International Series of Numerical Mathematics, Vol. 157, 233–243

On Some General Inequalities Related to Jensen's Inequality

Milica Klaričić Bakula, Marko Matić and Josip Pečarić

Abstract. We present several general inequalities related to Jensen's inequality and the Jensen-Steffensen inequality. Some recently proved results are obtained as special cases of these general inequalities.

Mathematics Subject Classification (2000). 26D15.

Keywords. Convex functions, Jensen's inequality, Jensen-Steffensen inequality, Slater's inequality, Mercer's inequality.

1. Introduction

Let the real function φ be defined on some nonempty interval I of the real line $\mathbb{R}$. We say that φ is *convex* on I if

$$\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

An important property of convex functions is the existence of the left and the right derivative on the interior $\mathring{I}$ of I (see [11]). If $\varphi : I \to \mathbb{R}$ is convex then for any $x \in \mathring{I}$ the left derivative $\varphi'_-(x)$ and the right derivative $\varphi'_+(x)$ are increasing on $\mathring{I}$ and

$$\varphi'_-(x) \le \varphi'_+(x) \quad \text{for all } x \in \mathring{I}.$$

It can be also proved that for any convex function $\varphi : I \to \mathbb{R}$ the inequalities

$$\varphi(z) + c(z)(y-z) \le \varphi(y), \quad c(z) \in \left[\varphi'_-(z), \varphi'_+(z)\right] \tag{1.1}$$

$$\varphi(y) \le \varphi(z) + c(y)(y-z), \quad c(y) \in \left[\varphi'_-(y), \varphi'_+(y)\right] \tag{1.2}$$

hold for all $y, z \in \mathring{I}$.

One consequence of (1.1) and (1.2) is that $\varphi : I \to \mathbb{R}$ is convex if and only if there is at least one line of support for φ at each $x_0 \in \mathring{I}$. Furthermore, φ is differentiable if and only if the line of support at $x_0 \in \mathring{I}$ is unique. In this case, the line of support is

$$A(x) = \varphi(x_0) + \varphi'(x_0)(x - x_0).$$

There are many known inequalities for convex functions, but surely the most important of them is Jensen's inequality. In its integral form it is stated as follows (see [10, p. 45]).

Theorem A. (*Jensen*) *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space with* $0 < \mu(\Omega) < \infty$*, and let* $u : \Omega \to I,\ I \subset \mathbb{R}$*, be a function from* $L^1(\mu)$*. Then for any convex function* $\varphi : I \to \mathbb{R}$ *the inequality*

$$\varphi\left(\frac{1}{\mu(\Omega)}\int_\Omega u d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega (\varphi \circ u)\, d\mu \tag{1.3}$$

holds.

One of the inequalities which are strongly related to Jensen's inequality is the Jensen-Steffensen inequality for convex functions. An integral version was proved by Steffensen, but here we consider a variant given by R.P. Boas in [3].

Theorem B. (*Steffensen-Boas*) *Let* $f : [\alpha, \beta] \to (a, b)$ *be a continuous and monotonic function, where* $-\infty < \alpha < \beta < +\infty$ *and* $-\infty \leq a < b \leq +\infty$*, and let* $\varphi : (a, b) \to \mathbb{R}$ *be a convex function. If* $\lambda : [\alpha, \beta] \to \mathbb{R}$ *is either continuous or of bounded variation satisfying*

$$(\forall x \in [\alpha, \beta]) \quad \lambda(\alpha) \leq \lambda(x) \leq \lambda(\beta), \qquad \lambda(\beta) - \lambda(\alpha) > 0, \tag{1.4}$$

then

$$\varphi\left(\frac{\int_\alpha^\beta f(t)\,\mathrm{d}\lambda(t)}{\int_\alpha^\beta \mathrm{d}\lambda(t)}\right) \leq \frac{\int_\alpha^\beta \varphi(f(t))\,\mathrm{d}\lambda(t)}{\int_\alpha^\beta \mathrm{d}\lambda(t)}. \tag{1.5}$$

In [7] a couple of companion inequalities to Jensen's inequality in its discrete and integral form were proved. The main result in its discrete form is stated as follows.

