Chapter 1

Linear Operators and Spectra

This chapter recalls some basic concepts of the theory of linear operators in normed spaces, with emphasis on Hilbert spaces. It also fixes some notation and introduces the concept of a spectrum along with various proofs. Compact operators are discussed. The readers are supposed to have had a first contact with functional analysis.

1.1 Bounded Operators

Let $\mathbb{F}$ denote either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. For $z \in \mathbb{C}$, let $\overline{z}$ denote its complex conjugate. As usual in mathematics, *iff* will be an abbreviation for "if and only if."

Definition 1.1.1. A *linear operator* between the vector spaces X and Y is a transformation $T : \text{dom}\, T \subset X \to Y$, for which its domain $\text{dom}\, T$ is a vector subspace and $T(\xi + \alpha\eta) = T(\xi) + \alpha T(\eta)$, for all $\xi, \eta \in \text{dom}\, T$ and all scalar $\alpha \in \mathbb{F}$.

Note that $T(0) = 0$ for any linear operator T, and that the set of linear operators with the same domain and codomain is a vector space with pointwise operations; frequently $T(\xi)$ will also be denoted by $T\xi$. Simple examples of linear operators are the *identity operator* $\mathbf{1} : X \to X$, with $\mathbf{1}(\xi) = \xi$, and the *null* (or *zero*) *operator* $T\xi = 0$, $\forall \xi$.

In many cases it is imperative to consider domains dense in another set; so throughout this text the notation $A \sqsubseteq B$ will indicate that A is a dense subset of B, with respect to the appropriate topology. The natural numbers $\{1, 2, 3, \dots\}$ will be denoted by $\mathbb{N}$ and the term *enumerable* indicates the cardinality $\aleph_0$ of the set of natural numbers, while *countable* refers to finite numbers (including zero); so, *uncountable* indicates that something is infinite and with cardinality different from $\aleph_0$.

$\mathcal{N}, \mathcal{B}, \mathcal{H}$ always denote a normed space, a Banach space and a Hilbert space, respectively. In any metric space, the sphere, open and closed balls centered at ξ and of radius $r > 0$ will be denoted by $S(\xi; r)$, $B(\xi; r)$ and $\overline{B}(\xi; r)$, respectively. If A is a subset of a vector space, then $\mathrm{Lin}(A)$ denotes the linear subspace spanned by A.

Example 1.1.2. Let $\phi \in \mathrm{L}^\infty_\mu(\Omega)$, with μ being σ-finite. Then the multiplication operator by ϕ, defined by $\mathcal{M}_\phi : \mathrm{L}^p_\mu(\Omega) \to \mathrm{L}^p_\mu(\Omega)$,

$$(\mathcal{M}_\phi \psi)(t) := \phi(t)\psi(t), \quad \psi \in \mathrm{L}^p_\mu(\Omega),$$

is a linear operator $\forall\, 1 \le p \le \infty$. Note that $(\mathcal{M}_\phi \psi) \in \mathrm{L}^p_\mu$ for $\psi \in \mathrm{L}^p_\mu$.

Remark 1.1.3. The notation of the Banach spaces $\mathrm{L}^p_\mu(\Omega)$, $1 \le p \le \infty$, is standard. In case $\Omega \subset \mathbb{R}^n$ and the measure is Lebesgue measure, the simplified notation $\mathrm{L}^p(\Omega)$ will be employed.

Example 1.1.4. Let X and Y be compact metric spaces and $u : Y \to X$ continuous. Then $T_u : C(X) \to C(Y)$, $(T_u\psi)(y) = \psi(u(y))$, is a linear operator.

Exercise 1.1.5. Let $T : \mathrm{dom}\, T \subset X \to Y$ be a linear operator. Verify the following items:

a) The range of T, $\mathrm{rng}\, T := T(\mathrm{dom}\, T) \subset Y$, and the *kernel* (or *null space*) of T, $\mathrm{N}(T) := \{\xi \in \mathrm{dom}\, T : T\xi = 0\}$, are vector spaces.
b) If the dimension $\dim(\mathrm{dom}\, T) = n < \infty$, then $\dim(\mathrm{rng}\, T) \le n$.
c) The inverse operator of T, $T^{-1} : \mathrm{rng}\, T \to \mathrm{dom}\, T$, exists if, and only if, $T\xi = 0 \Rightarrow \xi = 0$ and, in case it exists, it is also a linear operator.
d) If T, S are invertible linear operators, then $(TS)^{-1} = S^{-1}T^{-1}$ (by supposing, of course, that the operations are well posed).

A rich theory is obtained through the fusion of linear operators with the natural topology generated by norms. The next result is an example of such fusion; it shows that if a linear operator is continuous at some point of its domain, then it is uniformly continuous on its whole domain.

Theorem 1.1.6. *Let $T : \mathcal{N}_1 \to \mathcal{N}_2$ be a linear operator. Then the following assertions are equivalent:*

i) $\sup_{\|\xi\| \le 1} \|T\xi\| < \infty$.
ii) $\exists C > 0$ *such that* $\|T\xi\| \le C\|\xi\|$, $\forall \xi \in \mathcal{N}_1$.
iii) T *is uniformly continuous.*
iv) T *is continuous.*
v) T *is continuous at zero* (*i.e., the null vector*).

Proof. i) $\Longrightarrow$ ii) Let $C = \sup_{\|\xi\| \le 1} \|T\xi\|$. If $0 \ne \xi \in \mathcal{N}_1$, then $\|T(\xi/\|\xi\|)\| \le C$, i.e., $\|T\xi\| \le C\|\xi\|$, $\forall \xi \in \mathcal{N}_1$.

ii) $\Longrightarrow$ iii) If $\xi, \eta \in \mathcal{N}_1$, then $\|T\xi - T\eta\| = \|T(\xi - \eta)\| \le C\|\xi - \eta\|$.

iii) $\Longrightarrow$ iv) and iv) $\Longrightarrow$ v) are obvious.

v) $\Longrightarrow$ i) Since T is continuous at zero, there exists $\delta > 0$ with $\|T\xi\| \le 1$ if $\|\xi\| \le \delta$. Thus, if $\|\xi\| \le 1$, it follows that $\|\delta\xi\| \le \delta$ and $\|T(\delta\xi)\| \le 1$; therefore, $\|T\xi\| \le 1/\delta$, and i) holds. □

Definition 1.1.7. A continuous linear operator is also called *bounded*, and the set of bounded linear operators from $\mathcal{N}_1$ to $\mathcal{N}_2$ will be denoted by $\mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$. The notation $\mathrm{B}(\mathcal{N})$ will also be used as an abbreviation of $\mathrm{B}(\mathcal{N}, \mathcal{N})$.

Note the distinct use of the term *bounded linear operator* compared to the use in *bounded application* in general, i.e., one with bounded range; in the latter sense every linear (nonzero) operator is not bounded; verify this.

Example 1.1.8. The operator T_u in Example 1.1.4 is continuous, since for all $\psi \in C(X)$ one has $\|T_u\psi\|_\infty = \sup_{t\in Y} |\psi(u(t))| \le \sup_{t\in X} |\psi(t)| = \|\psi\|_\infty$, and T_u is bounded by Theorem 1.1.6(ii).

Exercise 1.1.9. Let X and Y be finite-dimensional vector spaces and $T : X \to Y$ a linear operator. Choose bases in X and Y and show that T can be represented by a matrix, and discuss how the matrix that represents T changes if other bases are considered.

Proposition 1.1.10. *If* $T : \mathcal{N}_1 \to \mathcal{N}_2$ *be linear and* $\dim \mathcal{N}_1 < \infty$, *then* T *is bounded.*

Proof. Consider in $\mathcal{N}_1$ the norm $|||\xi||| = \|\xi\| + \|T\xi\|$; then there exists $C > 0$ such that $|||\xi||| \le C\|\xi\|$, because all norms on finite-dimensional vector spaces are equivalent. Hence, $\|T\xi\| \le |||\xi||| \le C\|\xi\|$ and T is bounded. □

Example 1.1.11. For $1 \le p < \infty$, $l^p(\mathbb{N})$ denotes the Banach space of sequences $\xi = (\xi_j)_{j\in\mathbb{N}}$ so that $\|\xi\|_p = \left(\sum_j |\xi_j|^p\right)^{1/p} < \infty$. For $p = \infty$ the space $l^\infty(\mathbb{N})$ carries the norm $\|\xi\|_\infty = \sup_j |\xi_j|$. Similarly one defines $l^p(\mathbb{Z})$, $1 \le p \le \infty$.

Let $T : \left\{(\xi_n) \in l^p(\mathbb{N}) : \sum_n |n^2\xi_n|^p < \infty\right\} \to l^p(\mathbb{N})$, with $1 \le p < \infty$, $T(\xi_n) = (n^2\xi_n)$; this operator is linear, but is not continuous, since if $\{e_n\}_{n=1}^\infty$ denotes the canonical basis of $l^p(\mathbb{N})$, i.e., $e_n = (\delta_{j,n})_j$, then $e_n/n \to 0$, while Te_n does not converge to zero. Another argument: T is not bounded since $\|e_n\|_p = 1$ and $\|Te_n\|_p = n^2, \forall n$.

Example 1.1.12 (Shifts). The *right* (*left*) *shift* operator in $l^p(\mathbb{Z})$, $1 \le p \le \infty$, is defined by $S_r : l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$ (resp. S_l), $\eta = S_r\xi$ (resp. $\eta = S_l\xi$), with $\eta_j = \xi_{j-1}$ (resp. $\eta_j = \xi_{j+1}$), $j \in \mathbb{Z}$. Note that the shift operator in $l^p(\mathbb{Z})$ is a bijective isometry (i.e., an isometric mapping), so bounded. They are also defined on $l^p(\mathbb{N})$ in an analogous way, but if $\eta = S_r\xi$ then it is defined $\eta_1 = 0$; these operators are also bounded, but S_r in $l^p(\mathbb{N})$ is not onto, although it is isometric.

Note that $\mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$ is a vector space with pointwise operations, and it turns out that

$$\|T\| := \sup_{\substack{\xi\in\mathcal{N}_1 \\ \|\xi\|\le 1}} \|T\xi\|$$

is a norm on $\mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$. In fact, if $T \in \mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$, $\|T\| = 0 \iff T\xi = 0$, $\forall \xi \in \mathcal{N}_1$, that is, $T = 0$; $\|\alpha T\| = |\alpha| \|T\|$ is immediate; if $S \in \mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$, then

$$\|T + S\| = \sup_{\|\xi\| \le 1} \|T\xi + S\xi\| \le \sup_{\|\xi\| \le 1} (\|T\xi\| + \|S\xi\|) \le \|T\| + \|S\|.$$

If a topology is not explicitly given in $\mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$, it is supposed that the topology is the one induced by this norm.

Exercise 1.1.13. a) If $T \in \mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$, check that

$$\|T\| = \inf_{C > 0} \{\|T\xi\| \le C\|\xi\|, \ \forall \xi \in \mathcal{N}_1\} = \sup_{\|\xi\| = 1} \|T\xi\| = \sup_{\xi \ne 0} \frac{\|T\xi\|}{\|\xi\|}.$$

b) If T, S are bounded linear operators and TS (the composition, but usually called product of operators) is defined, show that TS is bounded and $\|TS\| \le \|T\| \|S\|$. Therefore, if T^n (nth iterate of T) is defined, then $\|T^n\| \le \|T\|^n$.

Example 1.1.14. The zero operator is the unique operator whose norm is zero, and for the identity operator $\|\mathbf{1}\| = 1$ (with $\mathcal{N} \ne \{0\}$).

Example 1.1.15. Let X be the vector space of polynomials in $C[0,1]$ and $D : X \hookleftarrow$ the differential operator $(Dp)(t) = p'(t)$, $p \in X$. This operator is linear and does not belong to $\mathrm{B}(X)$, since if $p_n(t) = t^n$, then for all $n \ge 1$ one has $(Dp_n)(t) = nt^{n-1}$, $\|p_n\|_\infty = 1$, while $\|Dp_n\|_\infty = n$.

Example 1.1.16. The operator $\mathcal{M}_\phi$, with $\phi \in \mathrm{L}^\infty_\mu(\Omega)$ (see Example 1.1.2) is bounded in $\mathrm{L}^p_\mu(\Omega), 1 \le p \le \infty$, and $\|\mathcal{M}_\phi\| = \|\phi\|_\infty$ ($= \sup \mathrm{ess}\, |\phi|$).

Proof. It will be supposed that $\|\phi\|_\infty \ne 0$ and demonstrated for $1 \le p < \infty$. The cases $p = \infty$ and $\|\phi\|_\infty = 0$ are left as exercises. If $\|\psi\|_p = 1$, then by

$$\|\mathcal{M}_\phi \psi\|_p^p = \int_\Omega |\phi(t)|^p |\psi(t)|^p d\mu(t) \le \|\phi\|_\infty^p \|\psi\|_p^p,$$

one gets that $\mathcal{M}_\phi$ is bounded and $\|\mathcal{M}_\phi\| \le \|\phi\|_\infty$.

Let $0 < \theta < \|\phi\|_\infty$; then there exists a measurable set A, with $0 < \mu(A) < \infty$ (recall that μ is σ-finite) obeying $\|\phi\|_\infty \ge |\phi(t)| > \theta$, $\forall t \in A$. Thus, χ_A, the characteristic function of A (i.e., $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$), belongs to $\mathrm{L}^p_\mu(\Omega)$ and

$$\|\mathcal{M}_\phi \chi_A\|_p^p = \int_A |\phi(t)|^p |\chi_A(t)|^p d\mu(t) \ge \theta^p \|\chi_A\|_p^p;$$

so $\|\mathcal{M}_\phi\| \ge \theta$ and, therefore, $\|\mathcal{M}_\phi\| = \|\phi\|_\infty$. □

Example 1.1.17. Let $K : (\Omega, \mathcal{A}, \mu) \times (\Omega, \mathcal{A}, \mu) \to \mathbb{F}$ measurable (σ-finite space) and suppose that there exists $C > 0$ with

$$\int_\Omega |K(x,y)| d\mu(x) \le C, \quad \text{for } y \ \ \mu - \text{a.e.}$$

Then, $T_K : \mathrm{L}^1_\mu(\Omega) \hookleftarrow$ given by

$$(T_K\psi)(x) = \int_\Omega K(x,y)\psi(y)d\mu(y), \quad \psi \in \mathrm{L}^1_\mu(\Omega),$$

is bounded and $\|T_K\| \le C$.

Proof. If $\psi \in \mathrm{L}^1_\mu(\Omega)$ then

$$|(T_K\psi)(x)| \le \int_\Omega |K(x,y)\psi(y)|d\mu(y);$$

thus, $\|T_K\psi\|_1 = \int_\Omega |(T_K\psi)(x))|d\mu(x) \le \iint |K(x,y)|\ |\psi(y)|d\mu(y)d\mu(x)$. By the Fubini Theorem it is found that

$$\|T_K\psi\|_1 \le \iint_{\Omega\times\Omega} |K(x,y)|d\mu(x)\ |\psi(y)|d\mu(y) \le C\|\psi\|_1.$$

Therefore $\|T_K\| \le C$. □

Exercise 1.1.18. Let $(e_n)_{n=1}^\infty$ be the usual basis of $l^2(\mathbb{N})$ and $(\alpha_n)_{n=1}^\infty$ a sequence in $\mathbb{F}$. Show that the operator $T : l^2(\mathbb{N}) \hookleftarrow$ with $Te_n = \alpha_n e_n$ is bounded if, and only if, $(\alpha_n)_{n=1}^\infty$ is a bounded sequence. Verify that, in this case, $\|T\| = \sup_n |\alpha_n|$.

Exercise 1.1.19. Let $C^1(0,1)$ be the set of continuously differentiable real functions on $(0,1)$, as a subspace of $\mathrm{L}^2(0,1)$ (i.e., use the norm of L^2). Apply the differential operator $(D\psi)(t) = \psi'(t)$, $D : C^1(0,1) \to \mathrm{L}^2(0,1)$, to functions $\psi_n(t) = \sin(n\pi t)$ and conclude that D is not bounded.

Exercise 1.1.20. Show that the differential operator $D : C^\infty[a,b] \hookleftarrow$ is not bounded for any norm on $C^\infty[a,b]$.

The next result gives a simple answer to an important question. Under which conditions $\mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$ is a Banach space?

Theorem 1.1.21. *If $\mathcal{N}$ is a normed space and $\mathcal{B}$ a Banach space, then* $\mathrm{B}(\mathcal{N}, \mathcal{B})$ *is Banach.*

Proof. Let $(T_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathrm{B}(\mathcal{N}, \mathcal{B})$. Since for each $\xi \in \mathcal{N}$ one has $\|T_n\xi - T_k\xi\| \le \|T_n - T_k\|\|\xi\|$, then $(T_n\xi)$ is Cauchy in $\mathcal{B}$ and converges to $\eta \in \mathcal{B}$. Define $T : \mathcal{N} \to \mathcal{B}$ by $T\xi = \eta$, which is clearly linear. It will be shown that this operator is bounded and $T_n \to T$ in $\mathrm{B}(\mathcal{N}, \mathcal{B})$.

