Ordinal- and Continuous-Response Stochastic Volatility Models for Price Changes: An Empirical Comparison

Claudia Czado, Gernot Müller and Thi-Ngoc-Giau Nguyen

Abstract Ordinal stochastic volatility (OSV) models were recently developed and fitted by Müller & Czado (2009) to account for the discreteness of financial price changes, while allowing for stochastic volatility (SV). The model allows for exogenous factors both on the mean and volatility level. A Bayesian approach using Markov Chain Monte Carlo (MCMC) is followed to facilitate estimation in these parameter driven models. In this paper the applicability of the OSV model to financial stocks with different levels of trading activity is investigated and the influence of time between trades, volume, day time and the number of quotes between trades is determined. In a second focus we compare the performance of OSV models and SV models. The analysis shows that the OSV models which account for the discreteness of the price changes perform quite well when applied to such data sets.

1 Introduction

Modeling price changes in financial markets is a challenging task especially when models have to account for salient features such as fat tail distributions and volatility clustering. An additional difficulty is to allow for the discreteness of price changes. These are still present after the US market graduation to decimalization of possible tick sizes. Recently, Müller & Czado (2009) introduced the class of ordinal stochastic volatility (OSV) models, which utilizes the advantages of continuous-response stochastic volatility (SV) models (see Ghysels et al. (1996) and more lately Shephard (2006)) such as fat tails and persistence through autoregressive terms in the volatility process, while adjusting for the discreteness of the price changes.

OSV models are based on a threshold approach, where the hidden continuous process follows a SV model, thus providing a more realistic extension of the ordered

Claudia Czado, Gernot Müller and Thi-Ngoc-Giau Nguyen
Zentrum Mathematik, Technische Universität München, 85747 Garching, Germany,
e-mail: cczado@ma.tum.de, mueller@ma.tum.de, ntngiau2002@yahoo.com

T. Kneib, G. Tutz (eds.), *Statistical Modelling and Regression Structures*,
DOI 10.1007/978-3-7908-2413-1_16,

probit model suggested by Hausman et al. (1992). In addition we allow for exogenous variables both on the mean and variance level of the hidden process. Parameter estimation in OSV models using maximum likelihood is not feasible, since first the hidden SV process has no closed form of the likelihood and second the threshold approach induces the need to evaluate multidimensional integrals with dimension equal to the length of the financial time series. Therefore Müller & Czado (2009) follow a Bayesian approach. Here Markov Chain Monte Carlo (MCMC) methods allow for sampling from the posterior distributions of model parameters and the hidden process variables.

While Müller & Czado (2009) provided the model specification, developed and implemented the necessary estimation techniques, this paper explores the applicability of the OSV model to financial stocks with different levels of trading activity. In particular, we investigate which exogenous factors such as volume, daytime, time elapsed between trades and the number of quotes between trades have influence on the mean and variance level of the hidden process and thus on the discrete price changes. A second focus of this paper is to compare the performance of the OSV and SV models when these are fitted to such discrete price changes.

Alternative discrete price change models are based on rounding and decomposition ideas. Following the rounding approach Harris (1990) models discrete prices by assuming constant variances of the underlying efficient price, while Hasbrouck (1999a) models efficient prices for bid and ask prices separately using GARCH dynamics for the volatility of the efficient price processes. Hasbrouck (1999a) proposes to use non-Gaussian, non-linear state space estimation of Kitagawa (1987). Other works of Manrique & Shephard (1997), Hasbrouck (1999), Hasbrouck (2003) and Hasbrouck (2004) also use MCMC techniques for estimation.

Decomposition models for discrete price changes assume that the price change is a product of usually three random variables: a price change indicator, the direction of the price change, and the size of the price change. Rydberg & Shephard (2003) and Liesenfeld et al. (2006) follow this approach. Russell & Engle (2005) introduce a joint model of price changes and time elapsed between trades (duration) where price changes follow an autoregressive conditional multinomial (ACM) model and durations the autoregressive conditional duration (ACD) model of Engle & Russell (1998). A common feature of these models is that the time dependence is solely induced by lagged endogenous variables, while our OSV specification allows for parameter driven time dynamics.

The paper is organized as follows: Section 2 introduces the OSV and SV model specifications and summarizes their estimation using MCMC methods. It also considers the problem of model selection among OSV, among SV and between OSV and SV models. The data application to three NYSE stocks with different trading levels from the TAQ data base are given in Section 3. Special emphasis is given to model interpretation and model selection. The paper closes with a summary and outlines further research.

2 Ordinal- and Continuous-Response Stochastic Volatility Models

In this section we recall the OSV and SV model specifications and briefly summarize MCMC techniques which have been developed to estimate these models. Furthermore, we discuss methods of model selection within and between the two model classes.

2.1 OSV and SV Model Specification and Interpretation

As introduced by Müller & Czado (2009) we consider the following stochastic volatility model for an ordinal valued time series $\{Y_{t_i}, i = 1, \ldots, I\}$, where t_i, $i = 1, \ldots, I$ denote the possibly unequally spaced observation times. In this model the response Y_{t_i} with K possible values is viewed as a censored observation from a hidden continuous variable $Y^*_{t_i}$ which follows a stochastic volatility model, i.e.

$$\begin{aligned} Y_{t_i} = k \;\Leftrightarrow\; Y^*_{t_i} &\in [c_{k-1}, c_k), \\ Y^*_{t_i} &= \mathbf{x}'_{t_i}\beta + \exp(h^*_{t_i}/2)\varepsilon^*_{t_i}, \\ h^*_{t_i} &= \mathbf{z}'_{t_i}\alpha + \phi(h^*_{t_{i-1}} - \mathbf{z}'_{t_{i-1}}\alpha) + \sigma\eta^*_{t_i}, \end{aligned} \tag{1}$$

where $c_0 = -\infty < c_1 < \cdots < c_{K-1} < c_K = +\infty$ are unknown threshold parameters (also called cutpoints). Moreover, $\mathbf{x}_{t_i}$ and $\mathbf{z}_{t_i}$ are p and q dimensional covariate vectors on the hidden mean and log volatility level, respectively. Associated with these covariate vectors are unknown regression parameters β and α, respectively. The parameter ϕ is an unknown autocorrelation parameter and σ^2 an unknown variance parameter on the hidden log volatility scale. The error variables $\varepsilon^*_{t_i}$ and $\eta^*_{t_i}$ are assumed to be i.i.d. standard normal, with independence also between $\{\varepsilon^*_{t_i}, i = 1, \ldots, I\}$ and $\{\eta^*_{t_i}, i = 1, \ldots, I\}$. For t_0 we assume $\mathbf{z}_0 := (0, \ldots, 0)'$ and that h^*_0 follows a known distribution. Finally, for identifiability reasons we have to fix a threshold parameter, and hence we set $c_1 = 0$. The model specified by (1) is abbreviated by $OSV(X_1, \ldots, X_p; Z_1, \ldots, Z_q)$, where $(X_1, \ldots, X_p)$ and $(Z_1, \ldots, Z_q)$ represent the names of the covariates with corresponding observation vectors $\mathbf{x}_{t_i}$ and $\mathbf{z}_{t_i}$ at time t_i, respectively.

