Chapter 9

Splitting and gap theorems in the presence of a Poincaré–Sobolev inequality

9.1 Splitting theorems

Up to now, we have been using Theorem 4.5 to show that solutions of a differential problem of the type

$$\begin{cases} \psi\Delta\psi + a\left(x\right)\psi^2 \geq -A\left|\nabla\psi\right|^2, \\ \psi \geq 0 \end{cases}$$

have to be identically zero. The aim of this section is to present a geometrical problem in which the second alternative of Theorem 4.5 does actually occur, that is, ψ becomes a positive solution of the linear equation

$$\Delta\psi + a\left(x\right)\psi = 0.$$

We shall focus our attention on splitting-type theorems depending on spectral and Ricci curvature bounds. To say that the complete manifold $(M, \langle,\rangle)$ *splits* usually means that M is isometric to the Riemannian direct product

$$\left(N_1 \times N_2, (,)_{N_1} + (,)_{N_2}\right)$$

for suitable complete Riemannian manifolds $\left(N_i, (,)_{N_i}\right)$, $i = 1, 2$. Manifolds that do not split are said to be irreducible. By way of example, the fundamental structure theorem by G. de Rham asserts that any complete, simply connected manifold splits, according to its holonomy, into simply connected, geodesically complete factors $(\mathbb{R}^n, \mathrm{can})$, $\left(N_2, (,)_{N_2}\right), \dots, \left(N_k, (,)_{N_k}\right)$; furthermore, the decomposition is uniquely determined and the complete manifolds $\left(N_i, (,)_{N_i}\right)$ are irreducible. See, e.g., [142].

Actually, the results we present in this section involve with a more general notion of splitting which allows warped factors. Accordingly, we say that $(M, \langle,\rangle)$ splits even if it is isometric to the warped product

$$\left(N_1 \times N_2, (,)_{N_1} + f^2\,(,)_{N_2}\right)$$

where $\left(N_i, (,)_{N_i}\right)$, $i = 1, 2$, are complete Riemannian manifolds and $f \in C^\infty\left(N_1\right)$ is a suitable positive function.

Our purpose is to give a somewhat simplified proof of the following result due to P. Li and J. Wang, [103]. It extends previous work by X. Wang, [161] on the structure of conformally compact Einstein manifolds.

Theorem 9.1. *Let $(M, \langle , \rangle)$ be a complete manifold of dimension $m \geq 3$ satisfying*

$$\lambda_1 = \lambda_1 (-\Delta_M) > 0.$$

Suppose also

$$^M\mathrm{Ric} \geq -\left(\frac{m-1}{m-2}\right)\lambda_1. \tag{9.1}$$

Then, either

(1) $(M, \langle , \rangle)$ *has only one non-parabolic end,*

or

(2) $(M, \langle , \rangle)$ *splits as the warped product* $\mathbb{R} \times \Sigma$ *with metric*

$$\langle , \rangle = dt^2 + \cosh^2\left(t\sqrt{\tfrac{\lambda_1}{m-2}}\right)(,) \tag{9.2}$$

where $(\Sigma, (,))$ is a compact, isometrically imbedded hypersurface of $(M, \langle , \rangle)$ satisfying

$$^\Sigma\mathrm{Ric} \geq -\lambda_1. \tag{9.3}$$

Remark 9.2. According to Lemma 7.13 above, the presence of a Sobolev–Poincaré inequality implies that non-parabolic ends are precisely the infinite volume ends.

The proof of Theorem 9.1 has its root in the classical splitting theorem by J. Cheeger and D. Gromoll, [32]. This latter states that if a complete manifold $(M, \langle , \rangle)$ of non-negative Ricci curvature contains a line, i.e., a minimizing geodesic $\gamma : \mathbb{R} \to M$, then M splits as the Riemannian product $\mathbb{R} \times N$ where $N \subset M$ is a totally geodesic hypersurface. Note that in the case where M has more than one end, N is necessarily compact. The (very) original argument supplied by Cheeger-Gromoll relies on the existence of a harmonic function u of distance-type, i.e., satisfying the condition $|\nabla u| = 1$. A substantial part of the proof is devoted to showing that, under the assumptions of the theorem, the Busemann function corresponding to (a half-line in) γ has the desired properties. Note that the integral curves of the gradient vector field ∇u are geodesic lines of M, so that, in particular, ∇u is complete. Moreover the level sets of u are smooth hypersurfaces with Gauss map ∇u. Let N be such a level set. Using the flow ϕ_t of ∇u through N, one establishes a smooth diffeomorphism between $\mathbb{R} \times N$ and M. Finally, one observes that ϕ_t is, in fact, a Riemannian isometry with respect to the product metric on $\mathbb{R} \times N$. Indeed, obviously, $(\phi_t)_* \left(\frac{d}{dt}\right) = \nabla u$ which is a unit vector field of M.

Moreover, using the harmonic function u in the Weitzenbock formula and recalling the refined Kato inequality yield

$$|\mathrm{Hess}\,(u)|^2 = \frac{m}{m-1}\,|\nabla\,|\nabla u||^2 = 0,$$

i.e., ∇u is a parallel (hence a Killing) vector field, and N is totally geodesic. Thus, by the very definition of Lie derivative, for every $V, W \in T_xM$, it holds that

$$\frac{d}{dt}\left\langle (\phi_t)_* V, (\phi_t)_* W \right\rangle = 2\mathrm{Hess}\,(u)\left((\phi_t)_* V, (\phi_t)_* W\right) = 0$$

showing that, obviously, ϕ_t is a one-parameter group of isometries of M. In particular, for any orthonormal basis $e_1, \dots, e_{m-1}$ of $T_xN = \langle \nabla u\,(x)\rangle^{\perp} \subset T_xM$, the tangent vectors $(\phi_t)_* e_1, \dots, (\phi_t)_* e_{m-1} \in \langle \nabla u\,(\phi_t\,(x))\rangle^{\perp}$ remain orthonormal in $T_{\phi_t(x)}M$. This completes the proof.

We note in passing that a differentiable splitting can be obtained by simply assuming that there exists a smooth function u without critical points such that $|\nabla u|$ is constant on the level sets of u. Indeed, one considers the flow ϕ_t of the unit vector field $\nabla u/|\nabla u|$, which, by the assumption that $|\nabla u| = \alpha(u)$, moves level sets of u onto level sets of u. Therefore, having chosen a level set Σ_o, the map $\phi : \mathbb{R} \times \Sigma_o \to M$ realizes the splitting.

A generalization of this kind of argument led, first M. Cai and G.J. Galloway, [23], and later X. Wang, [161], and P. Li and J. Wang, [103], to obtain the following

Theorem 9.3. *Let $(M, \langle , \rangle)$ be a complete manifold of dimension $m \geq 2$ and Ricci curvature satisfying*

$${}^M\mathrm{Ric} \geq -\rho$$

for some constant $\rho > 0$. Suppose that $u \in C^\infty(M)$ is a non-constant harmonic function such that, setting

$$\psi = |\nabla u| \in Lip_{loc}(M),$$

it holds that

$$\psi\Delta\psi + \rho\psi^2 = \frac{1}{m-1}\,|\nabla\psi|^2 \quad on\ M, \tag{9.4}$$

in the weak sense. Then, the level sets of u are smooth hypersurfaces, and $(M, \langle , \rangle)$ splits as the warped product $\mathbb{R} \times \Sigma_0$ with metric

$$\langle \,, \rangle = dt^2 + w(t)\,(\,,)$$

for a suitable level set Σ_0 of u endowed with the inherited metric, and where

$$w(t) = \left\{ \frac{C_1 \exp\left(t\sqrt{\frac{\rho}{m-1}}\right) + C_2 \exp\left(-t\sqrt{\frac{\rho}{m-1}}\right)}{C_1 + C_2} \right\}^2, \tag{9.5}$$

for some non-negative constants C_1 and C_2 which are not both zero. Moreover, if M has at least two ends, then Σ_0 is compact and M has exactly two ends. If both ends of M have infinite volume, we may chose Σ_0 in such a way that

$$w(t) = \cosh^2\left(t\sqrt{\tfrac{\rho}{m-1}}\right). \tag{9.6}$$

If one end of M has finite volume, then, up to replacing t with $-t$, we have

$$w(t) = \exp\left(2t\sqrt{\tfrac{\rho}{m-1}}\right). \tag{9.7}$$

Remark 9.4. Since here $A = -1/(m-1)$, according to Lemma 4.12, for every $p > (m-2)/2(m-1)$, we have $\psi^p \in W^{1,2}_{loc}(M)$ and

$$\nabla\psi^p = p\psi^{p-1}\nabla\psi \in L^2_{loc}. \tag{9.8}$$

This fact will be crucial in the arguments below.