Theorem C. (*Matić, Pečarić*) *Let* $\varphi : C \to \mathbb{R}$ *be a convex function defined on an open convex subset* C *in a normed real linear space* X*. For the given vectors* $\boldsymbol{x}_i \in C,\ i = 1, 2, \ldots, n$*, and a nonnegative real* n*-tuple* $\boldsymbol{p}$ *such that* $P_n = \sum_{i=1}^n p_i > 0$ *let*

$$\overline{\boldsymbol{x}} = \frac{1}{P_n}\sum_{i=1}^n p_i \boldsymbol{x}_i, \qquad \overline{y} = \frac{1}{P_n}\sum_{i=1}^n p_i \varphi(\boldsymbol{x}_i).$$

If $\boldsymbol{c}, \boldsymbol{d} \in C$ *are arbitrarily chosen vectors, then*

$$\varphi(\boldsymbol{c}) + a^*(\boldsymbol{c}; \overline{\boldsymbol{x}} - \boldsymbol{c}) \leq \overline{y} \leq \varphi(\boldsymbol{d}) + \frac{1}{P_n}\sum_{i=1}^n p_i a^*(\boldsymbol{x}_i; \boldsymbol{x}_i - \boldsymbol{d}). \tag{1.6}$$

Also, when φ is strictly convex we have equality in the first inequality in (1.6) *if and only if $\boldsymbol{x}_i = \boldsymbol{c}$ for all indices i with $p_i > 0$, while equality holds in the second inequality in* (1.6) *if and only if $\boldsymbol{x}_i = \boldsymbol{d}$ for all indices i with $p_i > 0$.*

In the rest of the paper without any loss of generality for the convex function $\varphi : (a,b) \to \mathbb{R}$ we denote

$$\varphi'(x) := \varphi'_+(x), \quad x \in (a,b).$$

Theorem D. (*Klaričić, Matić, Pečarić*) *Let $\varphi : (a,b) \to \mathbb{R}, -\infty \leq a < b \leq +\infty,$ be a convex function and $\boldsymbol{p} \in \mathbb{R}^n$ $(n \geq 2)$ such that*

$$0 \leq P_k = \textstyle\sum_{i=1}^k p_i \leq P_n, \; k = 1, \ldots, n, \quad P_n > 0. \tag{1.7}$$

Then for any $\boldsymbol{x} \in (a,b)^n$ such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \quad \text{or} \quad x_1 \geq x_2 \geq \cdots \geq x_n$$

the inequalities

$$\varphi(c) + \varphi'(c)(\overline{x} - c) \leq \frac{1}{P_n}\sum_{i=1}^n p_i \varphi(x_i) \leq \varphi(d) + \frac{1}{P_n}\sum_{i=1}^n p_i \varphi'(x_i)(x_i - d) \tag{1.8}$$

hold for all $c, d \in (a,b)$.

Under the stated assumptions on $\boldsymbol{x}$ and $\boldsymbol{p}$ the inequalities in (1.8) are valid for all $c, d \in (a,b)$, so in the first inequality in (1.8) we may choose $c = \overline{x}$ thus obtaining the discrete Jensen-Steffensen inequality. Moreover, the choice $c = \overline{x}$ is the best possible since

$$\varphi(c) + \varphi'(c)(\overline{x} - c) \leq \varphi(\overline{x})$$

for all $c \in (a,b)$.

The integral version of Theorem D, stated in Theorem E, has been also proved in [6].

Theorem E. (*Klaričić, Matić, Pečarić*) *Suppose that f, φ and λ are as in Theorem B. Then $\overline{x}$ and $\overline{y}$ given by*

$$\overline{x} = \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta f(t)\, \mathrm{d}\lambda(t),$$

$$\overline{y} = \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta \varphi(f(t))\, \mathrm{d}\lambda(t)$$

are well defined and $\overline{x} \in (a,b)$. Furthermore, if $\varphi'(f)$ and λ have no common discontinuity points, then the inequalities

$$\begin{aligned} &\varphi(c) + \varphi'(c)(\overline{x} - c) \\ &\leq \overline{y} \leq \varphi(d) + \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta \varphi'(f(t))[f(t) - d]\, \mathrm{d}\lambda(t) \end{aligned} \tag{1.9}$$

hold for each $c, d \in (a,b)$.

In [9] the following theorem was proved.

