Chapter 1

Curves and Gradients in Metric Spaces

As we briefly discussed in the introduction, the notion of gradient flows in a metric space $\mathscr{S}$ relies on two elementary but basic concepts: the metric derivative of an absolutely continuous curve with values in $\mathscr{S}$ and the upper gradients of a functional defined in $\mathscr{S}$. The related definitions are presented in the next two sections (a more detailed treatment of this topic can be found for instance in [20]); the last one deals with curves of maximal slope.
When $\mathscr{S}$ is a Banach space and its distance is induced by the norm, one can expect that curves of maximal slope could also be characterized as solutions of (doubly, if $\mathscr{S}$ is not Hilbertian) nonlinear (sub)differential inclusions: this aspect is discussed in the last part of this chapter.
Throughout this chapter (and in the following ones of this first part)

$$(\mathscr{S}, d) \quad \text{will be a given } \textit{complete metric space}; \tag{1.0.1}$$

we will denote by (a, b) a generic open (possibly unbounded) interval of $\mathbb{R}$.

1.1 Absolutely continuous curves and metric derivative

Definition 1.1.1 (Absolutely continuous curves). *Let $(\mathscr{S}, d)$ be a complete metric space and let $v : (a, b) \to \mathscr{S}$ be a curve; we say that v belongs to $AC^p(a, b; \mathscr{S})$, for $p \in [1, +\infty]$, if there exists $m \in L^p(a, b)$ such that*

$$d(v(s), v(t)) \le \int_s^t m(r)\, dr \qquad \forall\, a < s \le t < b. \tag{1.1.1}$$

In the case $p = 1$ we are dealing with absolutely continuous *curves and we will denote the corresponding space simply with $AC(a, b; \mathscr{S})$.*

We recall also that a map $\varphi : (a,b) \to \mathbb{R}$ is said to have *finite pointwise variation* if

$$\sup \left\{ \sum_{i=1}^{n-1} |\varphi(t_{i+1}) - \varphi(t_i)| : \ a < t_1 < \cdots < t_n < b \right\} < +\infty. \tag{1.1.2}$$

It is well known that any bounded monotone function has finite pointwise variation and that any function with finite pointwise variation can be written as the difference of two bounded monotone functions.

Any curve in $AC^p(a,b;\mathscr{S})$ is uniformly continuous; if $a > -\infty$ (resp. $b < +\infty$) we will denote by $v(a+)$ (resp. $v(b-)$) the right (resp. left) limit of v, which exists since $\mathscr{S}$ is complete. The above limit exist even in the case $a = -\infty$ (resp. $b = +\infty$) if $v \in AC(a,b;\mathscr{S})$. Among all the possible choices of m in (1.1.1) there exists a minimal one, which is provided by the following theorem (see [7, 8, 20]).

Theorem 1.1.2 (Metric derivative). *Let $p \in [1,+\infty]$. Then for any curve v in $AC^p(a,b;\mathscr{S})$ the limit*

$$|v'|(t) := \lim_{s\to t} \frac{d(v(s),v(t))}{|s-t|} \tag{1.1.3}$$

exists for $\mathscr{L}^1$-a.e. $t \in (a,b)$. Moreover the function $t \mapsto |v'|(t)$ belongs to $L^p(a,b)$, it is an admissible integrand for the right hand side of (1.1.1), *and it is minimal in the following sense:*

$$\begin{gathered} |v'|(t) \le m(t) \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t \in (a,b), \\ \textit{for each function } m \textit{ satisfying } (1.1.1). \end{gathered} \tag{1.1.4}$$

Proof. Let $(y_n) \subset \mathscr{S}$ be dense in $v((a,b))$ and let $\mathsf{d}_n(t) := d(y_n, v(t))$. Since all functions d_n are absolutely continuous in (a,b) the function

$$\mathsf{d}(t) := \sup_{n\in\mathbb{N}} |\mathsf{d}_n'(t)|$$

is well defined $\mathscr{L}^1$-a.e. in (a,b). Let $t \in (a,b)$ be a point where all functions d_n are differentiable and notice that

$$\liminf_{s\to t} \frac{d(v(s),v(t))}{|s-t|} \ge \sup_{n\in\mathbb{N}} \liminf_{s\to t} \frac{|\mathsf{d}_n(s) - \mathsf{d}_n(t)|}{|s-t|} = \mathsf{d}(t).$$

This inequality together with (1.1.1) shows that $\mathsf{d} \le m$ $\mathscr{L}^1$-a.e., therefore $\mathsf{d} \in L^p(a,b)$. On the other hand the definition of d gives

$$d(v(s),v(t)) = \sup_{n\in\mathbb{N}} |\mathsf{d}_n(s) - \mathsf{d}_n(t)| \le \int_s^t \mathsf{d}(r)\,dr \qquad \forall s,\, t \in (a,b),\ s \le t,$$

and therefore

$$\limsup_{s\to t} \frac{d(v(s),v(t))}{|s-t|} \le \mathsf{d}(t)$$

at any Lebesgue point t of d. □

In the next remark we deal with the case when the target space is a dual Banach space, see for instance [13].

Remark 1.1.3 (Derivative in Banach spaces). Suppose that $\mathscr{S} = \mathscr{B}$ is a *reflexive Banach space* (respectively: a *dual Banach space*): then a curve v belongs to $AC^p(a,b;\mathscr{S})$ if and only if it is differentiable (resp. weakly*-differentiable) at $\mathscr{L}^1$-a.e. point $t \in (a,b)$, its derivative v' belongs to $L^p(a,b;\mathscr{B})$ (resp. to $L^p_{w^*}(a,b;\mathscr{B})$) and

$$v(t) - v(s) = \int_s^t v'(r)\,dr \quad \forall\, a < s \le t < b. \tag{1.1.5}$$

In this case,

$$\|v'(t)\|_{\mathscr{B}} = |v'|(t) \quad \mathscr{L}^1\text{-a.e. in } (a,b). \tag{1.1.6}$$

Lemma 1.1.4 (Lipschitz and arc-length reparametrizations). *Let v be a curve in $AC(a,b;\mathscr{S})$ with length $L := \int_a^b |v'|(t)\,dt$.*

