Chapter 4

Metric Foliations in Space Forms

We have so far focused our attention mostly on the base space B of a Riemannian submersion $M \to B$, in particular when searching for new metrics of nonnegative curvature on B. It is also interesting to look at the total space of the fibration. The very fact that there exists a Riemannian submersion from M (or more generally, that M admits a metric foliation) is a sign that the space possesses a fair amount of symmetry. One therefore expects those Riemannian manifolds with the largest amount of symmetry – namely, space forms – to be the ones that display the most variety as far as these foliations are concerned. Surprisingly, a complete classification of metric foliations on spaces of constant curvature is not yet available. There does, however, exist a classification of metric *fibrations*, at least in nonnegative curvature, which will be described in this chapter.

4.1 Isoparametric foliations

Recall from Section 1.4 that the mean curvature of a metric foliation on M is the one-form κ given by $\kappa(e) = \operatorname{tr} S_{e^{\mathbf{h}}}$, $e \in TM$. It essentially measures the infinitesimal rate of change of the volume form of the leaves in horizontal directions. To see this, assume the foliation is oriented (which is always the case, at least up to a cover of M). Let ω denote the form on M that is locally the metric dual of $U_1 \wedge \cdots \wedge U_k$, where $U_1, \dots, U_k$ is a local oriented orthonormal basis of the vertical distribution; i.e.,

$$\omega(E_1, \dots, E_k) = \det(\langle U_i, E_j \rangle), \qquad E_j \in \mathfrak{X}M.$$

We denote by $\omega^{\mathbf{v}}$ the restriction of ω to the vertical distribution. With this notation, we have:

Proposition 4.1.1 (Rummler [112]). *The vertical restriction of the Lie derivative of ω in a horizontal direction $X \in \mathfrak{X}M$ satisfies*

$$(\mathcal{L}_X\omega)^{\mathbf{v}} = -\kappa(X)\omega^{\mathbf{v}}.$$

Proof. If $U_1, \dots, U_k$ denotes an oriented local orthonormal frame, then

$$\begin{aligned}(\mathcal{L}_X\omega)(U_1, \dots, U_k) &= X\omega(U_1, \dots, U_k) - \sum_{i=1}^{k} \omega(U_1, \dots, [X, U_i]^{\mathbf{v}}, \dots, U_k) \\ &= -\sum_{i=1}^{k} \omega(U_1, \dots, [X, U_i]^{\mathbf{v}}, \dots, U_k).\end{aligned}$$

Furthermore, we may replace $[X, U_i]^{\mathbf{v}}$ by its projection onto U_i, which equals

$$\langle [X, U_i], U_i \rangle U_i = -\langle \nabla_{U_i} X, U_i \rangle U_i = \langle S_X U_i, U_i \rangle U_i,$$

so that

$$(\mathcal{L}_X\omega)(U_1, \dots, U_k) = -(\operatorname{tr} S_X)\omega(U_1, \dots, U_k),$$

as claimed. □

Definition 4.1.1. A metric foliation is said to be *isoparametric* if its mean curvature form is basic.

By definition, a 1-form κ is basic if its metrically dual vector field is basic; i.e., if $\kappa(X)$ is locally constant along leaves for basic X. In view of Proposition 4.1.1, this amounts to saying that the infinitesimal rate of change of the volume of leaves in basic directions is independent from the point on the leaf at which it is measured. For example, any homogeneous foliation is isoparametric. This follows from Proposition 2.3.4, which actually asserts a stronger property, namely that for basic X and left-invariant U, $S_X U$ is left-invariant. The converse is not true in general: If M is an open manifold of nonnegative curvature with soul S, and if the metric projection $M \to S$ has totally geodesic fibers, then the resulting fibration is isoparametric, but not homogeneous unless M splits locally isometrically as a product over S (one way to see this is to notice that if $M \to S$ is homogeneous, then by Proposition 2.3.4, $A_X Y$ is left-invariant for basic X and Y, and in particular has constant norm along geodesics emanating from the soul; it must then be identically zero by Proposition 3.6.1. Now apply Theorem 2.2.2). A typical example is $TS^n = SO(n+1) \times_{SO(n)} \mathbb{R}^n \to S^n$. When the foliation is one-dimensional, however, the converse is true under weak curvature restrictions, see also [134], [55]:

Proposition 4.1.2. *Any one-dimensional isoparametric Riemannian foliation $\mathcal{F}$ with complete leaves on a manifold with curvature bounded below is locally homogeneous; i.e., $\mathcal{F}$ is locally generated by a Killing field.*

Proof. Since the mean curvature form κ is basic, so is $d\kappa$, and by Proposition 1.4.1, the function

$$\operatorname{div} A_X Y = -\frac{1}{2} d\kappa(X,Y)$$

is then constant along leaves for basic X and Y. If T is a local unit vertical field, this translates into $\langle \nabla_T A_X Y, T\rangle$ being constant, since $\langle \nabla_Z A_X Y, Z\rangle$ is always zero for horizontal Z. Thus, if $c : \mathbb{R} \to M$ is a unit-speed curve parametrizing a complete leaf, then $\langle A_X Y \circ c, \dot c\rangle'$ is constant, and $\langle A_X Y \circ c, \dot c\rangle$ is an affine function. It must then be constant by O'Neill's formula, if the curvature is bounded below. Consequently, $d\kappa(X,Y) = 0$, and κ is closed, because

$$d\kappa(X,T) = X\kappa(T) - T\kappa(X) - \kappa[X,T] = -T\kappa(X) = 0.$$

Thus, $\kappa = d\phi$ locally, for a function ϕ that is constant along leaves. Set $L := e^{-\phi}$, $U := LT$. Then U is Killing, since

$$\langle \nabla_X U, X\rangle = L\langle \nabla_X T, X\rangle = L\langle \nabla_T X, X\rangle = 0,$$

and

$$\langle \nabla_X U, T\rangle + \langle \nabla_T U, X\rangle = XL\langle T,T\rangle - \langle U, \nabla_T X\rangle = XL + L\kappa(X) = 0. \qquad \square$$

The relevance of isoparametric foliations for space forms is illustrated by the following:

Theorem 4.1.1 ([63]). *Any metric foliation of a space form of nonnegative curvature is isoparametric.*

Proof. Let x be horizontal, and consider a Riemannian submersion that locally defines the foliation in a neighborhood of the footpoint of x. We will prove a stronger assertion, namely that if λ is an eigenvalue of S_x, then it is also an eigenvalue (of equal multiplicity) of $S_{\tilde x}$, for any horizontal $\tilde x$ with $\pi_*\tilde x = \pi_* x$. Denote by γ (resp. $\tilde\gamma$) the geodesic with initial tangent vector x (resp. $\tilde x$), and consider the Jacobi field J along γ with $J(0) = u$, $J'(0) = -S_x u = -\lambda u$, where u denotes a unit λ-eigenvector of S_x. Notice that $J = \phi E$, where E is the parallel field along γ with $E(0) = u$, and ϕ is the solution of the O.D.E.

$$\phi'' + c\phi = 0, \qquad \phi(0) = 1, \quad \phi'(0) = -\lambda,$$

with c denoting the curvature of the space. Assume for now that $\lambda \neq 0$ if $c = 0$. Then $J(l) = 0$ for some $l \in \mathbb{R}$. But J is by definition projectable, so that $\pi_* J$ is Jacobi along $\pi \circ \gamma$ by Theorem 1.6.1. By Lemma 1.6.1, there exists a unique Jacobi field $\tilde J$ along $\tilde\gamma$ with $\pi_*\tilde J = \pi_* J$ and $\tilde J(l) = 0$. In particular, $\tilde J(0)$ must be vertical (because J is), so that $\tilde J'^{\mathbf{v}}(0) = -S_{\tilde x}\tilde J(0)$. This, together with the fact that $\tilde J(l) = 0$ implies that $\tilde J = \phi\tilde E$ for some parallel field E. It follows that $\tilde J(0)$ is a λ-eigenvector of $S_{\tilde x}$. Let k denote the multiplicity of λ as an eigenvalue of S_x. Since for any Jacobi field Y along $\pi \circ \gamma$ that vanishes at 0 and l there exists a unique projectable Jacobi field J along γ with $J(0)$ in the λ-eigenspace of S_x, l is a conjugate point of $\pi \circ \gamma$ of order $\le k$. Conversely, if J is a projectable Jacobi field along γ with $J'(0) = -S_x J(0)$ and $J(0) \neq 0$, then $\pi_* J$ is a nontrivial Jacobi field that vanishes at 0 and l (if $\pi_* J \equiv 0$, then J is vertical; i.e., J is a holonomy field, and since $J(l) = 0$, $J \equiv 0$). Thus, l is a conjugate point of order k. Applying Lemma 1.6.1 again, we see that the multiplicity of λ as an eigenvalue of $S_{\tilde{x}}$ is also k. This establishes the theorem in all cases except perhaps when $c = \lambda = 0$. But then it must also be true in the latter case. □

Corollary 4.1.1. *A one-dimensional metric foliation of a space form of nonnegative curvature is locally homogeneous.*

Proof. This follows immediately from Theorem 4.1.1 and Proposition 4.1.2. Notice that if the space is simply connected, then the Killing field is globally defined. □

Even though the above is not necessarily true in constant negative curvature (see Examples and Remarks (i) below), a slightly more general result does hold, cf. also [55]:

Theorem 4.1.2. *A one-dimensional metric foliation of a hyperbolic space form is either locally homogeneous or flat.*

Proof. We first claim that in constant (not necessarily negative) curvature c, the A-tensor vanishes everywhere as soon as it vanishes at any one point p. To see this, it is enough to show that it is zero along any horizontal geodesic γ emanating from p, since for basic X, Y, $A_X Y$ has constant norm along leaves by O'Neill's formula. An equivalent claim is that the totally geodesic hypersurface $\exp_p(\mathcal{H}_p)$ is horizontal everywhere, or alternatively, that any parallel vector field E along γ with $E(0)$ horizontal remains horizontal for all t. But if J is a holonomy Jacobi field along γ, then

$$\langle J, E\rangle'' = \langle J'', E\rangle = -\langle R(J, \dot{\gamma})\dot{\gamma}, E\rangle = -c\langle J, E\rangle.$$

The claim now follows from the initial conditions, because $\langle J, E\rangle(0) = 0$ by assumption, and

$$\langle J, E\rangle'(0) = \langle J', E\rangle(0) = -\langle (S_{\dot{\gamma}} + A^*_{\dot{\gamma}})J, E\rangle(0) = -\langle A^*_{\dot{\gamma}} J, E\rangle(0) = 0$$

if $A_p \equiv 0$.

Resuming the proof of the theorem, assume that the foliation is not flat. By the above claim, there exist at any point p basic vector fields X, Y with $A_X Y \neq 0$ at p. Theorem 1.5.1 then implies that

$$R^{\mathbf{v}}(X, Y)X = -(\nabla^{\mathbf{v}}_X A)_X Y + 2S_X A_X Y,$$

so that if T is a local unit field spanning the vertical distribution, then

$$0 = \langle R(X, Y)X, T\rangle = -\langle (\nabla^{\mathbf{v}}_X A)_X Y, T\rangle + 2\langle S_X A_X Y, T\rangle. \tag{4.1.1}$$

Now, the first term on the right in (4.1.1) is locally constant along leaves, since it can be written as

$$\begin{aligned}\langle(\nabla^{\mathbf{v}}_X A)_X Y, T\rangle &= \langle\nabla_X(A_X Y), T\rangle - \langle A_{\nabla_X X} Y, T\rangle - \langle A_X \nabla_X Y, T\rangle \\ &= X\langle A_X Y, T\rangle - \langle A_{\nabla_X X} Y, T\rangle - \langle A_X \nabla_X Y, T\rangle,\end{aligned}$$

where both $\nabla^{\mathbf{h}}_X X$ and $\nabla^{\mathbf{h}}_X Y$ are basic, whereas

$$TX\langle A_X Y, T\rangle = [T, X]\langle A_X Y, T\rangle + XT\langle A_X Y, T\rangle = 0$$

because $[T, X]$ is vertical. Thus, $\langle S_X A_X Y, T\rangle$ is constant along leaves by (4.1.1), so that $\kappa(X)$ is also constant, since

$$\langle S_X A_X Y, T\rangle = \langle A_X Y, S_X T\rangle = \langle A_X Y, T\rangle\kappa(X).$$

Summarizing, $\kappa(X)$ is locally constant for all X in a nonempty open subset of basic fields along the leaf through p, and κ is therefore basic. $\square$

Examples and Remarks 4.1.1. (i) In a space of constant curvature c, any submanifold with flat normal bundle is locally a leaf of a (flat) metric foliation, as remarked in Examples 2.2.1(ii). This foliation cannot be extended to the whole space if $c > 0$ by Theorem 2.2.2. For $c = 0$, it can iff L is totally geodesic. When $c < 0$, however, there is no such rigidity. This also shows that Theorem 4.1.1 does not hold in this case: Consider for example a line in hyperbolic space. Deform the line in a neighborhood of some point so that it is no longer totally geodesic there, but do it slightly so that the exponential map of the normal bundle is still one-to-one. Exponentiating parallel sections of the normal bundle then yields a metric foliation of hyperbolic space that is not isoparametric.