Theorem F. (*Pečarić*) *Suppose that* φ *is convex on* (a,b) *and* $a < x_1 \leq \cdots \leq x_n < b$. *If* $p_1, \ldots, p_n$ *are real numbers such that the conditions* (1.7) *hold and if*

$$\sum\nolimits_{i=1}^{n} p_i \varphi'(x_i) \neq 0, \qquad \widetilde{x} = \frac{\sum\nolimits_{i=1}^{n} p_i x_i \varphi'(x_i)}{\sum\nolimits_{i=1}^{n} p_i \varphi'(x_i)} \in (a,b),$$

then

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi(\boldsymbol{x}_i) \leq \varphi(\widetilde{x}).$$

In paper [8] A. Mercer proved the following variant of Jensen's inequality:

$$\varphi\left(x_1 + x_n - \sum_{i=1}^{n} w_i x_i\right) \leq \varphi(x_1) + \varphi(x_n) - \sum_{i=1}^{n} w_i \varphi(x_i), \tag{1.10}$$

which holds whenever φ is a convex function on an interval containing the n-tuple $\boldsymbol{x}$ such that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and where $\boldsymbol{w}$ is a positive n-tuple with $\sum_{i=1}^{n} w_i = 1$. His result was generalized for weights satisfying the conditions as in the Jensen-Steffensen inequality in [1], and two alternative proofs of (1.10) were given in [13] and [2].

2. The results

The goal of this paper is to obtain Mercer-type variants of Theorems C, D, E and F.

In the following with $(\Omega, \mathcal{A}, \mu)$ we denote a measure space with $0 < \mu(\Omega) < \infty$ and for $a, b, m, M \in \mathbb{R}$ we always assume $\infty \leq a < m < M < b \leq \infty$.

Theorem 1. *Let* $\varphi : (a,b) \to \mathbb{R}$ *be a convex function and let* $u : \Omega \to [m,M]$ *be a measurable function such that* $\varphi' \circ u$ *belongs to* $L^1(\mu)$. *Then the inequalities*

$$\begin{aligned}
&\varphi(c) + \varphi'(c)\left(m + M - c - \frac{1}{\mu(\Omega)} \int_\Omega u \, d\mu\right) \\
&\leq \varphi(m) + \varphi(M) - \frac{1}{\mu(\Omega)} \int_\Omega (\varphi \circ u) \, d\mu \qquad (2.1) \\
&\leq \varphi(d) + \varphi'(m)(m-d) + \varphi'(M)(M-d) - \frac{1}{\mu(\Omega)} \int_\Omega (u(t) - d)(\varphi' \circ u) \, d\mu
\end{aligned}$$

hold for all $c, d \in [m,M]$.

Proof. We prove the first inequality in (2.1).

For all $u(t) \in [m,M]$, $t \in \Omega$, we can write

$$u(t) = \lambda_t m + (1 - \lambda_t) M, \quad \lambda_t \in [0,1]$$

hence

$$(\varphi \circ u)(t) = \varphi(\lambda_t m + (1 - \lambda_t) M) \leq \lambda_t \varphi(m) + (1 - \lambda_t) \varphi(M)$$

for all $t \in \Omega$. Also

$$\begin{aligned}\varphi(m+M-u(t)) &= \varphi((1-\lambda_t)m+\lambda_t M) \le (1-\lambda_t)\varphi(m)+\lambda_t\varphi(M)\\ &= \varphi(m)+\varphi(M)-[\lambda_t\varphi(m)+(1-\lambda_t)\varphi(M)]\\ &\le \varphi(m)+\varphi(M)-(\varphi\circ u)(t).\end{aligned}$$

If in (1.1) we choose $z=c$ and $y=m+M-u(t)$ we obtain

$$\begin{aligned}&\varphi(c)+\varphi'(c)(m+M-u(t)-c)\\ &\le \varphi(m+M-u(t)) \le \varphi(m)+\varphi(M)-(\varphi\circ u)(t).\end{aligned} \tag{2.2}$$

Integrating over Ω and dividing by $\mu(\Omega)$ we obtain

$$\begin{aligned}&\varphi(c)+\varphi'(c)\left(m+M-c-\frac{1}{\mu(\Omega)}\int_\Omega u\,d\mu\right)\\ &\le \frac{1}{\mu(\Omega)}\int_\Omega \varphi(m+M-u(t))\,d\mu \le \varphi(m)+\varphi(M)-\frac{1}{\mu(\Omega)}\int_\Omega(\varphi\circ u)\,d\mu.\end{aligned}$$

Now it remains to prove the second inequality in (2.1). Let $d, u(t) \in [m, M]$, $t \in \Omega$.

