. Namely, for any $\mathscr{H} \in C^\infty(G)^G$ let $\mathscr{H}_1$ and $\mathscr{H}_2$ denote the G-invariant functions on D given by

$$\mathscr{H}_1(A, B) := \mathscr{H}(A) \quad \text{and} \quad \mathscr{H}_2(A, B) := \mathscr{H}(B). \tag{12}$$

The two Abelian Poisson algebras on $P(\mu_0)$ are provided by

$$\mathscr{C}^a := \{\hat{\mathscr{H}}_1 \,|\, \mathscr{H} \in C^\infty(G)^G\}, \quad \mathscr{C}^b := \{\hat{\mathscr{H}}_2 \,|\, \mathscr{H} \in C^\infty(G)^G\}. \tag{13}$$

Note also that D itself can be identified as the space of flat connections on Σ modulo the "based gauge transformations" defined by maps $\eta \in C^\infty(\Sigma, G)$ for which $\eta(p_0) = e$ for a fixed point p_0 on the boundary of the removed disc. The matrices A and B represent the holonomies of the flat connections along the standard generators of the fundamental group $\pi_1(\Sigma, p_0)$.

3.2 Symplectic Action of the Mapping Class Group on $P(\mu_0)$

Let us consider the (orientation-preserving) mapping class group of the one-holed torus,

$$\mathrm{MCG}^+(\Sigma) \equiv \pi_0(\mathrm{Diff}^+(\Sigma)), \tag{14}$$

whose elements are equivalence classes of orientation-preserving diffeomorphisms up to homotopy. It is known that the mapping class groups acts by structure preserving smooth maps on every reduced phase space $P(\mu_0)$ (10), where "structure preserving" means symplectomorphism whenever $P(\mu_0)$ is a smooth manifold. The origin of the mapping class group action is especially clear in the setting of flat connections, where it arises from the pull-back of the connection 1-forms by diffeomorphisms. However, it is also possible to directly describe the mapping class group action on $P(\mu_0)$ by taking advantage of the quasi-Hamiltonian formalism.

For the one-holed torus there exists a (geometrically engendered) isomorphism

$$\mathrm{MCG}^+(\Sigma) \simeq SL(2, \mathbb{Z}). \tag{15}$$

The infinite discrete group $SL(2, \mathbb{Z})$ is generated by two elements S and T subject to the relations

$$S^2 = (ST)^3, \quad S^4 = 1. \tag{16}$$

As concrete matrices, one may take

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{17}$$

which actually represent the action of corresponding mapping classes on the standard basis of the homology group $H_1(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}^2$. The mapping class of T is known as a Dehn twist and that of S as the standard orientation-preserving duality generator "exchanging" the standard homology cycles. By arguments detailed in [4, 7], it is natural to associate to S and T the following diffeomorphisms S_D and T_D of the double:

$$S_D(A,B) := (B^{-1}, BAB^{-1}), \quad T_D(A,B) := (AB, B). \tag{18}$$

It is not difficult to check that

$$S_D^* \omega = \omega, \quad S_D \circ \Psi_g = \Psi_g \circ S_D, \quad \mu \circ S_D = \mu, \tag{19}$$

and similar relations hold for T_D as well, i.e., both S_D and T_D are automorphisms of the internally fused double. Moreover, one finds that S_D and T_D satisfy

$$S_D^2 = (S_D \circ T_D)^3, \quad S_D^4 = Q, \tag{20}$$

where Q is the central element of the group of automorphisms of the double given by

$$Q(A,B) = \Psi_{\mu(A,B)^{-1}}(A,B). \tag{21}$$

It is an immediate consequence of the above relations that S_D and T_D descend to maps S_P and T_P on any reduced phase space $P(\mu_0)$ (10), and these maps generate an $SL(2,\mathbb{Z})$ action on $P(\mu_0)$. Indeed, Q descends to the trivial identity map id_P on $P(\mu_0)$, and thus (20) implies the identities

$$S_P^2 = (S_P \circ T_P)^3, \quad S_P^4 = \mathrm{id}_P. \tag{22}$$

The resulting $SL(2,\mathbb{Z})$ action preserves the (stratified) symplectic structure on $P(\mu_0)$.

Finally, consider the action of S_P on the two Abelian Poisson algebras $\mathscr{C}^a$ and $\mathscr{C}^b$ displayed in (13). For any $\mathscr{H} \in C^\infty(G)^G$, define $\mathscr{H}^\sharp \in C^\infty(G)^G$ by

$$\mathscr{H}^\sharp(g) := \mathscr{H}(g^{-1}). \tag{23}$$

Then the following identities hold:

$$\hat{\mathscr{H}}_2 \circ S_P = \hat{\mathscr{H}}_1 \quad \text{and} \quad \hat{\mathscr{H}}_1 \circ S_P = \hat{\mathscr{H}}_2^\sharp, \quad \forall \mathscr{H} \in C^\infty(G)^G. \tag{24}$$

In this way, S_P exchanges the elements $\hat{\mathscr{H}}_2$ of $\mathscr{C}^b$ with the elements $\hat{\mathscr{H}}_1$ of $\mathscr{C}^a$.

4 Compactified Ruijsenaars–Schneider $\mathrm{III_b}$ System

In [13] Ruijsenaars studied, among others, a particular real form of the complex trigonometric Ruijsenaars–Schneider system whose Hamiltonian exhibits periodic dependence both on the particle-positions and on the conjugate momenta. This system is termed the $\mathrm{III_b}$ system, where the label "b" indicates the bounded nature of the underlying phase space. The $\mathrm{III_b}$ Hamiltonian given by (25) below is formally integrable since it admits the sufficient number of constants of motion in involution. However, true integrability holds only after compactifying the local phase space, whereby the Hamiltonian flows become complete. Here, we first summarize the definition of the local $\mathrm{III_b}$ system and then present its compactification. Although the content of this section can be found in [13], too, for the sake of readability we display all definitions in a self-contained manner.

The many-body interpretation of the $\mathrm{III_b}$ system is based on the Hamiltonian

$$H_y^{\mathrm{loc}}(\delta,\Theta) \equiv \sum_{j=1}^{n} \cos p_j \prod_{k\neq j}^{n} \left[1 - \frac{\sin^2 y}{\sin^2(x_j - x_k)}\right]^{\frac{1}{2}}, \tag{25}$$

where $\delta_j = e^{\mathrm{i}2x_j}$ $(j = 1,\dots,n)$ are interpreted as the positions of n "particles" moving on the circle and the canonically conjugate momenta p_j encode the compact variables $\Theta_j = e^{-\mathrm{i}p_j}$; the index k in the product runs over $\{1,2,\dots,n\}\setminus\{j\}$. To guarantee the reality of H_y^{loc} on a non-trivial connected open domain, one may require to have $|y| < |x_j - x_k| < \pi - |y|$ for all $j \neq k$, and consistency then enforces the real coupling constant $y \neq 0$ to satisfy

$$0 < |y| < \pi/n. \tag{26}$$

We impose the center of mass condition $\prod_{j=1}^n \delta_j = \prod_{j=1}^n \Theta_j = 1$, and parametrize the variables so that the local phase space of the system gets identified with

$$M_y^{\mathrm{loc}} \equiv \mathscr{P}_y^0 \times \mathbb{T}_{n-1}, \tag{27}$$

where $\mathbb{T}_{n-1}$ is the $(n-1)$-torus and $\mathscr{P}_y^0$ is the interior of the polytope

$$\mathscr{P}_y := \left\{ (\xi_1,\dots,\xi_{n-1}) \in \mathbb{R}^{n-1} \,\middle|\, \xi_j \geq |y|,\ j = 1,\dots,n-1,\ \sum_{j=1}^{n-1} \xi_j \leq \pi - |y| \right\}. \tag{28}$$

Using the $n\times n$ matrix $E_{j,j}$ having 1 in the jj position and the identity matrix 1_n, we introduce

$$H_k := E_{k,k} - E_{k+1,k+1}, \quad \lambda_k := \sum_{j=1}^{k} E_{j,j} - \frac{k}{n} 1_n, \quad k = 1,\dots,n-1. \tag{29}$$

Then, for $\xi \in \mathscr{P}_y^0$ and $\tau = (\tau_1,\dots,\tau_{n-1}) = (e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_{n-1}}) \in \mathbb{T}_{n-1}$, we define the diagonal $SU(n)$ matrices

$$\delta(\xi) := \exp\big(-2\mathrm{i}\sum_{k=1}^{n-1} \xi_k \lambda_k\big), \quad \Theta(\tau) := \exp\big(-\mathrm{i}\sum_{k=1}^{n-1} \theta_k H_k\big). \tag{30}$$

The choice of $\mathscr{P}_y^0$ as the domain of the particle-positions ξ guarantees the positivity of the expressions under the square root in (25). In terms of the variables $(\xi,\tau)\in\mathscr{P}_y^0\times\mathbb{T}_{n-1}$, the symplectic form of the system reads

$$\Omega^{\mathrm{loc}} := \frac{1}{2}\mathrm{tr}\left(\delta^{-1}d\delta\wedge\Theta^{-1}d\Theta\right) = \mathrm{i}\sum_{k=1}^{n-1} d\xi_k\wedge\tau_k^{-1}d\tau_k = \sum_{k=1}^{n-1} d\theta_k\wedge d\xi_k. \tag{31}$$

Note that for any diagonal matrix $\mathscr{D}$ (like δ,Θ etc), we apply the notation $\mathscr{D}=\mathrm{diag}(\mathscr{D}_1,\ldots,\mathscr{D}_n)$.

The Hamiltonian (25) admits $(n-1)$ Poisson commuting constants of motion given by independent spectral invariants of the following $SU(n)$-valued local Lax matrix:

$$L_{\mathrm{loc}}^y(\xi,\tau)_{jl} := \frac{e^{\mathrm{i}y}-e^{-\mathrm{i}y}}{e^{\mathrm{i}y}\delta_j(\xi)\delta_l(\xi)^{-1}-e^{-\mathrm{i}y}} W_j(\xi,y)W_l(\xi,-y)\Theta_l(\tau)\Delta_l(\tau)\Delta_j(\tau)^{-1}. \tag{32}$$

Here we use the positive functions

$$W_j(\xi,y) := \prod_{k\neq j}^{n}\left[\frac{e^{\mathrm{i}y}\delta_j(\xi)-e^{-\mathrm{i}y}\delta_k(\xi)}{\delta_j(\xi)-\delta_k(\xi)}\right]^{\frac{1}{2}}, \tag{33}$$

and $\Delta(\tau) := \mathrm{diag}(\tau_1,\ldots,\tau_{n-1},1)$. The Hamiltonian (25) is recovered from the local Lax matrix as the real part of the trace

$$H_y^{\mathrm{loc}}(\delta(\xi),\Theta(\tau)) = \mathrm{Re}\,\mathrm{tr}\left(L_{\mathrm{loc}}^y(\xi,\tau)\right). \tag{34}$$

Ruijsenaars [13] realized that the flows of H_y^{loc} and of its commuting family are not complete on M_y^{loc}, and then completed the local phase space in the way described below.

Let us consider the symplectic manifold $(\mathbb{C}P(n-1),\chi_0\omega_{\mathrm{FS}})$, where

$$\chi_0 := \pi - n|y|, \tag{35}$$

and ω_{FS} is the standard Fubini-Study symplectic form. It is convenient to identify the complex projective space $\mathbb{C}P(n-1)$ as the factor space $S_{\chi_0}^{2n-1}/U(1)$ with

$$S_{\chi_0}^{2n-1} = \left\{(u_1,\ldots,u_n)\in\mathbb{C}^n \mid \sum_{k=1}^{n}|u_k|^2=\chi_0\right\}. \tag{36}$$

Let $\mathbb{C}P(n-1)_0$ be the open dense submanifold of $\mathbb{C}P(n-1)$ where none of the homogeneous coordinates can vanish. By utilizing the canonical projection $\pi_{\chi_0}: S_{\chi_0}^{2n-1}\to\mathbb{C}P(n-1)$, we define a diffeomorphism $\mathscr{E}: M_y^{\mathrm{loc}}\to\mathbb{C}P(n-1)_0$ by the formula

$$\mathscr{E}(\xi,\tau) := \pi_{\chi_0}(\tau_1\sqrt{\xi_1-|y|},\ldots,\tau_{n-1}\sqrt{\xi_{n-1}-|y|},\sqrt{\xi_n-|y|}) \tag{37}$$

with $\xi_n := \pi - \sum_{k=1}^{n-1} \xi_k$. By using that $\pi_{\chi_0}^*(\chi_0 \omega_{\rm FS}) = {\rm i} \sum_{k=1}^n d\bar{u}_k \wedge du_k$, one sees that $\mathscr{E}$ is a symplectomorphism

$$\mathscr{E}^*(\chi_0 \omega_{\rm FS}) = \Omega^{\rm loc}. \tag{38}$$

Thus we can identify $(M_y^{\rm loc}, \Omega^{\rm loc})$ with the dense open submanifold $\mathbb{C}P(n-1)_0$ of the compact phase space $(\mathbb{C}P(n-1), \chi_0 \omega_{\rm FS})$. The crucial fact is that, by means of this identification, the local Lax matrix $L^y_{\rm loc}$ extends to a smooth (even real-analytic) matrix function on $\mathbb{C}P(n-1)$. This fact is actually not difficult to verify [4, 13]. From now on we denote the resulting "global Lax matrix" as L^y. Since $L^y \in C^\infty(\mathbb{C}P2(n-1), SU(n))$ satisfies

$$L^y \circ \mathscr{E} = L^y_{\rm loc}, \tag{39}$$

it follows that all the smooth spectral invariants of $L^y_{\rm loc}$ (like the Hamiltonian (34)) extend to smooth functions on the compactified phase space $\mathbb{C}P(n-1)$. The corresponding Hamiltonian flows are automatically complete on $\mathbb{C}P(n-1)$, simply since every smooth vector field has complete flows on a compact manifold. By definition, the compactified $\mathrm{III_b}$ system is the integrable system on the phase space $(\mathbb{C}P(n-1), \chi_0 \omega_{\rm FS})$ whose commuting Hamiltonians are generated by the Lax matrix L^y.

5 Self-Duality of the $\mathrm{III_b}$ System from Reduction

The compactified $\mathrm{III_b}$ system, encapsulated by the triple

$$(\mathbb{C}P(n-1), \chi_0 \omega_{\rm FS}, L^y), \tag{40}$$

possesses two distinguished Abelian Poisson algebras of observables. The first Abelian algebra is generated by the "global particle-position variables" $\mathscr{J}_k$ defined by

$$\mathscr{J}_k \circ \pi_{\chi_0}(u) = |u_k|^2 + |y|, \qquad k = 1, \dots, n-1. \tag{41}$$

The terminology is justified by the identity $\mathscr{J}_k(\mathscr{E}(\xi, \tau)) = \xi_k$. The $\mathscr{J}_k$ are the components of the toric moment map

$$\mathscr{J} := (\mathscr{J}_1, \dots, \mathscr{J}_{n-1}) : \mathbb{C}P(n-1) \to \mathbb{R}^{n-1} \tag{42}$$

that generates the so-called rotational action of the torus $\mathbb{T}_{n-1}$ on $(\mathbb{C}P(n-1), \chi_0 \omega_{\rm FS})$. Its image is the closed polytope $\mathscr{P}_y$ (28). The other distinguished Abelian algebra is spanned by the action-variables furnished by certain spectral functions of the global Lax matrix L^y.

In the rest of this section we take

$$G := SU(n), \quad \langle X, Y \rangle := -\frac{1}{2} {\rm tr}\,(XY), \quad \forall X, Y \in \mathscr{G}. \tag{43}$$

Define the polytope $\mathscr{P}_0$ similarly to (28) and also define $\delta(\xi)$ like in (30) for any $\xi \in \mathscr{P}_0$. It is well-known that any $g \in G$ is conjugate to a matrix $\delta(\xi)$ for a unique $\xi \in \mathscr{P}_0$, and g is regular (has n distinct eigenvalues) if and only if the corresponding ξ belongs to the interior $\mathscr{P}_0^0$ of $\mathscr{P}_0$. Therefore we can uniquely define a G-invariant (i.e. conjugation invariant) function Ξ_k on G by requiring that

$$\Xi_k(\delta(\xi)) = \xi_k, \quad \forall \xi \in \mathscr{P}_0, \quad k = 1, \dots, n-1. \tag{44}$$

The "spectral function" Ξ_k is continuous on G and its restriction to the dense open submanifold of regular elements, G_{reg}, belongs to $C^\infty(G_{\mathrm{reg}})^G$.

It was shown in [13], and follows readily from our Theorem 5.1 given below, that the global Lax matrix L^y takes values in G_{reg} and the functions

$$\mathscr{I}_k := \Xi_k \circ L^y \tag{45}$$

can serve as action-variables of the compactified $\mathrm{III_b}$ system. In fact, these functions Poisson commute and their Hamiltonian flows are 2π-periodic. The image of the toric moment map

$$\mathscr{I} := (\mathscr{I}_1, \dots, \mathscr{I}_{n-1}) : \mathbb{C}P(n-1) \to \mathbb{R}^{n-1} \tag{46}$$

is the same polytope $\mathscr{P}_y$ as the image of moment map $\mathscr{J}$.

One can check that the spectral functions satisfy

$$\Xi_k^\sharp = \Xi_{n-k}, \tag{47}$$

where we applied the definition (23). Thus, if we define the spectral Hamiltonians α_k and β_k on D by

$$\alpha_k(A,B) := \Xi_k(A) \quad \text{and} \quad \beta_k(A,B) := \Xi_k(B), \tag{48}$$

then (18) implies the identities $\beta_k \circ S_D = \alpha_k$ and $\alpha_k \circ S_D = \beta_{n-k}$. Although they are not globally C^∞, α_k and β_k descend to "reduced spectral Hamiltonians" $\hat{\alpha}_k$ and $\hat{\beta}_k$ on any reduced phase space $P(\mu_0)$ obtained from the double. As special cases of (24), with the $SL(2,\mathbb{Z})$ generator S_P they satisfy

$$\hat{\beta}_k \circ S_P = \hat{\alpha}_k \quad \text{and} \quad \hat{\alpha}_k \circ S_P = \hat{\beta}_{n-k}, \quad \forall k = 1, \dots, n-1. \tag{49}$$

Having the necessary preliminaries at hand, the principal result of our paper [4] can be summarized as follows.

Theorem 5.1. *For the particular moment map value*

$$\mu_0 = \mathrm{diag}(e^{2\mathrm{i}y}, \dots, e^{2\mathrm{i}y}, e^{2(1-n)\mathrm{i}y}), \qquad 0 < |y| < \pi/n, \tag{50}$$

the "constraint surface" $\mu^{-1}(\mu_0)$ *lies in* $G_{\rm reg} \times G_{\rm reg}$ *and the reduced phase space* $(P(\mu_0), \hat{\omega})$ *is a smooth manifold symplectomorphic to* $(\mathbb{C}P(n-1), \chi_0 \omega_{\rm FS})$. *The maps*

$$\hat{\alpha} := (\hat{\alpha}_1, \dots, \hat{\alpha}_{n-1}) : P(\mu_0) \to \mathbb{R}^{n-1} \quad \textit{and} \quad \hat{\beta} : (\hat{\beta}_1, \dots, \hat{\beta}_{n-1}) : P(\mu_0) \to \mathbb{R}^{n-1} \tag{51}$$

are toric moment maps generating two effective Hamiltonian actions of $\mathbb{T}_{n-1}$ *on* $(P(\mu_0), \hat{\omega})$. *The images of both* $\hat{\alpha}$ *and* $\hat{\beta}$ *yield the polytope* $\mathscr{P}_y$ *(28), and there exists a symplectomorphism*

$$f_\beta : \mathbb{C}P(n-1) \to P(\mu_0) \tag{52}$$

that satisfies

$$\hat{\beta}_k \circ f_\beta = \mathscr{J}_k \quad \textit{and} \quad \hat{\alpha}_k \circ f_\beta = \mathscr{I}_k, \qquad \forall k = 1, \dots, n-1. \tag{53}$$

Combining Theorem 5.1 with the generalities reviewed in Sect. 3.2, we obtain the following important result.

Corollary 5.1. *The symplectomorphisms* $f_\beta^{-1} \circ S_P \circ f_\beta$ *and* $f_\beta^{-1} \circ T_P \circ f_\beta$ *generate an* $SL(2, \mathbb{Z})$ *action on the compactified* $\mathrm{III_b}$ *phase space* $(\mathbb{C}P(n-1), \chi_0 \omega_{\rm FS})$. *The mapping class duality symplectomorphism*

$$\mathfrak{S} := f_\beta^{-1} \circ S_P \circ f_\beta \tag{54}$$

acts by exchanging the particle-positions $\mathscr{J}_k$ *with the action-variables* $\mathscr{I}_k$ *according to*

$$\mathscr{J}_k \circ \mathfrak{S} = \mathscr{I}_k, \quad \textit{and} \quad \mathscr{I}_k \circ \mathfrak{S} = \mathscr{J}_{n-k}, \qquad \forall k = 1, \dots, n-1. \tag{55}$$

For the sake of completeness, let us also present the explicit formula of our map f_β. For this, we introduce a unitary matrix $g_y(\xi)$ for each $\xi \in \mathscr{P}_y^0$ by

$$\begin{aligned} g_y(\xi)_{jn} &:= -g_y(\xi)_{nj} := v_j(\xi, y), \quad \forall j = 1, \dots, n-1, \quad g_y(\xi)_{nn} := v_n(\xi, y), \\ g_y(\xi)_{jl} &:= \delta_{jl} - \frac{v_j(\xi, y) v_l(\xi, y)}{1 + v_n(\xi, y)}, \quad \forall j, l = 1, \dots, n-1, \end{aligned} \tag{56}$$

where $v_j(\xi, y) := \left[\frac{\sin y}{\sin ny}\right]^{\frac{1}{2}} W_j(\xi, y)$ using (33).

Theorem 5.2. *Applying the previous notations, the map* $f_0 : \mathbb{C}P(n-1)_0 \to P(\mu_0)$ *defined by*

$$(f_0 \circ \mathscr{E})(\xi, \tau) := p\left(g_y(\xi)^{-1} \Delta(\tau) L^y_{\rm loc}(\xi, \tau) \Delta(\tau)^{-1} g_y(\xi), g_y(\xi)^{-1} \delta(\xi) g_y(\xi)\right) \tag{57}$$

is a diffeomorphism from $\mathbb{C}P(n-1)_0$ *onto a dense open submanifold of* $P(\mu_0)$. *This map is symplectic,* $f_0^* \hat{\omega} = \chi_0 \omega_{\mathrm{FS}}$, *and it extends to a global diffeomorphism* $f_\beta : \mathbb{C}P(n-1) \to P(\mu_0)$.

The map f_β that extends f_0 automatically has the properties mentioned in Theorem 5.1 above. The statement that f_0 is symplectic and that it extends to a global diffeomorphism were quite non-trivial to prove. In [4][2] the extended map f_β was also given explicitly by making use of a covering of $\mathbb{C}P(n-1)$ by n coordinate patches and giving f_β explicitly on each patch.

To conclude this section, we remind that an integrable many-body system is self-dual in the sense of Ruijsenaars if there exists a symplectomorphism that exchanges its particle-position variables with the action-variables. Hence the message of equation (55) is that *our mapping class symplectomorphism* $\mathfrak{S}$ *(54) qualifies as a self-duality symplectomorphism in the sense of Ruijsenaars.* In fact, we have also checked that $\mathfrak{S}$ coincides precisely with the self-duality symplectomorphism of the $\mathrm{III_b}$ system constructed originally by a very different (non-geometric, direct) method in [13].

6 Further Results and Open Problems

This section contains a collection of remarks concerning the results of [4] and open problems.

First of all, let us recall that every quasi-Hamiltonian reduction of the internally fused double represents the moduli space of flat connections on the one-holed torus Σ with fixed conjugacy class of the holonomy around the hole. This is also the classical phase space of the Chern–Simons field theory on the three-dimensional manifold $[0,1] \times \Sigma$ with corresponding boundary condition. Therefore, our results outlined in the previous section prove the Chern–Simons interpretation of the $\mathrm{III_b}$ system and that of its self-duality, confirming the conjectures of Gorsky and his collaborators [6, 8].

In addition to the coupling constant, y, a second parameter, Λ, can be introduced into the $\mathrm{III_b}$ system by replacing the symplectic form (31) by $\Lambda \Omega^{\mathrm{loc}}$. This parameter, which is important at the quantum mechanical level, can be incorporated into the reduction approach by taking the invariant scalar product on $su(n)$ to be $-\frac{\Lambda}{2}\mathrm{tr}$ instead of (43). The quantum mechanics of the $\mathrm{III_b}$ system was studied by van Diejen and Vinet [15], who diagonalized the relevant commuting difference operators using Macdonald polynomials; see also our note [5] where we reproduced the joint spectrum of the action-variables by a simple argument. The Hilbert space of the

[2]The correspondence $L^y_{\mathrm{loc}}(\xi,\tau) \equiv \Delta(\tau)^{-1} L^{\mathrm{loc}}_y(\delta(\xi), \rho(\tau)^{-1}) \Delta(\tau)$ between the respective notations should be noted for those wish to see the details in [4].

Chern–Simons theory can be always equipped with a representation of the mapping class group [16], and it could be interesting to elaborate this representation in the specific case of the $\mathrm{III_b}$ system by building on the work [15].

Ruijsenaars [13] also considered an anti-symplectic involution $\mathfrak{R}$ on $\mathbb{C}P(n-1)$ that enjoys

$$\mathscr{J}_k \circ \mathfrak{R} = \mathscr{I}_k, \quad \mathscr{I}_k \circ \mathfrak{R} = \mathscr{J}_k, \quad k = 1, \ldots, n-1, \tag{58}$$

and is given by $\mathfrak{R} = \hat{C} \circ \mathfrak{S}$ where $\hat{C}$ is the complex conjugation involution. We have shown [4] that $\mathfrak{R}$ arises from the map R_D of the double of $SU(n)$ defined by

$$R_D := \rho_D \circ S_D^2, \qquad \rho_D(A,B) := (\bar{B}, \bar{A}), \qquad \forall (A,B) \in D. \tag{59}$$

Although R_D is not quite an automorphism of D, it descends to a map R_P on any reduced phase space $P(\mu_0)$ with diagonal constant matrix μ_0. (If μ_0 and μ_0' are conjugate then $P(\mu_0)$ and $P(\mu_0')$ are naturally equivalent, and therefore one may take μ_0 diagonal without loss of generality.) The involution R_P reverses the sign of the induced Poisson structure on $P(\mu_0)$, and together with S_P and T_P it generates a $GL(2,\mathbb{Z})$ action on $P(\mu_0)$.

Let $\mathscr{Z}$ be the center of the group G. Notice that $\mathscr{Z} \times \mathscr{Z}$ acts on the internally fused double $D = G \times G$ by the automorphisms

$$(z_1, z_2) : (A,B) \mapsto (z_1 A, z_2 B), \quad \forall (z_1, z_2) \in \mathscr{Z} \times \mathscr{Z}. \tag{60}$$

This action descends to the reduced phase space $P(\mu_0)$, and in the special case $G = SU(n)$ and μ_0 (50) it gives rise to the $\mathbb{Z}_n \times \mathbb{Z}_n$ action on $\mathbb{C}P(n-1)$ used in some considerations in [13].

The reader is invited to study [4] for further results, which include for example the factorization of S_D as a product of three Dehn twist automorphisms of the double, where the Dehn twist automorphisms themselves are realized in terms of certain quasi-Hamiltonian flows.

It could be worthwhile to explore the structure of the stratified symplectic spaces $P(\mu_0)$ in general, and to possibly uncover new integrable systems on them. Some sort of trigonometric spin Ruijsenaars–Schneider systems are expected to arise in this way, which might be integrable analogously to spin Sutherland systems [11].

Finally, the most intriguing open problem stems from the fact that a reduction treatment of the self-dual hyperbolic Ruijsenaars–Schneider system (the one which is related for example to sine-Gordon solitons) is still missing. Presently we do not know what master phase space should give this system upon reduction. Is it possible to construct such a master phase space? Of course, there exist other important variants of the Ruijsenaars–Schneider system (BC_n case [10], elliptic systems) that should be further studied as well.

Acknowledgements We wish to thank the organizers, V. Dobrev in particular, for the pleasant atmosphere that we enjoyed at the LT-9 workshop in Varna. This work was supported in part by the Hungarian Scientific Research Fund (OTKA) under the grant K 77400.

References

1. Alekseev, A., Malkin, A., Meinrenken, E.: Lie group valued moment maps. J. Diff. Geom. **48**, 445–495 (1998). arXiv:dg-ga/9707021
2. Duistermaat, J.J., Grünbaum, F.A.: Differential equations in the spectral parameter. Commun. Math. Phys. **103**, 177–240 (1986)
3. Fehér, L., Klimčík, C.: Poisson-Lie interpretation of trigonometric Ruijsenaars duality. Commun. Math. Phys. **301**, 55–104 (2011). arXiv:0906.4198 [math-ph]
4. Fehér, L., Klimčík, C.: Self-duality of the compactified Ruijsenaars–Schneider system from quasi-Hamiltonian reduction. Nucl. Phys. B **860**, 464–515 (2012). arXiv:1101.1795 [math-ph]
5. Fehér, L., Klimčík, C.: On the spectra of the quantized action variables of the compactified Ruijsenaars–Schneider system. Theor. Math. Phys. **171**, 704–714 (2012). arXiv:1203.2864 [math-ph]
6. Fock, V., Gorsky, A., Nekrasov, N., Rubtsov, V.: Duality in integrable systems and gauge theories. JHEP **0007**, 028 (2000). arXiv:hep-th/9906235
7. Goldman, W.: The modular group action on real $SL(2)$-characters of a one-holed torus. Geom. Topology **7**, 443–486 (2003). arXiv:math/0305096
8. Gorsky, A., Nekrasov, N.: Relativistic Calogero-Moser model as gauged WZW theory. Nucl. Phys. B **436**, 582–608 (1995). arXiv:hep-th/9401017
9. Huebschmann, J.: Poisson geometry of certain moduli spaces. Red. Circ. Mat. Palermo II Suppl. **39**, 15–35 (1996); and references therein
10. Pusztai, B.G.: The hyperbolic $BC(n)$ Sutherland and the rational $BC(n)$ Ruijsenaars–Schneider–van Diejen models: Lax matrices and duality. Nucl. Phys. B **856**, 528–551 (2012) arXiiv:1109.0446 [math-ph]
11. Reshetikhin, N.: Degenerate Integrability of spin Calogero-Moser systems and the duality with the spin Ruijsenaars systems. Lett. Math. Phys. **63**, 55–71 (2003). arXiv:math/0202245
12. Ruijsenaars, S.N.M.: Action-angle maps and scattering theory for some finite-dimensional integrable systems I. The pure soliton case. Commun. Math. Phys. **115**, 127–165 (1988)
13. Ruijsenaars, S.N.M.: Action-angle maps and scattering theory for some finite-dimensional integrable systems III. Sutherland type systems and their duals. Publ. RIMS **31**, 247–353 (1995)
14. Ruijsenaars, S.N.M.: Systems of Calogero-Moser type. In: Proceedings of the 1994 CRM–Banff Summer School 'Particles and Fields', pp. 251–352. Springer, New York (1999)
15. van Diejen, J.F., Vinet, L.: The quantum dynamics of the compactified trigonometric Ruijsenaars–Schneider model. Commun. Math. Phys. **197**, 33–74 (1998)
16. Witten, E.: Quantum field theory and the Jones polynomial. Commun. Math. Phys. **121**, 351–399 (1989)

Hilbert Space Decomposition for Coulomb Blockade in Fabry–Pérot Interferometers

Lachezar S. Georgiev

Abstract We show how to construct the thermodynamic grand potential of a droplet of incompressible fractional quantum Hall liquid, formed inside of an electronic Fabry–Pérot interferometer, in terms of the conformal field theory disk partition function for the edge states in presence of Aharonov–Bohm flux. To this end we analyze in detail the algebraic structure of the edge states' Hilbert space and identify the effect of the variation of the flux. This allows us to compute, in the linear response approximation, all thermodynamic properties of the conductance in the regime when the Coulomb blockade is softly lifted by the change of the magnetic flux due to the weak coupling between the droplet and the two quantum point contacts.

1 The FQHE Fabry–Pérot Interferometer

The electronic version [1] of the famous optical Fabry–Pérot interferometer, which we will analyze here, is constructed by two quantum point contacts (QPC) inside of an incompressible fractional quantum Hall (FQH) bar [2–4]. In the weak-backscattering regime, small gate voltages on the QPCs create constrictions inside the incompressible FQH liquid and facilitate tunneling of non-Abelian quasiparticles along the QPCs. However, this regime is unstable in the sense of the renormalization group flow, i.e., even a small number of quasiparticles tunneling along the QPCs at low T significantly renormalizes the tunneling amplitudes thus intensifying tunneling and eventually the two QPCs pinch off, which corresponds to the strong backscattering regime that is already a stable fixed point of the renormalization group flow.

L.S. Georgiev (✉)
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria
e-mail: lgeorg@inrne.bas.bg

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_31,

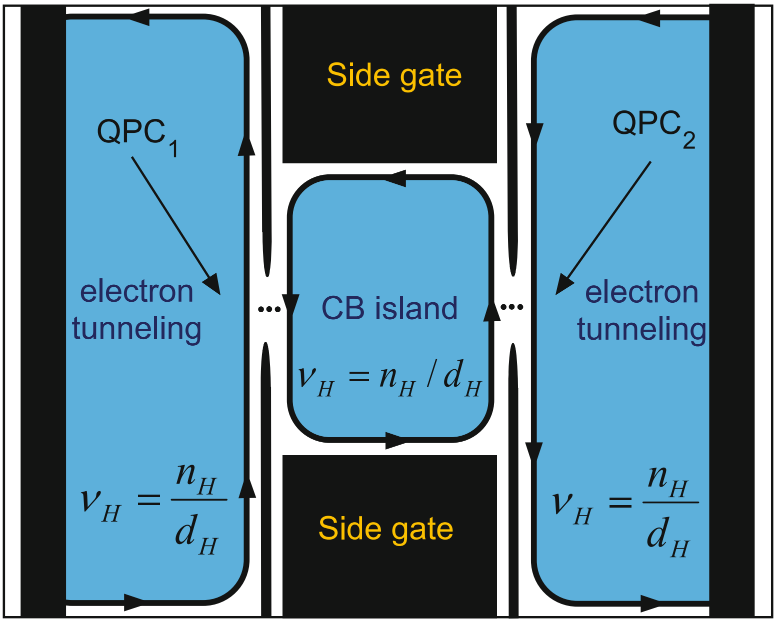


Fig. 1 A FQH bar with two QPCs in the strong backscattering regime in which single electrons could tunnel, if small bias is present, between the three disconnected liquids producing discrete peaks in the conductance. The side gates' voltage could change the area of the CB island varying in this way the flux through the island

In the strong backscattering regime, when the QPCs gate voltages are big enough that the two constrictions are completely pinched-off, the two-dimensional electron gas is split into three disconnected FQH liquids forming a *Coulomb blockade* (CB) island in the middle, see Fig. 1. Only electrons could tunnel between the disconnected parts of the interferometer and the main mechanism at low temperature and low bias is through *single electron tunneling*. The conductance in the CB regime is determined in the following steps (sequential tunneling through CB island): first one electron tunnels from the left FQH liquid through the left QPC to the island, then the electron which is accommodated at the edge of the CB island is transported along the edge and then it tunnels through the right QCP to the right FQH liquid. Using the Landauer formula one can see [5] that the CB conductance is

$$G_{\rm CB}(T,\phi) = \left(\frac{h}{e^2}\right) \frac{G_L G_R}{G_L + G_R} G_{\rm is}(T,\phi), \tag{1}$$

where the CB island's conductance $G_{\rm is}$ depends on the magnetic flux $\phi = B.A$: for most values of the flux we have Coulomb blockade ($G = 0$) and for special discrete values of flux we have conductance peaks [6–10]. The tunneling conductances of the two QPCs are independent of the flux and vanishing at low-temperature as $G_{L,R} \propto T^{4\Delta-2}$ where Δ is the scaling dimension of the electron operator.

2 Coulomb Blockade Island'S Conductance: The CFT Point of View

An interesting observation in this setup is that the conductance of the CB island can be explicitly computed at finite temperature within the framework of the conformal field theory [5]. This is due to the Einstein's relation [5, 11], which expresses the conductance $\sigma(0)$ in terms of the charge stiffness (or, thermodynamic density of states)

$$\sigma(0) = e^2 D \left. \frac{\partial n}{\partial \mu} \right|_T, \tag{2}$$

where D is the diffusion coefficient, μ is the chemical potential, n is the electron density and the thermodynamic derivative is at constant temperature.

The diffusion coefficient is usually related to the relaxation time [11], for normal conductors, however, for a *ballistic one-dimensional channel*, such as the FQH edge, the relaxation time must be replaced by the *time-of-flight* τ_f and the diffusion coefficient could be written as follows [11]

$$D_{\mathrm{bal}} = v_F^2 \tau_f, \quad \tau_f = L/(2v_F) \ \Rightarrow \ D = L v_F/2,$$

where v_F is the Fermi velocity at the edge and L is the circumference or length of the edge. According to (2) the charge stiffness can be computed as a derivative of the thermodynamic average of the particle number. To this end we shall use the Grand canonical partition function for a disk-shaped CB island derived within the CFT framework [5, 12]

$$Z_{\mathrm{disk}}(\tau, \zeta) = \mathrm{tr}_{\mathscr{H}_{\mathrm{edge}}} \, \mathrm{e}^{-\beta(H-\mu N)} = \mathrm{tr}_{\mathscr{H}_{\mathrm{edge}}} \, \mathrm{e}^{2\pi i \tau (L_0 - c/24)} e^{2\pi i \zeta Q}, \tag{3}$$

where the Hamiltonian of the disk $H = \hbar \frac{2\pi v_F}{L} \left(L_0 - \frac{c}{24} \right)$ is related to the zero mode of the Virasoro stress-tensor, c is the Virasoro central charge [12], v_F is the Fermi velocity of the edge states and L is the circumference of the disk; the particle number $Q \equiv N = \sqrt{\nu_H} J_0$ is proportional to the zero mode of the $\widehat{u(1)}$ current and ν_H is the FQH filling factor.

The Hilbert space $\mathscr{H}_{\mathrm{edge}}$ for the edge-states depends on the number and type of the residual quasiparticles which might be localized in the bulk when the magnetic field varies slightly around the value corresponding to the plateau of the Hall conductance. The thermodynamic parameters, such as the temperature and the chemical potential are related to the *modular parameters* τ and ζ on the torus introduced in a standard way for the rational CFTs [12]

$$\tau = i\pi \frac{T_0}{T}, \quad T_0 = \frac{\hbar v_F}{\pi k_B L}, \quad \zeta = i \frac{1}{2\pi k_B T} \mu. \tag{4}$$

2.1 CFT Disk Partition Function in Presence of AB Flux

When magnetic field threading the CB disk or the area of the disk are changed the effect on the one-dimensional edge state's system[1] is through the variation of the Aharonov–Bohm (AB) flux. As can be seen in [13] introducing AB flux changes the boundary conditions of the electron field operator and naturally twists the $\widehat{u(1)}$ current and the Virasoro stress tensor. The ultimate effect of this twisting on the partition function is that it simply shifts the modular parameter as follows $\zeta \to \zeta + \phi\tau$, i.e. the partition function in presence of AB flux ϕ is

$$Z^{\phi}_{\text{disk}}(\tau,\zeta) = Z_{\text{disk}}(\tau,\zeta+\phi\tau). \tag{5}$$

The Grand potential on the edge [14]

$$\Omega(T,\mu) = -k_B T \ln Z_{\text{disk}}(\tau,\zeta) \tag{6}$$

can be used to compute the particle density in the usual way

$$\langle n\rangle_{\beta,\mu} = -\frac{k_B T}{L}\frac{\partial}{\partial\mu}\ln Z_{\text{disk}}(\tau,\zeta) = \frac{1}{L}\langle J_0\rangle_{\beta,\mu} \tag{7}$$

where $\beta = (k_B T)^{-1}$ is the inverse temperature and the thermal average is as usual

$$\langle A\rangle_{\beta,\mu} = Z^{-1}_{\text{disk}}(\tau,\zeta)\,\text{tr}_{\mathscr{H}_{\text{edge}}}\, A\, \text{e}^{2\pi i\tau(L_0-c/24)} e^{2\pi i\zeta J_0}. \tag{8}$$

2.2 Coulomb Island'S Conductance

In order to obtain the charge stiffness of the CB island, we need to differentiate the particle density which, according to (3) and (8), is related to the thermodynamic averages of the zero mode of the $\widehat{u(1)}$ current

$$\left\langle \frac{\partial n}{\partial\mu}\right\rangle_{\beta,\mu} = \frac{1}{L k_B T}\left(\langle J_0^2\rangle_{\beta,\mu} - (\langle J_0\rangle_{\beta,\mu})^2\right). \tag{9}$$

On the other hand, the Grand potential on the edge $\Omega(T,\mu)$ depends on the AB flux ϕ threading the edge because of (4) and (5) and the second derivative with respect to ϕ is

[1]due to the incompressibility of the FQH droplet, the states in the bulk are localized and the only states capable of carrying electric current are living on the edge which is a one-dimensional channel

$$\frac{\partial^2 \Omega}{\partial \phi^2} = -\frac{(h v_F/L)^2}{k_B T}\left(\langle J_0^2\rangle_{\beta,\mu} - (\langle J_0\rangle_{\beta,\mu})^2\right). \tag{10}$$

Comparing (9) with (10) we conclude [5] that *edge conductance is exactly proportional to the magnetic susceptibility* $\kappa(T,\phi) = -(e/h)^2\partial^2\Omega(T,\phi)/\partial\phi^2$, i.e.

$$G_{\mathrm{is}}(T,\phi) = \frac{\sigma_{\mathrm{is}}(0)}{L} = -\frac{L}{2v_F}\left(\frac{e}{h}\right)^2 \frac{\partial^2\Omega(T,\phi)}{\partial\phi^2} \tag{11}$$

This beautiful result, which relates a non-equilibrium quantity, such as the CB islands' conductance G_{is}, to an equilibrium one expressed as a derivative of the Grand potential Ω, is valid within the Kubo linear response regime, characterized by the conditions $G_{L,R} \ll e^2/h$, which is used in the derivation [11] of the Einstein's relation.