The proof of Theorem 9.3 is reached by means of two main steps: first, we show a differential splitting of the original manifold and, next, the metric rigidity of the differential decomposition.

We need to recall some facts of both topological and analytical nature. Although they are quite simple, we provide a proof for the sake of completeness.

Lemma 9.5. *Let Σ be a non-compact, connected manifold. Then, every compact set $[a,b] \times K$ of $\mathbb{R} \times \Sigma$ has a connected complement in $\mathbb{R} \times \Sigma$. In particular, $\mathbb{R} \times \Sigma$ has only one end.*

Proof. Let $P_j = (t_j, x_j) \in \mathbb{R} \times \Sigma - [a,b] \times K$, $j = 1, 2$. We show that there is a continuous path γ in $\mathbb{R} \times \Sigma - [a,b] \times K$ connecting P_1 to P_2. Roughly speaking, γ is obtained by circumnavigating around $[a,b] \times K$. Four possibilities can occur: (i) $t_1, t_2 \in [a,b]$ and $x_1, x_2 \notin K$; (ii) $x_1, x_2 \in K$ and $t_1, t_2 \notin [a,b]$; (iii) up to interchanging P_1 with P_2, $t_1 \in [a,b]$, $x_1 \notin K$ and $t_2 \notin [a,b]$, $x_2 \in K$; (iv) $t_1, t_2 \notin [a,b]$ and $x_1, x_2 \notin K$. We limit ourselves to consider case (i), the other cases being similar and left to the reader. Fix $\bar{t} \notin [a,b]$ and define the paths $\Gamma_j(s) = (\bar{t}s + t_j(1-s)) \times x_j$, $j = 1, 2$. Clearly, $\Gamma_j(s)$ lies in the complement of $[a,b] \times K$ and satisfies $\Gamma_j(0) = P_j$, $\Gamma_j(1) = (\bar{t}, x_j)$. Next, observe that Σ is connected and locally path connected, hence a path connected space. Choose a path σ in Σ satisfying $\sigma(0) = x_1$, $\sigma(1) = x_2$ and define $\Gamma_3(s) = \bar{t} \times \sigma(s)$. Since $\Gamma_3(s) \in \mathbb{R} \times \Sigma - [a,b] \times K$ and $\Gamma_3(0) = \Gamma_1(1)$, $\Gamma_3(1) = \Gamma_2(1)$, we can form a new path in $\mathbb{R} \times \Sigma - [a,b] \times K$ from $\gamma(0) = P_1$ to $\gamma(1) = P_2$ by setting $\gamma = \Gamma_1 * \Gamma_3 * \overline{\Gamma_2}$, where $*$ means juxtaposition and $\overline{\Gamma_2}$ is nothing but Γ_2 taken with opposite orientation. $\square$

Lemma 9.6. *Let Σ be a closed connected submanifold of a connected Riemannian manifold $(M, \langle\,,\rangle)$. The normal exponential map $\exp^\perp : T^\perp\Sigma \to M$, defined as the restriction of $\exp$ to the normal bundle of Σ, is onto M.*

Proof. Since M is complete, given $x \in M$, a standard compactness argument shows that there exists $x_o \in \Sigma$ such that $d(x, x_o) = \mathrm{dist}(x, \Sigma)$. Again by completeness of M there exists a minimizing geodesic γ with $\gamma(0) = x_o$, $\gamma(1) = x$ and $\mathrm{length}(\gamma) = d(x, x_o)$. Since, $\exp^{\perp}$ is a local diffeomorphism of a neighborhood of the zero section of $T^{\perp}\Sigma$ onto its image, and by Gauss' lemma geodesics normal to Σ locally minimize the distance from Σ, then $\gamma'(0) \perp T_{x_o}\Sigma$ and by definition $\exp^{\perp}(\gamma'(0)) = \gamma(1) = x$. □

Finally we shall need the following simple ODE result.

Lemma 9.7. *Let $\rho_1 > 1$, $\rho_2 > 0$ and $C_1, C_2 \geq 0$. Then, the solution of the differential equation*

$$\beta\beta'' - \rho_1 (\beta')^2 + \rho_2\beta^2 = 0$$

is given by

$$\beta(t) = \left\{C_1 \exp\left(t\sqrt{\rho_2(\rho_1 - 1)}\right) + C_2 \exp\left(-t\sqrt{\rho_2(\rho_1 - 1)}\right)\right\}^{-\frac{1}{\rho_1 - 1}}.$$

We are now in a position to prove Theorem 9.3.

Proof of Theorem 9.3. For the sake of clarity, we divide the argument into several steps.

Step 1. The function $\psi = |\nabla u|$ satisfies $\psi > 0$ on M. In particular, ψ is smooth and the level sets of u are smooth hypersurfaces of M with (Gauss map) unit normal $\nabla u / |\nabla u|$.

Indeed, let

$$g = \psi^{\frac{m-2}{m-1}}.$$

We insert the test function

$$(\psi + \epsilon)^{-\frac{m}{m-1}}\lambda, \quad \text{with } \lambda \in Lip_c(M)$$

in the weak formulation of (9.4), and perform a computation, similar to that carried out in the proof of Theorem 4.5, to obtain

$$\begin{aligned}\int \psi^{-\frac{1}{m-1}}\left(\frac{\psi}{\psi+\epsilon}\right)^{\frac{m}{m-1}}\langle\nabla\psi, \nabla\lambda\rangle &= \int \rho\lambda\psi^2(\psi+\epsilon)^{-\frac{m}{m-1}} \\ &\quad -\frac{m}{m-1}\int \lambda\left(\psi^{-\frac{m}{2(m-1)}}|\nabla\psi|\right)^2\left(\frac{\psi}{\psi+\epsilon}\right)^{\frac{m}{m-1}}\frac{\epsilon}{\psi+\epsilon}.\end{aligned}$$

Using (9.8) with $p = 1 - \frac{m}{2(m-1)}$ we may let $\epsilon \to 0$ and apply the dominated convergence theorem to deduce that the function g satisfies

$$\Delta g + \frac{m-2}{m-1}\rho g = 0$$

weakly on M. Thus, the desired positivity of ψ follows from a local Harnack inequality (see, e.g., [60]).

Step 2. Let $\{e_i\}$ be a local orthonormal frame of M such that

$$e_1 = \frac{\nabla u}{|\nabla u|} = \frac{\nabla u}{\psi}.$$

Then, denoting by u_{ij} the coefficients of $\operatorname{Hess}(u)$ with respect to this frame, it holds that

$$\begin{aligned} u_{1j} &= -(m-1)\,\mu\,\delta_{1j}, \quad j = 1,\dots,m, \\ u_{ij} &= \mu\,\delta_{ij}, \quad i,j = 2,\dots,m, \end{aligned} \tag{9.9}$$

for some smooth function μ. Equivalently,

$$\begin{aligned} &\operatorname{Hess}(u)(e_1,\cdot) = -(m-1)\,\mu\,\langle e_1,\cdot\rangle \quad &&\text{on } TM, \\ &\operatorname{Hess}(u)(\cdot,\cdot) = \mu\,\langle\cdot,\cdot\rangle &&\text{on } \langle e_1\rangle^{\perp}. \end{aligned} \tag{9.10}$$

In particular, ψ is constant on each path-component of the smooth, level hypersurfaces $\{u = \text{const.}\}$ of u.