To interpret such a model, denote the mean and variance of the hidden process at t_i by μ_{t_i} and $\sigma^2_{t_i}$, respectively. As μ_{t_i} is increased holding $\sigma^2_{t_i}$ fixed, we see that the probability of a large (small) category is increased (decreased). For fixed μ_{t_i}, we see that if $\sigma^2_{t_i}$ is increased the probability of extreme categories is increased. These two situations are illustrated in Figure 1.

Furthermore, the OSV model allows to quantify the probability $p^k_{t_i} := P(Y_{t_i} = k)$ for observing a specific category k at time t_i. This probability is given by

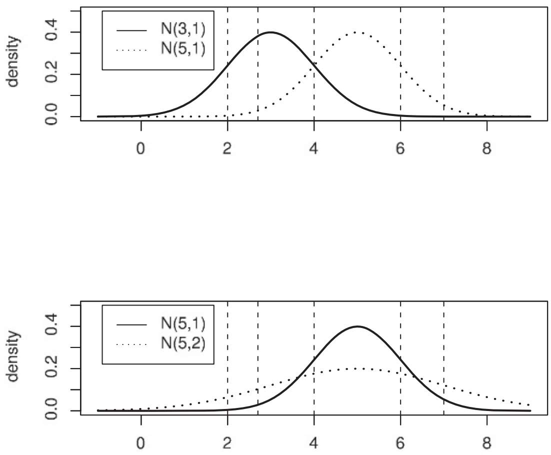

Fig. 1 Category probabilities (visualized as area under the curve between adjacent threshold bounds) as mean and variance of a hidden normally distributed random variable vary

$$p_{t_i}^k = \begin{cases} \Phi\left((c_1 - \mathbf{x}_{t_i}'\beta)/\exp(h_{t_i}^*/2)\right) & \text{for } k=1, \\ \Phi\left((c_k - \mathbf{x}_{t_i}'\beta)/\exp(h_{t_i}^*/2)\right) - \Phi\left((c_{k-1} - \mathbf{x}_{t_i}'\beta)/\exp(h_{t_i}^*/2)\right) & \\ & \text{for } k=2,\dots,K-1, \\ 1 - \Phi\left((c_{K-1} - \mathbf{x}_{t_i}'\beta)/\exp(h_{t_i}^*/2)\right) & \text{for } k=K, \end{cases}$$

where $\Phi()$ denotes the cumulative distribution function of a standard normal random variable. Therefore the model is able to identify time points where there is a large probability of extreme small or large category labels. Note that no symmetry assumptions about the occurrence of large/small categories are present in the model specification.

We conclude this subsection by presenting the ordinary stochastic volatility model. For a real valued time series $\{Y_{t_i}^c, i = 1,\dots,I\}$ the ordinary SV model is specified by

$$\begin{aligned} Y_{t_i}^c &= \mathbf{x}_{t_i}'\beta + \exp(h_{t_i}/2)\varepsilon_{t_i} \\ h_{t_i} &= \mathbf{z}_{t_i}'\alpha + \phi(h_{t_{i-1}} - \mathbf{z}_{t_{i-1}}'\alpha) + \sigma\eta_{t_i}, \end{aligned} \quad (2)$$

where $\mathbf{x}_{t_i}, \beta, \mathbf{z}_{t_i}, \alpha, \phi$ and σ^2 are specified as in the OSV model. The error variables ε_{t_i} and η_{t_i} are assumed to be i.i.d. standard normal, with independence also between $\{\varepsilon_{t_i}, i = 1,\dots,I\}$ and $\{\eta_{t_i}, i = 1,\dots,I\}$. Analogously to the OSV case, the model specified by (2) is denoted by $SV(X_1,\dots,X_p;Z_1,\dots,Z_q)$.

In our application we use $OSV(X_1,\dots,X_p;Z_1,\dots,Z_q)$ models for the category labels of the associated price change classes, whereas $SV(X_1,\dots,X_p;Z_1,\dots,Z_q)$ models are applied to the observed price changes directly.

2.2 Bayesian Inference for OSV and SV Models

Bayesian inference for the SV models was thoroughly investigated in Chib et al. (2002). They used an estimation procedure based on a state space approximation which we just briefly recall. Obviously, in model (2) one can equivalently write

$$\log\left(Y_{t_i}^c - \mathbf{x}_{t_i}'\beta\right)^2 = h_{t_i} + \log\varepsilon_{t_i}^2.$$

Kim et al. (1998) have shown that the distribution of $\log\varepsilon_{t_i}^2$ can be approximated very well by a seven-component mixture of normals. In particular, one can assume $\log\varepsilon_{t_i}^2 \approx \sum_{k=1}^7 q_k u_{t_i}^{(k)}$ where $u_{t_i}^{(k)}$ is normally distributed with mean m_k and variance v_k^2 independent of t_i. Moreover, the random variables $\{u_{t_i}^{(k)} \,|\, i=1,\dots,I,\ k=1,\dots,7\}$ are independent. The quantity q_k denotes the probability that the mixture component k occurs. These probabilities are also independent of t and are given in Table 1 of Chib et al. (2002) together with the corresponding means and variances. Let $s_{t_i} \in \{1,\dots,7\}$ denote the component of the mixture that occurs at time t_i and let $\pi(s_{t_i})$ denote the prior for s_{t_i}, where $\pi(s_{t_i}{=}k) = q_k$. Then, by setting $\tilde{Y}_{t_i}^c := \log\left(Y_{t_i}^c - \mathbf{x}_{t_i}'\beta\right)^2$, one arrives at

$$\tilde{Y}_{t_i}^c = h_{t_i} + u_{t_i}^{(s_{t_i})}$$

which, together with the second equation of (2), gives the desired state space representation.

The inference for the OSV models is even more complicated, since a straightforward extension of the algorithm by Chib et al. (2002) shows an unacceptable bad mixing of the chains. Therefore, Müller & Czado (2009) developed a grouped-move multigrid Monte Carlo (GM-MGMC) algorithm which exhibits fast convergence of the produced Markov chains. Since the SV model given by (2) is a submodel of the OSV model, we use the same sampling scheme also for the SV model, of course reduced by the sampling of the cutpoints which do not appear in the SV model, and the variables $Y_{t_i}^*$, $i=1,\dots,I$, which are observed in the SV case.

Each iteration of the GM-MGMC sampler consists of three parts. In the first part, the parameter vector β is drawn in a block update from a $(p+1)$-variate normal distribution, the latent variables $Y_{t_i}^*$, $i=1,\dots,I$, from truncated univariate normals, and the cutpoints c_k, $k=2,\dots,K-1$, from uniform distributions. In the second part, the grouped move step is performed. Here one draws a transformation element γ^2 from a Gamma distribution and updates β, $(Y_{t_1}^*,\dots,Y_{t_I}^*)$, and $\mathbf{c}$ by multiplication by the element $\gamma = \sqrt{\gamma^2}$. The third part starts with computation of the state space approximation, i.e. by computing $\tilde{Y}_{t_i}^* = \log(Y_{t_i}^* - \mathbf{x}_{t_i}'\beta)^2$ for $i=1,\dots,I$. Then s_{t_i}, $i=1,\dots,I$, are updated in single updates, and (α,ϕ,σ) by a Metropolis-Hastings step. Finally, the log volatilities $h_{t_1}^*,\dots,h_{t_I}^*$ are drawn in one block using the simulation smoother of De Jong & Shephard (1995). For more details on the updates we refer to Müller & Czado (2009).