2.3 Disk Partition Functions for FQH Droplets

To compute the partition function for the edge of a disk FQH sample we need some knowledge of the structure of the underlying CFT. The rational CFT for a FQH state always contains a $\widehat{u(1)}$ current algebra which is completely determined by the filling factor $\nu_H = n_H/d_H$. This current algebra always contributes a $c = 1$ stress-tensor to the Virasoro algebra due to the Sugawara contribution [12]. There is in general, a neutral Virasoro generator $T^{(0)}(z)$ as well, defined by $T(z) - T^{(c)}(z) = T^{(0)}(z)$ whose central charge must be positive.

The electron field operator naturally decomposes into a charged $\widehat{u(1)}$ part and a neutral component which must be a primary field of the neutral Virasoro algebra. From the electron CFT dimension $\Delta_{\mathrm{el}} = \frac{d_H}{2n_H} + \Delta^{(0)}$ we see that its statistical angle $\theta/\pi = 2\Delta_{\mathrm{el}} = 2\Delta^{(0)} + \frac{d_H}{n_H}$, which must be an odd integer, imposes certain conditions on the structure of the CFT. In particular, the electron field operator must have a non-trivial neutral component when $n_H > 1$, hence the neutral Virasoro algebra must be non-trivial, too. This also implies that the charged and neutral parts of the RCFT are not completely independent and therefore the partition function will not be simply a product of charged and neutral partition functions—instead there are pairing rules for the admissible combinations of charged and neutral characters.

3 Decomposable Subalgebra and $\mathbb{Z}_{n_H}$ Grading

In this section we will consider in more detail the algebraic structure of the rational CFT corresponding to a general FQH state on a disk.

We start by noting that the $\widehat{u(1)}$ part[2] of the electron field operator, constructed as a $\widehat{u(1)}$ vertex exponent [15], with a charge parameter determined by the filling factor,

$$: \psi_{\rm el}(z)\overline{\Psi^{(0)}}(z) : \simeq : \mathrm{e}^{-i\frac{1}{\sqrt{\nu_H}}\phi^{(c)}(z)} :, \tag{12}$$

of a chiral boson normalized by

$$\langle \phi^{(c)}(z)\phi^{(c)}(w)\rangle = -\ln(z-w), \tag{13}$$

certainly commutes with all neutral field operators. However, the vertex exponent (12) has in general a non-integer statistical angle $\theta/\pi = d_H/n_H$ and is not local for $n_H > 1$. Therefore it does not belong to the chiral (super)algebra $\mathscr{A}$ and cannot be used to decompose the latter.

The way out of this locality problem is to consider the n_H-th power of the vertex exponent (12)

$$: \exp\left(-i\frac{n_H}{\sqrt{\nu_H}}\phi^{(c)}(z)\right) : = : \exp\left(-i\sqrt{n_H d_H}\phi^{(c)}(z)\right) : \tag{14}$$

which still commutes with all neutral field operators but is local because its statistics is $\theta/\pi = n_H d_H$, so that it does belong to $\mathscr{A}$. It is worth stressing that the $\widehat{u(1)}$ vertex operator (14) together with all neutral generators of $\mathscr{A}$ generates a *decomposable chiral subalgebra* $\mathscr{A}_D$ of the original chiral superalgebra $\mathscr{A}$

$$\mathscr{A}_D = \widehat{u(1)}_m \otimes \mathscr{A}^{(0)} \subset \mathscr{A}. \tag{15}$$

We use the notation $\widehat{u(1)}_m$ to denote the rational extension [12, 15] of the $\widehat{u(1)}$ current algebra with the pair of vertex exponents $: \mathrm{e}^{\pm i\sqrt{m}\phi(z)} :$ with $m = n_H d_H$.

Because the decomposable subalgebra $\mathscr{A}_D$ misses only the powers of the full electron operator $\psi_{\rm el}^s$ with $s = 0, \ldots, n_H - 1$, the original superalgebra $\mathscr{A}$ can be naturally represented as the following *direct sum decomposition*

$$\mathscr{A} = \bigoplus_{s=0}^{n_H-1} \psi_{\rm el}^s \mathscr{A}_D. \tag{16}$$

Due to the orthogonality of the different powers of the electron field, following from the $\widehat{u(1)}$ charge conservation,

$$\left\langle \psi_{\rm el}^s \mathscr{A}_D, \psi_{\rm el}^{s'} \mathscr{A}_D \right\rangle = \left\langle \psi_{\rm el}^s, \psi_{\rm el}^{s'} \right\rangle \langle \mathscr{A}_D, \mathscr{A}_D\rangle = 0 \quad \text{if} \quad s \neq s',$$

where $\langle \ldots, \ldots \rangle$ denotes the scalar product, it appears that the decomposition in (16) is in fact a $\mathbb{Z}_{n_H}$-graded direct sum decomposition.

[2]this part can be considered as the result of the fusion of the full electron operator with its neutral component $\Psi^{(0)}(z)$

The virtue of having a decomposable subalgebra is that it defines the following *dual algebra inclusion*

$$\mathscr{A}_D \subset \mathscr{A} \subset \mathscr{A}^* \subset \mathscr{A}_D^*, \tag{17}$$

which simplifies the construction of the representation spaces. It follows from (17) that all representations of $\mathscr{A}$ are also representations of $\mathscr{A}_D$ and at the same time that not all representations of $\mathscr{A}_D$ are true representations of $\mathscr{A}$.

Given that the decomposable algebra (15) is simply a tensor product, its irreducible representations (IR) are labeled by pairs of quantum numbers (l, Λ), where l is the electric charge of the bulk quasiparticles in such units that $Q_{\text{el}}(\text{bulk}) = l/d_H$, and Λ is the (total) neutral topological charge of the bulk quasiparticles. Then, it follows from (16) that all IRs of $\mathscr{A}$ are direct sums of IRs of $\mathscr{A}_D$, corresponding to the orbit of the simple current's action, hence we shall be labeling the irreducible representations of $\mathscr{A}$ by the same pair (l, Λ), corresponding to the $s = 0$ component in (16).

As follows from (17), not all representations of $\mathscr{A}_D$ are true representations of the original superalgebra $\mathscr{A}$. In order to identify the *physical* excitations, corresponding to the true representations of $\mathscr{A}$ we will require that they are local with respect to the electron field. The locality principle implies that those IRs of $\mathscr{A}_D$ which are local with respect to the electron are also IRs of $\mathscr{A}$. To formulate more precisely the locality requirement let us consider the decomposition of the electron field and an arbitrary excitation labeled by (l, Λ) into $\widehat{u(1)}$ and neutral parts

$$\begin{aligned} \text{electron:} \qquad & \psi_{\text{el}}(z) = :\mathrm{e}^{-i\frac{d_H}{\sqrt{n_H d_H}}\phi^{(c)}(z)}: \Psi_{\omega}^{(0)}(z) \\ \text{excitation:} \qquad & \psi_{l,\Lambda}(z) = :\mathrm{e}^{i\frac{l}{\sqrt{n_H d_H}}\phi^{(c)}(z)}: \Psi_{\Lambda}^{(0)}(z), \end{aligned}$$

where the $\widehat{u(1)}$ boson is normalized as in (13), the electric charge is related to the $\widehat{u(1)}$ label l by $Q_{\text{el}}(l) = l/d_H$, so that the electric charge label of the electron is $l = -d_H$, and ω denotes the (nontrivial) neutral topological charge of the electron.

Now, to identify the *physical* excitations within the extended dual algebra $\mathscr{A}_D^*$ we require local operator product expansion (OPE) of the excitation with respect to the electron, i.e., we require the power of the coordinate distance $(z-w)$ in the short-distance OPE to be integer

$$\psi_{\text{el}}(z)\psi_{l,\Lambda}(w) \underset{z\to w}{\simeq} (z-w)^{-\frac{l}{n_H}+Q_\omega(\Lambda)} \quad :\mathrm{e}^{i\frac{l-d_H}{\sqrt{n_H d_H}}\phi^{(c)}(z)}: \quad \Psi_{\omega*\Lambda}^{(0)}(w),$$

where $Q_\omega(\Lambda)$ is the (neutral) monodromy charge defined by the following combination of conformal dimensions $\Delta_{\Lambda'}$ of the neutral Virasoro IRs

$$Q_\omega(\Lambda) \equiv \Delta_{\omega*\Lambda} - \Delta_\Lambda - \Delta_\omega \ \mathrm{mod}\ \mathbb{Z}, \quad \left(\Delta_\omega = \Delta^{(0)}\right). \tag{18}$$

Thus, the locality condition implies that the physical excitations (respectively, the true IRs of $\mathscr{A}$) must satisfy the following $\mathbb{Z}_{n_H}$ *pairing rule* which selects the admissible pairs (l,Λ) of charged and neutral quantum numbers

$$n_H Q_\omega(\Lambda) \equiv l \bmod n_H. \tag{19}$$

The representation spaces of $\mathscr{A}_D = \widehat{u(1)}_m \otimes \mathscr{A}^{(0)}$ labeled by the pairs (l,Λ) which obey the PR (19) (that guarantees these pairs are true representations of the original algebra $\mathscr{A}$) are naturally tensor products of the representation spaces $\mathscr{H}_l^{(c)}$ for the $\widehat{u(1)}$ current algebra and those, $\mathscr{H}_\Lambda^{(0)}$, for the neutral Virasoro algebra, i.e.

$$\mathscr{H}_{l,\Lambda}^D = \mathscr{H}_l^{(c)} \otimes \mathscr{H}_\Lambda^{(0)}, \tag{20}$$

which explains why we looked for a decomposable subalgebra.

The representation spaces $\mathscr{H}_{l,\Lambda}^{\mathscr{A}}$ for the original algebra $\mathscr{A}$ can be obtained by the action of $\mathscr{A}$ over the lowest-weight state $|l,\Lambda\rangle$. Because of the decomposition (16) this space has a natural direct sum decomposition into representation space $\mathscr{H}_{l,\Lambda}^D$ for the decomposable subalgebra

$$\mathscr{H}_{l,\Lambda}^{\mathscr{A}} = \mathscr{A}|l,\Lambda\rangle = \bigoplus_{s=0}^{n_H-1} \psi_{\mathrm{el}}^s \mathscr{A}^D |l,\Lambda\rangle = \bigoplus_{s=0}^{n_H-1} \mathscr{J}^s \mathscr{H}_{l,\Lambda}^D,$$

where $\mathscr{J} \simeq \psi_{\mathrm{el}}^*(0)$ is the simple current [12] representing the action of the electron field operator over the lowest-weight states, i.e.

$$\mathscr{J}|l,\Lambda\rangle = |l+d_H, \omega * \Lambda\rangle,$$

which means that the simple current $\mathscr{J}$ acts on lowest-weight states by fusion—the $\widehat{u(1)}$ charge is simply shifted by the electric charge of the electron, while the neutral Virasoro topological charges are fused with that of the electron.

Taking into account (20) we finally obtain the representation space for $\mathscr{A}$

$$\mathscr{H}_{l,\Lambda}^{\mathscr{A}} = \bigoplus_{s=0}^{n_H-1} \mathscr{J}^s \left(\mathscr{H}_l^{(c)} \otimes \mathscr{H}_\Lambda^{(0)} \right) = \bigoplus_{s=0}^{n_H-1} \mathscr{H}_{l+sd_H}^{(c)} \otimes \mathscr{H}_{\omega^s * \Lambda}^{(0)}. \tag{21}$$

The benefit of this representation of the Hilbert space for a general FQH disk is that its $\widehat{u(1)}$ part $\mathscr{H}_l^{(c)}$, which is the edge-states' space of the Luttinger liquid, is completely determined by the filling factor ν_H and the neutral part $\mathscr{H}_\Lambda^{(0)}$ is what distinguishes between FQH states with the same filling factor but different universality classes.

4 The RCFT Partition Function for a General FQH Disk

Now that we know the general structure of the Hilbert space for an arbitrary FQH disk state we can obtain the corresponding structure of the partition function by plugging (21) into (3). Notice however, that the $\widehat{u(1)}_m$ representation spaces $\mathscr{H}_l^{(c)}$ entering (21) correspond to $m = n_H d_H$ and therefore the electric charge operator Q could be represented in terms of $\widehat{u(1)}_m$ number operator $N = J_0/\sqrt{m}$, i.e.

$$Q = \sqrt{\frac{n_H}{d_H}} J_0 = \sqrt{\frac{n_H}{d_H}} \sqrt{n_H d_H} N = n_H N. \tag{22}$$

Therefore, using the properties of the trace as well as the structure of the Hilbert space (21), we obtain the main result—the partition function for a general FQH disk can be represented as a sum of n_H products of $\widehat{u(1)}$ and neutral partition functions

$$Z_{l,\Lambda}(\tau,\zeta) = \sum_{s=0}^{n_H-1} K_{l+sd_H}(\tau, n_H\zeta; n_H d_H)\, \mathrm{ch}_{\omega^s * \Lambda}(\tau), \tag{23}$$

where the $\widehat{u(1)}$ partition functions $K_{l+sd_H}(\tau, n_H\zeta; n_H d_H)$ are expressed as Luttinger liquid partition functions for $m = n_H d_H$ in the notation of [15]

$$K_l(\tau,\zeta;m) = \frac{\mathrm{CZ}(\tau,\zeta)}{\eta(\tau)} \sum_{n=-\infty}^{\infty} q^{\frac{m}{2}\left(n+\frac{l}{m}\right)^2} \mathrm{e}^{2\pi i \zeta \left(n+\frac{l}{m}\right)}. \tag{24}$$

The absolute temperature and the Bolzmann factor $e^{-\beta}$ are related to the modular parameter τ

$$q = \mathrm{e}^{-\beta \Delta\varepsilon} = \mathrm{e}^{2\pi i \tau}, \quad \Delta\varepsilon = \hbar \frac{2\pi v_F}{L}, \tag{25}$$

where $\Delta\varepsilon$ is the non-interacting energy spacing, v_F is the Fermi velocity on the edge and L is the circumference of the disk. The Dedekind function η and Cappelli–Zemba factors [16] entering (24) are explicitly given by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \quad \mathrm{CZ}(\tau,\zeta) = \mathrm{e}^{-\pi \nu_H \frac{(\mathrm{Im}\,\zeta)^2}{\mathrm{Im}\,\tau}}.$$

It is worth stressing that the $\widehat{u(1)}$ partition functions (24) are completely explicit and totally determined by the filling factor's numerator n_H and denominator d_H. The extra n_H in front of ζ in the Luttinger-liquid partition function $K_{l+sd_H}(\tau, n_H\zeta; n_H d_H)$ appears due to the relation (22).

The neutral partition functions, which are known mathematically as the characters of the representations $\mathscr{H}_\Lambda^{(0)}$ of the neutral Virasoro algebra with central charge $c-1$, are defined as usual as the trace over the representation space [12]

$$\mathrm{ch}_{\Lambda}(\tau) = \mathrm{tr}_{\mathscr{H}_{\Lambda}^{(0)}} q^{L_0^{(0)} - \frac{c-1}{24}}.$$

The neutral topological charge of the electron is denoted by ω and $\omega * \Lambda$ in (23) denotes the fusion of the topological charges of the electron and the bulk quasiparticles. Unlike the charged-part partition functions the neutral ones are not completely determined by the filling factor, though their structure is almost fixed by the neutral weights ω, Λ and their fusion rules, thus representing more subtle topological properties of the FQH universality class. Fortunately, for most of the FQH universality classes these functions are explicitly known.

5 Application: Coulomb Blockade in the $\mathbb{Z}_3$ Read–Rezayi State

The structure of the partition function (23), in which the $\widehat{u(1)}$ part is explicitly separated, is very convenient for the computation of the CB peaks for a FQH island at finite temperature since the variation of the AB flux ϕ changes only the $\widehat{u(1)}$ partition functions (24) because of (5). Consider, for example a CB island in which the FQH state is the $\mathbb{Z}_3$ Read–Rezayi (parafermion) state [17, 18], characterized by $n_H = 3$, $d_H = 5$, i.e. $\nu_H = 3/5$. The decomposable chiral subalgebra is $\widehat{u(1)}_{15} \times \mathscr{W}_3$, where $\mathscr{W}_3$ is the $\mathbb{Z}_3$ parafermion algebra of Fateev–Zamolodchikov [19]. The neutral part of the electron operator has a topological charge $\omega = \psi_1,\ \omega^2 = \psi_2$ given by the parafermion currents. As a simple illustration of the entire procedure let us consider the case when there are no quasiholes in the bulk, which corresponds to $l = 0, \Lambda = 0$. The partition function (23) takes the form

$$Z_{0,0}(\tau,\zeta) = K_0(\tau;3\zeta;15)\mathrm{ch}_{00}(\tau) + K_5(\tau;3\zeta;15)\mathrm{ch}_{01}(\tau) + K_{-5}(\tau;3\zeta;15)\mathrm{ch}_{02}(\tau)$$

where the K functions are defined in (24), the Bolzmann factor q is defined in (25) and the neutral partition functions are defined by

$$\mathrm{ch}_{0,l}(\tau) = q^{-\frac{1}{30}} \sum_{n_1,\, n_2 \geq 0}^{(l)} \frac{q^{\frac{2}{3}\left(n_1^2 + n_1 n_2 + n_2^2\right)}}{(q)_{n_1}(q)_{n_2}}, \quad (q)_n = \prod_{j=1}^{n} (1 - q^j).$$

and the sum $\sum^{(l)}$ is restricted by the condition $n_1 + 2n_2 = l \bmod 3$. Introducing AB flux as in (5) and plugging the partition function with flux into (11) we calculate numerically the conductance of the CB island at temperature $T = 0.5T_0$ as the flux is varied, see Fig. 2.

Under the assumption that the neutral and charged modes propagate with the same Fermi velocity we see that the CB peaks are clustered in bunches of three, separated by flux period $\Delta\phi_1 = 1$ inside the bunch, and separated by a flux period

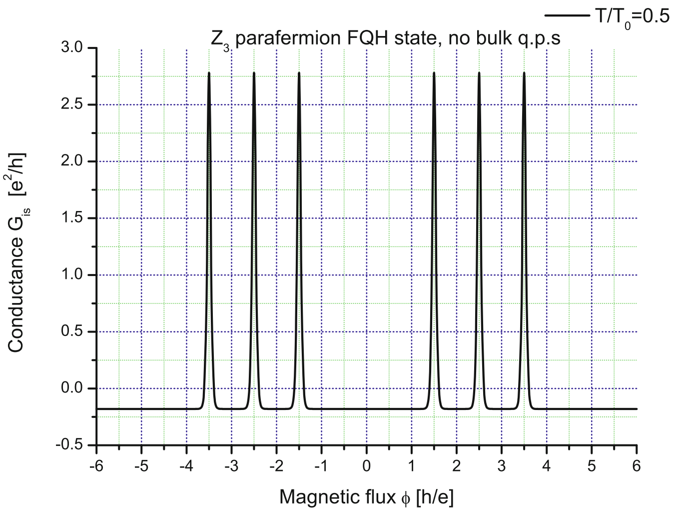


Fig. 2 Coulomb blockade peaks of the conductance, appearing when AB flux is varied, for the $\mathbb{Z}_3$ parafermion FQH island without non-trivial quasiparticles in the bulk at temperature $T/T_0 = 0.5$

$\Delta\phi_2 = 3$ between the bunches, which is in agreement with the previous results at zero temperature [7, 9, 10].

Most of the characteristics of the CB peaks, such as the height, the width and the periods, can be derived asymptotically at very low temperatures [5].

Acknowledgements I would like to thank Andrea Cappelli, Guillermo Zemba, Ady Stern, Pasquale Sodano and Reinhold Egger for many useful discussions as well as INFN-Firenze and the Galileo Galilei Institute for Theoretical Physics for hospitality and support. The author has been supported as a Research Fellow by the Alexander von Humboldt Foundation as well as by ESF and by the Bulgarian NSF grant DO 02-257.

References

1. Ji, Y., Chung, Y., Sprinzak, D., Heiblum, M., Mahalu, D., Shtrikman, H.: An electronic Mach–Zehnder interferometer. Nature **422**, 415 (2003)
2. Wen, X.-G.: Topological orders in rigid states. Int. J. Mod. Phys. **B4**, 239 (1990)
3. Froehlich, J., Zee, A.: Large scale physics of the quantum Hall fluid. Nucl. Phys. B **364**, 517 (1990)
4. Cappelli, A., Trugenberger, C.A., Zemba, G.R.: Infinite symmetry in the quantum Hall effect. Nucl. Phys. **B396**, 465 (1993)
5. Georgiev, L.S.: Thermal broadening of the Coulomb blockade peaks in quantum Hall interferometers. EPL **91**, 41001 (2010). http://xxx.lanl.gov/abs/arXiv:1003.4871
6. Stern, A., Halperin, B.I.: Proposed experiments to probe the non-abelian $\nu = 5/2$ quantum Hall state. Phys. Rev. Lett. **94**, 016802 (2006). http://xxx.lanl.gov/abs/cond-mat/0508447

7. Ilan, R., Grosfeld, E., Stern, A.: Coulomb blockade as a probe for non-Abelian statistics in Read–Rezayi states. Phys. Rev. Lett. **100**, 086803 (2008). http://xxx.lanl.gov/abs/arXiv:0705.2187
8. Cappelli, A., Georgiev, L.S., Zemba, G.R.: Coulomb blockade in hierarchical quantum Hall droplets. J. Phys. A Math. Theor. **42**, 222001 (2009). http://xxx.lanl.gov/abs/arXiv:0902.1445
9. Bonderson, P., Nayak, C., Shtengel, K.: Coulomb blockade doppelgangers in quantum Hall states. Phys. Rev. B **81**, 165308 (2010). http://xxx.lanl.gov/abs/arXiv:0909.1056
10. Cappelli, A., Viola, G., Zemba, G.R.: Chiral partition functions of quantum Hall droplets. Ann. Phys. **325**, 465 (2010). http://xxx.lanl.gov/abs/atXiv:0909.3588
11. Di Ventra, M.: Electrical Transport in Nanoscale Systems. Cambridge University Press, Cambridge (2008)
12. Di Francesco, P., Mathieu, P., Sénéchal, D.: Conformal Field Theory. Springer, New York (1997)
13. Georgiev, L.S.: A universal conformal field theory approach to the chiral persistent currents in the mesoscopic fractional quantum Hall states. Nucl. Phys. **B 707**, 347–380 (2005). http://xxx.lanl.gov/abs/hep-th/0408052
14. Kubo, R., Toda, M., Hashitsume, N.: Statistical Physics II. Springer, Berlin (1985)
15. Furlan, P., Sotkov, G., Todorov, I.: Two-dimensional conformal quantum field theory. Riv. Nuovo Cim. **12**(6), 1 (1989)
16. Cappelli, A., Zemba, G.R.: Modular invariant partition functions in the quantum Hall effect. Nucl. Phys. **B490**, 595 (1997). http://xxx.lanl.gov/abs/hep-th/9605127
17. Read, N., Rezayi, E.: Beyond paired quantum hall states: Parafermions and incompressible states in the first excited landau level. Phys. Rev. **B59**, 8084 (1998)
18. Cappelli, A., Georgiev, L.S., Todorov, I.T.: Parafermion Hall states from coset projections of Abelian conformal theories. Nucl. Phys. **B 599 [FS]**, 499–530 (2001). http://xxx.lanl.gov/abs/hep-th/0009229
19. Fateev, V., Zamolodchikov, A.: Conformal quantum field theory models in two dimensions having Z_3 symmetry. Nucl. Phys. **B280**, 644 (1987)

Group Classification of Variable Coefficient KdV-like Equations

Olena Vaneeva

Abstract The exhaustive group classification of the class of KdV-like equations with time-dependent coefficients $u_t + uu_x + g(t)u_{xxx} + h(t)u = 0$ is carried out using equivalence based approach. A simple way for the construction of exact solutions of KdV-like equations using equivalence transformations is described.

1 Introduction

A number of physical processes are modeled by generalizations of the well-known equations of mathematical physics such as, e.g., the KdV and mKdV equations, the Kadomtsev–Petviashvili equation, which contain time-dependent coefficients. That is why last decade these equations do attract attention of researchers. A number of the papers devoted to the study of variable coefficient KdV or mKdV equations with time-dependent coefficients were commented in [10]. In the majority of papers the results were obtained mainly for the equations which are reducible to the standard KdV or mKdV equations by point transformations. Unfortunately equivalence properties are neglected usually and finding of exact solutions is reduced to complicated calculations of systems involving a number of unknown functions using computer algebra packages. It is shown in [10, 12] that the usage of equivalence transformations allows one to obtain the results in a much simpler way.

In this paper this fact is reaffirmed via presentation the correct group classification of a class of variable coefficient KdV equations using equivalence based

O. Vaneeva (✉)
Institute of Mathematics of the National Academy of Sciences of Ukraine,
3 Tereshchenkivs'ka Street, 01601 Kyiv-4, Ukraine
e-mail: vaneeva@imath.kiev.ua

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_32,

approach. Namely, we investigate Lie symmetry properties and exact solutions of variable coefficient KdV equations of the form

$$u_t + uu_x + g(t)u_{xxx} + h(t)u = 0, \qquad (1)$$

where g and h are arbitrary smooth functions of the variable t, $g \neq 0$. It is shown in Sect. 2 that using equivalence transformations the function h can be always set to the zero value and therefore the form of h does not affect results of group classification. The group classification of class (1) with $h = 0$ is carried out in [10]. So, using the known classification list and equivalence transformations we present group classification of the initial class (1) without direct calculations.

An interesting property of class (1) is that it is normalized, i.e., all admissible point transformations within this class are generated by transformations from the corresponding equivalence groups. Therefore, there are no additional equivalence transformations between cases of the classification list, which is constructed using the equivalence relations associated with the corresponding equivalence group. In other words, the same list represents the group classification result for the corresponding class up to the general equivalence with respect to point transformations.

Recently the authors of [3] obtained a partial group classification of class (1) (the notation a and b was used there instead of h and g, respectively). The reason of failure was neglecting an opportunity to use equivalence transformations. This is why only some cases of Lie symmetry extensions were found, namely the cases with $h = \mathrm{const}$, $h = 1/t$ and $h = 2/t$.

In fact the group classification problem for class (1) up to its equivalence group is already solved since this class is reducible to class (1) with $h = 0$ whose group classification is carried out in [10]. Using the known classification list and equivalence transformations we present group classifications of class (1) without the simplification of both equations admitting extensions of Lie symmetry algebras and these algebras themselves by equivalence transformations. The extended classification list can be useful for applications and convenient to be compared with the results of [3].

Note that in [1, 4] group classifications for more general classes that include class (1) were carried out. Nevertheless those results obtained up to very wide equivalence group seem to be inconvenient to derive group classification for class (1).

2 Equivalence Transformations

An important step under solving a group classification problem is the construction of the equivalence group of the class of differential equations under consideration. The usage of transformations from the related equivalence group often gives an opportunity to essentially simplify a group classification problem and to present the final results in a closed and concise form. Moreover, sometimes this appears to be a crucial point in the exhaustive solution of such problems [2, 12–14].

There exist several kinds of equivalence groups. The *usual equivalence group* of a class of differential equations consists of the nondegenerate point transformations in the space of independent and dependent variables and arbitrary elements of the class such that the transformation components for the variables do not depend on arbitrary elements and each equation from the class is mapped by these transformations to equations from the same class. If any point transformation between two fixed equations from the class belongs to its (usual) equivalence group then this class is called *normalized*. See theoretical background on normalized classes in [8, 9].

We find the equivalence group $G_1^{\sim}$ of class (1) using the results obtained in [10] for more general class of variable coefficient KdV-like equations. Namely, in [10] a hierarchy of normalized subclasses of the general third-order evolution equations was constructed. The equivalence group for normalized class of variable coefficient KdV equations

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)x + l(t) = 0, \qquad (2)$$

as well as criterion of reducibility of equations from the this class to the standard KdV equation were found therein.

The equivalence group $G^{\sim}$ of class (2) consists of the transformations

$$\tilde{t} = \alpha(t), \quad \tilde{x} = \beta(t)x + \gamma(t), \quad \tilde{u} = \theta(t)u + \varphi(t)x + \psi(t), \qquad (3)$$

where α, β, γ, θ, φ and ψ run through the set of smooth functions of t, $\alpha_t\beta\theta \neq 0$. The arbitrary elements of (2) are transformed as follows

$$\tilde{f} = \frac{\beta}{\alpha_t\theta} f, \quad \tilde{g} = \frac{\beta^3}{\alpha_t} g, \quad \tilde{h} = \frac{1}{\alpha_t}\left(h - \frac{\varphi}{\theta} f - \frac{\theta_t}{\theta}\right), \qquad (4)$$

$$\tilde{q} = \frac{1}{\alpha_t}\left(q - \frac{\varphi}{\theta} f + \frac{\beta_t}{\beta}\right), \quad \tilde{p} = \frac{1}{\alpha_t}\left(\beta p - \gamma q + \frac{\gamma\varphi - \beta\psi}{\theta} f + \gamma_t - \gamma\frac{\beta_t}{\beta}\right), \qquad (5)$$

$$\tilde{k} = \frac{1}{\alpha_t\beta}\left(\theta k - \varphi\alpha_t\tilde{h} - \varphi_t\right), \quad \tilde{l} = \frac{1}{\alpha_t}\left(\theta l - \gamma\alpha_t\tilde{k} - \psi\alpha_t\tilde{h} - \varphi p - \psi_t\right). \qquad (6)$$

We also adduce the criterion of reducibility of (2) to the standard KdV equation.

Proposition 1 ([10]). *An equation of form (2) is similar to the standard (constant coefficient) KdV equation if and only if its coefficients satisfy the condition*

$$s_t = 2gs^2 - 3qs + \frac{f}{g}k, \quad \text{where} \quad s := \frac{2q - h}{g} + \frac{f_t g - f g_t}{f g^2}. \qquad (7)$$

Class (1) is a subclass of class (2) singled out by the conditions $f=1$ and $p=q=k=l=0$. Substituting these values of the functions f,p,q,k and l to (7) we obtain the following assertion.

Corollary 1. *An equation from class (1) is reduced to the standard KdV equation by a point transformation if and only if there exist a constant c_0 and $\varepsilon\in\{0,1\}$ such that*

$$h=\frac{\varepsilon}{2}\frac{g}{\int g\,dt+c_0}-\frac{g_t}{g}. \tag{8}$$

As class (2) is normalized [10], its equivalence group $G^{\sim}$ generates the entire set of admissible (form-preserving) transformations for this class. Therefore, to describe the set of admissible transformations for class (1) we should set $\tilde f=f=1$, $\tilde p=p=\tilde q=q=\tilde k=k=\tilde l=l=0$ in (4)–(6) and solve the resulting equations with respect to transformation parameters. It appears that class (1) admits generalized extended equivalence group and it is normalized in generalized sense only.

Summing up the above consideration, we formulate the following theorem.

Theorem 1. *The generalized extended equivalence group $\hat G_1^{\sim}$ of class (1) consists of the transformations*

$$\tilde t=\alpha,\;\; \tilde x=\beta x+\gamma,\;\; \tilde u=\lambda(\beta u+\beta_t x+\gamma_t),\;\; \tilde h=\lambda h-2\lambda\frac{\beta_t}{\beta}-\lambda_t,\;\; \tilde g=\beta^3\lambda g.$$

Here α is an arbitrary smooth function of t with $\alpha_t\neq 0$, $\beta=(\delta_1\int e^{-\int h dt}dt+\delta_2)^{-1}$, $\gamma=\delta_3\int\beta^2 e^{-\int h dt}dt+\delta_4$; $\delta_1,\dots,\delta_4$ are arbitrary constants, $(\delta_1,\delta_2)\neq(0,0)$ and $\lambda=1/\alpha_t$.

The usual equivalence group $G_1^{\sim}$ of class (1) is the subgroup of the generalized extended equivalence group $\hat G_1^{\sim}$, which is singled out with the condition $\delta_1=\delta_3=0$.

The parameterization of transformations from $\hat G_1^{\sim}$ by the arbitrary function $\alpha(t)$ allows us to simplify the group classification problem for class (1) via reducing the number of arbitrary elements. For example, we can gauge arbitrary elements via setting either $h=0$ or $g=1$. Thus, the gauge $h=0$ can be made by the equivalence transformation

$$\hat t=\int e^{-\int h(t)\,dt}dt,\quad \hat x=x,\quad \hat u=e^{\int h(t)\,dt}u, \tag{9}$$

that connects (1) with the equation $\hat u_{\hat t}+\hat u\hat u_{\hat x}+\hat g(\hat t)\hat u_{\hat x\hat x\hat x}=0$. The new arbitrary element $\hat g$ is expressed via g and h in the following way:

$$\hat g(\hat t)=e^{\int h(t)\,dt}g(t).$$

This is why without loss of generality we can restrict the study to the class

$$u_t+uu_x+g(t)u_{xxx}=0, \tag{10}$$

since all results on symmetries and exact solutions for this class can be extended to class (1) with transformations of the form (9).

The equivalence group for class (10) can be obtained from Theorem 1 by setting $\tilde{h} = h = 0$. Note that class (10) is normalized in the usual sense.

Theorem 2 ([10]). *The equivalence group $G_0^{\sim}$ of class (10) is formed by the transformations*

$$\tilde{t} = \frac{at+b}{ct+d}, \quad \tilde{t} = \frac{e_2x+e_1t+e_0}{ct+d},$$

$$\tilde{u} = \frac{e_2(ct+d)u - e_2cx - e_0c + e_1d}{\varepsilon}, \quad \tilde{g} = \frac{{e_2}^3}{ct+d}\frac{g}{\varepsilon},$$

where a, b, c, d, e_0, e_1 and e_2 are arbitrary constants with $\varepsilon = ad - bc \neq 0$ and $e_2 \neq 0$, the tuple (a,b,c,d,e_0,e_1,e_2) is defined up to nonzero multiplier and hence without loss of generality we can assume that $\varepsilon = \pm 1$.

3 Lie Symmetries

The group classification of class (10) up to $G_0^{\sim}$-equivalence is carried out in [10] in the framework of classical approach [5,6]. The result reads as follows.

The kernel of the maximal Lie invariance algebras of equations from class (10) coincides with the one-dimensional algebra $\langle \partial_x \rangle$. All possible $G_0^{\sim}$-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by the cases 1–4 of Table 1.

For any equation from class (1) there exists an imaged equation in class (10) with respect to transformation (9). The equivalence group $G_0^{\sim}$ of class (10) is induced by the equivalence group $\hat{G}_1^{\sim}$ of class (1) which, in turn, is induced by the equivalence group $G^{\sim}$ of class (2). These guarantee that Table 1 presents also the group classification list for class (1) up to $\hat{G}_1^{\sim}$-equivalence (resp. for the class (2) up to $G^{\sim}$-equivalence). As all of the above classes are normalized, we can state that we

Table 1 The group classification of the class $u_t + uu_x + g\,u_{xxx} = 0$, $g \neq 0$

N	$g(t)$	Basis of $A^{\max}$
0	$\forall$	∂_x
1	t^n	$\partial_x,\quad t\partial_x + \partial_u,\quad 3t\partial_t + (n+1)x\partial_x + (n-2)u\partial_u$
2	e^t	$\partial_x,\quad t\partial_x + \partial_u,\quad 3\partial_t + x\partial_x + u\partial_u$
3	$e^{\delta \arctan t}\sqrt{t^2+1}$	$\partial_x,\quad t\partial_x + \partial_u,\quad 3(t^2+1)\partial_t + (3t+\delta)x\partial_x + ((-3t+\delta)u + 3x)\partial_u$
4	1	$\partial_x,\quad t\partial_x + \partial_u,\quad 3t\partial_t + x\partial_x - 2u\partial_u,\quad \partial_t$

Here n, δ are arbitrary constants, $n \geq 1/2$, $n \neq 1$, $\delta \geq 0 \bmod G_0^{\sim}$.

obtain Lie symmetry classifications of these classes up to general point equivalence. This leads to the following assertion.

Corollary 2. *An equation from class (1) (resp. (2)) admits a four-dimensional Lie invariance algebra if and only if it is reduced by a point transformation to constant coefficient KdV equation, i.e., if and only if condition (8) (resp. (7)) holds.*

To derive the group classification of class (1) which is not simplified by equivalence transformations, we first apply transformations from the group $G_0^{\sim}$ to the classification list presented in Table 1 and obtain the following extended list:

0. arbitrary $\hat g$: $\langle\partial_{\hat x}\rangle$;
1. $\hat g = c_0(a\hat t+b)^n(c\hat t+d)^{1-n}$, $n\neq 0,1$: $\langle\partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ X_3\rangle$, where

$$X_3 = 3(a\hat t+b)(c\hat t+d)\partial_{\hat t}+(3ac\hat t+ad(n+1)+bc(2-n))\hat x\partial_{\hat x}$$
$$+\left[3ac\hat x-(3ac\hat t+ad(2-n)+bc(n+1))\hat u\right]\partial_{\hat u};$$

2. $\hat g = c_0(c\hat t+d)\exp\left(\dfrac{a\hat t+b}{c\hat t+d}\right)$: $\langle\partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ X_3\rangle$, where

$$X_3 = 3(c\hat t+d)^2\partial_{\hat t}+(3c(c\hat t+d)+\varepsilon)\hat x\partial_{\hat x}+\left[3c^2\hat x+(\varepsilon-3c(c\hat t+d))\hat u\right]\partial_{\hat u};$$

3. $\hat g = c_0 e^{\delta\arctan\left(\frac{a\hat t+b}{c\hat t+d}\right)}\sqrt{(a\hat t+b)^2+(c\hat t+d)^2}$: $\langle\partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ X_3\rangle$, where

$$X_3 = 3\left((a\hat t+b)^2+(c\hat t+d)^2\right)\partial_{\hat t}+(3a(a\hat t+b)+3c(c\hat t+d)+\varepsilon\delta)\hat x\partial_{\hat x}$$
$$+\left(3(a^2+c^2)\hat x-(3a(a\hat t+b)+3c(c\hat t+d)-\varepsilon\delta)\hat u\right)\partial_{\hat u};$$

4a. $\hat g = c_0$: $\langle\partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ \partial_{\hat t},\ 3\hat t\partial_{\hat t}+\hat x\partial_{\hat x}-2\hat u\partial_{\hat u}\rangle$;
4b. $\hat g = c\hat t+d$, $c\neq 0$: $\langle\partial_{\hat x},\ \hat t\partial_{\hat x}+\partial_{\hat u},\ 3(c\hat t+d)\partial_{\hat t}+2c\hat x\partial_{\hat x}-c\hat u\partial_{\hat u},\ X_4\rangle$, where

$$X_4 = (c\hat t+d)^2\partial_{\hat t}+c(c\hat t+d)\hat x\partial_{\hat x}+c(c\hat x-(c\hat t+d)\hat u)\partial_{\hat u}.$$

Here c_0, a, b, c, d and δ are arbitrary constants, $(a^2+b^2)(c^2+d^2)\neq 0$, $\varepsilon = ad-bc$, $c_0\neq 0$.

Then we find preimages of equations from the class $\hat u_{\hat t}+\hat u\hat u_{\hat x}+\hat g(\hat t)\hat u_{\hat x\hat x\hat x}=0$ with arbitrary elements collected in the above list with respect to transformation (9). The last step is to transform basis operators of the corresponding Lie symmetry algebras. The results are presented in Table 2.

It is easy to see that Table 2 includes all cases presented in [3] as particular cases.

Table 2 The group classification of the class $u_t + uu_x + gu_{xxx} + hu = 0$, $g \neq 0$

N	$h(t)$	$g(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_x
1	$\forall$	$c_0 T_t (aT+b)^n (cT+d)^{1-n}$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 3T_t^{-1}(aT+b)(cT+d)\partial_t + \big[3acT + ad(n+1) + bc(2-n)\big]x\partial_x + \Big(3acxT_t - \big[3acT + 3hT_t^{-1}(aT+b)(cT+d) + ad(n+1) + bc(2-n)\big]u\Big)\partial_u$
2	$\forall$	$c_0 T_t (cT+d)\exp\left(\frac{aT+b}{cT+d}\right)$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 3T_t^{-1}(cT+d)^2\partial_t + (3c(cT+d)+\varepsilon)x\partial_x + \left[3c^2xT_t + \left(\varepsilon - 3(c(cT+d) + hT_t^{-1}(cT+d)^2\right)u\right]\partial_u$
3	$\forall$	$c_0 T_t e^{\delta \arctan\left(\frac{aT+b}{cT+d}\right)} G(t)$	$\partial_x,\ T\partial_x + T_t\partial_u,\ 3T_t^{-1}G^2\partial_t + \big[3a(aT+b) + 3c(cT+d) + \varepsilon\delta\big]x\partial_x + \big[3(a^2+c^2)xT_t - \left(3a(aT+b) + 3c(cT+d) - \varepsilon\delta + 3hT_t^{-1}G^2\right)u\big]\partial_u$
4a	$\forall$	$c_0 T_t$	$\partial_x,\ T\partial_x + T_t\partial_u,\ T_t^{-1}(\partial_t - hu\partial_u),\ 3TT_t^{-1}\partial_t + x\partial_x - (2 + 3TT_t^{-1}h)u\partial_u$
4b	$\forall$	$(cT+d)T_t$	$\partial_x,\ T\partial_x + T_t\partial_u,\ T_t^{-1}(cT+d)^2\partial_t + c(cT+d)x\partial_x + [c^2xT_t - (cT+d)(c + T_t^{-1}(cT+d)h)u]\partial_u,\ 3T_t^{-1}(cT+d)\partial_t + 2cx\partial_x - (c + 3T_t^{-1}(cT+d)h)u\partial_u$

Here $T = \int e^{-\int h(t)\,dt}dt$, $T_t = e^{-\int h(t)\,dt}$, $G = \sqrt{(aT+b)^2 + (cT+d)^2}$; n c_0, a, b, c, d and δ are arbitrary constants, $(a^2+b^2)(c^2+d^2) \neq 0$, $\varepsilon = ad - bc$, $c_0 \neq 0$, $n \neq 0, 1$. In the case (4b) $c \neq 0$.

