Chapter 2

Weak Solutions, A Priori Estimates

The fundamental laws of continuum mechanics interpreted as infinite families of integral identities introduced in Chapter 1, rather than systems of partial differential equations, give rise to the concept of weak (or variational) solutions that can be vastly extended to extremely divers physical systems of various sorts. The main stumbling block of this approach when applied to the field equations of fluid mechanics is the fact that the available *a priori estimates* are not strong enough in order to control the flux of the total energy and/or the dissipation rate of the kinetic energy. This difficulty has been known since the seminal work of Leray [132] on the incompressible NAVIER-STOKES SYSTEM, where the validity of the so-called energy equality remains an open problem, even in the class of suitable weak solutions introduced by Caffarelli et al. [37]. The question is whether or not the rate of decay of the kinetic energy equals the dissipation rate due to viscosity as predicted by formula (1.39). It seems worth noting that certain weak solutions to *hyperbolic conservation laws* indeed dissipate the kinetic energy whereas classical solutions of the same problem, provided they exist, do not. On the other hand, however, we are still very far from complete understanding of possible singularities, if any, that may be developed by solutions to dissipative systems studied in fluid mechanics. The problem seems even more complex in the framework of compressible fluids, where Hoff [113] showed that singularities survive in the course of evolution provided they were present in the initial data. However, it is still not known if the density may develop “blow up” (gravitational collapse) or vanish (vacuum state) in a finite time. Quite recently, Brenner [28] proposed a daring new approach to fluid mechanics, where at least some of the above mentioned difficulties are likely to be eliminated.

Given the recent state of the art, we anticipate the hypothetical possibility that the weak solutions may indeed dissipate more kinetic energy than indicated by (1.33), thereby replacing the classical expression of the entropy production rate (1.39) by an *inequality*

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta\Big). \tag{2.1}$$

Similarly to the theory of hyperbolic systems, the entropy production rate σ is now to be understood as a non-negative measure on the set $[0,T] \times \overline{\Omega}$, whereas the term

$$\int_0^T \int_\Omega \sigma\varphi \ \mathrm{d}x \text{ is replaced by } \langle\sigma;\varphi\rangle_{[\mathcal{M}^+;C]([0,T]\times\overline{\Omega})} \text{ in (1.40).}$$

Although it may seem that changing *equation* to mere *inequality* may considerably extend the class of possible solutions, it is easy to verify that inequality (2.1) reduces to the classical formula (1.39) as soon as the weak solution is regular and satisfies the global energy balance (1.36). By a regular solution we mean that all state variables ϱ, $\mathbf{u}$, ϑ are continuously differentiable up to the boundary of the space-time cylinder $[0,T] \times \overline{\Omega}$, possess all the necessary derivatives in $(0,T) \times \Omega$, and ϱ, ϑ are strictly positive. Indeed if ϑ is smooth we are allowed to use the quantity $\vartheta\varphi$ as a test function in (1.40) to obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \varrho s\Big(\partial_t\vartheta + \mathbf{u}\cdot\nabla_x\vartheta\Big)\varphi \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \varrho s\vartheta\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \mathbf{q}\cdot\nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t + \langle\sigma;\vartheta\varphi\rangle + \int_0^T \int_\Omega \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\vartheta\varphi \ \mathrm{d}x \ \mathrm{d}t \\
&= -\int_0^T \int_\Omega \varrho\mathcal{Q}\varphi \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}$$

for any $\varphi \in C_c^\infty((0,T)\times\overline{\Omega})$. Moreover, as ϱ, $\mathbf{u}$ satisfy the equation of continuity (1.22), we get

$$\begin{aligned}
&\int_0^T \int_\Omega \varrho s\Big(\partial_t\vartheta + \mathbf{u}\cdot\nabla_x\vartheta\Big)\varphi \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_\Omega \varrho s\vartheta\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big) \ \mathrm{d}x \ \mathrm{d}t \\
&\quad = -\int_0^T \int_\Omega \varrho\vartheta\Big(\partial_t s + \mathbf{u}\cdot\nabla_x s\Big)\varphi \ \mathrm{d}x \ \mathrm{d}t \\
&\quad = -\int_0^T \int_\Omega \varrho\Big(\partial_t e + \mathbf{u}\cdot\nabla_x e\Big)\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega p\mathrm{div}_x\mathbf{u}\varphi \ \mathrm{d}x,
\end{aligned}$$

where we have used Gibbs' relation (1.2). Consequently, we deduce

$$\begin{aligned}
&\int_\Omega \varrho e(\varrho,\vartheta)(t_2) \ \mathrm{d}x - \int_\Omega \varrho e(\varrho,\vartheta)(t_1) \ \mathrm{d}x \\
&\quad = \int_{t_1}^{t_2} \int_\Omega \Big(\varrho\mathcal{Q} - p\mathrm{div}_x\mathbf{u}\Big) \ \mathrm{d}x \ \mathrm{d}t + \int_{t_1}^{t_2} \int_\Omega \Big(\vartheta\sigma + \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\vartheta\Big) \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}$$

for $0 < t_1 \leq t_2 < T$.

Conversely, since regular solutions necessarily satisfy the kinetic energy equation (1.33), we can use the total energy balance (1.36) in order to conclude that

$$\int_\Omega \varrho e(\varrho,\vartheta)(t_2)\ \mathrm{d}x - \int_\Omega \varrho e(\varrho,\vartheta)(t_1)\ \mathrm{d}x = \int_{t_1}^{t_2}\int_\Omega \Big(\varrho\mathcal{Q} + \mathbb{S}:\nabla_x\mathbf{u} - p\mathrm{div}_x\mathbf{u}\Big)\ \mathrm{d}x\ \mathrm{d}t;$$

whence, by means of (2.1),

$$\sigma = \frac{1}{\vartheta}\Big(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\vartheta\Big) \text{ in } [t_1,t_2]\times\overline{\Omega}.$$

Note that our approach based on postulating *inequality* (2.1), together with *equality* (1.36) is reminiscent of the concept of *weak solutions with defect measure* elaborated by DiPerna and Lions [64] and Alexandre and Villani [5] in the context of Boltzmann's equation. Although uniqueness in terms of the data is probably out of reach of such a theory, the piece of information provided is sufficient in order to study the qualitative properties of solutions, in particular, the long-time behavior and singular limits for several scaling parameters tending to zero. Starting from these ideas, we develop a thermodynamically consistent mathematical model based on the *state variables* $\{\varrho,\mathbf{u},\vartheta\}$ and enjoying the following properties:

- The problem admits global-in-time solutions for any initial data of finite energy.
- The changes of the total energy of the system are only due to the action of the external source terms represented by $\mathbf{f}$ and $\mathcal{Q}$. In the absence of external sources, the total energy is a constant of motion.
- The total entropy is increasing in time as soon as $\mathcal{Q}\geq 0$, the system evolves to a state maximizing the entropy.
- Weak solutions coincide with classical ones provided they are smooth, notably the entropy production rate σ is equal to the expression on the right-hand side of (2.1).

2.1 Weak formulation

For reader's convenience and future use, let us summarize in a concise form the *weak formulation* of the problem identified in Chapter 1. The problem consists of finding a trio $\{\varrho,\mathbf{u},\vartheta\}$ satisfying a family of integral identities referred to in the future as a NAVIER-STOKES-FOURIER SYSTEM. We also specify the minimal regularity of solutions required, and interpret formally the integral identities in terms of standard partial differential equations provided all quantities involved in the weak formulation are smooth enough.