For the Bayesian approach one also has to specify the prior distributions for $\mathbf{c}$, β, h_0^*, α, ϕ, and σ. Assuming prior independence the joint prior density can be written as

$$\pi(\mathbf{c},\beta,h_0^*,\alpha,\phi,\sigma) = \pi(\mathbf{c})\pi(\beta)\pi(h_0^*)\pi(\alpha_1)\cdots\pi(\alpha_q)\pi(\phi)\pi(\sigma).$$

For β a multivariate normal prior distribution is chosen, for h_0^* the Dirac measure at 0, and for the remaining parameters uniform priors. In particular,

$$\begin{aligned} \pi(\mathbf{c}) &= \mathbf{I}_{\{0<c_2<\cdots<c_{K-1}<C\}}, & \pi(\beta) &= N_{p+1}(\beta\,|\,\mathbf{b}_0,B_0),\\ \pi(h_0^*) &= \mathbf{I}_{\{h_0^*=0\}}, & \pi(\alpha_j) &= \mathbf{I}_{(-C_\alpha,C_\alpha)}(\alpha_j), \quad j=1,\ldots,q,\\ \pi(\phi) &= \mathbf{I}_{(-1,1)}(\phi), & \pi(\sigma) &= \mathbf{I}_{(0,C_\sigma)}(\sigma), \end{aligned}$$

where $C>0$, $C_\alpha>0$, and $C_\sigma>0$ are (known) hyperparameters, as well as the mean vector $\mathbf{b}_0$ and the covariance matrix B_0.

2.3 Model Selection

We now look at some criteria for model selection among OSV models, among SV models, and between OSV and SV models.

Model Selection Between OSV Models

We consider a model specification to be reasonable when credible intervals do not contain zero for all parameters. However model selection among such reasonable models is difficult since the likelihood cannot be evaluated simply for OSV models, thus the often used deviance information criteria (DIC) of Spiegelhalter et al. (2002) or score measures discussed in Gneiting & Raftery (2007) cannot be computed directly. Therefore we consider the following simple model selection criteria.

To choose among OSV models we first derive estimates of the ordinal categories for each t_i based on the MCMC iteration values. Note that the hidden volatility for each t_i is updated in each MCMC iteration, but we use only the average value of the log volatility estimates at t_i over all MCMC iterations. These averages are denoted by $\hat{h}_{t_i}^*$ and are used to derive fitted values for the hidden process. Let $\beta^r,\alpha^r,\sigma^r,\phi^r$ and c_k^r, $k=2,\ldots,K-1$ denote the rth MCMC iterate of β,α,σ,ϕ and c_k, $k=2,\ldots,K-1$, respectively for $r=1,\ldots,R$. The estimated log volatilities $\hat{h}_{t_i}^*$ allow to derive fitted hidden process variables $y_{t_i}^{*r}$ defined by

$$y_{t_i}^{*r} := \mathbf{x}_{t_i}'\beta^r + \exp(\hat{h}_{t_i}^*/2)\varepsilon_{t_i}^{*r},$$

where $\varepsilon_{t_i}^{*r}$ are i.i.d. standard normal observations. Finally find category k such that $y_{t_i}^{*r}\in[c_{k-1}^r,c_k^r)$ and set

$$y_{t_i}^r := k.$$

The ordinal category at time t_i is now fitted by the empirical median of $\{y^r_{t_i}, r = 1,\dots,R\}$, which we denote as $\hat{y}_{t_i}$.

To construct interval estimates for the ordinal categories we define

$$y^{*r}_{t_i,1-\alpha} := \mathbf{x}'_{t_i}\beta^r + \exp(\hat{h}^*_{t_i}/2)z_{1-\alpha},$$
$$y^{*r}_{t_i,\alpha} := \mathbf{x}'_{t_i}\beta^r - \exp(\hat{h}^*_{t_i}/2)z_{\alpha},$$

where z_δ denotes the δ quantile of a standard normally distributed random variable. Then we find categories $k_{1-\alpha}$ such that $y^{*r}_{t_i,1-\alpha} \in [c^r_{k-1}, c^r_k)$ and k_α such that $y^{*r}_{t_i,\alpha} \in [c^r_{k-1}, c^r_k)$, respectively, and set

$$y^r_{t_i,1-\alpha} := k_{1-\alpha} \quad \text{and} \quad y^r_{t_i,\alpha} := k_\alpha.$$

The interval estimate for a category at a time t_i is now defined as the interval $[\hat{y}_{t_i,\alpha}, \hat{y}_{t_i,1-\alpha}]$ where $\hat{y}_{t_i,\alpha}$ and $\hat{y}_{t_i,1-\alpha}$ denote the empirical medians of $\{y^r_{t_i,\alpha}, r = 1,\dots,R\}$ and $\{y^r_{t_i,1-\alpha}, r = 1,\dots,R\}$, respectively.

Alternatively we could consider a $100(1-\alpha)\%$ credible interval, which is given by $[\hat{y}^B_{t_i,\alpha}, \hat{y}^B_{t_i,1-\alpha}]$, where $\hat{y}^B_{t_i,\alpha}$ $(\hat{y}^B_{t_i,1-\alpha})$ denotes the empirical α $(1-\alpha)$ quantile of $\{y^r_{t_i}, r = 1,\dots,R\}$. Since the fitted category $y^r_{t_i}$ of the rth MCMC iterate takes on only a few values, the empirical α and $(1-\alpha)$ quantiles are not well defined. Therefore we will not follow this approach.

To choose among several OSV specifications we now count the times the observed category coincides with the fitted category as well as how many times the interval estimate covers the observed category. We choose the model with the highest correctly fitted and covered categories as the best model. Note that the observed coverage percentage is not identical with $100(1-\alpha)$ for the α value used in the construction of the interval estimates, since category values for different time points are dependent.

Model Selection Between SV Models

For the SV models we follow a similar approach as for the OSV models. First let $\hat{h}^c_{t_i}$ denote the average value of the log volatility estimates at time t_i over all MCMC iterations. Again let β^r, α^r, σ^r, and ϕ^r denote the rth MCMC iterate of β, α, σ, and ϕ for $r = 1,\dots,R$ for the SV model, respectively. Define

$$y^{c,r}_{t_i} := \mathbf{x}'_{t_i}\beta^r + \exp(\hat{h}^c_{t_i}/2)\varepsilon^r_{t_i}$$
$$y^{c,r}_{t_i,1-\alpha} := \mathbf{x}'_{t_i}\beta^r + \exp(\hat{h}^c_{t_i}/2)z_{1-\alpha}$$
$$y^{c,r}_{t_i,\alpha} := \mathbf{x}'_{t_i}\beta^r - \exp(\hat{h}^c_{t_i}/2)z_{\alpha},$$

where $\varepsilon^r_{t_i}$ are i.i.d. standard normal. Now determine the median of $\{y^{c,r}_{t_i}, r = 1,\dots,R\}$, $\{y^{c,r}_{t_i,1-\alpha}, r = 1,\dots,R\}$ and $\{y^{c,r}_{t_i,\alpha}, r = 1,\dots,R\}$, and denote them by $\hat{y}^c_{t_i}, \hat{y}^c_{t_i,1-\alpha}$ and $\hat{y}^c_{t_i,\alpha}$, respectively. Since $\hat{y}^c_{t_i}$ is real-valued, it is not informative to count the times the observed value is equal the fitted value $\hat{y}^c_{t_i}$ for all t_i. Hence, we only count the number of times the observed value is covered by the interval $[\hat{y}^c_{t_i,\alpha}, \hat{y}^c_{t_i,1-\alpha}]$ for all t_i.