4 Generation of Exact Solutions

A number of recent papers concern the construction of exact solutions to different classes of KdV- or mKdV-like equations using, e.g., such methods as "generalized (G'/G)-expansion method", "Exp-function method", "Jacobi elliptic function expansion method", etc. A number of references are presented in [10]. Almost in all cases exact solutions were constructed only for equations which are reducible to the standard KdV or mKdV equations by point transformations and usually these were only solutions similar to the well-known one-soliton solution. In this section we show that the usage of equivalence transformations allows one to obtain more results in a simpler way. This approach is used also in [11].

The N-soliton solution of the KdV equation in the canonical form

$$U_t - 6UU_x + U_{xxx} = 0 \tag{11}$$

was constructed as early as in the seventies by Hirota [7]. The two-soliton solution of (11) has the form

$$U = -2\frac{\partial^2}{\partial x^2}\ln\left(1 + b_1 e^{\theta_1} + b_2 e^{\theta_2} + Ab_1b_2e^{\theta_1+\theta_2}\right), \tag{12}$$

where a_i, b_i are arbitrary constants, $\theta_i = a_i x - a_i^3 t$, $i = 1,2$; $A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2$.

Combining the simple transformation $\hat{u} = -6U$ that connects (11) with the KdV equation of the form

$$\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} + \hat{u}_{\hat{x}\hat{x}\hat{x}} = 0 \qquad (13)$$

and transformation (9), we obtain the formula

$$u = -6e^{-\int h(t)dt}\, U\left(\int e^{-\int h(t)\,dt}dt, x\right)$$

for generation of exact solutions for the equations of the general form

$$u_t + uu_x + e^{-\int h(t)\,dt}u_{xxx} + h(t)u = 0. \qquad (14)$$

These equations are preimages of (13) with respect to transformation (9). Here h is an arbitrary nonvanishing smooth function of the variable t.

The two-soliton solution (12) leads to the following solution of (14)

$$u = 12e^{-\int h(t)dt}\frac{\partial^2}{\partial x^2}\ln\left(1 + b_1 e^{\theta_1} + b_2 e^{\theta_2} + Ab_1 b_2 e^{\theta_1+\theta_2}\right), \qquad (15)$$

where a_i, b_i are arbitrary constants, $\theta_i = a_i x - a_i^3 \int e^{-\int h(t)\,dt}dt$, $i = 1,2$; $A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2$. In a similar way one can construct N-soliton, rational and other types of solutions for equations from class (14) using known solutions of classical KdV equation.

5 Conclusion

In this paper group classification problem for class (1) is carried out with respect to the corresponding equivalence group using equivalence based approach. Using the normalization property it is proved that this classification coincides with the one carried out up to general point equivalence. The classification list extended by equivalence transformations is also presented. Such list is convenient for further applications.

It is shown that the usage of equivalence groups is a crucial point for exhaustive solution of the problem. Moreover, equivalence transformations allow one to construct exact solutions of different types in a much easier way than by direct solving. These transformations can also be utilized to obtain conservation laws, Lax pairs and other related objects for equations reducible to well-known equations of mathematical physics by point transformations without direct calculations.

Acknowledgements The author thanks the Organizing Committee and especially Prof. Vladimir Dobrev for hospitality and giving an opportunity to give a talk. Her participation in the Workshop was supported by the Abdus Salam International Centre for Theoretical Physics. The author is also grateful to Prof. Roman Popovych for useful discussions and valuable comments.

References

1. Güngör, F., Lahno, V.I., Zhdanov, R.Z.: J. Math. Phys. **45**, 2280–2313 (2004)
2. Ivanova, N.M., Popovych, R.O., Sophocleous, C.: Lobachevskii J. Math. **31**, 100–122 (2010)
3. Johnpillai, A.G., Khalique, C.M.: Appl. Math. Comput. **216**, 3761–3771 (2010)
4. Magadeev, B.A.: Algebra i Analiz **5**, 141–156 (1993) (in Russian); translation in St. Petersburg Math. J. **5**, 345–359 (1994)
5. Olver, P.: Applications of Lie Groups to Differential Equations. Springer, New York (1986)
6. Ovsiannikov, L.V.: Group Analysis of Differential Equations. Academic Press, New York (1982)
7. Polyanin, A.D., Zaitsev, V.F.: Handbook of Nonlinear Partial Differential Equations. Chapman & Hall/CRC Press, Boca Raton (2004)
8. Popovych, R.O.: Collection of Works of Institute of Mathematics, vol. 3(2), pp. 239–254. Institute of Mathematics, Kyiv, Ukraine (2006)
9. Popovych, R.O., Kunzinger, M., Eshraghi, H.: Acta Appl. Math. **109**, 315–359 (2010)
10. Popovych, R.O., Vaneeva, O.O.: Commun. Nonlinear Sci. Numer. Simulat. **15**, 3887–3899 (2010)
11. Tang, X.Y., Zhao, J., Huang, F., Lou, S.Y.: Stud. Appl. Math. **122**, 295–304 (2009)
12. Vaneeva, O.O.: Commun. Nonlinear Sci. Numer. Simulat. **17**, 611–618 (2012)
13. Vaneeva, O.O., Johnpillai, A.G., Popovych, R.O., Sophocleous, C.: J. Math. Anal. Appl. **330**, 1363–1386 (2007)
14. Vaneeva, O.O., Popovych, R.O., Sophocleous, C.: Acta Appl. Math. **106**, 1–46 (2009)

A New Diffeomorphism Symmetry Group of Magnetohydrodynamics

Asher Yahalom

Abstract Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Yahalom (A four function variational principle for Barotropic magnetohydrodynamics, EPL 89, 34005 (2010) has shown that barotropic magnetohydrodynamics is mathematically equivalent to a four function field theory defined a by a Lagrangian for some topologies. The four functions include two surfaces whose intersections consist the magnetic field lines, the part of the velocity field not defined by the comoving magnetic field and the density. This Lagrangian admits a newly discovered group of Diffeomorphism Symmetry. I discuss the symmetry group and derive the related Noether current.

1 Introduction

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Eulerian variational principles for non-magnetic fluid dynamics were first introduced by Davydov [2]. Following the work of Davydov, Zakharov and Kuznetsov [10] suggested an Eulerian variational principle for magnetohydrodynamics. However, the variational principle suggested by Zakharov and Kuznetsov contained *two* more functions than the standard formulation of magnetohydrodynamics with a total sum of *nine* variational variables. Another Eulerian variational principle for magnetohydrodynamics was introduced independently by Calkin [1] in a work that preceded Zakharov and Kuznetsov paper by seven years. However, Calkin's variational principle also depends on as much as eleven variational variables. The situation was somewhat

A. Yahalom
Ariel University Center of Samaria, Ariel, Israel
e-mail: asya@ariel.ac.il

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_33,

improved when Vladimirov and Moffatt [6] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. Their variational principle contained only three more functions in addition to the seven variables which appear in the standard equations of magnetohydrodynamics which are the magnetic field $\mathbf{B}$ the velocity field $\mathbf{v}$ and the density ρ. Kats [3] has generalized Moffatt's work for compressible non barotropic flows but without reducing the number of functions and the computational load. The current paper will discuss only continuous flows due to space limitations, possible extensions of the current formalism to discontinuous flows will be discussed in a future paper. Sakurai [4] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [5]. In a work Yahalom and Lynden-Bell [7, 9] have combined the Lagrangian of Sturrock [5] with the Lagrangian of Sakurai [4] to obtain an Eulerian variational principle depending on only six functions. The vanishing of the variational derivatives of this Lagrangian entail all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints.

The non-singlevaluedness of the functions appearing in the reduced representation of barotropic magnetohydrodynamics was discussed in particular with connection to the topological invariants of magnetic and cross helicities. It was shown that flows with non trivial topologies which have non zero magnetic or cross helicities can be adequately described by the functions of the reduced representation provided that some of them are non-single valued [7, 9]. The cross helicity per unit flux was shown to be equal to the discontinuity of the function ν, this discontinuity was shown to be a conserved quantity along the flow. The magnetic helicity per unit flux was shown to be equal to the discontinuity of another function ζ.

In a more recent work [8] the number of needed functions was further reduced and it was shown that magnetohydrodynamics is mathematically equivalent to a four function field theory defined a by a Lagrangian.

In the current paper I show that the four function Lagrangian [8] admits a newly discovered group of diffeomorphism symmetry. The relevant Noether current and conservation laws of the newly discovered group of diffeomorphism symmetry are discussed.

The plan of this paper is as follows. First I introduce the standard notations and equations of barotropic magnetohydrodynamics. Next I introduce the potential representation of the magnetic field $\mathbf{B}$ and the velocity field $\mathbf{v}$. This is followed by a review of the Eulerian variational principle developed by Yahalom and Lynden-Bell [7, 9]. After those introductory sections I will present the four function Eulerian variational principles for non-stationary magnetohydrodynamics [8]. Finally I will discuss the newly discovered group of diffeomorphism symmetry and the relevant Noether current and conservation laws associated with the group of diffeomorphism symmetry.

2 The Standard Formulation of Barotropic Magnetohydrodynamics

The standard set of equations solved for barotropic magnetohydrodynamics are given below:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = -\nabla p(\rho) + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}. \tag{4}$$

The following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and ∇ has its standard meaning in vector calculus. $\mathbf{B}$ is the magnetic field vector, $\mathbf{v}$ is the velocity field vector and ρ is the fluid density. Finally $p(\rho)$ is the pressure which we assume depends on the density alone (barotropic case). The justification for those equations and the conditions under which they apply can be found in standard books on magnetohydrodynamics (see for example [5]). Equation (1) describes the fact that the magnetic field lines are moving with the fluid elements ("frozen" magnetic field lines), (2) describes the fact that the magnetic field is solenoidal, (3) describes the conservation of mass and equation (4) is the vector Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term:

$$\mathbf{J} = \frac{\nabla \times \mathbf{B}}{4\pi}, \tag{5}$$

is the electric current density which is not connected to any mass flow. The number of independent variables for which one needs to solve is seven $(\mathbf{v}, \mathbf{B}, \rho)$ and the number of equations (1), (3), (4) is also seven. Notice that (2) is a condition on the initial $\mathbf{B}$ field and is satisfied automatically for any other time due to (1). Also notice that $p(\rho)$ is not a variable rather it is a given function of ρ.

3 Potential Representation of Vector Quantities of Magnetohydrodynamics

It was shown in [9] that $\mathbf{B}$ and $\mathbf{v}$ can be represented in terms of five scalar functions $\alpha, \beta, \chi, \eta, \nu$. Following Sakurai [4] the magnetic field takes the form:

$$\mathbf{B} = \nabla \chi \times \nabla \eta. \tag{6}$$

Hence **B** satisfies automatically (2) for co-moving χ and η surfaces and is orthogonal to both $\nabla\chi$ and $\nabla\eta$. The above expression can also describe a magnetic field with non-zero magnetic helicity as was demonstrated in [9]. Moreover, the velocity **v** can be represented in the following form:

$$\mathbf{v} = \nabla\nu + \alpha\nabla\chi + \beta\nabla\eta. \tag{7}$$

this is a generalization of the Clebsch representation for magnetohydrodynamics.

4 The Action of Barotropic Magnetohydrodynamics

It was shown in [9] (Eq. (4.15) of [9], notice also the change in notation $\mathcal{L}$ here instead of $\hat{\mathcal{L}}$ used previously) that the action of barotropic magnetohydrodynamics takes the form:

$$\begin{aligned} A &\equiv \int \mathcal{L} d^3x dt, \\ \mathcal{L} &\equiv -\rho\left[\frac{\partial\nu}{\partial t} + \alpha\frac{\partial\chi}{\partial t} + \beta\frac{\partial\eta}{\partial t} + \varepsilon(\rho) + \frac{1}{2}(\nabla\nu + \alpha\nabla\chi + \beta\nabla\eta)^2\right] \\ &\quad -\frac{1}{8\pi}(\nabla\chi\times\nabla\eta)^2, \end{aligned} \tag{8}$$

in which $\varepsilon(\rho)$ is the specific internal energy. Taking the variational derivatives to zero for arbitrary variations leads to the following set of six equations:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{9}$$

$$\frac{d\chi}{dt} = 0, \tag{10}$$

$$\frac{d\eta}{dt} = 0, \tag{11}$$

$$\frac{d\nu}{dt} = \frac{1}{2}\mathbf{v}^2 - w, \tag{12}$$

in which w is the specific enthalpy.

$$\frac{d\alpha}{dt} = \frac{\nabla\eta\cdot\mathbf{J}}{\rho}, \tag{13}$$

$$\frac{d\beta}{dt} = -\frac{\nabla\chi\cdot\mathbf{J}}{\rho}. \tag{14}$$

In all the above equations **B** is given by (6) and **v** is given by (7). The mass conservation equation (3) is readily obtained. Now one needs to show that also (1) and (4) are satisfied.

It can be easily shown that provided that **B** is in the form given in (6), and (10), (11) are satisfied, then (1) are satisfied.

It was shown in [9] that a velocity field given by (7), such that the equations for $\alpha, \beta, \chi, \eta, \nu$ satisfy the corresponding equations (9), (10), (11), (12), (13), (14) must satisfy Euler's equations. This proves that the barotropic Euler equations can be derived from the action given in (8) and hence all the equations of barotropic magnetohydrodynamics can be derived from the above action without restricting the variations in any way except on the relevant boundaries and cuts.

5 A Simpler Action for Barotropic Magnetohydrodynamics

Can we obtain a further reduction of barotropic magnetohydrodynamics? Can we formulate magnetohydrodynamics with less than the six functions $\alpha, \beta, \chi, \eta, \nu, \rho$? The answer is yes, in fact four functions χ, η, ν, ρ will suffice. To see this we may write the two equations (10), (11) as equations for α, β that is:

$$\frac{d\chi}{dt} = \frac{\partial \chi}{\partial t} + \mathbf{v} \cdot \nabla \chi = \frac{\partial \chi}{\partial t} + (\nabla \nu + \alpha \nabla \chi + \beta \nabla \eta) \cdot \nabla \chi = 0,$$
$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla \eta = \frac{\partial \eta}{\partial t} + (\nabla \nu + \alpha \nabla \chi + \beta \nabla \eta) \cdot \nabla \eta = 0, \tag{15}$$

in which we have used (7). Solving for α, β we obtain:

$$\alpha[\chi, \eta, \nu] = \frac{(\nabla \eta)^2 (\frac{\partial \chi}{\partial t} + \nabla \nu \cdot \nabla \chi) - (\nabla \eta \cdot \nabla \chi)(\frac{\partial \eta}{\partial t} + \nabla \nu \cdot \nabla \eta)}{(\nabla \eta \cdot \nabla \chi)^2 - (\nabla \eta)^2 (\nabla \chi)^2}$$
$$\beta[\chi, \eta, \nu] = \frac{(\nabla \chi)^2 (\frac{\partial \eta}{\partial t} + \nabla \nu \cdot \nabla \eta) - (\nabla \eta \cdot \nabla \chi)(\frac{\partial \chi}{\partial t} + \nabla \nu \cdot \nabla \chi)}{(\nabla \eta \cdot \nabla \chi)^2 - (\nabla \eta)^2 (\nabla \chi)^2}. \tag{16}$$

Hence α and β are not free variables any more, but depend on χ, η, ν. Moreover, the velocity **v** now depends on the same three variables χ, η, ν:

$$\mathbf{v} = \nabla \nu + \alpha[\chi, \eta, \nu] \nabla \chi + \beta[\chi, \eta, \nu] \nabla \eta. \tag{17}$$

Since **v** is given now by (17) it follows that the two equations (10), (11) are satisfied identically and need not be derived from a variational principle. The above equation can be somewhat simplified resulting in:

$$\mathbf{v} = \nabla \nu + \frac{1}{\mathbf{B}^2} \left[\frac{\partial \eta}{\partial t} \nabla \chi - \frac{\partial \chi}{\partial t} \nabla \eta + \nabla \nu \times \mathbf{B} \right] \times \mathbf{B}$$
$$= \frac{1}{\mathbf{B}^2} \left[\left(\frac{\partial \eta}{\partial t} \nabla \chi - \frac{\partial \chi}{\partial t} \nabla \eta \right) \times \mathbf{B} + \mathbf{B} (\nabla \nu \cdot \mathbf{B}) \right] \equiv \mathbf{v}_\perp + \mathbf{v}_\parallel \tag{18}$$

Hence the velocity $\mathbf{v}$ is partitioned naturally into two components one which is parallel to the magnetic field and another one which is perpendicular to it. Inserting the velocity representation (18) into (16) will lead to the result:

$$\alpha = \frac{\nabla\eta \cdot (\mathbf{B} \times (\mathbf{v} - \nabla\nu))}{\mathbf{B}^2}$$
$$\beta = -\frac{\nabla\chi \cdot (\mathbf{B} \times (\mathbf{v} - \nabla\nu))}{\mathbf{B}^2}. \tag{19}$$

The reader should notice that the above quantities become singular for $\mathbf{B} = 0$, hence the formalism is only adequate for describing flows for which $\mathbf{B} \neq 0$. If the magnetic field vanishes at infinity this is not an obstacle from the present formalism point of view since the velocity field need not be defined at infinity where there is no flow. For flow without magnetic fields the present formalism is not appropriate and other variational economic formalisms may be suggested. Finally (16) should be substituted into (8) to obtain a Lagrangian density $\mathcal{L}$ in terms of χ, η, ν, ρ:

$$\mathcal{L}[\chi,\eta,\nu,\rho] = \rho\left[\frac{1}{2}\mathbf{v}^2 - \frac{d\nu}{dt} - \varepsilon(\rho)\right] - \frac{\mathbf{B}^2}{8\pi} \tag{20}$$

where $\mathbf{v}$ is given by (18) and $\mathbf{B}$ by (6). Or more explicitly as:

$$\mathcal{L}[\chi,\eta,\nu,\rho] = \frac{1}{2}\frac{\rho}{(\nabla\chi\times\nabla\eta)^2}\left[\nabla\eta\frac{\partial\chi}{\partial t} - \nabla\chi\frac{\partial\eta}{\partial t} + (\nabla\chi\times\nabla\eta)\times\nabla\nu\right]^2$$
$$-\rho\left[\frac{\partial\nu}{\partial t} + \frac{1}{2}(\nabla\nu)^2 + \varepsilon(\rho)\right] - \frac{(\nabla\chi\times\nabla\eta)^2}{8\pi}. \tag{21}$$

It is shown in [8] by variational analysis that indeed all the needed equations can be derived from the above Lagrangian.

6 Diffeomorphism Symmetry and Noether Currents

This Lagrangian density admits an infinite symmetry group of transformations of the form:

$$\hat{\eta} = \hat{\eta}(\chi,\eta), \quad \hat{\chi} = \hat{\chi}(\chi,\eta), \tag{22}$$

provided that the absolute value of the Jacobian of these transformation is unity:

$$\left|\frac{\partial(\hat{\eta},\hat{\chi})}{\partial(\eta,\chi)}\right| = 1. \tag{23}$$

In particular the Lagrangian density admits an exchange symmetry:

$$\hat{\eta} = \chi, \quad \hat{\chi} = \eta. \tag{24}$$

Consider the following transformations:

$$\hat{\eta} = \eta + \delta\eta(\chi,\eta), \qquad \hat{\chi} = \chi + \delta\chi(\chi,\eta), \tag{25}$$

in which $\delta\eta, \delta\chi$ are considered small in some sense. Inserting the above quantities into (23) and keeping only first order terms we arrive at:

$$\partial_\eta \delta\eta + \partial_\chi \delta\chi = 0. \tag{26}$$

This equation can be solved as follows:

$$\delta\eta = \partial_\chi \delta f, \quad \delta\chi = -\partial_\eta \delta f, \tag{27}$$

in which $\delta f = \delta f(\chi,\eta)$ is an arbitrary small function.

The variational derivative of the Lagrangian $L = \int d^3x\mathcal{L}$ in which $\mathcal{L}$ is given in (21) with respect to χ and η was calculated in [8] (31) and (32) (in action form). Assuming that the relevant equations of motion and boundary conditions hold, we have:

$$\delta_\chi L = -\frac{d\int d^3x\rho\alpha\delta\chi}{dt}, \qquad \delta_\eta L = -\frac{d\int d^3x\rho\beta\delta\eta}{dt} \tag{28}$$

The sum of the above two expressions is:

$$\delta_\chi L + \delta_\eta L = -\frac{d}{dt}\int d^3x\rho(\alpha\delta\chi + \beta\delta\eta), \tag{29}$$

If we choose variations $\delta\chi, \delta\eta$ such that (27) hold, then the variations are symmetry variations and the total variation $\delta_\chi L + \delta_\eta L$ vanishes. For this case we have:

$$\frac{d}{dt}\int d^3x\rho(\alpha\partial_\eta\delta f - \beta\partial_\chi\delta f) = 0. \tag{30}$$

Hence the quantity $\delta G = \int d^3x\rho(\alpha\partial_\eta\delta f - \beta\partial_\chi\delta f)$ is conserved. Using the comoving magnetic metage μ defined in [9] (4.23), (6.25), this can be written as:

$$\delta G = \int d\chi d\eta d\mu(\alpha\partial_\eta\delta f - \beta\partial_\chi\delta f) = \int d\chi d\mu\alpha\delta f\Big|_{\eta 1}^{\eta 2} - \int d\eta d\mu\beta\delta f\Big|_{\chi 1}^{\chi 2}$$
$$+\int d\chi d\eta d\mu\delta f(\partial_\chi\beta - \partial_\eta\alpha). \tag{31}$$

Since δf is arbitrary, we arrive at the following conserved current.

$$J = \int d\mu(\partial_\chi\beta - \partial_\eta\alpha). \tag{32}$$

7 Conclusion

We have discussed the significance of the magnetohydrodynamics diffeomorphism symmetry group and in particular have shown the existence of the related conserved Noether current. Future research will be oriented towards understanding the physical consequences of this new conservation law.

References

1. Calkin, M.: An action prinicple for magnetohydrodynamics. Can. J. Phys. **41**, 2241–2251 (1963)
2. Davydov, B.: Doklady Akad. Nauk **69**, 165–168 (1949) (in Russian)
3. Kats, A.V.: Variational principle in canonical variables, Weber transformation, and complete set of the local integrals of motion for dissipation-free magnetohydrodynamics. (Los Alamos Archives physics-0212023) *JETP Lett.* **77**, 657 (2003)
4. Sakurai, T.: A new approach to the force-free field and its application to the magnetic field of solar active regions. Pub. Ast. Soc. Jpn. **31**, 209 (1979)
5. Sturrock, P.A.: Plasma Physics. Cambridge University Press, Cambridge (1994)
6. Vladimirov, V.A., Moffatt, H.K.: On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part 1. fundamental principles. J. Fluid. Mech. **283**, 125–139 (1995)
7. Yahalom, A.: Using Magnetic Fields to Optimize Material Flow - a Variational Approach to Magnetohydrodynamics. In: Proceedings of the Israeli - Russian Bi-National Workshop, Jerusalem, Israel, June 2007
8. Yahalom, A.: A four function variational principle for barotropic magnetohydrodynamics. EPL **89**, 34005 (2010). doi: 10.1209/0295-5075/89/34005. [Los-Alamos Archives - arXiv:0811.2309]
9. Yahalom, A., Lynden-Bell, D.: Simplified variational principles for barotropic magnetohydrodynamics. (Los-Alamos Archives - physics/0603128). J. Fluid Mech. **607**, 235–265 (2008)
10. Zakharov, V.E., Kuznetsov, E.A.: Variational Principle and Canonical Variables in Magnetohydrodynamics. Doklady Akad. Nauk **194**(6), 1288–1289 (1970) (in Russian). The paper was translated to English: Zakharov, V.E., Kuznetsov, E.A.: Variational principle and canonical variables in magnetohydrodynamics. Sov. Phys. Dokl. **15**(10), 913–914 (1971)

Invariance Properties of the Exceptional Quantum Mechanics (F_4) and Its Generalization to Complex Jordan Algebras (E_6)

Sultan Catto, Yoon S. Choun, and Levent Kurt

Abstract We consider a case in which the octonionic observables form a Jordan Algebra. Then the automorphism group turns out to be an exceptional group F_4 or E_6 and we are led to a gauge field theory of quarks and leptons based on exceptional groups. Some relations of octonion and split octonion algebras and their relation to algebra of quarks are explicitly shown.

1 Introduction

The symmetries of the hadronic spectrum and the hadronic decays have uncovered a colored quark substructure. Weak and electromagnetic interactions showed us that quarks behave like leptons and a local field theory of both leptons and quarks makes sense. Strong interactions are well described by a local gauge theory based on the exact color group while weak and electromagnetic interactions are unified within a gauge field theory of the spontaneously broken local flavor group. The symmetries between leptons and quarks led us to the notion that these fundamental fermions belong to the same multiplet of a unifying group. The successful candidates for such a unifying group have turned out to be subgroups of the exceptional group E_6. On the other hand, the only non-trivial generalizations of the Hilbert space ofQuantum Mechanics and the algebra of observables involve algebraicand

S. Catto (✉)
Physics Department, The Graduate School and University Center, The City University of New York, New York, NY 10010, USA

The Rockefeller University, 1230 York Avenue, New York, NY 10021-6399, USA

Y.S. Choun • L. Kurt
Physics Department, The Graduate School and University Center, The City University of New York, New York, NY 10010, USA

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_34,

geometrical structures connected with the exceptional groups F_4 and E_6. These unique and intrinsically finite structures exhibit an exact color symmetry originating in octonions [1] that go far in the building of these exotic structures.

2 The Setting

In the case of the usual Quantum Mechanics transition amplitudes are invariant under unitary transformations up to a phase, as

$$< \alpha' \mid \beta' >=< \alpha \mid \beta > \tag{1}$$

when

$$\mid \alpha' >= U \mid \alpha >, \quad \mid \beta' >= U \mid \beta >, \quad UU^\dagger = 1 \tag{2}$$

Then the projection operators transform as

$$P'_\alpha = UP_\alpha U^\dagger, \quad P'_\beta = UP_\beta U^\dagger. \tag{3}$$

The observables Ω that are linear combinations of projection operators also transform in the same way

$$\Omega' = U\Omega U^\dagger \tag{4}$$

so that the Jordan product Ω of two observables Ω_1 and Ω_2 also transforms like a projection operator, since

$$\begin{aligned} \Omega' &= \Omega'_1 \cdot \Omega'_2 = \tfrac{1}{2}(\Omega'_1\Omega'_2 + \Omega'_2\Omega'_1) \\ &= \tfrac{1}{2}U(\Omega_1\Omega_2 + \Omega_2\Omega_1)U^\dagger = U\Omega U^\dagger \end{aligned} \tag{5}$$

It follows that, in a n-dimensional Hilbert space, with $n \times n$ hermitian matrices associated with observables and projection operators for states, the automorphism group of the Jordan algebra of observables is $U(n)$ or $SU(n)$. In order to find the invariance group of octonionic Quantum Mechanics we must find the automorphism group of the exceptional Jordan algebra. This was shown to be the group F_4 by Chevalley and Schafer [2] more than a decade after the discovery of exceptional Jordan algebras.

3 Exceptional Quantum Mechanics (F_4)

The infinitesimal action on F_4 on an element J of the Jordan algebra can be written simply.

If H_1 and H_2 are traceless octonionic hermitian matrices we have [3]

$$\delta J = [H_1, J, H_2] \tag{6}$$

as the associator bracket.

The transformation property of the projection operators P_α for states α is obtained by putting $J = P_\alpha$. Let us show that this gives the unitary group in the associative case. Let

$$iH = -\frac{1}{4}[H_1, H_2], \quad H = H^\dagger \tag{7}$$

Then, using $P^2 = P$ we can write

$$\delta J = i[H, J] \tag{8}$$

Exponentiation gives

$$J' = J + [iH, J] + \tfrac{1}{2!}[iH, [iH, J]] + \ldots. = e^{iH}\, J\, e^{-iH} \tag{9}$$

which shows that J is transformed by a unitary matrix. In the octonionic case the finite transformation of F_4 is given by the series

$$J' = J + [H_1, J, H_2] + \frac{1}{2!}[H_1, [H_1, J, H_2], H_2] + \ldots. \tag{10}$$

which only involves the Jordan product and can no longer be written in the form (9). When H_1 involves only one octonion and H_2 is a purely scalar matrix then (10) can be integrated in the form (9) with iH replaced by an antihermitian octonionic matrix involving one octonion only. It is seen that the full group is determined by the traceless hermitian matrices H_1 and H_2 so that it has 52 parameters. The invariants under the F_4 transformation are

$$I_1 = \mathrm{Tr}J, \tag{11}$$

$$I_2 = \mathrm{Tr}J^2, \tag{12}$$

$$I_3 = \mathrm{Det}J = \frac{1}{3}\mathrm{Tr}(J \cdot J \times J) \tag{13}$$

An irreducible representation of F_4 is obtained by taking $I_1 = 0$. It corresponds to traceless Jordan matrices. We have seen that octonionic Quantum Mechanics based on real octonions provides us automatically with a finite Hilbert space with F_4 symmetry. Since F_4 has $SU(3) \times SU(3)^c$ as a maximal subgroup we have a fundamental justification for the color degree of freedom and the SU(3) flavor [4]. Under this group we have the decomposition

$$26 = (8,1) + (3,3) + (\bar{3}, \bar{3}) \tag{14}$$

The color singlet part which is a SU(3) flavor octet lies in an ordinary quantum mechanical space with SU(3) symmetry involving one of the octonionic imaginary units while the colored degrees of freedom involve the remaining six imaginary units.

The behavior of various states under the color group is best seen if we use split octonion units defined by

$$u_0 = \frac{1}{2}(1 + ie_7), \quad u_0^* = \frac{1}{2}(1 - ie_7) \tag{15}$$

$$u_j = \frac{1}{2}(e_j + ie_{j+3}), \quad u_j^* = \frac{1}{2}(e_j - ie_{j+3}), \quad (j = 1,2,3) \tag{16}$$

The automorphism group of the octonion algebra is the 14 parameter exceptional group G_2. The imaginary octonion units e_α $(\alpha = 1,\ldots,7)$ fall into its seven-dimensional representation.

Under the $SU(3)^c$ subgroup of G_2 that leaves e_7 invariant u_0 and u_0^* are singlets while u_j and u_j^* correspond respectively to the representations (3) and $(\bar{3})$.

The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner:

$$u_0^2 = u_0, \quad u_0 u_0^* = 0 \tag{17}$$

$$u_0 u_j = u_j u_0^* = u_j, \quad u_0^* u_j = u_j u_0 = 0 \tag{18}$$

$$u_i u_j = -u_j u_i = \varepsilon_{ijk} u_k^* \tag{19}$$

$$u_i u_j^* = -u_0 \delta_{ij} \tag{20}$$

together with the complex conjugate equations. Here one sees the virtue of octonion multiplication. If we consider the direct products

$$3 \times 3 = \bar{3} + 6, \tag{21}$$

$$3 \times \bar{3} = 1 + 8 \tag{22}$$

for $SU(3)^c$, then these equations show that octonion multiplication gets rid of (6) in 3×3, while it gets rid of (8) in $3 \times \bar{3}$. Combining (19) and (20) we find

$$(u_i u_j) u_k = -\varepsilon_{ijk} u_0^* \tag{23}$$

Thus, the octonion product leaves only the color singlet part in $3 \times \bar{3}$ and $3 \times 3 \times 3$, so that it is a natural algebra for colored quarks.

4 Complex Jordan Algebras (E_6)

We can now consider the general element F of the Jordan algebra with complex components. It can be decomposed as follows

$$F = u_0 L + u_0^* L^T + u_j^* Q_j + u_j R_j^* \tag{24}$$

Here L, Q_j, R_j are 3×3 complex matrices, T denotes transposition and Q_i and R_i are antisymmetric so that

$$Q_j = -Q_j^T, \quad R_j = -R_j^T \tag{25}$$

If we associate L with the ($\bar{3}$,3) representation of a group $SU(3)\times SU(3)$, Q_j with (3,1) and R_j^* with (1,$\bar{3}$), then, together with the color index j we find that F has the $SU(3)\times SU(3)\times SU(3)^c$ decomposition

$$F = (\bar{3},3,1^c) + (3,1,3^c) + (1,\bar{3},\bar{3}^c) \tag{26}$$

F is the 27-dimensional representation of the exceptional group E_6. The color singlet part L can be associated with the lepton matrix L, that is, in terms of the $SU(3)\times SU(3)$ flavor group the leptons fall in a (3×3) matrix that represents the $(\bar{3},3)$:

$$(\bar{3},3) : L^{(e)} = \begin{pmatrix} \hat{N}_R & \hat{\tau}_R & \hat{e}_R \\ \hat{\tau}_L & \hat{\nu}_L & \beta_L^e \\ e_L & \nu_L^e & \alpha_L^e \end{pmatrix} \tag{27}$$

Meanwhile, Q_j and R_j^* respectively are associated with left-handed quarks and right-handed antiquarks: the (3,1) quarks and the $(1,\bar{3})$ antiquarks are

$$(3,1) : \begin{pmatrix} u_L^i \\ d_L^i \\ b_L^i \end{pmatrix}, (1,\bar{3}) : \begin{pmatrix} \hat{u}_R^i \\ \hat{d}_R^i \\ \hat{b}_R^i \end{pmatrix} \tag{28}$$

It follows that lepton and colored quark fields can be combined in a complex Jordan matrix of the form (24) which is hermitian with respect to octonionic conjugation only, so that

$$\bar{F}^T = u_0^* L^T + u_0 L - u_j^* Q_j^T - u_j R_j^{*T} = F \tag{29}$$

The $\bar{27}$ representation of E_6 corresponds to F^*.

The E_6 transformation of F involves three traceless real octonionic Jordan matrices H_1, H_2, H_3 and we have [5]

$$\delta F = [H_1, F, H_2] + iH_3 \cdot F \tag{30}$$

This shows that F_4 may be treated as a subgroup of E_6 and that E_6 has $3 \times 26 = 78$ real parameters. The Freudenthal product of F_1 and F_2 projects out the $\bar{27}$ representation out of the symmetric product of 27×27, so that we can write

$$F_1 \times F_2 = F_3^*, \quad F_1^* \times F_2^* = F_3 \tag{31}$$

Another E_6 invariant operation is the triple product defined by

$$D = \{ABC\} = (A \cdot B) \cdot C + A \cdot (B \cdot C) - (A \cdot C) \cdot B \tag{32}$$

(see also Murat Günaydin [6], where the use of the quadratic Jordan approach and the Jordan triple product in the formulation of quantum mechanics including the octonionic case is also discussed) so that for ordinary matrices over reals or complex numbers one finds

$$\{ABC\} = \frac{1}{2}(ABC + CBA) = D \tag{33}$$

D is obviously hermitian if A,B and C are hermitian. Special case will be

$$\{ABA\} = ABA = 2(A \cdot B) \cdot A - (A \cdot A) \cdot B \tag{34}$$

Hence

$$Det\{ABA\} = (DetA)^2 DetB \tag{35}$$

which is true for all Jordan algebras including the exceptional cases. With this, we now have

$$\{F_1^* F_2 F_3^*\} = F_4^*, \quad \{F_1 F_2^* F_3\} = F_4 \tag{36}$$

are also invariant if F_1, F_2, F_3, F_4 transform like (27). Finally we can construct the invariant

$$(F_1, F_2) = \mathrm{Tr}(F_1 \cdot F_2^*) \tag{37}$$

It follows that, with three Jordan matrices F_1, F_2, F_3 we can associate the invariant

$$(F_3, F_1 \times F_2) = \mathrm{Tr}(F_3 \cdot F_1 \times F_2) \tag{38}$$

Given one F, Tr F is not E_6 invariant. But we can construct 4 invariant real quantities I_2, I_3, I_3' and I_4 defined by

$$I_2 = (F, F^*), \quad I_3 + iI_3 = (F, F \times F) = 3\mathrm{Det}F \tag{39}$$

$$I_4 = (F \times F, F^* \times F^*) \tag{40}$$

A geometry which generalizes the projective geometry of the Moufang plane can be based on the complex matrices F. It is called the geometry of complex octonionic planes [7]. A generalized point (or state) is defined by S such that

$$S \times S = 0 \tag{41}$$

The distance d_{12} between points S_1 and S_2 or the transition probability Π_{12} is given by

$$\Pi_{12} = \cos^2 d_{12} = (S_1, S_2) = \mathrm{Tr}(S_1 \cdot S_2^*) \tag{42}$$

and is E_6 invariant.

It is possible to associate idempotent projection operators with such states and generalize the quantum mechanical formalism. The geometry is more complicated than the Moufang geometry and all its quantum mechanical implications have not yet been worked out [8, 9]. However, the existence of this E_6 invariant exotic geometry and its close correspondence with the phenomenological symmetries of quarks and leptons provides a strong motivation for the reformulation of the properties of the complex octonionic planes in purely quantum mechanical terms.

5 Summary

If it turns out that F_4 or E_6 describe correctly the internal symmetries of fundamental fields we may seek the origin of these symmetries in the properties of unique finite Hilbert spaces associated with exotic geometries.

Acknowledgements The development of the ideas presented here is due in a large part to stimulating discussions with our colleagues and friends including Vladimir Akulov, Vladimir Dobrev, Francesco Iachello, Ramzi Khuri, Pierre Ramond and late Feza Gürsey. One of us (SC) would like to thank Professor Dobrev for invitation to give a talk Varna on the subject of this paper. This work was supported in part by DOE contracts No. DE-AC-0276-ER 03074 and 03075; NSF Grant No. DMS-8917754.

References

1. Catto, S., Choun, Y.: Acta Polytechnica **51**, 77 (2011)
2. Chevalley, C., Schafer, R.D.: Proc. Natl. Acad. Sci. USA **36**, 137 (1950)
3. Schafer, R.D.: An Introduction to Non-associative Algebras. Academic Press, New York (1966)
4. Günaydin, M., Gürsey, F.: Phys. Rev. **D9**, 3387 (1974)
5. Gürsey, F., Ramond, P., Sikivie, P.: Phys. Lett. **60B**, 177 (1976)
6. Günaydin, M.: The exceptional superspace and the quadratic jordan formulation of quantum mechanics. In: Schwarz, J.H. (ed.) Elementary Particles and the Universe: Essays in Honor of Murray Gell-Mann, pp. 99–119. Cambridge University Press, Cambridge (1991)
7. Faulkner, J.R.: Memoirs of the American Mathematical Society, vol. 104. American Mathematical Society, Providence, RI (1970)
8. Gürsey, F., Marchildon, L.: J. Math. Phys. **19**, 942 (1977)
9. Kac, V.: Comm. Math. Phys. **53**, 31 (1977)

Matrix Superpotentials

Yuri Karadzhov

Abstract We present a collection of matrix valued shape invariant potentials which give rise to new exactly solvable problems of SUSY quantum mechanics. It includes all irreducible matrix superpotentials of the generic form $W = kQ + \frac{1}{k}R + P$ where k is a variable parameter, Q is the unit matrix multiplied by a real valued function of independent variable x, and P, R are hermitian matrices depending on x. In particular we recover the Pron'ko-Stroganov "matrix Coulomb potential" and all known scalar shape invariant potentials of SUSY quantum mechanics. In addition, five new shape invariant potentials are presented.

1 Introduction

Invented by E. Witten [1] as a toy model supersymmetric quantum mechanics (SSQM) became a fundamental field including many interesting external and internal problems. In particular the SSQM presents powerful tools for explicit solution of quantum mechanical problems using the shape invariance approach [2]. Unfortunately, the number of problems satisfying the shape invariance condition is rather restricted. However, such problems include practically all cases when the related Schrödinger equation is exactly solvable and has an explicitly presentable potential. Well known exceptions are exactly solvable Schrödinger equations with Natanzon potentials [3] which are formulated in terms of implicit functions. The list of shape invariant potentials depending on one variable can be found in [4].

An interesting example of QM problem which admits a shape invariant supersymmetric formulation was discovered by Pron'ko and Stroganov [5] who studied

Y. Karadzhov (✉)
Institute of Mathematics, National Academy of Sciences of Ukraine,
3 Tereshchenkivs'ka Street, Kyiv-4, 01601, Ukraine
e-mail: yuri.karadzhov@gmail.com

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_35,

a motion of a neutral non-relativistic fermion which interacts anomalously with the magnetic field generated by a thin current carrying wire.