Given $\varepsilon > 0$ there exists $N(\varepsilon)$ such that, if $n, k \ge N(\varepsilon)$, then $\|T_n - T_k\| < \varepsilon$. By the continuity of the norm it follows that

$$\|T_n\xi - T\xi\| = \lim_{k\to\infty} \|T_n\xi - T_k\xi\| \le \varepsilon\|\xi\|, \qquad n \ge N(\varepsilon),$$

and $(T_n - T) \in \mathrm{B}(\mathcal{N}, \mathcal{B})$ with $\|T_n - T\| \le \varepsilon$. Since $\mathrm{B}(\mathcal{N}, \mathcal{B})$ is a vector space, and $T = T_n + (T - T_n)$, then $T \in \mathrm{B}(\mathcal{N}, \mathcal{B})$. The inequality $\|T_n - T\| \le \varepsilon$, valid for all $n \ge N(\varepsilon)$, shows that $T_n \to T$ and $\mathrm{B}(\mathcal{N}, \mathcal{B})$ is complete. □

Exercise 1.1.22. Suppose that $T_n \to T$ in $\mathrm{B}(\mathcal{N})$ and $\xi_n \to \xi$ in $\mathcal{N}$. Show that $T_n\xi_n \to T\xi$.

Exercise 1.1.23. Let $T \in \mathrm{B}(\mathcal{B})$. Show that, for all $t \in \mathbb{F}$, the operator e^{tT} defined by the series

$$e^{tT} := \sum_{j=0}^{\infty} \frac{(tT)^j}{j!}$$

belongs to $\mathrm{B}(\mathcal{B})$ and $\|e^{tT}\| \le e^{|t|\|T\|}$.

Exercise 1.1.24. Let $T \in \mathrm{B}(\mathcal{B})$, with $\|T\| < 1$. Show that the operator defined by the series $S = \sum_{j=0}^{\infty} T^j$ belongs to $\mathrm{B}(\mathcal{B})$ and $S = (\mathbf{1} - T)^{-1}$.

Uniformly continuous functions on metric spaces have uniformly continuous extensions to the closure of their domains; in the case of linear operators there is an analogous result, which is a consequence of the uniform continuity of bounded operators (Theorem 1.1.6).

Definition 1.1.25. If $\mathcal{N}$ is a normed space, then the Banach space $\mathrm{B}(\mathcal{N}, \mathbb{F})$ will be denoted by $\mathcal{N}^*$ and termed *dual space* of $\mathcal{N}$. Each element of $\mathcal{N}^*$ is called a continuous *linear functional* on $\mathcal{N}$ (Why is $\mathcal{N}^*$ complete?).

Remark 1.1.26. a) Recall that by the Hahn-Banach theorem $\mathcal{N}^*$ separates points of $\mathcal{N}$, that is, if $\eta \neq \xi \in \mathcal{N}$, then there exists $f \in \mathcal{N}^*$ with $f(\xi) \neq f(\eta)$. In particular, if $f(\xi) = 0$ for all $f \in \mathcal{N}^*$, then $\xi = 0$.

b) The Hahn-Banach theorem can also be used to prove the converse of Theorem 1.1.21, so that $\mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$ is complete iff $\mathcal{N}_2$ is a Banach space.

Example 1.1.27. The integral on $C[a, b]$ is an element of the dual of $C[a, b]$, since $\psi \mapsto \int_a^b \psi(t)\, dt$ is linear and continuous. In fact, every finite Borel (complex) measure μ over $[a, b]$ defines an element of the dual of $C[a, b]$ through the integral $\psi \mapsto \int_a^b \psi(t)\, d\mu(t)$, because

$$\left| \int_a^b \psi(t)\, d\mu(t) \right| \le \|\psi\|_\infty\, |\mu|([a, b]).$$

Example 1.1.28 (Unbounded functional). Consider the linear functional

$$f : C[-1, 1] \subset \mathrm{L}^1[-1, 1] \to \mathbb{F}, \quad f(\psi) = \psi(0).$$

Pick a function $\psi \in C[-1, 1]$ with $\psi(-1) = \psi(1) = 0$ and $\psi(0) \neq 0$. For each $n \ge 2$, set $\psi_n(t) = \psi(nt)$ if $|t| \le 1/n$, and equal to zero otherwise. Note that $\|\psi_n\|_1 = \int_{-1}^1 |\psi_n(t)|\, dt = \|\psi\|_1/n$, which converges to zero for $n \to \infty$. However, $f(\psi_n) = \psi(0) \neq 0$ for all n, and f is not continuous.

Example 1.1.29. Let $1 < p < \infty$ and $1/p + 1/q = 1$. Each $\phi \in \mathrm{L}^q_\mu(\Omega)$ defines an element of the dual of $\mathrm{L}^p_\mu(\Omega)$, since by Hölder inequality the product $\phi\psi \in \mathrm{L}^1_\mu(\Omega)$, for all $\psi \in \mathrm{L}^p_\mu(\Omega)$, and

$$\psi \mapsto \int_\Omega \phi\psi d\mu$$

is linear and bounded with norm $\leq \|\phi\|_q$ (again by Hölder). Hence, $\mathrm{L}^q_\mu(\Omega) \subset \mathrm{L}^p_\mu(\Omega)^*$. The proof is found in books on Integration Theory that $\mathrm{L}^p_\mu(\Omega)^* = \mathrm{L}^q_\mu(\Omega)$, for $1 < p < \infty$ and, if the measure μ is σ-finite, one also has $\mathrm{L}^1_\mu(\Omega)^* = \mathrm{L}^\infty_\mu(\Omega)$.

Exercise 1.1.30. Show that the dual of l^p is l^q, with $1 < p < \infty$ and $1/p + 1/q = 1$.

Theorem 1.1.31 (Uniform Boundedness Principle). *Any family of operators* $\{T_\alpha\}_{\alpha\in J}$ *in* $\mathrm{B}(\mathcal{B},\mathcal{N})$ *so that, for each* $\xi \in \mathcal{B}$,

$$\sup_{\alpha\in J} \|T_\alpha \xi\| < \infty,$$

satisfies $\sup_{\alpha\in J} \|T_\alpha\| < \infty$.

Proof. Put $E_k = \{\xi \in \mathcal{B} : \|T_\alpha\xi\| \leq k, \forall \alpha \in J\}$, which is a closed set; indeed, since T_α is continuous, it is the intersection of the closed sets $T_\alpha^{-1}\overline{B}(0;k)$ for all $\alpha \in J$. Since $\mathcal{B} = \bigcup_{k=1}^\infty E_k$, by the Baire theorem there exists E_m with nonempty interior. Let $B_\mathcal{B}(\xi_0;r)$ $(r > 0)$ be an open ball contained in E_m; then, for any $\alpha \in J$ one has $\|T_\alpha\xi\| \leq m$ for all $\xi \in B_\mathcal{B}(\xi_0;r)$.

If $\xi \in \mathcal{B}$, $\|\xi\| = 1$, it is found that $\eta = \xi_0 + r\xi/2$ belongs to $B_\mathcal{B}(\xi_0;r)$ and

$$\|T_\alpha\xi\| = \frac{2}{r}\,\|T_\alpha\eta - T_\alpha\xi_0\| \leq \frac{2}{r}\,(\|T_\alpha\eta\| + \|T_\alpha\xi_0\|) \leq \frac{4m}{r};$$

thus $\|T_\alpha\xi\| \leq 4m/r$ for all $\alpha \in J$ and $\|\xi\| = 1$; it then follows that $\sup_\alpha \|T_\alpha\| \leq 4m/r < \infty$. □

Corollary 1.1.32. *A subset* $H \subset \mathcal{B}^* = \mathrm{B}(\mathcal{B},\mathbb{F})$ *is bounded if, and only if, for all* $\xi \in \mathcal{B}$, $\sup_{f\in H} |f(\xi)| < \infty$.

Proof. If H is bounded, then $M = \sup_{f\in H} \|f\| < \infty$ and for all $\xi \in \mathcal{B}$ one has $\sup_{f\in H} |f(\xi)| \leq M\|\xi\| < \infty$. To show the other statement, by using the notation presented in the uniform boundedness principle, it is enough to consider H as the family T_α in the Banach space $\mathcal{B}^*$. □

Corollary 1.1.33 (Banach-Steinhaus Theorem). *Let* $(T_n)_{n=1}^\infty$ *be a sequence in* $\mathrm{B}(\mathcal{B},\mathcal{N})$ *so that for each* $\xi \in \mathcal{B}$ *there exists the limit*

$$T\xi := \lim_{n\to\infty} T_n\xi.$$

Then $\sup_n \|T_n\| < \infty$ *and* T *is a bounded operator in* $\mathrm{B}(\mathcal{B},\mathcal{N})$.

Proof. Clearly T is linear. Since for all $\xi \in \mathcal{B}$ there exists $\lim_{n\to\infty} T_n\xi$, then $\sup_n \|T_n\xi\| < \infty$, and by the uniform boundedness principle one has $\sup_n \|T_n\| < \infty$. By the definition of T it follows that

$$\|T\xi\| \leq (\sup_n \|T_n\|)\,\|\xi\|, \qquad \forall \xi \in \mathcal{B}$$

and, therefore, T is bounded. □

Example 1.1.34. Let $\mathcal{N}$ be the normed space of the elements $\xi = (\xi_j) \in l^\infty(\mathbb{N})$ that have just a finite number of nonzero entries ξ_j. Define $T_n : \mathcal{N} \to l^\infty$ by $T_n\xi = (n\xi_n)_{j\in\mathbb{N}}$. Then $T_n \in \mathrm{B}(\mathcal{N}, l^\infty)$ for all n, and for each $\xi \in \mathcal{N}$ there exists the limit $\lim_{n\to\infty} T_n\xi = 0$, but $\lim_{n\to\infty} \|T_n\| = \infty$. This shows that the conclusions of the Banach-Steinhaus theorem (and of the uniform boundedness principle) may fail if the domain of the operators is not complete.

Exercise 1.1.35. Let $S_l : l^2(\mathbb{N}) \hookleftarrow$ be the shift

$$S_l(\xi_1, \xi_2, \xi_3, \dots) = (\xi_2, \xi_3, \xi_4, \dots)$$

and $T_n = S_l^n$. Find $\|T_n\xi\|$, and the limit operator described in the Banach-Steinhaus theorem.

Proposition 1.1.36. *Let $\{T_\alpha\}_{\alpha\in J}$ be a family in $\mathrm{B}(\mathcal{B}, \mathcal{N})$ with*

$$\sup_{\alpha\in J} \|T_\alpha\| = \infty.$$

Then the set $\mathcal{I} = \{\xi \in \mathcal{B} : \sup_\alpha \|T_\alpha\xi\| < \infty\}$ is meager in $\mathcal{B}$ (that is, it is a subset of a countable union of closed subsets of $\mathcal{B}$ with empty interior).

Proof. By using the notation of the proof of the uniform boundedness principle, one has $\mathcal{I} = \cup_{k=1}^\infty E_k$, and by that proof it follows that the interior of every E_k is empty, since if not one would get $\sup_{\alpha\in J} \|T_\alpha\| < \infty$. Since E_k is closed, then $\mathcal{I}$ is meager. □

Denote $C_p[0, 2\pi] = \{\psi \in C[0, 2\pi] : \psi(0) = \psi(2\pi)\}$, which is a closed subspace of $C[0, 2\pi]$, so it is Banach, and

$$(\mathrm{F}\psi)_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-int}\psi(t)\, dt, \qquad \psi \in C_p[0, 2\pi].$$

Corollary 1.1.37. *The set of elements $\psi \in C_p[0, 2\pi]$ whose Fourier series $\sum_{n\in\mathbb{Z}} (\mathrm{F}\psi)_n\, e^{int}$ converges for $t = 0$ is meager.*

Proof. By working with trigonometric relations it is found that, for each N, the partial sum $(S_N\psi)(t) = \sum_{|n|\le N} (\mathrm{F}\psi)_n\, e^{int}$ can be written in the form

$$(S_N\psi)(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin[(2N+1)(t-s)/2]}{\sin[(t-s)/2]} \psi(s)\, ds.$$

Note that $f_N : C_p[0, 2\pi] \to \mathbb{C}$, $f_N(\psi) = (S_N\psi)(0)$, is an element of the dual of $C_p[0, 2\pi]$; thus, in order to conclude this proof it is enough to show that $\sup_N \|f_N\| = \infty$ and use Proposition 1.1.36 with f_N represented by T_α.

Consider $\phi_N(t) = \sin[(2N+1)t/2]$, an element of $C_p[0,2\pi]$ with norm equal to 1; thus

$$\begin{aligned} f_N(\phi_N) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin^2[(2N+1)s/2]}{\sin(s/2)}\, ds \\ &\geq \frac{1}{\pi}\int_0^{2\pi} \frac{\sin^2[(2N+1)s/2]}{s}\, ds \\ &= \frac{1}{\pi}\int_0^{(2N+1)\pi} \frac{\sin^2 u}{u} du \\ &\geq \frac{1}{\pi}\sum_{n=1}^{2N+1}\int_{(n-1)\pi}^{n\pi} \frac{\sin^2 u}{n\pi} du = \frac{1}{2\pi}\sum_{n=1}^{2N+1}\frac{1}{n}. \end{aligned}$$

Since the harmonic series is divergent, one concludes that $\lim_{N\to\infty}\|f_N\| = \infty$, and the proof is complete. □

Exercise 1.1.38. Verify that $C_p[0,2\pi]$ is a Banach space, and also the validity of the expression for a partial sum for the Fourier series used in the proof of Corollary 1.1.37.

Now the famous Riesz representation theorem of Hilbert spaces $\mathcal{H}$, which shows that every Hilbert space is naturally identified to its dual, is recalled and demonstrated. In order to fix notation, remember that an inner product in a vector space X is a map $(\xi,\eta) \mapsto \langle \xi,\eta\rangle$, $X\times X \to \mathbb{F}$, so that for any $\xi,\eta,\zeta \in X$ and $\alpha \in \mathbb{F}$ it satisfies:

i) $\langle \alpha\xi+\eta,\zeta\rangle = \bar{\alpha}\langle\xi,\zeta\rangle + \langle\eta,\zeta\rangle$,
ii) $\langle\xi,\eta\rangle = \overline{\langle\eta,\xi\rangle}$,
iii) $\langle\xi,\xi\rangle \geq 0$, and $\langle\xi,\xi\rangle = 0$ iff $\xi = 0$.

In an inner product space one has the induced norm $\|\xi\| := \sqrt{\langle\xi,\xi\rangle}$, so that the Cauchy-Schwarz $|\langle\xi,\eta\rangle| \leq \|\xi\|\|\eta\|$ and triangular $\|\xi+\eta\| \leq \|\xi\| + \|\eta\|$ inequalities always hold.

Exercise 1.1.39. Show that equality in Cauchy-Schwarz occurs iff $\{\xi,\eta\}$ is linearly dependent, while equality in the triangular occurs iff either $\xi = 0$ or $\eta = t\xi$ for some $t \geq 0$.

Let $\{\xi_\alpha\}_{\alpha\in J}$ be an orthonormal set in $\mathcal{H}$. One of the advantages of the presence of an inner product in a Hilbert space $\mathcal{H}$ is the existence of orthonormal basis of $\mathcal{H}$, that is, if $\overline{\mathrm{Lin}(\{\xi_\alpha\}_{\alpha\in J})} = \mathcal{H}$. The following facts illustrate such advantages quite well. For each $\xi \in \mathcal{H}$, the Bessel inequality

$$\|\xi\|^2 \geq \sum_{\alpha\in J} |\langle\xi_\alpha,\xi\rangle|^2$$

holds; in particular, $\langle\xi_\alpha,\xi\rangle \neq 0$ only for a countable number of indices $\alpha \in J$. Furthermore, the following assertions are equivalent:

i) $\{\xi_\alpha\}_{\alpha\in J}$ is an orthonormal basis of $\mathcal{H}$.

ii) If $\xi \in \mathcal{H}$, then the Fourier series of ξ, with respect to $\{\xi_\alpha\}_{\alpha\in J}$, converges in $\mathcal{H}$ for ξ (and independent of the sum order), that is,

$$\xi = \sum_{\alpha\in J} \langle \xi_\alpha, \xi\rangle \, \xi_\alpha, \qquad \forall \xi \in \mathcal{H}.$$

iii) [Parseval Identity] For all $\xi \in \mathcal{H}$,

$$\|\xi\|^2 = \sum_{\alpha\in J} |\langle \xi_\alpha, \xi\rangle|^2.$$

Furthermore, if $\{\xi_\alpha\}_{\alpha\in J}$ is an orthonormal basis and $\eta = \sum_{\alpha\in J}\langle \xi_\alpha, \eta\rangle \xi_\alpha$, then

$$\langle \xi, \eta\rangle = \sum_{\alpha} \langle \xi, \xi_\alpha\rangle \, \langle \xi_\alpha, \eta\rangle.$$

Theorem 1.1.40 (Riesz Representation). *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}^*$ its dual. The map $\gamma : \mathcal{H} \to \mathcal{H}^*$, $\gamma(\xi) = f_\xi$, for $\xi \in \mathcal{H}$, given by*

$$\gamma(\xi)(\eta) = f_\xi(\eta) = \langle \xi, \eta\rangle, \qquad \forall \eta \in \mathcal{H},$$

is an antilinear (i.e., $\alpha\xi \mapsto \overline{\alpha}\xi$, $\forall \alpha \in \mathbb{F}$) and onto isometry on $\mathcal{H}^*$.