Model Selection Between OSV and SV Models

The coverage percentage by the interval estimate for the OSV and SV, respectively, is used as a measure how good the model explains the observed values. A larger percentage gives a better fit.

3 Application

In this section we investigate the applicability of the OSV model to financial stocks with different levels of trading activity, and determine the influence of time between trades, volume, day time and the number of quotes between trades. Moreover, we compare the performance of OSV models and SV models using suitable model selection criteria.

3.1 Data

To investigate the gain of the OSV model over a corresponding SV model for the price changes we selected three stocks traded at the NYSE, reflecting stocks which are traded at a low, medium and high level. We chose the Fremont General Corporation (FMT), the Agilent Technologies (Agilent) and the International Business Machine Cooperation (IBM) from the TAQ data base for a low, medium and high level of trading, respectively. The data was collected between November 1-30, 2000 excluding November 23, 24 (thanksgiving).

Table 1 contains trading characteristics for the three stocks during the investigated time period. The absolute values of extremal price changes increase as trading activity increases (cf. rows 'price diff. between t_{i-1} and t_i'), indicating a higher volatility for more frequently traded stocks. As expected, the median and maximum time between trades decreases as the level of trading increases. For the number of quotes between trades we see a different behavior; while the medium number of quotes remains constant, the maximal number of quotes is the same for FMT and IBM, while it is lower for Agilent. Finally, Agilent has the highest maximum volume per trade among these three stocks.

To illustrate the discreteness of the observed price changes we recorded the number of occurrences of tick changes of size $\leq -3/16, -2/16, -1/16, 0, 1/16, 2/16, \geq 3/16$ together with their percentages in Table 2. For each of the tick change size we associate a category label (necessary for the OSV formulation) also given in Table 2. We see that the observed price changes are quite symmetric around 0 during the investigated time period and that a zero price change is observed most often.

The considered OSV and SV models allow for covariates on the mean and volatility level. To get an idea of possible day time effects we report the corresponding observed median values of price, price change, time between trades, number of quotes

Table 1 Observed characteristics of the FMT, Agilent and IBM stocks between Nov. 1 - 30, 2000

		minimum	median	maximum
FMT	price (dollar)	2 7/16	4 5/16	5 5/16
	price diff. between t_{i-1} and t_i (dollar)	−4/16	0	2/16
	time diff. between t_{i-1} and t_i (seconds)	0	192	4001
	number of quotes between t_{i-1} and t_i	0	1	24
	volume per trade	100	1000	122400
Agilent	price (dollar)	38 1/16	46 3/16	53 15/16
	price diff. between t_{i-1} and t_i (dollar)	−11/16	0	8/16
	time diff. between t_{i-1} and t_i (seconds)	0	11	276
	number of quotes between t_{i-1} und t_i	0	1	14
	volume per trade	100	500	247000
IBM	price (dollar)	91 10/16	99 7/16	104 5/16
	price diff. between t_{i-1} and t_i (dollar)	−13/16	0	14/16
	time diff. between t_{i-1} and t_i (seconds)	0	7	150
	number of quotes between t_{i-1} and t_i	0	1	24
	volume per trade	100	1000	225000

Table 2 Observed price changes together with category label, frequency and relative frequency in percent for the FMT, Agilent and IBM stocks from Nov. 1-30, 2000

	price difference	$\leq -3/16$	$-2/16$	$-1/16$	0	1/16	2/16	$\geq 3/16$
FMT	category	1	2	3	4	5	6	7
	frequency	3	25	229	755	227	28	0
	rel. freq. (%)	0.2	2.0	18.1	59.6	17.9	2.2	0.0
Agilent	category	1	2	3	4	5	6	7
	frequency	196	939	4662	16599	4747	863	216
	rel. freq. (%)	0.7	3.3	16.5	58.8	16.8	3.1	0.8
IBM	category	1	2	3	4	5	6	7
	frequency	585	3090	10251	22286	11161	2546	613
	rel. freq. (%)	1.2	6.1	20.3	44.1	22.1	5.0	1.2

and volume in Table 3. All stocks show larger (smaller) time intervals between trades during midday (opening and closing times), however the median price change is constant over the day time indicating no effect on the mean level of the hidden process. With regard to the volatility we also recorded the minimal and maximal price changes during trading hours in Table 4. Here we see less changes for different trading hours for the FMT and Agilent stocks compared to the IBM stock. This may indicate a day time effect on the volatility level for IBM stocks, which is detected by a corresponding OSV model specification.

Comparing Table 3 with Table 4 we might identify covariates on the volatility level. For example the median volume value exhibits a similar pattern as the pattern of volatility changes for the FMT and IBM stocks, indicating that volume has some explanatory power for the volatility of the price changes. For Agilent stocks the patterns of volume and volatility of the price changes do not match as well. For the other covariates the identification is less pronounced, so we consider them all as potentially useful covariates and let the statistical models identify them.

Table 3 Observed median number of price, price change, time between trades, number of quotes between trades and volume for different trading hours of the FMT, Agilent and IBM stock between Nov. 1 -30, 2000

	day time	9:30-10	10-11	11-12	12-1	1-2	2-3	3-4
FMT	price (dollar)	4 8/16	4 3/16	4 4/16	4 4/16	4 8/16	4 10/16	4 3/16
	price diff. (dollar)	0	0	0	0	0	0	0
	time diff. (sec.)	89	176	210	256	182.5	208	165
	no. of quotes	1	1	1	2	1	1	1
	volume	1000	1000	1000	1000	1000	1000	1000
Agilent	price (dollar)	46 8/16	46 9/16	46 2/16	46 4/16	45 14/16	46 2/16	46 4/16
	price diff. (dollar)	0	0	0	0	0	0	0
	time diff. (sec.)	7	10	11	12	12	11	10
	no. of quotes	1	1	1	1	1	1	1
	volume	600	600	500	500	500	500	500
IBM	price (dollar)	99 3/16	99 9/16	99 8/16	99 11/16	99 11/16	99 7/16	99 4/16
	price diff. (dollar)	0	0	0	0	0	0	0
	time diff. (sec.)	6	6	7	9	9	7	6
	no. of quotes	1	1	1	1	1	1	1
	volume	1300	1000	800	600	700	800	1000

Table 4 Minimal and maximal price changes for different trading hours of the FMT, Agilent and IBM stock between Nov. 1-30, 2000

	day time	9:30-10	10-11	11-12	12-1	1-2	2-3	3-4
FMT	min. price change	−2/16	−2/16	−2/16	−4/16	−3/16	−2/16	−2/16
	max. price change	2/16	2/16	2/16	2/16	2/16	2/16	2/16
Agilent	min. price change	−6/16	−5/16	−4/16	−11/16	−5/16	−4/16	−8/16
	max. price change	8/16	6/16	5/16	5/16	4/16	7/16	7/16
IBM	min. price change	−13/16	−8/16	−4/16	−9/16	−4/16	−5/16	−8/16
	max. price change	14/16	10/16	5/16	10/16	4/16	6/16	9/16

3.2 OSV Models

As response we choose the category corresponding to the price change at trading time t_i, denoted by y_{t_i}. To model a possibly present dependence between the current price change category and the previous one, we use the lagged price change as a covariate on the mean level and denote it by LAG1 (no other covariates turned out to be significant for the mean level in our analysis). In addition, we allow for an intercept parameter on the mean level. For possible covariates on the volatility level we use volume (V), daytime (D), time elapsed between trades (T) and the number of quotes between trades (Q). For numerical stability we use centered and standardized versions of these variables. For reasons of identifiability, no intercept is included in the term $\mathbf{z}_{t_i}'\alpha$.