The supersymmetric approach to the Pron'ko–Stroganov (PS) problem was first applied in paper [6] with using the momentum representation. In paper [7] this problem was solved using its shape invariance in the coordinate representation. Recently a relativistic generalization of the PS problem was proposed [8] which can also be integrated using its supersymmetry with shape invariance.

The specificity of the PS problem is that it is formulated using a *matrix superpotential* while in the standard SSQM the superpotential is simply a scalar function. Matrix superpotentials themselves were discussed in many papers, see, e.g., [9–11, 13] but this discussion was actually reduced to analysis of particular examples. In paper [11] a certain class of such superpotentials was described which however was *ad hoc* restricted to 2×2 matrices which depend linearly on the variable parameter. Thus, in contrast to the case of scalar superpotentials, the class of known matrix potentials includes only few examples which are important but rather particular, while the remaining part of this class is still "terra incognita". It seems to be interesting to extend our knowledge of these potentials since this way it is possible to find new systems of Schrödinger equations which are exactly integrable.

2 The Spectral Problem

Let's start with a spectral problem

$$H_k \psi = E_k \psi, \tag{1}$$

where H_k—Hamiltonian with a matrix potential, E_k and ψ—its eigenvalues and eigenfunctions correspondingly. In the Schrödinger equation, Hamiltonian has the form

$$H_k = -\frac{\partial^2}{\partial x^2} + V_k(x), \tag{2}$$

where $V_k(x)$—matrix potential dependent on the parameter k and the variable x.

Let us assume that Hamiltonian can be factorized in the following way

$$H_k = a_k^\dagger a_k + c_k, \tag{3}$$

where c_k—scalar function of k, that vanishes with a corresponding member in the Hamiltonian. From here on, we will drop the sign of unit matrix I and write c_k instead of $c_k I$. The operators a_k and $a_k^\dagger$ can be considered of the form

$$a_k = \frac{\partial}{\partial x} + W_k(x), a_k^\dagger = -\frac{\partial}{\partial x} + W_k(x), \tag{4}$$

where W_k—hermitian matrix, that is called a superpotential.

As W_k is hermitian, operators a_k and $a_k^\dagger$ are hermitian-conjugate. It allows to find the ground state of the spectral problem (1), by simply solving the first order differential equation. Indeed, multiplying the expression

$$a_k^\dagger a_k \psi = 0 \tag{5}$$

on the left by the hermitian-conjugate spinor $\psi^\dagger$ and integrating it on the real line $\mathbb{R}$ we get

$$||a_k \psi||_2 = 0, \tag{6}$$

where $||\cdot||_2$ denotes the norm in $L_2(\mathbb{R})$. Hence,

$$a_k \psi = 0. \tag{7}$$

The square-integrable function $\psi_k^0(x)$, that is a normalized solution to the (7), is an eigenfunction of the Hamiltonian, that corresponds to the eigenvalue $E_k^0 = c_k$, and is called a ground state of the system (1).

Suppose the system satisfies the shape-invariance condition:

$$H_k^+ = H_{k+1}, \tag{8}$$

where H_k^+ is a Hamiltonian's superpartner, that is defined by the following formula:

$$H_k^+ = a_k a_k^\dagger + c_k. \tag{9}$$

This condition allows to fully discover the spectrum by a series of algebraic operations, knowing the eigenvalue of the system $\psi_k^0(x)$. Indeed, if we use the condition (8), it is easy to show that the function

$$\psi_k^1(x) = \frac{a_k^\dagger \psi_{k+1}^0(x)}{||a_k^\dagger \psi_{k+1}^0(x)||_2} \tag{10}$$

is a Hamiltonian's eigenfunction with an eigenvalue $E_k^1 = c_{k+1}$. It is called the first exited state of the system (1). Analogously, by induction we prove that the function

$$\psi_k^n(x) = \frac{a_k^\dagger a_{k+1}^\dagger \cdots a_{k+n-1}^\dagger \psi_{k+n}^0(x)}{||a_k^\dagger a_{k+1}^\dagger \cdots a_{k+n-1}^\dagger \psi_{k+n}^0(x)||_2} \tag{11}$$

is a Hamiltonian's eigenfunction with an eigenvalue $E_k^n = c_{k+n}$. It is called the nth exited state of the system (1).

It will be useful to denote

$$C_k = c_{k+1} - c_k, \tag{12}$$

then energy for nth exited states is expressed by formula

$$E_k^n = E_k^0 + \sum_{i=0}^{n-1} C_{k+i}. \tag{13}$$

Consequently, if the system of Schrödinger equations (1) satisfies the shape-invariance condition, it can be integrated explicitly.

3 The Classification Problem

As it was mentioned shape invariant potentials lead to exactly integrable systems of Schrödinger equations. Let us state the problem to find all Hamiltonians that allow factorization (3) and satisfy shape-invariance condition (8). In terms of the superpotential these conditions can be written through a single equation

$$W_k^2 + W_k' = W_{k+1}^2 - W_{k+1}' + C_k, \tag{14}$$

where C_k described by formula (12). Thus, to solve the given problem, it is enough to find all the superpotentials, that satisfy the (14).

We will consider the superpotentials of the form

$$W_k = kQ + P + \frac{1}{k}R, \tag{15}$$

where k is a variable parameter, Q is the unit matrix multiplied by a real valued function of independent variable x, and P, R are hermitian matrices depending on x.

We are interested in the irreducible superpotentials, i.e. the ones that can not be reduced to a block-diagonal form by means of an unitary transformation, that does not depend on variable x. Because if the considered superpotential is reducible, the problem is divided into a set of similar problems of a smaller dimension.

In the following section the equation describing unknown matrices P, Q and R is obtained, given the corresponding superpotentials satisfy the (14).

4 The Determining Equations

A system of determining equations can be obtained by substituting the expression (15), for the superpotential, into the (14) and separating the variables. Hence:

$$Q' = Q^2 + \nu, \tag{16}$$

$$P' = QP + \mu, \tag{17}$$

$$R' = 0, \tag{18}$$

$$R^2 = \omega^2, \tag{19}$$

$$\{P, R\} + \lambda = 0, \tag{20}$$

$$C_k = 2\mu + (2k+1)\nu - \frac{\lambda}{k(k+1)} + \frac{(2k+1)\omega^2}{k^2(k+1)^2}, \tag{21}$$

where $\nu, \mu, \omega, \lambda$—arbitrary real constants.

5 Superpotentials

Solving the system of determining (16)–(21) we obtain that the only irreducible superpotentials of the form (15) are of dimension 1×1 or 2×2.

While one-dimensional superpotentials are completely recover the well known list from the paper [4] two-dimensional superpotentials are new. They are presented in the list below:

$$W_k = \left((1-2\mu)\sigma^3 - 2k - 1\right)\frac{1}{2x} + \frac{\omega}{2k+1}\sigma^1, \tag{22}$$

$$W_k = \lambda\left(-k - \mu\exp(-\lambda x)\sigma^1 - \frac{\omega}{k}\sigma^3\right), \tag{23}$$

$$W_k = \lambda\left(k\tan\lambda x - \mu\sec\lambda x\sigma^3 + \frac{\omega}{k}\sigma^1\right), \tag{24}$$

$$W_k = \lambda\left(-k\coth\lambda x - \mu\,\mathrm{csch}\,\lambda x\sigma^3 - \frac{\omega}{k}\sigma^1\right), \quad \mu > 0,\ \omega > 0, \tag{25}$$

$$W_k = \lambda\left(-k\tanh\lambda x - \mu\,\mathrm{sech}\,\lambda x\sigma^1 - \frac{\omega}{k}\sigma^3\right), \tag{26}$$

These superpotentials are defined up to translations $x \to x + c$, $k \to k + \gamma$, and up to unitary transformations $W_k \to U_a W_k U_a^\dagger$ where $U_1 = \sigma^1, U_2 = \frac{1}{\sqrt{2}}(1 \pm \mathrm{i}\sigma^2)$ and $U_3 = \sigma^3$. In particular these transformations change signs of parameters μ and ω, thus without loss of generality we can set

$$\omega > 0, \quad \mu < 0 \tag{27}$$

in all superpotentials (22)–(26). Zero values of these parameters are excluded if superpotentials (22)–(25) are irreducible.

Conditions (27) can be imposed also for superpotential (25). To unify some following calculations we prefer to fix the signs of μ and k in the way indicated in (25).

If $\mu = 0$ and $\omega = 1$ then operator (22) coincides with the well known superpotential for PS problem, but for $\mu \neq 0$ superpotential (22) is not equivalent to it. The other found superpotentials are new also and make it possible to formulate consistent, exactly solvable problems for Schrödinger equation with matrix potential. The corresponding potentials V_k can be found starting with (22)–(25) and using definition

$$V_k = W_k^2 - W_k' \tag{28}$$

6 Conclusion

Generalizing the supersymmetric PS problem we find a family of matrix potentials for Shrödinger equation satisfying the shape invariance condition. In this way we find five exactly solvable problems for systems of coupled Shrödinger equations. The related matrix superpotentials are given by (22)–(26).

Let us stress that we present the completed classification of shape invariant superpotentials of the generic form (15) where P and R are hermitian matrices of arbitrary finite dimension and Q is proportional to the unit matrix. Namely, we show that such objects can be reduced to direct sums of known scalar superpotentials and superpotentials presented in Sect. 5.

The found superpotentials include parameters λ, k, μ and ω whose possible values are restricted but quite arbitrary. Moreover, parameters ω in (22) and μ in (23) can be reduced to unity by scaling and shifting the independent variable x correspondingly.

Superpotential (22) is a slightly generalized effective superpotential for the PS problem. Moreover, these superpotentials coincide for a particular value $\mu = 0$ of arbitrary parameter μ. However, if $\mu \neq 0$ superpotential (22) is not equivalent to the superpotential appearing in the PS problem and corresponds to a more general interaction in the initial three-dimension problem.

At the best of our knowledge the remaining superpotentials (23)–(26) are new. The related Schrödinger equations can be integrated using tools of the SUSY quantum mechanics.

The problem of classification of matrix superpotentials (15) with generic hermitian matrices Q, P and R is a subject of our contemporary research.

Acknowledgements The author thanks the Organizing Committee and especially Prof. Vladimir Dobrev for hospitality and giving an opportunity to give a talk. His participation in the Workshop was supported by the Abdus Salam International Centre for Theoretical Physics. The author is also grateful to Prof. Anatoly Nikitin for useful discussions and valuable comments.

References

1. Witten, E.: Nucl. Phys. B **185**, 513 (1981); **202**, 253 (1982)
2. Gendenshtein, L.: JETP Lett. **38**, 356 (1983)

3. Natanzon, G.A.: Vestnik Leningrad Univ. **10**, 22 (1971); Teor. Mat. Fiz. **38**, 146 (1979)
4. Cooper, F., Khare, A., Sukhatme, U.: Phys. Rep. **251**, 267 (1995)
5. Pron'ko, G.P., Stroganov, Y.G.: Sov. Phys. JETP **45**, 1075 (1977)
6. Voronin, A.I.: Phys. Rev. A **43**, 29 (1991)
7. Hau, L.V., Golovchenko, G.A., Burns, M.M.: Phys. Rev. Lett. **74**, 3138 (1995)
8. Ferraro, E., Messina, N., Nikitin, A.G.: Phys. Rev. A **81**, 042108 (2010)
9. Andrianov, A.A., Ioffe, M.V.: Phys. Lett. B **255**, 543 (1991); Andrianov, A.A., Ioffe, M.V., Spiridonov, V.P., Vinet, L.: Phys. Lett. B **272**, 297 (1991)
10. Andrianov, A.A., Cannata, F., Ioffe, M.V., Nishnianidze, D.N.: J. Phys. A Math. Gen. **30**, 5037 (1997)
11. Fukui, T.: Phys. Lett. A **178**, 1 (1993)
12. Ioffe,M.V, Kuru, S., Negro, J., Nieto, L.M.: J. Phys. A **39**, 6987 (2006)
13. de Lima Rodrigues, R., Bezerra, V.B., Vaidyac, A.N.: Phys. Lett. A **287**, 45 (2001)
14. Golub, Gene H., Van Loan, Charles F.: Matrix Computations, 3rd edn. Johns Hopkins, Baltimore, MD (1996). ISBN 978-0-8018-5414-9

Part VIII
Various Mathematical Results

On Finite W-Algebras for Lie Superalgebras in the Regular Case

Elena Poletaeva and Vera Serganova

Abstract We study finite W-algebras corresponding to the regular nilpotent orbits for classical Lie superalgebras of Type I. In the case when the Lie superalgebra has defect 1 we give a complete description of the finite W-algebras. We also present some partial results for the case $\mathfrak{gl}(n|n)$ and formulate a general conjecture about the structure of these algebras.

1 Introduction

Finite W-algebras for semi-simple Lie algebras were introduced by A. Premet [8] (see also [6]). In the case of Lie superalgebras, finite W-algebras were studied by mathematicians and physicists in the following works [1, 3, 10, 12].

In this paper we study the case when the corresponding nilpotent element is regular. Recall that an old result of B. Kostant states that if $\mathfrak{g}$ is a reductive Lie algebra and e is a regular nilpotent element, then the finite W-algebra for $\mathfrak{g}$ coincides with the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ [5]. It is clear that this does not hold for Lie superalgebras, since the finite W-algebra must have a non-trivial odd part, and $Z(\mathfrak{g})$ is even.

We consider the case when $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a classical simple Lie superalgebra, i.e. $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}$ has an invariant symmetric bilinear form $(,)$.

E. Poletaeva (✉)
Department of Mathematics, University of Texas-Pan American, Edinburg, TX 78539, USA
e-mail: elenap@utpa.edu

V. Serganova
Department of Mathematics, University of California, Berkeley, CA 94720, USA
e-mail: serganov@math.berkeley.edu

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_36,

Let $e \in \mathfrak{g}_{\bar{0}}$ be an even nilpotent element, and we fix an sl_2-triple f,h,e. The linear operator ad_h defines a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus \mathfrak{g}_i$. Let

$$\mathfrak{m} = \oplus_{i \leq -2} \mathfrak{g}_i \oplus l,$$

where l is a maximal isotropic subspace in $\mathfrak{g}_{-1}$ with respect to the (super)skew-symmetric bilinear form $\omega(x,y) = (e,[x,y])$. Let $\chi \in \mathfrak{g}^*$ be defined by the formula $\chi(x) = (e,x)$, and C_χ denote the one-dimensional $\mathfrak{m}$-module with character χ. The generalized Whittaker module Q_χ is by definition the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} C_\chi$ and $H_\chi = \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}$ is by definition the finite W-algebra associated to the nilpotent element e.

Certain results of A. Premet can be easily generalized for classical Lie superalgebras. For example, Kazhdan filtration can be defined exactly as in the Lie algebra case. Recall that if $\mathfrak{n} \subset \mathfrak{g}$ is an ad_h-invariant subspace such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}$, then

$$H_\chi = \{X \in U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m} \cong S(\mathfrak{n}) \mid gXv = \chi(g)Xv \text{ for all } g \in \mathfrak{m}\},$$

where $v \in C_\chi$. For any $y \in \mathfrak{n}$ let $wt(y)$ be the weight of y with respect to ad_h and $Deg(y) = wt(y) + 2$. The degree function Deg induces a $\mathbb{Z}$-grading on $S(\mathfrak{n})$. The following result is true and can be proved exactly as in [8].

Theorem 1. *Deg defines a filtration on H_χ, and the associated graded superalgebra $Gr(H_\chi)$ is supercommutative.*

One of the important results of A. Premet is that $Gr(H_\chi)$ is isomorphic to $S(\mathfrak{g}^e)$, where $\mathfrak{g}^e = \mathrm{Ker}(ad_e)$. In order to prove this he introduced the map $P : H_\chi \to S(\mathfrak{g}^e)$, namely he proved that for $X \in H_\chi \subset S(\mathfrak{n})$ the term $P(X)$ of the highest degree and of the highest weight belongs to $S(\mathfrak{g}^e)$. For Lie superalgebras one can construct the similar map if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even. We proved in certain cases that P is an isomorphism of vector spaces if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even. In an arbitrary case when $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even we can prove that P is injective, but we did not complete the proof that it is surjective. The original proof of A. Premet is based on the similar result in characteristic p-case. Since recently Wang and Zhao proved Kac-Weisfeiler conjecture for classical Lie superalgebras [11], we think that it is possible to use their result for the proof of surjectivity of P.

In the case, when $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, there exists an odd element $\theta \in H_\chi$ such that $\theta^2 = 1$ and θ is induced by an element in $l^\perp \cap \mathfrak{g}_{-1}$.

We hope that understanding of finite W-algebras for regular elements might help in studying representations of $\mathfrak{g}$ which is very difficult even in finite-dimensional case. The difficulty is partially due to the fact that $Z(\mathfrak{g})$ is not Noetherian.

There is a natural stratification of the spectrum of maximal ideals in $Z(\mathfrak{g})$, called the degree of atypicality. The number of strata minus 1 is the important invariant of the superalgebra called the defect $\mathrm{def}\mathfrak{g}$ of $\mathfrak{g}$. For instance,

$$\mathrm{def}(\mathfrak{sl}(m|n)) = \min(m,n), \quad \mathrm{def}(\mathfrak{psl}(n|n)) = n,$$

$$\mathrm{def}(\mathfrak{osp}(2m+1|2n)) = \mathrm{def}(\mathfrak{osp}(2m|2n)) = \min(m,n).$$

The exceptional Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ all have defect one (see the definitions of these superalgebras in [4]).

It is interesting that if e is regular, then $\dim(\mathfrak{g}^e)_{\bar{1}} = 2\mathrm{def}\mathfrak{g}$ or $2\mathrm{def}\mathfrak{g}+1$. More precisely, $\dim(\mathfrak{g}^e)_{\bar{1}} = 2\mathrm{def}\mathfrak{g}$ if $\mathfrak{g}$ is one of the following superalgebras:

$$\mathfrak{sl}(m|n), \mathfrak{osp}(2m+1|2n), m \geq n, \mathfrak{osp}(2m|2n), m \leq n, G(3),$$

and $\dim(\mathfrak{g}^e)_{\bar{1}} = 2\mathrm{def}\mathfrak{g}+1$ if $\mathfrak{g}$ is one of the following superalgebras:

$$\mathfrak{osp}(2m+1|2n), m < n, \mathfrak{osp}(2m|2n), m > n, D(2,1;\alpha), F(4).$$

It is interesting to find a conceptual explanation of this phenomenon.

2 The Case of Defect One

Recall that a classical simple Lie superalgebra $\mathfrak{g}$ is of Type I if it admits a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1$, which is compatible with the $\mathbb{Z}_2$-grading. Note that there are two Lie superalgebras of Type I and defect one: $\mathfrak{g} = \mathfrak{sl}(1|n)$ and $\mathfrak{g} = \mathfrak{osp}(2|2n)$. We consider in detail the case when $\mathfrak{g} = \mathfrak{sl}(1|n)$. The case of $\mathfrak{g} = \mathfrak{osp}(2|2n)$ is similar.

Let $\mathfrak{g} = \mathfrak{sl}(1|n)$. We use the following notations for some elementary matrices in $\mathfrak{gl}(1|n)$, which are clear from the picture below.

$$\left(\begin{array}{c|ccccc} h_0 & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\ \hline \xi_1 & h_1 & e_1 & \cdots & \cdots & \cdots \\ \xi_2 & f_1 & h_2 & e_2 & \cdots & \cdots \\ \xi_3 & \cdots & f_2 & h_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & e_{n-1} \\ \xi_n & \cdots & \cdots & \cdots & f_{n-1} & h_n \end{array}\right)$$

Let $\mathfrak{sl}(2) = < e, h, f >$, where e is a regular nilpotent element:

$$e = \sum_{i=1}^{n-1} e_i, \quad h = \mathrm{diag}(0|n-1, n-3, \cdots, 3-n, 1-n), \quad f = \sum_{i=1}^{n-1} i(n-i) f_i.$$

h defines a $\mathbb{Z}$-grading of $\mathfrak{gl}(1|n)$ whose degrees on the elementary matrices are

$$\left(\begin{array}{c|ccccc} 0 & 1-n & 3-n & \cdots & n-3 & n-1 \\ \hline n-1 & 0 & 2 & 4 & \cdots & 2n-2 \\ n-3 & -2 & 0 & 2 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 3-n & 4-2n & \cdots & \cdots & 0 & 2 \\ 1-n & 2-2n\cdots & \cdots & \cdots & -2 & 0 \end{array}\right)$$

Let $c = \mathrm{diag}(n|1,\cdots,1)$ be a central element of $\mathfrak{g}_{\bar{0}}$. Note that $\dim(\mathfrak{g}^e) = (n|2)$, and

$$\mathfrak{g}^e =< e, e^2, \cdots, e^{n-1}, c \mid \xi_1, \mu_n >, \quad \mathfrak{m} = (\bigoplus_{j\leq -2} \mathfrak{g}_j) \bigoplus l.$$

Note that $l = 0$ if n is odd, and $\dim l = 1$ if n is even. The result does not depend on a choice of l. Let $l =< \xi_{k+1} >$ if $n = 2k$, then $\mathfrak{m}$ is generated by the following elements:

$$f_1, f_2, \cdots, f_{n-1};$$

$$\mu_1, \cdots, \mu_{k-1}, \xi_{k+1}, \cdots, \xi_n, \text{ if } n = 2k;$$

$$\mu_1, \cdots, \mu_k, \xi_{k+2}, \cdots, \xi_n, \text{ if } n = 2k+1.$$

Note that

$$\chi(f_1) = \cdots = \chi(f_{n-1}) = 1,$$
$$\chi(\mu_1) = \cdots = \chi(\mu_{k-1}) = \chi(\xi_{k+1}) = \cdots = \chi(\xi_n) = 0 \text{ if } n = 2k,$$
$$\chi(\mu_1) = \cdots = \chi(\mu_k) = \chi(\xi_{k+2}) = \cdots = \chi(\xi_n) = 0 \text{ if } n = 2k+1.$$

H_χ has two odd generators: X and Y. To describe them recall that $\mathfrak{g} = \mathfrak{sl}(1|n)$ admits a $\mathbb{Z}$-grading

$$\mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1.$$

Fix an *adh*-homogeneous bases:

$$B(\mathfrak{m}_{-1}) \text{ of } \mathfrak{m}\cap\mathfrak{g}^{-1}, B(\mathfrak{m}_1) \text{ of } \mathfrak{m}\cap\mathfrak{g}^1, B(\mathfrak{n}_{-1}) \text{ of } \mathfrak{n}\cap\mathfrak{g}^{-1}, B(\mathfrak{n}_1) \text{ of } \mathfrak{n}\cap\mathfrak{g}^1;$$

$$B(\mathfrak{g}^{\pm 1}) \text{ of } \mathfrak{g}^{\pm 1}.$$

Let $\mathbb{C}_\chi =< v >$. Set

$$Yv = (\prod_{x\in B(\mathfrak{m}_1)} x)(\prod_{y\in B(\mathfrak{n}_{-1})} y)v,$$

$$Xv = (\prod_{y\in B(\mathfrak{m}_{-1})} y)(\prod_{x\in B(\mathfrak{n}_1)} x)v.$$

Then Xv and Yv are both Whittaker vectors, and hence $X, Y \in H_\chi$.

In order to describe even generators of H_χ, recall that the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie (super)algebra $\mathfrak{g}$ is generated by the so-called Casimir elements. In the case when $\mathfrak{g} = \mathfrak{gl}(m|n)$ the Casimir elements Ω_k for $k \geq 2$ are defined as follows (see [9]). Let

$$p(E_{ij}) = p(i) + p(j),$$

where E_{ij} is an elementary matrix and

$$p(i) = \begin{cases} \bar{0} & \text{if } 1 \leq i \leq m \\ \bar{1} & \text{if } m+1 \leq i \leq m+n. \end{cases} \tag{1}$$

$$\Omega_k := \sum_{i_1, i_2, \ldots, i_k} (-1)^{p(i_2)+\cdots+p(i_k)} E_{i_1 i_2} E_{i_2 i_3} \ldots E_{i_k i_1}.$$

In the case when $\mathfrak{g} = \mathfrak{sl}(1|n)$, H_χ has n even generators: Ω_k, $k = 2, \ldots n$ and c. There exists a bijective map

$$P : H_\chi \longrightarrow S(\mathfrak{g}^e)$$

given as follows:

$$P(X) = \mu_n, \quad P(Y) = \xi_1, \quad P(c) = c,$$

$$(-1)^{k+1} P\left(\frac{1}{k}\Omega_k\right) = e^{k-1} \text{ for } k = 2, \ldots, n.$$

Let

$$\tilde{\Omega} = \prod_{y \in B(\mathfrak{g}^{-1})} ad_y\Big(\prod_{x \in B(\mathfrak{g}^1)} x\Big).$$

Then $\tilde{\Omega}$ is an element of $Z(\mathfrak{g})$ and has degree n. Under Harish-Chandra homomorphism it goes to the polynomial $(h_0 - h_1) \cdots (h_0 - h_n)$ [9]. One can also characterize $\tilde{\Omega}$ by the following property: it is the element of $Z(\mathfrak{g})$ of minimal degree such that it is zero on all atypical irreducible representations.

Theorem 2. *Let $\mathfrak{g}$ be a Lie superalgebra of Type I and defect 1 (i.e. $\mathfrak{g} = \mathfrak{sl}(1|n)$ or $\mathfrak{osp}(2|2n-2)$.) Let n be the rank of $\mathfrak{g}_{\bar{0}}$, c be a central element of $\mathfrak{g}_{\bar{0}}$ and $\Omega_2, \ldots, \Omega_n$ be the first $n-1$ Casimir elements in $Z(\mathfrak{g})$. Then H_χ is a finite extension of $\mathbb{C}[c, \Omega_2, \ldots, \Omega_n]$ with odd generators X, Y and defining relations*

$$X^2 = Y^2 = 0, [c, X] = X, [c, Y] = -Y,$$

$$[\Omega_i, X] = [\Omega_i, Y] = 0, i = 2, \ldots, n, XY + YX = \tilde{\Omega}.$$

Note that in this case $H_\chi \cong U(\mathfrak{g}^e)$.

3 The Case of $\mathfrak{gl}(2|2)$

Let $\mathfrak{g} = \mathfrak{gl}(2|2)$. Consider the following elementary matrices in $\mathfrak{g}$:

$$\left(\begin{array}{cc|cc} h_1 & e_1 & y_1 & y_3 \\ f_1 & h_2 & \mu_1 & y_2 \\ \hline x_1 & x_3 & h_3 & e_2 \\ \xi_1 & x_2 & g_1 & h_4 \end{array}\right).$$

Let $\mathfrak{sl}(2) =< e,h,f >$, where e is a regular nilpotent element:

$$e = e_1 + e_2, \quad h = \mathrm{diag}(1,-1|1,-1), \quad f = f_1 + f_2.$$

h defines an even $\mathbb{Z}$-grading of $\mathfrak{gl}(2|2)$ whose degrees on the elementary matrices are

$$\left(\begin{array}{cc|cc} 0 & 2 & 0 & 2 \\ -2 & 0 & -2 & 0 \\ \hline 0 & 2 & 0 & 2 \\ -2 & 0 & -2 & 0 \end{array}\right)$$

Let $z = \mathrm{diag}(1,1|1,1)$, $c = \mathrm{diag}(1,1|-1,-1)$. Note that $\dim(\mathfrak{g}^e) = (4|4)$, and

$$\mathfrak{g}^e =< e_1, e_2, z, c \mid x_3, y_3, x_1 + x_2, y_1 + y_2 >, \quad \mathfrak{m} = \mathfrak{g}_{-2},$$

$\mathfrak{m}$ is generated by ξ_1, μ_1, f_1, g_1. Note that

$$\chi(f_1) = -\chi(g_1) = 1, \quad \chi(\mu_1) = \chi(\xi_1) = 0.$$

H_χ has four even generators C_1, C_2, z, c and four odd generators $X, Y, \tilde{X}_1, \tilde{Y}_1$.
Set $z = 0$, i.e. consider $\mathfrak{g} = \mathfrak{pgl}(2|2)$. Let

$$\begin{aligned} X &= x_1 + x_2, \quad Y = y_1 + y_2, \\ C_1 &= e_1 + \frac{1}{4}(h_1 - h_2)^2 - x_2 y_1, \\ C_2 &= -e_2 + \frac{1}{4}(h_3 - h_4)^2 - y_2 x_1. \end{aligned}$$

We define X_1 and Y_1 from the following equations

$$\begin{aligned} [X, C_1] = [X, C_2] = X_1 + \tfrac{1}{4} X, \\ [Y, C_1] = [Y, C_2] = -Y_1 + \tfrac{1}{4} Y. \end{aligned}$$

Explicitly,

$$\begin{aligned} X_1 = x_3 + \tfrac{1}{2}(h_1 - h_2)x_1 + \tfrac{1}{2}(h_1 + h_2 + 2h_3)x_2, \\ Y_1 = y_3 - \tfrac{1}{2}(h_3 - h_4)y_1 - \tfrac{1}{2}(2h_1 + h_3 + h_4)y_2. \end{aligned}$$

Next, we change X_1 and Y_1 to $\tilde{X}_1$ and $\tilde{Y}_1$ as follows

$$\begin{aligned} \tilde{X}_1 &= X_1 + \frac{1}{4} c X, \\ \tilde{Y}_1 &= Y_1 + \frac{1}{4} c Y. \end{aligned}$$

There is a bijective map

$$P : H_\chi \longrightarrow S(\mathfrak{g}^e)$$

given as follows:

$$\begin{aligned} P(X) &= x_1 + x_2, \quad P(Y) = y_1 + y_2, \\ P(\tilde{X}_1) &= x_3, \quad P(\tilde{Y}_1) = y_3, \\ P(C_1) &= e_1, \quad P(C_2) = -e_2, \quad P(c) = c. \end{aligned}$$

Note that the commutators of odd generators $X, Y, \tilde{X}_1, \tilde{Y}_1$ are in $Z(\mathfrak{g})$. The nonzero commutation relations between the odd generators are

$$[\tilde{X}_1, Y] = [\tilde{Y}_1, X] = \frac{1}{2}\Omega_2, \quad [\tilde{X}_1, \tilde{Y}_1] = \frac{1}{3}\Omega_3.$$

Let

$$E_1 = C_1 - C_2 + XY, \quad \tilde{E}_1 = C_1 + C_2 + \tfrac{1}{8}c^2. \tag{2}$$

Note that $E_1 = \frac{1}{2}\Omega_2 \in Z(\mathfrak{g})$.

Thus the nonzero commutation relations between even generators $E_1, \tilde{E}_1, c$ and odd generators $X, Y, \tilde{X}_1, \tilde{Y}_1$ are

$$\begin{gathered} [\tilde{E}_1, X] = -2\tilde{X}_1 - X, \quad [\tilde{E}_1, Y] = 2\tilde{Y}_1 - Y, \\ [\tilde{E}_1, \tilde{X}_1] = -\tilde{E}_1 X - (c+1)\tilde{X}_1 + \tfrac{1}{4}c^2 X, \quad [\tilde{E}_1, \tilde{Y}_1] = \tilde{E}_1 Y + (c-1)\tilde{Y}_1 - \tfrac{1}{4}c^2 Y, \\ [X, c] = 2X, \quad [Y, c] = -2Y, \quad [\tilde{X}_1, c] = 2\tilde{X}_1, \quad [\tilde{Y}_1, c] = -2\tilde{Y}_1. \end{gathered}$$

4 The Case of $\mathfrak{gl}(\mathbf{n}|\mathbf{n})$

Let $\mathfrak{g} = \mathfrak{gl}(n|n)$. Consider the following elementary matrices in $\mathfrak{g}$:

$$\left(\begin{array}{ccccc|ccccc}
h_1 & e_1 & \cdots & \cdots & \cdots & y_1 & y_{n+1} & \cdots & \cdots & \cdots \\
f_1 & h_2 & e_2 & \cdots & \cdots & \mu_1 & y_2 & y_{n+2} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & f_{n-2} & h_{n-1} & e_{n-1} & \cdots & \cdots & \mu_{n-2} & y_{n-1} & y_{2n-1} \\
\cdots & \cdots & \cdots & f_{n-1} & h_n & \cdots & \cdots & \cdots & \mu_{n-1} & y_n \\
\hline
x_1 & x_{n+1} & \cdots & \cdots & \cdots & h_{n+1} & e_n & \cdots & \cdots & \cdots \\
\xi_1 & x_2 & x_{n+2} & \cdots & \cdots & g_1 & h_{n+2} & e_{n+1} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \xi_{n-2} & x_{n-1} & x_{2n-1} & \cdots & \cdots & g_{n-2} & h_{2n-1} & e_{2n-2} \\
\cdots & \cdots & \cdots & \xi_{n-1} & x_n & \cdots & \cdots & \cdots & g_{n-1} & h_{2n}
\end{array}\right)$$

Let $\mathfrak{sl}(2) =< e,h,f >$, where e is a regular nilpotent element:

$$e = e_1 + e_2 + \cdots + e_{2n-2}, \quad f = \sum_{i=1}^{n-1} i(n-i)(f_i + g_i),$$
$$h = \mathrm{diag}(n-1, n-3, \cdots, 3-n, 1-n | n-1, n-3, \cdots, 3-n, 1-n).$$

h defines an even $\mathbb{Z}$-grading of $\mathfrak{gl}(n|n)$ whose degrees on the elementary matrices are

$$\left(\begin{array}{ccccc|ccccc} 0 & 2 & \cdots & 2n-4 & 2n-2 & 0 & 2 & \cdots & 2n-4 & 2n-2 \\ -2 & 0 & 2 & \cdots & 2n-4 & -2 & 0 & 2 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 4-2n & \cdots & -2 & 0 & 2 & 4-2n & \cdots & -2 & 0 & 2 \\ 2-2n & 4-2n & \cdots & -2 & 0 & 2-2n & 4-2n & \cdots & -2 & 0 \\ \hline 0 & 2 & \cdots & 2n-4 & 2n-2 & 0 & 2 & \cdots & 2n-4 & 2n-2 \\ -2 & 0 & 2 & \cdots & 2n-4 & -2 & 0 & 2 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 4-2n & \cdots & -2 & 0 & 2 & 4-2n & \cdots & -2 & 0 & 2 \\ 2-2n & 4-2n & \cdots & -2 & 0 & 2-2n & 4-2n & \cdots & -2 & 0 \end{array}\right)$$

Let $z = \mathrm{diag}(1, \cdots, 1 | 1, \cdots, 1)$, $c = \mathrm{diag}(1, \cdots, 1 | -1, \cdots, -1)$. Note that $\dim(\mathfrak{g}^e) = (2n|2n)$, and

$$\mathfrak{g}^e_{\bar{0}} =< (e_1 + \cdots + e_{n-1})^i, (e_n + \cdots + e_{2n-2})^i, z, c >,$$
$$\mathfrak{g}^e_{\bar{1}} =< (x_1 + \cdots + x_n), (y_1 + \cdots + y_n), (x_{n+1} + \cdots + x_{2n-1})^i, (y_{n+1} + \cdots + y_{2n-1})^i >,$$

where $i = 1, \cdots, n-1$, and the powers are considered for the corresponding $n \times n$ matrices.

$$\mathfrak{m} = \bigoplus_{i=2}^{n} \mathfrak{g}_{2-2i},$$

and it is generated by ξ_i, μ_i, f_i, g_i for $i = 1, \cdots, n-1$.
Note that

$$\chi(f_i) = -\chi(g_i) = 1,$$
$$\chi(\mu_i) = \chi(\xi_i) = 0, \text{ for } i = 1, \cdots, n-1.$$

It follows from the work of physicists [1] who use truncated super-Yangians to study finite W-algebras for $\mathfrak{gl}(m|n)$, that H_χ has $2n$ even generators: z, c and $C_1^i, C_2^i, i = 1, \cdots, n-1$, which correspond to the first $n-1$ Casimir elements for the upper and lower $\mathfrak{sl}(n)$, respectively, and $2n$ odd generators: $X, Y, \tilde{X}_i, \tilde{Y}_i$, $i = 1, \cdots, n-1$.
Set $z = 0$, i.e. consider $\mathfrak{g} = \mathfrak{pgl}(n|n)$. Let

$$X = x_1 + x_2 + \ldots + x_n, \quad Y = y_1 + y_2 + \ldots + y_n,$$

$$C_1^1 = (e_1 + e_2 + \ldots + e_{n-1}) + \frac{1}{2n} \sum_{1 \le i < j \le n} (h_i - h_j)^2 - x_2 y_1 - x_3(y_1 + y_2) - \ldots - x_n(y_1 + y_2 + \ldots + y_{n-1}),$$

$$C_2^1 = -(e_n + \ldots + e_{2n-2}) + \frac{1}{2n} \sum_{n+1 \le i < j \le 2n} (h_i - h_j)^2 - y_2 x_1 - y_3(x_1 + x_2) - \ldots - y_n(x_1 + x_2 + \ldots + x_{n-1}).$$

Let $X_0 = X, Y_0 = Y$. We define X_i and Y_i for $i = 1, \ldots n-1$ from the equations

$$[X_i, C_1^1] = X_{i+1} + (\frac{n-1}{2n})X_i,$$

$$[Y_i, C_1^1] = -Y_{i+1} + (\frac{n-1}{2n})Y_i.$$

Next, we change X_i to $\tilde{X}_i$ and Y_i to $\tilde{Y}_i$ for $i = 1, 2, \ldots, n-1$ as follows:

$$\tilde{X}_i = \textstyle\sum_{k=0}^{i} \binom{i}{k} \frac{1}{(2n)^k} c^k X_{i-k}, \quad \tilde{Y}_i = \sum_{k=0}^{i} \binom{i}{k} \frac{1}{(2n)^k} c^k Y_{i-k}. \tag{3}$$

Then the commutators for odd generators $X, Y, \tilde{X}_i, \tilde{Y}_i$ for $i = 1, 2, \ldots, n-1$ are in $Z(\mathfrak{g})$.

Conjecture 1. There exists a bijective map

$$P : H_\chi \longrightarrow S(\mathfrak{g}^e)$$

given as follows:

$$P(X) = x_1 + \ldots + x_n, \quad P(Y) = y_1 + \ldots + y_n, \quad P(c) = c,$$

$$P(\tilde{X}_i) = (x_{n+1} + \ldots + x_{2n-1})^i, \quad P(\tilde{Y}_i) = (y_{n+1} + \ldots + y_{2n-1})^i,$$

$$P(C_1^i) = (e_1 + \ldots + e_{n-1})^i, \quad P(C_2^i) = (-1)^i (e_n + \ldots + e_{2n-2})^i,$$

where $i = 1, \ldots, n-1$.

Remark. In the case when $\mathfrak{g} = \mathfrak{gl}(n|n)$, one can define elements $E_i, \tilde{E}_i$ for $i = 1, \ldots, n-1$, analogously to (2), so that $E_i \in Z(\mathfrak{g})$. In particular,

$$E_1 = C_1^1 - C_2^1 + XY, \quad \tilde{E}_1 = C_1^1 + C_2^1 + \frac{1}{4n} c^2.$$

Then $E_1 = \frac{1}{2}\Omega_2 \in Z(\mathfrak{g})$, and the following relations are satisfied

$$[X_i, C_1^1] = [X_i, C_2^1], \quad [Y_i, C_1^1] = [Y_i, C_2^1],$$

$$[X, c] = 2X, \quad [Y, c] = -2Y, \quad [\tilde{X}_i, c] = 2\tilde{X}_i, \quad [\tilde{Y}_i, c] = -2\tilde{Y}_i,$$

$$[\tilde{E}_1, \tilde{X}_i] = -2\tilde{X}_{i+1} - \tilde{X}_i, \quad [\tilde{E}_1, \tilde{Y}_i] = 2\tilde{Y}_{i+1} - \tilde{Y}_i, \quad i = 0, \ldots, n-2. \tag{4}$$

Note that one can obtain the elements $\tilde{X}_i$ and $\tilde{Y}_i$, where $i = 1, \ldots, n-1$, defined in (3) from relations (4) by induction.

In the general case of $\mathfrak{gl}(m|n)$, we have the following

Conjecture 2. Let $\mathfrak{g}=\mathfrak{gl}(m|n)$, and $d=\min(m,n)$. Then

(1) there exist odd generators X_i, Y_i, of H_χ for $i=1,\dots,d$ such that

$$[X_i,X_j]=[Y_i,Y_j]=0 \text{ for } 1\le i,j\le d, \quad [X_i,Y_j]\in Z(\mathfrak{g}),$$

(2) the center of H_χ is $Z(\mathfrak{g})$.

5 Truncated Super-Yangians

It was observed by physicists that the finite W-algebra based on $\mathfrak{gl}(m|n)$ is a truncation of the super-Yangian $Y(\mathfrak{gl}(m|n))$ [1]. Recall that for a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, the Yangian of $\mathfrak{g}$ is an infinite-dimensional Hopf algebra $Y(\mathfrak{g})$. It is a deformation of the universal enveloping algebra of the Lie algebra of polynomial currents of $\mathfrak{g}$ [7].