We consider two cases.

Case 1. $u(t) \ge d$. From (1.2) we have

$$\begin{aligned}\varphi(m)-\varphi(d) &\le \varphi'(m)(m-d),\\ \varphi(M)-(\varphi\circ u)(t) &\le \varphi'(M)(M-u(t)),\end{aligned}$$

hence

$$\begin{aligned}&\varphi(m)+\varphi(M)-(\varphi\circ u)(t)\\ &= \varphi(d)+\varphi(m)-\varphi(d)+\varphi(M)-(\varphi\circ u)(t)\\ &\le \varphi(d)+\varphi'(m)(m-d)+\varphi'(M)(M-u(t))\\ &= \varphi(d)+\varphi'(m)(m-d)+\varphi'(M)(M-d)-\varphi'(M)(u(t)-d).\end{aligned} \tag{2.3}$$

Since φ is convex the derivative φ' is nondecreasing and we know that from $u(t) \le M$ follows $(\varphi'\circ u)(t) \le \varphi'(M)$, hence (2.3) implies

$$\begin{aligned}&\varphi(m)+\varphi(M)-(\varphi\circ u)(t)\\ &\le \varphi(d)+\varphi'(m)(m-d)+\varphi'(M)(M-d)-(\varphi'\circ u)(t)(u(t)-d).\end{aligned} \tag{2.4}$$

Case 2. $u(t) < d$. Similarly as in the previous case we can write

$$\begin{aligned}&\varphi(m)+\varphi(M)-(\varphi\circ u)(t)\\ &= \varphi(d)+\varphi(m)-(\varphi\circ u)(t)+\varphi(M)-\varphi(d)\\ &\le \varphi(d)+\varphi'(m)(m-u(t))+\varphi'(M)(M-d)\\ &= \varphi(d)+\varphi'(m)(m-d)+\varphi'(M)(M-d)+\varphi'(m)(d-u(t)).\end{aligned}$$

From $m \leq u(t)$ we have $\varphi'(m) \leq (\varphi' \circ u)(t)$, hence

$$\begin{aligned}
&\varphi(m) + \varphi(M) - (\varphi \circ u)(t) \\
&\leq \varphi(d) + \varphi'(m)(m-d) + \varphi'(M)(M-d) + (\varphi' \circ u)(t)(d - u(t)) \\
&= \varphi(d) + \varphi'(m)(m-d) + \varphi'(M)(M-d) - (\varphi' \circ u)(t)(u(t) - d),
\end{aligned}$$

which is again (2.4).

In other words, for any $d, u(t) \in [m, M]$ the inequality in (2.4) holds. Integrating (2.4) over Ω and dividing by $\mu(\Omega)$ we obtain the second inequality in (2.1). The proof is complete. □

Corollary 1. *Let $\varphi : (a,b) \to \mathbb{R}$ be a convex function. If $\boldsymbol{p} \in \mathbb{R}_+^n$ and $\boldsymbol{x} \in [m, M]^n$ then the inequalities*

$$\begin{aligned}
&\varphi(c) + \varphi'(c)\left(m + M - c - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \\
&\leq \varphi(m) + \varphi(M) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) \qquad (2.5) \\
&\leq \varphi(d) + \varphi'(m)(m-d) + \varphi'(M)(M-d) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi'(x_i)(x_i - d)
\end{aligned}$$

hold for all $c, d \in [m, M]$.

Proof. This is a straightforward consequence of Theorem 1. We simply choose

$$\begin{aligned}
\Omega &= \{1, 2, \ldots, n\}, \\
\mu(\{i\}) &= p_i, \quad i = 1, 2, \ldots, n, \\
u(i) &= x_i, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

□

Corollary 2. *The following inequalities are valid under the assumptions of Corollary 1:*

$$\begin{aligned}
0 &\leq \varphi(m) + \varphi(M) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) - \varphi(\overline{x}) \\
&\leq \varphi'(m)(m - \overline{x}) + \varphi'(M)(M - \overline{x}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi'(x_i)(x_i - \overline{x}),
\end{aligned}$$

where $\overline{x} = m + M - 1/P_n \sum_{i=1}^{n} p_i x_i$.