(a) *For every $\varepsilon > 0$ there exists a strictly increasing absolutely continuous map*

$$\mathsf{s}_\varepsilon : (a,b) \to (0, L_\varepsilon) \quad \textit{with } \mathsf{s}_\varepsilon(a+) = 0,\ \mathsf{s}_\varepsilon(b-) = L_\varepsilon := L + \varepsilon(b-a), \tag{1.1.7}$$

and a Lipschitz curve $\hat v_\varepsilon : (0, L_\varepsilon) \to \mathscr{S}$ such that

$$v = \hat v_\varepsilon \circ \mathsf{s}_\varepsilon, \quad |\hat v_\varepsilon'| \circ \mathsf{s}_\varepsilon = \frac{|v'|}{\varepsilon + |v'|} \in L^\infty(a,b). \tag{1.1.8}$$

The map s_ε admits a Lipschitz continuous inverse $\mathsf{t}_\varepsilon : (0, L_\varepsilon) \to (a,b)$ with Lipschitz constant less than ε^{-1}, and $\hat v_\varepsilon = v \circ \mathsf{t}_\varepsilon$.

(b) *There exists an increasing absolutely continuous map*

$$\mathsf{s} : (a,b) \to [0, L] \quad \textit{with } \mathsf{s}(a+) = 0,\ \mathsf{s}(b-) = L, \tag{1.1.9}$$

and a Lipschitz curve $\hat v : [0, L] \to \mathscr{S}$ such that

$$v = \hat v \circ \mathsf{s}, \quad |\hat v'| = 1 \quad \mathscr{L}^1\textit{-a.e. in } [0, L]. \tag{1.1.10}$$

Proof. Let us first consider the case (a) with $\varepsilon > 0$; we simply define

$$\mathsf{s}_\varepsilon(t) := \int_a^t \big(\varepsilon + |v'|(\theta)\big)\,d\theta, \quad t \in (a,b); \tag{1.1.11}$$

s_ε is strictly increasing with $\mathsf{s}_\varepsilon' \ge \varepsilon$, $\mathsf{s}_\varepsilon\big((a,b)\big) = (0, L_\varepsilon)$, its inverse map $\mathsf{t}_\varepsilon : (0, L_\varepsilon) \to (a,b)$ satisfies a Lipschitz condition with constant $\le \varepsilon^{-1}$, and

$$\mathsf{t}_\varepsilon' \circ \mathsf{s}_\varepsilon = \frac{1}{\varepsilon + |v'|} \quad \mathscr{L}^1\text{-a.e. in } (a,b).$$

Setting $\hat v^\varepsilon := v \circ \mathsf{t}_\varepsilon$, for every choice of $t_i = \mathsf{t}_\varepsilon(s_i)$ with $0 < s_1 < s_2 < L_\varepsilon$ we have

$$\begin{aligned} d(\hat v_\varepsilon(s_1), \hat v_\varepsilon(s_2)) = d(v(t_1), v(t_2)) &\le \int_{t_1}^{t_2} |v'|(t)\,dt \\ &\le \mathsf{s}_\varepsilon(t_2) - \mathsf{s}_\varepsilon(t_1) - \varepsilon(t_2 - t_1) = s_2 - s_1 - \varepsilon(t_2 - t_1), \end{aligned} \tag{1.1.12}$$

so that $\hat v_\varepsilon$ is 1-Lipschitz and can be extended to $[0, L_\varepsilon]$ since $\hat v_\varepsilon(0+) = v(a+)$ and $\hat v_\varepsilon(L_\varepsilon -) = v$; dividing the above inequality by $s_2 - s_1$ and passing to the limit as $s_2 \to s_1$ we get the bound

$$|\hat v'_\varepsilon| \circ \mathsf{s}_\varepsilon \le 1 - \frac{\varepsilon}{\varepsilon + |v'|} = \frac{|v'|}{\varepsilon + |v'|} \quad \mathscr{L}^1\text{-a.e. in } (a,b). \tag{1.1.13}$$

On the other hand,

$$\begin{aligned} d(v(t_2), v(t_1)) = d(\hat v_\varepsilon(s_2), \hat v_\varepsilon(s_1)) &\le \int_{s_1}^{s_2} |\hat v'_\varepsilon|(s)\, ds \\ = \int_{t_1}^{t_2} |\hat v'_\varepsilon|(\mathsf{s}_\varepsilon(t))\, \mathsf{s}'_\varepsilon(t)\, dt &\le \int_{t_1}^{t_2} \big(|\hat v'_\varepsilon| \circ \mathsf{s}_\varepsilon\big)\big(\varepsilon + |v'|\big)\, dt. \end{aligned} \tag{1.1.14}$$

By (1.1.4) we obtain

$$|v'| \le \big(|\hat v'_\varepsilon| \circ \mathsf{s}_\varepsilon\big)\big(\varepsilon + |v'|\big) \quad \mathscr{L}^1\text{-a.e. in } (a,b),$$

which, combined with the converse inequality (1.1.13), yields (1.1.8).

(b) We define $\mathsf{s} := \mathsf{s}_0$ for $\varepsilon = 0$ by (1.1.11) and we consider the left continuous, increasing map

$$\mathsf{t}(s) := \min\big\{t \in [a,b] : \mathsf{s}(t) = s\big\}, \quad s \in [0, L],$$

which satisfies $\mathsf{s}(\mathsf{t}(s)) = s$ in $[0, L]$. Moreover, still denoting by v its continuous extension to the closed interval $[a,b]$, we observe that

$$\mathsf{t}(\mathsf{s}(t)) \le t, \quad v(\mathsf{t}(\mathsf{s}(t))) = v(t) \quad \forall\, t \in [a,b], \tag{1.1.15}$$

since

$$d(v(\mathsf{t}(\mathsf{s}(t))), v(t)) = \int_{\mathsf{t}(\mathsf{s}(t))}^{t} |v'|(\theta)\, d\theta = \mathsf{s}(t) - \mathsf{s}(t) = 0.$$

Defining $\hat v := v \circ \mathsf{t}$ as above, (1.1.12) (with $\varepsilon = 0$) shows that $\hat v$ is 1-Lipschitz and (1.1.15) yields $v = \hat v \circ \mathsf{s}$. Finally, (1.1.14) shows that $|\hat v'| \circ \mathsf{s} = 1$ $\mathscr{L}^1$-a.e. in (a,b). □

1.2 Upper gradients

In this section we define a kind of "modulus of the gradient" for real valued functions defined on metric spaces, following essentially the approach of [92, 51].