Remark 1.1.41. This theorem implies that each element of $\mathcal{H}^*$ is identified to a unique $\xi \in \mathcal{H}$, via f_ξ, and $\|f_\xi\| = \|\xi\|$; one then says such ξ represents f_ξ. Note that two distinct notations for this map were introduced: $\gamma(\xi)$ and f_ξ; this is convenient in certain situations.

Proof. If $\xi = 0$, clearly $f_\xi = 0$. If $\xi \in \mathcal{H}$, then f_ξ is a linear functional and $|f_\xi(\eta)| = |\langle \xi, \eta\rangle| \le \|\xi\|\|\eta\|$, so that $f_\xi \in \mathcal{H}^*$ with $\|f_\xi\| \le \|\xi\|$. In view of $\|\xi\|^2 = f_\xi(\xi) \le \|f_\xi\|\|\xi\|$ one has $\|f_\xi\| \ge \|\xi\|$. Hence $\|f_\xi\| = \|\xi\|$, and the map γ is an isometry, obviously antilinear (linear in the real case). Then we only need to show that every $f \in \mathcal{H}^*$ is of the form f_ξ for some $\xi \in \mathcal{H}$. If $f = 0$, then $f = f_\xi$ for $\xi = 0$. If $f \neq 0$, since the kernel $\mathrm{N}(f)$ is a proper closed vector subspace (since f is continuous) of $\mathcal{H}$, it is found that

$$\mathcal{H} = \mathrm{N}(f) \oplus \mathrm{N}(f)^\perp,$$

and there exists $\zeta \in \mathrm{N}(f)^\perp$ with $\|\zeta\| = 1$. Now, by noticing that the vector $(f(\eta)\zeta - f(\zeta)\eta) \in \mathrm{N}(f)$, for all $\eta \in \mathcal{H}$ (this remark is simple but essential in this proof), one concludes that

$$\langle \zeta, f(\eta)\zeta - f(\zeta)\eta\rangle = 0, \qquad \forall \eta \in \mathcal{H},$$

that is, $f(\eta) = \langle \overline{f(\zeta)}\zeta, \eta\rangle$. Therefore, $f = \gamma(\overline{f(\zeta)}\zeta)$. □

Exercise 1.1.42. If $f \in \mathcal{H}^*$, what is the dimension of $\mathrm{N}(f)^\perp$?

Example 1.1.43. The hypothesis that the inner product space is complete can not be dispensed with in Theorem 1.1.40. Consider the subspace $\mathcal{N}$ of $l^2(\mathbb{N})$ whose elements have just a finite number of nonzero entries; then $f : \mathcal{N} \to \mathbb{F}$, $f(\eta) = \sum_{j=1}^{\infty} \eta_j/j$, belongs to $\mathcal{N}^*$, but there is no $\xi \in \mathcal{N}$ with $f = f_\xi$, since the vector $(1, 1/2, 1/3, \dots) \notin \mathcal{N}$.

Now a simple and useful technical result, although it is restricted to complex inner product spaces, as illustrated by Example 1.1.45.

Lemma 1.1.44. *Let $(X, \langle\cdot,\cdot\rangle)$ be a complex inner product space. If $T : X \hookleftarrow$ is a linear operator and $\langle T\xi, \xi\rangle = 0$ for all $\xi \in X$, then $T = 0$. Hence, if T, S are linear operators and $\langle T\xi, \xi\rangle = \langle S\xi, \xi\rangle$ for all $\xi \in X$, then $T = S$.*

Proof. For all $\alpha \in \mathbb{C}$ and any $\xi, \eta \in X$ one has

$$0 = \langle T(\alpha\xi + \eta), \alpha\xi + \eta\rangle = \bar{\alpha}\langle T\xi, \eta\rangle + \alpha\langle T\eta, \xi\rangle.$$

By picking, successively, $\alpha = 1$ and $\alpha = -i$ one obtains

$$\langle T\xi, \eta\rangle + \langle T\eta, \xi\rangle = 0 \text{ and } \langle T\xi, \eta\rangle - \langle T\eta, \xi\rangle = 0,$$

whose unique solution is $\langle T\xi, \eta\rangle = 0$, for all $\xi, \eta \in X$, that is, T is the zero operator. □

Example 1.1.45. Consider the rotation R by the right angle on $\mathbb{R}^2$, so that $R \neq 0$ while $\langle R\xi, \xi\rangle = 0$, $\forall\xi \in \mathbb{R}^2$.

Before closing this section, recall the parallelogram law

$$\|\xi + \eta\|^2 + \|\xi - \eta\|^2 = 2\|\xi\|^2 + 2\|\eta\|^2, \qquad \forall\xi, \eta \in X$$

as well as the polarization identity

$$\langle \xi, \eta\rangle = \frac{1}{4}\left(\|\xi + \eta\|^2 - \|\xi - \eta\|^2 + i\|\xi + i\eta\|^2 - i\|\xi - i\eta\|^2\right),$$

which hold in any (complex) inner product space.

1.2 Closed Operators

Before discussing closed operators it can be useful to recall the so-called open mapping theorem. A map between topological spaces is open if the image of every open subset is also open. There are invertible continuous maps that are not open, as shown by the following examples.

Example 1.2.1. The identity map between $\mathbb{R}^n$ with the discrete topology and $\mathbb{R}^n$ with the usual topology is continuous and invertible, but its inverse map is not continuous, that is, this bijective continuous map is not open.

Example 1.2.2. Let $X = [-1,0] \cup (1,2]$ in $\mathbb{R}$ and $\psi : X \to [0,4]$, $\psi(t) = t^2$. ψ is a continuous bijection, but its inverse $\psi^{-1} : [0,4] \to X$, given by

$$\psi^{-1}(t) = \begin{cases} -\sqrt{t} & \text{if } 0 \le t \le 1 \\ \sqrt{t} & \text{if } 1 < t \le 4 \end{cases},$$

is not continuous.

Exercise 1.2.3. Show that $T : l^1(\mathbb{N}) \hookleftarrow$ given by $T(\xi_1, \xi_2, \xi_3, \dots) = (\xi_1/1, \xi_2/2, \xi_3/3, \dots)$ is linear, continuous and invertible, but its inverse T^{-1}, defined on the range of T, is not a continuous operator.

Theorem 1.2.4 (Open Mapping). *If $T \in \mathrm{B}(\mathcal{B}_1, \mathcal{B}_2)$ with* rng $T = \mathcal{B}_2$*, then T is an open map.*

Proof. The following properties will be used, and only the last one is not immediate:

- **a)** for all $r, s > 0$ one has $TB(0;r) = \frac{r}{s} TB(0;s)$.
- **b)** for all $\xi \in \mathcal{B}_1$ and $r > 0$, one has $TB(\xi;r) = T\xi + TB(0;r)$ (sum of sets).
- **c)** if $B(0;\varepsilon) \subset \overline{TB(0;r)}$, then $B(0;\alpha\varepsilon) \subset \overline{TB(0;\alpha r)}$, for all $\alpha > 0$. Then if there is $r > 0$ so that $\overline{TB(0;r)}$ contains a neighborhood of the origin, then $\overline{TB(0;s)}$ contains a neighborhood of the origin for all $s > 0$ (note that such implications also hold without closures of the sets).
- **d)** if $B(\eta_0;\varepsilon) \subset \overline{TB(0;r)}$, then there exists $\delta > 0$ so that $B(0;\delta) \subset \overline{TB(0;r)}$ (note that it also holds without closure of the sets).

To prove the last property, pick $\xi_1 \in B(0;r)$ so that $\|\eta_1 - \eta_0\| < \varepsilon/2$, with $\eta_1 = T\xi_1$. Thus,

$$B(\eta_1;\varepsilon/2) \subset B(\eta_0;\varepsilon) \subset \overline{TB(0;r)},$$

and so

$$\begin{aligned} B(0;\varepsilon/2) &= B(\eta_1;\varepsilon/2) - \eta_1 \subset \{B(\eta_0;\varepsilon) - T\xi_1\} \\ &\subset \left\{\overline{TB(0;r)} - T\xi_1\right\} \subset \overline{T[B(0;r) - \xi_1]} \subset \overline{TB(0;2r)}\,. \end{aligned}$$

Then it follows that $B(0;\varepsilon/2) \subset \overline{TB(0;2r)}$ and, therefore, $B(0;\delta) \subset \overline{TB(0;r)}$ with $\delta = \varepsilon/4$, proving **d)**.

Lemma 1.2.5. *If $T \in \mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$ and there exists $r > 0$ so that the interior of $TB(0;r)$ is nonempty, then T is an open map.*

Proof. Since the interior of $TB(0;r) \neq \emptyset$, from the above properties it follows that for all $s > 0$, $TB(0;s)$ contains an open ball centered at the origin. To show that T is an open map it is enough to show that for all $\xi \in \mathcal{N}_1$ and all $s > 0$, $TB(\xi;s)$ contains a neighborhood of $T\xi$. In view of $TB(\xi;s) = T\xi + TB(0;s)$, one may consider $\xi = 0$ and verify that for all $s > 0$ the set $TB(0;s)$ contains a neighborhood of the origin, but this is exactly what was observed at the beginning of this proof. □

By this lemma, to prove the open mapping theorem it is enough to verify that there exists some $r > 0$ so that $TB(0;r)$ contains an open ball centered at the origin. Note that only from this point will the completeness of $\mathcal{B}_1, \mathcal{B}_2$ and that T is onto be used; the Baire theorem will be crucial.

Since T is onto $\mathcal{B}_2 = \bigcup_{n=1}^{\infty} \overline{TB(0;n)}$, and by the Baire theorem there is some m so that the interior of $\overline{TB(0;m)}$ is nonempty. By property **c)** it is possible to take $m = 1$.

By property **d)** one may suppose that there is $\delta > 0$ so that $B(0;\delta) \subset \overline{TB(0;1)}$. The goal now is to show that the relation $\overline{TB(0;1)} \subset TB(0;2)$ holds, which, by Lemma 1.2.5, proves the theorem.

Let $\eta \in \overline{TB(0;1)}$. Pick $\xi_1 \in B(0;1)$ with

$$(\eta - T\xi_1) \in B(0;\delta/2) \subset \overline{TB(0;1/2)}.$$

In the last step property **c)** was invoked. Pick now ξ_2 in $B(0;1/2)$ so that (again by **c)**)

$$(\eta - T\xi_1 - T\xi_2) \in B(0;\delta/2^2) \subset \overline{TB(0;1/2^2)}.$$

By induction, pick $\xi_n \in B(0;1/2^{n-1})$ satisfying

$$\left(\eta - \sum_{j=1}^{n} T\xi_j\right) \in B(0;\delta/2^n) \subset \overline{TB(0;1/2^n)}.$$

$(\sum_{j=1}^{n} \xi_j)_n$ is a Cauchy sequence and, since $\mathcal{B}_1$ is complete, there exists $\xi = \sum_{j=1}^{\infty} \xi_j$ and, by the continuity of the map T it follows that $\eta = T\xi$. Since $\|\xi\| < 2$, one gets $\overline{TB(0;1)} \subset TB(0;2)$. □

By the open mapping theorem the next result is evident; it is sometimes called the *inverse mapping theorem*.

Corollary 1.2.6. *If $T \in \mathrm{B}(\mathcal{B}_1, \mathcal{B}_2)$ is a bijection between $\mathcal{B}_1$ and $\mathcal{B}_2$, then T^{-1} is also a linear continuous map.*

Recall that the cartesian product $\mathcal{N}_1 \times \mathcal{N}_2$ of two normed spaces has a natural structure of vector space given by $\alpha(\xi,\eta) = (\alpha\xi, \alpha\eta)$, $\alpha \in \mathbb{F}$, and $(\xi_1,\eta_1)+(\xi_2,\eta_2) = (\xi_1+\xi_2, \eta_1+\eta_2)$; furthermore, this cartesian product becomes a normed space with the norm $\|(\xi,\eta)\| = (\|\xi\|^2_{\mathcal{N}_1} + \|\eta\|^2_{\mathcal{N}_2})^{\frac{1}{2}}$; such a norm is equivalent to $\|\xi\|_{\mathcal{N}_1} + \|\eta\|_{\mathcal{N}_2}$ and both may be employed.

Definition 1.2.7. The *graph* of a linear operator $T : \mathrm{dom}\, T \subset \mathcal{N}_1 \to \mathcal{N}_2$ is the vector subspace $\mathcal{G}(T) = \{(\xi, T\xi) : \xi \in \mathrm{dom}\, T\}$ of $\mathcal{N}_1 \times \mathcal{N}_2$. The *graph norm* of T on $\mathrm{dom}\, T$ is $\|\xi\|_T := \left(\|T\xi\|^2 + \|\xi\|^2\right)^{1/2}$.

Definition 1.2.8. A linear operator $T : \mathrm{dom}\, T \subset \mathcal{N}_1 \to \mathcal{N}_2$ is *closed* if for all convergent sequences $(\xi_n) \subset \mathrm{dom}\, T$, $\xi_n \to \xi \in \mathcal{N}_1$, with $(T\xi_n) \subset \mathcal{N}_2$ also convergent, $T\xi_n \to \eta$, then $\xi \in \mathrm{dom}\, T$ and $\eta = T\xi$. In other words, T is closed iff $\mathcal{G}(T)$ is a closed subspace of $\mathcal{N}_1 \times \mathcal{N}_2$.

Exercise 1.2.9. a) Show that $\mathcal{B}_1 \times \mathcal{B}_2$ with the norm $\|(\xi,\eta)\|$ defined above is a Banach space. b) Show that T is a closed operator iff dom T with the graph norm is a Banach space.

Exercise 1.2.10. Verify that $\mathcal{G}(T)$ is a vector subspace of $\mathcal{N}_1 \times \mathcal{N}_2$ and the equivalence quoted in the above definition of closed operator.

Remark 1.2.11. Pay attention to the difference between a continuous and a closed operator: a linear operator T is continuous if for $\xi_n \to \xi$ in dom T, then necessarily $T\xi_n \to T\xi$, while for a closed operator it is asked that if both $(\xi_n) \subset$ dom T and $(T\xi_n)$ are convergent, then necessarily $\xi = \lim_n \xi_n$ belongs to dom T and $T\xi_n \to T\xi$.

Exercise 1.2.12. Consider the linear operator $T : \mathrm{dom}\, T \subset \mathcal{N}_1 \to \mathcal{N}_2$, and let $\pi_1 : \mathcal{G}(T) \to \mathrm{dom}\, T$ and $\pi_2 : \mathcal{G}(T) \to \mathrm{rng}\ T$ be the natural projections $\pi_1(\xi, T\xi) = \xi$ and $\pi_2(\xi, T\xi) = T\xi$, for $\xi \in \mathrm{dom}\, T$. Show that such projections are continuous linear operators.

It is important to give conditions to guarantee that closed operators are continuous, since the requirement for being closed is in general easier to verify; the closed graph theorem, presented below, says that such concepts are equivalent for linear operators between Banach spaces.

A first result in this direction appears in:

Proposition 1.2.13. *Any operator* $T \in \mathrm{B}(\mathcal{B}_1, \mathcal{B}_2)$ *is closed.*

Proof. Let $\xi_n \to \xi$ with $T\xi_n \to \eta$. Since $\xi \in \mathrm{dom}\, T$ and T is continuous, then $T\xi_n \to T\xi = \eta$; thus T is closed. □

Exercise 1.2.14. If $\dim \mathcal{N}_1 < \infty$, show that every linear operator $T : \mathrm{dom}\, T \subset \mathcal{N}_1 \to \mathcal{N}_2$ is closed.

Example 1.2.15 (Bounded and nonclosed). Let $\mathbf{1} : \mathrm{dom}\, \mathbf{1} \to \mathcal{B}$, with dom $\mathbf{1}$ a proper dense subspace of $\mathcal{B}$, the identity operator $\mathbf{1}(\xi) = \xi$ for $\xi \in \mathrm{dom}\, \mathbf{1}$; such operator is bounded. Let $(\xi_n) \subset \mathrm{dom}\, \mathbf{1}$ with $\xi_n \to \xi \in \mathcal{B} \backslash \mathrm{dom}\, \mathbf{1}$. Since $\xi_n \to \xi$ and $\mathbf{1}(\xi_n) \to \xi$, but $\xi \notin \mathrm{dom}\, \mathbf{1}$, this operator is not closed. It is a rather artificial example, but it illustrates the difference between bounded and closed linear operators.