(ii) The Hopf fibrations with fibers S^1 and S^3 are homogeneous. Even though the higher-dimensional Hopf fibration $S^{15} \to S^8$ with fiber S^7 is isoparametric (in fact, it is totally geodesic) by Theorem 4.1.1, it is not homogeneous. Before arguing this, notice first that since the fibration is a fat bundle, it is *weakly substantial*; i.e., the image of the A-tensor equals the whole vertical distribution. This implies that it cannot be homogeneous. In fact, we claim that if a homogeneous submersion is weakly substantial, then the fiber is a Lie group (the fiber of the Hopf fibration is S^7, which of course is not a Lie group). To see this, let G be the group of isometries generating the fibration, so that a fiber has the form G/H, where H is the isotropy group at some point p. Consider $h \in H$. Since $\pi \circ h = \pi$ and since h preserves the vertical, and hence horizontal distributions, it must preserve basic fields; i.e., $h_* X = X \circ h$ for any basic field X (and in particular, h_* is the identity on the horizontal space at p). h_* must then also preserve the bracket of basic fields, so that $h_* A_x y = A_x y$ for any horizontal vectors x, y at p. Thus, h_* is the identity on the vertical space also, and since h is an isometry, it is the identity map. This means that H is trivial, and the fiber is G, as claimed.

(iii) Recall that given a metric foliation on M, a one-form α is basic if its metrically dual vector field $\alpha^\sharp$ is basic. This is easily seen to be equivalent to requiring that $\alpha(T) = 0$ and $d\alpha(T, E) = 0$ for any vertical field T and arbitrary field E. More generally, a differential form α on M is said to be *basic* if

$$\imath_T\alpha = 0, \quad \imath_T d\alpha = 0, \qquad \text{for vertical } T,$$

($\imath$ denotes interior multiplication). By definition, the differential of a basic form is again basic, so that d, when restricted to the algebra of basic forms, induces a so-called *basic cohomology* of the leaf space introduced by Reinhart [110]. A number of authors have studied this complex in an attempt to develop a basic Hodge theory and basic Laplacian, leading to a representation of basic cohomology classes by harmonic forms, see, e.g., [79] in the isoparametric case and [101] in the general case.

4.2 Metric fibrations of Euclidean space

Our next objective is to derive a classification of Riemannian submersions $\pi : \mathbb{R}^{n+k} \to B^n$. A simple, yet illustrative example to keep in mind throughout this discussion is the orbit fibration $\pi : \mathbb{R}^3 \to B^2 = \mathbb{R}^3/\mathbb{R}$, where $\mathbb{R}$ is the Lie group of isometries acting on $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ via glide-rotations; i.e.,

$$\begin{aligned} \mathbb{R} \times (\mathbb{C} \times \mathbb{R}) &\longrightarrow \mathbb{C} \times \mathbb{R}, \\ (t, (x, t_0)) &\longmapsto (e^{it}x, t_0 + t), \end{aligned}$$

cf. Examples and Remarks 1.5.1(iv). Notice that there is exactly one totally geodesic fiber, namely the z-axis. It turns out it is the fiber over the soul of the nonnegatively curved manifold B^2.

In general, if $\pi : \mathbb{R}^{n+k} \to B^n$ is a Riemannian submersion, then B^n has nonnegative curvature, and π factors as a fibration over the universal cover of B, followed by a covering map. Covering maps in nonnegative curvature are well understood (see, e.g., [37]), and we may therefore assume without loss of generality that B is simply connected. It follows from the long exact homotopy sequence of π that the fiber of the submersion is connected. Using the spectral sequence for the homology of the fibration, one concludes that B is contractible, cf. [70]. Since B is also a vector bundle over a soul, it must be diffeomorphic to Euclidean space, and any soul consists of a point.

Proposition 4.2.1. *If $\pi : \mathbb{R}^{n+k} \to B$ is a Riemannian submersion, then the fiber over any soul of B is an affine subspace.*

Proof. The general idea is to lift the soul construction to Euclidean space, cf. also [38]: Let $\{p\}$ denote a soul of B, $c : [0, \infty) \to B$ a ray emanating from p. Notice that any lift $\tilde{c}$ of c must also be a ray: this is of course trivial in this case, since any normal geodesic in Euclidean space is a ray, but is also true in general: otherwise, for some t, the line segment from $\tilde{c}(0)$ to $\tilde{c}(t)$ would be shorter than t, implying that $c = \pi(\tilde{c})$ is not minimal. If

$$B_c = \bigcup_{t>0} B_t(c(t))$$

is the open half-space determined by c from the soul construction, then $\pi(B_{\tilde{c}}) = B_c$, since π maps metric balls onto metric balls of the same radius. Denote by $\tilde{B}_c$ the union of all $B_{\tilde{c}}$, where $\tilde{c}$ ranges over all lifts of c, and by $\tilde{C}_c$ its complement in $\mathbb{R}^{n+k}$. $\tilde{C}_c$ is closed and convex (as an intersection of closed half-spaces), and by construction, $C_c \subset \pi(\tilde{C}_c)$, where C_c is the complement of B_c in B. The reverse inclusion also holds, for otherwise, one could find some $q \in \tilde{C}_c$ such that $\pi(q) \in B_c$; i.e., there would exist some t_0 such that $d(\pi(q), c(t_0)) < t_0$. But then the horizontal lift to q of a minimal geodesic from $\pi(q)$ to $c(t_0)$ is a curve of length less than t_0 connecting q to $\tilde{c}(t_0)$ for some lift $\tilde{c}$ of c, contradicting the fact that $q \in C_{\tilde{c}}$. Next, set

$$\tilde{C} := \bigcap_c \tilde{C}_c, \qquad C := \bigcap_c C_c,$$

where c ranges over all rays emanating from p. Just as above, one has that $\tilde{C}$ is a closed convex set of Euclidean space with $\tilde{C} = \pi^{-1}(C)$ and $\partial\tilde{C} = \pi^{-1}(\partial C)$. If $C = \{p\}$, then $\tilde{C} = \pi^{-1}(p)$ is a closed, connected, convex submanifold without boundary of Euclidean space, i.e., an affine subspace. If C has nonempty boundary, define sets

$$\tilde{C}^a = \{q \in \tilde{C} \mid d(q, \partial\tilde{C}) \geq a\}, \qquad C^a = \{q \in C \mid d(q, \partial C) \geq a\},$$

where $0 \leq a \leq \max\{d(q, \partial C) \mid q \in C\}$. Both sets are closed and totally convex by the results from Chapter 3. Furthermore, given any two points p_0, p_1 in B, the distance between them equals the distance between the fibers over them, as well as the distance between any point on one fiber and the other fiber. This is easily seen to imply that $\tilde{C}^a = \pi^{-1}(C^a)$. Iterating this procedure finitely many times until the set in the base no longer has boundary (and therefore equals $\{p\}$) lets us draw the same conclusion as when C consists of the single point p, thereby establishing the result. $\square$

The above proposition allows us to strengthen Theorem 4.1.1 in the case of a fibration of Euclidean space:

Proposition 4.2.2. *The mean curvature form κ of a Riemannian submersion $\pi : \mathbb{R}^{n+k} \to B^n$ is basic and exact; i.e., there exists a function $f : B \to \mathbb{R}$ such that $\kappa = d(f \circ \pi)$.*

Proof. The fact that κ is basic follows from Theorem 4.1.1, so we only have to show that it is closed. Since κ vanishes when applied to vertical vectors, $d\kappa(U, V) = 0$ for vertical U, V. Furthermore, for basic X, the bracket $[X, U]$ is vertical, so that

$$d\kappa(X, U) = X\kappa(U) - U\kappa(X) = -U\kappa(X) = 0,$$

because κ is basic. It remains to show that $d\kappa(X,Y) = 0$ for basic X, Y, or equivalently (by Proposition 1.4.1), that the vertical field A_XY has vanishing divergence. Now, this divergence is the one induced by the fiber metric, since $\langle\nabla_Z A_XY, Z\rangle = -\langle A_XY, \nabla_Z Z\rangle = 0$ for basic Z. Furthermore, the divergence is constant along fibers because κ, and hence also $d\kappa$, is basic. Denote by F the totally geodesic fiber over the soul, and consider a minimal segment c from F to some fiber L at distance l from F. The horizontal lifts of $\pi \circ c$ induce a holonomy diffeomorphism $h : F \to L$, and by Lemma 1.4.2, the derivative of h at any point q of F satisfies $h_*u = J(l)$, where J is the holonomy field along the line $t \mapsto \exp(tZ_q)$, with $J(0) = u$ (here, Z denotes the basic field along F that extends $\dot c(0)$). Now, F is totally geodesic, so that $J'(0) = -A^*_Z u$, and $J(t) = E + tF$, where E and F are the parallel fields with $E(0) = u$ and $F(0) = -A^*_Z u$. In particular E and F are mutually orthogonal, so that

$$|h_*u|^2 = |u|^2 + l^2|A^*_Z u|^2.$$

Thus, the norm of h_* is bounded below by 1, and since $|A_XY|$ is constant along fibers, it is also bounded above by some constant. It follows that if $B_r \subset L$ denotes the h-image of a ball of radius r in F about some point, then $\operatorname{vol} B_r \geq a \cdot r^k$ and $\operatorname{vol} \partial B_r \leq b \cdot r^{k-1}$ for some positive constants a and b. If N_r denotes the outward unit normal field to ∂B_r, then Stokes' Theorem implies

$$a \cdot |\operatorname{div} A_XY| \cdot r^k \leq \left| \int_{B_r} \operatorname{div} A_XY \right| = \left| \int_{\partial B_r} \langle A_XY, N_r\rangle \right| \leq b \cdot |A_XY| \cdot r^{k-1},$$

so that $\operatorname{div} A_XY \equiv 0$ if the above inequality is to hold for all $r > 0$. $\square$

Up to a congruence of $\mathbb{R}^{n+k}$, the totally geodesic fiber F is $\{0\} \times \mathbb{R}^k \subset \mathbb{R}^{n+k}$. Normalize the function $f : B^n \to \mathbb{R}$ from Proposition 4.2.2 so that it equals zero at $\pi(F)$. If ω is the vertical volume form from Proposition 4.1.1, define the *holonomy form* of the fibration to be the k-form η given by

$$\eta := e^{-(f\circ\pi)}\omega.$$

It can alternatively be described as follows: Consider an oriented orthonormal parallel basis $E_1, \dots, E_k$ of vector fields along F, and extend them radially via holonomy diffeomorphisms from F; i.e., define vector fields U_i on $\mathbb{R}^{n+k}$ by

$$U_i(x,u) := ||\big(E_i(0,u) - A^*_{\mathcal{J}_{(0,u)}x} E_i(0,u)\big), \qquad i = 1, \dots, k, \tag{4.2.1}$$

where $||$ denotes parallel translation from $(0,u)$ to (x,u), and $\mathcal{J}_{(0,u)}$ is the canonical isomorphism of Euclidean space with its tangent space at $(0,u)$. Thus, for a line c emanating orthogonally from F, $U_i \circ c$ is the holonomy Jacobi field that equals E_i at 0. The relation between the U_i and η is given by the following:

Lemma 4.2.1. $\eta^\sharp = U_1 \wedge \cdots \wedge U_k$.

