Chapter VI.

Meromorphic functions of several variables

This chapter presents the analogues of the Mittag-Leffler and Weierstrass theorems for functions of several complex variables. To this end it develops fundamental methods of multivariable complex analysis that reach far beyond the applications we are going to give here. – *Meromorphic functions* of several variables are defined as local quotients of holomorphic functions (VI.2); the definition requires some information on zero sets of holomorphic functions (VI.1). After introducing *principal parts* and *divisors* we formulate the main problems that arise: *To find a meromorphic function with i) a given principal part* (first Cousin problem) *ii) a given divisor* (second Cousin problem); *iii) to express a meromorphic function as a quotient of globally defined holomorphic functions* (Poincaré problem). These problems are solved on *polydisks* – bounded or unbounded, in particular on the whole space – in VI.6–8. The essential method is a constructive solution of the *inhomogeneous Cauchy-Riemann equations* (VI.3 and 5) based on the one-dimensional inhomogeneous Cauchy formula – see Chapter IV.2. Along the way, various extension theorems for holomorphic functions are proved (VI.1 and 4). Whereas the first Cousin problem can be completely settled by these methods, the second requires additional topological information which is discussed in VI.7, and for the Poincaré problem one needs some facts on the ring of convergent power series which we only quote in VI.8.

The main results on principal parts and divisors go back to P. Cousin 1895 and H. Poincaré 1883. It was remarked somewhat later (Gronwall 1917) that the second Cousin problem meets with a topological obstruction whose nature was finally cleared up by K. Oka in 1939 [Ok, Ra]. It was also Oka who solved these problems on the class of domains where they are most naturally posed: on *domains of holomorphy* of which polydisks are the simplest example [Ok]. The solution of the inhomogeneous Cauchy-Riemann equation in VI.5 is given by a method due to S. Bochner [Bo]; the result is usually referred to as Dolbeault's lemma, because P. Dolbeault exploited it systematically in his study of complex manifolds. The connection of compactly supported solutions and holomorphic extension theorems was discovered by L. Ehrenpreis in 1961 [Eh]; the *Kugelsatz* in VI.4 is due to F. Hartogs. Hartogs' work of 1906 ff [Ha] can be seen as the beginning of modern complex analysis in n variables. The language of *cocycles* and their solutions that we have used throughout the last sections was worked out, in this context, by H. Cartan and J. P. Serre around 1950. All the above problems can be solved on arbitrary *plane* domains [FL1].

A modern comprehensive exposition of the theory on general domains of holomorphy – even on *Stein manifolds* – can be found in [GR], [Hö], and [Ra]. Our presentation follows Hörmander and Range. For historical aspects see also [Li].

1. Zero sets of holomorphic functions

Let f be a function holomorphic on some subdomain G of $\mathbb{C}^n$; we assume $f \not\equiv 0$. The zero set

$$V(f) = \{\mathbf{z} : f(\mathbf{z}) = 0\}$$

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of f is clearly a relatively closed nowhere dense subset of G. We can say more:

Proposition 1.1. *$G \setminus V(f)$ is connected.*

This will be deduced from the following fundamental result:

Theorem 1.2 (Riemann's extension theorem). *If a function h is holomorphic on $G \setminus V(f)$ and locally bounded on G, then it extends to a holomorphic function $\widehat{h}$ on all of G:*

$$\widehat{h} \in \mathcal{O}(G), \quad \widehat{h}|G \setminus V(f) = h.$$

Proof: Since $V(f)$ is nowhere dense in G, the extension $\widehat{h}$, if it exists, is uniquely determined by h. So we only have to find a local holomorphic extension (near an arbitrary point $\mathbf{z}_0 \in V(f)$). We choose coordinates $\mathbf{z} = (\mathbf{z}', z_n)$, $\mathbf{z}' \in \mathbb{C}^{n-1}$, $z_n \in \mathbb{C}$, such that $\mathbf{z}_0 = 0 = (0', 0)$ and $f(0', z_n) \not\equiv 0$. Because f is continuous, there are polydisks $D' \subset \mathbb{C}^{n-1}$ and $D_n \subset \mathbb{C}$ with centres $0'$ and 0, respectively, with the following properties: $D = D' \times D_n \subset\subset G$ and $f(\mathbf{z}) \neq 0$ on $D' \times \partial D_n$. Therefore, for each $\mathbf{z}' \in D'$, the function $z_n \mapsto f(\mathbf{z}', z_n)$ is holomorphic on D_n and not identically zero; its zeros are consequently isolated in D_n. This shows that the function $h(\mathbf{z}', z_n)$ is, for fixed $\mathbf{z}' \in D'$, holomorphic in z_n, as long as $(\mathbf{z}', z_n) \notin V(f)$, and bounded on D_n; Riemann's extension theorem in one variable yields a holomorphic (in z_n) extension $\widehat{h}(\mathbf{z}', z_n)$ to all of D_n. It remains to show that $\widehat{h}$ is holomorphic on D as a function of $\mathbf{z} = (\mathbf{z}', z_n)$. But this follows from the Cauchy integral representation

$$\widehat{h}(\mathbf{z}', z_n) = \frac{1}{2\pi i} \int\limits_{\partial D_n} \frac{h(\mathbf{z}', \zeta_n)}{\zeta_n - z_n}\, d\zeta_n \ :$$

the right hand side is holomorphic in $(\mathbf{z}', z_n)$. □

The proof of Prop. 1.1 is now easy: if $G \setminus V(f)$ could be decomposed into two open non-empty sets U_0 and U_1,

$$G \setminus V(f) = U_0 \cup U_1, \quad U_0 \cap U_1 = \varnothing,$$

the function $f = 0$ on U_0 and $= 1$ on U_1 would be holomorphic on $G \setminus V(f)$ but clearly not holomorphically extendible to G.

We will consider a slightly more general situation.

Definition 1.1.

i. *An analytic hypersurface S of a domain G is a non-empty subset $S \subset G$ with the following property: for each $\mathbf{z}_0 \in G$ there exists an open neighbourhood U of $\mathbf{z}_0$ and a holomorphic function f on U, nowhere $\equiv 0$, such that*

$$S \cap U = V(f) = \{\mathbf{z} \in U : f(\mathbf{z}) = 0\}.$$

ii. A subset $M \subset G$ is called thin, if it is relatively closed in G and if for each $\mathbf{z} \in M$ there is a neighbourhood U and a holomorphic function $f \not\equiv 0$ on U with $M \cap U \subset V(f)$.