Corollary 3. *Suppose that the conditions of Corollary* 1 *are satisfied and additionally assume*

$$\sum\nolimits_{i=1}^{n} p_i\varphi'(x_i) \neq P_n\left[\varphi'(m)+\varphi'(M)\right],$$

$$\widetilde{x} = \frac{P_n\left[m\varphi'(m)+M\varphi'(M)\right]-\sum\nolimits_{i=1}^{n} p_i x_i\varphi'(x_i)}{P_n\left[\varphi'(m)+\varphi'(M)\right]-\sum\nolimits_{i=1}^{n} p_i\varphi'(x_i)} \in [m,M].$$

Then

$$\varphi(m)+\varphi(M)-\frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi(x_i) \leq \varphi(\widetilde{x}).$$

The inequalities obtained in Corollary 2 and 3 are the Mercer-type variants of the corresponding inequalities given in [4] and [12].

Theorem 2. *Let $\varphi:(a,b)\to\mathbb{R}$ be a convex function and $\boldsymbol{w}\in\mathbb{R}^l$ such that*

$$0 \leq W_k = \sum_{i=1}^{k} w_i \leq W_l,\ k=1,\ldots,l, \quad W_l > 0.$$

Let $\boldsymbol{\xi}\in[m,M]^l$ be such that $\xi_1\leq\xi_2\leq\cdots\leq\xi_l$ or $\xi_1\geq\xi_2\geq\cdots\geq\xi_l$. Then the inequalities

$$\begin{aligned}
&\varphi(c)+\varphi'(c)\left(m+M-c-\frac{1}{W_l}\sum_{i=1}^{l} w_i\xi_i\right)\\
&\leq \varphi(m)+\varphi(M)-\frac{1}{W_l}\sum_{i=1}^{l} w_i\varphi(\xi_i) \qquad (2.6)\\
&\leq \varphi(d)+\varphi'(m)(m-d)+\varphi'(M)(M-d)-\frac{1}{W_l}\sum_{i=1}^{l} w_i\varphi'(\xi_i)(\xi_i-d)
\end{aligned}$$

hold for all $c,d\in[m,M]$.

Proof. For $n=l+2$ we define

$$\begin{array}{cccccc}
x_1=m, & x_2=\xi_1, & x_3=\xi_2, & \ldots & x_{n-1}=\xi_l, & x_n=M\\
p_1=1, & p_2=-w_1/W_l, & p_2=-w_2/W_l, & \ldots & p_{n-1}=-w_l/W_l, & p_n=1
\end{array}.$$

It is obvious that $x_1\leq x_2\leq\cdots\leq x_n$ if $\xi_1\leq\xi_2\leq\cdots\leq\xi_l$ or $x_1\geq x_2\geq\cdots\geq x_n$ if $\xi_1\geq\xi_2\geq\cdots\geq\xi_l$ and that

$$0 \leq P_k = \sum_{i=1}^{k} p_i \leq P_n, \quad k=1,2,\ldots,n, \quad P_n = 1 > 0,$$

hence we can apply Theorem D on φ, $\boldsymbol{x}$ and $\boldsymbol{p}$ thus obtaining (2.6). □

Note that under the conditions of Theorem 2 we also have

$$\overline{\xi}=m+M-\frac{1}{W_l}\sum_{i=1}^{l}w_i\xi_i\in[m,M],$$

which means that in (2.6) we can choose $c=\overline{\xi}$ in which case the first inequality in (2.6) becomes the generalized Mercer inequality as it was stated in [1]. Mercer's inequality itself can be obtained in the same way as a special case of Corollary 1.