Proof. Since η and the dual of $U_1 \wedge \cdots \wedge U_k$ are both vertical forms, it suffices to show that at any point p,

$$\eta(v_1, \dots, v_k) = \langle U_1(p) \wedge \cdots \wedge U_k(p), v_1 \wedge \cdots \wedge v_k \rangle,$$

where $v_1, \dots, v_k$ denotes a positively oriented orthonormal basis of the fiber at p; equivalently, that $e^{-(f \circ \pi)} = \omega(U_1, \dots, U_k)$. Now by definition, both functions are constant equal to 1 along F. Furthermore, if X is the tangent field of a horizontal geodesic emanating from F, then

$$X\big(e^{-(f\circ\pi)}\big) = -e^{-(f\circ\pi)} X(f \circ \pi) = -e^{-(f\circ\pi)} \kappa(X),$$

whereas by Proposition 4.1.1,

$$\begin{aligned} X\big(\omega(U_1, \dots, U_k)\big) &= \mathcal{L}_X \big(\omega(U_1, \dots, U_k)\big) = (\mathcal{L}_X \omega)(U_1, \dots, U_k) \\ &= -\omega(U_1, \dots, U_k)\kappa(X), \end{aligned}$$

using the fact that $[X, U_i] = 0$. The claim clearly follows. $\square$

Lemma 4.2.1 says that the k-form $U_1 \wedge \cdots \wedge U_k$ is holonomy-invariant in the sense that the wedge product of holonomy fields is independent of the chosen horizontal path. We will soon see that the vector fields U_i are in fact global Killing fields that generate the isometric action. In the special case of a one-dimensional fibration, it is easy to see that U is a Killing field, i.e., that the assignment $z \mapsto \nabla_z U$ is skew-adjoint: If $T = U/|U|$ is the unit field in direction U, then $U = e^{-(f\circ\pi)}T$, so that, for horizontal X,

$$\langle \nabla_X U, X \rangle = -\langle \nabla_X X, U \rangle = 0,$$

whereas

$$\langle \nabla_U U, U \rangle = \frac{1}{2} U\big(e^{-2(f\circ\pi)}\big) = 0$$

since $\pi_* U = 0$. Finally,

$$\langle \nabla_X U, T \rangle + \langle \nabla_T U, X \rangle = X\big(e^{-(f\circ\pi)}\big) + e^{-(f\circ\pi)}\kappa(X) = 0$$

by Lemma 4.2.1. Thus, U is Killing.

For simplicity of notation, we will, for the remainder of the section, identify Euclidean space with its tangent space at any point. Thus, the vector field U_i from (4.2.1) becomes a map from $\mathbb{R}^{n+k}$ to itself, and the holonomy form (or rather its dual $\eta^\sharp$) is a map from $\mathbb{R}^{n+k}$ to $\Lambda_k(\mathbb{R}^{n+k})$. We say a map from $\mathbb{R}^{n+k}$ to a vector space is *polynomial of degree at most* r if each component function $\phi : \mathbb{R}^{n+k} \to \mathbb{R}$ of this map in some basis is a polynomial of degree at most r in the usual sense. For example, each vector field U_i is polynomial of degree at most 1 along any affine subspace $\mathbb{R}^n \times \{u\}$, since the map $x \mapsto A_x^* E$ is linear. It follows that $\eta^\sharp$ is polynomial of degree at most k along these subspaces. In fact, it is polynomial along *any* horizontal space, not just those based at F; this will be a key point in the forthcoming classification of metric fibrations on Euclidean space:

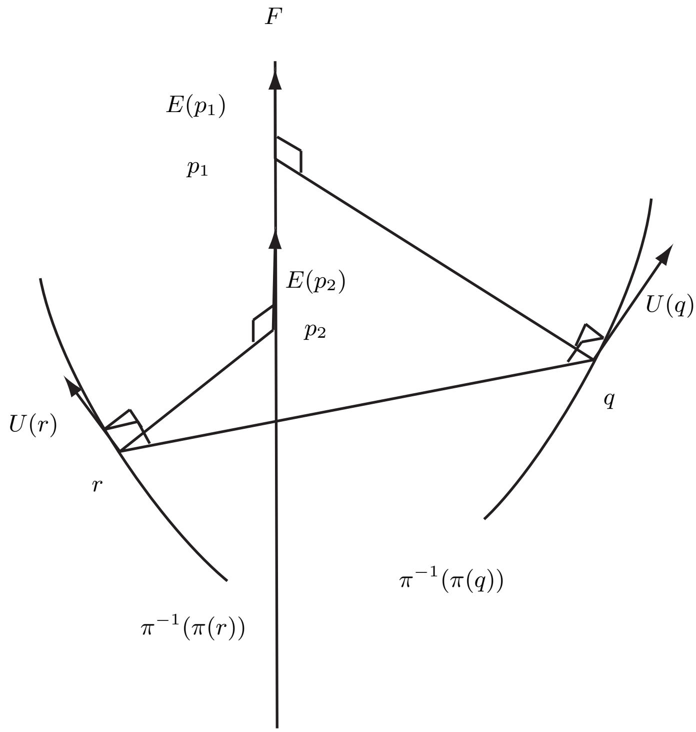


Figure 4.1: Holonomy invariance in dimension one.

Lemma 4.2.2. *$\eta^\sharp$ is polynomial of degree at most k on every affine horizontal subspace.*

Proof. Consider $p \in \mathbb{R}^{n+k}$, and a point q on the horizontal subspace H through p. By Lemma 4.2.1, $\eta^\sharp$ is holonomy-invariant, so that

$$\eta^\sharp(q) = \bigwedge_i U_i(q) = \bigwedge_i \big(U_i(p) - (A^*_{q-p} + S_{q-p})U_i(p)\big].$$

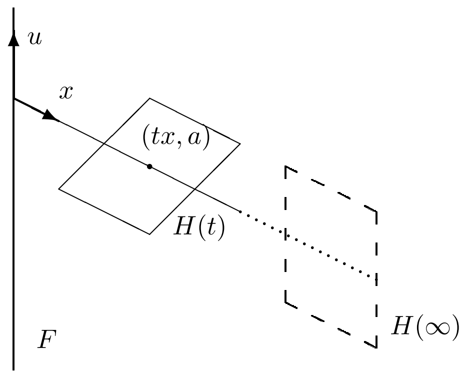
Thus, after translating the origin to p, it suffices to show that the map

$$x \longmapsto \bigwedge_i (E_i - A^*_x E_i - S_x E_i)$$

is polynomial of degree at most k. This in turn follows from the fact that $x \mapsto A^*_x E + S_x E$ is linear. $\square$

Lemma 4.2.3. *$\eta^\sharp$ is polynomial along every affine plane through a point $(0, a) \in F$ spanned by a horizontal x and a vertical u in the image of A_x.*

Proof. Normalize x to have length 1, and denote by $H(t)$ the horizontal space at (tx, a) for $t \geq 0$; i.e., at distance t from F along the (horizontal) line in direction x through $(a, 0)$. By Lemma 4.2.2, $\eta^\sharp$ is polynomial on $H(t)$. Now, x is in $H(t)$ for all t. By a continuity argument, the claim follows once we establish that u is in $H(\infty)$, where the latter denotes the limit of $H(t)$ as $t \to \infty$. In fact, $H(\infty)$ is the direct sum of the kernel of A_x and the image of A_x, because of the form of holonomy fields in Euclidean space: the holonomy field J that equals E at time 0 is $J(t) = E - tA_x^* E$ (where E is extended to be parallel). Now, let t go to infinity, to conclude that the vertical space at infinity is spanned by $(\ker + \operatorname{Im})A_x^*$. Equivalently, the horizontal space at infinity is spanned by $(\ker + \operatorname{Im})A_x$. $\square$



Since F is totally geodesic, any vector x in its normal bundle ν may be extended by parallel translation to a horizontal vector field along F. Such a field will be denoted by the same lowercase letter x to distinguish it from the uppercase notation X for basic fields. Thus, the former are the parallel sections for the usual Euclidean connection $\nabla^{\mathbf{h}}$ on ν, whereas the latter represent those that are parallel for the Bott connection ∇^B from (1.3.3). The connection difference form $\Omega = \nabla^{\mathbf{h}} - \nabla^B$ is then the 1-form on F with values in the bundle of skew-symmetric endomorphisms of ν given by

$$\Omega(u)x = -A_x^* u, \qquad u \in TF, \quad x \in \nu.$$

At this point, it is convenient to simplify matters by getting rid of the "translational" part of the submersion, which is *grosso modo* the kernel of A^*: for a point p in the fiber F, denote by $\mathcal{A}_p$ the (affine) space at p spanned by all integrability fields $A_x y$. Define the kernel of A^* to be the union over $p \in F$ of $\mathcal{A}_p^\perp$. We then have the following

Proposition 4.2.3. *π factors as an orthogonal projection $\mathbb{R}^{n+k-l} \times \mathbb{R}^l \to \mathbb{R}^{n+k-l} \times \{0\}$ followed by a Riemannian submersion $\pi' : \mathbb{R}^{n+k-l} \to B$, where the fiber F' of π' over the soul of B is spanned by the image of A; i.e., for any $p \in F'$, $F' = \mathcal{A}_p$. Furthermore, given parallel sections x, y of the normal bundle ν' of F', $A_x y$ is a parallel vector field along F'.*

Proof. Let x be a vector in the normal bundle ν of F at $(0,a)$, and u a vector in the image of A_x. By Lemma 4.2.3, the holonomy form is polynomial along the plane through $(0,a)$ spanned by x and u, and therefore so is its derivative in direction x. The restriction of this derivative to the line $t \mapsto \gamma_u(t) := (0, a+tu)$ is given by

$$\nabla_x \eta^\sharp = -\sum_i E_1 \wedge \cdots \wedge A_x^* E_i \wedge \cdots \wedge E_k.$$

Now, the E_j are parallel, and $A_x^* E_i$ is horizontal and bounded in norm. Since a bounded polynomial is constant, we conclude that each $A_x^* E_i$ is parallel along γ_u, or equivalently,

$$(A_x y \circ \gamma_u)' \equiv 0, \qquad u \in \operatorname{Im} A_x. \tag{4.2.2}$$

Thus, the image of A_x, though *a priori* not of constant rank along F, is totally geodesic, and consists of a disjoint union of affine subspaces. Next, let $u \in \ker A_x^*$. We claim that $\dot\gamma_u(t) \in \ker A_x^*$ for all t. To see this, consider the variation $V(t,s) = \exp_{su} tx$, which projects down to a variation $W = \pi \circ V$ on the quotient. The Jacobi field $Y(t) = W_* D_2(t,0)$ induced by W satisfies $Y(0) = 0$, and

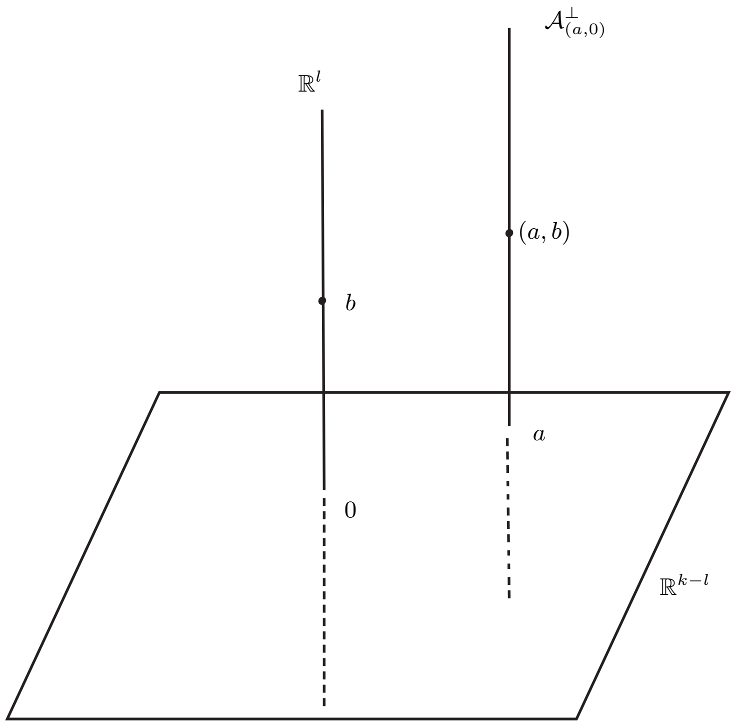
$$Y'(0) = \pi_* \nabla_{D_1(0,0)} (V_* D_2)^{\mathbf{h}} = -\pi_* \nabla^{\mathbf{h}}_{D_1(0,0)} (V_* D_2)^{\mathbf{v}} = \pi_* A_x^* u = 0.$$