Analytic hypersurfaces are clearly closed in G, hence thin. The same proof as above carries Prop. 1.1 over to thin sets:

Proposition 1.3. *If M is a thin subset of a domain G then $G \setminus M$ is connected.* □

Exercises

1. The $2n$-dimensional (Lebesgue-)measure of a thin set is zero. Proof! Hint: Apply the Weierstrass preparation theorem.
2. The function z_2^{-1} cannot be holomorphically extended to all of $\mathbb{C}^2$. Is it locally integrable? Is it locally square integrable?

2. Meromorphic functions

Meromorphic functions of one complex variable were defined, in Chapter II, as functions which are holomorphic up to isolated singularities; the singularities were required to be poles. Since, in more than one variable, there are no isolated singular points, we use the alternative characterisation of meromorphic functions as local quotients of holomorphic functions for our definition.

Definition 2.1. *A meromorphic function on a domain $G \subset \mathbb{C}^n$ is a pair (f, M), where M is a thin set in G and f a holomorphic function on $G \setminus M$ with the following property: for each point $\mathbf{z}_0 \in G$ there is a neighbourhood U of $\mathbf{z}_0$ and there are holomorphic functions g and h on U, such that $V(h) \subset M$ and*

$$f(\mathbf{z}) = \frac{g(\mathbf{z})}{h(\mathbf{z})} \quad \text{for } \mathbf{z} \in U \setminus M. \tag{1}$$

Examples:

a) $f = g/h$, $M = V(h)$, where g and h are holomorphic on G and $h \not\equiv 0$, is a meromorphic function.

b) In particular, holomorphic functions are meromorphic.

The representation (1) is of course not unique. A closer study of the ring of convergent power series allows to define a representation (1) by a reduced fraction which is essentially unique. We do not need that here. We therefore introduce, for general (f, M), the set

$$P = \{\mathbf{z} \in G : h(\mathbf{z}) = 0 \text{ for all } (g, h) \text{ with } (1)\}$$

Then P is obviously contained in M and closed, and f can be extended to a holomorphic function $\widehat{f}$ on $G \setminus P$. We identify (f, M) with $(\widehat{f}, P)$.

Definition 2.2. *The set P as defined above is called the polar set of f.*

Since P is uniquely determined by the holomorphic function f on $G \setminus M$, we will denote meromorphic functions (f, M) simply by f, assuming, if necessary, that f is holomorphically continued to all of $G \setminus P$. It can even be shown that the polar set is a hypersurface or empty, but this requires a more detailed study of the ring of convergent power series. The polar set can alternatively be described as the set of points where f is unbounded.

We continue our examples:

c) The function $f(z_1, z_2) = z_1/z_2$ is meromorphic in $\mathbb{C}^2$, with the polar set $P = V(z_2) = \{(z_1, z_2) : z_2 = 0\}$. Note that f is unbounded at all points of P, but that $|f(z_1, z_2)| \to \infty$ only for $(z_1, z_2) \to (a, 0)$ with $a \neq 0$. This is a general phenomenon: a meromorphic function need not yield a continuous map into the Riemann sphere $\widehat{\mathbb{C}}$ – except in the case of one variable.

d) The most common way of defining a meromorphic function is the following: Let $\{U_i : i \in I\}$ be an open covering of G and $g_i, h_i \in \mathcal{O}(U_i)$ satisfy

i. h_i nowhere $\equiv 0$

ii. $g_i h_j \equiv g_j h_i$ on $U_{ij} = U_i \cap U_j$.

Then, setting

$$f = \frac{g_i}{h_i} \text{ on } U_i,$$

we obtain a well-defined meromorphic function f on G. It is holomorphic outside the thin set

$$M = \{\mathbf{z} : h_i(\mathbf{z}) = 0 \text{ for all } i \text{ with } \mathbf{z} \in U_i\}.$$

Let us now state the identity theorem for meromorphic functions:

Proposition 2.1. *If two meromorphic functions f_1 and f_2 on G coincide on a non-empty open set where both are holomorphic, they are identical on G.*

Proof: $G \setminus (P_1 \cup P_2)$, where P_j is the polar set of f_j, is connected; consequently, $f_1 \equiv f_2$ there. This says that $(f_1, P_1 \cup P_2)$ and $(f_2, P_1 \cup P_2)$ are the same meromorphic function and implies, in fact, that $P_1 = P_2$. □

Addition or multiplication of meromorphic functions at the points where they are holomorphic clearly yield meromorphic functions: the polar set of the resulting function is contained in the union of the polar sets of the summands resp. factors. Also, *if f is meromorphic and does not vanish identically, $1/f$ is again a meromorphic function.* Namely, let $f = g/h$ on an open connected set $U \subset G$; then $g \not\equiv 0$, and $1/f = h/g$

defines a meromorphic function on U. Since G can be covered by open sets as above, we are in the situation of example d) to define $1/f$.

Note that the polar set Q of $1/f$ is locally contained in the zero set $V(g)$, and that

$$Q \cap (G \setminus P) = V(f),$$

where P is the polar set of f. All this is summed up in

Proposition 2.2. *The meromorphic functions on a domain G form a field, denoted by $\mathcal{M}(G)$.* □

At this point we ask an important – and deep – question: *Is $\mathcal{M}(G)$ the quotient field of $\mathcal{O}(G)$?* The answer is positive for polydisks (including $\mathbb{C}^n$) and more general classes of domains, but not for arbitrary domains.

Exercises

1. Consider the function $f(z_1, z_2) = z_1/z_2$. Prove: the set of accumulation points of sequences $f(\mathbf{z}_j)$, $\mathbf{z}_j \to 0$, is the Riemann sphere.
2. (A more precise description of the above situation) Let $f\colon \mathbb{C}^2 \setminus \{0\} \to \widehat{\mathbb{C}}$ be given by $f(\mathbf{z}) = z_1/z_2$. Let $M \subset (\mathbb{C}^2 \setminus \{0\}) \times \widehat{\mathbb{C}}$ be its graph. Consider M as a subset of $\mathbb{C}^2 \times \widehat{\mathbb{C}}$ and show that its closure $\overline{M}$ is $M \cup (\{0\} \times \widehat{\mathbb{C}})$. Introduce homogeneous coordinates ζ_1, ζ_2 on $\widehat{\mathbb{C}}$ and describe $\overline{M}$ by a homogeneous quadratic polynomial in $z_1, z_2, \zeta_1, \zeta_2$.