Exercise 1.2.16. If $\mathcal{N} \subset \mathcal{B}$, show that $T \in \mathrm{B}(\mathcal{N}, \mathcal{B})$ is closed if, and only if, $\mathcal{N}$ is a Banach space.

Remark 1.2.17. If $T \in \mathrm{B}(\mathcal{N}_1, \mathcal{B}_2)$ with $\mathcal{N}_1 \subset \mathcal{B}_1$, then its unique continuous linear extension $\overline{T} : \overline{\mathcal{N}_1} \to \mathcal{B}_2$ is a closed operator (Proposition 1.2.13). Then, every continuous linear operator is "basically" closed, and the artificiality in Example 1.2.15 is unavoidable.

Example 1.2.18 (Unbounded and closed). Let $C^1[0,\pi] \subset C[0,\pi]$ (both with the uniform convergence topology) be the subspace of continuously differentiable functions on $[0,\pi]$ and $D : C^1[0,\pi] \to C[0,\pi]$, $(D\psi)(t) = \psi'(t)$. D is not continuous, since the sequence $\psi_n(t) = \sin(nt)/n \to 0$, while $(D\psi_n)(t) = \cos(nt)$ does not converge uniformly to zero. However, this operator is closed. In fact, if $\psi_n \to \psi$ and $D\psi_n = \psi_n' \to \varphi$, then, as these limits are uniform,

$$\int_0^t \varphi(s)\,ds = \int_0^t \lim_{n\to\infty} \psi_n'(s)\,ds = \lim_{n\to\infty} \int_0^t \psi_n'(s)\,ds = \psi(t) - \psi(0).$$

Thus, $\psi \in \operatorname{dom} D = C^1[0,\pi]$ and $(D\psi)(t) = \varphi(t)$, $\forall t$, and D is closed.

Exercise 1.2.19. From Example 1.2.18, show that if $(\psi_j)_{j=1}^\infty \subset C^1[0,\pi]$ is such that the series $\psi(t) = \sum_{j=1}^\infty \psi_j(t)$ and $\varphi(t) = \sum_{j=1}^\infty \psi_j'(t)$ converge uniformly, then ψ is continuously differentiable and $\varphi = \psi'$.

Example 1.2.20 (Unbounded and nonclosed). Let dom T be the set of continuous functions in $\mathrm{L}^1[-1,1]$ and $(T\psi)(t) = \psi(0)$, $\forall t$, as element of $\mathrm{L}^1[-1,1]$. This operator is neither continuous nor closed, since $\psi_n(t) = e^{-|t|n} \to 0$ in $\mathrm{L}^1[-1,1]$, while $(T\psi_n)(t) = 1$, $\forall t$, for all n. Note that it has no closed extensions.

Theorem 1.2.21 (Closed Graph). *If $T : \mathcal{B}_1 \to \mathcal{B}_2$ is a linear operator, then T is continuous if, and only if, T is closed.*

Proof. One of the assertions of the closed graph theorem was already discussed; it is only needed to show that, under such conditions, if the linear operator T is closed, then it is bounded; the open mapping theorem will be used.

By hypotheses $\mathcal{G}(T)$ is closed in $\mathcal{B}_1 \times \mathcal{B}_2$, then $\mathcal{G}(T)$ is also a Banach space. The projection operators π_1 and π_2 (see Exercise 1.2.12) are both linear and continuous. Moreover, π_1 is a bijection between the Banach spaces $\mathcal{G}(T)$ and $\mathcal{B}_1$; thus, by the open mapping theorem, its inverse $\pi_1^{-1} : \mathcal{B}_1 \to \mathcal{G}(T)$ is continuous. Since T is the composition

$$T = \pi_2 \circ \pi_1^{-1},$$

it follows that it is a bounded operator. □

Example 1.2.22 (Unbounded and closed). It is essential that the operator range is a complete space. The operator $T^{-1} : \operatorname{rng} T \to l^1(\mathbb{N})$ in Exercise 1.2.3 has closed graph but is not continuous.

Remark 1.2.23. One could imagine that a linear operator is not closed because its domain was chosen too small, and by considering the closure $\overline{\mathcal{G}(T)}$ in $\mathcal{N}_1 \times \mathcal{N}_2$ a closed operator would result. This may not work, since $\overline{\mathcal{G}(T)}$ is not necessarily the graph of an operator; see Example 1.2.20 where the point $(0,1)$ belongs to $\overline{\mathcal{G}(T)}$, however it is not of the form $(0, S0)$ for any linear operator S.

Exercise 1.2.24. Let E be a subspace of $\mathcal{N}_1 \times \mathcal{N}_2$. Show that E is the graph of a linear operator if, and only if, E does not contain any element of the form $(0,\eta)$, with $\eta \neq 0$.

Definition 1.2.25.

(a) The linear operators T, for which $\overline{\mathcal{G}(T)}$ is the graph of a linear extension $\overline{T}$ of T, are called *closable operators* and $\overline{T}$ is the *closure* of T (see Proposition 1.2.27).

(b) If the operator $T : \overline{\text{dom}}\ T \sqsubseteq \mathcal{N}_1 :\rightarrow \mathcal{N}_2$ is closed, a subspace $\mathcal{D} \subset \text{dom}\ T$ is called a *core* of T if $\overline{T|_{\mathcal{D}}} = T$, that is, if the closure of the restriction $T|_{\mathcal{D}}$ is T.

Exercise 1.2.26. Show that X is a core of the closed operator T iff $\{(\xi, T\xi) : \xi \in X\}$ is dense in $\mathcal{G}(T)$.

If the linear operator $T : \text{dom}\ T \subset \mathcal{N}_1 \rightarrow \mathcal{N}_2$ is closable, then

$$\mathcal{D} = \{\xi \in \mathcal{N}_1 : \exists(\xi_n) \subset \text{dom}\ T, \xi_n \rightarrow \xi \text{ and exists } \eta \in \mathcal{N}_2 \text{ with } T\xi_n \rightarrow \eta\}$$

is a subset of all closed extensions of T. Define dom $\tilde{T} = \mathcal{D}$ and, for $\xi \in \mathcal{D}$, $\tilde{T}\xi := \eta$, and note that, by construction, $\mathcal{G}(\tilde{T})$ is closed in $\mathcal{N}_1 \times \mathcal{N}_2$, and so $\tilde{T}$ is closed. Note also that $\mathcal{G}(\tilde{T}) = \overline{\mathcal{G}(T)}$. Therefore $\tilde{T}$ is the closure of T, that is, $\tilde{T} = \overline{T}$. In summary:

Proposition 1.2.27. *If $T : \text{dom}\ T \subset \mathcal{N}_1 \rightarrow \mathcal{N}_2$ is closable, then $\overline{\mathcal{G}(T)}$ is the graph of its closure $\overline{T}$, which is the smallest closed extension of T.*

Exercise 1.2.28. Show that T is a closed operator acting in $\mathcal{H}$ iff dom T with the *graph inner product of* T, given by $\langle \eta, \xi \rangle_T := \langle T\eta, T\xi \rangle + \langle \eta, \xi \rangle$, is a Hilbert space. This inner product generates a graph norm (Definition 1.2.7) and the corresponding orthogonality will be denoted by $\perp_T$.

1.3 Compact Operators

The compact operators have some similarities with operators on finite-dimensional spaces and so the theory presents several technical simplifications. These operators are important in many applications, sometimes as integral operators, a historically important example of compact operator.

It is convenient to recall some definitions and properties – in the form of exercises – of metric spaces theory. A set A in the metric space (X, d) is *relatively compact*, or *precompact*, if its closure $\overline{A}$ is compact. A is *totally bounded* if, for all $\varepsilon > 0$, A is in the finite union of open balls in X with radii ε; so, any totally bounded set is also bounded.

Exercise 1.3.1. Show that if $A \subset (X, d)$ is precompact, then A is totally bounded and, so, bounded.

Exercise 1.3.2. If $A \subset (X, d)$ is totally bounded, show that, for all $\varepsilon > 0$, A is in the union of a finite number of open balls of radii ε centered at points of A. Conclude then that a totally bounded set is separable with the induced topology, that is, it contains a countable dense subset.

Lemma 1.3.3. *Any totally bounded subset of a complete metric space is precompact.*

Proof. Let A be a totally bounded set; then its closure is also totally bounded (from a cover of balls, the family of balls with the same centers but with double radii covers the closure of the set). Since this set is in a complete metric space, to show that its closure is compact it is enough to check that every sequence $(\xi_n) \subset \overline{A}$ has a Cauchy subsequence. Such a set being totally bounded, there is a subsequence $(\xi_{1,n})$ of (ξ_n) contained in an open ball of radius 1. In the same way, there exists a subsequence $(\xi_{2,n})$ of $(\xi_{1,n})$ contained in an open ball of radius 1/2; it is possible to construct subsequences $(\xi_{k,n})_{n\geq 1}$ of $(\xi_{k-1,n})_{n\geq 1}$ contained in some open ball of radius $1/k$, for all $k \in \mathbb{N}$. To finish the proof note that $(\xi_{k,k})_{k\geq 1}$ is a Cauchy subsequence of the original sequence. □

Definition 1.3.4. A linear operator $T : \mathcal{N}_1 \to \mathcal{N}_2$ is *compact*, also called *completely continuous*, if the range $T(A)$, of any bounded set $A \subset \mathcal{N}_1$ is precompact in $\mathcal{N}_2$. The set of such compact operators will be denoted by $\mathrm{B}_0(\mathcal{N}_1, \mathcal{N}_2)$ (or $\mathrm{B}_0(\mathcal{N})$ in case $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}$).

Remark 1.3.5. Equivalently, $T : \mathcal{N}_1 \to \mathcal{N}_2$ linear is compact if $(T\xi_n)$ has a convergent subsequence in $\mathcal{N}_2$ for every bounded sequence $(\xi_n) \subset \mathcal{N}_1$. Verify this!

Exercise 1.3.6. If $\dim \mathcal{N} = \infty$, show that the identity operator $\mathbf{1} : \mathcal{N} \hookleftarrow$ is not compact (use, for instance, Riesz's Lemma 1.6.2).

Proposition 1.3.7. *Let $\mathcal{N}_1, \mathcal{N}_2$ be normed spaces and $T, S : \mathcal{N}_1 \to \mathcal{N}_2$ linear operators. Then:*

i) $\mathrm{B}_0(\mathcal{N}_1, \mathcal{N}_2)$ *is a vector subspace of* $\mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$.

ii) *If T is compact and S bounded, then TS and ST are compact operators* (*suppose all operations are well posed*).

Proof. i) Let $T \in \mathrm{B}_0(\mathcal{N}_1, \mathcal{N}_2)$; since $T(S(0;1))$ is precompact, it is bounded. Thus, $T \in \mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$. The proof that $\mathrm{B}_0(\mathcal{N}_1, \mathcal{N}_2)$ is a vector subspace is left to the readers.

ii) If A is a bounded set, then $S(A)$ is also bounded and, so, $T(S(A))$ is precompact. Therefore, TS is compact.

Given a bounded set A, the range by T of any sequence $(\xi_n) \subset A$ has a convergent subsequence $(T\xi_{n_j})$, since T is compact. S being continuous, $(ST\xi_{n_j})$ is also convergent. Therefore, $ST(A)$ is precompact and ST is a compact operator. □

Remark 1.3.8. A map between metric spaces is *compact* if the range of bounded sets is precompact; the Dirichlet function $h : \mathbb{R} \to \mathbb{R}$, $h(t) = 1$ if $t \in \mathbb{Q}$ and $h(t) = 0$ otherwise, is compact, but not continuous in any point of its domain (cf. Proposition 1.3.7).

Important examples of compact operators are the finite-rank operators.

Definition 1.3.9. $T \in \mathrm{B}(\mathcal{N}_1, \mathcal{N}_2)$ is of *finite rank* if $\dim \operatorname{rng} T < \infty$. The vector space of finite rank operators between these spaces will be denoted by $\mathrm{B}_\mathrm{f}(\mathcal{N}_1, \mathcal{N}_2)$ (it will also be used the obvious notation $\mathrm{B}_\mathrm{f}(\mathcal{N})$).

Proposition 1.3.10. *All finite rank operators are compact. In particular* $\mathcal{N}^* = \mathrm{B}_0(\mathcal{N}, \mathbb{F})$.

Proof. Let $T \in \mathrm{B_f}(\mathcal{N}_1, \mathcal{N}_2)$ and $A \subset \mathcal{N}_1$ a bounded set. Since T is a bounded operator, $T(A)$ is bounded and its closure $\overline{T(A)}$ is a closed and bounded set and, in view of $\dim \operatorname{rng} T < \infty$, it follows that $\overline{T(A)}$ is a compact set. □

Lemma 1.3.11. *If $T \in \mathrm{B_0}(\mathcal{N}_1, \mathcal{N}_2)$, then $T(\mathcal{N}_1)$ is separable.*

Proof. Since $\mathcal{N}_1 = \bigcup_{j=1}^{\infty} B(0;j)$, then for $T : \mathcal{N}_1 \to \mathcal{N}_2$, $\operatorname{rng} T = \bigcup_{j=1}^{\infty} T(B(0;j))$. In order to conclude the lemma, it is sufficient to show that for each $j \in \mathbb{N}$ the set $TB(0;j)$ has a countable dense subset. If T is compact, $TB(0;j)$ is totally bounded; thus, for each $m \in \mathbb{N}$ it can be covered by a finite number of open balls of radii $1/m$, centered at points of $TB(0;j)$. The union of the centers of such open balls for all $m \in \mathbb{N}$ is a dense countable set of $TB(0;j)$. □

Exercise 1.3.12. Let $T : \mathcal{N}_1 \to \mathcal{N}_2$ linear. Show that it is compact if, and only if, $TB(0;1)$ is precompact in $\mathcal{N}_2$.

Theorem 1.3.13. $\mathrm{B_0}(\mathcal{N}, \mathcal{B})$ *is a closed subspace of* $\mathrm{B}(\mathcal{N}, \mathcal{B})$*; therefore,* $\mathrm{B_0}(\mathcal{N}, \mathcal{B})$ *is a Banach space.*

Proof. Let $(T_n) \subset \mathrm{B_0}(\mathcal{N}, \mathcal{B})$, with $T_n \to T$ in $\mathrm{B}(\mathcal{N}, \mathcal{B})$. It will be shown that for all $r > 0$ the set $TB(0;r)$ is totally bounded and, therefore, precompact by Lemma 1.3.3. From this it follows that T is also a compact operator.

Given $\varepsilon > 0$, there is n such that $\|T_n - T\| < \varepsilon/r$. Since T_n is compact, the set $T_nB(0;r)$ is totally bounded and, so, it is in the union of certain balls $B(T_n\xi_1;\varepsilon), B(T_n\xi_2;\varepsilon), \ldots, B(T_n\xi_m;\varepsilon)$, with $\xi_j \in B(0;r)$, for all $1 \le j \le m$. Hence, if $\xi \in B(0;r)$ there is one of these ξ_j such that $T_n\xi \in B(T_n\xi_j;\varepsilon)$. From this

$$\begin{aligned} \|T\xi - T\xi_j\| &\le \|T\xi - T_n\xi\| + \|T_n\xi - T_n\xi_j\| + \|T_n\xi_j - T\xi_j\| \\ &< \|T - T_n\|\|\xi\| + \varepsilon + \|T_n - T\|\|\xi_j\| \\ &< \frac{\varepsilon}{r}r + \varepsilon + \frac{\varepsilon}{r}r = 3\varepsilon, \end{aligned}$$

showing that $TB(0;r) \subset \bigcup_{j=1}^{m} B(T_n\xi_j;3\varepsilon)$. Therefore $TB(0;r)$ is totally bounded for all $r > 0$. □

Corollary 1.3.14. *If $(T_n) \subset \mathrm{B_f}(\mathcal{N}, \mathcal{B})$ and $T_n \to T$ in $\mathrm{B}(\mathcal{N}, \mathcal{B})$, then the operator T is compact.*

Proof. Combine Proposition 1.3.10 and Theorem 1.3.13. □

Recall that a sequence $(\xi_n) \subset \mathcal{N}$ *converges weakly* to $\xi \in \mathcal{N}$ if $\lim_{n\to\infty} f(\xi_n) = f(\xi)$ for all $f \in \mathcal{N}^*$, and that all weakly convergent sequences are bounded. $\xi_n \xrightarrow{\mathrm{w}} \xi$ and $\mathrm{w} - \lim \xi_n = \xi$ will be used to indicate that (ξ_n) converges weakly to ξ. The convergence of (ξ_n) to ξ in the norm of $\mathcal{N}$ will be called *strong convergence* and indicated by $\xi_n \to \xi$, $\xi_n \xrightarrow{\mathrm{s}} \xi$ and $\mathrm{s} - \lim \xi_n = \xi$.