Thus, Y is identically zero, or equivalently, the parallel field x is actually basic along γ_u, so that $A_x^* \dot\gamma_u = -(x \circ \gamma_u)' \equiv 0$. This establishes the claim. The latter in turn implies that the image of A has constant rank: in fact, it says that for any point p in F, $\mathcal{A}_p^\perp = \ker A_p^*$ is totally geodesic since it is the intersection over all unit horizontal x at p of the kernel of A_x^*. Now, up to congruence, $\mathcal{A}_0$ is $\mathbb{R}^{k-l} \times \{0\}$ for some integer l by (4.2.2). It follows that for any $(a,b) \in \mathbb{R}^l \times \mathbb{R}^{k-l} = F$, $\mathcal{A}_{(a,b)}^\perp = \ker A_{(a,b)}^* = \{a\} \times \mathbb{R}^l$, since $\mathcal{A}_{(a,0)}^\perp = \{a\} \times \mathbb{R}^l$. Thus, $\mathcal{A}_{(a,b)} = \mathbb{R}^{k-l} \times \{b\}$, and F splits isometrically as $\mathbb{R}^{k-l} \times \mathbb{R}^l$ with the image of A tangent to the first factor, and the kernel of A^* tangent to the second one. This splitting extends to all of Euclidean space, since the kernel of A^* is invariant under parallel translation along horizontal lines γ that intersect F, thereby establishing the first part of the proposition. After factoring out an orthogonal projection, we now have a submersion $\pi' : \mathbb{R}^{n+k-l} \to B$ where the fiber F' over the soul of B is spanned by the image of the A-tensor. An argument similar to the one that led to (4.2.2) then implies that each integrability field is parallel along any line in F', thereby concluding the proof. $\square$

We are now in a position to classify metric fibrations of Euclidean spaces:

Theorem 4.2.1. *Let $\pi : \mathbb{R}^{n+k} \to B^n$ be a Riemannian submersion with connected fibers. Then there exists an orthogonal representation $\phi : \mathbb{R}^k \to SO(n)$, such that, up to congruence, π is the orbit fibration of the free isometric group action ψ of $\mathbb{R}^k$ on $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ given by*

$$\psi(v)(x,u) = (\phi(v)x, u+v), \qquad u, v \in \mathbb{R}^k, \quad x \in \mathbb{R}^n.$$



Conversely, of course, given any homomorphism $\phi : \mathbb{R}^k \to SO(n)$, the orbits of the free isometric action ψ described above form a metric fibration of $\mathbb{R}^{n+k}$. Before going into the proof of the theorem, it may be useful to give a rough description of the main idea involved: If we identify the trivial rank n normal bundle ν of F with $\mathbb{R}^n$ by means of parallel translation, then the bundle of skew-adjoint endomorphisms of ν is simply $\mathfrak{so}(n)$. Similarly, TF is identifiable with F via parallel translation. The connection difference form $\Omega = \nabla^{\mathbf{h}} - \nabla^B$ can then be viewed as a linear map $\Omega : F = \mathbb{R}^k \to \mathfrak{so}(n)$. Proposition 4.2.3 now implies that Ω is a Lie algebra homomorphism. The corresponding Lie group homomorphism turns out to be the representation ϕ in the theorem.

Proof. In general, it is a standard fact that if ∇^1 and ∇^2 are connections on a vector bundle with connection difference 1-form $\Omega = \nabla^1 - \nabla^2$, then the curvature tensors of these connections satisfy

$$R^1 - R^2 = d_{\nabla^2}\Omega + [\Omega, \Omega], \tag{4.2.3}$$

where d_∇ denotes the exterior derivative operator associated to ∇; i.e.,

$$d_\nabla \Omega(U, V) = \nabla_U \Omega(V) - \nabla_V \Omega(U) - \Omega[U, V], \tag{4.2.4}$$

cf. [106]. Now, both the Bott and the Euclidean connections on ν are flat (since they admit globally parallel sections), so that if $\Omega = \nabla^{\mathbf{h}} - \nabla^B$, then (4.2.3) becomes

$$d_{\nabla^{\mathbf{h}}}\Omega = -d_{\nabla^B}\Omega = [\Omega, \Omega]. \tag{4.2.5}$$

If U, V are parallel vector fields on F, and x is a parallel section of ν, then by Proposition 4.2.3, $A_x^* V$ is a parallel section of ν, so that

$$\big(\nabla_U \Omega(V)\big)x = \nabla_U\big(\Omega(V)x\big) = -\nabla_U(A_x^* V) = 0.$$

(4.2.4) then implies that $d_{\nabla^{\mathbf{h}}}\Omega = 0$, and (4.2.5) that $[\Omega, \Omega] = 0$. $F = \mathbb{R}^k$ will be identified with its tangent space at any point via parallel translation, and similarly, sections of the normal bundle of F will be viewed as maps from $\mathbb{R}^k$ to $\mathbb{R}^n$. The restriction of Ω to $0 \in \mathbb{R}^k$ then defines a linear map from $\mathbb{R}^k$ to $\mathfrak{so}(n)$, which we denote by the same letter. The fact that $[\Omega, \Omega] = 0$ now implies that it is a Lie algebra homomorphism. Let $\phi : \mathbb{R}^k \to SO(n)$ denote the corresponding Lie group homomorphism, and for horizontal $x \in \mathbb{R}^n$, consider the section X of ν given by $X(u) = \phi(u)x$, for $u \in \mathbb{R}^k$. If v, $w \in \mathbb{R}^k$, then

$$(\nabla_w X)(v) = \frac{d}{dt}_{|0}(t \mapsto \phi(v + tw)x) = \frac{d}{dt}_{|0}(t \mapsto \phi(tw) \cdot \phi(v)x = \Omega(w)X(v),$$

so that X is the basic field along F with $X(0) = x$. Thus, the fiber through any point $(x, u) \in \mathbb{R}^n \times \mathbb{R}^k$ can be described as the set of all $\big(X(u+v), u+v\big)$ where X is the basic field with $X(u) = x$ and v ranges over $\mathbb{R}^k$. This completes the proof of the theorem, since the free action ψ in the statement satisfies

$$\psi(v)(x, u) = (\phi(v)x, u + v) = (\phi(u+v)\phi(-u)x, u+v) = (X(u+v), u+v).$$

Here, we have used the fact that $X(0) = \phi(-u)x$, which follows from $X(u) = \phi(u)X(0) = x$. $\square$

It was already observed in Example 2.3.1 that along any given fiber of π, there exists a point-wise orthonormal basis of Killing fields. This in turn implies that the fibers are flat submanifolds of $\mathbb{R}^{n+k}$. From the above description of the action ψ, they can be viewed as generalized helices.

The soul argument no longer works of course for metric foliations, since one has no global complete quotient space. Using different methods, it was shown in [62] that they are also homogeneous, at least for leaves of dimension less than three.

In [24], Boltner studies the so-called *equidistant* foliations of Euclidean space. These are singular metric foliations in the sense that leaves need not share the same dimension, but on the other hand, they are required to be imbedded submanifolds, and furthermore globally equidistant; i.e., the distance function from a fixed leaf is constant when restricted to a leaf. The latter condition guarantees that the space B of leaves inherits a metric space structure, and is in fact an Alexandrov space of nonnegative curvature as defined for example in [29], with the projection $\mathbb{R}^{n+k} \to B$ a submetry. Just as in the fibration case, B is shown to have a one-point set that is totally convex, the preimage of which is an affine subspace. The foliation is not, however, necessarily homogeneous.

4.3 Metric foliations of spheres

We now consider a k-dimensional metric foliation $\mathcal{F}$ of the Euclidean sphere $M = S^{n+k}$. All local results and most global ones actually hold on any complete space form of positive curvature, since such a folation can be lifted to the universal cover. Nevertheless, we shall assume for the sake of simplicity that M is a sphere.

According to Theorem 1.8.1, there is a single dual leaf, so that the dual distribution at any point consists of the whole tangent space. This suggests that the A-tensor is highly nontrivial. We begin with the following

Lemma 4.3.1. *If x is a nonzero horizontal vector, then $A_x^* u \neq 0$ for any eigenvector u of S_x.*

Proof. If not, then the holonomy field J along $t \mapsto \gamma_x(t) := \exp(tx)$ that equals u when $t = 0$ satisfies $J'(0) = -S_x J(0) - A_x^* J(0) = -\lambda J(0)$ for some scalar λ. Then $J = (\cos - \lambda \sin)E$, where E is the parallel field along γ_x with $E(0) = u$. This contradicts the fact that J can never vanish. $\square$

As a consequence, the A-tensor cannot vanish at any single point of M.

Definition 4.3.1. $\mathcal{F}$ is said to be *substantial* along a leaf L if there exists a normal vector $x \in \mathcal{H}_p$ at some $p \in L$ such that $A_x : \mathcal{H}_p \to \mathcal{V}_p$ is onto, or equivalently, if A_x^* is one-to-one.

Of course, if A_x^* is one-to-one, then it remains so for all x in an open dense subset of $\mathcal{H}_p$. Furthermore, this condition is independent of the point p in L, since $A_X Y$ has constant norm along L for basic X, Y by O'Neill's curvature formula. Now, Theorem 1.5.1 implies in our present context that

$$(\nabla_z^{\mathbf{v}} A)_x y = S_z A_x y - S_y A_z x - S_x A_y z, \qquad x, y, z \in \mathcal{H}. \tag{4.3.1}$$

In particular, if $x = z = \dot{c}(t)$ is the tangent field of a horizontal geodesic c, and Y is horizontally parallel along c, then

$$(A_{\dot{c}} Y)'^{\mathbf{v}} = 2 S_{\dot{c}} A_{\dot{c}} Y,$$

so that the kernel of $A_{\dot{c}}$ is horizontally parallel, and $A_{\dot{c}}$ has constant rank. Thus, if $\mathcal{F}$ is substantial along a leaf L, then it remains so along all leaves in an open, dense subset of M.

Proposition 4.3.1. *If $k \leq 3$, then $\mathcal{F}$ is substantial everywhere.*

Proof. Although the argument requires considering several cases (and is therefore fairly long), it always relies in an essential way on Lemma 4.3.1. Let $p \in M$, L the leaf through p. We may assume that $S_x \neq 0$ for any nonzero x, for otherwise the claim follows from Lemma 4.3.1. Thus, the linear map $x \mapsto S_x$ from $\mathcal{H}_p$ to the space of self-adjoint endomorphisms of $\mathcal{V}_p$ is one-to-one, and in particular, $n \leq k(k+1)/2$. On the other hand, $n+k$ must be odd – the tangent bundle of an even-dimensional sphere admits no proper subbundles – so the only remaining possibilities are (k, n) equaling (2,3), (3,6), (3,4), (2,1), or (3,2). In the first three cases, where $n \geq k$, consider, for $u \in \mathcal{V}_p = \mathbb{R}^k$, the skew-adjoint endomorphism A_u of $\mathcal{H}_p = \mathbb{R}^n$ given by $A_u x = A_x^* u$ for $x \in \mathbb{R}^n$. We claim that for any nonzero u,

$$\operatorname{rank} A_u > n - k. \tag{4.3.2}$$

In particular, A_u is nonzero if $u \neq 0$, so that

$$\dim E = k, \qquad E = \{A_u \mid u \in \mathbb{R}^k\}. \tag{4.3.3}$$

Thus, $\mathcal{V}_p = \mathbb{R}^k$ is spanned by all $A_x y$, $x, y \in \mathcal{H}_p$. To see why (4.3.2) holds, assume to the contrary that A_v has rank $\leq n-k$ for some nonzero $v \in \mathcal{V}_p$; then A_v has nullity $\geq k$, and the space $W_v = \{S_x \mid x \in \ker A_v\}$ has dimension at least k by Lemma 4.3.1 again. But W_v must then intersect the space of self-adjoint endomorphisms of $\mathcal{V}_p$ that have v as eigenvector, since the latter, as a subspace of the space of all self-adjoint endomorphisms, has codimension $k-1$. In other words, there exists a nonzero x such that v is an eigenvector of S_x and $A_x^* v = 0$, contradicting Lemma 4.3.1.

An equivalent way of saying that $\mathcal{F}$ is substantial along L is that there exists a vector $x \in \mathcal{H}_p$ that is not annihilated by any nonzero element of E from (4.3.3); i.e., $A_u x \neq 0$ for any nonzero $u \in \mathcal{V}_p$. The case $(k, n) = (2, 3)$ then follows, since a two-dimensional space E of skew-adjoint endomorphisms of $\mathbb{R}^3$ cannot annihilate all of $\mathbb{R}^3$. Although this can easily be argued directly, we will prove it instead in the setting that will be used in the other cases: consider the real projective space $\mathbb{R}P^2$ of the three-dimensional vector space $\mathfrak{o}(3)$ of all skew-adjoint endomorphisms of $\mathbb{R}^3$; since any nonzero element of $\mathfrak{o}(3)$ has nullity 1, the subset $\bar{E}$ of $\mathbb{R}P^2 \times \mathbb{R}^3$ consisting of all $([\alpha], u)$, where $\alpha \in E \setminus \{0\}$ and $u \in \ker \alpha$, is a smooth line bundle over a curve in $\mathbb{R}P^2$. The projection $\pi_2 : \bar{E} \to \mathbb{R}^3$ onto the second factor has as image the set of points in $\mathbb{R}^3$ annihilated by E, and the latter has therefore measure zero.