Corollary 4. *The following inequalities are valid under the assumptions of Theorem 2:*

$$\begin{aligned}0&\leq\varphi(m)+\varphi(M)-\frac{1}{W_l}\sum_{i=1}^{l}w_i\varphi(\xi_i)-\varphi\left(\overline{\xi}\right)\\&\leq\varphi'(m)\left(m-\overline{\xi}\right)+\varphi'(M)\left(M-\overline{\xi}\right)-\frac{1}{W_l}\sum_{i=1}^{l}w_i\varphi'(\xi_i)\left(\xi_i-\overline{\xi}\right),\end{aligned}$$

where

$$\overline{\xi}=m+M-\frac{1}{W_l}\sum_{i=1}^{l}w_i\xi_i.$$

Corollary 5. *Suppose that the conditions of Theorem 2 are satisfied and additionally assume*

$$\sum\nolimits_{i=1}^{l}w_i\varphi'(\xi_i)\neq W_l\left[\varphi^{'}(m)+\varphi^{'}(M)\right],$$

$$\widetilde{\xi}=\frac{W_l\left[m\varphi^{'}(m)+M\varphi^{'}(M)\right]-\sum\nolimits_{i=1}^{l}w_i\xi_i\varphi^{'}(\xi_i)}{W_l\left[\varphi'(m)+\varphi'(M)\right]-\sum\nolimits_{i=1}^{l}w_i\varphi'(\xi_i)}\in(m,M).$$

Then

$$\varphi(m)+\varphi(M)-\frac{1}{W_l}\sum_{i=1}^{l}w_i\varphi(\xi_i)\leq\varphi\left(\widetilde{\xi}\right).$$

The inequalities given in Corollary 4 are the Mercer type variants of a result from [5] and the inequality given in Corollary 5 is the Mercer type variant of Theorem F.

Now we prove the integral case of Theorem 2.

Theorem 3. *Suppose that $f:[\alpha,\beta]\to[m,M],\varphi,\lambda,\overline{x}$ and $\overline{y}$ are all as in Theorem E and additionally assume that φ is continuously differentiable. Then the inequalities*

$$\begin{aligned}&\varphi(c)+\varphi'(c)(m+M-c-\overline{x})\leq\varphi(m)+\varphi(M)-\overline{y}\\&\leq\varphi(d)+\varphi'(m)(m-d)+\varphi'(M)(M-d)\\&-\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}\varphi'(f(t))[f(t)-d]\,\mathrm{d}\lambda(t)\end{aligned}\qquad(2.7)$$

hold for each $c,d\in[m,M]$.

Proof. Suppose that f is nondecreasing (for f nonincreasing the proof is analogous). For arbitrary $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$ such that $\tilde{\alpha} < \alpha$ and $\tilde{\beta} > \beta$ we define a new function $\tilde{f} : [\tilde{\alpha}, \tilde{\beta}] \to [m, M]$ by

$$\tilde{f}(t) = \begin{cases} m + \frac{f(\alpha)-m}{\alpha-\tilde{\alpha}}(t-\tilde{\alpha}), & t \in [\tilde{\alpha}, \alpha], \\ f(t), & t \in [\alpha, \beta], \\ M + \frac{M-f(\beta)}{\tilde{\beta}-\beta}(t-\tilde{\beta}), & t \in [\beta, \tilde{\beta}]. \end{cases}$$

It can be easily seen that the function $\tilde{f}$ is continuous and nondecreasing.

Next we define two new functions $\tilde{\lambda}_s : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ and $\tilde{\lambda}_c : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ by

$$\tilde{\lambda}_s(t) = \begin{cases} 1, & t = \tilde{\alpha}, \\ 0, & t \in (\tilde{\alpha}, \tilde{\beta}), \\ -1, & t = \tilde{\beta}, \end{cases}$$

and

$$\tilde{\lambda}_c(t) = \begin{cases} 1, & t \in [\tilde{\alpha}, \alpha], \\ \frac{\lambda(\beta)-\lambda(t)}{\lambda(\beta)-\lambda(\alpha)}, & t \in [\alpha, \beta], \\ 0, & t \in [\beta, \tilde{\beta}]. \end{cases}$$

Note that for any function $g : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ continuous at the points $\tilde{\alpha}$ and $\tilde{\beta}$ we have

$$\begin{aligned} \int_{\tilde{\alpha}}^{\tilde{\beta}} g(t)\,\mathrm{d}\tilde{\lambda}_s(t) &= g(\tilde{\alpha})[\tilde{\lambda}_s(\tilde{\alpha}+0) - \tilde{\lambda}_s(\tilde{\alpha})] + g(\tilde{\beta})[\tilde{\lambda}_s(\tilde{\beta}) - \tilde{\lambda}_s(\tilde{\beta}-0)] \\ &= -g(\tilde{\alpha}) - g(\tilde{\beta}). \end{aligned} \tag{2.8}$$