Let $\phi : \mathscr{S} \to (-\infty, +\infty]$ be an extended real functional, with proper effective domain

$$D(\phi) := \{v \in \mathscr{S} : \phi(v) < +\infty\} \ne \varnothing. \tag{1.2.1}$$

If $\mathscr{S}$ is a vector space and ϕ is differentiable, then $|\nabla\phi|$ has the following natural variational characterization:

$$g \geq |\nabla\phi| \quad \Leftrightarrow \quad \begin{array}{c} \big|(\phi\circ v)'\big| \leq g(v)|v'| \\ \text{for every regular curve } v:(a,b)\to\mathscr{S}. \end{array} \tag{1.2.2}$$

We want to define a notion of "upper gradient" g for ϕ modeled on (1.2.2). A first possibility is to use an integral formulation of (1.2.2) along absolutely continuous curves.

Definition 1.2.1 (Strong upper gradients, [92, 51]). *A function $g:\mathscr{S}\to[0,+\infty]$ is a* strong upper gradient *for ϕ if for every absolutely continuous curve $v\in AC(a,b;\mathscr{S})$ the function $g\circ v$ is Borel and*

$$\big|\phi(v(t))-\phi(v(s))\big| \leq \int_s^t g(v(r))|v'|(r)\,dr \quad \forall\, a<s\leq t<b. \tag{1.2.3}$$

In particular, if $g\circ v|v'|\in L^1(a,b)$ then $\phi\circ v$ is absolutely continuous and

$$|(\phi\circ v)'(t)| \leq g(v(t))|v'|(t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(a,b). \tag{1.2.4}$$

We also introduce a weaker notion, based on a pointwise formulation:

Definition 1.2.2 (Weak upper gradients). *A function $g:\mathscr{S}\to[0,+\infty]$ is a* weak upper gradient *for ϕ if every curve $v\in AC(a,b;\mathscr{S})$ such that*

(i) *$g\circ v|v'|\in L^1(a,b)$;*

(ii) *$\phi\circ v$ is $\mathscr{L}^1$-a.e. equal in (a,b) to a function φ with finite pointwise variation in (a,b);*

we have

$$|\varphi'(t)| \leq g(v(t))|v'|(t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(a,b). \tag{1.2.5}$$

In this case, if $\phi\circ v\in AC(a,b)$ then $\varphi=\phi\circ v$ and (1.2.3) holds.

Remark 1.2.3 (Approximate derivative). Condition (ii) of Definition 1.2.2 is equivalent to say that $\phi\circ v$ has *essential bounded variation* in (a,b). Accordingly, condition (1.2.5) could be stated without any reference to φ by replacing $\varphi'(t)$ with the *approximate derivative* of $\phi\circ v$ (see Definition 5.5.1).

Among all the possible choices for an upper gradient of ϕ, we recall the definition of the local and global slopes (see also [51], [64]):

Definition 1.2.4 (Slopes). *The* local and global slopes *of ϕ at $v\in D(\phi)$ are defined by*

$$|\partial\phi|(v) := \limsup_{w\to v}\frac{\big(\phi(v)-\phi(w)\big)^+}{d(v,w)}, \quad \mathfrak{l}_\phi(v) := \sup_{w\neq v}\frac{\big(\phi(v)-\phi(w)\big)^+}{d(v,w)}. \tag{1.2.6}$$

Theorem 1.2.5 (Slopes are upper gradients). *The function $|\partial\phi|$ is a weak upper gradient for ϕ. If ϕ is d-lower semicontinuous then $\mathfrak{l}_\phi$ is a strong upper gradient for ϕ.*

Proof. In order to show that $|\partial\phi|$ is a weak upper gradient we consider an absolutely continuous curve $v:(a,b)\to\mathscr{S}$ satisfying the assumptions of Definition 1.2.2; we introduce the set

$$A := \big\{t\in(a,b):\ \phi(v(t))=\varphi(t),\ \varphi \text{ is differentiable at } t,\ \exists\,|v'|(t)\big\}$$

and we observe that $(a,b)\setminus A$ is $\mathscr{L}^1$-negligible.

If the derivative of φ vanishes at $t\in A$ then (1.2.5) is surely satisfied, therefore it is not restrictive to consider points $t\in A$ such that $\varphi'(t)\neq 0$. In order to fix the ideas, let us suppose that $t\in A$ and $\varphi'(t)>0$; since $d(v(s),v(t))\neq 0$ when $s\in A\setminus\{t\}$ belongs to a suitable neighborhood of t we have

$$\begin{aligned}|\varphi'(t)|=\varphi'(t)&=\lim_{s\uparrow t,s\in A}\frac{\phi(v(t))-\phi(v(s))}{t-s}=\lim_{s\uparrow t,s\in A}\frac{\phi(v(t))-\phi(v(s))}{d(v(s),v(t))}\frac{d(v(s),v(t))}{t-s}\\&\le\limsup_{s\uparrow t,s\in A}\frac{\phi(v(t))-\phi(v(s))}{d(v(s),v(t))}\lim_{s\uparrow t,s\in A}\frac{d(v(s),v(t))}{t-s}\le|\partial\phi|(v(t))\,|v'|(t).\end{aligned}$$

In order to check the second part of the Theorem, we notice first that $v\mapsto\mathfrak{l}_\phi(v)$ is lower semicontinuous in $\mathscr{S}$. Indeed, if $w\neq v$ and $v_h\to v$ then $w\neq v_h$ for h large enough and therefore

$$\liminf_{h\to\infty}\mathfrak{l}_\phi(v_h)\ge\liminf_{h\to\infty}\frac{\big(\phi(v_h)-\phi(w)\big)^+}{d(v_h,w)}\ge\frac{\big(\phi(v)-\phi(w)\big)^+}{d(v,w)}.$$

By taking the supremum w.r.t. w the lower semicontinuity follows.