Next, let $k = 3$ and $n = 6$. By (4.3.2), any nonzero $\alpha \in E$ has nullity at most 2; thus, any given α is either invertible or has two-dimensional kernel. If no α is invertible, then as above, the subset $\bar{E}$ of $\mathbb{R}P^5 \times \mathbb{R}^6$ consisting of all pairs $([\alpha], u)$ with $\alpha \in E \setminus \{0\}$ and $u \in \ker \alpha$ is a plane bundle over a two-dimensional projective space, and E cannot annihilate a set of dimension greater than 4. So assume some $\alpha \in E$ is invertible. Recall the canonical isomorphism $\Lambda_2(\mathbb{R}^{2n}) \cong \mathfrak{o}(2n)$ that maps $u \wedge v$ to the skew-adjoint transformation $w \mapsto \langle v, w\rangle u - \langle u, w\rangle v$, and let $\bar{\alpha} \in \Lambda_2(\mathbb{R}^{2n})$ denote the bivector associated to $\alpha \in \mathfrak{o}(2n)$. Notice that α is singular iff $f(\alpha) = 0$, where f is the Pfaffian, $f(\alpha) = \star\bar{\alpha}^n/n$. Thus, $f(\alpha)$ is a homogeneous cubic polynomial in the components of α relative to any given basis of E, and the annihilating set $f^{-1}(0)$ is a cone over a manifold of dimension ≤ 1. The set $\bar{E}$ above is then a plane bundle over this manifold, and cannot annihilate all of $\mathbb{R}^6$.

We next consider the case $k = 3$ and $n = 4$. If the Pfaffian is not identically zero, then the claim follows as above, so we only need to show that f cannot be trivial. In that situation,

$$0 = 2f(\alpha) = \star\bar{\alpha} \wedge \bar{\alpha}, \qquad \alpha \in E,$$

(i.e., $\bar{\alpha}$ is decomposable), and by polarization, $\bar{\alpha}\wedge\bar{\beta} = 0$ for any α, $\beta \in E$. Consider a basis α_i of E, $1 \leq i \leq 3$. Since $\bar{\alpha}_1 \wedge \bar{\alpha}_2 = 0$, they share a common factor, and we may write

$$\bar{\alpha}_1 = \epsilon_0 \wedge \epsilon_1, \qquad \bar{\alpha}_2 = \epsilon_0 \wedge \epsilon_2$$

for some independent one-forms ϵ_0, ϵ_1, ϵ_2 on $\mathbb{R}^4$. Now, $\bar{\alpha}_3$ may or may not lie in the span of $\epsilon_i \wedge \epsilon_j$, $0 \leq i < j \leq 2$. In the former case, consider any ϵ_3 that does not belong to the span of ϵ_i, $0 \leq i \leq 2$. If e_i denotes the basis dual to ϵ_i, then all of E annihilates e_3, which contradicts Lemma 4.3.1. In the latter case, $\bar{\alpha}_3 = \beta \wedge \epsilon_3$, and since it shares a common factor with α_1 and α_2,

$$\bar{\alpha}_3 = (s_0\epsilon_0 + s_1\epsilon_1) \wedge \epsilon_3 = (t_0\epsilon_0 + t_2\epsilon_2) \wedge \epsilon_3.$$

It follows that $s_0 = t_0$, and $s_1 = t_2 = 0$; i.e., $\bar{\alpha}_3$ is a multiple of $\epsilon_0 \wedge \epsilon_3$, and no nonzero element of E annihilates the vector e_0 of the dual basis.

Finally, the last two cases cannot occur by [90] (cf. also [91]), where Molino provides a classification of Riemannian foliations of codimension $k \leq 3$ on compact, simply connected manifolds. In our situation, this also follows by a direct argument: the case $(k, n) = (2, 1)$ may be ruled out since otherwise $A \equiv 0$, contradicting Lemma 4.3.1. Next, consider $k = 3$, $n = 2$. At any point, the image of the A-tensor is one-dimensional, and the claim again follows from Lemma 4.3.1, if we can establish that for some nonzero x, S_x has an eigenvector orthogonal to that image; i.e., if given any two-dimensional subspace E of self-adjoint endomorphisms of $\mathbb{R}^3$ and any plane P through 0 in $\mathbb{R}^3$, some element in $E^* = E \setminus \{0\}$ has an eigenvector in P. We will argue this by contradiction: if not, then each element of E^* has three distinct eigenvalues, thus defining continuous functions $\lambda_i : E^* \to \mathbb{R}$ with $\lambda_1 < \lambda_2 < \lambda_3$. Similarly, we can find continuous unit eigenvector fields $X_i : E^* \to \mathbb{R}^3 \setminus \{0\}$, $SX_i(S) = \lambda_i(S)X_i(S)$ for $S \in E^*$, with image contained in one of the two open half-spaces with boundary P. But $-S$ has eigenvalues $-\lambda_1(S) > -\lambda_2(S) > -\lambda_3(S)$, so that $X_1(-S) = X_3(S)$, $X_2(-S) = X_2(S)$, and $X_3(-S) = X_1(S)$. Thus, $X_1 \wedge X_2 \wedge X_3(-S) = -X_1 \wedge X_2 \wedge X_3(S)$, which is impossible since E^* is connected. □

From now on, we assume, unless otherwise specified, that the leaf dimension k is no larger than 3. Let U denote a connected open set such that the restriction

$\mathcal{F}_{|U}$ is given by the fibers of a Riemannian submersion $\pi : U \to B$, and consider the space $\mathfrak{A}$ of *integrability fields* spanned by all $A_X Y$ on U where X, Y are elements of the space $\mathfrak{B}$ of basic fields on U. Our next endeavor is to show that $\mathfrak{A}$ is a Lie algebra. Notice first that by Proposition 1.5.1,

$$\pi_* A_X^* A_X Y = \frac{1}{3}\big(\pi_* R(X,Y)X - R^B(\pi_* X, \pi_* Y)\pi_* X\big),$$

so that $A_X^* A_X Y \in \mathfrak{B}$ if X, $Y \in \mathfrak{B}$, and thus,

$$T\langle A_X Y, A_X Z\rangle = 0, \qquad T \text{ vertical}, \quad X, Y, Z \in \mathfrak{B}. \tag{4.3.4}$$

Lemma 4.3.2. *If X, $Y \in \mathfrak{B}$, then $S_X A_X Y \in \mathfrak{A}$.*

Proof. (4.3.1) implies

$$2\langle S_X A_X Y, A_X Y\rangle = \frac{1}{2} X|A_X Y|^2 - \langle A_X \nabla_X^{\mathbf{h}} Y, A_X Y\rangle - \langle A_Y \nabla_X X, A_Y X\rangle,$$

which is constant along leaves by (4.3.4), since $TX = XT - [T, X]$, and $[T, X]$ is vertical. By polarization,

$$T\langle S_X A_X Y, A_X Z\rangle = 0, \qquad T \text{ vertical}, \quad X, Y, Z \in \mathfrak{B}. \tag{4.3.5}$$

Consider a leaf L in U. Since $\mathcal{F}$ is substantial, we may assume that A_X is onto L. Using (4.3.4) and (4.3.5), we can find $Y_1, \dots, Y_k \in \mathfrak{B}$ such that $A_X Y_{i|L}$ is an orthonormal frame of eigenvector fields of S_X with constant eigenvalues λ_i along L. Then for any basic Y, the restriction $A_X Y_{|L}$ is a constant linear combination $\sum_i \alpha_i A_X Y_i$, with $\alpha_i = \langle A_X Y, A_X Y_i\rangle$, and $S_X A_X Y = A_X Z$, where $Z = \sum_i \alpha_i \lambda_i Y_i \in \mathfrak{B}$. Thus, $S_X A_X Y \in \mathfrak{A}$. $\square$

Proposition 4.3.2. $\mathfrak{A} \oplus \mathfrak{B}$ *is a Lie algebra that contains* $\mathfrak{A}$ *as an ideal.*

Proof. For X, $Y \in \mathfrak{B}$, $T \in \mathfrak{A}$, the Jacobi identity implies

$$\begin{aligned} 2[A_X Y, T] &= \big[[X,Y]^{\mathbf{v}}, T\big] = \big[[X,Y], T\big] - \big[[X,Y]^{\mathbf{h}}, T\big] \\ &= \big[X, [Y,T]\big] - \big[Y, [X,T]\big] - \big[[X,Y]^{\mathbf{h}}, T\big], \end{aligned}$$

and it remains to show that

$$[Y, T] \in \mathfrak{A}, \qquad Y \in \mathfrak{B}, T \in \mathfrak{A}. \tag{4.3.6}$$

Now, by (4.3.1),

$$[X, A_X Y] = \nabla_X^{\mathbf{v}} A_X Y + S_X A_X Y = 3 S_X A_X Y + A_X \nabla_X^{\mathbf{h}} Y - A_Y \nabla_X^{\mathbf{h}} X,$$

and using Lemma 4.3.2, we conclude that $\big[X, [X,Y]^{\mathbf{v}}\big] \in \mathfrak{A}$. Thus, by polarization,

$$\big[X, [Y,Z]^{\mathbf{v}}\big] + \big[Y, [X,Z]^{\mathbf{v}}\big] \in \mathfrak{A}. \tag{4.3.7}$$

Furthermore,

$$\big[Y, [Z, X]^{\mathbf{v}}\big] + \big[Y, [X, Z]^{\mathbf{v}}\big] = 0, \tag{4.3.8}$$

and

$$\big[Z, [X, Y]^{\mathbf{v}}\big] + \big[Y, [X, Z]^{\mathbf{v}}\big] = -\big[Z, [Y, X]^{\mathbf{v}}\big] - \big[Y, [Z, X]^{\mathbf{v}}\big] \in \mathfrak{A} \tag{4.3.9}$$

by (4.3.7). Adding (4.3.7) through (4.3.9) then implies

$$\big(\circlearrowleft \big[X, [Y, Z]^{\mathbf{v}}\big]\big) + 3\big[Y, [X, Z]^{\mathbf{v}}\big] \in \mathfrak{A}, \tag{4.3.10}$$

where $\circlearrowleft$ denotes cyclic summation. Now, $\big[X, [Y, Z]^{\mathbf{v}}\big]$ is vertical by the Jacobi identity, so that

$$\begin{aligned} \circlearrowleft \big[X, [Y, Z]^{\mathbf{v}}\big] &= \circlearrowleft \big[X, [Y, Z]\big] - \circlearrowleft \big[X, [Y, Z]^{\mathbf{h}}\big] = - \circlearrowleft \big[X, [Y, Z]^{\mathbf{h}}\big]^{v} \\ &= -2 \circlearrowleft A_X [Y, Z]^{\mathbf{h}} \in \mathfrak{A}, \end{aligned}$$

which, together with (4.3.10), proves (4.3.6). □

It follows from Proposition 4.3.2 that the restriction $\mathfrak{A}_L$ of $\mathfrak{A}$ to a leaf L in U is a Lie algebra with dimension $k \le \dim \mathfrak{A}_L \le \binom{n}{2}$. We now improve on this estimate:

Lemma 4.3.3. *$\langle T_1, T_2\rangle$ is constant along L for any $T_i \in \mathfrak{A}_L$. In particular, $\mathfrak{A}_L$ has dimension k.*

Proof. It must be shown that $\langle A_{Z_1} Z_2, A_{Z_3} Z_4\rangle$ is constant along L for any $Z_i \in \mathfrak{B}$. Since A_X is onto $\mathcal{V}_L$ for an open dense subset of basic fields X along L, we may assume that the Z_i belong to a subspace H of basic fields along L, of dimension $3 \le \dim H = m+1 \le 4$, such that $A_{X_0}(H) = \mathcal{V}_L$ for some $X_0 \in H$. By (4.3.4), there exist linearly independent $X_1, \dots, X_m$ such that $\{A_{X_0} X_i \mid i \le k\}$ is an orthonormal basis of $\mathcal{V}_L$. Using skew-symmetry of A, it suffices to show that $\langle A_{X_i} X_j, A_{X_0} X_l\rangle$ is constant for $0 \le i < j \le m$, $1 \le l \le k$. Now, by (4.3.4), this is true if $i = 0$, or $i = l$, or $j = l$. The other cases then follow from (4.3.4) together with the fact that $A_{X_i} X_j$ has constant length: for example, when $k = 3$, then

$$\langle A_{X_1} X_2, A_{X_0} X_3\rangle^2 = |A_{X_1} X_2|^2 - \langle A_{X_1} X_2, A_{X_0} X_1\rangle^2 - \langle A_{X_1} X_2, A_{X_0} X_2\rangle^2$$

is constant. □

We are now in a position to prove the main result of this section. The argument will make use of the following classical theorem, a proof of which can be found in [116]:

Theorem 4.3.1 (Fundamental theorem for submanifolds). *Let M_i, $i = 1, 2$, denote k-dimensional Riemannian submanifolds of the simply connected spaceform Q_c^{n+k} of constant curvature c, $h : M_1 \to M_2$ an isometry. Let $E(\nu_i)$ denote the total space of the normal bundle ν_i of M_i in Q_c, and suppose there exists a linear* *bundle isometry $H : E(\nu_1) \to E(\nu_2)$ covering h, such that H preserves the normal connections $\nabla_i^{\mathbf{h}}$ on ν_i and the second fundamental forms S^i of M_i; i.e.,*

$$\nabla^{\mathbf{h}}_{2\,T}(HX) = H\nabla^{\mathbf{h}}_{1\,T}X, \qquad h_*S^1_X T = S^2_{HX}h_*T$$

for any sections T and X of the tangent and normal bundle respectively of M_1. Then there exists an isometry $\tilde{h}$ of Q_c such that $\tilde{h}_{|M_1} = h$, and the restriction of $\tilde{h}_*$ to $E(\nu_1)$ equals H.