3. The inhomogeneous Cauchy-Riemann equation in dimension 1

Holomorphic functions of n variables are solutions of the homogeneous Cauchy-Riemann equations

$$\frac{\partial f}{\partial \bar{z}_\nu} = 0, \qquad \nu = 1, \ldots, n. \tag{1}$$

In the next sections we shall construct holomorphic or meromorphic functions with prescribed additional properties by the following method: We will, in a first step, construct a smooth but not holomorphic solution with the required additional properties, say f. Then f does not satisfy (1), that is

$$\frac{\partial f}{\partial \bar{z}_\nu} = f_\nu \neq 0.$$

In a second step we will find a solution of the inhomogeneous Cauchy-Riemann system

$$\frac{\partial u}{\partial \bar{z}_\nu} = f_\nu, \tag{2}$$

such that $f - u$ still has the required properties; the function $f - u$ is then clearly holomorphic. – This method is based on a careful study of (2); note that (2) can only be solved if the right-hand side satisfies the *integrability condition*

$$\frac{\partial f_\nu}{\partial \bar{z}_\mu} = \frac{\partial f_\mu}{\partial \bar{z}_\nu}, \quad \nu, \mu = 1, \ldots, n. \tag{3}$$

This condition is automatically fulfilled if the f_ν are given as above.

The main work will be done in one variable, so from now on we take $n = 1$. The integrability condition (3) is then empty.

Theorem 3.1. *Let $D' \subset\subset D$ be two disks in $\mathbb{C}$ and $G \subset \mathbb{R}^k$ a domain. There is a linear operator*

$$T\colon \mathcal{C}^\infty(D \times G) \to \mathcal{C}^\infty(D' \times G) \tag{4}$$

with

$$\frac{\partial Tf}{\partial \bar{z}} = f|D' \times G \tag{5}$$

and

$$\frac{\partial}{\partial t_j} Tf = T\frac{\partial f}{\partial t_j}, \quad j = 1, \ldots, k. \tag{6}$$

(We have denoted the variable in $\mathbb{C}$ by z, the variables in $\mathbb{R}^k$ by t_j. $\mathcal{C}^\infty$ is the space of smooth - i.e. infinitely differentiable - functions. – The theorem holds with the same proof for any pair of domains $D' \subset\subset D \subset \mathbb{C}$, but we only need it in the above situation.)

Proof: 0) We choose a smooth real-valued function φ with compact support in D, $\varphi \equiv 1$ on D', and define, for $f \in \mathcal{C}^\infty(D \times G)$, $z \in D'$, $\mathbf{t} \in G$:

$$Tf(z, \mathbf{t}) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi(\zeta) f(\zeta, \mathbf{t})}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}. \tag{7}$$

This operator will be shown to have the required properties.

1) The substitution $w = \zeta - z$ leads to

$$Tf(z, \mathbf{t}) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\varphi(w + z) f(w + z, \mathbf{t})}{w} \, dw \wedge d\overline{w}, \tag{8}$$

which immediately shows that Tf is smooth on $\mathbb{C} \times G$; differentiation under the integral (7) yields (6).

2) The inhomogeneous Cauchy formula – see IV.2 – gives for the compactly supported function φf

$$\varphi(z) f(z, \mathbf{t}) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial/\partial\bar{\zeta}[\varphi(\zeta) f(\zeta, \mathbf{t})]}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}; \tag{9}$$

a boundary integral does not occur because $\varphi(\zeta) f(\zeta, \mathbf{t}) \equiv 0$ for $|\zeta|$ large enough. – On the other hand, differentiation of (8) with respect to $\bar{z}$ leads to

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} T f(z, \mathbf{t}) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial/\partial\bar{z}[\varphi(z+w) f(z+w, \mathbf{t})]}{w} \, dw \wedge d\overline{w} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial/\partial\bar{\zeta}[\varphi(\zeta) f(\zeta, \mathbf{t})]}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}. \end{aligned} \tag{10}$$

Comparison of (9) and (10) shows the claim: for $z \in D'$ one has

$$f(z, \mathbf{t}) = \varphi(z) f(z, \mathbf{t}) = \frac{\partial}{\partial \bar{z}} T f(z, \mathbf{t}). \qquad \square$$

Remark: *If f is holomorphic in some of the parameters, then Tf is holomorphic in the same parameters.*

In fact, assume f holomorphic in t_0 – so now G is a subdomain of $\mathbb{C} \times \mathbb{R}^\ell$ with coordinates t_0 and $\mathbf{t}$ – then in view of (6)

$$\frac{\partial}{\partial \bar{t}_0} T f(z, t_0, \mathbf{t}) = T \frac{\partial}{\partial \bar{t}_0} f(z, t_0, \mathbf{t}) = 0.$$

Exercises

1. Justify in detail the differentiation under the integral sign used in the proof of Thm. 3.1.

4. The Cauchy-Riemann equations with compact support

We will solve, for $n = 1, 2, \ldots$, the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial \bar{z}_\nu} = f_\nu, \quad \nu = 1, \ldots, n, \tag{1}$$

where the f_ν are smooth functions in $\mathbb{C}^n$ *with compact support* satisfying the *integrability condition*

$$\frac{\partial f_\nu}{\partial \bar{z}_\mu} = \frac{\partial f_\mu}{\partial \bar{z}_\nu}, \quad \nu, \mu = 1, \ldots, n. \tag{2}$$

The main result is

Theorem 4.1. (1) *has a smooth solution u. If $n > 1$, then u can be chosen with compact support.*

More precisely: *If $f_\nu \equiv 0, \nu = 1, \ldots, n$, outside a compact set K then there is, in case $n > 1$, a solution which vanishes on the unbounded component of the complement of K.* – Note that solutions are of course not unique: we can always add a holomorphic function to obtain a new solution from a given one.