Also, if λ is continuous on $[\alpha, \beta]$ then $\tilde{\lambda}_c$ is continuous on $[\tilde{\alpha}, \tilde{\beta}]$, and if λ is of bounded variation on $[\alpha, \beta]$ then $\tilde{\lambda}_c$ is of bounded variation on $[\tilde{\alpha}, \tilde{\beta}]$. This means that for any continuous and piecewise monotonic function $g : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ the integral $\int_{\tilde{\alpha}}^{\tilde{\beta}} g(t)\mathrm{d}\tilde{\lambda}_c(t)$ is well defined and

$$\begin{aligned} \int_{\tilde{\alpha}}^{\tilde{\beta}} g(t)\,\mathrm{d}\tilde{\lambda}_c(t) &= \int_{\tilde{\alpha}}^{\alpha} g(t)\,\mathrm{d}\tilde{\lambda}_c(t) + \int_{\alpha}^{\beta} g(t)\,\mathrm{d}\tilde{\lambda}_c(t) + \int_{\beta}^{\tilde{\beta}} g(t)\,\mathrm{d}\tilde{\lambda}_c(t) \\ &= \int_{\alpha}^{\beta} g(t)\,\mathrm{d}\tilde{\lambda}_c(t) = \int_{\alpha}^{\beta} g(t)\,\mathrm{d}\left[\frac{\lambda(\beta)-\lambda(t)}{\lambda(\beta)-\lambda(\alpha)}\right] \\ &= -\frac{1}{\lambda(\beta)-\lambda(\alpha)} \int_{\alpha}^{\beta} g(t)\,\mathrm{d}\lambda(t). \end{aligned} \tag{2.9}$$

Now we define $\tilde{\lambda} : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ by

$$\tilde{\lambda}(t) = \tilde{\lambda}_c(t) - \tilde{\lambda}_s(t), \quad t \in [\tilde{\alpha}, \tilde{\beta}].$$

From (2.8) and (2.9) we conclude that the integral $\int_{\tilde{\alpha}}^{\tilde{\beta}} g(t)\mathrm{d}\tilde{\lambda}(t)$ is well defined for any continuous and piecewise monotonic function $g : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ and

$$\begin{aligned}\int_{\tilde{\alpha}}^{\tilde{\beta}} g(t)\,\mathrm{d}\tilde{\lambda}(t) &= \int_{\tilde{\alpha}}^{\tilde{\beta}} g(t)\,\mathrm{d}\tilde{\lambda}_c(t) - \int_{\tilde{\alpha}}^{\tilde{\beta}} g(t)\,\mathrm{d}\tilde{\lambda}_s(t) \\ &= -\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} g(t)\,\mathrm{d}\lambda(t) + g(\tilde{\alpha}) + g(\tilde{\beta}). \qquad (2.10)\end{aligned}$$

We also have

$$\tilde{\lambda}(\tilde{\beta}) - \tilde{\lambda}(\tilde{\alpha}) = \tilde{\lambda}_c(\tilde{\beta}) - \tilde{\lambda}_c(\tilde{\alpha}) - \tilde{\lambda}_s(\tilde{\beta}) + \tilde{\lambda}_s(\tilde{\alpha}) = 0 - 1 + 1 + 1 = 1.$$

If we apply Theorem E on the functions $\tilde{f}, \varphi$ and $\tilde{\lambda}$ (we can do that even if the function $\tilde{\lambda}$ is neither continuous nor of bounded variation since all the integrals are well defined) we obtain

$$\begin{aligned}&\varphi(c) + \varphi'(c)(\tilde{x} - c) \\ &\leq \tilde{y} \leq \varphi(d) + \frac{1}{\tilde{\lambda}(\beta) - \tilde{\lambda}(\alpha)}\int_{\tilde{\alpha}}^{\tilde{\beta}} \varphi'\left(\tilde{f}(t)\right)\left[\tilde{f}(t) - d\right]\mathrm{d}\tilde{\lambda}(t)\end{aligned}$$