2.1.1 Equation of continuity

(i) Weak (renormalized) formulation:

$$\int_0^T \int_\Omega \varrho B(\varrho)\Big(\partial_t \varphi + \mathbf{u}\cdot\nabla_x \varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T \int_\Omega b(\varrho)\mathrm{div}_x \mathbf{u}\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_0 B(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x. \tag{2.2}$$

(ii) Admissible test functions:

$$b \in L^\infty \cap C[0,\infty),\ B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2}\ \mathrm{d}z, \tag{2.3}$$
$$\varphi \in C^1_c([0,T)\times\overline{\Omega}). \tag{2.4}$$

(iii) Minimal regularity of solutions required:

$$\varrho \geq 0,\ \varrho \in L^1((0,T)\times\Omega), \tag{2.5}$$
$$\varrho\mathbf{u} \in L^1((0,T)\times\Omega;\mathbb{R}^3),\ \mathrm{div}_x\mathbf{u} \in L^1((0,T)\times\Omega). \tag{2.6}$$

(iv) Formal interpretation:

$$\partial_t(\varrho B(\varrho)) + \mathrm{div}_x(\varrho B(\varrho)\mathbf{u}) + b(\varrho)\mathrm{div}_x\mathbf{u} = 0 \text{ in } (0,T)\times\Omega, \tag{2.7}$$
$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{2.8}$$

2.1.2 Balance of linear momentum

(i) Weak formulation:

$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\varphi + p\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x\varphi - \varrho\mathbf{f}\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\mathbf{u})_0\cdot\varphi(0,\cdot)\ \mathrm{d}x. \tag{2.9}$$

(ii) Admissible test functions:

$$\varphi \in C^1_c([0,T)\times\overline{\Omega};\mathbb{R}^3), \tag{2.10}$$

and either

$$\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of the complete slip boundary conditions,} \tag{2.11}$$

or

$$\varphi|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions.} \tag{2.12}$$

(iii) Minimal regularity of solutions required:

$$\varrho\mathbf{u} \in L^1((0,T)\times\Omega;\mathbb{R}^3),\ \varrho|\mathbf{u}|^2 \in L^1((0,T)\times\Omega), \tag{2.13}$$

$$p \in L^1((0,T)\times\Omega),\ \mathbb{S} \in L^1((0,T)\times\Omega;\mathbb{R}^{3\times 3}),\ \varrho\mathbf{f} \in L^1((0,T)\times\Omega;\mathbb{R}^3), \tag{2.14}$$

$$\nabla_x\mathbf{u} \in L^1(0,T;L^q(\Omega;\mathbb{R}^{3\times 3})),\ \text{for a certain } q > 1; \tag{2.15}$$

and, either

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of the complete slip boundary conditions,} \tag{2.16}$$

or

$$\mathbf{u}|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions.} \tag{2.17}$$

(iv) Formal interpretation:

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p = \mathrm{div}_x\mathbb{S} + \varrho\mathbf{f} \text{ in } (0,T)\times\Omega, \tag{2.18}$$

$$(\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0, \tag{2.19}$$

together with the complete slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\mathbb{S}\mathbf{n})\times\mathbf{n}|_{\partial\Omega} = 0, \tag{2.20}$$

or, alternatively, the no-slip boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{2.21}$$

2.1.3 Balance of total energy

(i) Weak formulation:

$$\int_0^T \int_\Omega \mathcal{E}(t)\ \mathrm{d}x\ \partial_t\psi(t)\ \mathrm{d}t = -\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\mathbf{f}(t) + \varrho\mathcal{Q}(t)\Big)\psi(t)\ \mathrm{d}x\ \mathrm{d}t - \psi(0)E_0 \tag{2.22}$$

$$\mathcal{E}(t) = \frac{1}{2}\varrho|\mathbf{u}|^2(t) + \varrho e(t) \text{ for a.a. } t \in (0,T). \tag{2.23}$$

(ii) Admissible test functions:

$$\psi \in C^1_c[0,T). \tag{2.24}$$

(iii) Minimal regularity of solutions required:

$$\mathcal{E},\ \varrho\mathbf{u}\cdot\mathbf{f},\ \varrho\mathcal{Q} \in L^1((0,T)\times\Omega). \tag{2.25}$$

(iv) Formal interpretation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{E}\ \mathrm{d}x = \int_\Omega \Big(\varrho\mathbf{u}\cdot\mathbf{f} + \varrho\mathcal{Q}\Big)\ \mathrm{d}x \text{ in } (0,T),\ \int_\Omega \mathcal{E}(0)\ \mathrm{d}x = E_0. \tag{2.26}$$

2.1.4 Entropy production

(i) Weak formulation:

$$\int_0^T \int_\Omega \varrho s\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t + \int_0^T \int_\Omega \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t + \langle\sigma;\varphi\rangle_{[\mathcal{M}^+;C]([0,T]\times\overline{\Omega})}$$
$$= -\int_\Omega (\varrho s)_0\varphi(0,\cdot)\ \mathrm{d}x - \int_0^T\int_\Omega \frac{\varrho}{\vartheta}\mathcal{Q}\varphi\ \mathrm{d}x\ \mathrm{d}t, \tag{2.27}$$

where $\sigma\in\mathcal{M}^+([0,T]\times\overline{\Omega})$,

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\vartheta\Big). \tag{2.28}$$

(ii) Admissible test functions

$$\varphi\in C^1_c([0,T)\times\overline{\Omega}). \tag{2.29}$$

(iii) Minimal regularity of solutions required:

$$\begin{gathered}\vartheta>0 \text{ a.a. on } (0,T)\times\Omega, \vartheta\in L^q((0,T)\times\Omega),\\ \nabla_x\vartheta\in L^q((0,T)\times\Omega;\mathbb{R}^3),\ q>1,\end{gathered} \tag{2.30}$$

$$\begin{gathered}\varrho s\in L^1((0,T)\times\Omega),\ \varrho s\mathbf{u},\ \frac{\mathbf{q}}{\vartheta}\in L^1((0,T)\times\Omega;\mathbb{R}^3),\\ \frac{\varrho}{\vartheta}\mathcal{Q}\in L^1((0,T)\times\Omega),\end{gathered} \tag{2.31}$$

$$\frac{1}{\vartheta}\mathbb{S}:\nabla_x\mathbf{u},\ \frac{1}{\vartheta^2}\mathbf{q}\cdot\nabla_x\vartheta\in L^1((0,T)\times\Omega). \tag{2.32}$$

(iv) Formal interpretation:

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\mathbf{u}) + \mathrm{div}_x\Big(\frac{\mathbf{q}}{\vartheta}\Big)$$
$$\geq \frac{1}{\vartheta}\Big(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\vartheta\Big) + \frac{\varrho}{\vartheta}\mathcal{Q} \text{ in } (0,T)\times\Omega, \tag{2.33}$$

$$\varrho s(0+,\cdot)\geq(\varrho s)_0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega}\leq 0. \tag{2.34}$$

2.1.5 Constitutive relations

(i) Gibbs' equation:

$$p=p(\varrho,\vartheta),\ e=e(\varrho,\vartheta),\ s=s(\varrho,\vartheta) \text{ a.a. in } (0,T)\times\Omega,$$

where

$$\vartheta Ds = De + pD\Big(\frac{1}{\varrho}\Big). \tag{2.35}$$

(ii) Newton's law:

$$\mathbb{S} = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big) + \eta \mathrm{div}_x \mathbf{u}\ \mathbb{I} \text{ a.a. in } (0,T)\times\Omega, \tag{2.36}$$

(iii) Fourier's law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta \text{ a.a. in } (0,T)\times\Omega. \tag{2.37}$$

2.2 A priori estimates

A priori estimates represent a corner stone of any mathematical theory related to a system of nonlinear partial differential equations. The remarkable informal rule asserts that "if we can establish *sufficiently strong* estimates for solutions of a nonlinear partial differential equation under the assumption that such a solution exists, then the solution does exist". *A priori* estimates are natural bounds imposed on the family of all admissible solutions through the system of equations they obey, the boundary conditions, and the given data. The modern theory of partial differential equations is based on function spaces, notably the Sobolev spaces, that have been identified by means of the corresponding *a priori* bounds for certain classes of elliptic equations.

Strictly speaking, *a priori* estimates are *formal*, being derived under the hypothesis that all quantities in question are smooth. However, as we shall see below, all bounds obtained for the NAVIER-STOKES-FOURIER SYSTEM hold even within the class of the weak solutions introduced in Section 2.1. This is due to the fact that all nowadays available *a priori* estimates follow from the physical principle of conservation of the total amount of certain quantities as mass and total energy, or they result from the dissipative mechanism enforced by means of the *Second law of thermodynamics*.