Proof: We define

$$u(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int\limits_{\mathbb{C}} \frac{f_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta}; \tag{3}$$

this is a smooth function on $\mathbb{C}^n$ satisfying – see the previous section –

$$\frac{\partial u}{\partial \bar{z}_1} = f_1.$$

In fact, (3) is the solution Tf_1 of section 3 constructed for a sufficiently large disk D' and $D = \mathbb{C}$. Now, for $\nu > 1$,

$$\frac{\partial u}{\partial \bar{z}_\nu} = \frac{1}{2\pi i} \int\limits_{\mathbb{C}} \frac{\partial f_1/\partial \bar{z}_\nu(\zeta, z_2, \ldots, z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta} = \frac{1}{2\pi i} \int\limits_{\mathbb{C}} \frac{\partial f_\nu/\partial \bar{\zeta}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta},$$

because of (2). Since f_ν has compact support, the last integral is, by the inhomogeneous Cauchy integral formula, $f_\nu(z_1, \ldots, z_n)$. So u solves (1). – Moreover, let the support of the functions f_ν be contained in a compact set K, and let U_0 be the unbounded component of $\mathbb{C}^n \setminus K$. If $n > 1$, there is an affine hyperplane E contained in U_0. The function u is holomorphic outside K, in particular its restriction to E is an entire function of $n - 1$ variables. Now, the integration in (3) only has to extend over a bounded domain (again because the support of f_1 is compact). This shows that $u(z) \to 0$ for $|z| \to \infty$. By Liouville's theorem, $u|E \equiv 0$. As E could be chosen arbitrarily in U_0, the identity theorem gives $u \equiv 0$ on U_0. □

Remark: For $n = 1$ the above solution need not have compact support – see Ex. 1.

A striking consequence of the above theorem is

Theorem 4.2 (Hartogs' Kugelsatz). *Let K be a compact subset of an open set $U \subset \mathbb{C}^n$, with $n > 1$, and suppose that $U \setminus K$ is connected. Then any function f holomorphic on $U \setminus K$ is the restriction to $U \setminus K$ of a holomorphic function $\widehat{f}$ on U.*

Proof: We choose a smooth real-valued function φ with compact support in U, $\varphi \equiv 1$ in a neighbourhood of K, and set, for $f \in \mathcal{O}(U \setminus K)$,

$$\tilde{f} = (1 - \varphi) f. \tag{4}$$

This function is smooth on all of U – where f is not defined, the right-hand side of (4) is 0. The derivatives

$$g_\nu = \frac{\partial \tilde{f}}{\partial \bar{z}_\nu} \tag{5}$$

are smooth in $\mathbb{C}^n$, with support contained in the support of φ, because outside that support $\tilde{f} = f$ is holomorphic. They obviously satisfy the integrability conditions (2) and therefore there exists a solution u of

$$\frac{\partial u}{\partial \bar{z}_\nu} = g_\nu, \tag{6}$$

which vanishes in the unbounded component of the complement of the support of φ. But the boundary of U belongs to that component. So

$$\widehat{f} = \tilde{f} - u \tag{7}$$

coincides with f on a non-empty open set in $U \setminus K$, hence on all of $U \setminus K$, and is holomorphic on U because of (5) and (6). □

The theorem applies in particular to a spherical shell: this explains the name.

Exercises

1. Let $f \in \mathcal{C}^\infty(\mathbb{C})$ be a smooth function with compact support; set $g = f_{\bar{z}}$. Compute the integral

$$\int_{\mathbb{C}} g(z)\, dx\, dy$$

with the help of Stokes' theorem. Use the information obtained that way to construct a compactly supported smooth function g in the plane that has no solution f of $f_{\bar{z}} = g$ with compact support.

5. The Cauchy-Riemann equations in a polydisk

The problem of solving the Cauchy-Riemann equations is more difficult if one drops the assumption of compact support; we study it here in polydisks. In all that follows, D, D', D_0, … will stand for polydisks in $\mathbb{C}^n$ centred at the origin. Let the functions f_ν, for $\nu = 1, \ldots, n$, be smooth on D and satisfy the integrability conditions

$$\frac{\partial f_\nu}{\partial \bar{z}_\mu} = \frac{\partial f_\mu}{\partial \bar{z}_\nu}, \quad \nu, \mu = 1, \ldots, n. \tag{1}$$

We want to solve

$$\frac{\partial u}{\partial \bar{z}_\nu} = f_\nu, \quad \nu = 1, \ldots, n, \tag{2}$$

by a smooth function u on D.

Proposition 5.1. *Let $D_0 \subset\subset D$ be relatively compact in D. Then (2) can be solved by a function $u \in \mathcal{C}^\infty(D_0)$.*

Proof: We denote by $A_k(D')$ the set of n-tuples of smooth functions

$$(f) = (f_1, \ldots, f_n)$$

on D' satisfying $f_\nu \equiv 0$ for $\nu > k$ and condition (1). Here, D' stands for any polydisk contained in D. Let us prove, for arbitrary pairs of polydisks $D_0 \subset\subset D'$, the claim in case the right-hand side of (2) belongs to $A_k(D')$; the case $k = n$ is what we want.

We proceed by induction with respect to k. If $k = 0$, then $(f) = (0)$, and the function $u \equiv 0$ solves. Now consider $(f) \in A_k(D')$, and assume the claim for $k - 1 \geqslant 0$. So

$$(f) = (f_1, \ldots, f_k, 0, \ldots, 0).$$

By (1) we have for $\mu > k$:

$$\frac{\partial f_k}{\partial \bar{z}_\mu} = \frac{\partial f_\mu}{\partial \bar{z}_k} \equiv 0,$$

since $f_\mu \equiv 0$. Hence f_k is holomorphic in the variables $z_{k+1}, \ldots, z_n$. We now choose the solution v of

$$\frac{\partial v}{\partial \bar{z}_k} = f_k \tag{3}$$

constructed in section 3 on a polydisk D'' with $D_0 \subset\subset D'' \subset\subset D'$:

$$v(z_1, \ldots, z_n) = Tf_k(z_1, \ldots, z_n).$$

Then v is holomorphic in $z_{k+1}, \ldots, z_n$, so

$$\frac{\partial v}{\partial \bar{z}_\mu} = 0 = f_\mu, \quad \mu > k. \tag{4}$$

The system $(g) = (g_1, \ldots, g_n)$ with

$$g_j = f_j - \frac{\partial v}{\partial \bar{z}_j}$$

belongs to $A_{k-1}(D'')$ and satisfies the integrability conditions: in fact, for $j > k$ we have $f_j = 0 = \partial v / \partial \bar{z}_j$, and for $j = k$ we have $g_k = 0$ in view of (3); finally (1) is satisfied for (g) because it is satisfied for (f) and for the derivatives of v. The induction hypothesis yields a solution $w \in \mathcal{C}^\infty(D_0)$ with

$$\frac{\partial w}{\partial \bar{z}_j} = g_j;$$

setting $u = v + w$ on D_0 we thus solve (2). – This concludes the induction and proves our claim. □