For all three stocks we ran a variety of models involving V, D, T and Q as well as quadratic functions of these. In the following we only present models where all covariates are significant, i.e. their individual 80% credible intervals do not contain zero. For all models we ran 20000 MCMC iterations of the GM-MGMC algorithm.

Appropriate burnin values were determined using trace plots. Furthermore, the estimated autocorrelations among the MCMC iterations suggested to take a subsample of every 20th iteration.

Fremont General Cooperation

The left panel of Table 5 presents, for three different OSV model specifications, the estimated posterior medians and means of each parameter together with a 80% credible interval for the subsampled MCMC iterations after burnin. Figure 2 shows estimated posterior densities for all parameters of the $OSV(1, LAG1; V, T)$ model. We see a symmetric behavior of the posteriors for the cutpoint parameters and regression parameters and slightly skewed distributions for σ and ϕ. The posterior density estimates for the remaining two OSV specifications show a similar behavior and are therefore omitted.

Interpreting the results for the OSV specifications, we see from the negative sign of LAG1, that an higher (lower) previous price change category decreases the probability of an higher (lower) current price change category, a fact which can be observed directly from the data, where often a positive price change is followed by a negative one and vice versa. A higher volume, a larger time interval between trades and a larger number of quotes increase the log volatility, thus the probability of observing an extreme positive or negative price change is increased.

It remains to choose among the three OSV specifications. Since the models $OSV(1, LAG1; V, T)$, $OSV(1, LAG1; T, Q)$ are nested within $OSV(1, LAG1; V, T, Q)$, the significance of the parameter estimates established by the credible intervals may lead to a slight preference for the $OSV(1, LAG1; V, T, Q)$ model specification. This is also confirmed, when we calculate the fitted price change categories (see Section 2.3) and compare them to the observed price categories. Moreover, we determine fitted interval bounds for the price change category and check how many times they are covering the observed price change category. The percentage of correctly fitted categories is 59.67%, 59.43% and 59.75% for the models $OSV(1, LAG1; V, T)$, $OSV(1, LAG1; T, Q)$ and $OSV(1, LAG1; V, T, Q)$, respectively. The corresponding values for the percentage of correctly covered categories are 96.45%, 96.37% and 96.53%, respectively. This may lead again to a slight preference for the large model.

Agilent Technologies

For the Agilent stock we found only a single OSV specification with significant parameter estimates, whose summary statistics are given in the left panel of Table 6. It is a different specification as for FMT stocks. The effect of the previous price change category for the Agilent stocks is similar to that one for the FMT stocks, and the autocorrelation of the hidden log volatilities is quite the same. A notable difference is the effect of the number of quotes between trades on the price change

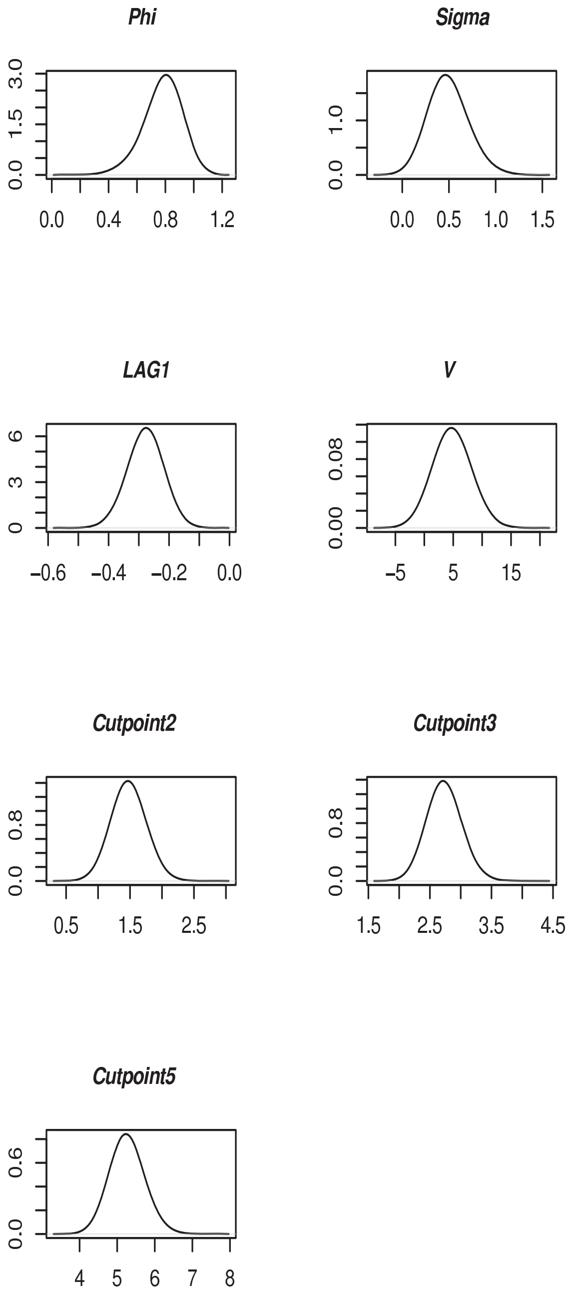

Fig. 2 Estimated posterior density for $OSV(1, LAG1; V, T)$ parameters for FMT stocks

Table 5 Estimated posterior means, medians and quantiles of three OSV (left panel) and three SV (right panel) model specifications with significant parameters fitted for FMT stocks based on the subsampled MCMC iterations after burnin

parameter	10%	90%	median	mean	10%	90%	median	mean
	$OSV(1, LAG1; V, T)$				$SV(1; V, T)$			
ϕ	0.64	0.89	0.80	0.78	0.75	0.79	0.77	0.77
σ	0.32	0.70	0.47	0.49	9.17	11.86	9.90	10.21
c_2	1.25	1.72	1.47	1.48				
c_3	2.50	2.98	2.72	2.73				
c_4	3.73	4.29	3.99	4.00				
c_5	4.88	5.68	5.24	5.26				
$\mathbf{1}$	4.11	4.86	4.46	4.47	$1.1 \cdot 10^{-6}$	$7.1 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$
LAG1	−0.33	−0.23	−0.28	−0.28				
V	1.86	7.83	4.66	4.79	14.14	24.96	19.48	19.51
T	3.45	9.46	6.32	6.39	25.09	36.75	31.09	31.04
	$OSV(1, LAG1; V, Q)$				$SV(1; V, Q)$			
ϕ	0.44	0.85	0.74	0.69	0.75	0.79	0.77	0.77
σ	0.39	0.93	0.58	0.63	9.18	11.82	9.94	10.19
c_2	1.47	2.00	1.72	1.73				
c_3	2.78	3.42	3.07	3.08				
c_4	4.01	4.73	4.34	4.36				
c_5	5.21	6.20	5.65	5.68				
$\mathbf{1}$	4.45	5.38	4.87	4.89	$1.1 \cdot 10^{-6}$	$7.1 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$
LAG1	−0.34	−0.24	−0.30	−0.29				
V	1.49	7.57	4.76	4.64	13.67	25.19	19.28	19.44
Q	1.72	6.84	4.24	4.27	15.44	26.29	21.36	21.10
	$OSV(1, LAG1; V, T, Q)$				$SV(1; V, T, Q)$			
ϕ	0.72	0.91	0.83	0.82	0.76	0.79	0.77	0.77
σ	0.26	0.58	0.40	0.42	9.12	11.71	9.88	10.11
c_2	1.01	1.58	1.24	1.27				
c_3	2.14	2.84	2.42	2.46				
c_4	3.38	4.13	3.71	3.74				
c_5	4.52	5.45	4.94	4.96				
1	3.77	4.65	4.18	4.19	$1.1 \cdot 10^{-6}$	$7.1 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$
LAG1	−0.33	−0.22	−0.27	−0.27				
V	1.87	7.54	4.58	4.59	13.82	24.53	19.71	19.35
T	1.55	6.42	4.02	4.00	15.06	26.38	20.73	20.74
Q	3.63	8.93	6.26	6.30	25.13	36.67	31.00	30.98