2.2.1 Total mass conservation

Taking $b \equiv 0$, $B = B(1) = 1$ in the renormalized equation of continuity (2.2) we deduce that

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x = M_0 \text{ for a.a. } t \in (0,T), \tag{2.38}$$

more specifically, for any $t \in (0,T)$ which is a Lebesgue point of the vector-valued mapping $t \mapsto \varrho(t,\cdot) \in L^1(\Omega)$. As a matter of fact, in accordance with the property of weak continuity in time of solutions to abstract balance laws discussed in Section 1.2, relation (2.38) holds for any $t \in [0,T]$ provided ϱ was redefined on a set of times of zero measure . Formula (2.38) rigorously confirms the intuitively obvious fact that the total mass M_0 of the fluid contained in a physical domain Ω is a constant of motion provided the normal component of the velocity field $\mathbf{u}$ vanishes on the boundary $\partial\Omega$.

2.2.2 Energy estimates

The balance of total energy expressed through (2.22) provides another sample of *a priori* estimates. Indeed assuming, for simplicity, that both $\mathbf{f}$ and $\mathcal{Q}$ are uniformly bounded we get

$$\Big| \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} + \varrho \mathcal{Q} \ \mathrm{d}x \Big| \\ \leq \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \sqrt{M_0} \|\sqrt{\varrho}\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^3)} + M_0 \|\mathcal{Q}\|_{L^\infty((0,T)\times\Omega)};$$

whence a straightforward application of Gronwall's lemma to (2.22) gives rise to

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \Big)(t) \ \mathrm{d}x \\ \leq c\Big(T, E_0, M_0, \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)}, \|\mathcal{Q}\|_{L^\infty((0,T)\times\Omega)}\Big). \tag{2.39}$$

In particular,

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_\Omega \varrho|\mathbf{u}|^2(t) \ \mathrm{d}x \leq c(\mathrm{data}), \tag{2.40}$$

where the symbol $c(\mathrm{data})$ denotes a generic positive constant depending solely on the *data*

$$T, E_0, M_0, \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)}, \|\mathcal{Q}\|_{L^\infty((0,T)\times\Omega)}, \ \text{and } S_0 = \int_\Omega (\varrho s)_0 \ \mathrm{d}x. \tag{2.41}$$

In order to get more information, we have to exploit the specific structure of the internal energy function e. In accordance with hypotheses (1.44), (1.50), (1.54), we have

$$\varrho e(\varrho,\vartheta) \geq a\vartheta^4 + \varrho \lim_{\vartheta\to 0} e_M(\varrho,\vartheta). \tag{2.42}$$

On the other hand, the molecular component e_M is given through (1.45), (1.46) in the degenerate area $\varrho > \overline{Z}\vartheta^{3/2}$, therefore

$$\lim_{\vartheta\to 0} e_M(\varrho,\vartheta) = \frac{3\varrho^{\frac{2}{3}}}{2} \lim_{\vartheta\to 0} \frac{\vartheta^{\frac{5}{2}}}{\varrho^{\frac{5}{3}}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big) = \frac{3\varrho^{\frac{2}{3}}}{2} \lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}}, \tag{2.43}$$

where, in accordance with (1.50),

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{2.44}$$

Consequently, going back to (2.42) we conclude

$$\varrho e(\varrho,\vartheta) \geq a\vartheta^4 + \frac{3p_\infty}{2}\varrho^{\frac{5}{3}}, \tag{2.45}$$

in particular, it follows from (2.39) that

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega \left(\vartheta^4 + \varrho^{\frac{5}{3}}\right)(t)\ \mathrm{d}x \le c(\text{data}). \tag{2.46}$$

It is important to note that estimate (2.46) yields a uniform bound on the pressure $p = p_M + p_R$. Indeed the pressure is obviously bounded in the degenerate area (1.49), where p_M satisfies (1.45) and the appropriate bound is provided by (2.39). Otherwise, using the hypothesis of thermodynamic stability (1.44), we obtain

$$0 \le p_M(\varrho,\vartheta) \le p_M(\overline{Z}\vartheta^{\frac{3}{2}},\vartheta) = \vartheta^{\frac{5}{2}}P(\overline{Z});$$

whence the desired bound follows from (2.46) as soon as Ω is bounded. Consequently, we have shown that the energy estimate (2.39) gives rise to

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega p(\varrho,\vartheta)(t)\ \mathrm{d}x \le c(\text{data}) \tag{2.47}$$

at least for a bounded domain Ω.

2.2.3 Estimates based on the Second law of thermodynamics

The *Second law of thermodynamics* asserts the irreversible transfer of the mechanical energy into heat valid for all physical systems. This can be expressed mathematically by means of the entropy production equation (2.27). In order to utilize this relation for obtaining *a priori* bounds, we introduce a remarkable quantity which will play a crucial role not only in the existence theory but also in the study of singular limits.

■ HELMHOLTZ FUNCTION:

$$H_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho\left(e(\varrho,\vartheta) - \overline{\vartheta}s(\varrho,\vartheta)\right), \tag{2.48}$$

where $\overline{\vartheta}$ is a positive constant.

Obviously, the quantity $H_{\overline{\vartheta}}$ is reminiscent of the *Helmholtz free energy* albeit in the latter $\overline{\vartheta}$ must be replaced by ϑ.

It follows from Gibbs' relation (2.35) that

$$\frac{\partial^2 H_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial\varrho^2} = \frac{1}{\varrho}\frac{\partial p(\varrho,\overline{\vartheta})}{\partial\varrho} = \frac{1}{\varrho}\frac{\partial p_M(\varrho,\overline{\vartheta})}{\partial\varrho}, \tag{2.49}$$

while

$$\frac{\partial H_{\overline{\vartheta}}(\varrho,\vartheta)}{\partial\vartheta} = \frac{\varrho}{\vartheta}(\vartheta-\overline{\vartheta})\frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} = 4a\vartheta^2(\vartheta-\overline{\vartheta}) + \frac{\varrho}{\vartheta}(\vartheta-\overline{\vartheta})\frac{\partial e_M(\varrho,\vartheta)}{\partial\vartheta}. \tag{2.50}$$

Thus, as a direct consequence of the hypothesis of thermodynamic stability (1.44), we thereby infer that

- $\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})$ is a strictly convex function, which, being augmented by a suitable affine function of ϱ, attains its global minimum at some positive $\overline{\varrho}$,
- the function $\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta)$ is decreasing for $\vartheta < \overline{\vartheta}$ and increasing for $\vartheta > \overline{\vartheta}$, in particular, it attains its (global) minimum at $\vartheta = \overline{\vartheta}$ for any fixed ϱ.

The total energy balance (2.22), together with the entropy production equation (2.27), gives rise to

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) \Big)(\tau) \ \mathrm{d}x + \overline{\vartheta}\sigma \Big[[0, \tau] \times \overline{\Omega} \Big]$$
$$= E_0 - \overline{\vartheta} S_0 + \int_0^\tau \int_\Omega \Big[\varrho \Big(\mathcal{Q} - \frac{\overline{\vartheta}}{\vartheta}\mathcal{Q} \Big) + \varrho \mathbf{f} \cdot \mathbf{u} \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{2.51}$$

for a.a. $\tau \in (0, T)$, where we have introduced the symbol $\sigma[Q]$ to denote the value of the measure σ applied to a Borel set Q.

Now suppose there exists a positive number $\overline{\varrho} > 0$ such that

$$\int_\Omega (\varrho - \overline{\varrho})(t) \ \mathrm{d}x = 0 \text{ for any } t \in [0, T].$$

Clearly, if Ω is a bounded domain, we have $\overline{\varrho} = M_0/|\Omega|$, where M_0 is the total mass of the fluid. Accordingly, relation (2.51) can be rewritten as

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■ TOTAL DISSIPATION BALANCE:

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big)(\tau) \ \mathrm{d}x + \overline{\vartheta}\sigma \Big[[0, \tau] \times \overline{\Omega} \Big]$$
$$= E_0 - \overline{\vartheta} S_0 - \int_\Omega \Big((\varrho_0 - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} + H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big) \ \mathrm{d}x$$
$$+ \int_0^\tau \int_\Omega \Big(\varrho \Big(\mathcal{Q} - \frac{\overline{\vartheta}}{\vartheta}\mathcal{Q} \Big) + \varrho \mathbf{f} \cdot \mathbf{u} \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{2.52}$$

for a.a. $\tau \in (0, T)$.