An approximation argument will finally give a solution of the Cauchy-Riemann system on the whole polydisk:

Theorem 5.2. *The system*

$$\frac{\partial u}{\partial \bar{z}_\nu} = f_\nu, \quad \nu = 1, \ldots, n, \tag{2}$$

with $f_\nu \in \mathcal{C}^\infty(D)$ has a solution $u \in \mathcal{C}^\infty(D)$ if and only if it satisfies the integrability conditions

$$\frac{\partial f_\nu}{\partial \bar{z}_\mu} = \frac{\partial f_\mu}{\partial \bar{z}_\nu}, \quad \nu, \mu = 1, \ldots, n. \tag{1}$$

Proof: We choose a sequence of polydisks

$$D_0 \subset\subset D_1 \subset\subset D_2 \subset\subset \ldots \subset\subset D$$

with

$$\bigcup_{\varkappa \geqslant 0} D_\varkappa = D$$

and functions $u'_\varkappa \in \mathcal{C}^\infty(D_\varkappa)$ which solve (2) on $D_\varkappa$. If the sequence $u'_\varkappa$ – which is defined on each fixed D_k for all $\varkappa \geqslant k$ – were convergent on D_k, we would simply define our solution u as the limit of that sequence. So our task is to modify the $u'_\varkappa$ in order to obtain a convergent sequence. We set $u_0 = u'_0$, $u_1 = u'_1$ and assume that we have found solutions $u_\varkappa$ of (2) on $D_\varkappa$, for $\varkappa = 1, \ldots, k$, such that

$$|u_\varkappa - u_{\varkappa-1}|_{D_{\varkappa-2}} < 2^{1-\varkappa}. \tag{5}$$

Here $|\cdot|_M$ denotes the supremum norm of a function on M. Now u'_{k+1} and u_k both solve (2) on D_k and consequently differ by a holomorphic function f_k on D_k. Power series development of f_k around 0 yields a polynomial p_k such that

$$|f_k - p_k|_{D_{k-1}} = |u'_{k+1} - u_k - p_k|_{D_{k-1}} < 2^{-k}.$$

Setting

$$u_{k+1} = u'_{k+1} - p_k,$$

we get a new solution of (2) on D_{k+1} which now satisfies (5) for $\varkappa = k+1$. Let us now define, for $z \in D$,

$$u(z) = \lim_{\varkappa \to \infty} u_\varkappa(z).$$

The limit is well-defined – see our remark above. Moreover, for $\varkappa \geqslant \lambda \geqslant k+2$ and $z \in D_k$

$$\begin{aligned}|u_\varkappa(z) - u_\lambda(z)| &\leqslant |u_\varkappa(z) - u_{\varkappa-1}(z)| + \cdots + |u_{\lambda+1}(z) - u_\lambda(z)| \\ &\leqslant 2^{-\varkappa+1} + \cdots + 2^{-\lambda} \leqslant 2 \cdot 2^{-\lambda};\end{aligned}$$

so the limit exists uniformly on D_k. This shows that u is a continuous function on D. Now, on D_k we have

$$u - u_k = \lim_{\varkappa \geqslant k+2} (u_\varkappa - u_k),$$

and the terms on the right-hand side are holomorphic on D_k – which implies that their uniform limit is holomorphic as well. Hence, $u - u_k$ and therefore u is smooth on D_k, and u thus solves (2) because it differs from the solution u_k by a holomorphic function. □

6. Principal parts: the first Cousin problem

Let $G \subset \mathbb{C}^n$ be a domain.

Definition 6.1. *A Cousin-I-distribution on G is a system (f_i, U_i), $i \in I$, where the U_i are open subsets of G which form a covering of G and the f_i are meromorphic functions on U_i such that*

$$f_{ij} = f_j - f_i \in \mathcal{O}(U_{ij}), \tag{1}$$

i.e. f_{ij} is holomorphic on $U_{ij} = U_i \cap U_j$.

If $U_{ij} = \varnothing$, condition (1) is of course void. The *principal part* of a meromorphic function can be defined as follows: *two meromorphic functions f and g have the same principal part if their difference is holomorphic.* In particular, their polar sets coincide in that case. So in the above definition, the principal parts of f_i and f_j coincide where both functions are defined. – Instead of Cousin-I-distribution we could also speak of a distribution of principal parts or simply of a principal part on G. Note that on $\mathbb{C}$ the Mittag-Leffler data define naturally a Cousin-I-distribution, and vice versa – see exercises.

Definition 6.2. *A solution of a Cousin-I-distribution (f_i, U_i) is a meromorphic function f on G such that $f - f_i$ is holomorphic on U_i for all i.*

In other words: f should be a meromorphic function with the given principal parts. The Mittag-Leffler theorem gives a solution of any Cousin-I-distribution in the complex plane. In general, on an arbitrary domain in $\mathbb{C}^n$, with $n > 1$, not every Cousin-I-distribution is soluble; for $n = 1$, it is – see [FL1]. We will show that any Cousin-I-distribution on a polydisk is soluble. So from now on, we will choose for G a (bounded or unbounded) polydisk $D \subset \mathbb{C}^n$. The main work will be done in proving the next theorem.

Theorem 6.1. *Let $\{U_i : i \in I\}$ be an open covering of the polydisk D. Suppose that for each pair of indices i, $j \in I$ with $U_{ij} \neq \varnothing$ a holomorphic function $g_{ij} \in \mathcal{O}(U_{ij})$ is given such that the following conditions are fulfilled:*

$$g_{ij} = -g_{ji} \tag{2}$$

$$g_{jk} - g_{ik} + g_{ij} = 0 \text{ on } U_{ijk}. \tag{3}$$

Then there are holomorphic functions $g_i \in \mathcal{O}(U_i)$ with

$$g_j - g_i = g_{ij} \text{ on } U_{ij}. \tag{4}$$

We have used the notation $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$. If $U_{ijk} = \varnothing$, then condition (3) is void.