Indeed, note that (9.4) comes from the Bochner formula for harmonic functions, once we replace the usual inequalities with the equality sign. In particular, it forces equality in the refined Kato inequality of Proposition 1.3. Thus, setting $M = (u_{ij}) \in M_m(\mathbb{R})$ and $y = (u_i) \in \mathbb{R}^m$, we deduce that

$$\|M\|^2 = \frac{m}{m-1}\frac{|My|^2}{|y|^2}.$$

Application of the algebraic Lemma 1.5 with $A = 1$ enables us to obtain (9.9). As for the second assertion, simply note that if σ is any curve in a path-component of $\{u = \text{const.}\}$, then $\dot\sigma$ is orthogonal to e_1 and, therefore, by (9.9),

$$\frac{d}{dt}|\nabla u| \circ \sigma = \operatorname{Hess}(u)(e_1,\dot\sigma) = 0.$$

As a matter of fact, using a similar argument together with Sard's theorem, one can show that each of the path components of a.e. level set of $|\nabla u|$ coincides with a path component of some level set of u.

Step 3. Let $\{e_i\}$ be as in Step 2. Then

$$D_{e_1} e_1 = 0,$$

so that the integral curves of the global unit vector field $e_1 = \nabla u/|\nabla u|$ are geodesics of M.

Indeed, recall that

$$\nabla |\nabla u| = \mathrm{Hess}\,(u)\,(e_1, \cdot)^{\#} \qquad (9.11)$$
$$= -\,(m-1)\,\mu e_1.$$

Therefore, by a direct computation we get

$$\begin{aligned} D_{e_1} e_1 &= \frac{1}{|\nabla u|} D_{\nabla u} \frac{\nabla u}{|\nabla u|} \\ &= \frac{1}{|\nabla u|} \left\{ -\frac{1}{|\nabla u|^2} \langle \nabla u, \nabla |\nabla u| \rangle \nabla u + \frac{1}{|\nabla u|} \mathrm{Hess}\,(u)\,(\nabla u, \cdot)^{\#} \right\} \\ &= \frac{1}{|\nabla u|} \left\{ -\mathrm{Hess}\,(u)\,(e_1, e_1)\, e_1 + \mathrm{Hess}\,(u)\,(e_1, \cdot)^{\#} \right\} \\ &= \frac{1}{|\nabla u|} \left\{ (m-1)\,\mu e_1 - (m-1)\,\mu e_1 \right\} = 0. \end{aligned}$$

Step 4. Let Σ_o be a connected component of a level set $\{u = u_o\}$ of u, and let ϕ_t be the flow of the vector field e_1. Then for every $x \in \Sigma_o$, $\phi_t(x)$ is defined for all t's, and $\phi : \mathbb{R} \times \Sigma_o \to M$ is a smooth diffeomorphism which realizes a differentiable splitting $\mathbb{R} \times \Sigma_o \underset{\mathrm{diff}}{\approx} M$. Moreover, if M has more than one end, then Σ_o is compact, and M has exactly two ends.

Indeed, according to the previous step, for every x in M the integral curve $t \to \phi_t(x)$ of e_1 is a geodesic. Since M is complete, the flow ϕ_t is defined for every t (this actually follows directly from the fact that e_1 has bounded length), and for every fixed t gives rise to a global diffeomorphism of M.

In particular, if $x \in \Sigma_o$, then $\phi_t(x)$ coincides with the normal exponential map, namely $\phi_t(x) = \exp^{\perp}(te_1(x))$, and according to Lemma 9.6 the map $\phi : \mathbb{R} \times \Sigma_o \to M$ is onto. We claim that it is also 1-1 so that it realizes a differentiable splitting $\mathbb{R} \times \Sigma_o \underset{\mathrm{diff}}{\approx} M$. Note that since for fixed t, $\phi_t : M \to M$ is a global diffeomorphism, if $x_1 \neq x_2$, then $\phi_t(x_1) \neq \phi_t(x_2)$. On the other hand, since ∇u never vanishes, u is strictly increasing along the integral curves of e_1. Therefore if $t_1 < t_2$, then $u(\phi_{t_1}(x)) < u(\phi_{t_2}(x))$, and $\phi_{t_1}(x) \neq \phi_{t_2}(x)$. Thus, if ϕ is not $1-1$ on $\mathbb{R} \times \Sigma_o$, there exist $x_1 \neq x_2 \in \Sigma_o$ and $t_1 \neq t_2$ such that $\phi_{t_1}(x_1) = \phi_{t_2}(x_2) = \bar{x}$. But then, assuming, e.g., that $t_1 < t_2$ we have $x_2 = \phi_{-t_2}(\bar{x}) = \phi_{-t_2+t_1}(\phi_{-t_1}(\bar{x})) = \phi_{-t_2+t_1}(x_1)$ and this is impossible since x_1 and x_2 belong to the same level set of u, and u increases along integral lines of ϕ_t.

Note that, since u increases along the integral curve $\phi_t(x)$, the image of a level set of u cannot intersect the same level set. Thus Σ_o is necessarily connected. Since $\Sigma_o = \{u = t_o\}$ was arbitrary, all level sets of u are in fact connected, and $|\nabla u|$ is constant on every such set.

The last assertion follows from Lemma 9.5 and the fact that the flow of ϕ determines the ends of M.

Step 5. The map $\phi : \mathbb{R} \times \Sigma_o \to M$ moves Σ_o onto any other level set of u.

Since $|\nabla u|$ is constant on level sets of u, there exists a function α defined on the interval $u(M)$ such that $|\nabla u| = \alpha(u)$. For fixed $x \in M$ and every $t \in \mathbb{R}$ we have $|\nabla u|(\phi_t(x)) = \alpha(u(\phi_t(x))$, and since the right-hand side is a continuous function of t, while $t \to u(\phi_t(x))$ is a continuous bijection of $\mathbb{R}$ onto its image, we deduce that the function α is continuous. Moreover

$$\begin{aligned} \frac{d}{ds} u \circ \phi_s(x) &= \left\langle \nabla u\left(\phi_s(x)\right), \dot{\phi}_s(x) \right\rangle \\ &= \left|\nabla u\left(\phi_s(x)\right)\right| \\ &= \alpha\left(u \circ \phi_s(x)\right). \end{aligned}$$

Whence, integrating and using the change of variable formula, we get

$$\int_{u(x)}^{u\circ\phi_t(x)} \frac{dy}{\alpha(y)} = t, \text{ on } \mathbb{R}.$$

In particular, the value $u(\phi_t(x))$ is independent of the point x in any fixed level set of u, showing that the image under ϕ_t of a level set of u is contained in a level set of u. The conclusion now follows from the fact that ϕ is a diffeomorphism of $\mathbb{R} \times \Sigma_o$ onto M.

Step 6. The smooth, positive function

$$\beta(t) = |\nabla u| \circ \phi_t(x) = \psi \circ \phi_t(x) \tag{9.12}$$

is independent of x varying in a given level set of u and satisfies the ODE

$$\beta'' - \frac{m}{m-1} \frac{(\beta')^2}{\beta} + \rho\beta = 0, \text{ on } \mathbb{R}. \tag{9.13}$$

The first assertion follows directly from the fact that ϕ_t moves level sets of u onto level sets of u, and that $|\nabla u|$ is constant on such level sets. As for the second assertion, direct computations that use (9.11) show

$$\begin{aligned} \beta'(t) &= \left\langle (\nabla\psi) \circ \phi_t(x), \frac{d}{dt}\phi_t(x) \right\rangle = \left\langle \nabla|\nabla u| \circ \phi_t(x), e_1 \right\rangle \\ &= \big(\operatorname{Hess}(u)(e_1, e_1)\big) \circ \phi_t(x) = -(m-1)\,\mu \circ \phi_t(x), \end{aligned} \tag{9.14}$$

and (again by (9.11))

$$|\nabla\psi|^2 \circ \phi_t(x) = (m-1)^2 \mu^2 \circ \phi_t(x),$$

so that, in particular,

$$\beta'(t)^2 = |\nabla\psi|^2 \circ \phi_t(x). \tag{9.15}$$

Moreover, since $t \to \phi_t(x)$ is a geodesic in M with tangent vector e_1,

$$\beta''(t) = \frac{d}{dt}\big(\langle \nabla\psi, e_1 \rangle (\phi_t(x))\big) = \operatorname{Hess}(\psi)(e_1, e_1)(\phi_t(x)). \tag{9.16}$$