Let now v be a curve in $AC(a,b;\mathscr{S})$ satisfying $\mathfrak{l}_\phi(v)|v'|\in L^1(a,b)$ and notice that $\mathfrak{l}_\phi(v)$ is lower semicontinuous, therefore Borel. We apply Lemma 1.1.4 with $\varepsilon=0$, and for the increasing and absolutely continuous map $\mathsf{s}:=\mathsf{s}_0:[a,b]\to[0,L]$ defined by (1.1.11) we set

$$\hat v(s):=v(\mathsf{t}(s)),\quad \varphi(s):=\phi(\hat v(s)),\quad g(s):=\mathfrak{l}_\phi(\hat v(s))\qquad s\in(0,L)$$

and we observe that for each couple $s_1,\ s_2\in(0,L)$ we have $(\varphi(s_1)-\varphi(s_2))^+\le g(s_1)|s_2-s_1|$, hence

$$|\varphi(s_1)-\varphi(s_2)|\le\max[g(s_1),g(s_2)]\,|s_2-s_1|. \tag{1.2.7}$$

The 1-dimensional change of variables formula gives

$$\int_0^L g(s)\,ds=\int_a^b\mathfrak{l}_\phi(v(t))|v'|(t)\,dt<+\infty, \tag{1.2.8}$$

therefore $g \in L^1(0, L)$ and (1.2.7) shows that φ belongs to the metric Sobolev space $W^{1,1}_m(0, L)$ in the sense of Hajłasz [91]. By a difference quotients argument this condition implies (see Lemma 1.2.6 below and [20]) that φ belongs to the conventional Sobolev space $W^{1,1}(0, L)$ and we simply have to check that φ coincides with its continuous representative. Since $\hat{v}$ is a Lipschitz map we immediately see that φ is lower semicontinuous in $(0, L)$: therefore continuity follows if we show that

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \varphi(s + r)\, dr \le \varphi(s) \quad \text{for all } s \in (0, L). \tag{1.2.9}$$

Invoking (1.2.7) we get

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \big(\varphi(s + r) - \varphi(s)\big)\, dr \le \limsup_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \big(\varphi(s + r) - \varphi(s)\big)^+ dr$$
$$\le \limsup_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} g(s + r)\, |r|\, dr \le \limsup_{\varepsilon \downarrow 0} \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} g(s + r)\, dr = 0.$$

Since $\phi(v(t)) = \phi(\hat{v}(\mathsf{s}(t))) = \varphi(\mathsf{s}(t))$, we obtain the absolute continuity of $\phi \circ v$; using the inequality $\mathfrak{l}_\phi(v) \ge |\partial\phi|(v)$ and the the fact that $|\partial\phi|$ is an upper gradient we conclude. □

Lemma 1.2.6. *Let φ, $g \in L^1(a, b)$ with $g \ge 0$ and assume that there exists a $\mathscr{L}^1$-negligible set $N \subset (a, b)$ such that*

$$|\varphi(s) - \varphi(t)| \le (g(s) + g(t))\, |s - t| \qquad \forall s,\, t \in (a, b) \setminus N.$$

Then $\varphi \in W^{1,1}(a, b)$ and $|\varphi'| \le 2g$ $\mathscr{L}^1$-a.e. in (a, b).

Proof. For every $\zeta \in C^\infty_c(a, b)$ we have

$$\begin{aligned} T(\zeta) := \int_a^b \varphi(t)\, \zeta'(t)\, dt &= \lim_{h \to 0} \int_a^b \varphi(t) \frac{\zeta(t + h) - \zeta(t)}{h}\, dt \\ &= \lim_{h \to 0} \int_a^b \frac{\varphi(t - h) - \varphi(t)}{h} \zeta(t)\, dt \le \limsup_{h \to 0} \int_a^b \big(g(t - h) + g(t)\big)|\zeta(t)|\, dt \\ &= 2 \int_a^b g(t)|\zeta(t)|\, dt \le 2\|g\|_{L^1(a,b)} \sup_{[a,b]} |\zeta|. \end{aligned}$$

We obtain from Riesz representation theorem that T can be represented by a signed measure λ in (a, b) having total variation less that $2\|g\|_{L^1(a,b)}$. Then, the inequality

$$\left| \int_a^b \zeta(t)\, d\lambda \right| \le 2 \int_a^b |\zeta(t)||g(t)|\, dt \qquad \forall \zeta \in C^\infty_c(a, b)$$

immediately gives that $|\lambda| \le 2|g|\mathscr{L}^1$. □

1.3 Curves of maximal slope

The notion of curves of maximal slope have been introduced (in a slight different form) in [64] and further developed in [65, 109]. Our presentation essentially follows the ideas of [8], combining them with the "upper gradient" point of view.

In order to motivate the main Definition 1.3.2 of this section, let us initially consider the finite dimensional case of the Euclidean space $\mathscr{S} := \mathbb{R}^d$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. The gradient $\nabla\phi$ of a smooth real functional $\phi : \mathscr{S} \to \mathbb{R}$ can be defined taking the derivative of ϕ along regular curves, i.e.

$$\boldsymbol{g} = \nabla\phi \quad \Leftrightarrow \quad \begin{array}{c} \big(\phi \circ v\big)' = \langle \boldsymbol{g}(v), v' \rangle \\ \text{for every regular curve } v : (0, +\infty) \to \mathscr{S}, \end{array} \tag{1.3.1}$$

and its modulus $|\nabla\phi|$ has the natural variational characterization (1.2.2). In this case, a steepest descent curve u for ϕ, i.e. a solution of the equation

$$u'(t) = -\nabla\phi(u(t)) \qquad t > 0, \tag{1.3.2}$$

can be characterized by the following two scalar conditions in $(0, +\infty)$

$$\big(\phi \circ u\big)' = -|\nabla\phi(u)|\,|u'|, \tag{1.3.3a}$$

$$|u'| = |\nabla\phi(u)|; \tag{1.3.3b}$$

in fact, (1.3.3a) forces the direction of the velocity u' to be opposite to the gradient one, whereas the modulus of u' is determined by (1.3.3b). (1.3.3a,b) are also equivalent, via Young inequality, to the single equation

$$\big(\phi \circ u\big)' = -\frac{1}{2}|u'|^2 - \frac{1}{2}|\nabla\phi(u)|^2 \quad \text{in } (0, +\infty). \tag{1.3.3c}$$