Theorem 4.3.2 (Gromoll-Grove, [56]). *When $k \leq 3$, any k-dimensional metric foliation of the Euclidean sphere S^{n+k} is homogeneous; specifically, it is the orbit foliation of a connected k-dimensional Lie subgroup of $SO(n+k+1)$.*

Proof. Consider a point p in the sphere, and the leaf L containing it. We begin by constructing a group of local isometries of L near p. These will then be extended to the whole ambient space. Denote by G the local Lie group of diffeomorphisms of some neighborhood of p in L generated by the flows of vector fields in $\mathfrak{A}_L$. There are neighborhoods U of e in G and V of p in L such that $\imath_p : U \to V$, $\imath_p(g) := g(p)$, is a diffeomorphism. According to the discussion in Section 2.3, a vector field on V belongs to $\mathfrak{A}_L$ iff it is $\imath_p$-related to a right invariant vector field of G. Denote by $\mathfrak{K}_L$ the algebra of vector fields on V that are $\imath_p$-related to *left* invariant vector fields of G; i.e.,

$$\mathfrak{K}_L = \{T \in \mathfrak{X}(M) \mid T = \imath_{p*}X \circ \imath_p^{-1},\ X \in \mathfrak{g}\}.$$

Since left and right invariant fields commute, $[\mathfrak{A}_L, \mathfrak{K}_L] = 0$. This implies that $\mathfrak{K}_L$ is an algebra of Killing fields: in fact, since $\mathfrak{A}_L$ contains a point-wise orthonormal basis of the vertical space, it suffices to check that the transformation $T \mapsto \nabla_T X$, $X \in \mathfrak{K}_L$, is skew-adjoint on these basis elements. But this is clear, since

$$\langle \nabla_T X, T\rangle = \langle \nabla_X T, T\rangle = \frac{1}{2}X\langle T, T\rangle = 0, \quad T \in \mathfrak{A}_L.$$

We next extend the isometries of $V \subset L$ generated by $\mathfrak{K}_L$ to (unique) leaf-preserving isometries of an open set in the sphere. Using the fact that a local Killing field has a unique global extension, the theorem then clearly follows. So consider such a local isometry ϕ, and extend it to a linear isometry Φ of the normal bundle of L near p by defining

$$\Phi X := X \circ \phi, \quad X \in \mathfrak{B}.$$

We claim that Φ preserves the normal connection: if $T \in \mathfrak{A}$ and $X \in \mathfrak{B}$, then $\nabla^{\mathbf{h}}_T X = -A^*_X T$ is basic by Lemma 4.3.3, and $\phi_* T = T \circ \phi$ because the algebras $\mathfrak{A}$ and $\mathfrak{K}_L$ commute. Thus,

$$\Phi(\nabla^{\mathbf{h}}_T X) = (\nabla^{\mathbf{h}}_T X) \circ \phi = \nabla^{\mathbf{h}}_{T\circ\phi} X = \nabla^{\mathbf{h}}_{\phi_* T} X = \nabla^{\mathbf{h}}_T (X \circ \phi) = \nabla^{\mathbf{h}}_T(\Phi X).$$

In the same way, Φ preserves the second fundamental form: Lemmas 4.3.2 and 4.3.3 imply that $S_X \mathfrak{A} \subset \mathfrak{A}$, so that

$$\phi_* S_X T = (S_X T) \circ \phi = S_{X\circ\phi}(T \circ \phi) = S_{\Phi X}\phi_* T.$$

The fundamental theorem for submanifolds then implies that ϕ extends to an isometry of a tubular neighborhood of V in the ambient space. Since this isometry must then locally be given by $\exp_V \circ \Phi \circ \exp_V^{-1}$, where $\exp_V$ is the exponential map of the normal bundle of V, it preserves leaves. $\square$

Little is known at present concerning metric foliations of spheres with higher-dimensional leaves. One remarkable fact is that when $k > 1$, they are always generalized Seifert fibrations, in the sense that all leaves are compact [52]. The latter are fairly similar to actual fibrations, at least from a homotopical point of view [75].

We end the section with a brief description of the foliations that can occur in Theorem 4.3.2. If $k = 1$, then $\mathcal{F}$ is the orbit foliation of a one-dimensional Lie subgroup G of $SO(n+2)$, and is therefore determined by a homomorphism $\phi : \mathbb{R} \to SO(n+2)$, $\phi(t) = e^{tM}$, where $M = \dot\phi(0) \in \mathfrak{o}(n+2)$. The skew-symmetric matrix M is similar, via an orthogonal matrix, to a block matrix of the form

$$\operatorname{diag}\{i\lambda_1, \dots, i\lambda_s, 0 \dots, 0\} := \begin{bmatrix} 0 & -\lambda_1 & & & & & & \\ \lambda_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & -\lambda_s & & & \\ & & & \lambda_s & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}$$

where $0 < \lambda_1 \le \cdots \le \lambda_s$. Since the action is free, M must actually have the form $\operatorname{diag}\{i\lambda_1, \dots, i\lambda_s\}$, and n is even, with $s = 1 + (n/2)$. Normalize M so that $\lambda_s = 1$. Then, up to congruence, G is a direct sum of rotations

$$\operatorname{diag}\{e^{i\lambda_1 t}, e^{i\lambda_2 t}, \dots, e^{it}\}, \qquad 0 < \lambda_1 \le \lambda_2 \le \cdots \le 1.$$

Notice there are always at least s compact leaves that are totally geodesic circles, namely the orbits of the odd standard basis vectors $\mathbf{e}_1, \mathbf{e}_3, \dots, \mathbf{e}_{n+1}$. All leaves are compact iff each λ_j is rational. Among these foliations, only one is a fibration, namely the Hopf fibration, corresponding to $\lambda_j = 1$ for all j.

Next, consider the case $k = 2$. Since the only two-dimensional subgroups of an orthogonal group are abelian, there can be no metric foliations of this dimension: for otherwise, there would exist independent M, N in the Lie algebra of G with vanishing bracket. Then M and N would share a common basis of complex eigenvectors, and such a vector would have the same orbit under the actions $t \mapsto e^{tM}$, $t \mapsto e^{tN}$, implying the action is not free. This also shows, incidentally, that there are no free $O(k)$-actions on spheres if $k > 3$, since the orthogonal Lie algebra is then no longer simple, and contains linearly independent commuting vectors.

When $k = 3$, the last argument implies that G has $SU(2)$ as universal cover. The classification in this case is obtained via representation theory, and we will limit ourselves to merely stating the result. The interested reader should consult [56] and [27] for further details. Let V_n denote the complex vector space of homogeneous polynomials p of degree n in two complex variables z_1, z_2,

$$p(z_1, z_2) = \sum_{k=0}^{n} a_k z_1^k z_2^{n-k},$$

and define an action ρ_n of $SU(2)$ on V_n by setting

$$(gp)(z) = p(zg), \qquad g \in SU(2), \quad p \in V_n, \quad z = (z_1, z_2),$$

with zg denoting matrix multiplication. Notice that ρ_1 is just the standard action of $SU(2)$ on $V_1 \cong \mathbb{C}^2$. The main result is that three-dimensional foliations of S^{n+3} exist precisely when $n = 4l$, and any such foliation is given by a direct sum $\rho_{n_1} \oplus \cdots \oplus \rho_{n_s}$ of irreducible representations of $SU(2)$, with n_j odd for all j, $1 \le n_1 \le \cdots \le n_s$, and $n_1 + \cdots + n_s = 2((n/4) + 1) - s$. Here again, only one is a fibration, namely the Hopf fibration with $n_1 = \cdots = n_s = 1$.

As far as metric *fibrations* of spheres are concerned, it follows from [28] that the fiber must be a homotopy sphere of dimension one, three, or seven. The first two cases are covered in Theorem 4.3.2, and the last one was solved by Wilking in [139], using Morse theoretical methods:

Theorem 4.3.3 (Gromoll-Grove, Wilking). *Any Riemannian submersion $S^{n+k} \to M^n$ of a Euclidean sphere is congruent to a Hopf fibration.*

In the special case of totally geodesic fibers, this result is due to Escobales [46] and Ranjan [107]. The extra assumption is quite strong, of course, and it is easy to see directly that in this case, M must be a rank one symmetric space: consider a point p in M. Local geodesic reflection in p of a curve c can be obtained by horizontally lifting that curve to the sphere, reflecting it in the fiber over p, and projecting back onto M. The first two steps preserve the length of c since the fiber is totally geodesic, so that geodesic reflection in M is distance non-increasing. It must then be an isometry, because its square is the identity. Thus, M is locally symmetric. If the fiber of π is connected, then M is simply connected, and hence globally symmetric. The rank statement follows from the fact that M has positive curvature by O'Neill's formula.

The discussion of foliations in space forms carried out in the last two sections raises several new questions: it would for example be interesting to determine how much of the rigidity that is apparent in constant nonnegative curvature carries over to more general manifolds, such as symmetric spaces. Most of the few known results deal with one-dimensional metric foliations: they have been shown to be homogeneous if the ambient space is a compact Lie group [94] with bi-invariant metric or $S^2 \times \mathbb{R}$ with the standard product metric [61]. The methods used in each case are specific to the situation at hand and do not easily generalize. In a related but slightly different direction, it is known that the same result holds for the Heisenberg group [95]; there are, however, noncompact Lie groups with left invariant metrics that admit one-dimensional metric foliations which are not homogeneous [135].

4.4 Geometry of the tangent bundle

In order to discuss metric foliations on a compact space form M of nonpositive curvature, some properties of the geodesic flow on the tangent bundle of M will be needed. The reader familiar with these concepts may skip this section without loss of continuity, and the one who wishes to study them in more detail is referred to [102] or [106].

Denote by $\pi : TM \longrightarrow M$ and $\tilde{\pi} : T^*M \longrightarrow M$ the bundle projections.

Definition 4.4.1. The fundamental 1-form θ on the co-tangent bundle T^*M is given by $\theta(\alpha) = \tilde{\pi}^*\alpha$, for $\alpha \in T^*M$.

Thus, for $\xi \in (T^*M)_\alpha$, $\theta(\alpha)(\xi) = \alpha(\tilde{\pi}_{*\alpha}\xi)$.

The Levi-Civita connection $\mathcal{H}$ of a Riemannian manifold M induces a bundle homomorphism $K : TTM \longrightarrow TM$ over $\pi : TM \longrightarrow M$, called the *connection map*, defined as follows: a vector $\xi \in TTM$ decomposes as $\xi^{\mathbf{h}} + \xi^{\mathbf{v}} \in \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \ker \pi_*$ and $\mathcal{H} = \ker K$. For $p \in M$, $u \in M_p$, and $\mathcal{V}_u = \imath_{*u}(M_p)$, with $\imath : M_p \hookrightarrow TM$ denoting inclusion, denote by $\mathcal{J}_u : M_p \longrightarrow (M_p)_u$ the isomorphism given by $\mathcal{J}_u w = \dot{\gamma}(0)$, $\gamma(t) = u + tw$. Then

$$K\xi = (\imath_* \circ \mathcal{J}_u)^{-1}\xi^{\mathbf{v}}. \tag{4.4.1}$$

Alternatively, for a vector field X on M and $u \in TM$,

$$\nabla_u X = KX_*u. \tag{4.4.2}$$

Since the restrictions $\pi_* : \mathcal{H}_u \longrightarrow M_{\pi(u)}$ and $K : \mathcal{V}_u \longrightarrow M_{\pi(u)}$ are both isomorphisms, the map

$$(\pi_*, K) : TTM \longrightarrow TM \oplus TM$$

is a bundle isomorphism over $\pi : TM \longrightarrow M$. In fact, its inverse $\mathcal{I} : TM \oplus TM \longrightarrow TTM$ is described as follows: for $u \in TM$, $w, z \in M_{\pi(u)}$, consider a curve $\gamma : I \to M$ with $\dot{\gamma}(0) = z$. If Z denotes the parallel field along γ with $Z(0) = u$ (i.e., Z is the horizontal lift to TM of γ starting at u), then

$$\mathcal{I}(z, w) = \dot{Z}(0) + \imath_*\mathcal{J}_u w.$$

We will routinely identify $(TM)_u$ with $M_{\pi(u)} \times M_{\pi(u)}$ via the isomorphism (π_*, K). The *Sasaki metric* $\langle\langle , \rangle\rangle$ on the manifold TM is that metric for which (π_*, K) becomes a linear isometry. It is a connection metric in the sense of Proposition 2.7.1.