There are also corresponding notions of convergence of a sequence (T_n) of bounded operators in $\mathrm{B}(\mathcal{N})$.

Definition 1.3.15. Let (T_n) be a sequence of operators in $B(\mathcal{N}_1, \mathcal{N}_2)$ and $T : \mathcal{N}_1 \to \mathcal{N}_2$ linear. One says that

a) T_n *converges uniformly*, or *in norm*, to T if

$$\|T_n - T\| \to 0.$$

The uniform convergence is denoted by $T_n \to T$ or $\lim_{n\to\infty} T_n = T$.

b) T_n *converges strongly* to T if

$$\|T_n\xi - T\xi\|_{\mathcal{N}_2} \to 0, \qquad \forall \xi \in \mathcal{N}_1.$$

The strong convergence of linear operators will be denoted by $T_n \xrightarrow{\text{s}} T$ or $\text{s} - \lim_{n\to\infty} T_n = T$.

c) T_n *converges weakly* to T if

$$|f(T_n\xi) - f(T\xi)| \to 0, \qquad \forall \xi \in \mathcal{N}_1,\ f \in \mathcal{N}_2^*.$$

The weak convergence of linear operators will be denoted by $T_n \xrightarrow{\text{w}} T$ or $\text{w} - \lim_{n\to\infty} T_n = T$.

Exercise 1.3.16. Show that in $B(\mathcal{N}_1, \mathcal{N}_2)$ the three kinds of limits defined above are well defined and unique (if they exist, of course). Moreover, verify that the uniform convergence $\Longrightarrow$ strong convergence $\Longrightarrow$ weak convergence, and with the same limits.

Example 1.3.17. Let $P_N : l^1(\mathbb{N}) \hookleftarrow$, $P_N\xi = (\xi_1, \xi_2, \dots, \xi_N, 0, 0, \dots)$, with $\xi = (\xi_1, \xi_2, \xi_3, \dots)$. Since $\|P_N\xi - \xi\| = \sum_{j=N+1}^{\infty} |\xi_j|$ it is found that $P_N \xrightarrow{\text{s}} \mathbf{1}$. On the other hand, $\|P_N\xi - \xi\| \le \|\xi\|$ and $\|P_N e_{(N+1)} - e_{(N+1)}\| = \|e_{(N+1)}\| = 1$, $\forall N$, and then (P_N) is not uniformly convergent ((e_j) is the canonical basis of $l^1(\mathbb{N})$). Adapt it to l^p, $1 < p < \infty$.

Exercise 1.3.18. Show that the sequence of operators $T_n : l^2(\mathbb{N}) \hookleftarrow$

$$T_n\xi = (\underbrace{0, 0, \dots, 0}_{n \text{ entries}}, \xi_{n+1}, \xi_{n+2}, \xi_{n+3}, \dots)$$

converges strongly to zero, but does not converge uniformly.

Exercise 1.3.19. Show that the sequence of operators $T_n : l^2(\mathbb{N}) \hookleftarrow$

$$T_n\xi = (\underbrace{0, 0, \dots, 0}_{n \text{ entries}}, \xi_1, \xi_2, \xi_3, \dots)$$

converges weakly to zero, but does not converge strongly.

As a reformulation of the Banach-Steinhaus theorem, one has (by using an obvious generalization of convergence of operators):

Proposition 1.3.20. *If (T_n) in $B(\mathcal{B}, \mathcal{N})$ converges strongly to the operator $T : \mathcal{B} \to \mathcal{N}$, then $T \in B(\mathcal{B}, \mathcal{N})$.*

Note that due to the Riesz representation Theorem 1.1.40, a sequence $(\xi_n) \subset \mathcal{H}$ converges weakly to ξ if, and only if,

$$\lim_{n\to\infty} \langle \eta, \xi_n \rangle = \langle \eta, \xi \rangle, \qquad \forall \eta \in \mathcal{H}.$$

Exercise 1.3.21. Show that every orthonormal sequence in a Hilbert space converges weakly to zero and has no strongly convergent subsequence.

Recall the Hilbert adjoint T^* of a bounded operator $T \in \mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$. It is the unique linear operator so that

$$\langle \xi, T\eta \rangle = \langle T^*\xi, \eta \rangle, \qquad \forall \xi \in \mathcal{H}_2, \eta \in \mathcal{H}_1.$$

Further, $T^* \in \mathrm{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $\|T^*\| = \|T\|$. The bounded linear operator T is self-adjoint if $T^* = T$. See a generalization of the concept of adjoint to certain unbounded operators in Definition 2.1.2. Finally, recall that an operator $P \in \mathrm{B}(\mathcal{H})$ is an *orthogonal projection* if it is self-adjoint and $P^2 = P$, and it projects onto the closed subspace rng P.

Proposition 1.3.22. *Let $T \in \mathrm{B}_0(\mathcal{H}_1, \mathcal{H}_2)$. If $\xi_n \xrightarrow{w} \xi$ in $\mathcal{H}_1$, then $T\xi_n \to T\xi$, i.e., a compact operator takes weakly convergent sequences to strongly convergent ones (this result also holds in normed spaces).*

Proof. Suppose $\xi_n \xrightarrow{w} \xi$ in $\mathcal{H}_1$. If $\eta \in \mathcal{H}_2$,

$$\langle \eta, T\xi_n \rangle = \langle T^*\eta, \xi_n \rangle \to \langle T^*\eta, \xi \rangle = \langle \eta, T\xi \rangle,$$

showing that $T\xi_n \xrightarrow{w} T\xi$. If $T\xi_n$ does not converge strongly to $T\xi$, there exists $\varepsilon > 0$ and a subsequence $(T\xi_{n_j})$ with $\|T\xi_{n_j} - T\xi\| \geq \varepsilon$. Since T is a compact operator, $T\xi_{n_j}$ has the strongly convergent subsequence and, necessarily, it converges to $T\xi$. The contradiction with the above inequality proves the proposition. □

In a Hilbert space the closure (with the usual norm of $\mathrm{B}(\mathcal{H})$) of the vector space of finite-rank operators coincides with the set of compact operators; to show this the following technical result will be useful. Remember that a Hilbert space is separable iff it has a countable orthonormal basis.

Lemma 1.3.23. *If $T \in \mathrm{B}_0(\mathcal{H}_1, \mathcal{H}_2)$, then* rng T *and* $\mathrm{N}(T)^\perp$ *are separable vector spaces.*

Proof. rng T is separable by Lemma 1.3.11. Let $\{e_\alpha\}_{\alpha\in J}$ be an orthonormal basis of $\mathrm{N}(T)^\perp$. If J is finite the result is clear.

Suppose that J is not finite; the goal is to show that J is enumerable. Every sequence $(e_{\alpha_j})_{j=1}^\infty$ of pairwise distinct elements of $\{e_\alpha\}_{\alpha\in J}$ weakly converges to zero (Exercise 1.3.21) and, by Proposition 1.3.22, $Te_{\alpha_j} \to 0$, for $j \to \infty$. Thus, for each $n \in \mathbb{N}$ there exists only a finite number of $\alpha \in J$ with $\|Te_\alpha\| \geq 1/n$. Hence, J is enumerable, for

$$J = \bigcup_{n=1}^\infty \{\alpha : \|Te_\alpha\| \geq 1/n\}.$$

Recall that $Te_\alpha \neq 0$, $\forall \alpha \in J$, since $e_\alpha \in \mathrm{N}(T)^\perp$. □

Remark 1.3.24. If $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a finite rank operator of rank $N < \infty$, then there exist vectors $\xi_1, \eta_1, \ldots, \xi_N, \eta_N$ so that

$$T\xi = \sum_{j=1}^{N} \langle \eta_j, \xi \rangle \, \xi_j,$$

the so-called *canonical form* of T. Indeed, if $\{\xi_1, \ldots, \xi_N\}$ is an orthonormal basis of rng T, then

$$T\xi = \sum_{j=1}^{N} \langle \xi_j, T\xi \rangle \, \xi_j = \sum_{j=1}^{N} \langle T^* \xi_j, \xi \rangle \, \xi_j;$$

now put $\eta_j = T^* \xi_j$.

Theorem 1.3.25. *An operator $T \in \mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$ is compact if, and only if, there is a sequence of finite rank operators $(T_n) \subset \mathrm{B_f}(\mathcal{H}_1, \mathcal{H}_2)$, which converges to T in $\mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof. If T is the limit of finite-rank operators, then T is compact by Corollary 1.3.14. Let $T \in \mathrm{B}_0(\mathcal{H}_1, \mathcal{H}_2)$ and P the orthogonal projection on $\mathrm{N}(T)^\perp$, so that $T = TP$. If $\dim \mathrm{N}(T)^\perp < \infty$ the result is clear; suppose then that $\dim \mathrm{N}(T)^\perp = \infty$ and pick an orthonormal basis $(e_j)_{j=1}^\infty$ of $\mathrm{N}(T)^\perp$, which is enumerable by Lemma 1.3.23. Denote by P_n the orthogonal projection on $\mathrm{Lin}(\{e_1, \ldots, e_n\})$. Thus, the operator $T_n = TP_n$ has finite rank. It will be shown that $T_n \to T$.

For each n there exists $\xi_n \in \mathcal{H}_1$, $\|\xi_n\| = 1$, with

$$\frac{1}{2}\|T - T_n\| \le \|(T - T_n)\xi_n\| = \|T(P - P_n)\xi_n\|.$$

Since $(P_n - P) \xrightarrow{\text{s}} 0$ and for all $\eta \in \mathcal{H}_1$,

$$|\langle \eta, (P - P_n)\xi_n \rangle| = |\langle (P - P_n)\eta, \xi_n \rangle| \le \|(P - P_n)\eta\|,$$

then $(P - P_n)\xi_n \xrightarrow{\text{w}} 0$. Since T is a compact operator, by Proposition 1.3.22 it follows that $T(P - P_n)\xi_n \to 0$ and, by the inequality above, it is found that $\|T - T_n\| \to 0$. □

Exercise 1.3.26. Let $T \in \mathrm{B}(\mathcal{H})$, with $\mathcal{H}$ separable. Show that there is a sequence (T_n) of finite rank operators which converges strongly to T, that is, $T_n \xrightarrow{\text{s}} T$.

Corollary 1.3.27. *Let $T \in \mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then T is compact if, and only if, its Hilbert adjoint T^* is compact.*

Proof. T is compact if, and only if, there exists a sequence $(T_n) \subset \mathrm{B_f}(\mathcal{H}_1, \mathcal{H}_2)$ so that $T_n \to T$. Since T_n^* has also finite rank and $\|T^* - T_n^*\| = \|(T - T_n)^*\| = \|T - T_n\|$, one concludes that T is compact if, and only if, T^* is compact. □

Proposition 1.3.28. *Let T be an operator in* $\mathrm{B}(\mathcal{H})$. *Then T is compact if, and only if, $(T\xi_n)$ is convergent in $\mathcal{H}$ for all weakly convergent sequences (ξ_n).*

Proof. If $\dim \mathcal{H} < \infty$ the proof is quite simple. Suppose that $\dim \mathcal{H} = \infty$. Taking into account the hypotheses and Proposition 1.3.22, it is enough to show that for each bounded sequence (ξ_n) in $\mathcal{H}$ the sequence $(T\xi_n)$ has a convergent subsequence. Since in a Hilbert space any bounded set has a weakly convergent sequence, (ξ_n) has a weakly convergent subsequence (ξ_{n_j}); by hypothesis, $(T\xi_{n_j})$ is convergent. Thus, the image of every bounded sequence admits a convergent subsequence, and so, T is a compact operator. □

Proposition 1.3.29. *Let $S_n, S \in \mathrm{B}(\mathcal{H})$ with $S_n \xrightarrow{\mathrm{s}} S$. If T is a compact operator, then $TS_n \to TS$ and $S_nT \to ST$ in the norm of* $\mathrm{B}(\mathcal{H})$.

Proof. By considering $S_n - S$ it is possible to suppose that $S = 0$. Since $\|T^* S_n^*\| = \|S_n T\|$, by Corollary 1.3.27, it is enough to prove that $S_n T \to 0$ uniformly. For each $\varepsilon > 0$ there is an operator $F_\varepsilon \in \mathrm{B_f}(\mathcal{H})$ so that $T = T_\varepsilon + F_\varepsilon$, and $\|T_\varepsilon\| < \varepsilon$. The last preparatory remark is that there exists $M > 0$ so that $\sup_n \|S_n\| \leq M$, a consequence of the Banach-Steinhaus theorem.

In view of

$$\begin{aligned}\|S_n T\| &\leq \|S_n(F_\varepsilon + T_\varepsilon)\| \\ &\leq \|S_n F_\varepsilon\| + \|T_\varepsilon\|\,\|S_n\| \\ &\leq \|S_n F_\varepsilon\| + \varepsilon\, M,\end{aligned}$$

it is sufficient to prove that $\|S_n F_\varepsilon\| \leq \varepsilon$ if n is large enough.

Write $F_\varepsilon(\cdot) = \sum_{j=1}^k \langle \eta_j, \cdot\rangle \xi_j$, $\eta_j \neq 0$. If $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ one has

$$\|S_n F_\varepsilon \xi\| \leq \sum_{j=1}^k |\langle \eta_j, \xi\rangle|\, \|S_n \xi_j\| \leq \sum_{j=1}^k \|\eta_j\|\, \|S_n \xi_j\|$$

and since $S_n \xrightarrow{\mathrm{s}} 0$ if n is large $\|S_n \xi_j\| < \varepsilon/(\|\eta_j\| k)$, $1 \leq j \leq k$. Thus, as required, $\|S_n F_\varepsilon\| \leq \varepsilon$ for n large enough. Thereby the proof of the proposition is complete. □

Example 1.3.30. Let $K : Q \to \mathbb{F}$ be continuous, with $Q = [a, b] \times [a, b]$. Then the integral operator $T_K : \mathrm{L}^2[a, b] \hookleftarrow$ given by

$$(T_K \psi)(t) = \int_a^b K(t, s)\psi(s)\, ds, \qquad \psi \in \mathrm{L}^2[a, b],$$

is compact.

Proof. For each $t \in [a,b]$ the function $s \mapsto K(t,s)$ is an element of $\mathrm{L}^2[a,b]$. Let $\psi \in B(0;R) \subset \mathrm{L}^2[a,b]$ and $M = \max_{(t,s)\in Q} |K(t,s)|$. For all $t \in [a,b]$ one has

$$|(T_K\psi)(t)| \leq \int_a^b |K(t,s)||\psi(s)|\,ds$$
$$\leq \left(\int_a^b |K(t,s)|^2\,ds\right)^{\frac{1}{2}} \|\psi\|_2 \leq M\sqrt{b-a}R,$$

and $T_K B(0;R)$ is a bounded set. This set is also equicontinuous, since for $\psi \in B(0;R)$,

$$|(T_K\psi)(t) - (T_K\psi)(r)| \leq \|K(t,\cdot) - K(r,\cdot)\|_2 \|\psi\|_2 \leq \varepsilon\sqrt{b-a}R,$$

if $|t-r| < \delta$. Hence, by the Ascoli theorem, $T_K B(0;R)$ is precompact in $(C[a,b], \|\cdot\|_\infty)$. Since $\|\phi\|_2 \leq \sqrt{b-a}\|\phi\|_\infty$, for all continuous ϕ (especially for $\phi = T_K\psi$), then $T_K B(0;R)$ is precompact in $\mathrm{L}^2[a,b]$. □

Exercise 1.3.31. Show that a precompact set (compact) in $(C[a,b], \|\cdot\|_\infty)$ is precompact (compact) in $\mathrm{L}^2[a,b]$. This occurs because the identity map $\mathbf{1} : (C[a,b], \|\cdot\|_\infty) \to \mathrm{L}^2[a,b]$ is continuous.

Example 1.3.32. Let $K \in \mathrm{L}^2(Q)$, with $Q = [a,b]\times[a,b]$. Then the integral operator $T_K : \mathrm{L}^2[a,b] \hookleftarrow$ given by $(T_K\psi)(t) = \int_a^b K(t,s)\psi(s)ds$, for $\psi \in \mathrm{L}^2[a,b]$, is compact.

Proof. Since the set of continuous functions on Q is dense in $\mathrm{L}^2(Q)$, there exists a sequence $K_n : Q \to \mathbb{F}$ of continuous functions so that $\|K - K_n\|_{\mathrm{L}^2(Q)} \to 0$. Thus, by defining $T_n : \mathrm{L}^2[a,b] \hookleftarrow$,

$$(T_n\psi)(t) = \int_a^b K_n(t,s)\psi(s)\,ds, \qquad \psi \in \mathrm{L}^2[a,b],$$

and using estimates similarly to those in preceding examples, one obtains $\|T_n\psi - T_K\psi\|_2 \leq \|K_n - K\|_{\mathrm{L}^2(Q)}\|\psi\|_2$, and $\|T_n - T_K\| \leq \|K_n - K\|_{\mathrm{L}^2(Q)}$, which vanishes as $n \to \infty$. By Example 1.3.30 each T_n is a compact operator, and so T_K is compact (Theorem 1.3.13). □

1.4 Hilbert-Schmidt Operators

One of the most important classes of compact operators on Hilbert spaces is constituted by the Hilbert-Schmidt operators, discussed in this section. Sometimes the shortest way to show that an operator on a Hilbert space is compact is to verify that it is Hilbert-Schmidt.