It is interesting to note that we can impose (1.3.3a,b) or (1.3.3c) as a system of differential inequalities in the couple (u, g), the first one saying that the function g is an upper bound for the modulus of the gradient (an "upper gradient", as we have seen in the previous section)

$$\big|\big(\phi \circ v\big)'\big| \le g(v)|v'| \quad \text{for every regular curve } v : (0, +\infty) \to \mathscr{S}, \tag{1.3.4a}$$

the second one imposing that the functional ϕ decreases along u as much as possible compatibly with (1.3.4a), i.e.

$$\big(\phi \circ u\big)' \le -g(u)|u'| \quad \text{in } (0, +\infty), \tag{1.3.4b}$$

and the last one prescribing the dependence of $|u'|$ on $g(u)$

$$|u'| = g(u) \quad \text{in } (0, +\infty), \tag{1.3.4c}$$

or even in a single formula

$$\big(\phi \circ u\big)' \le -\frac{1}{2}|u'|^2 - \frac{1}{2}g(u)^2 \quad \text{in } (0,+\infty). \tag{1.3.4d}$$

Whereas equations (1.3.1), (1.3.2) make sense only in a Hilbert-Riemannian framework, the formulation (1.3.4a,b,c,d) is of purely metric nature and can be extended to more general metric spaces $(\mathscr{S}, d)$, provided we understand $|u'|$ as the metric derivative of u. Of course, the concept of upper gradient provides only an upper estimate for the modulus of $\nabla\phi$ in the regular case, but it is enough to define steepest descent curves, i.e. curves which realize the minimal selection of $\frac{d}{dt}\phi(u(t))$ compatible with (1.2.4).

Remark 1.3.1 (p, q variants). Instead of (1.3.2) we can consider more general nonlinear coupling between time derivative and gradient, which naturally appears when a non euclidean distance in $\mathscr{S}$ is considered: in the last section of the present chapter we will briefly discuss the case of a Banach space.
In the easier Euclidean setting, the simplest generalization leads to an equation of the type

$$j(u'(t)) = -\nabla\phi(u(t)) \quad t > 0, \quad \text{with} \quad j(v) = \alpha(|v|)\frac{v}{|v|} \tag{1.3.5}$$

for a continuous, strictly increasing and surjective map $\alpha : [0,+\infty) \to [0,+\infty)$. In this case, the velocity u' still takes the opposite direction of $\nabla\phi(u)$ yielding (1.3.3a), but equation (1.3.3b) for its modulus is substituted by the monotone condition

$$\alpha(|u'|) = |\nabla\phi(u)|. \tag{1.3.6}$$

Introducing the strictly convex primitive function ψ of α and its conjugate ψ^*

$$\psi(z) := \int_0^z \alpha(r)\,dr, \quad \psi^*(z^*) := \max_{x\in[0,+\infty)} z^*x - \psi(x), \quad z, z^* \in [0,+\infty), \tag{1.3.7}$$

(1.3.5) is therefore equivalent to

$$\big(\phi \circ u\big)' \le -\psi(|u'|) - \psi^*(|\nabla\phi(u)|) \quad \text{in } (0,+\infty), \tag{1.3.8}$$

which, in the metric framework, could be relaxed to

$$\big(\phi \circ u\big)' \le -\psi(|u'|) - \psi^*(g((u)) \quad \text{in } (0,+\infty), \tag{1.3.9}$$

for an upper gradient g satisfying (1.3.4a).
Even if many results could be extended to this general situation (see [131]), for the sake of simplicity in the present book we will consider only a p, q-setting, where $p, q \in (1,+\infty)$ are conjugate exponents $p^{-1} + q^{-1} = 1$, corresponding to the choices

$$\alpha(z) := z^{p-1}, \quad \psi(z) = \frac{1}{p}z^p, \quad \psi^*(z^*) = \frac{1}{q}(z^*)^q,$$

and to the equation

$$j_p(u'(t)) = -\nabla\phi(u(t)), \quad j_p(v) := \begin{cases} |v|^{p-2}v & \text{for } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases} \tag{1.3.10}$$

Thus the idea is that (1.3.3a) is still imposed and (1.3.3b) is substituted by

$$|u'|^{p-1} = |\nabla\phi(u)| \quad \text{or, equivalently,} \quad |u'| = |\nabla\phi(u)|^{q-1} \tag{1.3.11}$$

and therefore, taking into account the strict convexity of $|\cdot|^p$, in the purely metric framework we end up with the inequality

$$\big(\phi\circ u\big)' \le -\frac{1}{p}|u'|^p - \frac{1}{q}g(u)^q \quad \text{in } (0,+\infty). \tag{1.3.12}$$

The limiting case $p = 1$ plays a crucial role in the modelization of rate–independent problems governed by *time-dependent* functionals ϕ_t, see [113, 131].