Proof: Let us choose a partition of unity subordinate to the covering, i.e. functions $\varphi_i \in \mathcal{C}^\infty(D)$, $0 \leqslant \varphi_i \leqslant 1$, $\operatorname{supp}\varphi_i \subset U_i$, such that the system of supports of the φ_i is locally finite, and

$$\sum_{i\in I} \varphi_i(z) \equiv 1. \tag{5}$$

Then for each i

$$h_i = \sum_{k\in I} \varphi_k g_{ki} \tag{6}$$

is a well-defined smooth function on U_i, and we have on U_{ij}

$$\frac{\partial h_i}{\partial \bar z_\nu} - \frac{\partial h_j}{\partial \bar z_\nu} = \sum_{k\in I} \frac{\partial \varphi_k}{\partial \bar z_\nu}(g_{ki} - g_{kj}) = \sum_{k\in I} \frac{\partial \varphi_k}{\partial \bar z_\nu} g_{ji} = \left(\frac{\partial}{\partial \bar z_\nu}\sum_{k\in I}\varphi_k\right) g_{ji} = 0 \tag{7}$$

because of (3) and (5). Hence, for each ν, the function

$$F_\nu(z) = \frac{\partial h_i}{\partial \bar z_\nu}(z), \quad z \in U_i, \tag{8}$$

is well-defined on all of D, and the F_ν satisfy, in view of their definition (8) as derivatives, the integrability conditions for the Cauchy-Riemann system. The previous section yields a smooth function $u \in \mathcal{C}^\infty(D)$ with

$$\frac{\partial u}{\partial \bar z_\nu} = F_\nu. \tag{9}$$

Now set on U_i

$$g_i = h_i - u. \tag{10}$$

Then $g_i \in \mathcal{O}(U_i)$ and

$$g_j - g_i = h_j - h_i = \sum_{k\in I} \varphi_k (g_{kj} - g_{ki}) = \left(\sum_{k\in I}\varphi_k\right) g_{ij} = g_{ij},$$

again by (2), (3) and (5). So the g_i solve (4). □

Let us point out that we only used solubility of the Cauchy-Riemann equations in the proof. It is worthwhile to state this explicitly as

Theorem 6.2. *If G is a domain where the Cauchy-Riemann equations for smooth data are always soluble, then for all data $\left(U_i,\, g_{ij}\right)$ as in Thm.* 6.1 *there are holomorphic functions $g_i \in \mathcal{O}\left(U_i\right)$ with $g_{ij} = g_j - g_i$.* □

It is equally worthwhile to introduce a new word in order to express the above results, and also for later use:

Definition 6.3. *The data $\left(U_i,\, g_{ij}\right)$ with the properties* (2) *and* (3) *of Thm.* 6.1 *is called an $\mathcal{O}$-cocycle; the $\left(U_i,\, g_i\right)$ satisfying* (4) *are an $\mathcal{O}$-solution of this cocycle.*

So we can say more conveniently: *An arbitrary $\mathcal{O}$-cocycle on a polydisk has an $\mathcal{O}$-solution; the same result holds on domains where the Cauchy-Riemann system is soluble.*

From here we deduce easily

Theorem 6.3. *Any Cousin-I-distribution on a polydisk – more generally: on a domain satisfying the assumption of Thm.* 6.2 *– is soluble.*

Proof: Let $(f_i,\, U_i)$ be such a distribution. Then the f_{ij} given by (1) define an $\mathcal{O}$-cocycle, which therefore has an $\mathcal{O}$-solution (g_i). Let us now set

$$f = f_i - g_i \text{ on } U_i. \tag{11}$$

In view of (4) the definition is independent of the choice of i and yields a meromorphic function on D which, by (11), solves the distribution. □

Exercises

1. Let $G \subset \mathbb{C}^n$ be an arbitrary domain and $f_j, j = 1, \ldots, n$, be smooth functions on G satisfying the integrability conditions (1) from section 5. Show that every point $a \in G$ has a neighbourhood U such that there is a smooth function u on U with

$$\frac{\partial u}{\partial \overline{z_j}} = f_j | U.$$

Choose an open covering U_i with corresponding solutions u_i of the above equation. Consider the differences $u_j - u_i$ on U_{ij}. From here, state and prove a converse to Thm. 6.2.

7. Divisors: the second Cousin problem

We now carry over Weierstrass' product theorem to higher dimensions. We shall even consider a slightly more general situation. In all that follows, G will be a domain in $\mathbb{C}^n$; later on it will be taken to be a polydisk. $\mathcal{M}^*(U)$ denotes the set of meromorphic functions on the open set U which nowhere vanish identically, $\mathcal{O}^*(U)$ is the set of holomorphic functions on U without zeros. Both sets are multiplicative groups: the groups of units in the rings $\mathcal{M}(U)$ resp. $\mathcal{O}(U)$.

Definition 7.1. *A divisor on G is a system*

$$\Delta = \big(U_i,\, f_i\big)_{i\in I},\ I \text{ an index set,}$$

where the $U_i \subset G$ form an open covering of G and the f_i are elements of $\mathcal{M}^*(U_i)$, such that for all $i,\, j \in I$ with $U_{ij} = U_i \cap U_j \neq \varnothing$ one has

$$\frac{f_j}{f_i} = g_{ij} \in \mathcal{O}^*(U_{ij})\,. \tag{1}$$

Two divisors Δ and Δ' are – by definition! – equal if their union $\Delta \cup \Delta'$ is a divisor.

Condition (1) means, intuitively, that on U_{ij} the polar sets of f_i and f_j coincide, including *multiplicities*, and that also the zero sets of f_i and f_j are identical, and that the two functions vanish there with the same multiplicity. By sticking to the above definition one circumvents the need to explicitly define the notion of multiplicity. – A *positive* divisor is a divisor given by holomorphic functions f_i satisfying (1).

If $f \not\equiv 0$ is a meromorphic function on G, then $(G,\, f)$ is a divisor. We call these divisors *principal* – notation: $\operatorname{div} f$. Now suppose that $\Delta = \big(U_i,\, f_i\big)$ is an arbitrary divisor. Then clearly,

$$\Delta = \operatorname{div} f, \qquad f \in \mathcal{M}^*(G)\,,$$

if and only if *the quotients f/f_i are holomorphic without zeros on U_i, for all i.* We define:

Definition 7.2. *A solution of Δ is a meromorphic function f with $\Delta = \operatorname{div} f$.*

Such a solution – if it exists – is obviously determined up to multiplication with an element of $\mathcal{O}^*(G)$. In the case of a positive divisor it is a holomorphic function with prescribed zeros (including multiplicities): that is just what is given in the Weierstrass product theorem.