Definition 1.4.1. An operator $T \in \mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$ is *Hilbert-Schmidt* if there is an orthonormal basis $\{e_j\}_{j\in J}$ of $\mathcal{H}_1$ with

$$\|T\|_{\mathrm{HS}} := \left(\sum_{j\in J} \|Te_j\|^2\right)^{\frac{1}{2}} < \infty.$$

The set of Hilbert-Schmidt operators between such Hilbert spaces will be denoted by $\mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ or, briefly, by $\mathrm{HS}(\mathcal{H})$ if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$.

Proposition 1.4.2. *Let* $T \in \mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$. *Then*

i) $\|T\|_{\mathrm{HS}}$ *does not depend on the orthonormal basis considered.*
ii) $T \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ *if, and only if, its adjoint* $T^* \in \mathrm{HS}(\mathcal{H}_2, \mathcal{H}_1)$. *Furthermore,* $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$.

Proof. If $\{e_j\}_{j\in J}$ and $\{f_k\}_{k\in K}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then, by Parseval,

$$\sum_{j\in J} \|Te_j\|^2 = \sum_{\substack{j\in J\\k\in K}} |\langle Te_j, f_k\rangle|^2 = \sum_{\substack{j\in J\\k\in K}} |\langle e_j, T^*f_k\rangle|^2 = \sum_{k\in K} \|T^*f_k\|^2.$$

Since such orthonormal bases are arbitrary $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$, and such values do not depend on the orthonormal bases considered. □

Corollary 1.4.3. *Let* S, T *be bounded operators between two Hilbert spaces. If one of them is Hilbert-Schmidt, then the product* TS *is also Hilbert-Schmidt* (*assuming the product is defined*).

Proof. If S is Hilbert-Schmidt, then for any orthonormal basis $\{e_j\}_{j\in J}$ of its domain

$$\|TS\|^2_{\mathrm{HS}} = \sum_{j\in J} \|TSe_j\|^2 \le \|T\|^2 \sum_{j\in J} \|Se_j\|^2 = \|T\|^2 \|S\|^2_{\mathrm{HS}},$$

and TS is Hilbert-Schmidt.

If the operator T is Hilbert-Schmidt, then by Proposition 1.4.2, one has that S^*T^* is Hilbert-Schmidt. Since $TS = (S^*T^*)^*$, then TS is Hilbert-Schmidt. □

Theorem 1.4.4. $\mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ *is a vector subspace of* $\mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$, *it is a Hilbert space with the norm* $\|\cdot\|_{\mathrm{HS}}$, *which is called Hilbert-Schmidt norm, and it is induced by the* (*Hilbert-Schmidt*) *inner product*

$$\langle T, S\rangle_{\mathrm{HS}} := \sum_{j\in J} \langle Te_j, Se_j\rangle, \qquad T, S \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2),$$

with $\{e_j\}_{j\in J}$ *being any orthonormal basis of* $\mathcal{H}_1$. *Furthermore, the inequality* $\|T\| \le \|T\|_{\mathrm{HS}}$ *holds.*

Proof. If $T, S \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$, then for any orthonormal basis $\{e_j\}_{j\in J}$ of $\mathcal{H}_1$ and all $\alpha \in \mathbb{F}$ one has (by Cauchy-Schwarz applied to the inner product $\sum_{j\in J} \|Te_j\|\|Se_j\|$ in l^2)

$$\begin{aligned}\|T + \alpha S\|_{\mathrm{HS}}^2 &\le \sum_{j\in J} \|Te_j\|^2 + |\alpha|^2 \sum_{j\in J} \|Se_j\|^2 + 2|\alpha| \sum_{j\in J} \|Te_j\|\|Se_j\| \\ &\le \left(\|T\|_{\mathrm{HS}} + |\alpha|\, \|S\|_{\mathrm{HS}}\right)^2,\end{aligned}$$

and so $\mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ is a vector space. From the same inequality it follows that $\|\cdot\|_{\mathrm{HS}}$ is a norm.

Now it will be verified that $\langle T, S\rangle_{\mathrm{HS}}$ is well posed and is independent of the orthonormal basis considered. By Cauchy-Schwarz

$$\begin{aligned}\sum_{j\in J} |\langle Te_j, Se_j\rangle| &\le \sum_{j\in J} \|Te_j\|\|Se_j\| \\ &\le \left(\sum_{j\in J} \|Te_j\|^2\right)^{\frac{1}{2}} \left(\sum_{j\in J} \|Se_j\|^2\right)^{\frac{1}{2}} \\ &= \|T\|_{\mathrm{HS}}\, \|S\|_{\mathrm{HS}},\end{aligned}$$

(note that this corresponds to $|\langle T, S\rangle_{\mathrm{HS}}| \le \|T\|_{\mathrm{HS}}\|S\|_{\mathrm{HS}}$) and the series defining $\langle T, S\rangle_{\mathrm{HS}}$ converges absolutely. By the polarization identity (or similarly to the proof of Proposition 1.4.2) it is found that

$$\sum_j \langle Te_j, Se_j\rangle = \sum_k \langle S^* f_k, T^* f_k\rangle,$$

for any orthonormal basis $\{f_k\}$ of $\mathcal{H}_2$; so $\langle T, S\rangle_{\mathrm{HS}}$ is independent of the orthonormal basis and, therefore, well posed. The properties of inner product are simple and left to the reader.

If $\xi \in \mathcal{H}_1$, $\|\xi\| = 1$, pick an orthonormal basis of $\mathcal{H}_1$ of the following form $\{\xi, \eta_l\}_l$. Thus, $\|T\xi\|^2 \le \sum_l \|T\eta_l\|^2 + \|T\xi\|^2 = \|T\|_{\mathrm{HS}}^2$, and so $\|T\| \le \|T\|_{\mathrm{HS}}$.

We only need to show that $\mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ is complete; for this, consider a Cauchy sequence $(T_n) \subset \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$. From the inequality $\|\cdot\|_{\mathrm{B}(\mathcal{H}_1,\mathcal{H}_2)} \le \|\cdot\|_{\mathrm{HS}}$ it is found that (T_n) is Cauchy in $\mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$ and, therefore, it converges to some $T \in \mathrm{B}(\mathcal{H}_1, \mathcal{H}_2)$. It will be shown that $T \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ and that $T_n \to T$ in this space.

For $\varepsilon > 0$, there exists $N(\varepsilon)$ with $\|T_n - T_m\|_{\mathrm{HS}}^2 < \varepsilon$ if $n, m \ge N(\varepsilon)$. Consider an orthonormal basis $\{e_j\}_{j\in J}$ of $\mathcal{H}_1$. If $F \subset J$ is finite,

$$\sum_{j\in F} \|T_n e_j - T_m e_j\|^2 \le \|T_n - T_m\|_{\mathrm{HS}}^2 < \varepsilon.$$

Taking $m \to \infty$ one obtains $\sum_{j\in F} \|(T_n - T)e_j\|^2 \le \varepsilon$, for all finite subsets F. Therefore, $\|T_n - T\|_{\mathrm{HS}}^2 = \sum_{j\in J} \|(T_n - T)e_j\|^2 \le \varepsilon$, so that $(T - T_n) \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$

and $(T_n - T) \to 0$ in this space. Since $\mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ is a vector space, then $T = (T - T_n) + T_n$ belongs to $\mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$, and this space is Hilbert. □

Exercise 1.4.5. Show that $\|\cdot\|_{\mathrm{HS}}$ is a norm and that $\|TS\|_{\mathrm{HS}} \le \|T\|_{\mathrm{HS}} \|S\|_{\mathrm{HS}}$.

At this point all the tools necessary to verify that Hilbert-Schmidt operators are compact are available.

Theorem 1.4.6. $\mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathrm{B}_0(\mathcal{H}_1, \mathcal{H}_2)$.

Proof. Let $T \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ and $(\xi_n) \subset \mathcal{H}_1$, with $\xi_n \xrightarrow{\mathrm{w}} \xi$. By Proposition 1.3.28, in order to show that T is compact it is sufficient to verify that $T\xi_n \to T\xi$. Note that, by linearity, it is sufficient to consider the case $\xi_n \xrightarrow{\mathrm{w}} 0$.

Let $\{e_j\}_{j\in J}$ be an orthonormal basis of $\mathcal{H}_2$. For each n it is known that the set $\{j \in J : \langle e_j, T\xi_n\rangle \neq 0\}$ is countable (if it is finite for all n the argument ahead is easily adapted) and, for notational simplicity, it will be denoted by the natural numbers. Thus,

$$\|T\xi_n\|^2 = \sum_{j=1}^{\infty} |\langle e_j, T\xi_n\rangle|^2 \le \sum_{j=1}^{N} |\langle T^* e_j, \xi_n\rangle|^2 + M \sum_{j=N+1}^{\infty} \|T^* e_j\|^2,$$

with $M = \sup_{n\in\mathbb{N}} \|\xi_n\|^2$ (M is finite since every weakly convergent sequence is bounded).

For $\varepsilon > 0$, pick N with $\sum_{j=N+1}^{\infty} \|T^* e_j\|^2 < \varepsilon/M$, which exists since $T^* \in \mathrm{HS}(\mathcal{H}_2, \mathcal{H}_1)$. Now, in view of $\xi_n \xrightarrow{\mathrm{w}} 0$, there exists K so that $\sum_{j=1}^{N} |\langle T^* e_j, \xi_n\rangle|^2 < \varepsilon$ if $n \ge K$. Thus, if $n \ge K$ one has $\|T\xi_n\|^2 < 2\varepsilon$, and one concludes that $T\xi_n \to 0$. □

Exercise 1.4.7. Let $T : l^2(\mathbb{N}) \hookleftarrow$ given by $(T\xi)_n = \sum_{j=1}^{\infty} a_{nj}\xi_j$, $n \in \mathbb{N}$, with $(a_{nj})_{n,j\in\mathbb{N}}$ an infinite matrix with $\sum_{n,j\in\mathbb{N}} |a_{nj}|^2 < \infty$. Show that T is a Hilbert-Schmidt operator and find its Hilbert-Schmidt norm.

The next lemma will be used in the important example ahead.

Lemma 1.4.8. *Let $\mathcal{H}_1 = \mathrm{L}^2_\mu(\Omega)$ and $\mathcal{H}_2 = \mathrm{L}^2_\nu(\Lambda)$ be separable spaces, with μ, ν σ-finite measures, and $\mathcal{H}_3 = \mathrm{L}^2_{\mu\times\nu}(\Omega \times \Lambda)$. Then, if (ψ_n) and (ϕ_j) are (countable) orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $(\overline{\psi_n}\phi_j)$ is an orthonormal basis of $\mathcal{H}_3$, which is also separable.*

Proof. By Fubini $(\overline{\psi_n}\phi_j)$ is an orthonormal set of $\mathcal{H}_3$. In order to prove this lemma it is enough to show that if $f \in \mathcal{H}_3$ satisfies $\langle f, \overline{\psi_n}\phi_j\rangle_{\mathcal{H}_3} = 0$, $\forall n, j$, then $f = 0$. For each $s \in \Lambda$, denote the function sector $f^s : \Omega \to \mathbb{F}$ by $f^s(t) = f(t,s)$, which belongs to $\mathcal{H}_1$ for s in a set of total measure ν, and for each n the function $F_n(s) = \langle \overline{f^s}, \psi_n\rangle_{\mathcal{H}_1}$ (it is measurable since ν is $\sigma-$finite), then $\langle f, \overline{\psi_n}\phi_j\rangle_{\mathcal{H}_3} = \langle F_n, \phi_j\rangle_{\mathcal{H}_2}$. Note that by Cauchy-Schwarz $\nu-$a.e. one has $|F_n(s)| \le \|f^s\|_{\mathcal{H}_1}$, so that $F_n \in \mathcal{H}_2$ for all n, in view of $\|F_n\|^2_{\mathcal{H}_2} \le \int_\Lambda \|f^s\|^2_{\mathcal{H}_1}\, d\nu(s) = \|f\|^2_{\mathcal{H}_3}$.

Thus, one obtains the condition $\langle F_n, \phi_j\rangle_{\mathcal{H}_2} = 0$, $\forall n, j$; since (ϕ_j) is a basis of $\mathcal{H}_2$, for all n one has $F_n(s) = 0$ ν-a.e. and, therefore, since (ψ_n) is a basis of $\mathcal{H}_1$, one finds that $f^s = 0$ (in $\mathcal{H}_1$) ν-a.e. Then the result $\|f\|_{\mathcal{H}_3} = 0$ follows. □

Example 1.4.9. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be as in Lemma 1.4.8. Then, the operator $T \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ if, and only if, there exits $K \in \mathcal{H}_3$ so that

$$(T\psi)(t) = (T_K\psi)(t) := \int_\Omega K(t,s)\psi(s)d\mu(s), \qquad \psi \in \mathcal{H}_1.$$

Furthermore, $\|T\|_{\mathrm{HS}} = \|K\|_{\mathcal{H}_3}$.

Proof. If (ψ_n) and (ϕ_j) are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then, by Lemma 1.4.8, $(\overline{\psi_n}\phi_j)$ is an orthonormal basis of $\mathcal{H}_3$. Suppose that $T = T_K$; then

$$\sum_n \|T_K\psi_n\|^2_{\mathcal{H}_2} = \sum_{n,j} |\langle T_K\psi_n, \phi_j\rangle_{\mathcal{H}_2}|^2 = \sum_{n,j} |\langle K, \overline{\psi_n}\phi_j\rangle_{\mathcal{H}_3}|^2 = \|K\|^2_{\mathcal{H}_3},$$

and so $T_K \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ and $\|T_K\|_{\mathrm{HS}} = \|K\|_{\mathcal{H}_3}$.

Pick $T \in \mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$. By using the above notation, one has

$$\sum_{n,j} |\langle \phi_j, T\psi_n\rangle_{\mathcal{H}_2}|^2 = \sum_n \|T\psi_n\|^2 = \|T\|^2_{\mathrm{HS}} < \infty,$$

consequently the function $K_0(t,s) = \sum_{n,j}\langle \phi_j, T\psi_n\rangle_{\mathcal{H}_2}\overline{\psi_n(s)}\phi_j(t)$ is well defined in the space $\mathcal{H}_3$; note that $\|K_0\|_{\mathcal{H}_3} = \|T\|_{\mathrm{HS}}$. It will be shown that $T = T_{K_0}$.

If $\psi \in \mathcal{H}_1$ and $\phi \in \mathcal{H}_2$, since T is bounded and the inner product is continuous,

$$\begin{aligned}
\langle \phi, T_{K_0}\psi\rangle_{\mathcal{H}_2} &= \int_\Lambda d\nu(t)\left(\overline{\phi(t)}\int_\Omega K_0(t,s)\psi(s)d\mu(s)\right) \\
&= \langle \phi\overline{\psi}, K_0\rangle_{\mathcal{H}_3} = \sum_{n,j}\langle \phi_j, T\psi_n\rangle_{\mathcal{H}_2}\langle \phi\overline{\psi}, \phi_j\overline{\psi_n}\rangle_{\mathcal{H}_3} \\
&= \sum_{n,j}\langle \phi_j, T\psi_n\rangle_{\mathcal{H}_2}\langle \phi, \phi_j\rangle_{\mathcal{H}_2}\langle \psi_n, \psi\rangle_{\mathcal{H}_1} \\
&= \left\langle \sum_j \langle \phi_j, \phi\rangle_{\mathcal{H}_2}\phi_j, \sum_n \langle \psi_n, \psi\rangle_{\mathcal{H}_1} T\psi_n \right\rangle_{\mathcal{H}_2} \\
&= \left\langle \phi, \sum_n \langle \psi_n, \psi\rangle_{\mathcal{H}_1} T\psi_n \right\rangle_{\mathcal{H}_2} = \left\langle \phi, T\sum_n \langle \psi_n, \psi\rangle_{\mathcal{H}_1} \psi_n \right\rangle_{\mathcal{H}_2} \\
&= \langle \phi, T\psi\rangle_{\mathcal{H}_2}.
\end{aligned}$$

Therefore, $T = T_{K_0}$. □

Remark 1.4.10. There is a family of compact operators in $\mathrm{B}(\mathcal{H})$ for each $1 \le p < \infty$, with certain norm $\|T\|_p < \infty$ (this norm is based on that of l^p); the Hilbert-Schmidt operators are obtained through $p = 2$. The case $p = 1$, discussed in Subsection 9.4.1, is important in mathematical physics, particularly in statistical mechanics and scattering theory, and such operators are called *trace class* ($\|T\|_1$ is a generalization of the trace of the absolute values of the entries of a matrix).