We observed that in the case when $\mathfrak{g}=\mathfrak{gl}(n|n)$, $\mathfrak{g}^e$ is isomorphic to the truncated Lie superalgebra of polynomial currents in $\mathfrak{gl}(1|1)$. Let

$$\mathfrak{gl}(1|1)=\left\{\left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22}\end{array}\right) \mid a_{ij}\in\mathbb{C}\right\}.$$

The isomorphism

$$\varphi:\mathfrak{g}^e\longrightarrow \mathfrak{gl}(1|1)\otimes\mathbb{C}[t]/(t^n)$$

is given as follows: for $i=1,\dots,n-1$

$$\varphi((e_1+\cdots+e_{n-1})^i)=E_{11}\otimes t^i, \quad \varphi((e_n+\cdots+e_{2n-2})^i)=E_{22}\otimes t^i,$$
$$\varphi((x_{n+1}+\cdots+x_{2n-1})^i)=E_{21}\otimes t^i, \quad \varphi((y_{n+1}+\cdots+y_{2n-1})^i)=E_{12}\otimes t^i,$$

$$\varphi\left(\frac{z+c}{2}\right)=E_{11},\ \ \varphi\left(\frac{z-c}{2}\right)=E_{22},\ \ \varphi(x_1+\cdots+x_n)=E_{21},\ \ \varphi(y_1+\cdots+y_n)=E_{12}.$$

Recall that the super-Yangian $Y(\mathfrak{gl}(m|n)$ of $\mathfrak{gl}(m|n)$ is an associative unital superalgebra over $\mathbb{C}$ with a countable set of generators

$$t_{ij}^{(1)},t_{ij}^{(2)},\dots, \text{ where } i,j=1,\dots,m+n$$

and the following defining relations

$$[t_{ij}^{(r+1)},t_{kl}^{(s)}]-[t_{ij}^{(r)},t_{kl}^{(s+1)}]=(-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)}(t_{kj}^{(r)}t_{il}^{(s)}-t_{kj}^{(s)}t_{il}^{(r)}),$$

where $r,s=0,1,\dots$ and $t_{ij}^{(0)}=\delta_{ij}$, and $p(i)$ is defined in (1), see [7].

It follows from [1] that in the case when $\mathfrak{g} = \mathfrak{gl}(n|n)$, the corresponding finite W-algebra is isomorphic to the truncated super-Yangian $Y(\mathfrak{gl}(1|1))/(n)$.

The finite W-algebras for $\mathfrak{g} = \mathfrak{gl}(m|n)$ were described as certain truncations of a shifted version of the Yangian $Y(\mathfrak{gl}(1|1))$ by J. Brown, J. Brundan and S. Goodwin in [2].

Acknowledgements The authors would like to thank the Mathematisches Forschungsinstitut Oberwolfach for the hospitality and support in the spring of 2010. They also thank Crystal Hoyt for stimulating discussions and pointing out [1] and [3].

E.P. thanks the organizers of the 9-th International Workshop "Lie Theory and Its Applications in Physics" (LT-9), 20–26 June 2011, Varna, Bulgaria, for the very interesting conference and for the hospitality.

References

1. Briot, C., Ragoucy, E.: J. Phys. A **36**(4), 1057–1081 (2003)
2. Brown, J., Brundan, J., Goodwin, S.: Principal W-algebras for *GL(m|n)*, e-print arXiv:1205.0992.
3. Brundan, J., Kleshchev, A.: Representations of shifted Yangians and finite W-algebras. Mem. Am. Math. Soc. **196**, 918 (2008)
4. Kac, V.G.: Adv. Math. **26**, 8–96 (1977)
5. Kostant, B.: Invent. Math. **48**, 101–184 (1978)
6. Losev, I.: J. Am. Math. Soc. **23**, 34–59 (2010)
7. Molev, A.: Yangians and classical Lie algebras. Mathematical Surveys and Monographs, vol. 143. Amer. Math. Soc., Providence, RI (2007)
8. Premet, A.: Adv. Math. **170**, 1–55 (2002)
9. Sergeev, A.: Represent. Theory **3**, 250–280 (1999)
10. Wang, W.: Field Inst. Commun. **59**, 71–105. Amer. Math. Soc., Providence, RI (2011)
11. Wang, W., Zhao, L.: Proc. Lond. Math. Soc. (3) **99**(1), 145–167 (2009)
12. Zhao, L.: e-print arXiv: 1012.2326

Young Tableaux and Homotopy Commutative Algebras

Michel Dubois-Violette and Todor Popov

Abstract A homotopy commutative algebra, or C_∞-algebra, is defined via the Tornike Kadeishvili homotopy transfer theorem on the vector space generated by the set of Young tableaux with self-conjugated Young diagrams $\{\lambda : \lambda = \lambda'\}$. We prove that this C_∞-algebra is generated in degree 1 by the binary and the ternary operations.

1 Introduction

We consider the 2-nilpotent graded Lie algebra $\mathfrak{g}$, with degree one generators in the finite dimensional vector space V over a field $\mathbb{K}$ of characteristic 0,

$$\mathfrak{g} = V \oplus [V, V].$$

Its Universal Enveloping Algebra (UEA) $U\mathfrak{g}$ arises naturally in physics as the subalgebra closed by the creation operators of the parastatistics algebra. The algebra of creation and annihilation parastatistics operators was introduced by H.S. Green [6], its defining relations generalize the canonical (anti)commutation relations.

As an UEA of a finite dimensional positively graded Lie algebra, $U\mathfrak{g}$ belongs to the class of Artin–Schelter regular algebras (see e.g. [5]). As every finitely generated graded connected algebra, $U\mathfrak{g}$ has a free minimal resolution which is canonically

M. Dubois-Violette
Laboratoire de Physique Théorique, UMR 8627, Université Paris XI, Bâtiment 210, F-91 405 Orsay Cedex, France
e-mail: Michel.Dubois-Violette@th.u-psud.fr

T. Popov (✉)
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tzarigradsko Chaussee, 1784 Sofia, Bulgaria
e-mail: tpopov@inrne.bas.bg

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_37,

built from the data of its Yoneda algebra $\mathscr{E} := \mathrm{Ext}_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})$. By construction the Yoneda algebra $\mathscr{E}$ is isomorphic (as algebra) to the cohomology of the Lie algebra (with coefficients in the trivial representation provided by the ground field $\mathbb{K}$)

$$\mathscr{E} = \mathrm{Ext}^{\bullet}_{U\mathfrak{g}}(\mathbb{K},\mathbb{K}) \cong H^{\bullet}(\mathfrak{g},\mathbb{K}) \tag{1}$$

the product on $\mathscr{E}$ being the super-commutative wedge product between cohomological classes in $H^{\bullet}(\mathfrak{g},\mathbb{K})$.

An important result due to Józefiak and Weyman [7] implies that a basis of the cohomology $\mathscr{E} = H^{\bullet}(\mathfrak{g},\mathbb{K})$ is indexed by Young tableaux with self-conjugated Young diagrams (i.e., symmetric with respect to the diagonal). On the other hand according to the homotopy transfer theorem due to Tornike Kadeishvili [8] the Yoneda algebra $\mathscr{E}$ is a C_∞-algebra.

The aim of this note is to describe the cohomology $H^{\bullet}(\mathfrak{g},\mathbb{K})$ (i.e., the vector space generated by the set of Young tableaux with self-conjugated Young diagrams $\{\lambda : \lambda = \lambda'\}$) with the C_∞-structure induced by the isomorphism (1) through the homotopy transfer.

Here we deal only with the parafermionic case corresponding to an (even) vector space V. To include the parabosonic degrees of freedom, one have to consider V in the category of vector superspaces. The supercase will be considered elsewhere.

2 Artin–Schelter Regularity

Let $\mathfrak{g}$ be the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ generated by the finite dimensional vector space V having Lie bracket

$$[x,y] := \begin{cases} x \wedge y & x,y \in V \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

We denote the Universal Enveloping Algebra $U\mathfrak{g}$ by PS and will refer to it as *parastatistics algebra* (by some abuse[1]). The parastatistics algebra $PS(V)$ generated in V is graded

$$PS(V) := U\mathfrak{g} = U(V \oplus \bigwedge^2 V) = T(V)/([[V,V],V]).$$

We shall write simply PS when the space of generators V is clear from the context.

Artin and Schelter [1] introduced a class of regular algebras sharing some "good" homological properties with the polynomial algebra $\mathbb{K}[V]$. These algebras were dubbed Artin–Schelter regular algebras (AS-regular algebra for short).

[1] Strictly speaking $PS(V)$ is the creation parastatistics algebra, closed by creation operators alone.

Definition 2.1 (AS-regular algebras). A connected graded algebra $\mathscr{A} = \mathbb{K} \oplus \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \dots$ is called Artin–Schelter regular of dimension d if

(i) $\mathscr{A}$ has finite global dimension d,
(ii) $\mathscr{A}$ has finite Gelfand–Kirillov dimension,
(iii) $\mathscr{A}$ is Gorenstein, i.e., $\mathrm{Ext}^i_{\mathscr{A}}(\mathbb{K}, \mathscr{A}) = \delta^{i,d}\mathbb{K}$.

A general theorem claims that the UEA of a finite dimensional positively graded Lie algebra is an AS-regular algebra of global dimension equal to the dimension of the Lie algebra [5]. Hence the parastatistics algebra PS is AS-regular of global dimension $d = \frac{\dim V(\dim V+1)}{2}$. In particular the finite global dimension of PS implies that the ground field $\mathbb{K}$ has a minimal resolution $P_\bullet$ by projective left PS-modules P_n

$$P_\bullet : \qquad 0 \to P_d \to \cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} \mathbb{K} \to 0. \tag{3}$$

Here $\mathbb{K}$ is a trivial left PS-module, the action being defined by the projection ε onto $PS_0 = \mathbb{K}$. Since PS is graded and, in the category of graded modules projective module, is the same as free module [2], we have $P_n \cong PS \otimes E_n$ where E_n are finite dimensional vector spaces.

The minimal projective resolution is unique (up to an isomorphism). Minimality implies that the complex $\mathbb{K} \otimes_{PS} P_\bullet$ has "zero differentials" hence

$$H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet = E_n.$$

One can calculate the derived functor $\mathrm{Tor}^{PS}_n(\mathbb{K}, \mathbb{K})$ using the resolution $P_\bullet$, it yields

$$\mathrm{Tor}^{PS}_n(\mathbb{K}, \mathbb{K}) = E_n. \tag{4}$$

The data of a minimal resolution of $\mathbb{K}$ by free PS-modules provides an easy way to find $\mathrm{Tor}^{PS}_n(\mathbb{K}, \mathbb{K})$. Conversely if the spaces $\mathrm{Tor}^{PS}_n(\mathbb{K}, \mathbb{K})$ are known, then one can construct a minimal free resolution of $\mathbb{K}$.

The Gorenstein property guarantees that when applying the functor $\mathrm{Hom}_{PS}(-, PS)$ to the minimal free resolution $P_\bullet$ we get another minimal free resolution $P^\bullet := \mathrm{Hom}_{PS}(P_\bullet, PS)$ of $\mathbb{K}$ by right PS-modules

$$P^\bullet : \quad 0 \leftarrow \mathbb{K} \leftarrow P_d' \leftarrow \cdots \leftarrow P_n' \leftarrow \cdots \leftarrow P_2' \leftarrow P_1' \leftarrow P_0' \leftarrow 0 \tag{5}$$

with $P_n' \cong E_n^* \otimes PS$. Note that by construction $E_n^* = \mathrm{Ext}^n_{PS}(\mathbb{K}, \mathbb{K})$, thus one has vector space isomorphisms [2]

$$E_n \cong E_n^* \cong \mathrm{Tor}^{PS}_n(\mathbb{K}, \mathbb{K}) \cong \mathrm{Ext}^n_{PS}(\mathbb{K}, \mathbb{K}). \tag{6}$$

The Gorenstein property is the analog of the Poincaré duality since it implies

$$E_{d-n}^* \cong E_n.$$

The finite global dimension d of PS and the Gorenstein condition imply that its Yoneda algebra

$$\mathscr{E}^\bullet := \mathrm{Ext}^\bullet_{PS}(\mathbb{K},\mathbb{K}) \cong \bigoplus_{n=0}^{d} E_n^*$$

is Frobenius [12].

3 Homology and Cohomology of $\mathfrak{g}$

Let us first recall that the standard Chevalley-Eilenberg chain complex $C_\bullet(\mathfrak{g}) = (U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g}, d_p)$ where the differential reads

$$\begin{aligned} d_p(u \otimes x_1 \wedge \ldots \wedge x_p) &= \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_p \\ &\quad + \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_p \end{aligned} \tag{7}$$

provides a non-minimal projective (in fact free) resolution of $\mathbb{K}$, $C(\mathfrak{g}) \xrightarrow{\varepsilon} \mathbb{K}$. With the latter resolution $C_\bullet(\mathfrak{g})$ one calculates homologies of the derived complex $\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})$

$$E_n = \mathrm{Tor}_n^{PS}(\mathbb{K},\mathbb{K}) \cong H_n(\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})) = H_n(\mathfrak{g},\mathbb{K}),$$

coinciding with the homologies $H_n(\mathfrak{g},\mathbb{K})$ of the Lie algebra $\mathfrak{g}$ with trivial coefficients. The derived complex $\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})$ is the chain complex with degrees $\bigwedge^\bullet \mathfrak{g} = \mathbb{K} \otimes_{PS} PS \otimes \bigwedge^\bullet \mathfrak{g}$ and differentials $\partial_p := id \otimes_{PS} d_p : \bigwedge^p \mathfrak{g} \to \bigwedge^{p-1} \mathfrak{g}$.

The differential ∂ is induced by the Lie bracket $[\cdot,\cdot] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ of the graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. It identifies a pair of degree 1 generators $e_i, e_j \in \mathfrak{g}_1$ with one degree 2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \mathfrak{g}_2$. The differential ∂_p is the extension of the mapping $\partial_2 := -[\cdot,\cdot]$ on the exterior powers $\bigwedge^p \mathfrak{g}$. In greater details the chain degrees read

$$\bigwedge^p \mathfrak{g} = \bigwedge^p (V \oplus \bigwedge^2 V) = \bigoplus_{s+r=p} \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V) \tag{8}$$

and differentials $\partial_{p=r+s} : \bigwedge^s(\bigwedge^2 V) \otimes \bigwedge^r(V) \to \bigwedge^{s+1}(\bigwedge^2 V) \otimes \bigwedge^{r-2}(V)$ are given by

$$\begin{aligned} \partial_p :\ & e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \otimes e_1 \wedge \ldots \wedge e_r \mapsto \\ & \sum_{i<j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \otimes e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_r. \end{aligned}$$

In duality, one has the cochain complex $\mathrm{Hom}_{PS}(C(\mathfrak{g}),\mathbb{K}) = (\bigwedge^\bullet \mathfrak{g}^*, \delta)$ calculating the cohomology[2]

$$E_n^* = \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K}) \cong H^n(\mathrm{Hom}_{PS}(C(\mathfrak{g}),\mathbb{K})) = H^n(\mathfrak{g},\mathbb{K}). \tag{9}$$

The coboundary map $\delta^p : \bigwedge^p \mathfrak{g}^* \to \bigwedge^{p+1}\mathfrak{g}^*$ is transposed to the differential ∂_{p+1}

$$\delta^p: \; e^*_{i_1 j_1} \wedge \ldots \wedge e^*_{i_s j_s} \otimes e^*_1 \wedge \ldots \wedge e^*_r \mapsto$$
$$\sum_{k=1}^{s} \sum_{i_k<j_k} (-1)^{i+j} e^*_{i_1 j_1} \wedge \ldots \wedge \hat{e}^*_{i_k j_k} \wedge \ldots \wedge e^*_{i_s j_s} \otimes e^*_{i_k} \wedge e^*_{j_k} \wedge e^*_1 \wedge \ldots \wedge \ldots \wedge e^*_r, \tag{10}$$

it is (up to a conventional sign) the extension of the dualization of the Lie bracket $\delta^1 := [\cdot,\cdot]^* : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ by the Leibniz rule (i.e., as derivation).

The algebra $\bigwedge^\bullet \mathfrak{g}^*$ is super-commutative (or graded-commutative) so $(\bigwedge^\bullet \mathfrak{g}^*, \delta)$ is a *(super-)commutative* DGA.

It is important that in the complexes $(\bigwedge^p \mathfrak{g}, \partial_p)$ and $(\bigwedge^p \mathfrak{g}^*, \delta^p)$ two different degrees are involved; one is the homological degree $p := r+s$ counting the number of $\mathfrak{g}$-generators, while the second is the tensor degree $t := 2s+r$ (also called weight). The differentials ∂ and δ preserve the tensor degree t but the spaces $H_n(\mathfrak{g},\mathbb{K})$ and $H^n(\mathfrak{g},\mathbb{K})$ are not homogeneous in t.

4 Littlewood Formula and *PS*

In this section we review the beautiful result of Józefiak and Weyman [7] giving a representation-theoretic interpretation of the Littlewood formula

$$\prod_i (1-x_i) \prod_{i<j} (1-x_i x_j) = \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} s_\lambda(x). \tag{11}$$

Here the sum is over the self-dual Young diagrams λ, $s_\lambda(x)$ stands for the Schur function and $r(\lambda)$ stands the rank of λ which is the number of diagonal boxes in λ.

An irreducible $GL(V)$-module V_λ is called Schur module, it has a basis labelled by semistandard Young tableaux which are fillings of the Young diagram λ with the numbers of the set $\{1,\ldots,\dim V\}$. The action of the linear group $GL(V)$ on the space V of the generators of the Lie algebra $\mathfrak{g}$ induces a $GL(V)$-action on the UEA $PS = U\mathfrak{g} \cong S(V \oplus \Lambda^2 V)$ and on the space $\bigwedge^\bullet \mathfrak{g} \cong \bigwedge^\bullet (V \oplus \bigwedge^2 V)$. The algebra $PS(V)$ has remarkable property, it is a model of the linear group $GL(V)$, in the sense that it contains every polynomial finite-dimensional irreducible representation V_λ of $GL(V)$ once and only once

$$PS(V) \cong \oplus_\lambda V_\lambda.$$

[2] In the presence of metric one has $\delta := \partial^*$ (see below)

A nice combinatorial proof of this fact was given by Chaturvedi [3]. The $GL(V)$-model $PS(V)$ enjoys the universal property that every parastatistics Fock representation specified by the parastatistics order $p \in \mathbb{N}_0$ is a factor of $PS(V)$ [4,10].

The differential ∂ commutes with the $GL(V)$ action and the homology $H_\bullet(\mathfrak{g},\mathbb{K})$ is also a $GL(V)$-module. The decomposition of the $GL(V)$-module $H_n(\mathfrak{g},\mathbb{K})$ into irreducible polynomial representations V_λ is given by the following theorem;

Theorem 4.1 (Józefiak and Weyman [7], Sigg [13]). *The homology $H_\bullet(\mathfrak{g},\mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ decomposes into irreducible $GL(V)$-modules*

$$H_n(\mathfrak{g},\mathbb{K}) = H_n(\textstyle\bigwedge^\bullet \mathfrak{g},\partial) \cong \mathrm{Tor}_n^{PS}(\mathbb{K},\mathbb{K})(V) \cong \displaystyle\bigoplus_{\lambda:\lambda=\lambda'} V_\lambda \tag{12}$$

where the sum is over self-conjugate Young diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

The data $H_n(\mathfrak{g},\mathbb{K}) = \mathrm{Tor}_n^{PS}(\mathbb{K},\mathbb{K})$ encodes the minimal free resolution $P_\bullet$ (cf. (3)).

The acyclicity of the complex $P_\bullet$ implies an identity about the $GL(V)$-characters

$$ch\,PS(V)\,.\,ch\left(\bigoplus_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} V_\lambda\right) = 1.$$

The character of a Schur module V_λ is the Schur function, $ch\,V_\lambda = s_\lambda(x)$. Due to the Poincaré–Birkhoff–Witt theorem $ch\,PS(V) = ch\,S(V \oplus \bigwedge^2 V)$ thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i<j} \frac{1}{(1-x_i x_j)} \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} s_\lambda(x) = 1. \tag{13}$$

But the latter identity is nothing but rewriting of the Littlewood identity (11). The moral is that the Littlewood identity reflects a homological property of the algebra PS, namely the above particular structure of the minimal projective (free) resolution of $\mathbb{K}$ by PS-modules.

5 Homotopy Algebras A_∞ and C_∞

Definition 5.2 (A_∞-algebra). A homotopy associative algebra, or A_∞-algebra, over $\mathbb{K}$ is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)

$$m_n : A^{\otimes n} \to A, \qquad \deg(m_n) = 2 - n \qquad n \geq 1$$

satisfying the Stasheff identities $\mathbf{SI(n)}$ for $n \geq 1$

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \qquad \mathbf{SI(n)}$$

where the sum runs over all decompositions $n = r + s + t$.

Here we assume the Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. We define the shuffle product $Sh_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A^{\otimes p+q}$ throughout the expression

$$(a_1 \otimes \ldots \otimes a_p) \sqcup (a_{p+1} \otimes \ldots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma)\, a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all (p,q)-shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q)$.

Definition 5.3 (C_∞-algebra [8]). A homotopy commutative algebra, or C_∞-algebra, is an A_∞-algebra $\{A, m_n\}$ such that each operation m_n vanishes on non-trivial shuffles

$$m_n\left((a_1 \otimes \ldots \otimes a_p) \sqcup (a_{p+1} \otimes \ldots \otimes a_n)\right) = 0, \quad 1 \le p \le n-1. \tag{14}$$

In particular for m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞-algebra such that $m_n = 0$ for $n \ge 3$ is a (super-)commutative DGA.

A morphism of two A_∞-algebras A and B is a family of graded maps $f_n : A^{\otimes n} \to B$ for $n \ge 1$ with $\deg f_n = 1-n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes r}) = \sum_{1 \le r \le n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r})$$

where the sum is on all decompositions $i_1 + \ldots + i_r = n$ and the sign on RHS is determined by $S = \sum_{k=1}^{r-1}(r-k)(i_k - 1)$. The morphism f is a *quasi-isomorphism of A_∞-algebras* if f_1 is a quasi-isomorphism. It is strict if $f_i = 0$ for all $i \neq 1$. The identity morphism of A is the strict morphism f such that f_1 is the identity of A.

A morphism of C_∞-algebras is a morphism of A_∞-algebras vanishing on non-trivial shuffles $f_n\left((a_1 \otimes \ldots \otimes a_p) \sqcup (a_{p+1} \otimes \ldots \otimes a_n)\right) = 0$, $1 \le p \le n-1$.

6 Homotopy Transfer Theorem

Lemma 6.1. *Every cochain complex (A,d) of vector spaces over a field $\mathbb{K}$ has its cohomology $H^\bullet(A)$ as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex (A,d) such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$ where H^n is the cohomology and B^n is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \to A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n \oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \underset{i}{\overset{p}{\rightleftarrows}} H^n$ are chain homomorphisms (satisfying the additional conditions $hh = 0$, $hi = 0$ and $ph = 0$). With these choices done the complex $(H^\bullet(A), 0)$ is a deformation retract of (A,d)

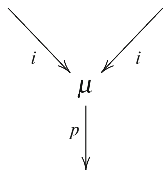
$$h \circlearrowright \quad (A,d) \underset{i}{\overset{p}{\rightleftarrows}} (H^\bullet(A),0), \qquad pi = Id_{H^\bullet(A)}, \qquad ip - Id_A = dh + hd.$$

Let now (A,d,μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d. The cochain complexes (A,d) and its contraction $H^\bullet(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞-structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$. For a friendly introduction to homotopy transfer theorems in much boarder context we send the reader to the textbook [11], see Chap. 9.

Theorem 6.2 (Kadeishvili [8]). *Let (A,d,μ) be a (commutative) DGA over a field $\mathbb{K}$. There exists a A_∞-algebra (C_∞-algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$-quasi-isomorphism $f_i : (\otimes^i H^\bullet(A), \{m_i\}) \to (A, \{d,\mu,0,0,\dots\})$ such that the inclusion $f_1 = i : H^\bullet(A) \to A$ is a cocycle-choosing homomorphism of cochain complexes. The differential m_1 on $H^\bullet(A)$ is zero ($m_1 = 0$) and m_2 is strictly associative operation induced by the multiplication on A. The resulting structure is unique up to quasi-isomorphism.*

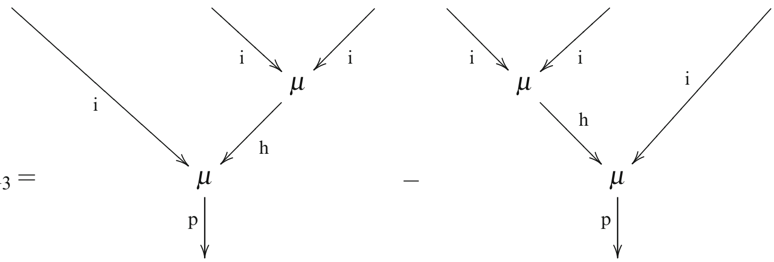
Kontsevich and Soibelman [9] gave an explicit expressions for the higher operations of the induced A_∞-structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion i, the root by the projection p the vertices by the product μ of the (commutative) DGA (A,d,μ) and the internal edges by the homotopy h. The C_∞-structure implies additional symmetries on trees. We will make use of the graphic representation for the binary operation on $H^\bullet(A)$

$m_2(x,y) := p\mu(i(x),i(y))$ or $m_2 =$



and the ternary one $m_3(x,y,z) = p\mu(i(x),h\mu(i(y),i(z))) - p\mu(h\mu(i(x),i(y)),i(z))$ being the sum of two planar binary trees with three leaves

$m_3 =$



Theorem 6.3. *The cohomology $H^\bullet(\mathfrak{g},\mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K},\mathbb{K})$ of the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \bigwedge^2 V$ is a homotopy commutative algebra which is generated in degree* 1 *(i.e., in $H^1(\mathfrak{g},\mathbb{K})$) by the operations m_2 and m_3.*

Sketch of the proof. Let us choose a metric $g(\cdot,\cdot) = \langle\cdot,\cdot\rangle$ on the vector space V and an orthonormal basis $\langle e_i, e_j\rangle = \delta_{ij}$. The choice induces a metric on $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$.

Due to the isomorphisms $\mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K}) \cong \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K})$ (see (6)) and $V \cong V^*$ the Theorem 4.1 implies the decomposition of $H^\bullet(\mathfrak{g},\mathbb{K})$ into irreducible $GL(V)$-modules

$$H^n(\mathfrak{g},\mathbb{K}) \cong H^n(\textstyle\bigwedge \mathfrak{g}^*,\delta) \cong \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K})(V^*) \cong \displaystyle\bigoplus_{\lambda:\lambda=\lambda'} V_\lambda$$

where the sum is over self-conjugate diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

In the presence of metric g the differential δ is identified with the adjoint of ∂, $\delta :\overset{g}{=} \partial^*$ while ∂ plays the role of a homotopy. In view of Lemma 6.1 we have the cohomology $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*,\delta^\bullet)$ as deformation retract of the complex $(\bigwedge^\bullet \mathfrak{g}^*,\delta^\bullet)$,

$$pi = Id_{H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)}, \quad ip - Id_{\bigwedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta, \quad \delta^* \overset{g}{=} \partial.$$

Here the projection p identifies the subspace $\ker\delta \cap \ker\delta^*$ with $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\mathrm{im}\delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

We apply the Kadeishvili homotopy transfer Theorem 6.2 for the commutative DGA $(\bigwedge^\bullet \mathfrak{g}^*,\mu,\delta^\bullet)$ and its deformation retract $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g},\mathbb{K})$ and conclude that the cohomology $H^\bullet(\mathfrak{g},\mathbb{K})$ is a C_∞-algebra.

The Kontsevich and Soibelman tree representations of the operations m_n provide explicit expressions. Let us take μ to be the super-commutative product $\wedge$ on the DGA $(\bigwedge^\bullet \mathfrak{g}^*,\delta^\bullet)$. The projection p maps onto the Schur modules V_λ with $\lambda = \lambda'$.

The binary operation on the degree 1 generators $e_i \in H^1(\mathfrak{g},\mathbb{K})$ is trivial, one gets

$$m_2(e_i,e_j) = p(e_i \wedge e_j) = 0 \quad p(V_{(1^2)}) = 0.$$

Hence $H^\bullet(\mathfrak{g},\mathbb{K})$ could not be generated in $H^1(\mathfrak{g},\mathbb{K})$ as algebra with product m_2.

The ternary operation m_3 restricted to $H^1(\mathfrak{g},\mathbb{K})$ is nontrivial, indeed one has

$$\begin{aligned} m_3(e_i,e_j,e_k) &= p\left\{e_i \wedge \partial(e_j \wedge e_k) - \partial(e_i \wedge e_j) \wedge e_k\right\} = p\left\{e_{ij} \wedge e_k - e_i \wedge e_{jk}\right\} \\ &= p\left\{(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) - e_{ki} \wedge e_j\right\} = e_{ik} \wedge e_j \in H^2(\mathfrak{g},\mathbb{K}) \end{aligned}$$

The completely antisymmetric combination in the brackets $(\dots)$ spans the Schur module $V_{(1^3)}$, $p(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$ yields a Jacobi-type identity. The monomials $e_{ij} \wedge e_k$ modulo $V_{(1^3)}$ span a Schur module $V_{(2,1)} \in H^2(\mathfrak{g},\mathbb{K})$ with basis in bijection with the semistandard Young tableaux $e_{ik} \wedge e_j \leftrightarrow \begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array}$ and $e_{ij} \wedge e_k \leftrightarrow \begin{array}{|c|c|}\hline i & k \\ \hline j \\ \cline{1-1}\end{array}$.

We check the symmetry condition on ternary operation m_3 in C_∞-algebra; indeed m_3 vanishes on the (signed) shuffles $Sh_{1,2}$ and $Sh_{2,1}$

$$m_3(e_i \sqcup\!\sqcup e_j \otimes e_k) = m_3(e_i,e_j,e_k) - m_3(e_j,e_i,e_k) + m_3(e_j,e_k,e_i) = 0 = m_3(e_i \otimes e_j \sqcup\!\sqcup e_k).$$

The operation m_n is bigraded by homological and tensor gradings of bidegree $(p,t) = (2-n,0)$. The bi-grading impose the vanishing of many higher products.

On the level of Schur modules the ternary operation glues three fundamental $GL(V)$-representations $V_\square$ into a Schur module $V_{(2,1)}$. By iteration of the process of gluing boxes we generate all elementary hooks $V_k := V_{(1^k,k+1)}$,

$$m_3(V_\square,V_\square,V_\square) = V_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}}, \quad m_3\left(V_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}},V_\square,V_\square\right) = V_{\begin{smallmatrix}\square&\square&\square\\ \square&&\\ \square&&\end{smallmatrix}}, \ldots, m_3(V_k,V_0,V_0) = V_{k+1}.$$

In our context the more convenient notation for Young diagrams is due to Frobenius: $\lambda := (a_1,\ldots,a_r|b_1,\ldots b_r)$ stands for a diagram λ with a_i boxes in the i-th row on the right of the diagonal, and with b_i boxes in the i-th column below the diagonal and the rank $r = r(\lambda)$ is the number of boxes on the diagonal.

For self-dual diagrams $\lambda = \lambda'$, i.e., $a_i = b_i$ we set $V_{a_1,\ldots,a_r} := V_{(a_1,\ldots,a_r|a_1,\ldots a_r)}$ when $a_1 > a_2 > \ldots > a_r \geq 0$ (and set the convention $V_{a_1,\ldots,a_r} := 0$ otherwise). Any two elementary hooks V_{a_1} and V_{a_2} can be glued together by the binary operation m_2, the decomposition of $m_2(V_{a_1},V_{a_2}) \cong m_2(V_{a_2},V_{a_1})$ is given by

$$m_2(V_{a_1},V_{a_2}) = V_{a_1,a_2} \oplus \left(\bigoplus_{i=1}^{a_2} V_{a_1+i,a_2-i}\right) \quad a_1 \geq a_2$$

where the "leading" term V_{a_1,a_2} has the diagram with minimal height. Hence any m_2-bracketing of the hooks $V_{a_1}, V_{a_2} \ldots, V_{a_r}$ yields[3] a sum of $GL(V)$-modules

$$m_2(\ldots m_2(m_2(V_{a_1},V_{a_2}),V_{a_3}),\ldots,V_{a_r}) = V_{a_1,\ldots,a_r} \oplus \ldots$$

whose module with minimal height is precisely $V_{a_1,\ldots,a_r}$. We conclude that all elements in the C_∞-algebra $H^\bullet(\mathfrak{g},\mathbb{K})$ can be generated in $H^1(\mathfrak{g},\mathbb{K})$ by m_2 and m_3. $\square$

Acknowledgements We are grateful to Jean-Louis Loday for many enlightening discussions and his encouraging interest. The work was supported by the French–Bulgarian Project Rila under the contract Egide-Rila N112.

[3] The operation m_2 is associative thus the result does not depend on the choice of the bracketing.

References

1. Artin, M., Schelter, W.F.: Graded algebras of global dimension 3. Adv. Math. **66**, 171–216 (1987)
2. Cartan, H.: Homologie et cohomologie d' une algèbre graduée. Séminaire Henri Cartan **11**, 1–20 (1958)
3. Chaturvedi, S.: Canonical partition functions for parastatistical systems of any order. Phys. Rev. E **54**, 1378–1382 (1996)
4. Dubois-Violette, M., Popov, T.: Homogeneous algebras, statistics and combinatorics. LMP **61**, 159–170 (2002)
5. Floystad, G., Vayne, J.E.: Artin–Schelter regular algebras of dimension five. Algebra, Geometry, and Mathematical Physics, Banach Center Publications, **93**, 19–39 (2011)
6. Green, H.S.: A generalized method of field quantization. Phys. Rev. **90**, 270–273 (1953)
7. Józefiak, T., Weyman, J.: Representation-theoretic interpretation of a formula of D. E. Littlewood. Math. Proc. Cambridge Phil. Soc. **103**, 193–196 (1988)
8. Kadeishvili, T.: The A_∞-algebra structure and cohomology of Hochschild and Harrison. Proc. Tbil. Math. Inst. **91**, 19–27 (1988)
9. Kontsevich, M., Soibelman, Y.: Deformations of algebras over operads and the Deligne conjecture. Math. Phys. Stud., vol. **21**, pp. 255–307. Kluwer Academic, Dordrecht (2000)
10. Loday, J.-L., Popov, T.: Parastatistics algebra, Young tableaux and the super plactic monoid. Int. J. Geom. Meth. Mod. Phys. **5**, 1295–1314 (2008)
11. Loday, J.-L., Vallette, B.: Algebraic Operads, Grundlehren Math. Wiss., vol. 346. Springer, Heidelberg (2012)
12. Lu, D.-M., Palmieri, J.H., Wu, Q.-S., Zhang, J.J.: A_∞-algebras for ring theorists. Algebra Colloquium, **11**(1), 91–128 (2004) arXiv[math.RA]0710.5492
13. Sigg, S.: Laplacian and homology of free 2-step nilpotent Lie algebras. J. Algebra **185**, 144–161 (1996)

Fixed Point Factorization

Elaine Beltaos

Abstract Fixed point factorization is a significant simplification of the modular data of an RCFT involving primaries fixed by simple-currents. The WZW models possess this feature. In the case of the A-series, fixed point factorization has been used to calculate NIM-rep coefficients, which has allowed for the computation of D-brane charges and charge groups in string theory. In this paper, we discuss fixed point factorization and its application to NIM-reps.

1 Introduction

The characters of an affine algebra obey a modularity property (see (1), (2)). This modularity is governed by two matrices S and T (the so-called modular data) corresponding to the transformations $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + 1$ resp., where τ is in the upper half plane. The S-matrix is the more important of the two.

In [16], Gannon–Walton found that the modular S-matrix entries $S_{\lambda\mu}$ for $A_r^{(1)}$ factored into S-matrix entries for a smaller-rank A-algebra, when at least one of λ or μ is fixed by a simple-current. This fixed point factorization was used later as a tool by Gaberdiel–Gannon to compute NIM-rep coefficients (a NIM-rep is a nonnegative integer representation of the fusion ring), which in turn allowed for the computation of D-brane charges and charge groups [10, 11]. Their result showed that the NIM-rep for the non-simply connected Lie group $SU(n)/\mathbb{Z}_d$ (where $d \neq 1$ is a divisor of n) coincides with the fusion ring of the simply connected $SU(n/d)$.

The existence and utility of the A-series fixed point factorization leads to the question of whether this feature is particular to the A-algebras or occurs in more

E. Beltaos (✉)
Grant MacEwan University, 10700 - 104 Avenue Edmonton, Alberta, T5J 4S2, Canada
e-mail: beltaose@macewan.ca

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_38,

(all?) RCFTs. Indeed, the WZW models do possess this feature [2].[1] In this paper, we present an example of fixed point factorization, and we apply our formula towards calculating the corresponding NIM-rep coefficients.

1.1 Notation

By $X_r^{(1)}$, where $X \in \{A,B,C,D\}$, we mean the nontwisted affine algebra with underlying simple finite dimensional algebra X_r. We let $P_+(X_r)$ (resp. $P_+^k(X_r^{(1)})$) denote the set of highest weights (resp. level k highest weights) for X_r (resp. $X_r^{(1)}$). We denote the n^{th} fundamental weight by Λ_n, $n = 0,\ldots,r$. We use $\mathbb{N}$ to denote the set of nonnegative integers, $\mathbb{H}$ to denote the upper half plane, and * to denote complex conjugation.

2 Modular Data

The characters ch_λ of an affine algebra $X_r^{(1)}$[2] (specialized to τ in the upper half plane) satisfy the modularity properties:

$$\text{ch}_\lambda(-1/\tau) = \sum_\mu S_{\lambda\mu}\text{ch}_\mu(\tau) \tag{1}$$

$$\text{ch}_\lambda(1+\tau) = \sum_\mu T_{\lambda\mu}\text{ch}_\mu(\tau), \tag{2}$$

where the sum is over all $\mu \in P_+^k$.

The coefficient matrices S and T defined by (1) and (2) generate a representation of the modular group $\text{SL}_2(\mathbb{Z})$ of 2×2 determinant 1 integer matrices, namely:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S \quad ; \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T.$$

The matrices S and T are called *modular data*; their entries lie in some cyclotomic field. Modular data satisfy several properties, including:

- T is diagonal and of finite order ($T^N = I$ for some positive integer N)
- S is unitary and symmetric ($SS^\dagger = I$, where † denotes complex conjugate transpose)
- $(ST)^3 = S^2 =: C$, where C is an order-two permutation matrix called charge-conjugation
- $S_{0\lambda} \geq S_{00} > 0$ for all $\lambda \in P_+^k$, where $0 = k\Lambda_0$ is the vacuum.

[1] The exceptional algebras $E_6^{(1)}$ and $E_7^{(1)}$ have nontrivial simple-currents. These are yet to be worked out, although current work suggests that fixed point factorization formulas should be attainable.

[2] For reasons of brevity, we focus here on the nontwisted case, although fixed point factorizations exist for the twisted algebras as well [2].

Modular data occur in many different areas in mathematics (e.g. finite groups [7]). For a more detailed introduction to modular data, see [12].

A key property of modular data is that the numbers defined by *Verlinde's formula*:

$$N_{\lambda\mu}^{\nu} := \sum_{\kappa \in P_+^k} \frac{S_{\lambda\kappa} S_{\mu\kappa} S_{\nu\kappa}^*}{S_{0\kappa}} \tag{3}$$

are nonnegative integers. The $N_{\lambda\mu}^{\nu}$ are called *fusion coefficients*; they are structure constants for a commutative, associative ring called the *fusion ring* (i.e. the ring multiplication is given by $x_\lambda * x_\mu = \sum_\nu N_{\lambda\mu}^{\nu} x_\nu$). Equation (3) says that the fusion matrices N_λ defined by $(N_\lambda)_{\mu,\nu} := N_{\lambda\mu}^{\nu}$ are simultaneously diagonalized by S, with eigenvalues $S_{\lambda\kappa}/S_{0\kappa}$. A nonnegative integer representation of a fusion ring is called *NIM-rep*. We will discuss NIM-reps in Sect. 4.

One of the simplest examples of modular data is for $A_1^{(1)}$. The S and T-matrices are

$$S_{\lambda\mu} = \sqrt{\frac{2}{k+2}} \sin\left[\pi \frac{(\lambda+1)(\mu+1)}{k+2}\right]$$

$$T_{\lambda\lambda} = \exp\left[-\frac{\pi i}{4}\right] \exp\left[\pi \mathrm{i} \frac{(\lambda+1)^2}{2(k+2)}\right],$$

where $0 \le \lambda, \mu \le k$, and the fusion coefficients are

$$N_{\lambda\mu}^{\nu} = \begin{cases} 1 & \text{if } \nu \equiv_2 \lambda+\mu \text{ and } |\lambda-\mu| \le \nu \le \min\{\lambda+\mu, 2k-\lambda-\mu\} \\ 0 & \text{else} \end{cases}.$$

More generally, a useful formula for the S-matrix is given by Kac–Peterson in [18]:

$$S_{\lambda\mu} = \kappa^{-r/2} s \sum_{w \in \overline{W}} (\det w) \exp\left[-2\pi i \frac{w(\overline{\lambda+\rho}) \cdot (\overline{\mu+\rho})}{\kappa}\right], \tag{4}$$

where $\overline{W}$ is the X_r Weyl group, $\overline{\rho} = (1, \ldots, 1)$ is the Weyl vector, κ and s are constants depending on r and k, and fusion coefficients can be calculated via the Kac–Walton formula [17, 19].

Comparing (4) with the Weyl character formula yields

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \overline{ch}_{\overline{\lambda}}\left(-2\pi i \frac{\overline{\mu+\rho}}{\kappa}\right) =: \chi_\lambda(\mu), \tag{5}$$

which relates ratios of the S-matrix to the finite dimensional simple characters at elements of finite order. We proceed with finding a fixed point factorization for $\chi_\lambda(\mu)$ as this is equivalent to finding one for the S-matrix. Equation (5) is the key to our fixed point factorization formulas, as well as the reason for the simplification of our problem to the fundamental representations.