Recalling (1.3.4a), (1.3.4d), and (1.3.12), we introduce the following definition:

Definition 1.3.2 (Curves of maximal slope). *We say that a locally absolutely continuous map $u : (a,b) \to \mathscr{S}$ is a* p-curve of maximal slope, *$p \in (1,+\infty)$ (we will often omit to mention p in the quadratic case), for the functional ϕ with respect to its upper gradient g, if $\phi\circ u$ is $\mathscr{L}^1$-a.e. equal to a non-increasing map φ and*

$$\varphi'(t) \le -\frac{1}{p}|u'|^p(t) - \frac{1}{q}g^q(u(t)) \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t \in (a,b). \tag{1.3.13}$$

Remark 1.3.3. Observe that (1.2.5) and (1.3.13) yield

$$|u'|^p(t) = g^q(u(t)) = -\varphi'(t) \quad \mathscr{L}^1\text{-a.e. in } (a,b), \tag{1.3.14}$$

in particular $u \in AC^p_{\rm loc}(a,b;\mathscr{S})$ and $g\circ u \in L^q_{\rm loc}(a,b)$. If u is a curve of maximal slope for ϕ with respect to a strong upper gradient g, then $\phi(u(t)) \equiv \varphi(t)$ is a locally absolutely continuous map in (a,b) and the energy identity

$$\frac{1}{p}\int_s^t |u'|^p(r)\,dr + \frac{1}{q}\int_s^t g^q(r)\,dr = \phi(u(s)) - \phi(u(t)) \tag{1.3.15}$$

holds in each interval $[s,t] \subset (a,b)$.

1.4 Curves of maximal slope in Hilbert and Banach spaces

We conclude this chapter dedicated to slopes and upper gradients by giving a closer look to the case when

$$\mathscr{S} = \mathscr{B} \text{ is a Banach space with norm } \|\cdot\|; \tag{1.4.1}$$

we denote by $\langle\cdot,\cdot\rangle$ the duality between $\mathscr{B}$ and its dual $\mathscr{B}'$ and by $\|\cdot\|_*$ the dual norm in $\mathscr{B}'$.
Let us first consider a C^1 functional $\phi:\mathscr{B}\to\mathbb{R}$: the chain rule (1.3.1) characterizes the Frèchet differential $D\phi:\mathscr{B}\to\mathscr{B}'$, which is defined by

$$\boldsymbol{g}=D\phi(v)\quad\Leftrightarrow\quad\lim_{w\to v}\frac{\phi(w)-\phi(v)-\langle\boldsymbol{g},w-v\rangle}{\|w-v\|}=0\quad\forall\,v\in\mathscr{B}.$$

Since the metric derivative $|v'|$ of a regular curve v coincides with the norm of the velocity vector $\|v'\|$, it is easy to show that upper gradients involve the dual norm of $D\phi(v)$: by (1.2.2) g is an upper gradient for ϕ iff

$$g\geq\|D\phi(v)\|_*\quad\forall\,v\in\mathscr{B}.\tag{1.4.2}$$

In this case, the steepest descent conditions (1.3.3a), (1.3.4b) become

$$\langle D\phi(u),u'\rangle=\big(\phi\circ u\big)'\leq-\|u'\|\,g(u)\leq-\|u'\|\,\|D\phi(u)\|_*,\tag{1.4.3}$$

whereas (1.3.3b) could take the more general p,q form (1.3.11) (but see also (1.3.6))

$$\|u'\|^{p-1}=\|D\phi(u)\|_*.\tag{1.4.4}$$

Combining (1.4.3) and (1.4.4) we end up with the doubly nonlinear differential inclusion

$$\mathfrak{J}_p(u'(t))\ni-D\phi(u(t))\quad t>0,\tag{1.4.5}$$

where $\mathfrak{J}_p:\mathscr{B}\to2^{\mathscr{B}'}$ is the p-duality map defined by

$$\xi\in\mathfrak{J}_p(v)\quad\Leftrightarrow\quad\langle\xi,v\rangle=\|v\|^p=\|\xi\|_*^q=\|v\|\,\|\xi\|_*,\tag{1.4.6}$$

which is single valued if the norm $\|\cdot\|$ of $\mathscr{B}$ is differentiable.

We want now to extend the previous considerations to a non-smooth setting. Recall that the *Fréchet subdifferential* $\partial\phi(v)\subset\mathscr{B}'$ of a functional $\phi:\mathscr{B}\to(-\infty,+\infty]$ at a point $v\in D(\phi)$ is defined by

$$\xi\in\partial\phi(v)\quad\Longleftrightarrow\quad\liminf_{w\to v}\frac{\phi(w)-(\phi(v)+\langle\xi,w-v\rangle)}{\|w-v\|_{\mathscr{B}}}\geq0.\tag{1.4.7}$$

As usual, $D(\partial\phi)$ denotes the subset of $\mathscr{B}$ given by all the elements $v\in D(\phi)$ such that $\partial\phi(v)\neq\varnothing$; $\partial\phi(v)$ is a (strongly) closed convex set and we will suppose that

$$\partial\phi(v)\quad\text{is weakly}^*\text{ closed}\quad\forall\,v\in D(\partial\phi);\tag{1.4.8}$$

(1.4.8) is surely satisfied if e.g. $\mathscr{B}$ is reflexive or ϕ is convex (see the next Proposition 1.4.4). $\partial^\circ\phi(v)$ is the subset of elements of minimal (dual) norm in $\partial\phi(v)$, which reduces to a single point if the dual norm of $\mathscr{B}$ is strictly convex. Notice that

$$|\partial\phi|(v)=\limsup_{w\to0}\frac{\phi(v)-\phi(v+w)}{\|w\|}\leq\limsup_{w\to0}\langle\xi,\frac{w}{\|w\|}\rangle\leq\|\xi\|_*\quad\forall\,\xi\in\partial\phi(v).$$

Therefore, if we extend the function $v \mapsto \|\partial^\circ\phi(v)\|_*$ to $+\infty$ outside of $D(\partial\phi)$ we have

$$|\partial\phi|(v) \le \|\partial^\circ\phi(v)\|_* \quad \forall\, v \in \mathscr{B}, \tag{1.4.9}$$

and we obtain from Theorem 1.2.5 that

$$\text{the map } \ v \mapsto \|\partial^\circ\phi(v)\|_* \quad \text{is a weak upper gradient for } \phi. \tag{1.4.10}$$

In the next proposition we characterize the ($\mathscr{L}^1$-a.e. differentiable) curves of maximal slope with respect to the upper gradient (1.4.10) as the solution of a suitable doubly nonlinear differential inclusion: in the case when $\mathscr{S}$ is a reflexive Banach space and ϕ is convex, these kind of evolution equations have been studied in [53, 52]; we refer to these contributions and to [148] for many examples of partial differential equations which can be studied by this abstract approach.