To treat the Hessian term on the right-hand side we write

$$\mathrm{Hess}\,(\psi)\,(e_1, e_1) = \Delta\psi - \sum_{j=2}^{m} \mathrm{Hess}\,(\psi)\,(e_j, e_j) \tag{9.17}$$

and claim that

$$\mathrm{Hess}\,(\psi)\,(e_j, e_j) = -\frac{(m-1)\,\mu^2}{\psi}. \tag{9.18}$$

Indeed, using (9.11) and the fact that $\{e_i\}$ is an orthonormal frame, we obtain, for every $j \geq 2$,

$$\begin{aligned} \mathrm{Hess}\,(\psi)\,(e_j, e_j) &= \mathrm{Hess}\,(|\nabla u|)\,(e_j, e_j) \\ &= \left\langle D_{e_j} \nabla\, |\nabla u|\,, e_j \right\rangle \\ &= e_j \left\langle \nabla\, |\nabla u|\,, e_j \right\rangle - \left\langle \nabla\, |\nabla u|\,, D_{e_j} e_j \right\rangle \\ &= e_j \left(-\,(m-1)\,\mu\delta_{1j} \right) + (m-1)\,\mu \left\langle e_1, D_{e_j} e_j \right\rangle \\ &= -\,(m-1)\,\mu \left\langle D_{e_j} e_1, e_j \right\rangle. \end{aligned} \tag{9.19}$$

On the other hand, using again (9.11), a direct computation yields

$$\begin{aligned} D_{e_j} e_1 &= |\nabla u|^{-1} D_{e_j} \nabla u - |\nabla u|^{-2}\, e_j(|\nabla u|) \nabla u \\ &= |\nabla u|^{-1}\, \mathrm{Hess}\,(u)\,(e_j, \cdot)^{\#} + |\nabla u|^{-2}\, \mathrm{Hess}(u)(e_j, e_1) \nabla u \\ &= \psi^{-1} \mathrm{Hess}\,(u)\,(e_j, \cdot)^{\#}. \end{aligned}$$

Inserting this latter into (9.19) and using (9.9) establishes equation (9.18).

Now, substituting (9.18) into (9.17), inserting the resulting equality into (9.16) and recalling (9.14) and (9.4), we conclude that

$$\begin{aligned} \beta''(t) &= \left(\Delta\psi + (m-1)^2 \frac{\mu^2}{\psi} \right) (\phi_t(x)) \\ &= \left(\Delta\psi + \frac{|\nabla\psi|^2}{\psi} \right) (\phi_t(x)) \\ &= \left(-\rho\psi + \frac{m}{m-1} \frac{|\nabla\psi|^2}{\psi} \right) (\phi_t(x)) \\ &= -\rho\beta + \frac{m}{m-1} \frac{(\beta')^2}{\beta}. \end{aligned} \tag{9.20}$$

Step 7. If Σ_0 is a level set of M and $\phi : \mathbb{R} \times \Sigma_0 \to M$ realizes the differentiable splitting, then

$$\phi^* \langle\, , \rangle = dt \otimes dt + \left(\frac{\beta(t)}{\beta(0)} \right)^{-2/(m-1)} (\, ,). \tag{9.21}$$

Obviously, $\phi_* \frac{\partial}{\partial t} = e_1 \circ \phi$ so that

$$\phi^* \langle \, , \rangle \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = 1. \tag{9.22}$$

Moreover, for every $V \in T\Sigma_0 = \langle e_1 \rangle^{\perp}$,

$$\left\langle \phi_* \frac{\partial}{\partial t}, \phi_* V \right\rangle = 0. \tag{9.23}$$

Indeed, if σ is any curve in Σ starting with velocity V, by definition of β and the fact that it is independent of the point $x \in \Sigma_0$, we have

$$\left.\frac{d}{ds}\right|_{s=0} (|\nabla u| \circ \phi_t) \circ \sigma(s) = \left.\frac{d}{ds}\right|_{s=0} \beta(t) = 0. \tag{9.24}$$

On the other hand, using (9.10) and (9.14), we have

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} |\nabla u| \circ (\phi_t \circ \sigma(s)) &= \operatorname{Hess}(u)(e_1, (\phi_t)_* V) \\ &= -(m-1)\, \mu \circ \phi_t \langle e_1, (\phi_t)_* V \rangle \\ &= \beta'(t) \langle e_1, (\phi_t)_* V \rangle . \end{aligned} \tag{9.25}$$

Since $\beta'(t) \neq 0$ by (9.14), combining (9.24) and (9.25) gives (9.23), and proves that $\phi^* \langle , \rangle$ takes the form

$$\phi^* \langle \, , \rangle = dt \otimes dt + (\phi_t)^* \langle \, , \rangle .$$

Thus, the desired conclusion will follow once we show that

$$(\phi_t)^* \langle \, , \rangle = \left(\frac{\beta(t)}{\beta(0)} \right)^{-2/(m-1)} (\, ,) .$$

By the definition of Lie derivative we have, for all vectors V, W tangent to Σ_0,

$$\langle D_V e_1, W \rangle + \langle D_W e_1, V \rangle = L_{e_1} \langle , \rangle (V, W) = \left.\frac{d}{ds}\right|_{s=0} (\phi_s^* \langle , \rangle)(V, W).$$

On the other hand, recalling that $|\nabla u|$ is locally constant on the level sets of u, so that

$$D_V e_1 = D_V \Big(\frac{\nabla u}{|\nabla u|} \Big) = \frac{1}{|\nabla u|} D_V \nabla u,$$

and using (9.10), we deduce that

$$\langle D_V e_1, W \rangle = \frac{1}{|\nabla u|} \operatorname{Hess}(u)(V, W) = \frac{\mu}{|\nabla u|} \langle V, W \rangle$$

so that

$$\left.\frac{d}{ds}\right|_{s=0}\left(\phi_s^*\left\langle\,,\right\rangle\right)=\frac{2\mu}{|\nabla u|}\left\langle\,,\right\rangle.$$

As a consequence, recalling the definition (9.12) of β, (9.14) and the group property of ϕ_t, we get

$$\begin{aligned}\frac{d}{dt}\left(\phi_t^*\left\langle\,,\right\rangle\right)&=\frac{2\mu\circ\phi_t}{|\nabla u|\circ\phi_t}\;\phi_t^*\left\langle\,,\right\rangle\\&=\frac{-2\beta'(t)}{m-1}\frac{1}{\beta(t)}\;\phi_t^*\left\langle\,,\right\rangle.\end{aligned}$$

Integrating over $[0,t]$ we finally obtain

$$\phi_t^*\left\langle\,,\right\rangle=\left(\frac{\beta(t)}{\beta(0)}\right)^{-\frac{2}{m-1}}\;\phi_0^*\left\langle\,,\right\rangle$$

as required.

Step 8. In view of (9.13) and by Lemma 9.7, the positive function $\beta(t)=\psi\circ\phi_t(x)$ has the expression

$$\beta(t)=\left\{C_1\exp\left(t\sqrt{\tfrac{\rho}{m-1}}\right)+C_2\exp\left(-t\sqrt{\tfrac{\rho}{m-1}}\right)\right\}^{-(m-1)},$$

for some non-negative constants C_1, C_2 which are independent of $x\in\Sigma_o$, since so is β, and cannot both vanish because $\psi=|\nabla u|>0$. Up to replacing t with $-t$, we may assume that $C_1>0$. Thus

$$w(t)=\left(\frac{\beta(t)}{\beta(0)}\right)^{-\frac{2}{m-1}}=\left\{\frac{C_1\exp\left(t\sqrt{\tfrac{\rho}{m-1}}\right)+C_2\exp\left(-t\sqrt{\tfrac{\rho}{m-1}}\right)}{C_1+C_2}\right\}^2,$$

as required to prove (9.5).