Divisors – more precisely, the $\big(U_i,\, f_i\big)$ satisfying (1) – are also called Cousin-II-distributions. The second Cousin problem can now be briefly stated:

Is each divisor on G a principal divisor?

The problem is by now completely understood for large classes of domains (domains of holomorphy); here we shall give a positive answer in the case of polydisks. Our discussion will show that certain topological conditions play a role, conditions which are fulfilled on a polydisk but not in general. The method of proof reaches beyond our immediate aim.

We start with a topological result:

Lemma 7.1. *Let $f\colon U \to \mathbb{C}^*$ be a continuous function in a simply connected domain $U \subset \mathbb{R}^k$. Then f has a continuous logarithm, i.e. there is a continuous function F on U with*

$$f = \exp F. \tag{2}$$

In fact, the universal cover of the punctured plane $\mathbb{C}^*$ is the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$, so f lifts to a continuous map F to $\mathbb{C}$.

Before proving the multiplicative analogue of Thm. 6.1 we introduce a convenient terminology:

Definition 7.3.

i. An $\mathcal{O}^*$-cocycle in a domain G is a data $g = \big(U_i, g_{ij}\big)$, where the U_i, $i \in I$, are an open covering of G and $g_{ij} \in \mathcal{O}^*(U_{ij})$ such that

$$g_{ij} = {g_{ji}}^{-1}, \tag{3}$$

$$g_{jk}{g_{ik}}^{-1}g_{ij} = 1 \text{ on } U_{ijk}, \tag{4}$$

provided $U_{ijk} \neq \varnothing$.

ii. An $\mathcal{O}^*$-solution h of g is a system $h_i \in \mathcal{O}^*(U_i)$ of holomorphic functions with

$$g_{ij} = \frac{h_j}{h_i} \text{ on } U_{ij}.$$

If we replace $\mathcal{O}^*$ by $\mathcal{C}^*$, the space of continuous functions without zeros, we obtain in the same way the notion of a $\mathcal{C}^*$-cocycle and a $\mathcal{C}^*$-solution. Since an $\mathcal{O}^*$-cocycle is also a $\mathcal{C}^*$-cocycle, we can speak of $\mathcal{C}^*$-solutions of an $\mathcal{O}^*$-cocycle.

The analogue of Thm. 6.1 and 6.2 now comes up with a surprise:

Theorem 7.2. *Let D be a polydisk – or, more generally, a domain where the Cauchy-Riemann system is always soluble. Then an $\mathcal{O}^*$-cocycle has an $\mathcal{O}^*$-solution if and only if it has a $\mathcal{C}^*$-solution.*

Proof: Let $\big(U_i, g_{ij}\big)$ be the given cocycle.

1) We first assume that the U_i are simply connected. By our assumption,

$$g_{ij} = c_j {c_i}^{-1} \tag{5}$$

with non-vanishing continuous functions c_i, c_j on U_i and U_j, respectively.
By Lemma 7.1, for each i,

$$c_i = \exp b_i,$$

b_i continuous on U_i. Now, on the intersection U_{ij}:

$$g_{ij} = \exp(b_j - b_i),$$

which means that

$$b_{ij} = b_j - b_i \tag{6}$$

is holomorphic. The b_{ij} are clearly an $\mathcal{O}$-cocycle and Thm. 6.1 provides us with a solution $a_i \in \mathcal{O}\,(U_i)$:

$$b_{ij} = a_j - a_i \tag{7}$$

on U_{ij}. Let us now set $h_i = \exp a_i$. Then

$$\frac{h_j}{h_i} = \exp(a_j - a_i) = \exp b_{ij} = \frac{c_j}{c_i} = g_{ij},$$

as required.

2) The general case will now be reduced to the above special case. Let the $\mathcal{O}^*$-cocycle $g = \big(U_i,\, g_{ij}\big)_{i\in I}$ be given with a $\mathcal{C}^*$-solution $c = \big(U_i,\, c_i\big)_{i\in I}$. We choose a covering V_α, $\alpha \in A$, of simply connected open sets which is *finer* than the covering by the U_i, i.e. each V_α is contained in some U_i, and there is a *refinement map* $\tau\colon A \to I$ such that always $V_\alpha \subset U_{\tau(\alpha)}$. We now set

$$g'_{\alpha\beta} = g_{\tau(\alpha)\tau(\beta)}; \quad c'_\alpha = c_{\tau(\alpha)},$$

restricted to $V_{\alpha\beta}$ and V_α resp. Then $\big(V_\alpha,\, g'_{\alpha\beta}\big)$ is again an $\mathcal{O}^*$-cocycle with a $\mathcal{C}^*$-solution $\big(V_\alpha,\, c'_\alpha\big)$, and by the first part of the proof we obtain an $\mathcal{O}^*$-solution $h'_\alpha \in \mathcal{O}^*(V_\alpha)$ with

$$g'_{\alpha\beta} = \frac{h'_\beta}{h'_\alpha}. \tag{8}$$

We now define $h_i \in \mathcal{O}^*(U_i)$ by

$$h_i = h'_\alpha g_{\tau(\alpha)i} \text{ on } U_i \cap V_\alpha. \tag{9}$$

On $U_i \cap V_{\alpha\beta}$ we have in view of (4):

$$h'_\alpha g_{\tau(\alpha)i} {h'_\beta}^{-1} g_{i\tau(\beta)} = g'_{\beta\alpha} g_{\tau(\alpha)\tau(\beta)} = g'_{\beta\alpha} g'_{\alpha\beta} = 1,$$

so (9) defines h_i on U_i uniquely. The definition yields on $U_{ij} \cap V_\alpha$

$$h_j {h_i}^{-1} = h'_\alpha g_{\tau(\alpha)j} {h'_\alpha}^{-1} g_{i\tau(\alpha)} = g_{\tau(\alpha)j} g_{i\tau(\alpha)} = g_{ij};$$

so (9) solves our problem. $\square$

As in the previous section we point out that, among the various properties of D, we only need the solubility of the Cauchy-Riemann equations for smooth data. We state this explicitly as