Recall that a vector field S on TM is called a *spray* on M if $\pi_* \circ S = 1_{TM}$ and $S \circ \mu_a = a\mu_{a*}S$, where μ_a denotes multiplication by $a \in \mathbb{R}$. The *geodesic spray* S is the unique horizontal spray on M; i.e., $S(u) = (u, 0)$, $u \in TM$. The integral curves of S are precisely the velocity fields $\dot\gamma : I \longrightarrow TM$ of geodesics $\gamma : I \longrightarrow M$ of M.

We shall denote by $\flat : TM \longrightarrow T^*M$, $\flat(u) = \langle u, \cdot\rangle$, and $\tilde\flat : TTM \longrightarrow T^*TM$ the musical isomorphisms induced by the metrics on M and TM respectively. The cotangent vector $\flat(u)$ is often written as $u^\flat$. The next proposition says that the geodesic spray is essentially the metric dual of the fundamental 1-form on T^*M:

Proposition 4.4.1. $S^{\tilde\flat} = \flat^*\theta$.

Proof. Let $u \in TM$, $\xi \in (TM)_u$. Since $\tilde\pi \circ \flat = \pi$ and S is horizontal,

$$\begin{aligned}\flat^*\theta(\xi) &= \theta(\flat_*\xi) = u^\flat(\tilde\pi_* \circ \flat_*\xi) = u^\flat(\pi_*\xi) = \langle u, \pi_*\xi\rangle = \langle \pi_*S(u), \pi_*\xi\rangle \\ &= \langle\langle S(u), \xi\rangle\rangle. \end{aligned}$$ □

Define a complex structure J on TTM by setting $J(u, w) = (-w, u)$; equivalently,

$$\pi_*J = -K, \qquad KJ = \pi_*. \tag{4.4.3}$$

Then the 2-form Ω, where

$$\Omega(\xi, \eta) := \langle\langle J\xi, \eta\rangle\rangle,$$

is a symplectic (i.e., nondegenerate) 2-form on TM, and if n is the dimension of M, then Ω^n equals $(-1)^{[n/2]}n!$ times the Sasaki metric volume element. On the other hand, $-d\theta$ is also a symplectic form, but one on T^*M rather than TM. The relation between the two is given by the following:

Proposition 4.4.2. $\Omega = -d(b^*\theta)$*; i.e.,* Ω *is the metric pullback of the canonical symplectic form* $-d\theta$ *on* T^*M*.*

Proof. Viewing the identity map 1_{TM} on TM as a vector field on M along $\pi : TM \longrightarrow M$, we have

$$\nabla_X(1_{TM}) = K(1_{TM})_*X = KX, \qquad X \in \mathfrak{X}TM.$$

Thus, if Y is another vector field on TM, then

$$\begin{aligned} -d(\flat^*\theta)(X, Y) &= -X\langle\pi_*S, \pi_*Y\rangle + Y\langle\pi_*S, \pi_*X\rangle + \langle\pi_*S, \pi_*[X, Y]\rangle \\ &= -\langle\nabla_X(1_{TM}), \pi_*Y\rangle - \langle 1_{TM}, \nabla_X\pi_*Y\rangle + \langle\nabla_Y(1_{TM}), \pi_*X\rangle \\ &\quad + \langle 1_{TM}, \nabla_Y\pi_*X\rangle + \langle 1_{TM}, \pi_*[X, Y]\rangle \\ &= -\langle KX, \pi_*Y\rangle + \langle KY, \pi_*X\rangle = \langle\pi_*JX, \pi_*Y\rangle + \langle KJX, KY\rangle \\ &= \langle\langle JX, Y\rangle\rangle, \end{aligned}$$

where we used (4.4.3) in the equality before last. □

Proposition 4.4.3. *If* $h : TM \longrightarrow \mathbb{R}$ *denotes the* energy function, $h(u) = (1/2)|u|^2$, *then* $i_S\Omega = dh$.

Proof. S is horizontal for the submersion $\pi : TM \longrightarrow M$, and if γ is an integral curve of S, then $\pi \circ \gamma$ is a geodesic of M. Thus, γ is a geodesic of the Sasaki metric, and S is an auto-parallel vector field. Given a vector field X on TM,

$$\begin{aligned} i_S\Omega(X) &= -d(\flat^*\theta)(S, X) = -S\langle\langle S, X\rangle\rangle + X\langle\langle S, S\rangle\rangle + \langle\langle S, [S, X]\rangle\rangle \\ &= -\langle\langle S, \nabla_S X - [S, X]\rangle\rangle + X\langle\langle S, S\rangle\rangle = -\langle\langle S, \nabla_X S\rangle\rangle + X\langle\langle S, S\rangle\rangle \\ &= \frac{1}{2} X\langle\langle S, S\rangle\rangle = X(h), \end{aligned}$$

since $\langle\langle S, S\rangle\rangle(u) = \langle \pi_* S, \pi_* S\rangle(u) = \langle u, u\rangle$. $\square$

Proposition 4.4.3 says that the geodesic spray is the *Hamiltonian* vector field of the energy function with respect to Ω.

Assume from now on that M is compact. Instead of working on TM, we shall restrict ourselves to the unit tangent bundle $T^1M = \{u \in TM \mid |u| = 1\}$, which has the advantage of being compact. We first describe the tangent space of this manifold at a given point:

Proposition 4.4.4. *If* $\imath : T^1M \hookrightarrow TM$ *denotes the inclusion map, then for* $u \in T^1M$,

$$\imath_*(T^1M)_u = \{\xi \in (TM)_u \mid \langle K\xi, u\rangle = 0\} = J \circ S(u)^\perp.$$

Alternatively, under the isomorphism (π_*, K), $\imath_*(T^1M)_u = (0, u)^\perp$. *In particular, there is a unique vector field on* T^1M *that is* $\imath$*-related to* S *(it will be denoted by* S *also).*

Proof. Since T^1M is the pre-image of 1 under the energy function h, the space $\imath_*(T^1M)_u$ is the kernel of h_*u, which by Proposition 4.4.3 equals $\{\xi \in (TM)_u \mid \Omega(S(u), \xi) = 0\}$. But $\Omega(S(u), \xi) = \langle\langle JS(u), \xi\rangle\rangle = \langle K\xi, u\rangle$ by (4.4.3). $\square$

We will denote by σ the restriction $\imath^*\flat^*\theta$ to T^1M of the 1-form $\flat^*\theta$ on TM. By Proposition 4.4.1, σ is the metric dual of the geodesic spray S on T^1M. Since the volume form of TM is

$$\bar{\omega} = \frac{(-1)^{[n/2]}}{n!}\Omega^n = \frac{(-1)^{n+[n/2]}}{n!} d(\flat^*\theta)^n,$$

the volume form of T^1M is

$$\omega = i_{JS}\imath^*\bar{\omega} = \frac{(-1)^{n+[n/2]}}{n!} i_{JS} d\sigma^n,$$

with i_{JS} denoting interior multiplication by JS. But $i_{JS}d\sigma^n = n(i_{JS}d\sigma) \wedge (d\sigma)^{n-1}$, and for $X \in \mathfrak{X}T^1M$,

$$i_{JS}d\sigma(X) = -\Omega(JS, X) = -\langle\langle J^2 S, X\rangle\rangle = \langle\langle S, X\rangle\rangle = \sigma(X).$$

Thus,

$$\omega = \frac{(-1)^{n+[n/2]}}{(n-1)!}\sigma \wedge (d\sigma)^{n-1}. \tag{4.4.4}$$

A 1-form α on an odd-dimensional manifold M^{2n-1} is said to be a *contact form* if $\alpha \wedge (d\alpha)^{n-1}$ is nowhere zero. (4.4.4) implies that the metric dual of the geodesic spray is a contact form on the unit tangent bundle.

Since T^1M is compact, S is complete, and its flow is a one parameter group $\{\phi_t\}_{t\in\mathbb{R}}$ of diffeomorphisms, called the *geodesic flow* of M. The volume form ω has finite integral over T^1M, and thus induces a probability measure on that space, called the *Liouville measure*.

Proposition 4.4.5. *The geodesic flow is measure-preserving; i.e., given $A \subset T^1M$, the volume of $\phi_t(A)$ is constant, $t \in \mathbb{R}$.*

Proof. The statement follows once we establish that $\mathcal{L}_S\omega = 0$, or using (4.4.4), that $\mathcal{L}_S\sigma = 0$. Now, $\mathcal{L}_S\sigma = i_S d\sigma + d i_S\sigma = i_S d\sigma$, because $i_S\sigma \equiv 1$. Given $X \in \mathfrak{X}T^1M$, $i_S d\sigma(X) = -\Omega(S, X) = -\langle\langle JS, X\rangle\rangle = 0$, since JS is orthogonal to T^1M. $\square$

Proposition 4.4.6. *Given $v \in T^1M$ and $(u, w) \in (T^1M)_v$, $\phi_{t*}(u, w) = (J(t), J'(t))$, where J is the Jacobi field along the geodesic $t \mapsto \exp(tv)$ with $J(0) = u$, $J'(0) = w$.*

Proof. Recall that for $v \in TM$, $\phi_t(v) = \dot{c}_v(t)$, where $c_v(t) = \exp(tv)$. Consider a curve $\gamma : I \longrightarrow T^1M$ with $\gamma(0) = v$, $\dot\gamma(0) = (u, w)$. Then

$$(t, s) \mapsto V(t, s) := \pi \circ \phi_t \circ \gamma(s) = \exp(t\gamma(s))$$

is a variation by geodesics of c_v. The corresponding Jacobi field $t \mapsto J(t) = V_*D_2(t, 0)$ is given by

$$J(t) = \pi_* \circ \phi_{t*}\dot\gamma(0) = \pi_* \circ \phi_{t*}(u, w),$$

and

$$J'(t) = \nabla_{D_1(t,0)}V_*D_2 = \nabla_{D_2(t,0)}V_*D_1.$$

But $V_*D_1(t, s) = \phi_t(\gamma(s))$, so

$$J'(t) = \nabla_{D(0)}(\phi_t \circ \gamma) = K(\phi_t \circ \gamma)_*D(0) = K \circ \phi_{t*}(u, w),$$

as claimed. $\square$

We end this section with two ergodic theorems that hold on measure spaces with a measure-preserving transformation, see [130]. In our context, with the transformation being the geodesic flow, they can be stated as follows:

Theorem 4.4.1. *Let A be a submanifold of the unit tangent bundle of M that is measure-invariant under the geodesic flow.*

1. (**Oseledets**) *For almost every $v \in A$, there exists a direct sum decomposition of the tangent space*

$$A_v = V^s(v) \oplus V^u(v) \oplus V^0(v)$$

of A at v, where for $\xi \neq 0$,

$$\xi \in V^s(v) \text{ iff } \lim_{t\to\pm\infty} \frac{1}{t} \ln |\phi_{t*}\xi| < 0,$$
$$\xi \in V^u(v) \text{ iff } \lim_{t\to\pm\infty} \frac{1}{t} \ln |\phi_{t*}\xi| > 0,$$
$$\xi \in V^0(v) \text{ iff } \lim_{t\to\pm\infty} \frac{1}{t} \ln |\phi_{t*}\xi| = 0.$$

2. (**Birkhoff**) *If $f : A \longrightarrow \mathbb{R}$ is integrable, then for a.e. $u \in A$,*

$$\tilde{f}(u) := \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\phi_s u) ds \text{ exists, and } \int_A f\omega = \int_A \tilde{f}\omega.$$

4.5 Compact space forms of nonpositive curvature

Although at the time of writing there does not seem to be a classification of metric foliations in space forms of curvature $\kappa \leq 0$, we will see that there are severe restrictions, at least in the compact case, cf. [81], [133]. The main tools used in the argument are the ergodic theorems introduced in the last section. So let M be a compact space of constant curvature $\kappa \leq 0$, and $\mathcal{F}$ a metric foliation on M. We begin by identifying the tangent space $\mathcal{H}_x$ of the horizontal bundle $\mathcal{H}$ at $x \in \mathcal{H}$. Notice that if $\mathcal{H}^1$ denotes the unit horizontal bundle, then for $x \in \mathcal{H}^1$, $\mathcal{H}^1_x = \mathcal{H}_x \cap (0, x)^\perp$ by Proposition 4.4.4.