categories. Here the parameter estimate has a negative sign, thus the probability of extreme price change categories is decreased when the number of quotes is increased.

Table 6 Estimated posterior means, medians and quantiles of the $OSV(1, LAG1; T, Q)$ (left panel) and $SV(1;T)$ fitted for Agilent stocks based on subsampled MCMC iterations after burnin

parameter	10%	90%	median	mean	10%	90%	median	mean
	$OSV(1, LAG1; T, Q)$				$SV(1;T)$			
ϕ	0.80	0.84	0.82	0.82	0.74	0.80	0.77	0.77
σ	0.46	0.52	0.49	0.49	9.12	10.37	9.67	9.72
c_2	1.28	1.33	1.30	1.30				
c_3	2.24	2.31	2.28	2.28				
c_4	3.42	3.52	3.47	3.47				
c_5	4.39	4.52	4.46	4.46				
c_6	5.36	5.56	5.47	5.47				
1	3.68	3.81	3.74	3.74	$1.1 \cdot 10^{-6}$	$7.1 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$	$4.1 \cdot 10^{-6}$
LAG1	−0.23	−0.21	−0.22	−0.22				
T	22.12	27.43	24.78	24.80	62.57	181.79	121.86	121.89
Q	−10.11	−5.25	−7.64	−7.64				

International Business Machines Cooperation

For the IBM stocks we have two OSV model specifications where all parameter estimates are significant (see the left panel of Table 7). The effect of the number of quotes is similar to that one of the Agilent stock. The full specification also includes a significantly negative daytime parameter, indicating a lower probability of extreme price change categories for later in the day than in the morning. This corresponds to the fact, that often the highest volatility during a day can be observed directly after opening of the exchange. The percentage of correctly fitted response categories is 41.54% for the $OSV(1, LAG1; V, Q)$ model compared to 41.48% for the $OSV(1, LAG1; V, T, Q, D)$ model. The percentage of correctly covered response categories is 92.94% for the for the $OSV(1, LAG1; V, Q)$ model compared to 92.93% for the $OSV(1, LAG1; V, T, Q, D)$ model. Hence, we prefer the simpler one of the two OSV model specifications.

In summary, we see that different OSV models are specified for the different stocks. Whereas there is a negative parameter estimate for the number of quotes between two subsequent trades of the Agilent and the IBM stock, the opposite is true for the less frequently traded FMT stock. Therefore the probability of extreme price changes seems to decrease for more frequently traded stocks when the number of quotes between trades increases, whereas this probability increases for less frequently traded stocks. In addition, the trading frequency influences the magnitude of autocorrelation present in the log volatilities. The highest autocorrelation was observed for the IBM stock. Daytime effects on the hidden volatility are not significant in our three preferred models. The effect of the time elapsed between trades on the log volatility is always positive. This indicates that larger time differences between two subsequent trades usually lead to a higher volatility. The positive regression coefficient for volume induces a larger volatility for larger volumes, which results in higher probabilities for the occurrence of extreme price change categories.

Table 7 Estimated posterior means, medians and quantiles of two OSV (left panel) and one SV (right panel) model specifications with significant parameters fitted for IBM stocks based on recorded MCMC iterations

parameter	10%	90%	median	mean	10%	90%	median	mean
		$OSV(1,LAG1;V,Q)$						
ϕ	0.93	0.94	0.94	0.94				
σ	0.20	0.23	0.21	0.21				
c_2	0.93	0.95	0.94	0.94				
c_3	1.65	1.69	1.67	1.67				
c_4	2.52	2.57	2.54	2.54				
c_5	3.33	3.40	3.37	3.37				
c_6	4.10	4.21	4.16	4.16				
1	3.04	3.12	3.08	3.08				
LAG1	−0.25	−0.24	−0.24	−0.24				
V	5.54	10.25	7.98	7.98				
Q	−9.17	−5.07	−7.12	−7.12				
		$OSV(1,LAG1;V,T,Q,D)$				$SV(1;V,T)$		
ϕ	0.92	0.94	0.93	0.93	0.67	0.69	0.68	0.68
σ	0.21	0.24	0.22	0.23	0.07	0.14	0.09	0.10
c_2	0.90	0.93	0.92	0.91				
c_3	1.66	1.70	1.68	1.68				
c_4	2.51	2.55	2.53	2.53				
c_5	3.34	3.41	3.38	3.38				
c_6	4.14	4.26	4.20	4.20				
1	3.04	3.12	3.08	3.08	$1.9\cdot10^{-5}$	$7.5\cdot10^{-4}$	$3.7\cdot10^{-4}$	$3.7\cdot10^{-4}$
LAG1	−0.25	−0.24	−0.24	−0.24				
V	5.22	9.76	7.43	7.46	0.54	9.60	4.91	4.99
T	33.36	38.85	36.08	36.02	23.15	37.59	29.28	29.99
Q	−8.37	−3.90	−6.06	−6.07				
D	−35.08	−22.16	−28.12	−28.26				

3.3 SV Models

For the SV setup we use the observed price changes as response and ignore their discrete nature. For each of the three stocks we investigated different SV specifications. A first difference to the OSV specifications are that none of the covariates LAG1, V, T, Q, and D for the mean level are significant. Therefore all SV models include only an intercept parameter in the mean level, which is significant but very close to zero. For the log volatilities we find significant covariates, which we present in the following. Again we ran 20000 MCMC iterations and determined appropriate burnin values and subsampling rates.

Fremont General Corporation

Three significant SV specifications were found for the FMT stocks and the results are summarized in the right panel of Table 5. The highest coverage percentage is achieved using the $SV(1;V,T,Q)$, which we select as best model among the SV models for the FMT stocks.

Agilent Technologies

For the Agilent stocks only a single SV specification produces significant parameter estimates and the results are presented in the right panel of Table 6. From this we see that only the time elapsed between trades has a significant effect on the price changes. A larger time interval between trades produces a larger volatility, i.e. extreme price changes become more likely.

International Business Machines Cooperation

For the frequently traded IBM stocks only the $SV(1;V,T)$ model produces significant posterior parameter estimates. The results presented in right panel of Table 7 show that both volume and time elapsed between trades increase the volatility, thus making more extreme price changes more likely.