Proposition 1.4.1 (Doubly nonlinear differential inclusions). *Let us consider a proper l.s.c. functional $\phi : \mathscr{B} \to (-\infty, +\infty]$ satisfying* (1.4.8) *and a curve $u \in AC^p(a, b; \mathscr{B})$ which is differentiable at $\mathscr{L}^1$-a.e. point of (a, b) (see Remark* 1.1.3*). If u is a p-curve of maximal slope for ϕ with respect to the weak upper gradient* (1.4.10)*, then*

$$\mathfrak{J}_p(u'(t)) \supset -\partial^\circ\phi(u(t)) \neq \varnothing \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t \in (a, b); \tag{1.4.11}$$

in particular, if the norm of $\mathscr{B}$ is differentiable, we have

$$\mathfrak{J}_p(u'(t)) = -\partial^\circ\phi(u(t)) \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t \in (a, b). \tag{1.4.12}$$

Conversely, if u satisfies (1.4.11) *and $\phi \circ u$ is ($\mathscr{L}^1$-a.e. equal to) a non increasing function, then u is a p-curve of maximal slope.*

Proof. Let us suppose that u is a p-curve of maximal slope for ϕ with respect to the upper gradient (1.4.10) and let φ be a non increasing map $\mathscr{L}^1$-a.e. equal to $\phi \circ u$ satisfying (1.3.13).
Then we can find a $\mathscr{L}^1$-negligible subset $N \subset (a, b)$ such that for every $t \in (a, b)\setminus N$ u and φ are differentiable at t, $\phi(u(t)) = \varphi(t)$, the inequality of (1.3.13) holds, and Definition (1.4.13) yields the chain rule

$$\varphi'(t) = \langle \xi, u'(t)\rangle \quad \forall\, \xi \in \partial^\circ\phi(u(t)). \tag{1.4.13}$$

It follows that for $t \in (a, b) \setminus N$

$$\langle \xi, u'(t)\rangle = \varphi'(t) \le -\frac{1}{p}\|u'(t)\|^p - \frac{1}{q}\|\xi\|_*^q \quad \forall\, \xi \in \partial^\circ\phi(u(t)), \tag{1.4.14}$$

which yields (1.4.11). When the norm of $\mathscr{B}$ is differentiable, the duality map $\mathfrak{J}_p$ is single-valued so that $\partial^\circ\phi$ contains at most one element: therefore (1.4.11) reduces to (1.4.12).
The converse implication follows by the same argument, since (1.4.11) and the chain rule (1.4.13) yields (1.3.13). □

Corollary 1.4.2 (Gradient flows in Hilbert spaces). *If $\mathscr{S} = \mathscr{B} = \mathscr{B}'$ is an Hilbert space, usually identified with its dual through the Risz isomorphism $\mathfrak{J}_2$, any 2-curve of maximal slope $u \in AC^2_{loc}(a, b; \mathscr{B})$ with respect to $\|\partial^\circ\phi(v)\|$ satisfies the gradient flow equation*

$$u'(t) = -\partial^\circ\phi(u(t)) \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (a, b). \tag{1.4.15}$$

Remark 1.4.3 (Non reflexive Banach spaces). The previous Proposition 1.4.1 strongly depends on the $\mathscr{L}^1$-a.e. differentiability of the considered curve and we have seen in Remark 1.1.3 that absolutely continuous curves enjoy this property if the underlying Banach space $\mathscr{B}$ satisfies the Radon-Nikodým property, e.g. if it is reflexive. One of the advantage of the purely metric formulation (1.3.13) is that it does not require any vector differentiability property of those curves and therefore it can be stated in any Banach space.

The next section will provide general existence and approximation results for curves of maximal slope with respect to the upper gradient $|\partial\phi|$: it is therefore important to know if $\|\partial^\circ\phi(v)\|_* = |\partial\phi|(v)$. In the following Proposition we deal with the case when ϕ is convex and l.s.c., proving in particular that $\|\partial^\circ\phi(v)\|_*$ is *a strong upper gradient* and coincides with $|\partial\phi|(v)$ and $\mathfrak{l}_\phi(v)$.

Proposition 1.4.4 (Slope and subdifferential of convex functions). *Let $\mathscr{B}$ be a Banach space and let $\phi : \mathscr{B} \to (-\infty, +\infty]$ be convex and l.s.c. Then*

$$\xi \in \partial\phi(v) \quad \Longleftrightarrow \quad \phi(w) - (\phi(v) + \langle \xi, w - v\rangle) \geq 0 \quad \forall\, w \in \mathscr{B} \tag{1.4.16}$$

for any $v \in D(\phi)$, the graph of $\partial\phi$ in $\mathscr{B} \times \mathscr{B}'$ is strongly-weakly* closed (in particular (1.4.8) *holds), with*

$$\xi_n \in \partial\phi(v_n),\ v_n \to v,\ \xi_n \rightharpoonup^* \xi \quad \Longrightarrow \quad \xi \in \partial\phi(v),\ \phi(v_n) \to \phi(v), \tag{1.4.17}$$

and

$$|\partial\phi|(v) = \min\Big\{\|\xi\|_* : \xi \in \partial\phi(v)\Big\} = \|\partial^\circ\phi(v)\|_* \quad \forall\, v \in \mathscr{B}. \tag{1.4.18}$$

Moreover

$$|\partial\phi|(v) = \mathfrak{l}_\phi(v) \quad \forall\, v \in \mathscr{B}, \tag{1.4.19}$$

so that, by Theorem 1.2.5, *$|\partial\phi|(v)$ is a strong upper gradient.*

Proof. The equivalence (1.4.16) and the identity (1.4.19) are simple consequence of the monotonicity of difference quotients of convex functions.