Step 9. Assume now that M has two ends. Note that if both the constants C_1 and C_2 are different from zero, then the positive function $\beta(t)$ tends to zero as $t\to\pm\infty$ and β attains its positive maximum B at some $t_0\in\mathbb{R}$. Without loss of generality, up to using the translated flow ϕ_{t+t_0}, we can assume $t_0=0$. Accordingly,

$$\beta(0)=B>0,\quad\beta'(0)=0$$

which implies that $\beta(t)$ has the expression

$$\beta(t)=B\cosh^{-(m-1)}\left(t\sqrt{\tfrac{\rho}{m-1}}\right)$$

and

$$w(t)=\left(\frac{\beta(t)}{\beta(0)}\right)^{-2/(m-1)}=\cosh^2\left(t\sqrt{\tfrac{\rho}{m-1}}\right).$$

On the other hand, if $C_2 = 0$, then

$$w(t) = \exp\left(2t\sqrt{\tfrac{\rho}{m-1}}\right).$$

Since Σ_0 is compact, in the former case both ends of M have infinite volume. In the latter case, $\operatorname{vol}((-\infty, 0] \times \Sigma_0) < +\infty$, and one of the ends of M has finite volume. □

In view of Theorem 9.3, the strategy of the proof of Theorem 9.1 is to use the assumptions on the geometry of M to produce a non-constant harmonic function satisfying the differential equation (9.4). This requires some preliminary results on the energy of a special class of harmonic functions considered in Section 7.1.

We recall briefly the construction. Let D be a relatively compact domain D with smooth boundary, and fix an exhaustion $\{D_n\}$ of M by relatively compact open domains with smooth boundary and $D \subset\subset D_n \subset\subset D_{n+1}$.

According to Proposition 7.10, if M has at least two non-parabolic ends E_1 and E_2, then the sequence of functions u_n which solve the boundary value problem

$$\begin{cases} \Delta u_n = 0 \ \text{ in } \ D_n, \\ u_n = 1 \ \text{ on } \ \partial D_n \cap E_1, \quad u_n = 0 \ \text{ on } \ \partial D_n \cap \big(M \setminus E_1\big) \end{cases}$$

has a subsequence which converges locally uniformly to a bounded harmonic function u with finite Dirichlet integral such that $0 < u < 1$, $\sup_{E_1} u = 1$, and $\inf_{E_2} u = 0$.

Lemma 9.8. *Maintaining the notation introduced above, let E be an end of M with respect to D, satisfying the Poincaré inequality*

$$0 < \lambda_1 = \lambda_1(-\Delta_E) = \inf \frac{\int_E |\nabla\varphi|^2}{\int_E \varphi^2} \tag{9.26}$$

where the infimum is taken over $C_c^\infty(E) \setminus \{0\}$. Let u be the harmonic function obtained with the approximation procedure described above, and let $0 < \delta < \sqrt{\lambda_1}$. Then, there exists a constant $C = C(u) > 0$ such that

$$\int_E e^{2\delta r}(u-\alpha)^2 \le \frac{C}{\left(\sqrt{\lambda_1} - \delta\right)^2}, \quad \textit{where } \alpha = \begin{cases} 1 & \textit{if } E = E_1, \\ 0 & \textit{otherwise.} \end{cases} \tag{9.27}$$

Remark 9.9. We observe that a similar statement, with appropriate values of α, holds for linear combinations of the harmonic functions as considered in the statement, that is, for harmonic functions obtained by the approximation procedure assigning to each u_n a constant boundary value on $E \cap \partial D_n$ (necessarily equal to zero if E is parabolic).

Proof. We consider the case where $E \neq E_1$, so that the approximating sequence u_n satisfies the boundary condition $u_n = 0$ on $E \cap \partial D_n$. The other case is dealt with by replacing u with $1-u$. Let $o \in D$ and R_0 such that $\bar{D} \subset B_{R_0}(o)$, fix $R \geq 2R_0$, and choose $n \in \mathbb{N}$ sufficiently large that $\bar{B}_R(o) \subset D_n$. We set $E_n = E \cap D_n$, $\partial E_n = E \cap \partial D_n$, $E(R) = E \cap B_R(o)$ and $\partial E(R) = \partial B_R(o) \cap E$. Let also $\varphi : M \to [0,1]$ be a smooth cut-off function such that

$$\varphi = 0 \text{ on } B_{R_0}, \quad \varphi = 1 \text{ off } B_R, \quad |\nabla \varphi| \leq \frac{2}{R-R_0} \text{ on } M.$$

Fix $0 < \delta < \sqrt{\lambda_1}$ and integrate over E_n to obtain

$$\begin{aligned}
\int_{E_n} \left|\nabla\left(\varphi e^{\delta r} u_n\right)\right|^2 &= \int_{E_n} u_n^2 \left|\nabla\left(\varphi e^{\delta r}\right)\right|^2 + 2\int_{E_n} u_n \varphi e^{\delta r} \left\langle \nabla u_n, \nabla\left(\varphi e^{\delta r}\right)\right\rangle \\
&\quad + \int_{E_n} \varphi^2 e^{2\delta r} \left|\nabla u_n\right|^2 \\
&= \int_{E_n} u_n^2 \left|\nabla\left(\varphi e^{\delta r}\right)\right|^2 + \int_{E_n} \varphi^2 e^{2\delta r} \left|\nabla u_n\right|^2 \\
&\quad + \frac{1}{2}\int_{E_n} \left\langle \nabla\left(u_n^2\right), \nabla\left(\varphi^2 e^{2\delta r}\right)\right\rangle.
\end{aligned}$$

Applying the divergence theorem, noting that u_n vanishes on ∂E_n, while $\phi^2 e^{2\delta r}$ is zero on ∂E, and using the harmonicity of u_n in E_n, we have

$$\begin{aligned}
\frac{1}{2}\int_{E_n} \left\langle \nabla\left(u_n^2\right), \nabla\left(\varphi^2 e^{2\delta r}\right)\right\rangle &= -\frac{1}{2}\int_{E_n} \varphi^2 e^{2\delta r} \Delta\left(u_n^2\right) \\
&= -\int_{E_n} \varphi^2 e^{2\delta r} u_n \Delta u_n - \int_{E_n} \varphi^2 e^{2\delta r} \left|\nabla u_n\right|^2 \\
&= -\int_{E_n} \varphi^2 e^{2\delta r} \left|\nabla u_n\right|^2,
\end{aligned}$$

which, inserted into the above identity, yields

$$\int_{E_n} \left|\nabla\left(\varphi e^{\delta r} u_n\right)\right|^2 = \int_{E_n} u_n^2 \left|\nabla\left(\varphi e^{\delta r}\right)\right|^2.$$

By Young's inequality, for every $\varepsilon > 0$ the right-hand side is estimated from above by

$$(1+\varepsilon)\,\delta^2 \int_{E_n} u_n^2 \varphi^2 e^{2\delta r} + \left(1+\varepsilon^{-1}\right) \int_{E_n} u_n^2 e^{2\delta r} \left|\nabla \varphi\right|^2,$$

so, using the Poincaré inequality to estimate the left-hand side from below and rearranging, we get

$$\left(\lambda_1 - (1+\varepsilon)\,\delta^2\right) \int_{E_n} u_n^2 \varphi^2 e^{2\delta r} \leq \left(1+\varepsilon^{-1}\right) \frac{4}{\left(R-R_0\right)^2} \int_{E(R)\setminus E(R_0)} u_n^2 e^{2\delta r}.$$

Choosing
$$\varepsilon = \frac{\sqrt{\lambda_1} - \delta}{\delta}$$
we obtain
$$\left(\sqrt{\lambda_1} - \delta\right)^2 \int_{E_n} u_n^2 \varphi^2 e^{2\delta r} \leq \frac{4}{(R - R_0)^2} \int_{E(R)\setminus E(R_0)} u_n^2 e^{2\delta r},$$
whence, recalling that φ is non-negative and identically equal to 1 outside B_R,
$$\left(\sqrt{\lambda_1} - \delta\right)^2 \int_{E_n \setminus E(R)} u_n^2 e^{2\delta r} \leq \frac{4}{(R - R_0)^2} \int_{E(R)\setminus E(R_0)} u_n^2 e^{2\delta r}.$$
Now, we choose $R = 2R_0$ and we let $n \to +\infty$ to get
$$\left(\sqrt{\lambda_1} - \delta\right)^2 \int_{E \setminus E(2R_0)} u^2 e^{2\delta r} \leq \frac{4}{R_0^2} \int_{E(2R_0)\setminus E(R_0)} u^2 e^{2\delta r},$$
and (9.27) follows. □