3.4 Comparison Between OSV and SV Models

We now compare all selected OSV and SV models by using the coverage percentages. These are reported in Table 8. We see that there is a clear preference for the OSV specifications for Agilent and IBM stocks, while for the FMT stock a slight preference for the SV specification is visible. A graphical illustration of this is given in Figure 3 where the interval estimates are plotted for the last 100 observations together with the observed values.

As a final comparison we estimate posterior densities of the volatilities for each price change category using the competing OSV and SV specifications for all three stocks. The corresponding plots are shown in Figure 4. The OSV specifications nicely identify different volatility patterns. In particular, extreme price categories correspond to larger volatilities. The competing SV specification for the IBM stocks shows a similar pattern. However, the SV specifications for the FMT and the Agilent stocks lead to quite different density estimates.

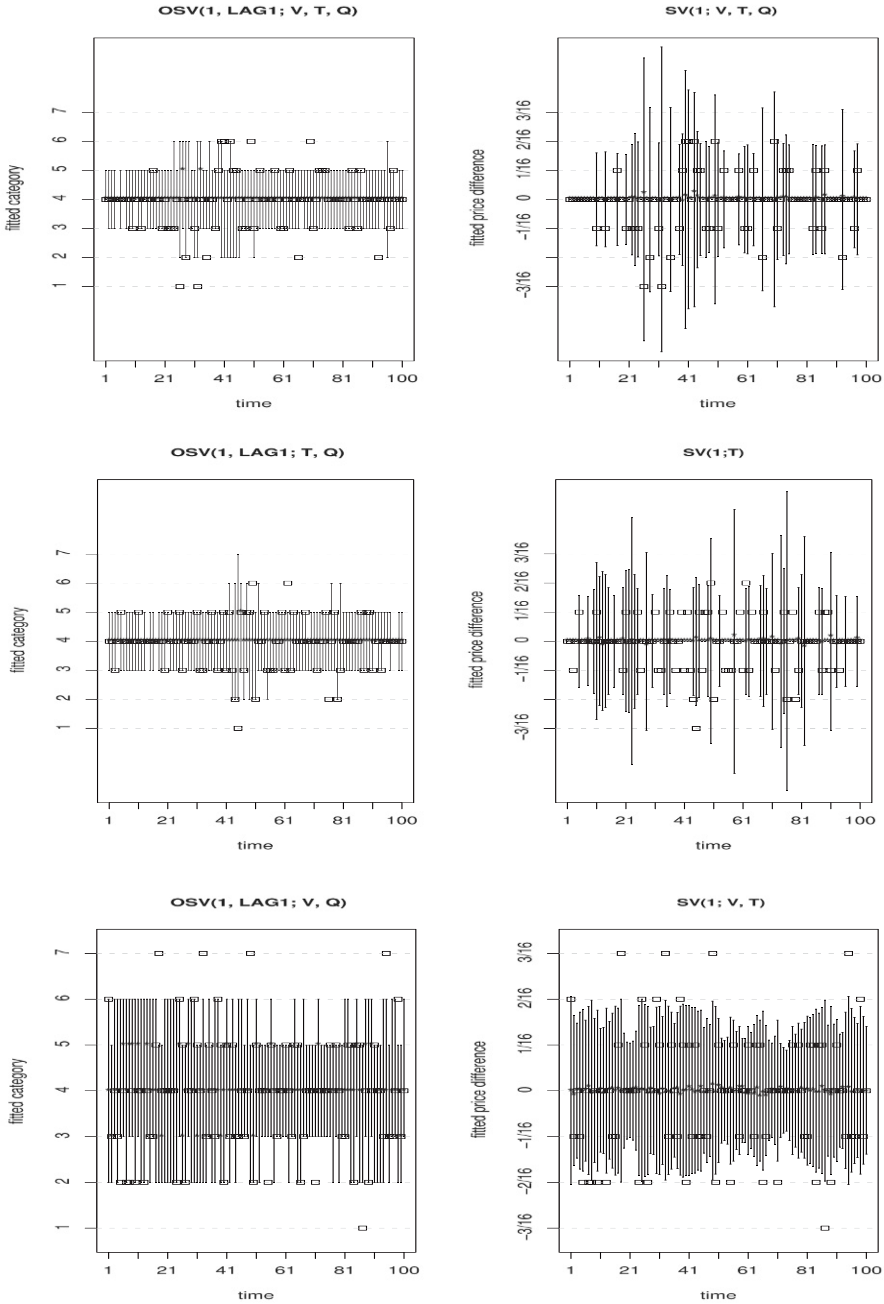

Fig. 3 Fitted categories and fitted price differences of OSV and SV model of the last 100 observations together with interval estimates for FMT (top row), Agilent (middle row) and IBM (bottom row) stocks, respectively