A *simple-current* of $X_r^{(1)}$, level k, is a weight $\nu \in P_+^k$ for which there exists a permutation J of P_+^k such that $N_{\nu,\lambda}^{\mu} = \delta_{\mu,J\lambda}$ with $\nu = J0$. These are precisely those ν such that $S_{0\nu} = S_{00}$. For the WZW models, simple-currents correspond to extended Dynkin diagram symmetries J [8], with the exception of $E_8^{(1)}$, level 2.[3] The permutation on P_+^k can be realized by labelling each node of the extended diagram with a Dynkin label of a weight λ—the diagram symmetry permutes the labels yielding $J\lambda$. We will refer to this permutation also as a simple-current. The set of simple-currents for $X_r^{(1)}$ forms an Abelian group $\mathscr{J}$, isomorphic to the centre of the (universal cover of) the Lie group (with exception $E_8^{(1)}$ level 2). For $A_r^{(1)}$, $\mathscr{J} \cong \mathbb{Z}_{r+1}$, for $B_r^{(1)}$ and $C_r^{(1)}$, $\mathscr{J} \cong \mathbb{Z}_2$ and for $D_r^{(1)}$, $\mathscr{J} \cong \mathbb{Z}_4$ (resp. $\mathbb{Z}_2 \times \mathbb{Z}_2$) if r is odd (resp. even). We define a *fixed point* of a simple-current as an element φ of P_+^k such that $J\varphi = \varphi$.

Closely related to modular data is the *modular invariant partition function* for the torus

$$\mathscr{Z}(\tau) = \sum_{\lambda,\mu \in P_+^k} M_{\lambda\mu}\chi_\lambda(\tau)\chi_\mu^*(\tau), \tag{6}$$

for $\tau \in \mathbb{H}$, describing how state-space decomposes into a finite sum $\oplus M_{\lambda\mu}\lambda \otimes \mu$ of $\mathscr{V} \otimes \mathscr{V}'$-modules, where $\mathscr{V}$, $\mathscr{V}'$ are the vertex operator algebras of holomorphic and anti-holomorphic fields resp. Uniqueness of the vacuum yields $M_{00} = 1$. Equivalently, we can consider the matrix M of coefficients, called a *modular invariant*. Thus a modular invariant is a matrix, indexed by P_+^k, satisfying the following properties:

$$\begin{aligned} M_{00} &= 1 \\ M_{\lambda\mu} &\in \mathbb{N} \text{ for all } \lambda,\mu \in P_+^k \\ SM = MS \quad &; \quad TM = MT, \end{aligned}$$

where S and T are the matrices defined in (1) and (2). The third property is modular invariance $\mathscr{Z}(-1/\tau) = \mathscr{Z}(1+\tau) = \mathscr{Z}(\tau)$.

It is not difficult to prove that for a given level k, there are finitely many modular invariants. The first modular invariant classification was achieved by Cappelli–Itzykson–Zuber, for $A_1^{(1)}$ [6] (for the "modern" approach, see [13]); the modular invariants display an A-D-E pattern. More modular invariant classifications followed (e.g. [14, 15]). Few complete modular invariant classifications exist, due to the existence of exceptional ($\mathscr{E}$-type) invariants occurring at low levels. However, for sufficiently high level k, modular invariants are generic and fall into two classes: the $\mathscr{A}$-type invariants, comprising the identity and its conjugations, and the $\mathscr{D}$-type invariants, comprising the simple-current invariants (and their conjugations), which are constructed in a natural way from the group of simple-currents. These have the form

$$M[J]_{\lambda\mu} = \sum_{\ell=1}^{n} \delta_{J^\ell\lambda\,\mu}\,\delta^{\mathbb{Z}}\left(Q_J(\lambda) + \ell r_J\right), \tag{7}$$

[3] But not all diagram symmetries are simple-currents.

where $\delta^{\mathbb{Z}}(x) = 1$ if $x \in \mathbb{Z}$ and 0 else, and $Q_J(\lambda)$ and r_J are rational numbers depending on the particular algebra. The matrix $M[J]$ is a modular invariant exactly when $T_{J0,J0}T^*_{00}$ is an n^{th} root of unity (where J has order n).

3 Fixed Point Factorisation

In this section, we give examples of fixed point factorization formulas for the algebras $A_r^{(1)}$ and $C_r^{(1)}$ (r odd).

A classical result (see e.g. [5]) is that the character ring of the finite dimensional simple algebra X_r is generated by the fundamental characters, that is, for each $\overline{\lambda} \in P_+(X_r)$, there is a polynomial $P_{\overline{\lambda}}$ such that

$$\overline{\text{ch}}_{\overline{\lambda}}(\overline{\mu}) = P_{\overline{\lambda}}(\overline{\text{ch}}_{\overline{\Lambda}_1}(\overline{\mu}), \ldots, \overline{\text{ch}}_{\overline{\Lambda}_r}(\overline{\mu})). \tag{8}$$

Recall (5) relating $X_r^{(1)}$ S-matrix ratios to simple finite dimensional characters of X_r. Putting these characters into (8) yields

$$\chi_\lambda(\mu) = P_{\overline{\lambda}}(\chi_{\Lambda_1}(\mu), \ldots, \chi_{\Lambda_r}(\mu)), \tag{9}$$

which means that to show that a fixed point factorization exists, it is enough to show that it exists at the fundamental weights.

3.1 The A-Series

This was the first incidence of fixed point factorization. This case was worked out by Gannon–Walton in [16], where they found a fixed point factorization for all $\lambda \in P_+^k$. We include the case that λ is a fundamental weight, as that is our main interest. Let $n := r+1$, and let Λ_ℓ be the ℓ^{th} fundamental weight for $A_r^{(1)}$. Let d be a proper divisor of n, and let φ be fixed by J^d, where J is a rotation of $2\pi/n$. Then φ is of the form $(\varphi_0, \ldots, \varphi_{d-1}, \ldots, \varphi_0, \ldots, \varphi_{d-1})$, i.e., $\frac{n}{d}$ copies of the truncated weight $(\varphi_0; \ldots, \varphi_{d-1}) =: \varphi'$, where φ' is a level $\frac{kd}{n}$ weight for $A_{d-1}^{(1)}$. The fixed point factorization is

$$\chi_{\Lambda_\ell}(\varphi) = \begin{cases} \chi'_{\Lambda'_{\ell d/n}}(\varphi') & \text{if } \frac{n}{d} | \ell \\ 0 & \text{if } \frac{n}{d} \nmid \ell \end{cases}, \tag{10}$$

where primes denote $A_{d-1}^{(1)}$ level $\frac{kd}{n}$ quantities.

3.2 The C-Series, r Odd

Let Λ_ℓ be the ℓ^{th} fundamental weight for $C_r^{(1)}$, where r is odd, and let φ be a fixed point for the order-two simple-current $(\lambda_0;\ldots,\lambda_r) \mapsto (\lambda_r;\ldots,\lambda_0)$. To each fixed point of J, associate the truncated weight $\varphi' = (\varphi_{\frac{r-1}{2}};\ldots,\varphi_0)$, which is a level $k/2$ weight for $C^{(1)}_{\frac{r-1}{2}}$. We have the formula

$$\chi_{\Lambda_\ell}(\varphi) = \begin{cases} (-1)^m \chi'_{\Lambda'_m}(\varphi') & \text{if } \ell = 2m \\ 0 & \text{else} \end{cases}, \tag{11}$$

where $m \in \{0,\ldots,\frac{r-1}{2}\}$, and Λ'_m is the m^{th} $C^{(1)}_{\frac{r-1}{2}}$ fundamental weight.

3.3 Further Fixed Point Factorizations

The formulas we presented above are among the cleanest of the fixed point factorization formulas. However, all of the formulas are elegant, involving only linear combinations of fundamental weight characters of the smaller-rank algebra, with coefficients ± 1. The table below lists the smaller-rank algebras involved in each case. We refer to the smaller-rank algebra for $\mathfrak{g}$ as its "fixed point factorization (FPF) algebra".

For the D-series, J_v is the order-two graph automorphism exchanging the 0 and 1, and the $r-1$ and r nodes, and J_s is the order-two (resp. four) automorphism $\lambda \mapsto (\lambda_r;\ldots,\lambda_0)$ (resp. $\lambda \mapsto (\lambda_{r-1};\lambda_r,\lambda_{r-2},\ldots,\lambda_0)$) if r is even (resp. odd). The B and C simple-currents are the only nontrivial extended diagram automorphisms.

4 The NIM-rep

Recall that the fusion coefficients defined by Verlinde's formula (3) are nonnegative integers, and they are structure constants for the fusion ring. A *NIM-rep* [1] is a nonnegative integer matrix representation of a fusion ring. That is: for each λ, assign a matrix $\mathcal{N}_\lambda$ with nonnegative integer entries, satisfying

$$\mathcal{N}_\lambda \mathcal{N}_\mu = \sum_\nu N_{\lambda\mu}^\nu \mathcal{N}_\nu. \tag{12}$$

We also require $\mathcal{N}_0 = I$ and $\mathcal{N}_{C\lambda} = \mathcal{N}_\lambda^t$, where ^{t} denotes transpose. Two NIM-reps $\mathcal{N}$ and $\mathcal{N}'$ are equivalent if there is a permutation matrix P such that $\mathcal{N}_\lambda = P\mathcal{N}'_\lambda P^{-1}$ for all $\lambda \in P_+^k$. See [12] for an introduction to NIM-reps.

The $X_r^{(1)}$ fusion ring is a homomorphic image of the X_r character ring. Thus the NIM-rep, as a representation of the fusion ring, is a structure-preserving map and respects (9). To determine a NIM-rep, it is therefore enough to know the $\mathscr{N}_{\Lambda_i}$ at the fundamental weights.

Every RCFT has a modular invariant and a NIM-rep. For example, the identity modular invariant has NIM-rep $\lambda \mapsto N_\lambda$ (the fusion matrices). The matrices $\mathscr{N}_\lambda$ commute and are normal, so there is a unitary matrix Ψ simultaneously diagonalizing the NIM-rep (note the similarity with (3)):

$$\mathscr{N}_{\lambda x}^{y} = \sum_{\mu} \Psi_{x\mu} \frac{S_{\lambda\mu}}{S_{0\mu}} \Psi_{y\mu}^{\dagger}, \tag{13}$$

where $\lambda \in P_+^k(X_r^{(1)})$, x and y are boundary states (physically these are the possible states of the endpoints of open strings), and the sum is over all *exponents* μ.

Recall the simple-current invariant (7) of a simple-current J. Its NIM-rep can be constructed as follows. For each $\lambda \in P_+^k$, define ord(λ) be the order of the stabilizer of λ in $\langle J \rangle$. The boundary states are then given by J-orbits of weights $[\lambda, i] = \{J^i\lambda \mid i = 0, \dots, n-1\}$, $1 \le j \le \text{ord}(\lambda)$, where n is the order of J. The multiplicity of λ is $M[J]_{\lambda\lambda}$. We will identify the multiset of exponents with the set $\{(\lambda, j) \mid \lambda \in P_+^k, 1 \le j \le M_{\lambda\lambda}\}$ (if $M[J]_{\lambda\lambda} = 0$, then λ does not appear in this set).

For example, consider the algebra $C_3^{(1)}$ at level $k = 4$. The order-two simple-current group is generated by the simple-current $J : (\lambda_0; \lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_3; \lambda_2, \lambda_1, \lambda_0)$. The set P_+^4 contains thirty five primaries, three of which are fixed points: namely (omitting the zeroth label) $(1,1,1)$, $(2,2,0)$ and $(0,0,2)$ (these all have order two). The entries of $M[J]$ lie in the set $\{0,1,2\}$ with $M_{\lambda\mu} = 2$ iff $\lambda = \mu = \varphi$ where φ is a fixed point. There are twenty two boundary states and exponents.

The boundary states are: $[(0,0,0)]$, $[(1,0,0)]$, $[(0,1,0)]$, $[(0,0,1)]$, $[(2,0,0)]$, $[(1,1,0)]$, $[(1,0,1)]$, $[(0,2,0)]$, $[(0,1,1)]$, $[(3,0,0)]$, $[(2,1,0)]$, $[(2,0,1)]$, $[(1,2,0)]$, $[(0,3,0)]$, $[(4,0,0)]$, $[(3,1,0)]$, $[(1,1,1),1]$, $[(1,1,1),2]$, $[(2,2,0),1]$, $[(2,2,0),2]$, $[(0,0,2),1]$, $[(0,0,2),2]$, and the exponents are: $(0,0,0)$, $(0,1,0)$, $(2,0,0)$, $(1,0,1)$, $(0,2,0)$, $(2,1,0)$, $(0,3,0)$, $(4,0,0)$, $(0,1,2)$, $(3,0,1)$, $(2,0,2)$, $(1,2,1)$, $(1,0,3)$, $(0,4,0)$, $(0,2,2)$, $(0,0,4)$, $((1,1,1),1)$, $((1,1,1),2)$, $((2,2,0),1)$, $((2,2,0),2)$, $((0,0,2),1)$, $((0,0,2),2)$.

The difficulty in computing NIM-rep coefficients $N_{\lambda[\varphi,i]}^{[\psi,j]}$ arises when both φ and ψ are fixed points of J (otherwise, the NIM-reps reduce to fusions, which are easy to calculate). However, fixed point factorization resolves this problem, reducing the NIM-reps to surprisingly simple expressions involving fusions.

We present here the case of $C_r^{(1)}$, where r is odd. Using fixed point factorization (11), we find these NIM-rep coefficients reduce to

$$\mathscr{N}_{\Lambda_{2m}[\varphi,i]}^{[\psi,j]} = \frac{1}{2}\left(N_{\Lambda_{2m}\varphi}^{\psi} + (-1)^{i+j+m} {N'}_{\Lambda'_m\varphi'}^{\psi'}\right), \tag{14}$$

and $\mathcal{N}_{\Lambda_\ell[\varphi,i]}^{[\psi,j]} = 1/2N_{\Lambda_\ell\varphi}^{\psi}$ if ℓ is odd. The Λ'_m refers to the m^{th} fundamental weight for $C_{\frac{r-1}{2}}^{(1)}$, level $k/2$, and φ' is the truncated fixed point. Formula (14) first appeared in [3] (some further NIM-reps have also been worked out there). The NIM-reps and applications in string theory for the remaining WZW models will follow in [4], from the fixed point factorizations in [2].

5 Concluding Remarks

The motivation for finding fixed point factorization formulas for the WZW models was the development of the tool towards calculating D-brane charges for non-simply connected Lie groups. The existence of a fixed point factorization for the WZW models leads to several questions. Foremost among them are which other RCFTS possess this feature, and what is a conceptual explanation for this phenomenon?

We also remark that the fixed point factorization algebras of Table 1 are in exact correspondence with the orbit Lie algebras described by Fuchs–Schellekens–Schweigert in [9] (given a simple finite dimensional or affine algebra $\mathfrak{g}$, its orbit Lie algebra $\breve{\mathfrak{g}}$ is obtained through a diagram-folding technique). They showed that the "twining characters" of a symmetrizable Kac–Moody algebra could be expressed in terms of ordinary characters of its orbit Lie algebra. It would be interesting to determine whether there is a connection between fixed point factorization and their work.

Acknowledgements We are grateful to the organizers of LT-9 for the opportunity to present our work and for hospitality during the workshop.

Table 1 Fixed point factorization algebras for the classical affine algebras

$X_r^{(1)}$, level k	Simple-current	FPF algebra	Level
$A_r^{(1)}$	J^d	$A_{d-1}^{(1)}$	$\frac{kd}{r+1}$
$B_r^{(1)}$	J	$A_{2(r-1)}^{(2)}$	k
$C_r^{(1)}$, r even	J	$A_{2(\frac{r}{2})}^{(2)}$	k
$C_r^{(1)}$, r odd	J	$C_{\frac{r-1}{2}}^{(1)}$	$\frac{k}{2}$
$D_r^{(1)}$	J_v	$C_{r-2}^{(1)}$	$\frac{k}{2}$
$D_r^{(1)}$, r even	J_s	$B_{\frac{r}{2}}^{(1)}$	$\frac{k}{2}$
$D_r^{(1)}$, r odd	J_s	$C_{\frac{r-3}{2}}^{(1)}$	$\frac{k}{4}$

References

1. Behrend, R.E., Pearce, P.A., Petkova, V.B., Zuber, J.B.: Boundary conditions in rational conformal field theories. Nucl. Phys. **B579**, 707–73 (2000)
2. Beltaos, E.: Fixed points and fusion rings. Part 1. J. High Energ Phys. **JHEP01**, 154 (2012)
3. Beltaos, E.: Fixed point factorization and NIM-reps for the affine Kac–Moody algebras, and the non-unitary W_3 minimal models, PhD Thesis, University of Alberta (2009)
4. Beltaos, E.: Fixed points and fusion rings. Part 2, in preparation
5. Bourbaki, N.: Groupes et Algèbres de Lie IV-VI. Hermann, Paris (1968)
6. Cappelli, A., Itzykson, C., Zuber, J.B.: The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories. Comm. Math. Phys. **113**, 1–26 (1987)
7. Coste, A., Gannon, T., Ruelle, P.: Finite group modular data. Nucl. Phys. **B581**, 679–717 (2000)
8. Fuchs, J.: Simple WZW currents. Comm. Math. Phys. **136**, 345–356 (1991)
9. Fuchs, J., Schellekens, B., Schweigert, C.: From Dynkin diagram symmetries to fixed point structures. Comm. Math. Phys. **180**, 39–97 (1996)
10. Gaberdiel, M.R., Gannon, T.: Twisted brane charges for non-simply connected groups. J. High Energ Phys. **JHEP035**, 1 (2007)
11. Gaberdiel, M.R., Gannon, T.: D-brane charges on non-simply connected groups. J. High Energ Phys. **JHEP04**, 030 (2004)
12. Gannon, T.: Modular data: the algebraic combinatorics of rational conformal field theory. J. Alg. Combin. **22**, 211–250 (2005)
13. Gannon, T.: The Cappelli-Itzykson-Zuber A-D-E classification. Rev. Math. Phys. **12**, 739–748 (2000)
14. Gannon, T.: The classification of affine SU(3) modular invariants revisited. Ann. Inst. Henri Poincaré Phys. Théor **65**, 15–55 (1996)
15. Gannon, T.: The classification of affine SU(3) modular invariant partition functions. Comm. Math. Phys. **161**, 233–264 (1994)
16. Gannon, T., Walton, M.: On fusion algebras and modular matrices. Comm. Math. Phys. **206**, 1–22 (1999)
17. Kac, V.G.: Infinite Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
18. Kac, V.G., Peterson, D. Infinite-dimensional Lie algebras, theta functions and modular forms. Adv. Math. **53**, 125–264 (1984)
19. Walton, M.: Algorithm for WZW fusion rules: a proof. Phys. Lett. **B241**, 365–368 (1990)

Differential Invariants of Second-Order Ordinary Differential Equations

M. Eugenia Rosado María

Abstract The notion of a differential invariant for systems of second-order differential equations σ on a manifold M with respect to the group of vertical automorphisms of the projection $p\colon \mathbb{R}\times M \to \mathbb{R}$, is defined and the Chern connection ∇^σ attached to a SODE σ allows one to determine a basis for second-order differential invariants of a SODE.

1 Introduction and Preliminaries

Geometry of second order ordinary differential equations (SODE for short) is a classical subject with an extensive knowledge. In [2], Chern associated a linear connection to each system of second-order ordinary differential equations, which has been studied by several authors from different points of view since then; e.g., see [1, 3, 6] among others. Below is shown how Chern's connection can be used in order to obtain a geometric basis for the algebra of differential invariants attached to a SODE in the sense of [5].

Let M be a connected manifold of class C^∞ and dimension n. Let $p\colon \mathbb{R}\times M \to \mathbb{R}$ be the projection $p(t,x)=t$. The bundle of r-jets of curves from $\mathbb{R}$ into M is denoted by $p^r\colon J^r(\mathbb{R},M)\to\mathbb{R}$, $r\geq 0$, with projections $p^r_s\colon J^r(\mathbb{R},M)\to J^s(\mathbb{R},M)$ for $r>s$, and $J^0(\mathbb{R},M)=\mathbb{R}\times M$. Let $j^r\gamma\colon \mathbb{R}\to J^r(\mathbb{R},M)$ denote the r-jet prolongation of the curve $\gamma\colon \mathbb{R}\to M$. Every coordinate system (x^i), $1\leq i\leq n$, on M induces a coordinate system $(t,x^i;\dot{x}^i,\ddot{x}^i)$ on $J^2(\mathbb{R},M)$ as follows:

$$\dot{x}^i\left(j^2_t\gamma\right)=\frac{d\left(x^i\circ\gamma\right)}{dt}(t),\quad \ddot{x}^i\left(j^2_t\gamma\right)=\frac{d^2\left(x^i\circ\gamma\right)}{dt^2}(t).$$

M.E. Rosado María (✉)
Departamento de Matemática Aplicada, Escuela Técnica Superior de Arquitectura, UPM, Avda. Juan de Herrera 4, 28040-Madrid, Spain
e-mail: eugenia.rosado@upm.es

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_39,

A system of second-order ordinary differential equations,

$$\ddot{x}^i = F^i\left(t, x^j, \dot{x}^j\right), \quad F^i \in C^\infty\left(J^1(\mathbb{R}, M)\right), \; 1 \le i \le n, \tag{1}$$

can also be viewed as a section $\sigma\colon J^1(\mathbb{R},M) \to J^2(\mathbb{R},M)$ of the projection p_1^2 by the formulas $\ddot{x}^i \circ \sigma = F^i$, $1 \le i \le n$. The correspondence $\sigma \leftrightarrow (F^i)_{i=1}^n$ is natural and bijective. In Sect. 2 the notion of a differential invariant for SODEs with respect to the group of p-vertical automorphisms of the submersion p, denoted by $\mathrm{Aut}^v(p)$ is introduced.

The main result in this paper states that invariant functions factor though the curvature mapping induced by the curvature K^σ of the splitting H^σ attached to a SODE σ (see the formulas (5) and (6) below), which almost coincides with the torsion tensor field of the Chern's connection.

2 Differential Invariants

A diffeomorphism Φ of $\mathbb{R} \times M$ is said to be a p-vertical automorphism of the submersion p if it is of the form $\Phi(t,x) = (t, \phi(t,x))$, $\forall (t,x) \in \mathbb{R} \times M$, where $\phi\colon \mathbb{R} \times M \to M$ is a smooth mapping. The set of p-vertical automorphism is a group with respect to composition of maps, which is denoted by $\mathrm{Aut}^v(p)$. For each $r \ge 0$, every $\Phi \in \mathrm{Aut}^v(p)$ induces a diffeomorphism $\Phi^{(r)}$ on $J^r(\mathbb{R},M)$ by setting

$$\Phi^{(r)}\left(j_t^r \gamma\right) = j_t^r\left(\phi \circ j^0 \gamma\right), \;\; \forall \gamma \in C^\infty(\mathbb{R}, M). \tag{2}$$

Let $(p_1^2)^r\colon J^r(p_1^2) \to J^1(\mathbb{R},M)$ be the r-jet bundle of the submersion p_1^2. According to (2), every $\Phi \in \mathrm{Aut}^v(p)$ induces diffeomorphisms $\Phi^{(1)}$ of $J^1(\mathbb{R},M)$ and $\Phi^{(2)}$ of $J^2(\mathbb{R},M)$, such that $p_1^2 \circ \Phi^{(2)} = \Phi^{(1)} \circ p_1^2$. Hence, for every $r \ge 0$, the pair $\Phi^{(2)}, \Phi^{(1)}$ induces a transformation $(\Phi^{(2)})^{(r)}\colon J^r(p_1^2) \to J^r(p_1^2)$ given by

$$(\Phi^{(2)})^{(r)}(j_\xi^r \sigma) = j^r_{\Phi^{(1)}(\xi)}(\Phi^{(2)} \circ \sigma \circ (\Phi^{(1)})^{-1}).$$

If $\mathscr{U} \subseteq J^r(p_1^2)$ is an open subset invariant under all these transformations, then a smooth function $\mathscr{I}\colon \mathscr{U} \to \mathbb{R}$ is said to be a *differential invariant* of order r with respect to the group $\mathrm{Aut}^v(p)$ if for every $\Phi \in \mathrm{Aut}^v(p)$ the following equation holds: $\mathscr{I} \circ (\Phi^{(2)})^{(r)} = \mathscr{I}$. We set $I(\sigma,\xi) = \mathscr{I}(j_\xi^r \sigma)$, $\xi \in J^1(\mathbb{R},M)$, for a given SODE σ on M, then the invariance condition above reads as:

$$I\left(\Phi \cdot \sigma, \Phi^{(1)}(\xi)\right) = I(\sigma, \xi), \quad \forall \xi \in J^1(\mathbb{R},M), \quad \forall \Phi \in \mathrm{Aut}^v(p).$$

Proposition A.1. *If $\Phi_t \in \mathrm{Aut}^v(p)$ is the flow of a p-vertical vector field X in $\mathfrak{X}(\mathbb{R} \times M)$, then $\Phi_t^{(2)}$ is the flow of a p^2-vertical vector field $X^{(2)} \in \mathfrak{X}(J^2(\mathbb{R},M))$ and*

$(\Phi_t^{(2)})^{(r)}$ *is the flow of a vector field* $(X^{(2)})^{(r)}$ *on* $J^r(p_1^2)$. *Every differential invariant of order r is a first integral of the distribution* $\mathscr{D}^{(r)}$ *on* $J^r(p_1^2)$ *spanned by all the r-jet prolongations* $(X^{(2)})^{(r)}$ *of p-vertical vector fields.*

3 The Chern Connection Attached to a SODE

As is known, $p_0^1\colon J^1(\mathbb{R},M)\to\mathbb{R}\times M$ is an affine bundle modelled over p'^*TM, $p'\colon\mathbb{R}\times M\to M$ denoting the projection $p'(t,x)=x$. In fact, if $v\in T_{x_0}M$, $j_{t_0}^1\gamma\in J^1(\mathbb{R},M)$ with $\gamma(t_0)=x_0$, then $v+j_{t_0}^1\gamma=j_{t_0}^1\gamma'$ is defined by the following two formulas: $\gamma'(t_0)=x_0$, $\gamma'_*(d/dt)_{t_0}=v+\gamma_*(d/dt)_{t_0}$. Hence, the following exact sequence holds:

$$0\to\left(p'\circ p_0^1\right)^*TM\xrightarrow{\varepsilon}V\left(p_0^1\right)\to T(J^1(\mathbb{R},M))\xrightarrow{(p_0^1)_*}(p_0^1)^*T\left(\mathbb{R}\times M\right)\to 0,\quad (3)$$

where $V\left(p_0^1\right)$ denotes the vector subbundle of p_0^1-vertical vectors and ε is defined by the directional derivative, i.e.,

$$\varepsilon\left(j_{t_0}^1\gamma,v\right)(f)=\partial/\partial t|_{t=0}f\left(tv+j_{t_0}^1\gamma\right),\ v\in T_{\gamma(t_0)}M,\ f\in C^\infty(J^1(\mathbb{R},M)).$$

Every SODE σ defines a vector field $X^\sigma\in\mathfrak{X}(J^1(\mathbb{R},M))$—called the dynamical flow in [6]—by $(X^\sigma)_\xi=(j^1\gamma)_*(d/dt)_{t_0}$, $\forall\xi\in(p^1)^{-1}(t_0)$, γ being the only solution to (1) satisfying the initial conditions $\gamma^i(t_0)=x^i(\xi)$, $d\gamma^i/dt(t_0)=\dot{x}^i(\xi)$, where $\gamma^i=x^i\circ\gamma$, $1\le i\le n$; its local expression is $X^\sigma=\frac{\partial}{\partial t}+\dot{x}^i\frac{\partial}{\partial x^i}+F^i\frac{\partial}{\partial\dot{x}^i}$.

The Lie derivative of the tensor field $J=\omega^i\otimes\frac{\partial}{\partial\dot{x}^i}$, $\omega^i=dx^i-\dot{x}^idt$ on $J^1(\mathbb{R},M)$ (cf. [6, formula (1.13)]) along X^σ is

$$L_{X^\sigma}J=-\omega^i\otimes X_i^\sigma+\varpi^j\otimes\frac{\partial}{\partial\dot{x}^j},$$

where $X_i^\sigma=\frac{\partial}{\partial x^i}+\frac{1}{2}\frac{\partial F^j}{\partial\dot{x}^i}\frac{\partial}{\partial\dot{x}^j}$ and $\varpi^j=d\dot{x}^j-F^jdt-\frac{1}{2}\frac{\partial F^j}{\partial\dot{x}^i}\omega^i$. Therefore, $L_{X^\sigma}J$ is diagonalizable with eigenvalues $0,+1,-1$, and multiplicities $1,n,n$, respectively (cf. [1, p. 6620]). If $T^0(J^1(\mathbb{R},M))$, $T^+(J^1(\mathbb{R},M))$, $T^-(J^1(\mathbb{R},M))$ are the corresponding vector subbundles of eigenvectors, then

$$\begin{aligned}T(J^1(\mathbb{R},M))&=T^0(J^1(\mathbb{R},M))\oplus T^-(J^1(\mathbb{R},M))\oplus T^+(J^1(\mathbb{R},M)),\\T^0(J^1(\mathbb{R},M))&=\langle X^\sigma\rangle,\\T^-(J^1(\mathbb{R},M))&=\langle X_i^\sigma\rangle,\\T^+(J^1(\mathbb{R},M))&=V\left(p_0^1\right)=\left\langle\frac{\partial}{\partial\dot{x}^i}\right\rangle.\end{aligned}$$

The epimorphism $(p_0^1)_*$ in (3) induces an isomorphism

$$(p_0^1)_*|_{T^0 \oplus T^-} : T^0(J^1(\mathbb{R},M)) \oplus T^-(J^1(\mathbb{R},M)) \xrightarrow{\cong} (p_0^1)^* T(\mathbb{R} \times M), \tag{4}$$

whose inverse mapping determines a section H^σ of $(p_0^1)_*$ given by

$$H^\sigma = dt \otimes X^\sigma + \omega^i \otimes X_i^\sigma. \tag{5}$$

Hence the exact sequence (3) splits and every $X \in T\left(J^1(\mathbb{R},M)\right)$ can uniquely be written as $X = X^v + X^h$, where

$$X^h = H^\sigma\left((p_0^1)_* X\right) \in T^0(J^1(\mathbb{R},M)) \oplus T^-(J^1(\mathbb{R},M)), \;\; X^v = X - X^h \in V(p_0^1).$$

The curvature form of the splitting (5); i.e.,

$$\begin{aligned} K^\sigma &\in \wedge^2 T^*\left(J^1(\mathbb{R},M)\right) \otimes V(p_0^1), \\ K^\sigma(X,Y) &= \left[X^h, Y^h\right]^v, \;\; \forall X,Y \in \mathfrak{X}(J^1(\mathbb{R},M)), \end{aligned} \tag{6}$$

is locally given as follows:

$$K^\sigma = -\left(P_j^k dt \wedge \omega^j + \sum_{i<j} T_{ij}^k \omega^i \wedge \omega^j\right) \otimes \frac{\partial}{\partial \dot{x}^k}, \tag{7}$$

where

$$\begin{aligned} T_{ij}^k &= \frac{1}{2}\left(\frac{\partial^2 F^k}{\partial x^i \partial \dot{x}^j} - \frac{\partial^2 F^k}{\partial x^j \partial \dot{x}^i} + \frac{1}{2}\left(\frac{\partial F^h}{\partial \dot{x}^i}\frac{\partial^2 F^k}{\partial \dot{x}^h \partial \dot{x}^j} - \frac{\partial F^h}{\partial \dot{x}^j}\frac{\partial^2 F^k}{\partial \dot{x}^h \partial \dot{x}^i}\right)\right), \\ P_j^k &= \frac{1}{2} X^\sigma\left(\frac{\partial F^k}{\partial \dot{x}^j}\right) - \frac{\partial F^k}{\partial x^j} - \frac{1}{4}\frac{\partial F^h}{\partial \dot{x}^j}\frac{\partial F^k}{\partial \dot{x}^h}. \end{aligned}$$

Let ∇^σ be the Chern connection attached to σ; see [1–3, 6] for the definition of the Chern connection attached to a SODE. The curvature K^σ of the splitting H^σ attached to σ essentially coincides with the torsion tensor field of the Chern connection. In fact, one has $T^\sigma = K^\sigma + dt \wedge \varpi^i \otimes X_i^\sigma$. The Chern connection ∇^σ is functorial with respect to $\mathrm{Aut}^v(p)$; i.e., $\Phi \cdot \nabla^\sigma = \nabla^{\Phi \cdot \sigma}$, $\forall \Phi \in \mathrm{Aut}^v(p)$, where $\Phi \cdot \nabla^\sigma$ is the linear connection defined by,

$$(\Phi \cdot \nabla^\sigma)_X Y = \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot X}\left((\Phi^{(1)})^{-1} \cdot Y\right)\right), \quad \forall X,Y \in \mathfrak{X}(J^1(\mathbb{R},M)),$$

and $\Phi \cdot \sigma$ is the SODE given by, $\Phi \cdot \sigma = \Phi^{(2)} \circ \sigma \circ (\Phi^{(1)})^{-1}$.

According to the formula (7) the tensor field K^σ defined in (6) takes values in the vector bundle $(p_0^1)^* \wedge^2 T^*(\mathbb{R} \times M) \otimes V(p_0^1)$. The *curvature mapping* $\mathscr{K} : J^2(p_1^2) \to (p_0^1)^* \wedge^2 T^*(\mathbb{R} \times M) \otimes V(p_0^1)$ is defined by setting $\mathscr{K}(j_\xi^2 \sigma) = (K^\sigma)_\xi$.

Coordinates are introduced in $(p_0^1)^* \wedge^2 T^*(\mathbb{R} \times M) \otimes V(p_0^1)$ by the formula

$$\eta = \left(y_i^j(\eta)\left(dt \wedge \omega^i\right)_{(t_0,x_0)} + \sum\nolimits_{h<i} y_{hi}^j(\eta)\left(\omega^h \wedge \omega^i\right)_{(t_0,x_0)}\right) \otimes \left(\frac{\partial}{\partial \dot{x}^j}\right)_\xi,$$

$\forall \eta \in \wedge^2 T^*_{(t_0,x_0)}(\mathbb{R}\times M)\otimes V_\xi(p_0^1)$ and the equations of $\mathscr{K}$ are the following:

$$t\circ\mathscr{K} = t,\ x^i\circ\mathscr{K} = x^i,\ \dot{x}^i\circ\mathscr{K} = \dot{x}^i,$$

$$y_a^i\circ\mathscr{K} = -\frac{1}{2}\left(\ddot{x}^i_{t\dot{a}} + \dot{x}^h\ddot{x}^i_{h\dot{a}} + \ddot{x}^h\ddot{x}^i_{\dot{h}\dot{a}}\right) + \ddot{x}^i_a + \frac{1}{4}\ddot{x}^k_{\dot{a}}\ddot{x}^i_{\dot{k}},$$

$$y_{ab}^k\circ\mathscr{K} = -\frac{1}{2}\left(\ddot{x}^k_{a\dot{b}} - \ddot{x}^k_{b\dot{a}} + \frac{1}{2}\left(\ddot{x}^h_{\dot{a}}\ddot{x}^k_{\dot{h}b} - \ddot{x}^h_{\dot{b}}\ddot{x}^k_{\dot{h}\dot{a}}\right)\right),\ a<b,$$

where $(t, x^i, \dot{x}^i, \ddot{x}^i, \ddot{x}^i_t, \ddot{x}^i_a, \ddot{x}^i_{\dot{a}}, \ddot{x}^i_{tt}, \ddot{x}^i_{ta}, \ddot{x}^i_{t\dot{a}}, \ddot{x}^i_{a\le b}, \ddot{x}^i_{a\dot{b}}, \ddot{x}^i_{\dot{a}\le\dot{b}})$ is the induced coordinate system on $J^2(p_1^2)$.

The curvature mapping is $\mathrm{Aut}^v(p)$-equivariant with respect to the natural actions, i.e., $\Phi\cdot\mathscr{K}(j^2_\xi\sigma) = \mathscr{K}(\Phi\cdot j^2_\xi\sigma)$, where the action on the left-hand side is defined by $\Phi\cdot\eta = \left(\wedge^2((\Phi^{(1)})^{-1})^*\otimes(\Phi^{(1)})_*\right)(\eta)$, for every vertical automorphism Φ, every $\eta\in(p_0^1)^*\wedge^2 T^*(\mathbb{R}\times M)\otimes V(p_0^1)$, and the action on the right-hand side is defined as follows: $\Phi\cdot j^2_\xi\sigma = j^2_{\Phi^{(1)}(\xi)}(\Phi\cdot\sigma)$.

4 Statement of the Main Result

The only first integrals of the distributions $\mathscr{D}^{(0)}$ and $\mathscr{D}^{(1)}$ are $(p^2)^*C^\infty(\mathbb{R})$ and $((p_1^2)^1)^*(p^1)^*C^\infty(\mathbb{R})$, respectively. In fact, from the general formulas of jet prolongation of vector fields (e.g., see [5]), one obtains

$$X = u^i\frac{\partial}{\partial x^i},\quad u^i\in C^\infty(\mathbb{R}\times M),$$

$$X^{(2)} = u^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial\dot{x}^i} + w^i\frac{\partial}{\partial\ddot{x}^i},$$

$$v^i = \frac{\partial u^i}{\partial t} + \frac{\partial u^i}{\partial x^h}\dot{x}^h,$$

$$w^i = \frac{\partial^2 u^i}{\partial t^2} + 2\frac{\partial^2 u^i}{\partial t\partial x^h}\dot{x}^h + \frac{\partial^2 u^i}{\partial x^h\partial x^k}\dot{x}^h\dot{x}^k + \frac{\partial u^i}{\partial x^h}\ddot{x}^h.$$

As the values of u^i and its derivatives can arbitrarily be taken at a given point $j^2_t\gamma\in J^2(\mathbb{R},M)$, one has

$$\mathscr{D}^{(0)} = \left\langle\frac{\partial}{\partial x^i},\frac{\partial}{\partial\dot{x}^i},\dot{x}^h\frac{\partial}{\partial\dot{x}^i}+\ddot{x}^h\frac{\partial}{\partial\ddot{x}^i},\frac{\partial}{\partial\ddot{x}^i},2\dot{x}^h\frac{\partial}{\partial\ddot{x}^i},\dot{x}^h\dot{x}^k\frac{\partial}{\partial\ddot{x}^i}\right\rangle = \left\langle\frac{\partial}{\partial x^i},\frac{\partial}{\partial\dot{x}^i},\frac{\partial}{\partial\ddot{x}^i}\right\rangle.$$

Hence, the only differential invariants of order 0 are the functions in $(p^2)^*C^\infty(\mathbb{R})$.