Lemma 9.10. *Assume that M has at least two non-parabolic ends, and that E is an end of M with respect to D satisfying the Poincaré inequality* (9.26)*. Let u be a harmonic function as in the statement of Lemma* 9.8*. Then there exists a constant $C = C\left(u, \sqrt{\lambda_1}\right) > 0$, such that for every sufficiently large R,*
$$\int_{E_R \setminus E_{R-1}} e^{2\sqrt{\lambda_1} r} (u - \alpha)^2 \leq C, \tag{9.28}$$
where α is as in Lemma 9.8*, namely, $\alpha = 1$ if u is the limit of a sequence of harmonic functions u_n equal to* 1 *on $E \cap \partial D_n$, and $\alpha = 0$ otherwise.*

Proof. Again we consider the case where $\alpha = 0$. Let $\varphi \in Lip_c(E)$. A straightforward computation shows the validity of the equality
$$\begin{aligned}\int_E \left|\nabla\left(e^{\sqrt{\lambda_1} r}\varphi u\right)\right|^2 = \int_E \Big\{ & e^{2\sqrt{\lambda_1} r} u^2 |\nabla\varphi|^2 + \lambda_1 e^{2\sqrt{\lambda_1} r}\varphi^2 u^2 \\ & + 2\sqrt{\lambda_1} e^{2\sqrt{\lambda_1} r}\varphi u^2 \langle\nabla\varphi, \nabla r\rangle + e^{2\sqrt{\lambda_1} r}\varphi^2 |\nabla u|^2 \\ & + 2e^{2\sqrt{\lambda_1} r}\varphi u \langle\nabla\varphi, \nabla u\rangle + 2\sqrt{\lambda_1} e^{2\sqrt{\lambda_1} r}\varphi^2 u \langle\nabla u, \nabla r\rangle \Big\}.\end{aligned}$$
The last three terms under the integral on the right-hand side can be written in the form
$$\left\langle \nabla u, \nabla\left(e^{2\sqrt{\lambda_1} r}\varphi^2 u\right)\right\rangle,$$
so that, integrating by parts and using the fact that u is harmonic, they cancel out and the above equality reduces to
$$\int_E \left|\nabla\left(\varphi e^{\sqrt{\lambda_1} r} u\right)\right|^2 = \int_E e^{2\sqrt{\lambda_1} r}\left(u^2 |\nabla\varphi|^2 + 2\sqrt{\lambda_1}\varphi u^2 \langle\nabla\varphi, \nabla r\rangle + \lambda_1 \varphi^2 u^2\right).$$

We now apply the Poincaré inequality to the left-hand side and rearrange to obtain

$$-2\sqrt{\lambda_1}\int_E e^{2\sqrt{\lambda_1}r}u^2\varphi\left\langle\nabla\varphi,\nabla r\right\rangle \leq \int_E e^{2\sqrt{\lambda_1}r}u^2\left|\nabla\varphi\right|^2. \tag{9.29}$$

We let $R_0 < R_1 < R$ and we choose φ to be given by the formula

$$\varphi\left(r\right) = \begin{cases} \frac{r(x)-R_0}{R_1-R_0} & \text{on } E(R_1)\setminus E(R_0), \\ \frac{R-r(x)}{R-R_1} & \text{on } E(R)\setminus E(R_1), \\ 0 & \text{elsewhere.} \end{cases}$$

Substituting into (9.29) we have

$$\begin{aligned} &\frac{2\sqrt{\lambda_1}}{\left(R-R_1\right)^2}\int_{E(R)\setminus E(R_1)} e^{2\sqrt{\lambda_1}r}\left(R-r\right)u^2 \qquad (9.30)\\ &\leq \frac{1}{\left(R-R_1\right)^2}\int_{E(R)\setminus E(R_1)} e^{2\sqrt{\lambda_1}r}u^2 + \frac{1}{\left(R_1-R_0\right)^2}\int_{E(R_1)\setminus E(R_0)} e^{2\sqrt{\lambda_1}r}u^2 \\ &\quad + \frac{2\sqrt{\lambda_1}}{\left(R_1-R_0\right)^2}\int_{E(R_1)\setminus E(R_0)} e^{2\sqrt{\lambda_1}r}\left(r-R_0\right)u^2. \end{aligned}$$

We fix $0 < t < R - R_1$ and we observe that

$$\frac{2t}{\left(R-R_1\right)^2}\int_{E(R-t)\setminus E(R_1)} e^{2\sqrt{\lambda_1}r}u^2 \leq \frac{2}{\left(R-R_1\right)^2}\int_{E(R-t)\setminus E(R_1)}\left(R-r\right)e^{2\sqrt{\lambda_1}r}u^2.$$

Thus, from (9.30) we get

$$\begin{aligned} \frac{2t\sqrt{\lambda_1}}{\left(R-R_1\right)^2}\int_{E(R-t)\setminus E(R_1)} e^{2\sqrt{\lambda_1}r}u^2 &\leq \frac{1}{\left(R-R_1\right)^2}\int_{E(R)\setminus E(R_1)} e^{2\sqrt{\lambda_1}r}u^2 \\ &\quad + \left[\frac{1}{\left(R_1-R_0\right)^2} + \frac{2\sqrt{\lambda_1}}{R_1-R_0}\right]\int_{E(R_1)\setminus E(R_0)} e^{2\sqrt{\lambda_1}r}u^2. \qquad (9.31) \end{aligned}$$

Let $R > \max\left\{2R_0, R_1 + 1/\sqrt{\lambda_1}\right\}$. We choose $R_1 = R_0 + 1$, $t = 1/\sqrt{\lambda_1}$ and define

$$g\left(R\right) = \int_{E(R)\setminus E(R_0+1)} e^{2\sqrt{\lambda_1}r}u^2.$$

The above inequality gives

$$\frac{2}{\left(R-R_0-1\right)^2}g\left(R-\frac{1}{\sqrt{\lambda_1}}\right) \leq \frac{1}{\left(R-R_0-1\right)^2}g\left(R\right) + 2A$$

for some $A = A\left(R_0, \sqrt{\lambda_1}, u\right) > 0$, and therefore

$$g\left(R\right) \leq \frac{1}{2}g\left(R+\frac{1}{\sqrt{\lambda_1}}\right) + A\left(R+\frac{1}{\sqrt{\lambda_1}}\right)^2. \tag{9.32}$$

Now we let k be a positive integer and we iterate (9.32) k-times to obtain

$$g(R) \leq \frac{1}{2^k} g\left(R + \frac{k}{\sqrt{\lambda_1}}\right) + AR^2 \sum_{i=1}^{k} \frac{\left(1 + \frac{i}{R\sqrt{\lambda_1}}\right)^2}{2^{i-1}}$$

and therefore

$$g(R) \leq \frac{1}{2^k} g\left(R + \frac{k}{\sqrt{\lambda_1}}\right) + CR^2$$

for some constant $C > 0$ which depends only on R_o, $\sqrt{\lambda_1}$ and u. We now use assumption (9.27) to show that

$$\lim_{k \to +\infty} \frac{1}{2^k} g\left(R + \frac{k}{\sqrt{\lambda_1}}\right) = 0.$$

Indeed, if $0 < \delta < \lambda_1$, applying (9.27) we have

$$\begin{aligned} g\left(R + \frac{k}{\sqrt{\lambda_1}}\right) &= \int_{B_{R + \frac{k}{\sqrt{\lambda_1}}} \setminus B_{R_0+1}} e^{2\sqrt{\lambda_1} r} u^2 \\ &= \int_{B_{R + \frac{k}{\sqrt{\lambda_1}}} \setminus B_{R_0+1}} e^{2\left(\sqrt{\lambda_1} - \delta\right) r} u^2 e^{2\delta r} \\ &\leq C \left(\sqrt{\lambda_1} - \delta\right)^{-2} e^{2\left(R + \frac{k}{\sqrt{\lambda_1}}\right)\left(\sqrt{\lambda_1} - \delta\right)}. \end{aligned}$$