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at least if Ω is a bounded domain. In contrast with (2.51), the quantity $H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}}{\partial \varrho}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$ at the left-hand side is obviously non-negative as a direct consequence of the hypothesis of thermodynamic stability.

Consequently, assuming $\mathcal{Q} \geq 0$, we can use (2.28), together with (2.52), in order to obtain

$$\int_0^T \int_\Omega \frac{1}{\vartheta} \Big(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \Big) \ \mathrm{d}x \leq c(\text{data}). \tag{2.53}$$

As the transport terms $\mathbb{S}$, $\mathbf{q}$ are given by (1.42), (1.43), notably they are linear functions of the affinities $\nabla_x \mathbf{u}$, $\nabla_x \vartheta$, respectively, we get

$$\int_0^T \int_\Omega \frac{\mu}{\vartheta} \left| \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \right|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c(\mathrm{data}), \tag{2.54}$$

and

$$\int_0^T \int_\Omega \frac{\kappa}{\vartheta^2} |\nabla_x \vartheta|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c(\mathrm{data}). \tag{2.55}$$

In order to continue, we have to specify the structural properties to be imposed on the transport coefficients μ and κ. In view of (1.52), it seems reasonable to assume that the heat conductivity coefficient $\kappa = \kappa_M + \kappa_R$ satisfies

$$\begin{gathered} 0 < \underline{\kappa}_M (1 + \vartheta^\alpha) \leq \kappa_M(\vartheta) \leq \overline{\kappa}_M (1 + \vartheta^\alpha), \\ 0 < \underline{\kappa}_R \vartheta^3 \leq \kappa_R(\vartheta) \leq \overline{\kappa}_R (1 + \vartheta^3), \end{gathered} \tag{2.56}$$

where $\underline{\kappa}_M$, $\overline{\kappa}_M$, $\underline{\kappa}_R$, $\overline{\kappa}_R$ are positive constants.

Similarly, the shear viscosity coefficient μ obeys

$$0 < \underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^\alpha) \tag{2.57}$$

for any $\vartheta \geq 0$, positive constants $\underline{\mu}$, $\overline{\mu}$, and a positive exponent α specified below. Note that κ_M, μ are not allowed to depend explicitly on ϱ – a hypothesis that is crucial in existence theory but entirely irrelevant in the study of singular limits. We remark that such a stipulation is physically relevant at least for gases (see Becker [20]) and certain liquids.

Keeping (2.56) in mind we deduce from (2.55) that

$$\int_0^T \int_\Omega \left(|\nabla_x \log(\vartheta)|^2 + |\nabla_x \vartheta^{\frac{3}{2}}|^2 \right) \, \mathrm{d}x \, \mathrm{d}t \leq c(\mathrm{data}). \tag{2.58}$$

Combining (2.58) with (2.46) we conclude that the temperature $\vartheta(t, \cdot)$ belongs to $W^{1,2}(\Omega)$ for a.a. $t \in (0, T)$, where the symbol $W^{1,2}(\Omega)$ stands for the *Sobolev space* of functions belonging with their gradients to the Lebesgue space $L^2(\Omega)$ (cf. the relevant part in Section 0.3). More specifically, we have, by the standard Poincaré's inequality (Theorem 10.14),

$$\| \vartheta^\beta \|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(\mathrm{data}) \text{ for any } 1 \leq \beta \leq \frac{3}{2}. \tag{2.59}$$

A similar estimate for $\log(\vartheta)$ is more delicate and is postponed to the next section.

From estimate (2.54) and Hölder's inequality we get

$$\left\| \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \right\|_{L^p(\Omega;\mathbb{R}^{3\times 3})}$$
$$\leq \left\| \sqrt{\frac{\vartheta}{\mu(\vartheta)}} \right\|_{L^q(\Omega)} \left\| \sqrt{\frac{\mu(\vartheta)}{\vartheta}} \Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \Big) \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$
$$\leq c \|(1 + \vartheta^{\frac{1-\alpha}{2}})\|_{L^q(\Omega)} \left\| \sqrt{\frac{\mu(\vartheta)}{\vartheta}} \Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \Big) \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$

provided

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{2}.$$

Thus we deduce from estimates (2.46), (2.54) that

$$\left\| \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \right\|_{L^2(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}))} \leq c(\mathrm{data}) \tag{2.60}$$

for

$$p = \frac{8}{5-\alpha}, \ 0 \leq \alpha \leq 1. \tag{2.61}$$

Similarly, in accordance with (2.59) and the standard embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ (see Theorem 0.4), we have

$$\| \vartheta \|_{L^3(0,T;L^9(\Omega))} \leq c(\mathrm{data}); \tag{2.62}$$

whence, following the arguments leading to (2.60),

$$\left\| \nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \, \mathbb{I} \right\|_{L^q(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}))} \leq c(\mathrm{data}) \tag{2.63}$$

for

$$q = \frac{6}{4-\alpha}, \ p = \frac{18}{10-\alpha}, \ 0 \leq \alpha \leq 1. \tag{2.64}$$

As we will see below, the range of suitable values of the parameter α in (2.61), (2.62) is subjected to further restrictions.

The previous estimates concern only certain components of the velocity gradient. In order to get uniform bounds on $\nabla_x \mathbf{u}$, we need the following version of *Korn's inequality* proved in Theorem 10.17 in the Appendix.

■ GENERALIZED KORN-POINCARÉ INEQUALITY:

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that r is a non-negative function such that*

$$0 < M_0 \leq \int_\Omega r \ \mathrm{d}x, \ \int_\Omega r^\gamma \ \mathrm{d}x \leq K \quad \textit{for a certain} \ \ \gamma > 1.$$

Then

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leq c(p, M_0, K)\Big(\Big\|\nabla_x \mathbf{v} + \nabla_x^\perp \mathbf{v} - \frac{2}{3}\mathrm{div}_x \mathbf{v}\ \mathbb{I}\Big\|_{L^p(\Omega;\mathbb{R}^3)} + \int_\Omega r|\mathbf{v}|\ \mathrm{d}x\Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^3)$, $1 < p < \infty$.

Applying Proposition 2.1 with $r = \varrho$, $\gamma = \frac{5}{3}$, $\mathbf{v} = \mathbf{u}$, we can use estimates (2.40), (2.46), (2.60), and (2.63) to conclude that

$$\| \mathbf{u} \|_{L^2(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} \leq c(\mathrm{data}) \text{ for } p = \frac{8}{5-\alpha}, \tag{2.65}$$

and

$$\| \mathbf{u} \|_{L^q(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} \leq c(\mathrm{data}) \text{ for } q = \frac{6}{4-\alpha},\ p = \frac{18}{10-\alpha}. \tag{2.66}$$

Estimates (2.65), (2.66) imply uniform bounds on the viscous stress tensor $\mathbb{S}$. To see this, write

$$\mu(\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big) = \sqrt{\vartheta\mu(\vartheta)}\sqrt{\frac{\mu(\vartheta)}{\vartheta}}\Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big),$$

where $\sqrt{\frac{\mu(\vartheta)}{\vartheta}}\Big(\nabla_x \mathbf{u} + \nabla_x^\perp \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\ \mathbb{I}\Big)$ admits the bound established in (2.54). On the other hand, in view of estimates (2.46), (2.62), ϑ is bounded in $L^{\frac{17}{3}}((0,T) \times \Omega)$. This fact combined with hypothesis (2.57) yields boundedness of $\sqrt{\vartheta\mu(\vartheta)}$ in $L^p((0,T)\times\Omega)$ for a certain $p > 2$. Assuming the bulk viscosity η satisfies

$$0 \leq \eta(\vartheta) \leq c(1 + \vartheta^\alpha), \tag{2.67}$$

with the same exponent α as in (2.57), we obtain

$$\| \mathbb{S} \|_{L^q(0,T;L^q(\Omega;\mathbb{R}^{3\times 3}))} \leq c(\mathrm{data}) \text{ for a certain } q > 1. \tag{2.68}$$

In a similar way, we can deduce estimates on the linear momentum and the kinetic energy. By virtue of the standard embedding relation $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $q \leq 3p/(3-p)$ (Theorem 0.4), we get

$$\| \mathbf{u} \|_{L^2(0,T;L^{\frac{24}{7-3\alpha}}(\Omega;\mathbb{R}^3))} + \| \mathbf{u} \|_{L^{\frac{6}{4-\alpha}}(0,T;L^{\frac{18}{4-\alpha}}(\Omega;\mathbb{R}^3))} \leq c(\mathrm{data}), \tag{2.69}$$

see (2.65), (2.66). On the other hand, by virtue of (2.40), (2.46),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho\mathbf{u}\|_{L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)} \leq c(\mathrm{data}). \tag{2.70}$$

Combining the last two estimates, we get

$$\| \varrho\mathbf{u}\otimes\mathbf{u} \|_{L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3}))} \leq c(\mathrm{data}) \text{ for a certain } q > 1, \tag{2.71}$$

provided

$$\alpha > \frac{2}{5}. \tag{2.72}$$

It is worth noting that (2.72) allows for the physically relevant exponent $\alpha = 1/2$ (cf. Section 1.4.4).