Exercise 1.4.11. Show that $\mathrm{HS}(\mathcal{H}_1, \mathcal{H}_2)$ is the closure of the set of finite rank operators with the norm $\|\cdot\|_{\mathrm{HS}}$.

Exercise 1.4.12. Fix $\eta \in \mathcal{H}$ with $\|\eta\| = 1$. Let $T_\eta : \mathcal{H} \to \mathcal{H}$ be defined by $T_\eta \xi = \langle \eta, \xi \rangle\, \eta$, $\xi \in \mathcal{H}$. Show that T_η is a linear Hilbert-Schmidt operator and find its norm $\|T\|_{\mathrm{HS}}$.

Exercise 1.4.13. Let $\mathcal{H}$ be separable and $T \in \mathrm{B}(\mathcal{H})$ an operator whose eigenvectors form an orthonormal basis (ξ_j) of $\mathcal{H}$, that is, for all j, $T\xi_j = \lambda_j \xi_j$, $\lambda_j \in \mathbb{F}$. Present conditions for $T \in \mathrm{HS}(\mathcal{H})$. Verify that on infinite-dimensional Hilbert spaces there always are compact operators that are not Hilbert-Schmidt.

Exercise 1.4.14. Are there sequences $(T_n) \subset \mathrm{HS}(\mathcal{H})$ that converge in $\mathrm{B}(\mathcal{H})$ but do not converge in $\mathrm{HS}(\mathcal{H})$?

1.5 The spectrum

Intuitively, the spectrum of a linear operator comprises of "the values in $\mathbb{C}$ this operator assumes;" the very definition of spectrum justifies this interpretation. The spectrum is a generalization of the set of eigenvalues of linear operators. The point is that, for a linear operator acting on a finite-dimensional space, the property of being injective is equivalent to being surjective; however, in infinite dimensions such properties are not equivalent and the definition of spectrum must be properly generalized. From now on, vector spaces are assumed complex.

The spectral question is directly related to the solvability and uniqueness of solutions of linear equations in Banach spaces, boundary problems, approximations of nonlinear problems by linear versions, stability and, in an essential way, to the mathematical apparatus of quantum mechanics.

Definition 1.5.1. Let $T : \mathrm{dom}\, T \subset \mathcal{B} \to \mathcal{B}$ be linear in the complex Banach space $\mathcal{B} \ne \{0\}$. The *resolvent set* of T, denoted by $\rho(T)$, is the set of $\lambda \in \mathbb{C}$ for which the *resolvent operator* of T at λ,

$$R_\lambda(T) : \mathcal{B} \to \mathrm{dom}\, T, \qquad R_\lambda(T) := (T - \lambda \mathbf{1})^{-1},$$

exists and is bounded, i.e., $R_\lambda(T)$ belongs to $\mathrm{B}(\mathcal{B})$.

Definition 1.5.2. The *spectrum* of T is the set $\sigma(T) = \mathbb{C} \backslash \rho(T)$.

Remark 1.5.3. a) If $T \in \mathrm{B}(\mathcal{B})$ and $(T - \lambda \mathbf{1})$ is one-to-one with range $\mathcal{B}$, then, by the Open Mapping Theorem 1.2.6, $R_\lambda(T) \in \mathrm{B}(\mathcal{B})$ and $\lambda \in \rho(T)$.

b) Every eigenvalue λ of T (i.e., there is an eigenvector $\xi \neq 0$ with $T\xi = \lambda\xi$) belongs to the spectrum of T, for $(T - \lambda\mathbf{1})$ is not invertible in this case.
c) Notation: if it is clear which operator T is involved, $R_\lambda = R_\lambda(T)$.

The definition of spectrum is not restricted to the real numbers in order to be nonempty for continuous operators (see Corollary 1.5.17). For example, if $\dim \mathcal{B} < \infty$, the spectrum is the set of its eigenvalues, but the rotation by a right angle operator on $\mathbb{R}^2$ has no real eigenvalue (check this!).

Exercise 1.5.4. Let $T : \mathcal{B} \hookleftarrow$ be linear with $\dim \mathcal{B} < \infty$. Show that $\sigma(T)$ is the set of eigenvalues of T and, by the fundamental theorem of algebra, conclude that $\sigma(T) \neq \emptyset$ in this case.

Exercise 1.5.5. Let $T : \mathrm{dom}\, T \subset \mathcal{B} \to \mathcal{B}$ be linear. Show that the eigenvectors $\{\xi_j\}_{j\in J}$ of T, corresponding to pairwise distinct eigenvalues $\{\lambda_j\}_{j\in J}$, form a linearly independent set of $\mathrm{dom}\, T$.

Proposition 1.5.6. *If $\sigma(T) \neq \mathbb{C}$, then T is a closed operator.*

Proof. Pick $\lambda_0 \in \rho(T)$; so $R_{\lambda_0}(T) \in \mathrm{B}(\mathcal{B})$. If $(\xi_n) \subset \mathrm{dom}\, T$ with $\xi_n \to \xi$ and $T\xi_n \to \eta$, then

$$R_{\lambda_0}(T)(\eta - \lambda_0\xi) = \lim_{n\to\infty} R_{\lambda_0}(T)(T\xi_n - \lambda_0\xi_n) = \lim_{n\to\infty} \xi_n = \xi;$$

hence $\xi \in \mathrm{dom}\, T$ and

$$\eta - \lambda_0\xi = (T - \lambda_0\mathbf{1})R_{\lambda_0}(T)(\eta - \lambda_0\xi) = (T - \lambda_0\mathbf{1})\xi.$$

Therefore $T\xi = \eta$ and T is closed. □

The converse of Proposition 1.5.6 may not hold:

Example 1.5.7. Let $D : \mathrm{dom}\, D = C^1[0,1] \subset C[0,1] \to C[0,1]$ and $(D\psi)(t) = \psi'(t)$, which is a closed and unbounded operator. If $\lambda \in \mathbb{C}$, the function $\psi_\lambda(t) = e^{\lambda t} \in \mathrm{dom}\, D$ and $D\psi_\lambda = \lambda\psi_\lambda$, showing that $\sigma(D) = \mathbb{C}$ and it is constituted exclusively of eigenvalues. Therefore $\rho(D) = \emptyset$.

Given an operator action, the spectrum may drastically depend on the domain assigned to it. This is illustrated by Examples 1.5.7 and 1.5.8.

Example 1.5.8. Let $\mathrm{dom}\, d = \{\psi \in (C^1[0,1], \|\cdot\|_\infty) : \psi(0) = 0\}$, $d : \mathrm{dom}\, d \to C[0,1]$, $(d\psi)(t) = \psi'(t)$, which is a closed and unbounded operator. If $\lambda \in \mathbb{C}$, the operator $W_\lambda : C[0,1] \to \mathrm{dom}\, d$, $(W_\lambda\phi)(t) = e^{\lambda t}\int_0^t e^{-\lambda s}\phi(s)\, ds$, $\phi \in C[0,1]$, is bounded and satisfies $(d - \lambda\mathbf{1})W_\lambda = \mathbf{1}$ (identity on $C[0,1]$) and $W_\lambda(d - \lambda\mathbf{1}) = \mathbf{1}$ (identity in $\mathrm{dom}\, d$). Therefore W_λ is the resolvent operator for d at λ and $\rho(d) = \mathbb{C}$, showing that $\sigma(d) = \emptyset$ (the resolvent W_λ was obtained by considering the solution of the differential equation $\psi' - \lambda\psi = \phi$ with $\psi(0) = 0$).

Below there are three useful identities involving resolvent operators; except the third one, the nomenclature is standard. The first identity relates the resolvent of a fixed operator at two points in its resolvent set; the second resolvent identity relates the resolvent of two different operators at a point in both resolvent sets; the third identity relates the difference of resolvents of two operators at a point in both resolvent sets with the difference at another point.

Proposition 1.5.9. *Let* $T : \mathrm{dom}\, T \subset \mathcal{B} \to \mathcal{B}$. *Then for any* $z, s \in \rho(T)$ *one has the first resolvent identity* (*also known as first resolvent equation*)

$$R_z(T) - R_s(T) = (z - s)R_z(T)R_s(T).$$

Furthermore, $R_z(T)$ *commutes with* $R_s(T)$.

Proof. Write

$$\begin{aligned} R_z - R_s &= R_z(T - s\mathbf{1})R_s - R_z(T - z\mathbf{1})R_s \\ &= R_z\left((T - s\mathbf{1}) - (T - z\mathbf{1})\right)R_s = (z - s)R_zR_s, \end{aligned}$$

which shows the first resolvent identity. The commutation claim is immediate from this relation. □

Exercise 1.5.10. For linear operators T, S acting in $\mathcal{B}$, with $\mathrm{dom}\, S \subset \mathrm{dom}\, T$, and $\lambda \in \rho(T) \cap \rho(S)$, verify the *second resolvent identity*

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(S - T)R_\lambda(S).$$

If $\mathrm{dom}\, T = \mathrm{dom}\, S$, such identity also equals $R_\lambda(S)(S - T)R_\lambda(T)$.

Proposition 1.5.11. *Let* S *and* T *be linear operators acting in* $\mathcal{B}$. *Then, for* $z, z_0 \in \rho(T) \cap \rho(S)$ *one has the third resolvent identity*

$$\begin{aligned} &R_z(T) - R_z(S) \\ &\quad = \left(\mathbf{1} + (z - z_0)R_z(T)\right)\left[R_{z_0}(T) - R_{z_0}(S)\right]\left(\mathbf{1} + (z - z_0)R_z(S)\right). \end{aligned}$$

Proof. By the first resolvent identity $R_z(T) = (\mathbf{1} + (z - z_0)R_z(T))R_{z_0}(T)$ and $R_z(S) = R_{z_0}(S)(\mathbf{1} + (z - z_0)R_z(S))$. By using such relations on the r.h.s. above one gets $R_z(T) - R_z(S)$. □

Theorem 1.5.12. *Let* $T : \mathrm{dom}\, T \subset \mathcal{B} \to \mathcal{B}$ *and* $\lambda_0 \in \rho(T)$. *Then for all* λ *in the disk* $|\lambda - \lambda_0| < 1/\|R_{\lambda_0}(T)\|$ *of the complex plane,* $R_\lambda(T) \in \mathrm{B}(\mathcal{B})$ *and*

$$R_\lambda(T) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}(T)^{j+1},$$

with an absolutely convergent series.

Proof. Note initially that $R_{\lambda_0}(T) \neq 0$, since it is the inverse of an operator. By the relation

$$\begin{aligned} T - \lambda\mathbf{1} &= T - (\lambda_0 + (\lambda - \lambda_0))\mathbf{1} \\ &= (T - \lambda_0\mathbf{1})\left[\mathbf{1} + (\lambda_0 - \lambda)R_{\lambda_0}\right], \end{aligned}$$

just formally it would follow that

$$R_\lambda = \left(\sum_{j=0}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}^j \right) R_{\lambda_0}.$$

It is left to justify this expression and show that it defines $(T - \lambda \mathbf{1})^{-1}$ in $\mathrm{B}(\mathcal{B})$. For $|\lambda - \lambda_0| < 1/\|R_{\lambda_0}(T)\|$ the series is absolutely convergent in $\mathrm{B}(\mathcal{B})$ and defines an operator satisfying

$$\begin{aligned} \left(\sum_{j=0}^{N} (\lambda - \lambda_0)^j R_{\lambda_0}^{j+1} \right) (T - \lambda \mathbf{1}) &= \sum_{j=0}^{N} (\lambda - \lambda_0)^j R_{\lambda_0}^{j+1} \left(T - (\lambda_0 + (\lambda - \lambda_0))\mathbf{1} \right) \\ &= \sum_{j=0}^{N} (\lambda - \lambda_0)^j R_{\lambda_0}^j - \sum_{j=0}^{N} (\lambda - \lambda_0)^{j+1} R_{\lambda_0}^{j+1} \\ &= \mathbf{1} - \left[(\lambda - \lambda_0) R_{\lambda_0} \right]^{N+1}. \end{aligned}$$

Now $\lim_{N\to\infty} \left[(\lambda - \lambda_0) R_{\lambda_0} \right]^N = 0$ in $\mathrm{B}(\mathcal{B})$, since $|\lambda - \lambda_0| < 1/\|R_{\lambda_0}(T)\|$; then $\left(\sum_{j=0}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}^{j+1} \right) (T - \lambda \mathbf{1}) = \mathbf{1}$. Similarly it is shown that

$$(T - \lambda \mathbf{1}) \left(\sum_{j=0}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}^{j+1} \right) = \mathbf{1}.$$

□

Corollary 1.5.13. *$\rho(T)$ is an open set and $\sigma(T)$ is a closed set of $\mathbb{C}$.*

Proof. One sees that $\rho(T)$ is open directly from Theorem 1.5.12, hence $\sigma(T)$ is closed. □

Corollary 1.5.14. *The map $\rho(T) \to \mathrm{B}(\mathcal{B})$ given by $\lambda \mapsto R_\lambda(T)$ is continuous and uniformly holomorphic, i.e., it has a derivative in $\mathrm{B}(\mathcal{B})$ defined by the limit*

$$\frac{dR_\lambda(T)}{d\lambda} := \lim_{h\to 0} \frac{R_{\lambda+h}(T) - R_\lambda(T)}{h} = R_\lambda(T)^2,$$

for all λ in a neighborhood of each point $\lambda_0 \in \rho(T)$.

Proof. By Theorem 1.5.12, if $\lambda_0 \in \rho(T)$ and $|\lambda - \lambda_0| < 1/\|R_{\lambda_0}(T)\|$,

$$\begin{aligned} \|R_\lambda(T) - R_{\lambda_0}(T)\| &\le \sum_{j=1}^{\infty} |\lambda - \lambda_0|^j \, \|R_{\lambda_0}(T)\|^{j+1} \\ &= |\lambda - \lambda_0| \, \|R_{\lambda_0}(T)\|^2 \sum_{j=0}^{\infty} |\lambda - \lambda_0|^j \, \|R_{\lambda_0}(T)\|^j \\ &= \frac{|\lambda - \lambda_0| \, \|R_{\lambda_0}(T)\|^2}{1 - |\lambda - \lambda_0| \, \|R_{\lambda_0}(T)\|} \longrightarrow 0 \ \text{ as } \ \lambda \to \lambda_0, \end{aligned}$$

showing that the map $\lambda \mapsto R_\lambda(T)$ in $\rho(T)$ is continuous.

By the first resolvent identity $(R_{\lambda+h} - R_\lambda)/h = R_{\lambda+h}R_\lambda$; taking $h \to 0$ and using the continuity shown above, it follows that the derivative exists and $dR_\lambda(T)/d\lambda = R_\lambda(T)^2$ holds. □

Corollary 1.5.15. *If both $\sigma(T)$ and $\rho(T)$ are nonempty, then*

$$\|R_\lambda(T)\| \geq 1/d(\lambda, \sigma(T))$$

for all $\lambda \in \rho(T)$ (with $d(\lambda, \sigma(T)) := \inf_{\mu\in\sigma(T)} |\mu - \lambda|$).

Proof. By Theorem 1.5.12, if $\lambda_0 \in \rho(T)$ and $\|R_{\lambda_0}(T)\| \, |\lambda - \lambda_0| < 1$, then $\lambda \in \rho(T)$. Thus, if $\lambda \in \sigma(T)$, necessarily $\|R_{\lambda_0}(T)\| \, |\lambda - \lambda_0| \geq 1$, that is,

$$\|R_{\lambda_0}(T)\| \geq \frac{1}{|\lambda - \lambda_0|}, \qquad \forall \lambda \in \sigma(T),$$

and (since $\sigma(T) \neq \emptyset$) the result follows. □

Now certain specific results on the spectrum of bounded operators will be discussed.

Corollary 1.5.16. *Let $T \in \mathrm{B}(\mathcal{B})$. If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$ and $\|R_\lambda(T)\| \to 0$ for $|\lambda| \to \infty$.*

Proof. Following the proof of the above theorem (write $T - \lambda\mathbf{1} = -\lambda(\mathbf{1} - T/\lambda)$), one concludes that the representation of $R_\lambda(T)$ by the series, called *Neumann's series* of T,

$$R_\lambda(T) = -\frac{1}{\lambda}\sum_{j=0}^{\infty}\left(\frac{T}{\lambda}\right)^j$$

is absolutely convergent if $|\lambda| > \|T\|$ and, in this case, that

$$\|R_\lambda(T)\| \leq 1/|\lambda| \sum_{j\geq 0}(\|T\|/\lambda)^j = 1/(|\lambda| - \|T\|).$$

It then follows that the spectrum $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$ and

$$\lim_{|\lambda|\to\infty} \|R_\lambda(T)\| = 0.$$ □

Corollary 1.5.17. *If $T \in \mathrm{B}(\mathcal{B})$, then $\sigma(T) \neq \emptyset$.*

Proof. If $f \in \mathrm{B}(\mathcal{B})^*$ (the dual of $\mathrm{B}(\mathcal{B})$) define $F : \rho(T) \to \mathbb{C}$ by $F(\lambda) = f(R_\lambda(T))$. Thus, by Corollary 1.5.14 it is found that

$$\frac{dF(\lambda)}{d\lambda} = \lim_{h\to 0}\frac{F(\lambda+h) - F(\lambda)}{h} = f\left(R_\lambda(T)^2\right),$$

which is continuous; hence, F is holomorphic in $\rho(T)$. By using the inequality $|F(\lambda)| \leq \|f\|\|R_\lambda(T)\|$ and Corollary 1.5.16, $\lim_{|\lambda|\to\infty} F(\lambda) = 0$.