Theorem 7.3. *If $G \subset \mathbb{C}^n$ is a domain where the Cauchy-Riemann system is soluble, then an $\mathcal{O}^*$-cocycle on G has an $\mathcal{O}^*$-solution if and only if it has a $\mathcal{C}^*$-solution.*

To express this more briefly we introduce

Definition 7.4. *A domain $G \subset \mathbb{C}^n$ has the Oka property if each $\mathcal{C}^*$-cocycle has a $\mathcal{C}^*$-solution.*

It now follows easily

Theorem 7.4. *Let G be a domain with the Oka property where the Cauchy-Riemann system is soluble. Then all divisors on G are principal.*

Proof: Let $\Delta = \big(U_i,\, f_i\big)$ be a divisor. By the Oka property there are continuous functions without zeros c_i on U_i such that

$$\frac{f_j}{f_i} =: g_{ij} = \frac{c_j}{c_i}.$$

The previous theorem gives us holomorphic functions $h_i \in \mathcal{O}^*(U_i)$ with

$$g_{ij} = \frac{h_j}{h_i}.$$

Then

$$f := \frac{f_i}{h_i} \text{ on } U_i$$

is a well-defined meromorphic function on G with divisor Δ. $\square$

In order to apply this to the polydisk we need

Theorem 7.5. *Polydisks have the Oka property.*

We do not give the – purely topological – proof, but refer the reader to [Ra]. The main consequence now is

Theorem 7.6. *All divisors on a polydisk are principal.* $\square$

8. Meromorphic functions revisited

We go back to the quotient representation of meromorphic functions. We have to use certain algebraic properties of the ring of convergent power series which can be deduced from the Weierstrass preparation theorem which has been proved in Chapter IV; the details of the deduction, however, will not be given in our book. So this is one instance where we rely on a bit more than just the previous arguments.

Let f be holomorphic in a domain G. If $\mathbf{z}_0 \in G$, then, as a consequence of Cauchy's integral formula, f can be developed into a convergent power series in $\mathbf{z} - \mathbf{z}_0$; let us call this series $f_{\mathbf{z}_0}$. We now need

Theorem 8.1. *The ring H of convergent power series of n variables is factorial.*

Moreover, again as a consequence of the Weierstrass preparation theorem, we have

Proposition 8.2. *If f and g are holomorphic and their power series $f_{\mathbf{z}_0}$ and $g_{\mathbf{z}_0}$ are coprime, then for all points $\mathbf{z}$ in a sufficiently small neighbourhood of $\mathbf{z}_0$ the series $f_{\mathbf{z}}$ and $g_{\mathbf{z}}$ are also coprime.*

For the proofs we refer to [Hö].

The upshot of the previous statements is: *A meromorphic function f can always locally be represented by quotients g/h of holomorphic functions with $g_{\mathbf{z}}$ and $h_{\mathbf{z}}$ coprime for all $\mathbf{z}$ in a sufficiently small open set.*

We apply this information to divisors. Let us first note that a divisor Δ given as

$$\Delta = \big(U_i,\, f_i\big)_{i\in I}$$

can equally well be given as

$$\Delta = \big(V_j,\, g_j\big)_{j\in J},$$

where the V_j are a refinement of the covering U_i and

$$g_j = f_{\sigma(j)}|V_j,$$

with $\sigma\colon J \to I$ a *refinement map*, i.e. $V_j \subset U_{\sigma(j)}$. Different refinement maps yield different g_j but the same divisor. So two divisors Δ and Γ can always be given by a Cousin-II-distribution defined over the same open covering: just pass to a common refinement of the original coverings! Now, if

$$\Delta = \big(U_i,\, f_i\big)_{i\in I},$$
$$\Gamma = \big(U_i,\, g_i\big)_{i\in I}$$

are given, we define their product as

$$\Delta\Gamma = \big(U_i,\, f_i g_i\big)_{i\in I}.$$

It is easy to check that this is again a divisor, and that the divisors on G form an abelian group under this multiplication; for instance

$$\Delta^{-1} = \big(U_i,\, 1/f_i\big)_{i\in I}.$$

Let now

$$\Delta = \big(U_i,\, f_i\big)_{i\in I}$$

be a divisor. We can choose the U_i so small that we have a representation

$$f_i = g_i/h_i, \quad g_i,\, h_i \in \mathcal{O}\,(U_i)$$

with $g_{i\mathbf{z}}$ and $h_{i\mathbf{z}}$ coprime for each $\mathbf{z} \in U_i$.

This implies that

$$\Delta_+ = \big(U_i,\, g_i\big)_{i\in I}$$

and

$$\Delta_- = \big(U_i,\, h_i\big)_{i\in I}$$

are again – necessarily positive – divisors. We show it for Δ_+:

On U_{ij} there are holomorphic non-vanishing functions $a_{ij} \in \mathcal{O}^*(U_{ij})$ with

$$f_j = a_{ij} f_i \text{ on } U_{ij}.$$

So

$$g_j = a_{ij}\frac{h_j}{h_i} g_i.$$

Since $g_{i\mathbf{z}}$ and $h_{i\mathbf{z}}$ are coprime, the right-hand side can only be holomorphic if h_j/h_i is holomorphic – without zeros, because we can apply the same argument to h_i/h_j. This shows

$$a_{ij}\frac{h_j}{h_i} \in \mathcal{O}^*(U_{ij})$$

and verifies our claim for Δ_+. Hence

Proposition 8.3. *Every divisor is the quotient of positive divisors.* □

Now let $f \not\equiv 0$ be a meromorphic function on a polydisk D. Its divisor decomposes

$$\operatorname{div} f = \Delta_+/\Delta_-$$

into the quotient of two positive divisors. Since these are principal – by what we know by now – we find holomorphic functions g and h on D with

$$\operatorname{div} g = \Delta_+, \quad \operatorname{div} h = \Delta_-.$$

This implies

$$\operatorname{div} f = \operatorname{div}(g/h),$$

and so f and g/h differ by a function $a \in \mathcal{O}^*(D)$.

$$f = a \cdot \frac{g}{h}.$$

Hence

Theorem 8.4. *Meromorphic functions on a polydisk are quotients of globally defined holomorphic functions; the field $\mathcal{M}(D)$ is the quotient field of the ring $\mathcal{O}(D)$.* □

Exercises

1. Use the Mittag-Leffler and Weierstrass theorems from Chapter III to explicitly solve the Cousin and Poincaré problems in the plane. (This has been mentioned in the main text without explanation of the details.)