2.2.4 Positivity of the absolute temperature

Our goal is to exploit estimate (2.58) in order to show

$$\int_0^T \int_\Omega \Big(|\log \vartheta|^2 + |\nabla_x \log \vartheta|^2 \Big) \ \mathrm{d}x \ \mathrm{d}t \leq c(\text{data}). \tag{2.73}$$

Formula (2.73) not only facilitates future analysis but is also physically relevant as it implies positivity of the absolute temperature with a possible exception of a set of Lebesgue measure zero.

In order to establish (2.73), we introduce the following version of *Poincaré's inequality* proved in Theorem 10.14 in the Appendix.

■ POINCARÉ'S INEQUALITY:

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $V \subset \Omega$ be a measurable set such that*

$$|V| \geq V_0 > 0.$$

Then there exists a positive constant $c = c(V_0)$ such that

$$\| \ v \ \|_{W^{1,2}(\Omega)} \leq c(V_0) \Big(\|\nabla_x v\|_{L^2(\Omega;\mathbb{R}^3)} + \int_V |v| \mathrm{d}x \Big)$$

for any $v \in W^{1,2}(\Omega)$.

In view of Proposition 2.2 the desired relation (2.73) will follow from (2.58) as soon as we show that the temperature ϑ cannot vanish identically in the physical domain Ω. As the hypothetical state of a system with zero temperature minimizes the entropy, it is natural to evoke the *Second law of thermodynamics* expressed in terms of the entropy balance (2.27).

The total entropy of the system $\int_\Omega \varrho s(\varrho, \vartheta) \ \mathrm{d}x$ is a non-decreasing function of time provided the heat source $\mathcal{Q}$ is non-negative. In particular,

$$\int_\Omega \varrho s(\varrho, \vartheta)(t, \cdot) \ \mathrm{d}x \geq \int_\Omega (\varrho s)_0 \ \mathrm{d}x \text{ for a.a. } t \in (0, T), \tag{2.74}$$

where we assume that the initial distribution of the entropy is compatible with that for the density, that means, $(\varrho s)_0 = \varrho_0 s(\varrho_0, \vartheta_0)$ for a suitable initial temperature distribution ϑ_0.

If $\varrho \geq \overline{Z}\vartheta^{\frac{3}{2}}$, meaning if (ϱ, ϑ) belong to the degenerate region introduced in (1.49), the pressure p and the internal energy e are interrelated through (1.45), (1.46). Then it is easy to check, by means of Gibbs' equation (2.35), that the specific entropy s can be written in the form $s = s_M + s_R$, where

$$s_M(\varrho, \vartheta) = S(Z),\ Z = \frac{\varrho}{\vartheta^{\frac{3}{2}}},\ S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2},\ Z \geq \overline{Z}. \tag{2.75}$$

The quantity

$$\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z}$$

plays a role of the specific heat at constant volume and is strictly positive in accordance with the hypothesis of thermodynamic stability (1.44). In particular, we can set

$$s_\infty = \lim_{Z\to\infty} S(Z) = \lim_{\vartheta\to 0} s_M(\varrho, \vartheta) \geq -\infty \text{ for any fixed } \varrho > 0. \tag{2.76}$$

Moreover, modifying S by a suitable additive constant, we can assume $s_\infty = 0$ in the case when the limit is finite.

In order to proceed we need the following assertion that may be of independent interest. The claim is that the absolute temperature ϑ must remain strictly positive at least on a set of positive measure.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that non-negative functions ϱ, ϑ satisfy*

$$0 < M_0 = \int_\Omega \varrho \ \mathrm{d}x,\ \int_\Omega \left(\vartheta^4 + \varrho^{\frac{5}{3}}\right) \mathrm{d}x \leq K,$$

and

$$\int_\Omega \varrho s(\varrho, \vartheta) \ \mathrm{d}x \geq S_0 > M_0 s_\infty \textit{ for a certain } S_0, \tag{2.77}$$

where $s_\infty \in \{0, -\infty\}$ is determined by (2.76).

Then there are $\underline{\vartheta} > 0$ and $V_0 > 0$, depending only on M_0, K, and S_0 such that

$$\left|\left\{x \in \Omega \ \middle|\ \vartheta(x) > \underline{\vartheta}\right\}\right| \geq V_0.$$

Proof. Arguing by contradiction we construct a sequence ϱ_n, ϑ_n satisfying (2.77) and such that

$$\begin{gathered}\varrho_n \to \varrho \text{ weakly in } L^{\frac{5}{3}}(\Omega),\ \textstyle\int_\Omega \varrho \ \mathrm{d}x = M_0,\\ \left|\left\{x \in \Omega \,|\, \vartheta_n > \frac{1}{n}\right\}\right| < \frac{1}{n}.\end{gathered} \tag{2.78}$$

In particular,

$$\begin{gathered}\vartheta_n \to 0 \text{ (strongly) in } L^p(\Omega) \text{ for any } 1 \leq p < 4,\\ \varrho_n s_R(\varrho_n, \vartheta_n) = \frac{4}{3}a\vartheta_n^3 \to 0 \text{ in } L^1(\Omega).\end{gathered} \tag{2.79}$$

Next we claim that

$$\limsup_{n\to\infty} \int_{\{\varrho_n \le \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n s_M(\varrho_n, \vartheta_n)\ \mathrm{d}x \le 0. \tag{2.80}$$

In order to see (2.80), we first observe that the specific (molecular) entropy s_M is increasing in ϑ; whence

$$s_M(\varrho,\vartheta) \le \begin{cases} s_M(\varrho,1) \text{ if } \vartheta < 1, \\ s_M(\varrho,1) + \int_1^\vartheta \frac{\partial s_M(\varrho,z)}{\partial z}\ \mathrm{d}z \le s_M(\varrho,1) + c\log\vartheta \text{ for } \vartheta \ge 1, \end{cases}$$

where we have used hypothesis (1.51). On the other hand, it follows from Gibbs' equation (2.35) that

$$\frac{\partial s_M(\varrho,\vartheta)}{\partial \varrho} = -\frac{1}{\varrho^2}\frac{\partial p_M(\varrho,\vartheta)}{\partial\vartheta};$$

whence

$$|s_M(\varrho,1)| \le c(\overline{Z})(1 + |\log(\varrho)|) \text{ for all } \varrho \le \overline{Z}.$$

Resuming the above inequalities yields

$$|s_M(\varrho,\vartheta)| \le c(1 + |\log(\varrho)| + |\log(\vartheta)|). \tag{2.81}$$

Returning to (2.80) we get

$$\begin{aligned} \int_{\{\varrho_n \le \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n s_M(\varrho_n,\vartheta_n)\ \mathrm{d}x &\le c \int_{\{\varrho_n \le \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n(1 + |\log(\varrho_n)| + |\log(\vartheta_n)|)\ \mathrm{d}x \\ &\le c(\overline{Z}) \int_\Omega (\vartheta_n^{\frac{3}{2}} + \vartheta_n^{\frac{3}{4}}\sqrt{\varrho_n}|\log(\sqrt{\varrho_n})| + \vartheta_n\sqrt{\vartheta_n}|\log(\sqrt{\vartheta_n})|)\ \mathrm{d}x \to 0, \end{aligned}$$

where we have used (2.78), (2.79).