For every $w \in \mathscr{B}$ the map $(v, \xi) \mapsto \phi(w) - \phi(v) - \langle \xi, w - v\rangle$ is upper-semicontinuous with respect to the strong-weak*-topology in the product $\mathscr{B} \times \mathscr{B}'$; thus by (1.4.16) the graph of $\partial\phi$ is closed in this topology; this shows the first implication of (1.4.17). the second one follows from (1.4.16), which yields

$$|\phi(v) - \phi(v_n)| \leq \|v_n - v\|\big(\|\xi_n\|_* + \|\xi\|_*\big).$$

The inequality

$$\frac{\phi(v)-\phi(v+w)}{\|w\|} \le \langle \xi, \frac{w}{\|w\|}\rangle \qquad \forall w \in \mathscr{B}\setminus\{0\}$$

yields that $\mathfrak{l}_\phi(v)$ can be estimated from above by $\|\xi\|_{\mathscr{B}'}$ for any $\xi \in \partial\phi$. Assuming that $\mathfrak{l}_\phi(v)$ is finite, to conclude the proof we need only to show the existence of $\xi \in \partial\phi(v)$ such that $\|\xi\|_{\mathscr{B}'} \le \mathfrak{l}_\phi(v)$. By definition we know that

$$-\mathfrak{l}_\phi(v)\|w\| \le \phi(v+w)-\phi(v) \quad \forall\, w \in \mathscr{B}, \tag{1.4.20}$$

i.e. the convex epigraph

$$\big\{(w,r) \in \mathscr{B}\times\mathbb{R} : r \ge \phi(v+w)-\phi(v)\big\}$$

of the function $w \mapsto \phi(v+w)-\phi(v)$ is disjoint from the open convex hypograph in $\mathscr{B}\times\mathbb{R}$

$$\big\{(w,r) \in \mathscr{B}\times\mathbb{R} : r < -\mathfrak{l}_\phi(v)\|w\|\big\}$$

Therefore we can apply a geometric version of Hahn-Banach theorem to obtain $\xi \in \mathscr{B}'$, $\alpha \in \mathbb{R}$ such that

$$-\mathfrak{l}_\phi(v)\|w\| \le \langle \xi, w\rangle + \alpha \le \phi(v+w)-\phi(v) \quad \forall\, w \in \mathscr{B}.$$

Taking $w = 0$ we get $\alpha = 0$; the first inequality shows that $\|\xi\|_{\mathscr{B}'} \le \mathfrak{l}_\phi(v)$ and the second one, according to (1.4.16), means that $\xi \in \partial\phi(v)$. □

The above results can be easily extended to C^1 perturbations of convex functions.

Corollary 1.4.5 (C^1-perturbations of convex functions). *Let us suppose that $\phi : \mathscr{B} \to (-\infty,+\infty]$ admits the decomposition $\phi = \phi_1+\phi_2$, where ϕ_1 is a proper, l.s.c., and convex functional, whereas $\phi_2 : \mathscr{B} \to \mathbb{R}$ is of class C^1. Then $\partial\phi = \partial\phi_1 + D\phi_2$ satisfies* (1.4.17) *and* (1.4.18)*, and $|\partial\phi|(v)$ is a strong upper gradient for ϕ.*

Proof. The sum rule $\partial\phi = \partial\phi_1 + D\phi_2$ follows directly from Definition (1.4.7) and the differentiability of ϕ_2.
In order to check the closure property (1.4.17), we observe that if $\xi_n \in \partial\phi(v_n)$ and $(v_n, \xi_n) \to (v, \xi)$ in the strong-weak* topology of $\mathscr{B}\times\mathscr{B}'$ then

$$\xi_n - D\phi_2(v_n) \in \partial\phi_1(v_n), \quad \xi_n - D\phi_2(v_n) \overset{*}{\rightharpoonup} \xi - D\phi_2(v) \in \partial\phi_2(v),$$

since $D\phi_2$ is continuous and ϕ_1 is convex: we obtain $\xi \in \partial\phi(v)$ and $\phi_1(v_n) \to \phi_1(v)$ which yield (1.4.17) being ϕ_2 continuous.

Finally, since we can add to ϕ_1 and subtract to ϕ_2 an arbitrary linear and continuous functional, in order to prove (1.4.18) it is not restrictive to suppose that $D\phi_2(v) = 0$; it follows that

$$
\begin{aligned}
|\partial\phi|(v) &= \limsup_{w\to v} \frac{\big(\phi(w)-\phi(v)\big)^+}{\|w-v\|} \\
&\geq \limsup_{w\to v} \frac{\big(\phi_1(w)-\phi_1(v)\big)^+}{\|w-v\|} - \limsup_{w\to v} \frac{|\phi_2(w)-\phi_2(w)|}{\|w-v\|} \\
&= |\partial\phi_1|(v) = \|\partial^\circ\phi_1(v)\|_* = \|\partial^\circ\phi(v)\|_*.
\end{aligned}
$$

Combining this inequality with the opposite one (1.4.9), we conclude. □

Let us rephrase the last conclusion of the previous Corollary, which is quite interesting in the case $\mathscr{B}$ does not satisfy the Radon-Nikodým property (as in many examples of rate-independent problems, see [114, 113]).

Remark 1.4.6 ("Upper" chain rule for (even non reflexive) Banach spaces). If $\phi : \mathscr{B} \to (-\infty, +\infty]$ is lower semicontinuou convex function (or a C^1 perturbation as in Corollary 1.4.5), v is a curve in $AC(a, b; \mathscr{B})$ with $\|\partial^\circ\phi\|_*\,|v'| \in L^1(a, b)$, then $\phi\circ v$ is absolutely continuous in (a, b); if $\mathscr{B}$ has the Radon-Nikodým property, then

$$\frac{d}{dt}\phi\circ v(t) = \langle\partial^\circ\phi(v(t)), v'(t)\rangle \quad \text{for } \mathscr{L}^1\text{-a.e. } t \text{ in } (a, b);$$

for general Banach spaces, one can always write the upper estimate

$$\Big|\frac{d}{dt}\phi\circ v(t)\Big| \leq \|\partial^\circ\phi(v(t))\|_*|v'|(t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t \text{ in } (a, b). \tag{1.4.21}$$

In the next chapter we will see how the last two proposition can be extended to a general class of functions defined on metric spaces and satisfying suitable geometric convexity conditions.