Lemma 4.5.1. $\mathcal{H}_x = \{(e, f) \in M_{\pi(x)} \times M_{\pi(x)} \mid f \in A_{e^{\mathbf{h}}} x - S_x e^{\mathbf{v}} + \mathcal{H}\}$.

Proof. Both spaces have the same dimension $2n - k$, so we only need to show that $\mathcal{H}_x$ is contained in the space on the right. Consider $\xi = (e, f) \in \mathcal{H}_x$ and a curve Z in $\mathcal{H}$ with $\dot{Z}(0) = \xi$. If $c := \pi \circ Z$, $p := c(0)$, then $\langle Z, U \circ c\rangle \equiv 0$ for any vertical field U, so that

$$\begin{aligned} 0 &= \langle Z, U \circ c\rangle'(0) = \langle Z', U \circ c\rangle(0) + \langle Z, (U \circ c)'\rangle(0) \\ &= \langle K\xi, U(p)\rangle + \langle x, \nabla_{\pi_\xi} U\rangle = \langle f, U(p)\rangle + \langle x, \nabla_e U\rangle \\ &= \langle f^{\mathbf{v}}, U(p)\rangle + \langle x, \nabla_{e^{\mathbf{h}}} U\rangle + \langle x, \nabla_{e^{\mathbf{v}}} U\rangle \\ &= \langle f^{\mathbf{v}}, U(p)\rangle - \langle A_{e^{\mathbf{h}}} x, U(p)\rangle + \langle S_x e^{\mathbf{v}}, U(p)\rangle. \end{aligned}$$

Thus, $f^{\mathbf{v}} = A_{e^{\mathbf{h}}} x - S_x e^{\mathbf{v}}$, as claimed. □

Consider $\mathcal{H}^1$ as a Riemannian submanifold of T^1M, where T^1M is endowed with the Sasaki metric, and observe that $\mathcal{H}^1$ is invariant under the geodesic flow, since a geodesic that starts out horizontally remains so.

Proposition 4.5.1. *The geodesic flow $\{\phi_t\}$ is measure-preserving on $\mathcal{H}^1$.*

Proof. Given $x \in \mathcal{H}^1$ and $\xi \in \mathcal{H}^1_x$, denote by Y_ξ the Jacobi field along the geodesic $t \mapsto \exp(tx)$ with $Y_\xi(0) = \pi_*\xi$, $Y'(0) = K\xi$. By Proposition 4.4.6, $\phi_{t*}\xi = (Y_\xi(t), Y'_\xi(t))$, after the usual identification of TTM with $TM \oplus TM$ via (π_*, K).

Consider first the negative curvature case, which we normalize so that $\kappa = -1$. Then $Y_\xi(t) = e^t E_1(t) + e^{-t}E_2(t)$ for some parallel fields E_i, and given $\eta \in \mathcal{H}^1_x$, we have

$$\langle \phi_{*t}\xi, \phi_{*t}\eta\rangle = \sum_{k=-2}^{2} a_k e^{kt} \tag{4.5.1}$$

for some constants a_k. But if ω is the volume element of $\mathcal{H}^1$ and ξ_i is a basis of $\mathcal{H}^1_x$, then

$$\phi_t^*\omega(\xi_1, \dots, \xi_{2n-k-1}) = (\det\langle \phi_{t*}\xi_i, \phi_{t_*}\xi_j\rangle)^{1/2} \tag{4.5.2}$$

must be constant by (4.5.1) and compactness of $\mathcal{H}^1$.

The flat case is similar: Jacobi fields now have the form $t \mapsto E_1(t) + tE_2(t)$, so that (4.5.2) becomes the square root of a polynomial in t. Compactness of $\mathcal{H}^1$ then forces it to be constant. □

Theorem 4.5.1. *Let M be a compact space form of curvature $\kappa \le 0$. If $\kappa < 0$, then M admits no metric foliations. If $\kappa = 0$, then any such foliation splits; i.e., it is locally congruent to a metric product foliation.*

Proof. We will show that the foliation is flat (and in particular, its orthogonal complement is a totally geodesic foliation). In negative curvature, the statement follows from the fact that compact manifolds of negative curvature admit no totally geodesic foliations [128], and in the flat case, from Theorem 2.2.2.

In the hyperbolic case, consider a point $x \in \mathcal{H}^1$ where the decomposition stated in Oseledets' ergodic theorem holds, so that $\mathcal{H}^1_x = V^s(x) \oplus V^u(x) \oplus V^0(x)$. We claim that

$$\begin{aligned} V^u(x) \subset \Delta &= \{(e,e) \mid e \in M_{\pi(x)}\}, \\ V^s(x) \subset \Delta^* &= \{(e,-e) \mid e \in M_{\pi(x)}\}. \end{aligned} \tag{4.5.3}$$

The arguments are similar in both instances, and we only prove the latter. Denote by $\mathsf{p}^u : \mathcal{H}^1_x \longrightarrow V^u(x)$ the projection. As pointed out earlier, if $\xi = (u,v) \in \mathcal{H}^1_x$, then $Y_\xi(t) = e^t E_1(t) + e^{-t}E_2(t)$, with E_i parallel, $E_1(0) = (u+v)/2$, $E_2(0) = (u-v)/2$. Then

$$\ln|\phi_{t*}\xi| = \ln\left(|Y_\xi|^2 + |Y'_\xi|^2\right)^{1/2}(t) = \ln\sqrt{2} + \frac{1}{2}\ln\left[e^{2t}|E_1|^2 + e^{-2t}|E_2|^2\right].$$

Notice that if $E_1 \ne 0$, then $(\ln|\phi_{*t}\xi|)/t \to 1$ as $t \to \infty$, so that $\mathsf{p}^u\xi \ne 0$. In other words, if $\xi \in V^s(x)$, then $E^1 = 0$, and $v = -u$ as claimed.

Now, consider any horizontal $0 \ne y \perp x$. By Lemma 4.5.1, $(0,y) \in \mathcal{H}^1_x$, and since $V^0(x)$ is spanned by $(0,x)$, $(0,y) = (e,e) + (f,-f) \in V^u(x) \oplus V^s(x)$ for some

e, f. But then $-f = e = y/2$, from which we conclude that $(y, y) \in \mathcal{H}^1_x$. Again by the lemma, $A_y x = 0$. Thus, $A_x = 0$ for almost every x, and by continuity, $\mathcal{F}$ is flat.

In the zero curvature case, choose $x \in \mathcal{H}^1$ as in the statement of Birkhoff's ergodic theorem, with $f : \mathcal{H}^1 \to \mathbb{R}$ given by $f(z) := |A^*_z|^2$. The result will follow once we show that if U is a unit vertical field along the geodesic c in direction x, then $(1/t)\int_0^t |A^*_{\dot c}U|^2 \to 0$ as $t \to \infty$. Assume first that U can be written as $J/|J|$ for some holonomy Jacobi field J along c. Since $J = E + tF$ for parallel fields E, F, we have

$$|A^*_{\dot c}U|^2 \le |A^*_{\dot c}U|^2 + |S_{\dot c}U|^2 = \frac{|J'|^2}{|J|^2} = \frac{|F|^2}{|E|^2 + 2t\langle E, F\rangle + t^2|F|^2},$$

and the claim certainly holds in this case. In general, if J_i, $i = 1, 2$, are holonomy fields with $J_i(0)$ orthonormal eigenvectors of $S_{\dot c(0)}$, then the angle $\measuredangle(J_1(t), J_2(t)) \to \measuredangle(J_1'(0), J_2'(0))$ as $t \to \infty$ by linearity of Jacobi fields in Euclidean space. It follows that there exists an orthonormal basis $\{u_i\}$ of eigenvectors of $S_{\dot c(0)}$ such that if J_i is the holonomy field with $J_i(0) = u_i$, then the angle between any two $J_i(t)$ and $J_k(t)$ lies in some fixed interval (α, β), for some $0 < \alpha < \beta < \pi$, and all $t > 0$. This in turn implies that U equals a functional linear combination $\sum f_i(J_i/|J_i|)$ with bounded f_i, and thus $(1/t)\int_0^t |A^*_{\dot c}U|^2 \to 0$. $\square$

It should be noted that the argument above extends with only minor modifications to compact locally homogeneous manifolds of negative curvature. It is therefore tempting to conjecture that there are no metric foliations on compact manifolds of negative curvature, especially in light of the following result (see [108]):

Theorem 4.5.2. *A compact manifold M with negative Ricci curvature admits no one-dimensional metric foliations.*

Proof. We begin by computing the divergence of the mean curvature vector field $Z = \nabla_T T$ of $\mathcal{F}$, where T is a (local) unit vertical field. Let $p \in M$, $\pi : U \to B$ a submersion defining $\mathcal{F}$ in a neighborhood U of p, and $\bar X_i$ local orthonormal fields on B with $\nabla_{\bar X_i}\bar X_j(\pi(p)) = 0$. Then the basic fields X_i on U that are π-related to $\bar X_i$ satisfy $\nabla^{\mathbf{h}}_{X_i} X_j(p) = 0$. Now,

$$\operatorname{div} Z = \sum_i \langle \nabla_{X_i}\nabla_T T, X_i\rangle + \langle \nabla_T\nabla_T T, T\rangle. \tag{4.5.4}$$

The second term on the right equals $-|\nabla_T T|^2$, whereas the first term may be rewritten as

$$\begin{aligned}\langle \nabla_{X_i}\nabla_T T, X_i\rangle &= X_i\langle \nabla_T T, X_i\rangle - \langle \nabla_T T, \nabla_{X_i} X_i\rangle \\ &= X_i\langle \nabla_T T, X_i\rangle = X_i\langle S_{X_i}T, T\rangle \\ &= \langle \nabla_{X_i}(S_{X_i}T), T\rangle + \langle S_{X_i}T, \nabla_{X_i}T\rangle \\ &= \langle \big(\nabla^{\mathbf{v}}_{X_i} S\big)_{X_i} T, T\rangle.\end{aligned}$$

Using (1.5.9), we obtain

$$\langle \nabla_{X_i} \nabla_T T, X_i \rangle = \langle R((T, X_i)X_i, T\rangle - |A^*_{X_i} T|^2 + |S_{X_i} T|^2.$$

Substituting in (4.5.4) and noticing that $|\nabla_T T|^2 = \sum_i |S_{X_i} T|^2$, we finally get

$$\operatorname{div} Z = \operatorname{Ric}(T) - |A^* T|^2, \tag{4.5.5}$$

with $|A^*T|$ denoting the norm of the operator $x \mapsto A^*_x T$. The theorem now clearly follows, since the divergence of Z integrates to zero over M. $\square$

One further consequence of (4.5.5) is that if the sectional curvature of a compact manifold M is nonpositive, then any one-dimensional metric foliation of M splits: In fact, both the A-tensor and the vertizontal curvatures must vanish, so that Theorem 2.2.2 applies.

In light of the above discussion, a negatively curved manifold M that admits a one-dimensional metric foliation cannot be compact. So what does M look like, topologically? If the curvature is a constant κ, then the answer is known: Namely, when $\kappa = 0$, M must be isometric to $\mathbb{R} \times_\Gamma \mathbb{R}^{n-1}$, where $\Gamma = \pi_1(M)$ acts diagonally by rigid motions. When $\kappa < 0$, M is diffeomorphic to $\mathbb{R} \times (\mathbb{R} \times_\Gamma \mathbb{R}^{n-2})$, with Γ as above, and in particular, M admits a flat metric. For a proof, the reader is referred to [10]. It should be noted, though, that this does not generalize to nonconstant negative curvature. For example, let S denote any compact surface with genus > 1, endowed with a hyperbolic metric, and let N denote the warped product $\mathbb{R} \times_{e^t} S$. Define a function $f : N \to \mathbb{R}$ by $f(t, p) = e^t$. Then the warped product $M = N \times_f \mathbb{R}$ has negative curvature, and since the hypersurfaces $N \times \{t\}$ are totally geodesic in M, their orthogonal complement are the leaves of a one-dimensional metric foliation on M. M, however, is diffeomorphic to $S \times \mathbb{R}^2$.