Finally, we have

$$\varrho s_M(\varrho,\vartheta) = \varrho S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big)$$

in the degenerate area $\varrho > \overline{Z}\vartheta^{\frac{3}{2}}$, and, consequently,

$$\begin{aligned} &\int_{\{\varrho_n > \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n s_M(\varrho_n,\vartheta_n)\ \mathrm{d}x \\ &\quad = \int_{\{Z\vartheta_n^{\frac{3}{2}} > \varrho_n > \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big)\ \mathrm{d}x + \int_{\{\varrho_n \ge Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big)\ \mathrm{d}x, \end{aligned}$$

where

$$\int_{\{Z\vartheta_n^{\frac{3}{2}} > \varrho_n \ge \overline{Z}\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big)\ \mathrm{d}x \le S(\overline{Z})Z \int_\Omega \vartheta_n^{\frac{3}{2}}\ \mathrm{d}x \to 0. \tag{2.82}$$

Combining (2.79–2.82), together with hypothesis (2.77), we conclude that

$$\liminf_{n\to\infty} \int_{\{\varrho_n > Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big)\ \mathrm{d}x > M_0 s_\infty \text{ for any } Z > \overline{Z}. \tag{2.83}$$

However, relation (2.83) leads immediately to contradiction as

$$\int_{\{\varrho_n > Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{\frac{3}{2}}}\Big)\ \mathrm{d}x \le S(Z) \int_{\{\varrho_n > Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x \to S(Z)M_0.$$

Indeed write $\int_\Omega \varrho_n\ \mathrm{d}x$ as $\int_{\{\varrho_n \le Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x + \int_{\{\varrho_n > Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x$, and observe that

$$0 \le \int_{\{\varrho_n \le Z\vartheta_n^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x = \int_{\{\varrho_n \le Z(\frac{1}{n})^{\frac{3}{2}}\}} \varrho_n\ \mathrm{d}x + \int_{\{\vartheta_n > \frac{1}{n}\}} \varrho_n\ \mathrm{d}x,$$

where the right-hand side tends to 0 by virtue of (2.77). □

By means of Proposition 2.2 and Lemma 2.1, it is easy to check that estimates (2.46), (2.58) give rise to (2.73).

2.2.5 Pressure estimates

The central problem of the mathematical theory of the Navier-Stokes-Fourier system is to control the pressure. Under the constitutive relations considered in this book, the pressure p is proportional to the volumetric density of the internal energy ϱe that is *a priori* bounded in $L^1(\Omega)$ uniformly with respect to time, see (2.45–2.47). This section aims to find *a priori* estimates for p in the *weakly closed* reflexive space $L^q((0,T)\times\Omega)$ for a certain $q > 1$. To this end, the basic idea is to "compute" p by means of the momentum equation (2.9) and use the available estimates in order to control the remaining terms. Such an approach, however, faces serious technical difficulties, in particular because of the presence of the time derivative $\partial_t(\varrho\mathbf{u})$ in the momentum equation. Instead we use the quantities

$$\varphi(t,x) = \psi(t)\phi(t,x), \text{ with } \phi = \mathcal{B}\Big[h(\varrho) - \frac{1}{|\Omega|}\int_\Omega h(\varrho)\ \mathrm{d}x\Big],\ \psi \in C_c^\infty(0,T) \tag{2.84}$$

as test functions in the momentum equation (2.9), where $\mathcal{B}$ is a suitable branch of the inverse div_x^{-1}.

There are several ways to construct the operator $\mathcal{B}$, here we adopt the formula proposed by Bogovskii (see Section 10.5 in the Appendix). In particular, the operator $\mathcal{B}$ enjoys the following properties.

■ BOGOVSKII OPERATOR $\mathcal{B} \approx \mathrm{div}_x^{-1}$:

(b1) Given

$$g \in C_c^\infty(\Omega), \ \int_\Omega g \ \mathrm{d}x = 0,$$

the vector field $\mathcal{B}[g]$ satisfies

$$\mathcal{B}[g] \in C_c^\infty(\Omega; \mathbb{R}^3), \ \mathrm{div}_x \mathcal{B}[g] = g. \tag{2.85}$$

(b2) For any non-negative integer m and any $1 < q < \infty$,

$$\| \ \mathcal{B}[g] \ \|_{W^{m+1,q}(\Omega;\mathbb{R}^3)} \leq c\|g\|_{W^{m,q}(\Omega)} \tag{2.86}$$

provided $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain, in particular, the operator $\mathcal{B}$ can be extended to functions $g \in L^q(\Omega)$ with zero mean satisfying

$$\mathcal{B}[g]|_{\partial\Omega} = 0 \text{ in the sense of traces.} \tag{2.87}$$

(b3) If $g \in L^q(\Omega)$, $1 < q < \infty$, and, in addition,

$$g = \mathrm{div}_x \mathbf{G}, \ \mathbf{G} \in L^p(\Omega; \mathbb{R}^3), \ \mathbf{G} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

then

$$\| \ \mathcal{B}[g] \ \|_{L^p(\Omega;\mathbb{R}^3)} \leq c\|\mathbf{G}\|_{L^p(\Omega;\mathbb{R}^3)}. \tag{2.88}$$

In order to render the test functions (2.84) admissible, we take

$$\varphi_\alpha(t,x) = \psi(t)[\phi]^\alpha(t,x), \text{ with } [\phi]^\alpha = \mathcal{B}\Big[h(\varrho) - \frac{1}{|\Omega|}\int_\Omega h(\varrho) \ \mathrm{d}x\Big]^\alpha, \ \psi \in C_c^\infty(0,T), \tag{2.89}$$

where h is a smooth bounded function, and the symbol $[v]^\alpha$ denotes convolution in the *time* variable t with a suitable family of regularizing kernels (see Section 10.1 in Appendix). Here, we have extended $h(\varrho)$ to be zero outside the interval $[0,T]$.

Since ϱ, $\mathbf{u}$ satisfy the renormalized equation (2.2), we easily deduce that

$$\partial_t\Big[h(\varrho)\Big]^\alpha + \mathrm{div}_x\Big[h(\varrho)\mathbf{u}\Big]^\alpha + \Big[(\varrho h'(\varrho) - h(\varrho))\mathrm{div}_x\mathbf{u}\Big]^\alpha = 0 \quad \text{for any } t \in (\alpha, T-\alpha) \text{ and a.a. } x \in \Omega, \tag{2.90}$$

in particular, from the properties (b2), (b3) we may infer that

$$\begin{aligned}\partial_t[\phi]^\alpha = &- \mathcal{B}\Big[\mathrm{div}_x(h(\varrho)\mathbf{u})\Big]^\alpha \\ &- \mathcal{B}\Big[\Big(\varrho h'(\varrho) - h(\varrho)\Big)\mathrm{div}_x\mathbf{u} - \frac{1}{|\Omega|}\int_\Omega \Big(\varrho h'(\varrho) - h(\varrho)\Big)\mathrm{div}_x\mathbf{u} \ \ \mathrm{d}x\Big]^\alpha\end{aligned} \tag{2.91}$$

(cf. Section 10.5 in Appendix).

By virtue of (2.86–2.88), we obtain

$$\| \, [\phi]^\alpha(t,\cdot) \, \|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leq c(p,\Omega) \| \, [h(\varrho)]^\alpha(t,\cdot) \, \|_{L^p(\Omega)}, \ 1 < p < \infty, \tag{2.92}$$

and

$$\| \, [\partial_t \phi]^\alpha(t,\cdot) \, \|_{L^p(\Omega;\mathbb{R}^3)} \leq c(p,s,\Omega) \, \| \, [h(\varrho)\mathbf{u}]^\alpha(t,\cdot) \, \|_{L^p(\Omega)} \tag{2.93}$$

$$+ \left\{ \begin{array}{c} \| \, [(\varrho h'(\varrho) - h(\varrho))\mathrm{div}\mathbf{u}]^\alpha(t,\cdot) \, \|_{L^{\frac{3p}{3+p}}(\Omega)} \text{ if } \frac{3}{2} < p < \infty, \\ \| [(\varrho h'(\varrho) - h(\varrho))\mathrm{div}\mathbf{u}]^\alpha(t,\cdot) \, \|_{L^s(\Omega)} \text{ for any } 1 < s < \infty \text{ if } 1 \leq p \leq \frac{3}{2}, \end{array} \right.$$

for any $t \in [\alpha, T-\alpha]$.