By computing the first jet prolongation of $X^{(2)}$, we obtain

$$
\begin{aligned}
(X^{(2)})^{(1)} &= u^i\frac{\partial}{\partial x^i} + v^i\frac{\partial}{\partial \dot{x}^i} + w^i\frac{\partial}{\partial \ddot{x}^i} + w^i_t\frac{\partial}{\partial \ddot{x}^i_t} + w^i_a\frac{\partial}{\partial \ddot{x}^i_a} + w^i_{\dot{a}}\frac{\partial}{\partial \ddot{x}^i_{\dot{a}}},\\
w^i_t &= \frac{\partial^3 u^i}{\partial t^3} + 2\frac{\partial^3 u^i}{\partial t^2\partial x^h}\dot{x}^h + \frac{\partial^3 u^i}{\partial t\partial x^h\partial x^k}\dot{x}^h\dot{x}^k + \frac{\partial^2 u^i}{\partial t\partial x^h}\ddot{x}^h\\
&\quad + \frac{\partial u^i}{\partial x^h}\ddot{x}^h_t - \frac{\partial u^a}{\partial t}\ddot{x}^i_a - \frac{\partial^2 u^a}{\partial t^2}\ddot{x}^i_{\dot{a}} - \frac{\partial^2 u^a}{\partial t\partial x^h}\dot{x}^h\ddot{x}^i_{\dot{a}},\\
w^i_a &= \frac{\partial^3 u^i}{\partial t^2\partial x^a} + 2\frac{\partial^3 u^i}{\partial t\partial x^a\partial x^h}\dot{x}^h + \frac{\partial^3 u^i}{\partial x^a\partial x^h\partial x^k}\dot{x}^h\dot{x}^k + \frac{\partial^2 u^i}{\partial x^a\partial x^h}\ddot{x}^h\\
&\quad + \frac{\partial u^i}{\partial x^h}\ddot{x}^h_a - \frac{\partial u^b}{\partial x^a}\ddot{x}^i_b - \frac{\partial^2 u^b}{\partial t\partial x^a}\ddot{x}^i_{\dot{b}} - \frac{\partial^2 u^b}{\partial x^a\partial x^h}\dot{x}^h\ddot{x}^i_{\dot{b}},\\
w^i_{\dot{a}} &= 2\frac{\partial^2 u^i}{\partial t\partial x^a} + 2\frac{\partial^2 u^i}{\partial x^a\partial x^h}\dot{x}^h + \frac{\partial u^i}{\partial x^h}\ddot{x}^h_{\dot{a}} - \frac{\partial u^r}{\partial x^a}\ddot{x}^i_{\dot{r}}.
\end{aligned}
$$

By collecting the derivatives of the functions u^i in this formula we obtain

$$
\begin{aligned}
(X^{(2)})^{(1)} &= u^r\frac{\partial}{\partial x^r} + \frac{\partial u^r}{\partial t}\chi^r_t + \frac{\partial u^r}{\partial x^a}\chi^r_a + \frac{\partial^2 u^r}{\partial t^2}\chi^r_{tt} + \frac{\partial^2 u^r}{\partial t\partial x^a}\chi^r_{ta} + \sum_{a\le b}\frac{\partial^2 u^r}{\partial x^a\partial x^b}\chi^r_{a\le b}\\
&\quad + \frac{\partial^3 u^r}{\partial t^3}\chi^r_{ttt} + \frac{\partial^3 u^r}{\partial t^2\partial x^a}\chi^r_{tta} + \sum_{a\le b}\frac{\partial^3 u^r}{\partial t\partial x^a\partial x^b}\chi^r_{t,a\le b}\\
&\quad + \sum_{a\le b\le c}\frac{\partial^3 u^r}{\partial x^a\partial x^b\partial x^c}\chi^r_{a\le b\le c},
\end{aligned}
$$

where

$$
\begin{aligned}
\chi^r_t &= \frac{\partial}{\partial \dot{x}^r} - \ddot{x}^i_r\frac{\partial}{\partial \ddot{x}^i_t},\\
\chi^r_a &= \dot{x}^a\frac{\partial}{\partial \dot{x}^r} + \ddot{x}^a\frac{\partial}{\partial \ddot{x}^r} + \ddot{x}^a_t\frac{\partial}{\partial \ddot{x}^r_t} + \ddot{x}^a_h\frac{\partial}{\partial \ddot{x}^r_h} - \ddot{x}^i_r\frac{\partial}{\partial \ddot{x}^i_a} - \ddot{x}^i_{\dot{r}}\frac{\partial}{\partial \ddot{x}^i_{\dot{a}}} + \ddot{x}^a_{\dot{b}}\frac{\partial}{\partial \ddot{x}^r_{\dot{b}}},\\
\chi^r_{tt} &= \frac{\partial}{\partial \ddot{x}^r} - \ddot{x}^i_{\dot{r}}\frac{\partial}{\partial \ddot{x}^i_t},\\
\chi^r_{ta} &= \dot{x}^a\left(2\frac{\partial}{\partial \ddot{x}^r} - \ddot{x}^i_{\dot{r}}\frac{\partial}{\partial \ddot{x}^i_t}\right) + \ddot{x}^a\frac{\partial}{\partial \ddot{x}^r_t} - \ddot{x}^i_{\dot{r}}\frac{\partial}{\partial \ddot{x}^i_a} + 2\frac{\partial}{\partial \ddot{x}^r_{\dot{a}}},\\
\chi^r_{a\le b} &= \frac{1}{1+\delta_{ab}}\left\{2\dot{x}^a\dot{x}^b\frac{\partial}{\partial \ddot{x}^r} + \ddot{x}^b\frac{\partial}{\partial \ddot{x}^r_a} + \ddot{x}^a\frac{\partial}{\partial \ddot{x}^r_b} - \ddot{x}^i_{\dot{r}}\left(\dot{x}^b\frac{\partial}{\partial \ddot{x}^i_a} + \dot{x}^a\frac{\partial}{\partial \ddot{x}^i_b}\right)\right.\\
&\quad \left. + 2\dot{x}^b\frac{\partial}{\partial \ddot{x}^r_{\dot{a}}} + 2\dot{x}^a\frac{\partial}{\partial \ddot{x}^r_{\dot{b}}}\right\},
\end{aligned}
$$

$$\chi^r_{ttt} = \frac{\partial}{\partial \ddot{x}^r_t},$$

$$\chi^r_{tta} = 2\dot{x}^a \frac{\partial}{\partial \ddot{x}^r_t} + \frac{\partial}{\partial \ddot{x}^r_a},$$

$$\chi^r_{t,a\le b} = \frac{2}{1+\delta_{ab}} \left\{ \dot{x}^a \dot{x}^b \frac{\partial}{\partial \ddot{x}^r_t} + \dot{x}^b \frac{\partial}{\partial \ddot{x}^r_a} + \dot{x}^a \frac{\partial}{\partial \ddot{x}^r_b} \right\},$$

$$\chi^r_{a\le b\le c} = \frac{2}{(1+\delta_{ab}+\delta_{bc})!} \left\{ \dot{x}^b \dot{x}^c \frac{\partial}{\partial \ddot{x}^r_a} + \dot{x}^a \dot{x}^c \frac{\partial}{\partial \ddot{x}^r_b} + \dot{x}^a \dot{x}^b \frac{\partial}{\partial \ddot{x}^r_c} \right\}.$$

Accordingly, $\partial/\partial x^r$, χ^r_t, χ^r_a, χ^r_{tt}, χ^r_{ta}, $\chi^r_{a\le b}$, χ^r_{ttt}, χ^r_{tta}, $\chi^r_{ta\le b}$, $\chi^r_{a\le b\le c}$ constitute a system of generators for the distribution $\mathscr{D}^{(1)}$. From the expressions above one obtains $\mathscr{D}^{(1)} = \langle \partial/\partial x^r, \partial/\partial \dot{x}^r, \partial/\partial \ddot{x}^r, \partial/\partial \ddot{x}^r_t, \partial/\partial \ddot{x}^r_a, \partial/\partial \ddot{x}^r_{\dot{a}} \rangle$. Therefore, the first-order differential invariants are the functions in $((p^2_1)^1)^*(p^1)^* C^\infty(\mathbb{R})$.

The formulas for $(X^{(2)})^{(2)}$ are similar but rather longer. The following result is obtained:

Theorem 4.1. *The rank of* $\mathscr{D}^{(2)}$ *is* 11 *if* $\dim M = 1$ *and* $\frac{1}{2}n\left(3n^2+11n+10\right)$ *if* $\dim M = n > 1$.

Theorem 4.2. *Every second-order differential invariant* $\mathscr{I} \colon J^2(p^2_1) \to \mathbb{R}$ *with respect to* $\mathrm{Aut}^v(p)$ *factors uniquely through the curvature mapping as follows:* $\mathscr{I} = \bar{\mathscr{I}} \circ \mathscr{K}$, *where* $\bar{\mathscr{I}} \colon (p^1_0)^* \wedge^2 T^*(\mathbb{R}\times M) \otimes V(p^1_0) \to \mathbb{R}$ *is an invariant smooth function under the natural action of* $\mathrm{Aut}^v(p)$.

Proof (Sketch of proof). The statement is a consequence of the following properties:

1) The curvature mapping is a surjective submersion.
2) For every $\eta \in \wedge^2 T^*_{(t_0,x_0)}(\mathbb{R}\times M) \otimes V_\xi(p^1_0)$ the fibre $\mathscr{K}^{-1}(\eta)$ is an affine subbundle over $J^1(p^2_1)$; hence the fibres of $\mathscr{K}$ are connected.
3) If we define $\tilde{\mathscr{D}}^{(2)}_{j^2_\xi \sigma} = \{(X^{(2)})^{(2)}_{j^2_\xi \sigma} \in \mathscr{D}^{(2)}_{j^2_\xi \sigma} : j^1_{j^0_{t_0}\gamma} X = 0\}$ for $\dim M \ge 2$, and $\tilde{\mathscr{D}}^{(2)}_{j^2_\xi \sigma} = \{(X^{(2)})^{(2)}_{j^2_\xi \sigma} \in \mathscr{D}^{(2)}_{j^2_\xi \sigma} : X^{(1)}_\xi = 0\}$ for $\dim M = 1$, then one has $\ker(\mathscr{K}_*)_{j^2_\xi \sigma} = \tilde{\mathscr{D}}^{(2)}_{j^2_\xi \sigma}$, $\xi = j^1_{t_0}\gamma$.
4) The curvature mapping is $\mathrm{Aut}^v(p)$-equivariant with respect to the natural actions.

The first and second properties directly follow from the equations of the curvature mapping. Moreover, we have

$$(X^{(2)})^{(2)}\left(y^i_j \circ \mathscr{K}\right) = \frac{\partial u^i}{\partial x^r}\left(y^r_j \circ \mathscr{K}\right) - \frac{\partial u^r}{\partial x^j}\left(y^i_r \circ \mathscr{K}\right),$$

$$(X^{(2)})^{(2)}\left(y^k_{ij} \circ \mathscr{K}\right) = \frac{\partial u^r}{\partial x^i}\left(y^k_{rj} \circ \mathscr{K}\right) + \frac{\partial u^k}{\partial x^r}\left(y^r_{ji} \circ \mathscr{K}\right) + \frac{\partial u^r}{\partial x^j}\left(y^k_{ir} \circ \mathscr{K}\right).$$

By evaluating these two formulas at $j_\xi^2\sigma$, we conclude $\tilde{\mathscr{D}}^{(2)}_{j_\xi^2\sigma} \subseteq \ker(\mathscr{K}_*)_{j_\xi^2\sigma}$ and from the Proposition A.1 and the first item above we have

$$\begin{aligned}\dim\ker(\mathscr{K}_*)_{j_\xi^2\sigma} &= \dim J^2(p_1^2) - \dim\left((p_0^1)^* \wedge^2 T^*(\mathbb{R}\times M)\otimes V(p_0^1)\right)\\ &= \frac{3}{2}n(n+2)(n+1)\\ &= \dim \tilde{\mathscr{D}}^{(2)}_{j_\xi^2\sigma}.\end{aligned}$$

5 Concluding Remarks

As the distribution $\mathscr{D}^{(2)}$ is involutive, the number of functionally independent second-order differential invariants is

$$\dim J^2(p_1^2) - rank\mathscr{D}^{(2)} = \begin{cases} \frac{1}{2}n^2(n-1)+1, & n \geq 2 \\ 2, & n = 1 \end{cases}$$

Reducing modulo $(p_0^1)_* T^0(J^1(\mathbb{R},M))$, the isomorphism (4) induces another isomorphism $\iota_1 : T^-(J^1(\mathbb{R},M)) \xrightarrow{\cong} (p_0^1)^*T(\mathbb{R}\times M)/(p_0^1)_*T^0(J^1(\mathbb{R},M))$. There is a natural embedding $(p')^*TM \hookrightarrow T(\mathbb{R}\times M)$ and pulling it back via p_0^1 one obtains another embedding $(p'\circ p_0^1)^*TM \hookrightarrow (p_0^1)^*T(\mathbb{R}\times M)$. By composing this latter embedding and the quotient map

$$(p_0^1)^*T(\mathbb{R}\times M) \to (p_0^1)^*T(\mathbb{R}\times M)/(p_0^1)_*T^0(J^1(\mathbb{R},M)),$$

an isomorphism $\iota_2 \colon (p'\circ p_0^1)^*TM \xrightarrow{\cong} (p_0^1)^*T(\mathbb{R}\times M)/(p_0^1)_*T^0(J^1(\mathbb{R},M))$ is deduced. From (7) it follows $i_{X^\sigma}K^\sigma = -P_j^h\omega^j \otimes \partial/\partial\dot{x}^h$, and an endomorphism $\tilde{K}^\sigma$ of $(p'\circ p_0^1)^*TM$ is defined: $\tilde{K}^\sigma = \varepsilon^{-1}\circ i_{X^\sigma}K^\sigma|_{T^-(M^1)}\circ(\iota_1)^{-1}\circ\iota_2$. Hence $\tilde{K}^\sigma\left(\partial/\partial x^j\right) = -P_j^h\partial/\partial x^h$. The coefficients of the characteristic polynomial of $\tilde{K}^\sigma$ determine n second-order invariants. This fact was early remarked in [4]. If $n = 1$ or $n = 2$ such invariants together with the function t exhaust a basis of second-order invariants, but this is no longer true for $n \geq 3$.

Acknowledgements Supported by Ministerio of Ciencia e Innovación of Spain (MICINN), under grant #MTM2008–01386.

References

1. Byrnes, G.B.: A complete set of Bianchi identities for tensor fields along the tangent bundle projection. J. Phys. A Math. Gen. **27**, 6617–6632 (1994)
2. Chern, S.-S.: Sur la géométrie d'un système d'équations différentielles du second ordre. Bull. Sci. Math. **63**, 206–212 (1939)
3. Crampin, M., Martínez, E., Sarlet, W.: Linear connections for systems of second-order ordinary differential equations. Ann. Inst. H. Poincaré Phys. Théor. **65**(2), 223–249 (1996)
4. Kosambi, D.D.: Systems of differential equations of the second order. Quart. J. Math. Oxford **6**, 1–12 (1935)
5. Kumpera, A.: Invariants différentiels d'un pseudogroupe de Lie. I, II. J. Diff. Geom. **10**, 289–345, 347–416 (1975)
6. Massa, E., Pagani, E.: Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian mechanics. Ann. Inst. H. Poincaré Phys. Théor. **61**(1), 17–62 (1994)

Some Properties of Harmonic Quasi-Conformal Mappings

Miljan Knežević

Abstract We are analyzing the properties of holomorphic functions and the hyperbolic metric to obtain some geometrically motivated inequalities for quasi-conformal and generalized harmonic mappings. Also, we are interested in which properties of hyperbolic harmonic mappings and the hyperbolic metric are essential for validity of some versions of the Ahlfors–Schwarz lemma.

1 Introduction

Euclidean harmonic mappings (shortly harmonic mappings), quasi-conformal mappings and generalized harmonic mappings occupy an important place in the geometric theory of functions of one or more complex variables and, also, in physics and in other areas. Motivated by the classical result in complex analysis, Ahlfors–Schwarz's lemma, using the method of comparison of some metrics, we tried to determine the conditions under which this lemma holds in the case of harmonic mappings in relation to the conformal metric given. In particular, we proved that every harmonic and quasi-conformal diffeomorphism of the unit disc is a quasi-isometry with respect to the hyperbolic distance (Wan, see [8]). Also, of the interest was to see if the appropriate version of the lemma is true for the harmonic mappings (Euclidean harmonic mappings). It was shown that every harmonic and quasi-conformal mapping of the unit disk into itself, up to a multiplicative positive constant that depends only on k, does not increase the corresponding hyperbolic distance. In addition, it was shown that in the case of the presence of conformal metric $ds^2 = \rho(w)|dw|^2$ in the image, whose Gaussian curvature $K(\rho)$ is bounded above by a negative constant $-a$, for some $a > 0$, any harmonic and k

M. Knežević (✉)
Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia
e-mail: kmiljan@math.rs

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_40,

quasi-conformal mapping of the unit disk into itself, up to a multiplicative constant that depends only on a and k, does not increase the distance induced by the metrics $ds^2 = \rho(w)|dw|^2$, with respect to the hyperbolic distance.

2 Definitions of Harmonic and Quasi-Conformal Mappings

Let R be an arbitrary Riemann surface with the conformal structure $\{(U_v, h_v)\}$. We denote by $z_v \in V_v = h_v(U_v) \subset \mathbf{C}$ local parameter on the surface R that corresponds to the local chart (U_v, h_v). Suppose that on the surface R is defined Riemann's metric $ds^2 = \rho_v(z_v)|dz_v|^2$, where $\rho_v \in C^2(V_v)$ is positive function, which is compatible with the conformal structure on R. Observe that in the neighborhood of any point of R, the metric is represented as the positive multiple of the Euclidean metric. Because of the conformal invariance of the corresponding representatives when changing local parameters, the previous metric on R we will call conformal metric, and, when it is possible, we will write $ds^2 = \rho(z)|dz|^2$ and the function $z \mapsto \rho(z)$ we call metric density function.

Definition 2.1. Let $ds^2 = \rho(z)|dz|^2$ be a conformal metric defined on a Riemann surface R. Regardless of the choice of local parameter, the Gaussian curvature of the metric $ds^2 = \rho(z)|dz|^2$ is given by the formula

$$K(\rho)(z) = -\frac{1}{2}\frac{(\triangle \log \rho)(z)}{\rho(z)}. \tag{1}$$

Example 2.1. Let $\mathbf{D}$ denote the unit disk in the complex plane. We analyze a conformal metric $ds^2 = \lambda(z)|dz|^2$ on $\mathbf{D}$, where the metric density $z \mapsto \lambda(z)$ is defined as

$$\lambda(z) = \left(\frac{2}{1-|z|^2}\right)^2, z \in \mathbf{D}. \tag{2}$$

Since $(\triangle \log \lambda)(z) = 2\lambda(z), z \in \mathbf{D}$, we get $K(\lambda)(z) = -1$, for all $z \in \mathbf{D}$. Furthermore, it is not difficult to show that for all $0 < r < 1$ holds $d_\lambda(0,r) = \log\frac{1+r}{1-r}$, where d_λ is corresponding distance that is induced by the metric $ds^2 = \lambda(z)|dz|^2$.

Conformal metric $ds^2 = \lambda(z)|dz|^2$ that is defined on the unit disk, with the density function as in relation (2), is called hyperbolic metric on $\mathbf{D}$. The appropriate metric density function is called hyperbolic density and the distance d_λ we will call the hyperbolic distance on the disk $\mathbf{D}$.

By using universal covering surface, every hyperbolic Riemann surface R could be provided by a conformal metric with the Gaussian curvature of -1. We will call that metric as hyperbolic metric on R.

Let R and S be Riemann surfaces.

Definition 2.2. We say that a C^2 mapping $f : R \to S$ is harmonic with respect to the conformal metric $ds^2 = \rho(w)|dw|^2$ given on S, if for each pair of local parameters

$z \in V = h(U) \subset \mathbf{C}$, on the surface R, and $w \in V' = g(W) \subset \mathbf{C}$, on the surface S, for which $f(U) \subset W$, the following assertion is valid

$$(f_z)_{\bar{z}}(z) + (\log \rho)_w(f(z))f_z(z)f_{\bar{z}}(z) = 0, z \in V, \tag{3}$$

where ρ is the corresponding representation of the conformal metric given in terms of the local parameter w.

In the case of mappings that act between domains in the complex plane and the presence of the Euclidean metric, the relation (3) defines an Euclidean harmonic mapping or, simply, a harmonic mapping. That is, if Ω is domain in $\mathbf{C}$, the mapping $f : z \mapsto f(z) \in \mathbf{C}$, $z \in \Omega$, is harmonic if $(f_z)_{\bar{z}}(z) = 0$, for all $z \in \Omega$. Observe that the previous fact about the harmonic mappings coincides with the well known Laplacian definition, since $\triangle f = 4(f_z)_{\bar{z}}(z)$.

The theory of generalized harmonic mappings, i.e. mappings that are harmonic with respect to the given conformal metric in the image, is closely related with the theory of holomorphic quadratic differentials (see [1,4,7]). Specifically, if $f : R \to S$ is harmonic mapping, where R and S are arbitrary Riemann surfaces, with respect to the conformal metric $ds^2 = \rho(w)|dw|^2$ on S, then f, on natural way, defines a family of holomorphic functions, in terms of the local parameters on R, and therefore a holomorphic quadratic differential.

Proposition A.1. *Suppose that R and S are Riemann surfaces. If $f : R \to S$ is harmonic mapping, with respect to the conformal metric $ds^2 = \rho(w)|dw|^2$ on S, then $\psi(z) = (\rho \circ f)(z)f_z(z)\overline{f_{\bar{z}}(z)}dz^2$ is a holomorphic quadratic differential on the surface R.*

Definition 2.3. The holomorphic quadratic differential $\psi(z) = (\rho \circ f)(z)f_z(z)$ $\overline{f_{\bar{z}}(z)}dz^2$, that is defined on R, we called Hopf differential of f and we write $\psi = \mathrm{Hopf}(f)$.

Observe that one could globally speak about the zeros of the "functions" f_z and $f_{\bar{z}}$ on the surface R. It is easy to prove that in the case of a generalized harmonic mapping the zeros of the functions f_z and $f_{\bar{z}}$, if they are not identically equal to zero, are isolated and of well defined order.

The following proposition is important in our approach and will be used as a motivation for a method of metrics comparison. The formula below is known as Bochner formula (see [7]).

Proposition A.2. *Let R and S be a Riemann surfaces and let $f : R \to S$ be orientation preserving harmonic mapping, with respect to the given conformal metric $ds^2 = \rho(w)|dw|^2$ on S. Denote by $ds^2 = \sigma(z)|dz|^2$ a conformal metric on R whose density, in terms of a local parameter z on R, is given by the formula $\sigma(z) = \rho(f(z))|f_z(z)|^2$. Then, independently of the choice of the local parameter z,*

$$K(\sigma)(z) = K(\rho)(f(z))(1 - |\mu(z)|^2), \tag{4}$$

where $\mu(z) = \frac{f_{\bar{z}}(z)}{f_z(z)}$.

It is easy to prove that, when $f_z(z) \neq 0$, the function $|\mu(z)| = \frac{|f_{\bar{z}}(z)|}{|f_z(z)|}$ is well defined on the surface R and that $|\mu(z)| < 1$. We called μ as a complex dilatation of the mapping f. Also, note that one could exclude the orientation preserving property of the mapping f in the Proposition A.2, so the Bochner formula (4) remains valid at a point z, where $f_z(z) \neq 0$.

Definition 2.4. Let R and S be a Riemann surfaces. Assume that the mapping $f : R \to S$ of the class C^2 is orientation preserving. If there is some $k \in [0,1)$, for which $|\mu(z)| \leq k$, independently of the choice of the local parameter z on R, then we say that the mapping f is k quasi-regular. In addition, if the mapping f is homeomorphism, then the mapping f is called k quasi-conformal. Obviously, 0 quasi-conformal mappings are conformal.

Since the mapping f is orientation preserving, the previous definition makes sense. Usually, the constant k is usually taken as a smallest constant for which the above inequality holds. Also, some authors use the constant $K = \frac{1+k}{1-k} \geq 1$ as a quasi-conformal constant.

3 Ahlfors–Schwarz Lemma

Let $D = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk in $\mathbf{C}$. Using the conformal automorphisms $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$, $a \in \mathbf{D}$, of the disk $\mathbf{D}$, one can define pseudo-hyperbolic distance on $\mathbf{D}$ by the formula

$$\delta(z_1, z_2) = |\phi_{z_1}(z_2)|,$$

where z_1, $z_2 \in \mathbf{D}$. The hyperbolic distance between the points z_1 and z_2 on the unit disk $\mathbf{D}$ is defined by

$$d_h(z_1, z_2) = \log \frac{1 + \delta(z_1, z_2)}{1 - \delta(z_1, z_2)}.$$

It is easy to verify that the distance d_λ on $\mathbf{D}$, that is induced by the hyperbolic metric $ds^2 = \lambda(z)|dz|^2$ (see formula (2)), is the same as the hyperbolic distance above.

Lemma 3.2 (Schwarz). *Let $f : \mathbf{D} \to \mathbf{D}$ be an analytic function and $f(0) = 0$. Then $|f(z)| \leq |z|$, $z \in \mathbf{D}$, and $|f'(0)| \leq 1$. If $|f(z)| = |z|$, for some $z \neq 0$, or $|f'(0)| = 1$, then $f(z) = e^{i\alpha} z$, for some $\alpha \in [0, 2\pi)$.*

First impression is that the Schwarz lemma has only analytic character, but Pick gives to the Schwarz lemma a geometric interpretation.

Lemma 3.3 (Schwarz–Pick). *Let f be an analytic function from the unit disk $\mathbf{D}$ into itself. Then f does not increase the corresponding hyperbolic (pseudo-hyperbolic) distances.*

Let $\Omega \subset \mathbf{C}$ be a domain in the complex plane. We define an ultra-hyperbolic metric on Ω.

Definition 3.5. A metric $ds^2 = \rho(z)|dz|^2$, $\rho : \Omega \to \mathbf{R}^+ \cup \{0\}$, is called an ultra-hyperbolic metric on Ω, if ρ is upper semi-continuous on Ω and if for all $z_0 \in \Omega$, for which $\rho(z_0) > 0$, there exits a function ρ_0, positive and of the class C^2 in some neighborhood V of z_0, such that $\rho(z) \geq \rho_0(z)$, $K(\rho_0)(z) \leq -1$, for all $z \in V$, and $\rho_0(z_0) = \rho(z_0)$.

By a simple argument one could prove that if ρ is an ultra-hyperbolic metric on Ω and $f : \mathbf{D} \to \Omega$ is analytic function, then the metric $ds^2 = \sigma(z)|dz|^2$, where $\sigma(z) = \rho(f(z))|f'(z)|^2$, $z \in \mathbf{D}$, is ultra-hyperbolic on the unit disk $\mathbf{D}$.

Lemma 3.4 (Ahlfors–Schwarz). *Let $ds^2 = \rho(w)|dw|^2$ be an ultra-hyperbolic metric defined on some domain $\Omega \subset \mathbf{C}$ and let $f : \mathbf{D} \to \Omega$ be an analytic function. Then $\rho(f(z))|f'(z)|^2 \leq \lambda(z)$, for all $z \in \mathbf{D}$.*

Thus, the hyperbolic metric is maximal ultra-hyperbolic metric on the unit disk $\mathbf{D}$. The following statement is essential for our approach.

Theorem 3.1. *Assume that $ds^2 = \sigma(z)|dz|^2$ and $ds^2 = \rho(z)|dz|^2$ are two conformal metrics on the unit disk $\mathbf{D}$. If $\sigma(z) \to +\infty$, when $|z| \to 1_-$, and if $K(\rho)(z) \leq K(\sigma)(z) < 0$, for all $z \in \mathbf{D}$, then $\rho(z) \leq \sigma(z)$, $z \in \mathbf{D}$. In particular, if $K(\rho)(z) \leq -1$, then*

$$\rho(z) \leq \lambda(z) = \left(\frac{2}{1-|z|^2}\right)^2, \ z \in \mathbf{D}. \tag{5}$$

Proof. Let $0 < r < 1$ and $\sigma_r(z) = \frac{1}{r^2}\sigma(\frac{z}{r})$, $z \in D_r = \{z \in \mathbf{C} : |z| < r\}$. We define $f_r(z) = \log \frac{\rho(z)}{\sigma_r(z)}$, $z \in D_r$. Since $\sigma_r(z) \to +\infty$, when $|z| \to r_-$, the function f_r takes its maximum on D_r at some point z_0. Therefore, $(\triangle f_r)(z_0) \leq 0$, i.e. $(\triangle \log \rho)(z_0) - (\triangle \log \sigma_r)(z_0) \leq 0$, that is equivalent to the

$$\sigma_r(z_0)K(\sigma_r)(z_0) \leq \rho(z_0)K(\rho)(z_0). \tag{6}$$

From (6), trivially, we get $\frac{\rho(z_0)}{\sigma_r(z_0)} \leq \frac{|K(\sigma_r)(z_0)|}{|K(\rho)(z_0)|} \leq 1$, which imply $f_r(z) \leq f_r(z_0) = \log \frac{\rho(z_0)}{\sigma_r(z_0)} \leq 0$, $z \in D_r$, i.e. $\frac{\rho(z)}{\sigma_r(z)} \leq 1$, $z \in D_r$. By taking the limit, when $r \to 1_-$, we finish the proof.

Observe that the statement of the previous theorem remains valid if we assume that the conformal metric $ds^2 = \sigma(z)|dz|^2$ satisfies $\sigma(z) \to +\infty$, when $|z| \to 1_-$, and $-1 \leq K(\sigma)(z) < 0$, $z \in \mathbf{D}$, and if the metric $ds^2 = \rho(z)|dz|^2$ is ultra-hyperbolic on the unit disk.

As we announced (see [4, 8]), taking into account a property that $\sigma(z) \to +\infty$, when $|z| \to 1_-$, of the conformal metric $ds^2 = \sigma(z)|dz|^2$, where $\sigma(z) = \lambda(f(z))|f_z(z)|^2$, $z \in \mathbf{D}$, by using Bochner formula and Lemma 3.4, we can prove Wan's result appropriated in the form of the Theorem 3.2. To see that $\sigma(z) \to +\infty$, when $|z| \to 1_-$, one could conclude from the fact that every harmonic diffeomorphism of the unit disc $\mathbf{D}$ have continuous extension from $\overline{\mathbf{D}}$ onto itself (see [2], Theorem 2.1) and from the important property which states that there is a positive constant c such that $|f_z(z)| \geq c$, for all $z \in D$ (see [5], Theorem 2B). Thus, we could prove the following:

Theorem 3.2 (Wan). *Every harmonic quasi-conformal diffeomorphism of the unit disk* **D** *onto itself, is a quasi-isometry of the unit disk* **D***, with respect to the hyperbolic metric.*

Proof. Consider a conformal metric $ds^2 = \sigma(z)|dz|^2$, where $\sigma(z) = \lambda(f(z))|f_z(z)|^2$, $z \in \mathbf{D}$. By the Bochner formula, we get $\frac{1}{2}(\triangle \log \sigma)(z) = \sigma(z)(1-|\mu(z)|^2)$, $z \in \mathbf{D}$, i.e. $K(\sigma)(z) = (|\mu(z)|^2 - 1)$, $z \in \mathbf{D}$. Since f is quasi-conformal, there exists $0 \le k < 1$ such that $|\mu(z)| \le k$, $z \in \mathbf{D}$. Thus, $0 > K(\sigma)(z) = (|\mu(z)|^2 - 1) \ge -1$, $z \in \mathbf{D}$, and since $\sigma(z) \to +\infty$, when $|z| \to 1_-$, by applying Theorem 3.1, we obtain $\lambda(z) \le \lambda(f(z))|f_z(z)|^2$, $z \in \mathbf{D}$.

Let $z_1, z_2 \in \mathbf{D}$, $w_1 = f(z_1)$, $w_2 = f(z_2)$ its images under f and $\Gamma : [0,1] \to \mathbf{D}$ be the geodesic arc, with respect to the hyperbolic metric, joining the points w_1 and w_2 in $\mathbf{D}$. If $\gamma = f^{-1} \circ \Gamma$, then

$$\begin{aligned} d_h(z_1,z_2) &\le \int_\gamma \sqrt{\lambda(z)}|dz| \le \int_\gamma \sqrt{\lambda(f(z))}|f_z(z)||dz| \\ &= \int_\gamma \sqrt{\lambda(f(z))}|f_z(z)|\frac{1-|\mu(z)|}{1-|\mu(z)|}|dz| \\ &\le \frac{1}{1-k}\int_\gamma \sqrt{\lambda(f(z))}|f_z(z)|(1-|\mu(z)|)|dz| \\ &\le \frac{1}{1-k}\int_\Gamma \sqrt{\lambda(w)}|dw| = \frac{1}{1-k} d_h(w_1,w_2). \end{aligned}$$

Thus,

$$(1-k)d_h(z_1,z_2) \le d_h(w_1,w_2). \tag{7}$$

On the other hand, let us consider the conformal metric $ds^2 = (1-k^2)\sigma(z)|dz|^2$, $z \in \mathbf{D}$. Then we have, $K((1-k^2)\sigma(z))(z) = \frac{|\mu(z)|^1-1}{1-k^2} \le \frac{k^2-1}{1-k^2} = -1$, $z \in \mathbf{D}$, and $(1-k^2)\sigma(z) = (1-k^2)\lambda(f(z))|f_z(z)|^2 \le \lambda(z)$, $z \in \mathbf{D}$. Thus, if $\gamma : [0,1] \to \mathbf{D}$ is the geodesic arc, with respect to the hyperbolic metric, joining the points z_1 and z_2 in $\mathbf{D}$ and $\Gamma = f \circ \gamma$, then

$$\begin{aligned} d_h(z_1,z_2) &= \int_\gamma \sqrt{\lambda(z)}|dz| \ge \sqrt{1-k^2}\int_\gamma \sqrt{\lambda(f(z))}|f_z(z)||dz| \\ &= \sqrt{1-k^2}\int_\gamma \sqrt{\lambda(f(z))}|f_z(z)|\frac{1+|\mu(z)|}{1+|\mu(z)|}|dz| \\ &\ge \frac{1-k^2}{1+k}\int_\gamma \sqrt{\lambda(f(z))}|f_z(z)|(1+|\mu(z)|)|dz| \\ &\ge (1-k)\int_\Gamma \sqrt{\lambda(w)}|dw| \ge (1-k)d_h(w_1,w_2). \end{aligned}$$

Therefore,

$$d_h(w_1,w_2) \le \frac{1}{1-k} d_h(z_1,z_2). \tag{8}$$

Now, the proof follows from (7) and (8).

Note that, on the right hand side in inequality (8), one could obtain better constant $\sqrt{\frac{1+k}{1-k}}$, which is obviously not larger than $\frac{1}{1-k}$. Also, it is easy to prove that the metric $ds^2 = \sigma(z)|dz|^2$, where $\sigma(z) = \lambda(f(z))|f_z(z)|^2$, $z \in \mathbf{D}$, is a complete metric on the unit disc $\mathbf{D}$, since $-1 \le K(\sigma)(z) = (|\mu(z)|^2 - 1) \le -(1-k^2)$ (see [3], Theorem 2.1).

4 Ahlfors–Schwarz Lemma for Harmonic Quasi-Conformal Mappings

Let f be a k quasi-conformal harmonic mapping from the unit disc $\mathbf{U}$ into itself, with complex dilatation μ, and let $\sigma(z) = \lambda(f(z))|f_z(z)|^2$, $z \in \mathbf{D}$. We show that (see [4])

$$K(\sigma)(z) = -\left(1 + |\mu(z)|^2 + 2\mathrm{Re}\left(\frac{(f(z))^2\,\overline{f_{\bar z}(z)}}{f_z(z)}\right)\right), \tag{9}$$

and, therefore,

$$-(1+|\mu(z)|)^2 \le K(\sigma)(z) \le -(1-|\mu(z)|)^2, \tag{10}$$

for all $z \in \mathbf{D}$. The inequality (10), obtained above, enables us to apply Lemma 3.4 and Theorem 3.1 to get the following proposition.

Proposition A.3. *Let f be a k quasi-conformal harmonic mapping from the unit disc $\mathbf{D}$ into itself. Then for all $z \in \mathbf{D}$ we have*

$$|f_z(z)| \le \frac{1}{1-k}\,\frac{1-|f(z)|^2}{1-|z|^2}.$$

Proof. Let us define $\sigma(z) = (1-k)^2\lambda(f(z))|f_z(z)|^2$, $z \in \mathbf{D}$. Since f is harmonic in $\mathbf{D}$, i.e. $(f_z)_{\bar z}(z) = 0$, $z \in \mathbf{D}$, then f_z is holomorphic and, by Lewy's theorem, does not vanish in $\mathbf{D}$, hence the mapping $z \mapsto \log|f_z(z)|$ is harmonic in $\mathbf{D}$. Therefore, $(\triangle \log\sigma)(z) = (\triangle \log(\lambda\circ f))(z)$, for all $z \in \mathbf{D}$. We also have

$$\begin{aligned}(\triangle \log(\lambda\circ f))(z) &= 4(\log(\lambda\circ f))_{z\bar z}(z)\\ &= \frac{8|f_z(z)|^2}{(1-|f(z)|^2)^2}\left(1+|\mu(z)|^2+2\mathrm{Re}\left(\frac{(f(z))^2\,\overline{f_z(z)}\,\overline{f_{\bar z}(z)}}{|f_z(z)|^2}\right)\right)\\ &= \frac{2\sigma(z)}{(1-k)^2}\left(1+|\mu(z)|^2+2\mathrm{Re}\left(\frac{(f(z))^2\,\overline{f_z(z)}\,\overline{f_{\bar z}(z)}}{|f_z(z)|^2}\right)\right).\end{aligned}$$

Hence,

$$K(\sigma)(z) = -\frac{1}{(1-k)^2}\left(1+|\mu(z)|^2+2\mathrm{Re}\left(\frac{(f(z))^2\,\overline{f_z(z)}\,\overline{f_{\bar z}(z)}}{|f_z(z)|^2}\right)\right),$$

for all $z \in \mathbf{D}$. Since $\left|\mathrm{Re}\left(\frac{(f(z))^2\,\overline{f_z(z)}\,\overline{f_{\bar z}(z)}}{|f_z(z)|^2}\right)\right| \le \left|\frac{(f(z))^2\,\overline{f_z(z)}\,\overline{f_{\bar z}(z)}}{|f_z(z)|^2}\right| \le |\mu(z)|$, we get

$$\mathrm{Re}\left(\frac{(f(z))^2\,\overline{f_z(z)}\,\overline{f_{\bar z}(z)}}{|f_z(z)|^2}\right) \ge -|\mu(z)|,$$

and, therefore, $K(\sigma)(z) \le -\frac{1}{(1-k)^2}\left(1+|\mu(z)|^2-2|\mu(z)|\right) = -\frac{(1-|\mu(z)|)^2}{(1-k)^2} \le -1$. By using Lemma 3.4, we obtain

$$(1-k)^2\lambda(f(z))|f_z(z)|^2 \le \lambda(z), \tag{11}$$

for all $z \in \mathbf{D}$. Now, the theorem follows easily from (11).

Following the same procedure as in the proof of Theorem 3.2, by considering the geodesic arcs, we obtain next result (see [4]).

Theorem 4.3. *Let f be a k quasi-conformal harmonic mapping from the unit disc $\mathbf{D}$ into itself. Then for any two points z_1 and z_2 in $\mathbf{D}$ we have*

$$d_h(f(z_1), f(z_2)) \le \frac{1+k}{1-k}\,d_h(z_1,z_2).$$

In order to get the opposite inequality as in Proposition A.3, we have to suppose that f is onto.

Theorem 4.4. *Let f be a k quasi-conformal harmonic mapping from the unit disc $\mathbf{D}$ onto itself. Then for all $z \in \mathbf{D}$ we have*

$$|f_z(z)| \ge \frac{1}{1+k}\,\frac{1-|f(z)|^2}{1-|z|^2}$$

and $d_h(f(z_1), f(z_2)) \ge \dfrac{1-k}{1+k} d_h(z_1,z_2)$.

In [4] we gave the complete proof of the Theorem 4.4. Also, the interested reader could see that, by using differen approach, we obtained the same versions of the theorems, mentioned in this paper, for the upper half plane $\mathbf{H}$ (see [6]). In addition, we gave the similar results for the metrics that satisfy some inequalities. We can stress that is of interest to describe the properties of quasi-conformal harmonic mappings between some other domains in $\mathbf{C}$. Also, it would be important to analyze a quasi-conformal harmonic mapping from the unit disk $\mathbf{D}$ into the unit ball in $\mathbf{R}^3$.

Acknowledgements This note was supported by the project ON174032 of the Serbian Ministry of Education, headed by professor M. Mateljević. In addition, I would also like to thank professor V. Dobrev and professor B. Dragovich since I had the opportunity to take a part at the LT conferences in Varna.

References

1. Gardiner, F.P.: Teichmüller Theory and Quadratic Differentials. Awiley-Interscience Publication, New York (1987)
2. Kalaj, D.: Univalent Harmonic Mappings Between Jordan Domains, vol. 69, br. 83, str. 108–112. Publications de l'Institut Mathematique (2001)
3. Kraus, D., Roth, O., Ruscheweyh, S.: A boundary version of Ahlfors Lemma, locally complete conformal metrics and conformally invariant reflection principles for analytic maps. J. d'Analyse Math. **101**(1), 219–256 (2007)
4. Knežević, M., Mateljević, M.: On the quasi-isometries of harmonic quasiconformal mappings. Journal of Mathematical Analysis and Applications, **334**(1), 404–413 (2007)
5. Mateljevic, M.: Dirichlet's principle, distortion and related problems for harmonic mappings, Publication l'Inst Math -Belgrade, nouvelle serie **75**(89), 147–171 (2004) (special number Quasiconformal and Harmonic mappings, special guest editor M. Mateljević), ISSN 0350–1302
6. Mateljević, M., Božin, V., Knežević, M.: Quasiconformality of harmonic mappings between Jordan domains. FILOMAT **24**(3), 111–122 (2010)
7. Schoen, R., Yau, S.T.: Lectures on Harmonic Maps, Conf. Proc. and Lect. Not. in Geometry and Topology, vol. II. International Press, Cambridge (1997)
8. Wan, T.: Conastant mean curvature surface, harmonic maps, and univrsal Teichmüller space. J. Diff. Geom. **35**, 643–657 (1992)

A Note on the Categorification of Lie Algebras

Isar Goyvaerts and Joost Vercruysse

Abstract In this short note we study Lie algebras in the framework of symmetric monoidal categories. After a brief review of the existing work in this field and a presentation of earlier studied and new examples, we examine which functors preserve the structure of a Lie algebra.

1 Introduction

Lie algebras have many generalizations such as Lie superalgebras, Lie color and (G,χ)-Lie algebras, braided Lie algebras, Hom–Lie algebras, Lie algebroids, etc.

Motivated by the way that the field of Hopf algebras benefited from the interaction with the field of monoidal categories (see e.g. [10]) on one hand, and the strong relationship between Hopf algebras and Lie algebras on the other hand, the natural question arose whether it is possible to study Lie algebras within the framework of monoidal categories, and whether Lie theory could also benefit from this viewpoint.

First of all, it became folklore knowledge that Lie algebras can be easily defined in any symmetric monoidal k-linear category over a commutative ring k, or (almost equivalently) in any symmetric monoidal additive category. Within this setting, many (but not all) of the above cited examples can already be recovered. We will treat slightly in more detail the examples of Lie superalgebras and Hom–Lie algebras in the second section.

I. Goyvaerts
Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium
e-mail: igoyvaer@vub.ac.be

J. Vercruysse (✉)
Département de Mathématiques, Université Libre de Bruxelles, Boulevard du Triomphe, B-1050 Bruxelles, Belgium
e-mail: jvercruy@ulb.ac.be

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_41,

As some examples, in particular Lie color algebras, do not fit into this theory, several attempts were made to define Lie algebras in any *braided*, rather than *symmetric* monoidal category. A reason to do this is that G-graded modules over any group G give rise to a monoidal category, whose center is a braided monoidal category that can be described as the category of Yetter–Drinfel'd modules over a Hopf algebra. In this way, Lie color algebras and (G,χ)-Lie algebras are recovered as a special case (see [9]). A slightly different point of view is advocated by Majid, whose motivation is to describe deformations of Lie algebras, that he calls braided Lie algebras, inside a braided monoidal category, such that the universal enveloping of this deformed Lie algebra encodes the same information as the deformed (quantum) enveloping algebra of the original Lie algebra (see [7]).