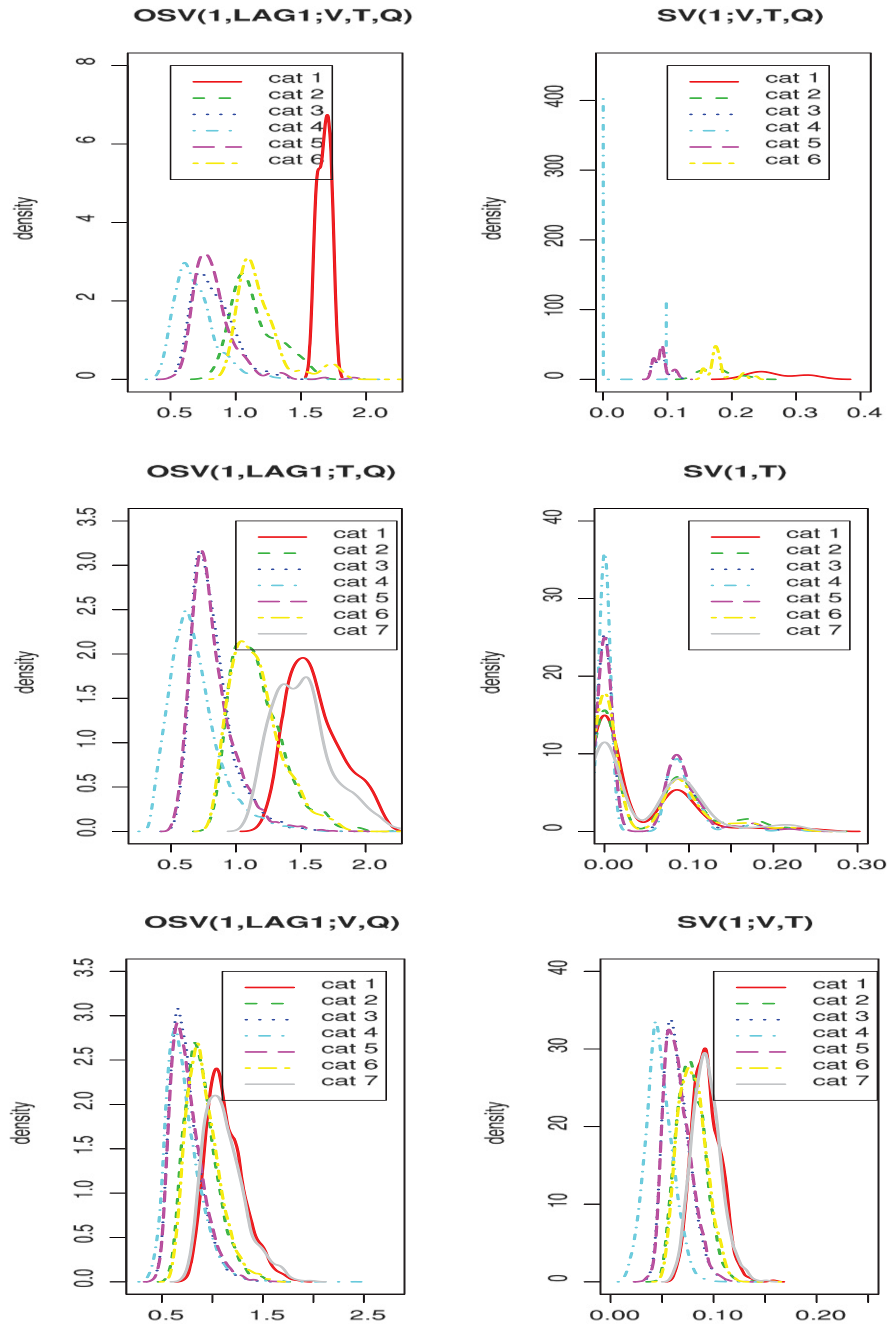

Fig. 4 Estimated posterior densities of the (hidden) volatilities for each category of OSV and SV model for FMT (top row), Agilent (middle row) and IBM (bottom row) stocks, respectively

Table 8 Percentage of correctly covered observations of different OSV and SV specifications for FMT, Agilent and IBM stocks

	OSV specifications		SV specifications	
FMT	$OSV(1, LAG1; V, T, Q)$	1223/1267 = 96.53%	$SV(1, LAG1; V, T, Q)$	1267/1267 = 100.00%
Agilent	$OSV(1, LAG1; T, Q)$	26738/28222 = 94.74%	$SV(1; T)$	20980/28222 = 74.34%
IBM	$OSV(1, LAG1; V, Q)$	46965/50532 = 92.94%	$SV(1; V, T)$	42811/50532 = 84.72%

4 Summary and Discussion

In this paper we presented the results of a Bayesian analysis of two model class specifications for financial price changes. Estimation is facilitated using MCMC methods. The OSV specification explicitly accounts for the discrete values of the price changes, while the SV specification ignores it. The OSV model captures the influence of the previous price change, whereas for the SV models this influence is not significant. In addition we see that volume, time between trades and the number of quotes between trades are important factors determining the volatility. Useful model specifications depend on the trading activity of the stock. In particular, a higher number of quotes between trades increases the volatility for less frequently traded stocks, whereas the opposite pattern is observed for stocks which are more frequently traded. As expected a larger duration between trades increases the volatility. A quadratic day time effect was not significant indicating that there was no strong volatility smile present in the data.

When comparing the OSV and SV models we see that the OSV models perform better (at least for the more frequently traded Agilent and IBM stocks) than the SV models with regard to the coverage proportion of interval estimates. However, more precise model comparison criteria for comparing non nested models with numerical intractable likelihoods in a Bayesian setup are needed and subject to current research. Finally, the OSV and SV model specifications lead to different density estimates for the volatility within the price change classes. However, the density estimates coming from the OSV specifications are quite convincing, since here extreme categories always come along with higher values of the volatility estimates.

Overall we conclude that the OSV models which account for the discreteness of the price changes perform quite well, when applied to data sets as considered in this analysis. Although it is computationally more involved to fit the OSV model to the data, the OSV model is tailored to the structure of ordinal-response data and hence most suitable for price changes.

Acknowledgements Claudia Czado is supported by the *Deutsche Forschungsgemeinschaft*.

References

Chib, S., Nardari, F. & Shephard, N. (2002). Markov chain Monte Carlo methods for stochastic volatility models, *Journal of Econometrics* **108**: 281–316.

De Jong, P. & Shephard, N. (1995). The simulation smoother for time series models, *Biometrika* **82**: 339–350.

Engle, R. F. & Russell, J. R. (1998). Autoregressive conditional duration: a new model for irregularly spaced transaction data, *Econometrica* **66**: 1127–1162.

Ghysels, E., Harvey, A. C. & Renault, E. (1996). Stochastic volatility, *in* C. R. Rao & G. S. Maddala (eds), *Statistical Methods in Finance*, North-Holland, Amsterdam, pp. 119–191.

Gneiting, T. & Raftery, A. E. (2007). Strictly proper scoring rules, prediction, and estimation, *Journal of the American Statistical Association* **102**: 359–378.

Harris, L. (1990). Estimation of stock variances and serial covariances from discrete observations, *Journal of Financial and Quantitative Analysis* **25**: 291–306.

Hasbrouck, J. (1999). Security bid/ask dynamics with discreteness and clustering, *Journal of Financial Markets* **2**: 1–28.

Hasbrouck, J. (1999a). The dynamics of discrete bid and ask quotes, *Journal of Finance* **54**: 2109–2142.

Hasbrouck, J. (2003). Markov chain Monte Carlo methods for Bayesian estimation of microstructure models (computational appendix to: Hasbrouck, J. (2004). Liquidity in the futures pits: inferring market dynamics from incomplete data, *Journal of Financial and Quantitative Analysis* **39**: 305–326), available at http://pages.stern.nyu.edu/∼jhasbrou.

Hasbrouck, J. (2004). Liquidity in the futures pits: inferring market dynamics from incomplete data, *Journal of Financial and Quantitative Analysis* **39**: 305–326.

Hausman, J. A., Lo, A. W. & MacKinlay, A. C. (1992). An ordered probit analysis of transaction stock prices, *Journal of Financial Economics* **31**: 319–379.

Kim, S., Shephard, N. & Chib, S. (1998). Stochastic volatility: likelihood inference and comparison with ARCH models, *Review of Economic Studies* **65**: 361–393.

Kitagawa, G. (1987). Non-Gaussian state space modeling of nonstationary time series, *Journal of the American Statistical Association* **82**: 1032–1041.

Liesenfeld, R., Nolte, I. & Pohlmeier, W. (2006). Modelling financial transaction price movements: a dynamic integer count data model, *Empirical Economics* **30**: 795–825.

Manrique, A. & Shephard, N. (1997). Likelihood analysis of a discrete bid/ask price model for a common stock quoted on the NYSE, *Nuffield College Economics papers*, W-15.

Müller, G. & Czado, C. (2009). Stochastic volatility models for ordinal valued time series with application to finance, *Statistical Modelling* **9**: 69–95.

Russell, J. R. & Engle, R. F. (2005). A discrete-state continuous-time model of financial transactions prices and times: the autoregressive conditional multinomial-autoregressive conditional duration model, *Journal of Business and Economic Statistics* **23**: 166–180.

Rydberg, T. H. & Shephard, N. (2003). Dynamics of trade-by-trade price movements: decomposition and models, *Journal of Financial Econometrics* **1**: 2–25.

Shephard, N. (2006). Stochastic volatility models, *in* S. N. Durlauf & L. E. Blume (eds), *New Palgrave Dictionary of Economics*, 2nd ed., Palgrave Macmillan. doi: 10.1007/s001090000086.

Spiegelhalter, D. J., Best, N. G., Carlin, B. P. & van der Linde, A. (2002). Bayesian measures of model complexity and fit, *Journal of the Royal Statistical Society B* **64**: 583–639.