Having completed the preliminary considerations we take the quantities φ_α specified in (2.89) as test functions in the momentum equation (2.9) to obtain

$$\int_0^T \left(\psi \int_\Omega p(\varrho,\vartheta)[h(\varrho)]^\alpha \ \mathrm{d}x \right) \mathrm{d}t = \sum_{j=1}^{5} I_j, \tag{2.94}$$

where

$$I_1 = \frac{1}{|\Omega|} \int_0^T \left(\psi \int_\Omega [h(\varrho)]^\alpha \int_\Omega p(\varrho,\vartheta) \ \mathrm{d}x \right) \mathrm{d}t,$$

$$I_2 = -\int_0^T \left(\psi \int_\Omega \varrho\mathbf{u} \cdot \partial_t [\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t,$$

$$I_3 = -\int_0^T \left(\psi \int_\Omega \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x [\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t,$$

$$I_4 = \int_0^T \left(\psi \int_\Omega \mathbb{S} : \nabla_x [\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t,$$

$$I_5 = -\int_0^T \left(\psi \int_\Omega \varrho\mathbf{f} \cdot [\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t,$$

and

$$I_6 = -\int_0^T \left(\psi' \int_\Omega \varrho\mathbf{u} \cdot [\phi]^\alpha \ \mathrm{d}x \right) \mathrm{d}t.$$

Now, our intention is to use the uniform bounds established in Section 2.2.3, together with the integral identity (2.94), in order to show that

$$\int_0^T \int_\Omega p(\varrho,\vartheta)\varrho^\nu \ \mathrm{d}x \ \mathrm{d}t \leq c(\text{data}) \text{ for a certain } \nu > 0. \tag{2.95}$$

To this end, the integrals $I_1, \dots, I_6$ are estimated by means of Hölder's inequality as follows:

$$|I_1| \leq \|\psi\|_{L^\infty(0,T)} \, \| \, [h(\varrho)]^\alpha \, \|_{L^1((0,T)\times\Omega)} \, \|p(\varrho,\vartheta)\|_{L^\infty(0,T;L^1(\Omega))},$$
$$|I_2| \leq \|\psi\|_{L^\infty(0,T)} \, \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))} \, \|\partial_t[\phi]^\alpha\|_{L^1(0,T;L^5(\Omega;\mathbb{R}^3))},$$

$$
\begin{aligned}
|I_3| &\le \|\psi\|_{L^\infty(0,T)} \, \|\varrho\mathbf{u}\otimes\mathbf{u}\|_{L^p((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \, \|\nabla_x[\phi]^\alpha\|_{L^{p'}((0,T)\times\Omega;\mathbb{R}^3)}, \\
&\qquad \text{where } p \text{ is the same as in (2.71)}, \\
|I_4| &\le \|\psi\|_{L^\infty(0,T)} \, \| \ \mathbb{S} \ \|_{L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \, \|\nabla_x[\phi]^\alpha\|_{L^{q'}((0,T)\times\Omega;\mathbb{R}^{3\times 3})}, \\
&\qquad \frac{1}{q}+\frac{1}{q'}=1, \ \text{with the same } q \text{ as in (2.68)}, \\
|I_5| &\le \|\psi\|_{L^\infty(0,T)} \, \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \|\varrho\|_{L^\infty(0,T;L^{\frac{5}{3}}(\Omega))} \, \|[\phi]^\alpha\|_{L^1(0,T;L^{\frac{5}{2}}(\Omega;\mathbb{R}^3))}, \\
|I_6| &\le \|\psi'\|_{L^1(0,T)} \, \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))} \, \| \ [\phi]^\alpha \ \|_{L^\infty(0,T;L^5(\Omega;\mathbb{R}^3))}.
\end{aligned}
$$

Furthermore, by virtue of the uniform bounds established in (2.92), (2.93), the above estimates are independent of the value of the parameter α, specifically,

$$
\begin{aligned}
|I_1| &\le \|\psi\|_{L^\infty(0,T)} \, \|h(\varrho)\|_{L^1((0,T)\times\Omega)} \, \|p(\varrho,\vartheta)\|_{L^\infty(0,T;L^1(\Omega))}, \\
|I_2| &\le \|\psi\|_{L^\infty(0,T)} \, \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))} \\
&\qquad \times \Big(\|h(\varrho)\mathbf{u}\|_{L^1(0,T;L^5(\Omega;\mathbb{R}^3))} + \|(\varrho h'(\varrho)-h(\varrho))\mathrm{div}_x\mathbf{u}\|_{L^1(0,T;L^{\frac{15}{8}}(\Omega))} \Big), \\
|I_3| &\le \|\psi\|_{L^\infty(0,T)} \, \|\varrho\mathbf{u}\otimes\mathbf{u}\|_{L^p((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \, \|h(\varrho)\|_{L^{p'}((0,T)\times\Omega)}, \\
&\qquad \text{with } p \text{ as in (2.71)}, \\
|I_4| &\le \|\psi\|_{L^\infty(0,T)} \, \|\mathbb{S}\|_{L^q((0,T)\times\Omega;\mathbb{R}^{3\times 3})} \, \|h(\varrho)\|_{L^{q'}((0,T)\times\Omega)}, \\
&\qquad \text{with } q \text{ as in (2.68)}, \\
|I_5| &\le \|\psi\|_{L^\infty(0,T)} \, \|\mathbf{f}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \|\varrho\|_{L^\infty(0,T;L^{\frac{5}{3}}(\Omega))} \, \|h(\varrho)\|_{L^1(0,T;L^{\frac{15}{11}}(\Omega))}, \\
|I_6| &\le \|\psi'\|_{L^1(0,T)} \, \|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))} \, \|h(\varrho)\|_{L^\infty(0,T;L^{\frac{15}{8}}(\Omega))}.
\end{aligned}
$$

Consequently, taking $h(\varrho) \approx \varrho^\nu$ in (2.94) for a sufficiently small $\nu > 0$ and sufficiently large values of ϱ, we can use estimates (2.46), (2.47), (2.68–2.71), together with the bounds on the integrals $I_1, \dots, I_6$ established above, in order to obtain the desired estimate (2.95).

Furthermore, as

$$
\underline{c}\varrho^{\frac{5}{3}} \le p_M(\varrho,\vartheta) \le \overline{c} \begin{cases} \varrho\vartheta \text{ for } \varrho \le \overline{Z}\vartheta^{\frac{3}{2}}, \\ \varrho^{\frac{5}{3}} \text{ for } \varrho \ge \overline{Z}\vartheta^{\frac{3}{2}}, \end{cases} \tag{2.96}
$$

estimate (2.95) implies

$$
\|\varrho\|_{L^{\frac{5}{3}+\nu}((0,T)\times\Omega)} \le c(\text{data}). \tag{2.97}
$$

Finally (2.97) together with (2.46) and (2.96) yields

$$
\|p_M(\varrho,\vartheta)\|_{L^p((0,T)\times\Omega)} \le c(\text{data}) \quad \text{for some } p > 1. \tag{2.98}
$$

2.2.6 Pressure estimates, an alternative approach

The approach to pressure estimates based on the operator $\mathcal{B} \approx \mathrm{div}_x^{-1}$ requires a certain minimal regularity of the boundary $\partial\Omega$. In the remaining part of this chapter, we briefly discuss an alternative method yielding uniform estimates in the interior of the physical domain together with equi-integrability of the pressure up to the boundary. In particular, the interior estimates may be of independent interest since they are sufficient for resolving the problem of global existence for the NAVIER-STOKES-FOURIER SYSTEM provided the equality sign in the total energy balance (2.22) is relaxed to inequality "$\leq$".