We will not discuss further these two last cited types of Lie algebras in this short note. Rather, we will study Lie algebras in a (possibly non-symmetric, possibly non-braided) monoidal category, such that the Lie algebra allows a local symmetry. That is, the Lie algebra possesses a self-invertible Yang-Baxter operator and the anti-symmetry and Jacobi identity are defined up to this Yang-Baxter operator.

2 Lie Algebras in Additive Monoidal Categories

Throughout, we will work in a symmetric monoidal and additive category. Without any change in the arguments, one can work in any k-linear symmetric monoidal category, where k is a commutative ring with characteristic different from 2.

Definition 2.1. Let $\mathscr{C} = (C,\otimes,I,a,l,r,c)$ be a symmetric monoidal additive category with associativity constraint a, left- and right unit constraints resp. l and r and symmetry c. A *Lie algebra* in $\mathscr{C}$ is a pair $(L,[-,-])$, where L is an object of $\mathscr{C}$ and $[-,-]: L\otimes L \to L$ is a morphism in $\mathscr{C}$ that satisfies the following two conditions

$$[-,-]\circ(id_{L\otimes L}+c_{L,L}) = 0_{L\otimes L,L}, \tag{1}$$

$$[-,-]\circ(id_L\otimes[-,-])\circ(id_{L\otimes(L\otimes L)}+t+w) = 0_{L\otimes(L\otimes L),L}, \tag{2}$$

where $t = c_{L\otimes L,L}\circ a^{-1}_{L,L,L}$ and $w = a_{L,L,L}\circ c_{L,L\otimes L}$.

Example 2.1. Let $\mathscr{M}_R = (\mathsf{Mod}(R),\otimes_R,R,a,l,r,c)$ be the Abelian, symmetric monoidal category of (right) R-modules over a commutative ring R ($\mathsf{Char}(R)\neq 2$) with trivial associativity and unit constraints and with symmetry $c=\tau$ (the flip). Taking a Lie algebra in $\mathscr{M}_R$, one obtains the classical definition of a Lie algebra over R.

Example 2.2. Let $\mathscr{C} = (\mathsf{Vect}^{\mathbb{Z}_2}(k),\otimes_k,k,a,l,r,c)$ be the Abelian, symmetric monoidal category of k-vector spaces ($\mathsf{Char}(k)\neq 2$) graded by $\mathbb{Z}_2$. We take the trivial associativity and unit constraints. The symmetry c is defined as follows: For any pair of objects (V,W) in $\mathscr{C}$; $c_{V,W}: V\otimes W \to W\otimes V; v\otimes w \mapsto (-1)^{|v||w|}w\otimes v$.

Taking a Lie algebra in $\mathscr{C}$, one recovers the definition of a Lie superalgebra (see also [9]).

Example 2.3. We now recall from [2], the construction of a non-trivial example of an Abelian, non-strict symmetric monoidal category (called the Hom-construction). Let $\mathscr{C}$ be a category. A new category $\mathscr{H}(\mathscr{C})$ is introduced as follows: objects are couples (M,μ), with $M \in \mathscr{C}$ and $\mu \in \mathsf{Aut}_{\mathscr{C}}(M)$. A morphism $f:\ (M,\mu) \to (N,\nu)$ is a morphism $f:\ M \to N$ in $\mathscr{C}$ such that $\nu \circ f = f \circ \mu$.

Now assume that $\mathscr{C} = (\mathscr{C}, \otimes, I, a, l, r, c)$ is a braided monoidal category. Then one easily verifies that $\mathscr{H}(\mathscr{C}) = (\mathscr{H}(\mathscr{C}), \otimes, (I,I), a, l, r, c)$ is again a braided monoidal category, with the tensor product defined by the following formula

$$(M,\mu) \otimes (N,\nu) = (M \otimes N, \mu \otimes \nu),$$

for (M,μ) and (N,ν) in $\mathscr{H}(\mathscr{C})$. On the level of morphisms, the tensor product is the tensor products of morphisms in $\mathscr{C}$. By deforming the category $\mathscr{H}(\mathscr{C})$, we obtain the category $\widetilde{\mathscr{H}}(\mathscr{C}) = (\mathscr{H}(\mathscr{C}), \otimes, (I,I), \tilde{a}, \tilde{l}, \tilde{r}, c)$ which is still a braided monoidal category (but no longer strict if $\mathscr{C}$ was strict). The associativity constraint $\tilde{a}$ is given by the formula

$$\tilde{a}_{M,N,P} = a_{M,N,P} \circ ((\mu \otimes N) \otimes \pi^{-1}) = (\mu \otimes (N \otimes \pi^{-1})) \circ a_{M,N,P},$$

for $(M,\mu), (N,\nu), (P,\pi) \in \mathscr{H}(\mathscr{C})$. The unit constraints $\tilde{l}$ and $\tilde{r}$ are given by

$$\tilde{l}_M = \mu \circ l_M = l_M \circ (I \otimes \mu)\ ;\ \tilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes I).$$

Now, A Lie algebra in $\widetilde{\mathscr{H}}(\mathscr{M}_R)$ is a triple $(L, [-,-], \alpha)$ with $(L,\alpha) \in \widetilde{\mathscr{H}}(\mathscr{M}_R)$, $[-,-]:\ L \otimes L \to L$ a morphism in $\widetilde{\mathscr{H}}(\mathscr{M}_R)$ (that is, $[\alpha(x) \otimes \alpha(y)] = \alpha[x,y]$), satisfying anti-symmetry and the so-called Hom–Jacobi identity;

$$[\alpha(x) \otimes [y \otimes z]] + [\alpha(y) \otimes [z \otimes x]] + [\alpha(z) \otimes [x \otimes y]] = 0,$$

We thus recover the definition of a Hom–Lie algebra (cf.[5]), where in this case α is a classical Lie algebra isomorphism.

Example 2.4. A Lie coalgebra in $\mathscr{C}$ is a Lie algebra in $\mathscr{C}^{op}$, the opposite category of $\mathscr{C}$. This means that a Lie coalgebra is a pair $(C, <->)$, where $<->: C \to C \otimes C$ is a map that satisfies the following two conditions

$$(id_{C\otimes C} + c_{C,C}) \circ <-> = 0;$$
$$(id_{C\otimes(C\otimes C)} + t + w) \circ (id_C \otimes <->) \circ <-> = 0.$$

Lie coalgebras were introduced by Michaelis [8].

Our next aim is to "free" the definition of Lie algebra of the global symmetry on our additive monoidal category.

Definition 2.2. Let $\mathscr{C} = (C, \otimes, I, a, l, r)$ be a (possibly non-symmetric) monoidal category and L an object in $\mathscr{C}$. A *self-invertible Yang-Baxter operator* on L is a morphism $c : L \otimes L \to L \otimes L$ that satisfies the following conditions:

$$c \circ c = L \otimes L; \tag{3}$$

$$a_{L,L,L} \circ (c \otimes L) \circ a^{-1}_{L,L,L} \circ (L \otimes c) \circ a_{L,L,L} \circ (c \otimes L)$$
$$= (L \otimes c) \circ a_{L,L,L} \circ (c \otimes L) \circ a^{-1}_{L,L,L} \circ (L \otimes c) \circ a_{L,L,L} \tag{4}$$

Given an object L in $\mathscr{C}$, together with a self-invertible Yang Baxter operator c as above, we can construct the following morphisms in $\mathscr{C}$:

$$t = t_c := a_{L,L,L} \circ (c \otimes L) \circ a^{-1}_{L,L,L} \circ (L \otimes c);$$
$$w = w_c := (L \otimes c) \circ a_{L,L,L} \circ (c \otimes L) \circ a^{-1}_{L,L,L}.$$

One can easily verify that $t \circ t = w$ and $t \circ w = id = w \circ t$.

Example 2.5. If $\mathscr{C}$ is a symmetric monoidal category, with symmetry $c_{X,Y} : X \otimes Y \to Y \otimes X$, for all $X, Y \in \mathscr{C}$, then $c_{L,L}$ is a self-invertible Yang-Baxter operator for $L \in \mathscr{C}$. Obviously, $c_{L,L}$ satisfies conditions (3); to see that $c_{L,L}$ also satisfies (4), one applies the hexagon condition in combination with the naturality of c. Moreover, $t_{c_{L,L}} = c_{L\otimes L,L} \circ a^{-1}_{L,L,L}$ and $w_{c_{L,L}} = a_{L,L,L} \circ c_{L,L\otimes L}$.

Definition 2.3. Let $\mathscr{C}$ be an additive, monoidal category, but not necessarily symmetric. A YB-Lie algebra in $\mathscr{C}$ is a triple $(L, \lambda, [-,-])$, where L is an object of $\mathscr{C}$, λ is a self-invertible Yang-Baxter operator for L in $\mathscr{C}$, and $[-,-] : L \otimes L \to L$ is a morphism in $\mathscr{C}$ that satisfies

$$[-,-] \circ (id_{L\otimes L} + \lambda) = 0_{L\otimes L,L}, \tag{5}$$

$$[-,-] \circ (id_L \otimes [-,-]) \circ (id_{L\otimes(L\otimes L)} + t_\lambda + w_\lambda) = 0_{L\otimes(L\otimes L),L}. \tag{6}$$

$$(id_L \otimes [-,-]) \circ t_\lambda \circ a_{L,L,L} = \lambda \circ ([-,-] \otimes id_L) \tag{7}$$

We call (6) the (right) λ-Jacobi identity for L. Equation (7) expresses the compatibility between the Lie bracket $[-,-]$ and the Yang-Baxter operator λ. Remark that in the case were $\lambda = c_{L,L}$ (see Example 2.5), this condition is automatically satisfied by the naturality of the symmetry $c_{-,-}$.

As for usual Lie algebras, the definition of a YB-Lie algebra is left–right symmetric, as follows from the following Lemma.

Lemma 2.1. *Let $(L, \lambda, [-,-])$ be a YB-Lie algebra in $\mathscr{C}$. Then L also satisfies the left λ-Jacobi identity, that is the following equation holds*

$$[-,-] \circ ([-,-] \otimes id_L) \circ a^{-1}_{L,L,L} \circ (id_{L\otimes(L\otimes L)} + t_\lambda + w_\lambda) = 0_{L\otimes(L\otimes L),L}.$$

Proof. Using (5) in the first equality, (7) in the second equality, $w_\lambda = t_\lambda^2 = t_\lambda^{-1}$ in the third equality and (6) in the last equality we find

$$\begin{aligned}
&[-,-] \circ ([-,-] \otimes id_L) \circ a^{-1}_{L,L,L} \circ (id_{L\otimes(L\otimes L)} + t_\lambda + w_\lambda) \\
&= -[-,-] \circ \lambda \circ ([-,-] \otimes id_L) \circ a^{-1}_{L,L,L} \circ (id_{L\otimes(L\otimes L)} + t_\lambda + w_\lambda) \\
&= -[-,-] \circ (id_L \otimes [-,-]) \circ t_\lambda \circ a_{L,L,L} \circ a^{-1}_{L,L,L} \circ (id_{L\otimes(L\otimes L)} + t_\lambda + w_\lambda) \\
&= -[-,-] \circ (id_L \otimes [-,-]) \circ (id_{L\otimes(L\otimes L)} + t_\lambda + w_\lambda) = 0
\end{aligned}$$
□

Example 2.6. Let $\mathscr{C}$ be any additive category, and consider the functor category $\mathsf{End}(\mathscr{C})$ of additive endofunctors on $\mathscr{C}$ and natural transformations between them. Recall that this is a monoidal category with the composition of functors as tensor product on objects and the Godement product as tensor product on morphisms. Moreover, $\mathsf{End}(\mathscr{C})$ inherits the additivity of $\mathscr{C}$. We will call a YB-Lie algebra in $\mathsf{End}(\mathscr{C})$ a *Lie monad* on $\mathscr{C}$.

Example 2.7. Let (B, μ_B) be an associative algebra in an additive, monoidal category $\mathscr{C}$ and suppose there is a self-invertible Yang-Baxter operator $\lambda : B\otimes B \to B\otimes B$ on B, such that the conditions hold:

$$\begin{aligned}
(B\otimes \mu_B) \circ a^{-1}_{B,B,B} \circ w_\lambda &= \lambda \circ (\mu_B \otimes B) \\
(\mu_B \otimes B) \circ t_\lambda \circ a_{B,B,B} &= \lambda \circ (B\otimes \mu_B)
\end{aligned} \tag{8}$$

Then we can consider a YB-Lie algebra structure on B, induced by the commutator bracket $[-,-]_B$ (defined by $[-,-]_B = \mu_B \circ (B\otimes B - \lambda)$). For example, If B is a braided Hopf algebra (or a braided bialgebra) in the sense of Takeuchi (see [10]) then B admits a Yang-Baxter operator λ that satisfies the diagrams (8). If λ is self-invertible, the commutator algebra of B is a YB-Lie-algebra in our sense. Moreover, the primitive elements of B can be defined as the equalizer $(P(B), eq)$ in the following diagram

$$P(B) \xrightarrow{\quad eq \quad} B \underset{\eta\otimes B + B\otimes\eta}{\overset{\Delta}{\rightrightarrows}} B\otimes B$$

where $\Delta : B \to B\otimes B$ is the comultiplication on B and $\eta : k \to B$ is the unit of B. One can show (see forthcoming [4]) that $P(B)$ is again a YB-Lie algebra.

3 Functorial Properties

In this section we study functors that send Lie algebras to Lie algebras.

Let $\mathscr{C} = (C, \otimes, I, a, l, r)$ and $\mathscr{D} = (D, \odot, J, a', l', r')$ be two additive, monoidal categories. For simplicity, we will suppose that $\mathscr{C}$ and $\mathscr{D}$ are strict monoidal, that is a, l, r and a', l', r' are identity natural transformations and will be omitted. By Mac Lane's coherence theorem, this puts no restrictions on the subsequent results.

Definition 3.4. A functor $F : \mathscr{C} \to \mathscr{D}$ will be called a *non-unital monoidal* functor, if there exists a natural transformation $\Psi_{X,Y} : FX \odot FY \to F(X \otimes Y)$ that satisfies the following condition

$$\Psi_{X\otimes Y,Z} \circ (\Psi_{X,Y} \odot FZ) = \Psi_{X,Y\otimes Z} \circ (FX \odot \Psi_{Y,Z}). \tag{9}$$

Lemma 3.2. *Let $(F, \Psi) : \mathscr{C} \to \mathscr{D}$ be a non-unital monoidal functor and use notation as above. Let $\lambda : L \otimes L \to L \otimes L$ be a self-invertible Yang-Baxter operator on $L \in \mathscr{C}$. Suppose that there exists a morphism $\lambda' : FL \otimes FL \to FL \otimes FL$ such that $\Psi_{L,L} \circ \lambda' = F(\lambda) \circ \Psi_{L,L}$. If $\Psi_{L\otimes L,L}$, $\Psi_{L,L} \odot FL$ and $\Psi_{L,L}$ are monomorphisms (e.g. Ψ is a natural monomorphism and the endofunctor $- \odot FL$ preserves monos), then λ' is a self invertible Yang-Baxter operator on FL.*

Proof. Using the compatibility between λ and λ' in the first equality, the naturality of Ψ in the second equality, (9) in the third equality, a repetition of the above arguments in the fourth equality, the Yang-Baxter identity for λ in the fifth equality, and a reverse computation in the last equality, we find

$$\begin{aligned}
&\Psi_{L\otimes L,L} \circ (\Psi_{L,L} \odot FL) \circ (\lambda' \odot FL) \circ (FL \odot \lambda') \circ (\lambda' \odot FL) \\
&\quad = \Psi_{L\otimes L,L} \circ (F(\lambda) \odot FL) \circ (\Psi_{L,L} \odot FL) \circ (FL \odot \lambda') \circ (\lambda' \odot FL) \\
&\quad = F(\lambda \otimes L) \circ \Psi_{L\otimes L,L} \circ (\Psi_{L,L} \odot FL) \circ (FL \odot \lambda') \circ (\lambda' \odot FL) \\
&\quad = F(\lambda \otimes L) \circ \Psi_{L,L\otimes L} \circ (FL \odot \Psi_{L,L}) \circ (FL \odot \lambda') \circ (\lambda' \odot FL) \\
&\quad = F(\lambda \otimes L) \circ F(L \otimes \lambda) \circ F(\lambda \otimes L) \circ \Psi_{L,L\otimes L} \circ (FL \odot \Psi_{L,L}) \\
&\quad = F(L \otimes \lambda) \circ F(\lambda \otimes L) \circ F(L \otimes \lambda) \circ \Psi_{L,L\otimes L} \circ (FL \odot \Psi_{L,L}) \\
&\quad = \Psi_{L\otimes L,L} \circ (\Psi_{L,L} \odot FL) \circ (FL \odot \lambda') \circ (\lambda' \odot FL) \circ (FL \odot \lambda')
\end{aligned}$$

As $\Psi_{L\otimes L,L}$ and $\Psi_{L,L} \odot FL$ are monomorphisms, we conclude from the computation above that λ' satisfies the Yang-Baxter identity. In a similar way, one proofs that λ' is self-invertible. □

Lemma 3.3. *Let $(F, \Psi) : \mathscr{C} \to \mathscr{D}$ be a non-unital monoidal functor and use notation as above. Let $\lambda : L \otimes L \to L \otimes L$ and $\lambda' : FL \otimes FL \to FL \otimes FL$ be $\mathscr{C}$ (resp. $\mathscr{D}$)-morphisms such that $\Psi_{L,L} \circ \lambda' = F(\lambda) \circ \Psi_{L,L}$. Then the following identities hold*

$$\Psi_{L\otimes L,L} \circ (\Psi_{L,L} \odot FL) \circ t_{\lambda'} = F(t_\lambda) \circ \Psi_{L,L\otimes L} \circ (FL \odot \Psi_{L,L}) \tag{10}$$

$$\Psi_{L\otimes L,L} \circ (\Psi_{L,L} \odot FL) \circ w_{\lambda'} = F(w_\lambda) \circ \Psi_{L,L\otimes L} \circ (FL \odot \Psi_{L,L}) \tag{11}$$

Proof. Let us proof (10), the proof of (11) is completely similar.

$$\begin{aligned}\Psi_{L\otimes L,L}\circ(\Psi_{L,L}\odot FL)\circ t_{\lambda'} &= \Psi_{L\otimes L,L}\circ(\Psi_{L,L}\odot FL)\circ(\lambda'\odot FL)\circ(FL\odot\lambda')\\ &= \Psi_{L\otimes L,L}\circ(F(\lambda)\odot FL)\circ(\Psi_{L,L}\odot FL)\circ(FL\odot\lambda')\\ &= F(\lambda\otimes L)\circ\Psi_{L\otimes L,L}\circ(\Psi_{L,L}\odot FL)\circ(FL\odot\lambda')\\ &= F(\lambda\otimes L)\circ\Psi_{L,L\otimes L}\circ(FL\odot\Psi_{L,L})\circ(FL\odot\lambda')\\ &= F(\lambda\otimes L)\circ\Psi_{L,L\otimes L}\circ(FL\odot F(\lambda))\circ(FL\odot\Psi_{L,L})\\ &= F(\lambda\otimes L)\circ F(L\otimes\lambda)\circ\Psi_{L,L\otimes L}\circ(FL\odot\Psi_{L,L})\\ &= F(t_\lambda)\circ\Psi_{L,L\otimes L}\circ(FL\odot\Psi_{L,L})\end{aligned}$$

We used the compatibility between λ and λ' in the second and fifth equality, the naturality of Ψ in the third and sixth equality and (9) in the fourth. □

Remark that the existence of the morphism λ' as in the above lemmata is guaranteed if F is a strong monoidal functor, as in this situation Ψ is invertible. Furthermore, if $\mathscr{C}$ and $\mathscr{D}$ are symmetric monoidal and we take λ and λ' induced by the symmetry of $\mathscr{C}$ and $\mathscr{D}$ respectively, then the compatibility condition between λ and λ' is automatically satisfied.

Theorem 3.1. *Let $(F,\Psi):\mathscr{C}\to\mathscr{D}$ be an additive non-unital monoidal functor and $(L,\lambda,[-,-])$ a YB-Lie algebra in $\mathscr{C}$. Suppose that there exists a self-invertible Yang-Baxter operator $\lambda':FL\otimes FL\to FL\otimes FL$ such that $\Psi_{L,L}\circ\lambda'=F(\lambda)\circ\Psi_{L,L}$. Then $(FL,\lambda',[-,-]')$ is a YB-Lie algebra in $\mathscr{D}$ with Lie-bracket given by*

$$[-,-]':FL\odot FL\xrightarrow{\ \Psi_{L,L}\ }F(L\otimes L)\xrightarrow{\ F([-,-])\ }FL\,.$$

Proof. Let us check that $[-,-]'$ is antisymmetric. Using the antisymmetry of $(L,[-,-])$ and compatibility between λ and λ' we obtain

$$\begin{aligned}[-,-]'\circ\lambda' &= F([-,-])\circ\Psi_{L,L}\circ\lambda' = F([-,-])\circ F(\lambda)\circ\Psi_{L,L}\\ &= F([-,-]\circ\lambda)\circ\Psi_{L,L} = -F([-,-])\circ\Psi_{L,L} = -[-,-]'.\end{aligned}$$

Next, let us check the Jacobi identity

$$\begin{aligned}&[-,-]'\circ(id_{FL}\odot[-,-]')\circ(id_{FL\odot(FL\odot FL)}+t_{\lambda'}+w_{\lambda'})\\ &= F([-,-])\circ\Psi_{L,L}\circ(id_{FL}\odot F([-,-]))\circ(id_{FL}\odot\Psi_{L,L})\circ(id+t_{\lambda'}+w_{\lambda'})\\ &= F([-,-])\circ F(id_L\odot[-,-])\circ\Psi_{L,L\otimes L}\circ(id_{FL}\odot\Psi_{L,L})\circ(id+t_{\lambda'}+w_{\lambda'})\\ &= F([-,-])\circ F(id_L\odot[-,-])\circ F(id_{L\otimes L\otimes L}+t_\lambda+w_\lambda)\circ\Psi_{L\otimes L,L}\circ(\Psi_{L,L}\odot id_{FL})\\ &= 0\end{aligned}$$

We used the naturality of Ψ in the second equality and Lemma 3.3 in the third equation and (6) in the last equality. □

Combining Theorem 3.1 with Lemma 3.2, we immediately obtain the following two satisfying corollaries, which allow to apply Theorem 3.1 in practical situations.

Corollary 3.1. *Let $(F,\Psi_0,\Psi):\mathscr{C}\to\mathscr{D}$ be an additive symmetric monoidal functor between additive symmetric monoidal categories. If $(L,[-,-])$ is a Lie algebra in $\mathscr{C}$, then $(FL,\Psi_{L,L}\circ[-,-])$ is a Lie algebra in $\mathscr{D}$.*

Corollary 3.2. *Let $(F,\Psi):\mathscr{C}\to\mathscr{D}$ be an additive (non-unital) strong monoidal functor between additive monoidal categories. If $(L,\lambda,[-,-])$ is a YB-Lie algebra in $\mathscr{C}$, then $(FL,\Psi_{L,L}^{-1}\circ F(\lambda)\circ\Psi_{L,L},F([-,-])\circ\Psi_{L,L})$ is a YB-lie algebra in $\mathscr{D}$.*

Example 3.8. Let us return to the case of Lie superalgebras, which are exactly Lie algebras in $\mathsf{Vect}^{\mathbb{Z}_2}(k)$ (cf. Example 2.2). It is well-known that this category is equivalent (even isomorphic) to the category $\mathscr{M}^{k[\mathbb{Z}_2]}$ of comodules over the group algebra $k[\mathbb{Z}_2]$, which is in fact a Hopf-algebra. Moreover, this equivalence of categories is an additive, monoidal equivalence, and even a symmetric one, taking into account the coquasitriangular structure on $k[\mathbb{Z}_2]$. By our Corollary 3.1, this implies that Lie algebras can be computed equivalently in $\mathsf{Vect}^{\mathbb{Z}_2}(k)$ as well as in $\mathscr{M}^{k[\mathbb{Z}_2]}$. In fact, Lie algebras in a general monoidal category $\mathscr{M}^H$ of comodules over a coquasitriangular bialgebra H have been studied in [3] and in [1], amongst others. Such a Lie algebra is a triple $(M,\rho_M,[-,-])$ with M a k-vectorspace, a coaction ρ on M and a k-linear map $[-,-]:M\otimes M\to M$ such that $([-,-]\otimes id_H)\circ\rho_{M\otimes M}=\rho_M\circ[-,-]$, which satisfy the condition (5):

$$[x\otimes y]=-[(y_{[0]}\otimes x_{[0]})\sigma(x_{[1]}\otimes y_{[1]})]$$

and (6):

$$\begin{aligned}&[x\otimes[y\otimes z]]+[z_{[0][0]}\otimes[x_{[0]}\otimes y_{[0]}]]\sigma(y_{[1]}\otimes z_{[1]})\sigma(x_{[1]}\otimes z_{[0][1]})\\&[y_{[0]}\otimes[z_{[0]}\otimes x_{[0][0]}]]\sigma(x_{[1]}\otimes y_{[1]})\sigma(x_{[0][1]}\otimes z_{[1]}).\end{aligned}$$

whenever $x,y,z\in M$ and where we used the Sweedler–Heynemann for comodules and $\sigma:H\otimes H\to k$ is the convolution invertible bilinear map from the coquasitriangular structure on H.

Example 3.9. Let us consider again the Hom-construction. It is proven in [2, Proposition 1.7] that the categories $\mathscr{H}(\mathscr{C})=(\mathscr{H}(\mathscr{C}),\otimes,(I,I),a,l,r,c)$ and $\tilde{\mathscr{H}}(\mathscr{C})=(\mathscr{H}(\mathscr{C}),\otimes,(I,I),\tilde{a},\tilde{l},\tilde{r},c)$ are isomorphic as monoidal categories. Let us briefly recall this isomorphism.

Let $F:\ \mathscr{H}(\mathscr{C})\to\widetilde{\mathscr{H}}(\mathscr{C})$ be the identity functor, and $\Psi_0:\ I\to I$ the identity. We define a natural transformation, by putting for all $M,N\in\mathscr{H}(\mathscr{C})$,

$$\Psi_{M,N}=\mu\otimes\nu:\ F(M)\otimes F(N)=M\otimes N\to F(M\otimes N)=M\otimes N.$$

Then (F, Ψ_0, Ψ) is a strict monoidal functor and it is clearly an isomorphism of categories. Moreover, if $\mathscr{C}$ is an additive category, then F is also an additive functor, so F preserves Lie algebras by Corollary 3.1 and YB-Lie algebras by Corollary 3.2. Let $((L,\alpha),[-,-])$ be a Lie algebra in $\mathscr{H}(\mathscr{C})$ i.e. $(L,[-,-])$ is a Lie algebra in $\mathscr{C}$ with a Lie algebra isomorphism α. Then $(F(L,\alpha),[-,-]')$ is a Lie algebra in $\widetilde{\mathscr{H}}(\mathscr{C})$. The inverse functor is also strict monoidal and additive, hence preserves Lie algebras. Consequently, Hom–Lie algebras, where α is a Lie algebra isomorphism, are nothing else than Lie algebras endowed with a Lie algebra isomorphism.

Example 3.10. Multiplier algebras serve as an important tool to study certain types of non-compact quantum groups, within the framework of multiplier Hopf algebras, see [11]. In [6] it was proven that the creation of the multiplier algebra of a non-degenerated idempotent (non-unital) k-algebra leads to a (symmetric) monoidal (additive) functor $(\mathbb{M}, \Psi_0, \Psi)$ on the category of these algebras. Hence the multiplier construction preserves Lie algebras by our Theorem 3.1. Moreover, as the monoidal product on the category of non-degenerated idempotent (non-unital) k-algebras is given by the monoidal product of underlying k-vectorspaces it follows that the multiplier construction also preserves the commutator Lie algebras associated to these algebras. Furthermore, the natural transformation Ψ is a natural monomorphism. Therefore, we can apply Lemma 3.2 and the functor $\mathbb{M}$ also preserves YB-Lie algebras.

Example 3.11. Let $\mathscr{C}$ be an additive monoidal category, and consider the additive monoidal category $\mathsf{End}(\mathscr{C})$ from Example 2.6. Consider the functor $\mathsf{E} : \mathscr{C} \to \mathsf{End}(\mathscr{C})$, that sends every object $X \in \mathscr{C}$ to the endofunctor $- \otimes X : \mathscr{C} \to \mathscr{C}$. Then E is an additive strong monoidal functor. By Corollary 3.2, a YB-Lie algebra in $\mathscr{C}$ leads to a YB-Lie algebra in $\mathsf{End}(\mathscr{C})$, i.e. to a Lie monad on $\mathscr{C}$.

Suppose now that $\mathscr{C}$ is a right closed monoidal category, i.e. every endofunctor $- \otimes X$ has a right adjoint, that we denote by $\mathsf{H}(X,-) : \mathscr{C} \to \mathscr{C}$. Then there exist natural isomorphisms $\pi_{Y,Z} : \mathsf{Hom}(Y, \mathsf{H}(X,Z)) \to \mathsf{Hom}(Y \otimes X, Z)$. One can proof (see e.g. [4]) that this isomorphism can be extended to an isomorphism

$$\mathsf{H}(X, \mathsf{H}(Y,-)) \cong \mathsf{H}(X \otimes Y, -)$$

in $\mathscr{C}$. Hence the contravariant functor $\mathsf{H} : \mathscr{C} \to \mathsf{End}(\mathscr{C})$ that sends an object $X \in \mathscr{C}$ to the endofunctor $\mathsf{H}(X,-)$ is a strong monoidal functor. As consequence, this functor sends a YB-Lie coalgebra in $\mathscr{C}$ to a Lie monad on $\mathscr{C}$. This idea is further explored in [4] to study dualities between infinite dimensional Hopf algebras and Lie algebras.

References

1. Bathurin, Y., Fischman, D., Montgomery, S.: On the generalized Lie structure of associative algebras. Israel J. Math. **96**, 27–48 (1996)
2. Caenepeel, S., Goyvaerts, I.: Monoidal Hom-Hopf algebras. Comm. Algebra **39**, 2216–2240 (2011)

3. Fischman, D., Montgomery, S.: A Schur double centralizer theorem for cotriangular Hopf algebras and generalized Lie algebras. J. Algebra **168**, 594–614 (1994)
4. Goyvaerts, I., Vercruysse, J.: On the duality of generalized Lie and Hopf algebras, in preparation
5. Hartwig, J., Larsson, D., Silvestrov, S.: Deformations of Lie algebras using σ-derivations. J. Algebra **295**, 314–361 (2006)
6. Janssen, K., Vercruysse, J.: Multiplier bi- and Hopf algebras. J. Algebra Appl. **9**, 275–303 (2010)
7. Majid, S.: Quantum and braided-Lie algebras. J. Geom. Phys. **13**, 307–356 (1994)
8. Michaelis, W.: Lie Coalgebras. Adv. Math. **38**, 1–54 (1980)
9. Pareigis, B.: On Lie algebras in the category of Yetter-Drinfeld modules. Appl. Categorical Struct. **6**, 151–175 (1998)
10. Takeuchi, M.: Survey of braided Hopf algebras. Contemp. Math. **267**, 301–323 (2000)
11. Van Daele, A.: Multiplier Hopf algebras. Trans. Am. Math. Soc. **342**(2), 917–932 (1994)

A Continuous Bialgebra Structure on a Loop Algebra

Rémi Léandre

Abstract We define on the set of Fourier series on a Lie algebra operations which give on it the structure of a continuous bialgebra.

1 Introduction

There are basically two theories of deformation:

(−) Deformation quantization (See [4, 15, 17] for reviews). People deform a convenient algebra of functions related to a symplectic structure or more generally to a Poisson structure on a manifold.
(−) Quantum groups (See [5]). People deform a coproduct instead of a product.

The tools of infinite dimensional analysis (Malliavin Calculus or white noise analysis) can be used for the theory of deformation quantization in infinite dimension [3, 6–9, 14].

This leads Léandre to use these tools to construct consistently some linear Poisson structures on some path spaces [10–12].

Moreover the Yang-Baxter equation in infinite dimension [1] was used to define formal Sklyanin Poisson structures on various path spaces [2, 16]. Léandre gives an analytic meaning to the simplest one of these Sklyanin Poisson structures [13].

Yang-Baxter equation plays a big role in the theory of Lie bialgebras. Let us recall a result of Drinfeld [5, Theorem 3.1].

Let $(g, [.])$ be a finite dimensional Lie algebra. Let $r \in \Lambda^2 g$. If r satisfies the Yang-Baxter equation, $\partial r(a) = [a \otimes 1 + 1 \otimes a, r]$ defines a Lie bialgebra structure on it (This means that the coproduct satisfies the co-Jacobi relation).

R. Léandre (✉)
Laboratoire de Mathématiques, Université de Franche Comté, 25030, Besançon, France
e-mail: Remi.leandre@univ-fcomte.fr

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics: IX International Workshop*, Springer Proceedings in Mathematics & Statistics 36,
DOI 10.1007/978-4-431-54270-4_42, © Springer Japan 2013

The simplest example in infinite dimension is given in [5, pp. 51–52]: in this case people consider as Lie algebra the set of Fourier polynomials in g. The object of this communication is to study it on Fourier series instead of Fourier polynomials!

2 The Formal Bialgebra Structure

We consider the complexified Lie algebra g of a compact Lie group endowed with its biinvariant metric $\|.\|$. We consider the set H_f of Fourier polynomials $g(.)$

$$g(s) = \sum_{n \geq 0} a_n \exp[2i\pi ns] \tag{1}$$

where only a finite number of a_n ($a_n \in g$!) are not equal to 0.

$H_f \otimes H_f$ consists of two parameters Fourier polynomials $g(.,.)$ with values in $g \otimes g$:

$$g(s,t) = \sum_{n \geq 0, m \geq 0} a_{n,m} \exp[2i\pi ns] \exp[2i\pi mt] \tag{2}$$

where only a finite number of $a_{n,m}$ belonging to $g \otimes g$ are not equal to 0.

Let x_i be an orthonormal basis of g. Classically H_f is a Lie bialgebra [5, p. 52]:

(-i) The Lie bracket is constructed as follows:

$$[g^1, g^2](s) = \sum [a^1_{n_1}, a^2_{n_2}] \exp[2i\pi n_1 s] \exp[2i\pi n_2 s] \tag{3}$$

(-ii) The Lie cobracket is given by the next formula:

$$\delta(g)(s,t) = \sum_{n, 0 \leq r \leq n-1, i} [x_i, a_n] \exp[2i\pi rs] \otimes x_i \exp[2i\pi(n-1-r)t] \tag{4}$$

Clearly $\delta(g)(.,.)$ belongs to $H_f \otimes H_f$.

3 The Continuous Bialgebra Structure

We consider the Sobolev space H_k $k \in N$ of elements $g(.)$

$$g(s) = \sum_{n \geq 0} a_n \exp[2i\pi ns] \tag{5}$$

such that

$$\|g(.)\|_k^2 = \sum \|a_n\|^2 (n^k + 1) < \infty \tag{6}$$

We consider $H_{\infty-} = \cap H_k$.

$H_k \otimes H_k$ is constituted of elements $g(.,.)$

$$g(s,t) = \sum_{n\geq 0, m\geq 0} a_{n,m} \exp[2i\pi ns] \exp[2i\pi mt] \tag{7}$$

such that

$$\|g(.,.)\|_k^2 = \sum \|a_{n,m}\|^2 (n^k+1)(m^k+1) < \infty \tag{8}$$

We consider

$$H_{\infty -} \otimes H_{\infty -} = \cap H_k \otimes H_k \tag{9}$$

Theorem 3.1. *The Lie bracket (3) can be defined as a continuous map from $H_{\infty -} \otimes H_{\infty -}$ into $H_{\infty -}$.*

Proof.

$$[g^1, g^2](s) = \sum_{n\geq 0} b_n \exp[2i\pi ns] \tag{10}$$

where

$$b_n = \sum_{n_1\geq 0, n_2 \geq 0, n_1+n_2=n} [a^1_{n_1}, a^2_{n_2}] \tag{11}$$

$$\|[g^1, g^2](.)\|_k^2 = \sum \|b_n\|^2 (n^k+1) \tag{12}$$

But

$$\|b_n\|^2 \leq \left(\sum_{n_1\geq 0, n_2\geq 0, n_1+n_2=n} \|a^1_{n_1}\| \|a^2_{n_2}\| \frac{(n_1^l+1)(n_2^l+1)}{(n_1^l+1)(n_2^l+1)} \right)^2 \tag{13}$$

But in the last sum

$$(n_1^l+1)(n_2^l+1) \geq C(n^l+1) \tag{14}$$

because $n_1 \geq 0, n_2 \geq 0$ and $n_1 + n_2 = n$. The Cauchy–Schwartz inequality shows that

$$\|b_n\|^2 \leq \frac{C}{n^{2l}+1} \|g^1(.)\|_{2l}^2 \|g^2(.)\|_{2l}^2 \tag{15}$$

We deduce that if l is big enough

$$\|[g^1, g^2](.)\|_k^2 \leq C \|g^1(.)\|_{2l}^2 \|g^2(.)\|_{2l}^2 \tag{16}$$

□

Theorem 3.2. *The Lie cobracket (4) can be defined as a continuous map from $H_{\infty -}$ into $H_{\infty -} \otimes H_{\infty -}$.*

Proof.

$$\delta g(s,t) = \sum_{n\geq 0} \sum_{n_1\geq 0, n_2\geq 0, n_1+n_2=n} b_n \exp[2i\pi n_1 s] \exp[2i\pi n_2 t] \tag{17}$$

By (4), we have

$$\|b_n\|^2 \leq C\|a_{n+1}\|^2 \tag{18}$$

Therefore

$$\|\delta g(.,.)\|_k^2 \leq C\sum_{n\geq 0}\sum_{n_1\geq 0, n_2\geq 0, n_1+n_2=n}\|a_{n+1}\|^2(n_1^k+1)(n_2^k+1) \tag{19}$$

Since $n_1 \geq 0, n_2 \geq 0$ and $n_1+n_2 = n$,

$$(n_1^k+1)(n_2^k+1) \leq C(n^{2k}+1) \tag{20}$$

Moreover there n ways to write $n_1+n_2 = n$ with $n_1 \geq 0, n_2 \geq 0$. Therefore

$$\|\delta g(.,.)\|_k^2 \leq C(\sum\|a_{n+1}\|^2(n^{2k+2}+1)) \leq C\|g(.)\|_{2k+2}^2 \tag{21}$$

□

References

1. Belavin, A., Drinfeld, V.: Triangle Equations and Simple Lie Algebras. Harwood Academic Publisher, Chur (1984)
2. Blaszak, M., Szablikowski, B.: J. Phys. A **40**, 404002 (2009)
3. Dito, G., Léandre, R.: J. Math. Phys. **48**, 023509 (2007)
4. Dito, G., Sternheimer, D.: In: Halbout, G. (ed.) Deformation Quantization, IRMA Lecture Math. Theor. Physics, vol. 1, pp. 9–54. Walter de Gruyter, Berlin (2002)
5. Etingof, P., Schiffman, O.: Lectures on Quantum Groups. International Press, Boston (1998)
6. Léandre, R.: In: Calogero, F., Françoise, J.P., Marrero, J.C., Manojlovic, N., Nunes da Costa, J. (eds.) Geometric Aspects of Integrable Systems. S.I.G.M.A. **3**, 027 (2007)
7. Léandre, R.: In: Scheutzow, M., Morters, P., Blath, J. (eds.) Trends in Stochastic Analysis (Festchrift in honour of H.v. Weizsaecker), pp. 283–303. L.M.S Lecture Note Series 353., Cambridge University Press, Cambridge (2009)
8. Léandre, R.: In: Gomez-Ullate, D., Hone, A., Lombardo, S., Puig i Sadurni, J. (eds.) NEEDS 2007". J. Nonlinear Math. Phys. **15**, 251–263 (2008)
9. Léandre, R.: In: Cattaneo, A., Dito, G., Kontsevich, M., Sternheimer, D. (eds.) Special issue on deformation quantization. S.I.G.M.A. **4**, 066 (2008)
10. Léandre, R.: In: Shu-Kun Lin (ed.) Feature papers: symmetry concepts and applications. Symmetry **1**, 55–63 (2009)
11. Léandre, R.: A Poisson structure in white-noise analysis. In: Kielanowski, P., Ali, S.T., Odzijewicz, A., Schlichenmaier, M., Voronov. Th. (eds.) A.I.P. Proceedings XXVIII workshop of geometrical method in physics, vol. 1191, pp. 123–127. A.I.P., Melville (2009)
12. Léandre, R.: In: Goldin, G., Kerner, R., Hounkonnou, M.N., Sinha, K. (eds.) Nonlinear and noncommutative mathematics: new developments and applications in quantum physics. Adv. Math. Phys. 146719 (2010)
13. Léandre, R.: In: Burdit, C., Navratil, O., Posta, S., Schnabl, M., Snobl, C. (eds.) Quantum theory and symmetrics **7**, J. Phys. Conf. Ser. **343**, 012066 (2012)
14. Léandre, R., Obame Nguema, B.: S.I.G.M.A. **8**, 011 (2012)
15. Maeda, Y.: Sugaku Expostions **16**, 224–255 (1991)
16. Semenov-Tyan-Shanskii, M.A.: Funct. Ana. Appl. **17**, 17–33 (1983)
17. Weinstein, A.: In: Séminaire Bourbaki, Astérisque, vol. 227, pp. 389–409. S.M.F., Paris (1994)