Therefore

$$\frac{1}{2^k} g\left(R + \frac{k}{\sqrt{\lambda_1}}\right) \leq C \left(\sqrt{\lambda_1} - \delta\right)^{-2} e^{2R\left(\sqrt{\lambda_1} - \delta\right)} e^{2\frac{k}{\sqrt{\lambda_1}}\left(\sqrt{\lambda_1} - \delta\right) - k \log 2} \to 0$$

as $k \to +\infty$, provided δ is sufficiently near to $\sqrt{\lambda_1}$. Recalling the definition of $g(R)$ we have thus proved

$$\int_{E(R) \setminus E(R_0+1)} e^{2\sqrt{\lambda_1} r} u^2 \leq \hat{C} R^2,\ R >> 1,$$

for some constant $\hat{C} = \hat{C}\left(u, R_0, \sqrt{\lambda_1}\right) > 0$ and therefore

$$\int_{E(R)} e^{2\sqrt{\lambda_1} r} u^2 \leq CR^2,\ \ R >> 1, \tag{9.33}$$

for some $C > 0$. To improve (9.33) we use again inequality (9.31). For R large enough, we choose $R_1 = R_0 + 1$, $t = R/2$ in (9.31) to obtain

$$R\sqrt{\lambda_1} \int_{E(\frac{R}{2}) \setminus E(R_0+1)} e^{2\sqrt{\lambda_1} r} u^2 \leq \int_{E(R) \setminus E(R_0+1)} e^{2\sqrt{\lambda_1} r} u^2 + 2AR^2$$

and then applying (9.33)

$$\int_{E(\frac{R}{2})\setminus E(R_0+1)} e^{2\sqrt{\lambda_1}r}u^2 \le CR$$

or, equivalently,

$$\int_{E(R)} e^{2\sqrt{\lambda_1}r}u^2 \le CR,\ R >> 1. \tag{9.34}$$

To conclude we set $R_1 = R - 4/\sqrt{\lambda_1}$, $t = 2/\sqrt{\lambda_1}$ in (9.31), and deduce that for sufficiently large R,

$$\begin{aligned}\int_{E(R-\frac{2}{\sqrt{\lambda_1}})\setminus E(R-\frac{4}{\sqrt{\lambda_1}})} & e^{2\sqrt{\lambda_1}r}u^2 \\ & \le \frac{1}{4}\int_{E(R)\setminus E(R-\frac{4}{\sqrt{\lambda_1}})} e^{2\sqrt{\lambda_1}r}u^2 + \frac{C}{R}\int_{E(R-\frac{2}{\sqrt{\lambda_1}})\setminus E(R_0)} e^{2\sqrt{\lambda_1}r}u^2\end{aligned}$$

for some constant $C = C\left(u, R_0, \sqrt{\lambda_1}\right) > 0$. Thus, using (9.34),

$$\int_{E(R-\frac{2}{\sqrt{\lambda_1}})\setminus E(R-\frac{4}{\sqrt{\lambda_1}})} e^{2\sqrt{\lambda_1}r}u^2 \le \frac{1}{3}\int_{E(R)\setminus E(R-\frac{2}{\sqrt{\lambda_1}})} e^{2\sqrt{\lambda_1}r}u^2 + C.$$

We iterate this inequality k-times to obtain, with the aid of (9.34),

$$\begin{aligned}\int_{E(R+\frac{2}{\sqrt{\lambda_1}})\setminus E(R)} e^{2\sqrt{\lambda_1}r}u^2 &\le C\sum_{i=1}^{k}\frac{1}{3^{i-1}} + \frac{1}{3^k}\int_{E(R+\frac{2k}{\sqrt{\lambda_1}})\setminus E(R+\frac{2(k-1)}{\sqrt{\lambda_1}})} e^{2\sqrt{\lambda_1}r}u^2 \\ &\le C + C\frac{1}{3^k}\left(R + \frac{2k}{\sqrt{\lambda_1}}\right),\end{aligned}$$

whence, letting $k \to +\infty$ we deduce that the integral on the left-hand side is bounded above by C. The required conclusion now follows at once. □

Lemma 9.11. *Let $(M, \langle,\rangle)$, u and E be as in the previous lemmas. Then, there exists a constant $C > 0$ independent of R such that, for R sufficiently large,*

$$\int_{E(R)} e^{2\sqrt{\lambda_1}r}\left|\nabla u\right|^2 \le CR. \tag{9.35}$$

In particular $\left|\nabla u\right|^2 \in L^1(E)$.

Proof. Let φ be the cut-off function

$$\varphi(x) = \begin{cases} r(x) - R + 1 & \text{on } E(R)\setminus E(R-1), \\ 1 & \text{on } E(R+1)\setminus E(R), \\ R - 2 - r(x) & \text{on } E(R+2)\setminus E(R+1), \\ 0 & \text{elsewhere.}\end{cases}$$

Consider the vector field

$$Z = \varphi^2 u \nabla u.$$

Using the divergence theorem, the fact that u is harmonic and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_E \varphi^2 |\nabla u|^2 &= -2\int_E \varphi u \langle \nabla\varphi, \nabla u\rangle - \int_E \varphi^2 u \Delta u \\ &\leq \frac{1}{2}\int_E \varphi^2 |\nabla u|^2 + 2\int_E |\nabla\varphi|^2 u^2.\end{aligned}$$

Thus, using (9.28) and the definition of φ,

$$\int_{E(R+1)\setminus E(R)} |\nabla u|^2 \leq 4\int_{E(R+2)\setminus E(R-1)} u^2 \leq Ce^{-2\sqrt{\lambda_1}(R-1)}.$$

It follows that

$$\int_{E(R+1)\setminus E(R)} e^{2\sqrt{\lambda_1}r} |\nabla u|^2 \leq e^{2\sqrt{\lambda_1}(R+1)} \int_{E(R+1)\setminus E(R)} |\nabla u|^2 \leq C.$$

We set $R = R_0 + i$ and we sum over $1 \leq i \leq k$ to get

$$\int_{E(R_0+k)\setminus E(R_0+1)} e^{2\sqrt{\lambda_1}r} |\nabla u|^2 \leq Ck \leq C\left(R_0 + k\right).$$

We have thus proved (9.35). The second assertion follows writing

$$\begin{aligned}\int_{E\setminus E(R_0)} |\nabla u|^2 &\leq \sum_k \int_{E(R_0+k+1)\setminus E(R_0+k)} e^{-2\sqrt{\lambda_1}(R_0+k)} \int_{\partial B_r} e^{2\sqrt{\lambda_1}r}|\nabla u|^2 \\ &\leq C\sum_k (R_0+k+1)e^{-2\sqrt{\lambda_1}(R_0+k)} < +\infty.\end{aligned}$$

□

We are now ready to give the

Proof of Theorem 9.1. Assume that M has at least two non-parabolic ends, and that $\lambda_1 = \lambda_1(-\Delta_M) > 0$. Then, there exists a non-constant harmonic function u on M which is obtained as the limit of a sequence $\{u_n\}$ as in the assumptions of Lemma 9.8. Having set $\psi = |\nabla u|$, and applying Lemma 9.11 to each end E of M with respect to D (note that $\lambda_1(-\Delta_E) \geq \lambda_1 > 0$ by domain monotonicity), we have the energy estimate

$$\int_{B_R} e^{2\sqrt{\lambda_1}r}\psi^2 \leq CR \tag{9.36}$$

with R sufficiently large. From Bochner's formula and assumption (9.1) on the Ricci curvature of M, we obtain

$$\psi\Delta\psi + \frac{m-1}{m-2}\lambda_1\psi^2 \geq \frac{1}{m-1}|\nabla\psi|^2. \tag{9.37}$$

Now, the operator

$$L = -\Delta - H\frac{m-1}{m-2}\lambda_1 \qquad \left(H = \frac{m-2}{m-1}\right)$$

clearly satisfies $\lambda_1 (L_M) = 0$. Furthermore, using Hölder inequality and (9.36),

$$\begin{aligned}\int_{B_R} \psi^{2\frac{m-2}{m-1}} &= \int_{B_R} \left(e^{2\sqrt{\lambda_1} r}\psi^2\right)^{\frac{m-2}{m-1}} e^{-2\frac{m-2}{m-1}\sqrt{\lambda_1} r} \\ &\leq CR^{\frac{m-2}{m-1}} \left(\int_{B_R} e^{-2(m-2)\sqrt{\lambda_1} r}\right)^{\frac{1}{m-1}}.\end{aligned}$$