If $\sigma(T) = \emptyset$, i.e., $\rho(T) = \mathbb{C}$, by continuity F is bounded in any ball in $\mathbb{C}$, and since it converges to zero for $|\lambda| \to \infty$, it is found that $F : \mathbb{C} \to \mathbb{C}$ is an entire and bounded function, hence constant by Liouville's Theorem. In view of $\lim_{|\lambda|\to\infty} F(\lambda) = 0$, one has $F(\lambda) = f(R_\lambda(T)) = 0$ for all $\lambda \in \mathbb{C}$, $f \in \mathrm{B}(\mathcal{B})^*$. By the Hahn-Banach Theorem one gets $R_\lambda(T) = 0$, $\forall \lambda \in \mathbb{C}$, but this can not occur, since $R_\lambda(T)$ is the inverse of some operator. This contradiction shows that $\sigma(T)$ is nonempty. □

Definition 1.5.18. The *spectral radius* of a bounded linear operator $T \in \mathrm{B}(\mathcal{B})$ is $r_\sigma(T) := \sup_{\lambda\in\sigma(T)} |\lambda|$.

The next result is the so-called *spectral radius formula* and is due to I. Gelfand, who has shown it in the context of Banach algebras, around 1940. This formula is a relation between a limit strongly related to the metric, and the spectral radius defined via the supremum of a set.

Theorem 1.5.19. *If $T \in \mathrm{B}(\mathcal{B})$, then $r_\sigma(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \le \|T\|$.*

Proof. Note, initially, that due to Corollary 1.5.16, $r_\sigma(T) \le \|T\|$. To demonstrate Theorem 1.5.19 we will use results from the Holomorphic Functions Theory combined with "any weakly convergent sequence is bounded," and the following simple observation: if $\lambda \in \mathbb{C}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its nth roots in $\mathbb{C}$, then

$$T^n - \lambda \mathbf{1} = (T - \lambda_1 \mathbf{1})(T - \lambda_2 \mathbf{1}) \cdots (T - \lambda_n \mathbf{1}).$$

This implies that $\lambda \in \sigma(T^n)$ if, and only if, $\lambda_j \in \sigma(T)$ for some $1 \le j \le n$. Hence, $\sigma(T^n) = \sigma(T)^n := \{\lambda^n : \lambda \in \sigma(T)\}$. From this relation one concludes that for all $n \in \mathbb{N}$ one has $r_\sigma(T) = r_\sigma(T^n)^{1/n} \le \|T^n\|^{1/n}$.

For each f in the dual of $\mathrm{B}(\mathcal{B})$, define $F : \rho(T) \to \mathbb{C}$ by $F(\lambda) = f(R_\lambda(T))$, which is a holomorphic function (see the proof of Corollary 1.5.17). If $|\lambda| > \|T\|$, by using the Neumann series

$$F(\lambda) = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} f(T^n),$$

and by the uniqueness of Laurent expansion the above series converge for all $\lambda \in \mathbb{C}$ in the region $|\lambda| > r_\sigma(T)$ (or Taylor expansion if the variable $s = 1/\lambda$, with $F(0) = 0$, is considered).

Given $\varepsilon > 0$, for $r_\sigma(T) < \alpha < r_\sigma(T) + \varepsilon$ and all $f \in \mathrm{B}(\mathcal{B})^*$, the series $\sum_{n=0}^{\infty} f(T^n/\alpha^n)$ converge. Thus, the sequence T^n/α^n converges weakly to zero in $\mathrm{B}(\mathcal{B})$; hence it is bounded and there exists $C = C(\alpha) > 0$ with

$$\|T^n/\alpha^n\| \le C \Longrightarrow \|T^n\|^{1/n} \le \alpha C^{1/n}, \qquad \forall n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} C^{1/n} = 1$, there is $N(\varepsilon) > 0$ such that

$$\|T^n\|^{\frac{1}{n}} < r_\sigma(T) + \varepsilon, \qquad \forall n \ge N(\varepsilon).$$

This relation, along with $r_\sigma(T) \le \|T^n\|^{1/n}$ verified above, show that $\lim_{n\to\infty} \|T\|^{1/n}$ exists and equals $r_\sigma(T)$. □

Exercise 1.5.20. If all pairs of the operators $\{T_1, \dots, T_n\} \subset \mathrm{B}(\mathcal{B})$ are *commuting*, i.e., $T_jT_k = T_kT_j$, $\forall j, k$, show that the *product* $T_1T_2\cdots T_n$ is invertible with bounded inverse if, and only if, each T_j is invertible in $\mathrm{B}(\mathcal{B})$.

Corollary 1.5.21. *If* $T \in \mathrm{B}(\mathcal{B})$, *then* $\sigma(T^n) = \sigma(T)^n$ *and* $r_\sigma(T^n) = r_\sigma(T)^n$.

Exercise 1.5.22. Present a proof of Corollary 1.5.21.

Example 1.5.23. Let $S_e : l^\infty(\mathbb{N}) \hookleftarrow$ be the shift operator

$$S_e(\xi_1, \xi_2, \xi_3, \dots) = (\xi_2, \xi_3, \xi_4, \dots).$$

Since $\|S_e\| = 1$, then $\sigma(S_e) \subset \overline{B}(0;1)$. Every $|\lambda| \le 1$ is an eigenvalue of S_e, for the equation $S_e\xi^\lambda = \lambda\xi^\lambda$ has the solution $\xi^\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots)$ in $l^\infty(\mathbb{N})$. Therefore $\sigma(S_e) = \overline{B}(0;1)$, $r_\sigma(S_e) = 1$, and every point of its spectrum is an eigenvalue.

Example 1.5.24. The Volterra operator $T : C[0,1] \hookleftarrow$, given by $(T\psi)(t) = \int_0^t \psi(s)\, ds$ has no eigenvalues. In fact, by the eigenvalue equation

$$(T\psi)(t) = \lambda\psi(t) = \int_0^t \psi(s)\, ds$$

one finds $\lambda\psi'(t) = \psi(t)$ (ψ is differentiable since it is the integral of a continuous function). If $\lambda = 0$ then $\psi = 0$ and zero is not an eigenvalue; if $\lambda \ne 0$, the solutions of this differential equation are $\psi(t) = C\exp(t/\lambda)$, and since $\psi(0) = 0$ it follows that the constant $C = 0$, and so $\psi = 0$ and no $\lambda \in \mathbb{C}$ is an eigenvalue of T.

From the inequality $|(T\psi)(t)| \le t\|\psi\|_\infty$ it is found, by induction, that

$$|(T^2\psi)(t)| \le \int_0^t s\|\psi\|_\infty\, ds = \frac{t^2}{2}\|\psi\|_\infty, \qquad |(T^n\psi)(t)| \le \frac{t^n}{n!}\|\psi\|_\infty.$$

Thus, $\|T^n\| \le 1/n!$ and $r_\sigma(T) \le \lim_{n\to\infty}(1/n!)^{1/n} = 0$. Therefore $r_\sigma(T) < \|T\|$, $\sigma(T) = \{0\}$ (since $\ne \emptyset$) and T has no eigenvalues.

Example 1.5.25. Let $\mathcal{M}_h$ on $\mathrm{L}^2[0,1]$, with $h(t) = t$. Then $\mathcal{M}_h$ has no eigenvalues, since from $\mathcal{M}_h\psi = \lambda\psi$ it follows that $(t - \lambda)\psi(t) = 0$, or $\psi(t) = 0$ for a.e. $t \ne \lambda$, i.e., $\psi = 0$ in $\mathrm{L}^2[0,1]$.

Exercise 1.5.26. Show that in Example 1.5.25 one has $\sigma(\mathcal{M}_h) = [0,1]$.

Exercise 1.5.27. If $T \in \mathrm{B}(\mathcal{B})$, show that $\lim_{|\lambda|\to\infty} \lambda R_\lambda(T) = -\mathbf{1}$.

Exercise 1.5.28. For $T \in \mathrm{B}(\mathcal{B})$, define $V(t) := e^{tT}$, $t \in \mathbb{R}$, as in Exercise 1.1.23. Show that: a) The map $t \mapsto V(t) \in \mathrm{B}(\mathcal{B})$ is continuous with $V(0) = \mathbf{1}$ and $V(t+s) = V(t)V(s)$. b) If $S \in \mathrm{B}(\mathcal{B})$ commutes with T, then it also commutes with $V(t), \forall t$. c) This map is uniformly holomorphic and $dV(t)/dt = TV(t)$. See related results in Section 5.2.

1.6 Spectra of Compact Operators

As expected, the spectral theory of compact linear operators has many similarities with the spectral theory on finite-dimensional spaces; for example, with the possible exception of zero, each eigenvalue of a compact operator has finite multiplicity. However, there are compact operators with no eigenvalues.

Example 1.6.1. Consider the operator $T : l^2(\mathbb{N}) \hookleftarrow$,

$$T(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_1/1, \xi_2/2, \xi_3/3, \dots).$$

T is compact and $0 \in \sigma(T)$ since T^{-1} is not bounded. However this operator has no eigenvalues (check this!).

The next lemma is a key tool to construct bounded sequences with no convergent subsequence in infinite-dimensional $\mathcal{N}$. Although there is no explicit notion of orthogonality, a geometric interpretation is important for turning its proof natural.

Lemma 1.6.2 (Riesz Lemma). *Let X be a proper closed vector subspace of a normed space $(\mathcal{N}, \|\cdot\|)$. Then, for each $0 < \alpha < 1$ there exists $\xi \in \mathcal{N} \setminus X$ with $\|\xi\| = 1$ and $\inf_{\eta \in X} \|\xi - \eta\| \geq \alpha$.*

Proof. Let $\zeta \in \mathcal{N} \setminus X$ and $c = \inf_{\eta \in X} \|\eta - \zeta\|$. Since X is closed, $c > 0$. Thus, for all $d > c$ there exists $\omega \in X$ with $c \leq \|\zeta - \omega\| \leq d$. The vector $\xi = (\zeta - \omega)/\|\zeta - \omega\|$ belongs to $\mathcal{N} \setminus X$ and $\|\xi\| = 1$. Moreover, for all $\eta \in X$ one has

$$\|\xi - \eta\| = \frac{1}{\|\zeta - \omega\|} \Big\| \zeta - (\omega + \|\zeta - \omega\| \eta) \Big\| \geq \frac{c}{\|\zeta - \omega\|} \geq \frac{c}{d}.$$

For $0 < \alpha < 1$ choose $d = c/\alpha$ and the result follows. □

Theorem 1.6.3. *The closed ball $\overline{B}(0;1)$ in a normed vector space $\mathcal{N}$ is compact if, and only if, $\dim \mathcal{N} < \infty$.*

Proof. If $\dim \mathcal{N} < \infty$, it is known that $\overline{B}(0;1)$ is compact. If $\dim \mathcal{N}$ is not finite, then Riesz's lemma will be used to construct a sequence in $\overline{B}(0;1)$ with no convergent subsequence.

Let $\xi_1 \in \mathcal{N}$, $\|\xi_1\| = 1$. By Riesz's lemma there exists $\xi_2 \in \mathcal{N}$, with $\|\xi_2\| = 1$, and $\|\xi_1 - \xi_2\| \geq 1/2$ (by choosing $\alpha = 1/2$ in Riesz's lemma). The vector space $\mathrm{Lin}(\{\xi_1, \xi_2\})$ is closed, since its dimension is finite. Again by Riesz's lemma, there exists $\xi_3 \in \mathcal{N}$, with $\|\xi_3\| = 1$, $\|\xi_3 - \xi_1\| \geq 1/2$ and $\|\xi_3 - \xi_2\| \geq 1/2$. In this way, a sequence $(\xi_n)_{n=1}^{\infty}$, $\|\xi_n\| = 1$, $\forall n$, and $\|\xi_j - \xi_k\| \geq 1/2$ for all $j \neq k$ is constructed. Since such sequence has no convergent subsequence , the closed ball $\overline{B}(0;1)$ is not compact. □

Proposition 1.6.4. *If $T \in \mathrm{B}_0(\mathcal{B})$, then every nonzero eigenvalue of T is of finite multiplicity, that is, $\dim \mathrm{N}(T - \lambda \mathbf{1}) < \infty$.*

Proof. Let B_1 be the closed ball centered at zero and radius 1 in the vector space $\mathrm{N}(T - \lambda \mathbf{1})$. It will be shown that B_1 is compact and, hence, $\dim \mathrm{N}(T - \lambda \mathbf{1}) < \infty$ by Theorem 1.6.3. Since T is compact, for a sequence $(\xi_n) \subset B_1$ $(T\xi_n = \lambda \xi_n)$, there is a convergent subsequence $(T\xi_{n_j})$ and, so, $(\xi_{n_j} = T\xi_{n_j}/\lambda)$ also converges to an element of B_1; hence that ball is compact. □

Exercise 1.6.5. Use the next argument as a variant of the proof of Proposition 1.6.4: suppose that B_1 is not compact; thus there exists a sequence $(\xi_n) \subset B_1$ with no convergent subsequence; use the compactness of T to reach a contradiction.

Proposition 1.6.6. *If $T \in \mathrm{B}_0(\mathcal{B})$, then for all $\varepsilon > 0$ the number of eigenvalues λ of T with $|\lambda| \geq \varepsilon$ is finite.*

Proof. Suppose that it is possible to choose $\varepsilon > 0$ so that there are infinitely many eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ of T with absolute values greater than or equal to ε. By Proposition 1.6.4 one may assume that such eigenvalues are pairwise distinct; denote by $\{\xi_j\}$ the respective eigenvectors. Recall that this set is linearly independent (Exercise 1.5.5).

Let $E_0 = \{0\}$ and $E_n = \mathrm{Lin}(\{\xi_1, \ldots, \xi_n\})$; note that such subspaces are closed for all n. By Riesz's Lemma 1.6.2 there exists a sequence $\{\eta_n\}$, $\eta_n \in E_n$, $\|\eta_n\| = 1$ and $\|\eta_n - \xi\| \geq 1/2$, $\forall \xi \in E_{n-1}$. The aim is to show that $\|T\eta_n - T\eta_m\| \geq \varepsilon/2$ for all distinct n, m, which then has no convergent subsequence, a contradiction with the compactness of T.

If $m < n$, then $T\eta_n - T\eta_m = \lambda_n \eta_n + [(T - \lambda_n \mathbf{1})\eta_n - T\eta_m]$. Clearly $T\eta_m \in E_m$ and, writing $\eta_n = \sum_{j=1}^n \alpha_j \xi_j$, one has

$$(T - \lambda_n \mathbf{1})\eta_n = \left[\sum_{j=1}^{n-1} \alpha_j (\lambda_j - \lambda_n) \xi_j \right] \in E_{n-1},$$

so that $\zeta_m := -[(T - \lambda_n \mathbf{1})\eta_n - T\eta_m]/\lambda_n$ belongs to the subspace E_{n-1}. Therefore, $\|T\eta_n - T\eta_m\| = |\lambda_n| \|\eta_n - \zeta_m\| \geq \frac{|\lambda_n|}{2} \geq \varepsilon/2$, and $\{T\eta_n\}$ has no convergent subsequence. □

From such propositions (and some simple extra argument) follows the important

Corollary 1.6.7. *Let $T \in \mathrm{B}_0(\mathcal{B})$ and Λ the set of eigenvalues of T. Then:*

i) *The unique possible accumulation point of Λ is zero.*
ii) *Λ is countable and, if $\lambda \neq 0$, then $\dim \mathrm{N}(T - \lambda \mathbf{1}) < \infty$.*
iii) *If Λ is an infinite set, then the eigenvalues of T can be ordered in a sequence converging to zero.*
iv) *If $\dim \mathcal{B} = \infty$, then zero belongs to the spectrum of T.*

Exercise 1.6.8. Present the details of the proof of Corollary 1.6.7.

Example 1.6.9. Any finite rank operator is compact and has finite spectrum.

Example 1.6.10. Consider the operator $T : l^2(\mathbb{N}) \hookleftarrow$,

$$T(\xi_1, \xi_2, \xi_3, \dots) = (\xi_1/1, \xi_2/2, \xi_3/3, \dots).$$

T is compact and zero is not an eigenvalue of T, however it belongs to its spectrum, since $\{1, 1/2, 1/3, \dots\}$ is a subset of $\sigma(T)$ (they are eigenvalues) and the spectrum is closed. It is also possible to infer directly that the resolvent operator $R_0(T)$ exists, with dense domain, but it is not bounded.