where

$$\begin{aligned}\tilde{x} &= \frac{1}{\tilde{\lambda}\left(\tilde{\beta}\right) - \tilde{\lambda}(\tilde{\alpha})}\int_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{f}(t)\,\mathrm{d}\tilde{\lambda}(t) = \int_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{f}(t)\,\mathrm{d}\tilde{\lambda}(t) \\ &= -\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} f(t)\,\mathrm{d}\lambda(t) + \tilde{f}(\tilde{\alpha}) + \tilde{f}(\tilde{\beta}) \\ &= m + M - \overline{x}\end{aligned}$$

and

$$\begin{aligned}\tilde{y} &= \frac{1}{\tilde{\lambda}\left(\tilde{\beta}\right) - \tilde{\lambda}(\tilde{\alpha})}\int_{\tilde{\alpha}}^{\tilde{\beta}} \varphi\left(\tilde{f}(t)\right)\mathrm{d}\tilde{\lambda}(t) = \int_{\tilde{\alpha}}^{\tilde{\beta}} \varphi\left(\tilde{f}(t)\right)\mathrm{d}\tilde{\lambda}(t) \\ &= \varphi(m) + \varphi(M) - \overline{y}.\end{aligned}$$

Now we have

$$\begin{aligned}&\varphi(c) + \varphi'(c)(m + M - c - \overline{x}) \leq \varphi(m) + \varphi(M) - \overline{y} \\ &\leq \varphi(d) + \int_{\tilde{\alpha}}^{\tilde{\beta}} \varphi'\left(\tilde{f}(t)\right)\left[\tilde{f}(t) - d\right]\mathrm{d}\tilde{\lambda}(t), \qquad (2.11)\end{aligned}$$

and if in the second inequality in (2.11) we apply (2.10) for the function $g : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ defined by

$$g(t) = \varphi'\left(\tilde{f}(t)\right)\left[\tilde{f}(t) - d\right]$$

we obtain (2.7). The proof is complete. □

References

[1] S. Abramovich, M. Klaričić Bakula, M. Matić, J. Pečarić, *A variant of Jensen-Steffensen's inequality and quasi-arithmetic means*, J. Math. Anal. Appl. **307**(3) (2005), 370–376.

[2] S. Abramovich, M. Klaričić Bakula, S. Banić, *A variant of Jensen-Steffensen's inequality for convex and superquadratic functions*, JIPAM **7** (2) (2006), Article 70.

[3] R.P. Boas, *The Jensen-Steffensen inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **302-319** (1970), 1–8.

[4] S.S. Dragomir, C.J. Goh, *A counterpart of Jensen's inequality*, Math. Comput. Modelling **24**(2) (1996), 1–11.

[5] N. Elezović, J. Pečarić, *A counterpart to Jensen-Steffensen's discrete inequality for differentiable convex mappings and applications in information theory*, Rad HAZU **481**, Mat. znan. 14 (2003), 25–28.

[6] M. Klaričić Bakula, M. Matić, J. Pečarić, *Generalizations of the Jensen-Steffensen and related inequalities*, submitted for publication.

[7] M. Matić, J. Pečarić, *Some companion inequalities to Jensen's inequality*, Math. Ineq. Appl. **3**(3) (2000), 355–368.

[8] A. McD. Mercer, *A variant of Jensen's inequality*, JIPAM **4** (4) (2003), Article 73.

[9] J. Pečarić, *A companion to Jensen-Steffensen's inequality*, J. Approx. Theory **44**(3) (1985), 289–291.

[10] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Inc., 1992.

[11] A.W. Roberts, D.E. Varberg, *Convex Functions*, Academic Press, 1973.

[12] M.L. Slater, *A companion inequality to Jensen's inequality*, J. Aprox. Theory **32**(2) (1981), 160–166.

[13] A. Witkowski, *A new Proof of the Monotonicity Property of Power Means*, JIPAM **5**(3) (2004), Article 73.

Milica Klaričić Bakula and Marko Matić
Department of Mathematics
Faculty of Natural Sciences, Mathematics and Education
University of Split
Teslina 12
21000 Split, Croatia
e-mail: milica@pmfst.hr
e-mail: mmatic@fesb.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz Baruna Filipovića 30
10000 Zagreb, Croatia
e-mail: pecaric@hazu.hr