Local pressure estimates. Similarly to the preceding part, the basic idea is to "compute" the pressure by means of the momentum equation (2.9). In order to do it locally, we introduce a family of test functions

$$\varphi(t,x) = \psi(t)\eta(x)(\nabla_x \Delta_x^{-1})[1_\Omega h(\varrho)], \tag{2.99}$$

where $\psi \in C_c^\infty(0,T)$, $\eta \in C_c^\infty(\Omega)$, $h \in C_c^\infty(0,\infty)$,

$$0 \leq \psi, \eta \leq 1, \text{ and } h(r) = r^\nu \text{ for } r \geq 1$$

for a suitable exponent $\nu > 0$. Here the symbol Δ_x^{-1} stands for the inverse of the Laplace operator on the whole space $\mathbb{R}^3$, specifically, in terms of the Fourier transform $\mathcal{F}_{x\to\xi}$,

$$\Delta_x^{-1}[v](x) = -\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{\mathcal{F}_{x\to\xi}[v]}{|\xi|^2}\Big], \tag{2.100}$$

see Sections 0.5 and 10.16.

Note that

$$\nabla_x\varphi = \psi\nabla_x\eta \otimes \nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] + \psi\eta\mathcal{R}[1_\Omega h(\varrho)],$$

where

$$\mathcal{R} = [\nabla_x \otimes \nabla_x]\Delta_x^{-1},\ \mathcal{R}_{i,j}[v](x) = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j\mathcal{F}_{x\to\xi}[v]}{|\xi|^2}\Big] \tag{2.101}$$

is a superposition of two Riesz maps. By virtue of the classical Calderón-Zygmund theory, the operator $\mathcal{R}_{i,j}$ is bounded on $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$. In particular, $\varphi \in L^q(0,T;W_0^{1,p}(\Omega;\mathbb{R}^3))$ whenever $h(\varrho) \in L^q(0,T;L^p(\Omega))$ for certain $1 \leq q \leq \infty$, $1 < p < \infty$, see Section 10.16 in Appendix.

Similarly, using the renormalized equation (2.2) with $b(\varrho) = h'(\varrho)\varrho - h(\varrho)$ we "compute"

$$\partial_t\varphi = \partial_t\psi\eta\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] + \psi\eta\Big(\nabla_x\Delta_x^{-1}\Big[1_\Omega(h(\varrho) - h'(\varrho)\varrho)\mathrm{div}_x\mathbf{u}\Big] - \nabla_x\Delta_x^{-1}[\mathrm{div}_x(1_\Omega h(\varrho)\mathbf{u})]\Big).$$

Let us point out that equation (2.2) holds on the whole space $\mathbb{R}^3$ provided $\mathbf{u}$ has been extended outside Ω and h replaced by $1_\Omega h(\varrho)$. Note that functions belonging to $W^{1,p}(\Omega)$ can be extended outside Ω to be in the space $W^{1,p}(\mathbb{R}^3)$ as soon as Ω is a bounded Lipschitz domain.

It follows from the above discussion that the quantity φ specified in (2.99) can be taken as a test function in the momentum equation (2.9), more precisely, the function φ, together with its first derivatives, can be approximated in the L^p-norm by a suitable family of regular test functions satisfying (2.10), (2.12). Thus we get

$$\int_0^T \int_\Omega \psi\eta\Big(ph(\varrho) - \mathbb{S} : \mathcal{R}[1_\Omega h(\varrho)]\Big) \ \mathrm{d}x \ \mathrm{d}t = \sum_{j=1}^{7} I_j, \tag{2.102}$$

where

$$I_1 = \int_0^T \int_\Omega \psi\eta\Big(\varrho\mathbf{u}\cdot\mathcal{R}[1_\Omega h(\varrho)\mathbf{u}] - (\varrho\mathbf{u}\otimes\mathbf{u}) : \mathcal{R}[1_\Omega h(\varrho)]\Big) \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_2 = -\int_0^T \int_\Omega \psi\eta \ \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\Big[1_\Omega(h(\varrho) - h'(\varrho)\varrho)\mathrm{div}_x\mathbf{u}\Big] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_3 = -\int_0^T \int_\Omega \psi\eta\varrho\mathbf{f}\cdot\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_4 = -\int_0^T \int_\Omega \psi p\nabla_x\eta\cdot\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_5 = \int_0^T \int_\Omega \psi\mathbb{S} : \nabla_x\eta\otimes\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \ \mathrm{d}x \ \mathrm{d}t,$$

$$I_6 = -\int_0^T \int_\Omega \psi(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\eta\otimes\nabla_x\Delta^{-1}[1_\Omega h(\varrho)] \ \mathrm{d}x \ \mathrm{d}t,$$

and

$$I_7 = -\int_0^T \int_\Omega \partial_t\psi \ \eta\varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}[1_\Omega h(\varrho)] \ \mathrm{d}x \ \mathrm{d}t.$$

Here, we have used the notation

$$\mathbb{A} : \mathcal{R} \equiv \sum_{i,j=1}^{3} A_{i,j}\mathcal{R}_{i,j}, \ \mathcal{R}[\mathbf{v}]_i \equiv \sum_{j=1}^{3} \mathcal{R}_{i,j}[v_j], \ i = 1,2,3.$$

Exactly as in Section 2.2.5, the integral identity (2.102) can be used to establish a bound

$$\int_0^T \int_K p(\varrho,\vartheta)\varrho^\nu \ \mathrm{d}x \ \mathrm{d}t \le c(\mathrm{data}, K) \text{ for a certain } \nu > 0, \tag{2.103}$$

and, consequently,

$$\int_0^T \int_K \varrho^{\frac{5}{3}+\nu} \, \mathrm{d}x \, \mathrm{d}t \le c(\mathrm{data}, K), \tag{2.104}$$

$$\int_0^T \int_K |p(\varrho,\vartheta)|^r \, \mathrm{d}x \, \mathrm{d}t \le c(\mathrm{data}, K) \text{ for a certain } r > 1 \tag{2.105}$$

for any compact $K \subset \Omega$.

Pressure estimates near the boundary. Our ultimate goal is to extend, in a certain sense, the local estimates established in Section 2.2.6 up to the boundary $\partial\Omega$. In particular, our aim is to show that the pressure is equi-integrable in Ω, where the bound can be determined in terms of the data. To this end, it is enough to solve the following auxiliary problem:

Given $q > 1$ arbitrary, find a function $\mathbf{G} = \mathbf{G}(x)$ such that

$$\mathbf{G} \in W_0^{1,q}(\Omega;\mathbb{R}^3), \ \mathrm{div}_x \mathbf{G}(x) \to \infty \ \textit{ uniformly for } \ \mathrm{dist}(x, \partial\Omega) \to 0. \tag{2.106}$$

If Ω is a bounded Lipschitz domain, the function $\mathbf{G}$ can be taken as a solution of the problem

$$\mathrm{div}_x \mathbf{G} = g \text{ in } \Omega, \ \mathbf{G}|_{\partial\Omega} = 0, \tag{2.107}$$

where

$$g = \mathrm{dist}^{-\beta}(x, \partial\Omega) - \frac{1}{|\Omega|} \int_\Omega \mathrm{dist}^{-\beta}(x, \partial\Omega) \, \mathrm{d}x, \text{ with } 0 < \beta < \frac{1}{q},$$

so that (2.106) is satisfied. Problem (2.107) can be solved by means of the operator $\mathcal{B}$ introduced in Section 2.2.5 as soon as Ω is a Lipschitz domain. For less regular domains, an explicit solution may be found by an alternative method (see Kukučka [126]).

Pursuing step by step the procedure developed in the preceding section we take the quantity

$$\varphi(t,x) = \psi(t)\mathbf{G}(x), \ \psi \in C_c^\infty(0,T),$$

as a test function in the momentum equation (2.9). Assuming $\mathbf{G}$ belongs to $W_0^{1,q}(\Omega;\mathbb{R}^3)$, with $q > 1$ large enough, we can deduce, exactly as in Section 2.2.6, that

$$\int_0^T \int_\Omega p(\varrho,\vartheta)\mathrm{div}_x \mathbf{G} \, \mathrm{d}x \, \mathrm{d}t \le c(\mathrm{data}). \tag{2.108}$$

Note that this step can be fully justified via a suitable approximation of $\mathbf{G}$ by a family of smooth, compactly supported functions. As $\mathrm{div}_x \mathbf{G}(x) \to \infty$ whenever $x \to \partial\Omega$, relation (2.108) yields equi-integrability of the pressure in a neighborhood of the boundary (cf. Theorem 0.8).