By the co-area formula, (9.1) and the Bishop comparison theorem we have

$$\int_{B_R} e^{-2(m-2)\sqrt{\lambda_1} r} = \int_0^R \int_{\partial B_t} e^{-2(m-2)\sqrt{\lambda_1} r} dt \leq C\int_0^R e^{2\sqrt{\lambda_1} t\left(\frac{m-1}{2\sqrt{m-2}} - m + 2\right)} dt.$$

Thus

$$\int_{B_R} \psi^{2\frac{m-2}{m-1}} \leq C \begin{cases} R & \text{for } m = 3, \\ R^{\frac{m-2}{m-1}} & \text{for } m \geq 4. \end{cases}$$

It both cases it follows that

$$\frac{r}{\int_{B_r} \psi^{2\frac{m-2}{m-1}}} \notin L^1(+\infty)$$

and therefore

$$\frac{1}{\int_{\partial B_r} \psi^{2\frac{m-2}{m-1}}} \notin L^1(+\infty).$$

Since $\psi \not\equiv 0$, applying Theorem 4.5 with

$$\beta = A = H - 1 = -\frac{1}{m-1},$$

we deduce that equality holds in (9.37). The result now follows from Theorem 9.3, noting that, since $\lambda_1 > 0$, the infinite volume ends are precisely the non-parabolic ends. □

9.2 Gap theorems

We have seen in Chapter 4 that vanishing theorems may be obtained by imposing spectral conditions, namely, letting ${}^H L$ be the Schrödinger operator $-\Delta - Ha(x)$, then $\lambda_1({}^H L_M) \geq 0$ implies that non-negative L^p solutions of the differential inequality

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geq 0$$

vanish identically provided the coefficients A, a and H satisfy suitable conditions. On the other hand, it was shown in Section 7.1 that the validity of Sobolev-type inequalities can be used to obtain spectral information on the operator ${}^{H}L$. In particular, if an appropriate integral norm of (the positive part of) the potential $Ha(x)$ is less than or equal to the L^2-Sobolev constant, then $\lambda_1({}^{H}L_M) \geq 0$ (see Lemma 7.33).

In this section we show that a direct use of a Poincaré–Sobolev inequality can be used to obtain similar gap theorems, requiring that the integral norm of $Ha(x)$ be smaller than a suitable multiple(> 1) of the Poincaré–Sobolev constant.

We are going to apply this result to the investigation, already considered in Section 7.4, of the topology at infinity of immersed submanifold, as well as to other geometric situations like characterization of space forms and gap theorems for harmonic maps.

Theorem 9.12. *Let $(M, \langle\, , \rangle)$ be a complete manifold and assume that, for some $0 \leq \alpha < 1$ and some non-negative function h, the inhomogeneous Sobolev–Poincaré-type inequality*

$$\int_M \left(|\nabla\varphi|^2 + h\varphi^2\right) \geq S\left(\alpha\right)^{-1} \left\{\int_M |\varphi|^{\frac{2}{1-\alpha}}\right\}^{1-\alpha} \tag{9.38}$$

holds for every $\varphi \in C_c^\infty(M)$ with a positive constant $S\left(\alpha\right) > 0$. Suppose that $\psi \in \mathrm{Lip}_{loc}\left(M\right)$ is a positive solution of

$$\psi\Delta\psi + a\left(x\right)\psi^2 + A\left|\nabla\psi\right|^2 \geq 0 \qquad \text{(weakly) on } M \tag{9.39}$$

satisfying

$$\int_{B_r} |\psi|^\sigma = o\left(r^2\right) \;\; \text{as } r \to +\infty \tag{9.40}$$

with $A \in \mathbb{R}$, $\sigma - A - 1 > 0$, $\sigma \neq 0$, and $a\left(x\right) \in C^0\left(M\right)$. Then

$$\left\| a_+\left(x\right) + \frac{4(\sigma - A - 1)}{\sigma^2} h \right\|_{L^{\frac{1}{\alpha}}(M)} \geq \frac{4(\sigma - A - 1)}{\sigma^2} S\left(\alpha\right)^{-1}. \tag{9.41}$$

Furthermore, if ψ is assumed to be non-negative and not identically zero, then (9.41) *holds under the further assumption that $\sigma > 0$.*

We remark for future use that in the case where $\alpha = 0$, then the constant $S(0)^{-1}$ coincides with the bottom of the L^2-spectrum $\lambda_1(L_M)$ of the Schrödinger operator $L = -\Delta + h$.

We also remark that in the case where $h = 0$, that is when a standard Sobolev inequality is assumed to hold, then the theorem states that there are no non-zero solutions of (9.39) satisfying (9.40) if the $L^{1/\alpha}$-norm of the coefficient a is smaller than a multiple, depending on the integrability exponent σ, of the Sobolev constant. Thus the result compares directly with Theorem 4.5, replacing the spectral assumption with the validity of a Sobolev–Poincaré inequality and a norm estimate on the potential.

Proof. Note that the conclusion certainly holds if either a_+ or h are not in $L^{1/\alpha}(M)$, so we assume that $a_+ + h \in L^{1/\alpha}$.

We consider the case where ψ is only assumed to be non-negative, and $0 < \sigma < 2$, the other cases being easier. Next, recall that, by (9.39), for every test function $0 \le \rho \in H^1_c$ we have

$$\int a_+\psi^2\rho \ge \int \psi\langle\nabla\psi, \nabla\rho\rangle + (1 - A)\rho|\nabla\psi|^2.$$

Let $\phi = \phi_r \in C^\infty_c(M)$ be a family of cut-off functions satisfying

$$\phi_r \equiv 1 \text{ on } B_r, \qquad \phi_r \equiv 0 \text{ off } B_{2r}, \qquad |\nabla\phi_r| \le \frac{4}{r} \text{ on } M,$$

and apply the above inequality to the test function $(\psi^2 + \eta)^{\frac{\sigma-2}{2}}\phi^2$ to obtain, after some manipulations,

$$\begin{aligned}\int a_+\phi^2(\psi^2+\eta)^{\frac{\sigma}{2}}\frac{\psi^2}{\psi^2+\eta} \ge 2\int (\psi^2+\eta)^{\frac{\sigma-2}{2}}\psi\phi\langle\nabla\phi,\nabla\psi\rangle \\ + \int (\psi^2+\eta)^{\frac{\sigma-2}{2}}\phi^2|\nabla\psi|^2\Big\{1 - A + (\sigma-2)\frac{\psi^2}{\psi^2+\eta}\Big\}.\end{aligned} \tag{9.42}$$

We use the fact that

$$0 \le \frac{\psi^2}{\psi^2+\eta} \le 1 \tag{9.43}$$

and that $\sigma - 2 < 0$, to estimate the LHS from above, and the second integral on the RHS from below. Also, by Young's inequality and (9.43), for every $\epsilon > 0$, the first integral on the RHS is estimated from below by

$$-\frac{1}{\epsilon}\int (\psi^2+\eta)^{\frac{\sigma}{2}}|\nabla\phi|^2 - \epsilon\int (\psi^2+\eta)^{\frac{\sigma-2}{2}}.$$

Inserting the resulting inequalities and rearranging, we conclude that

$$\begin{aligned}\int a_+\phi^2(\psi^2+\eta)^{\frac{\sigma}{2}} + \frac{1}{\epsilon}\int (\psi^2+\eta)^{\frac{\sigma}{2}}|\nabla\phi|^2 \\ \ge (\sigma - A - 1 - \epsilon)\int \phi^2(\psi^2+\eta)^{\frac{\sigma-2}{2}}|\nabla\psi|^2.\end{aligned} \tag{9.44}$$

Fix $\epsilon > 0$ small enough that $\sigma - A - 1 - \epsilon > 0$. As $\eta \searrow 0$, by dominated convergence, the LHS converges to

$$\int a_+\phi^2\psi^\sigma + \frac{1}{\epsilon}\int \psi^\sigma|\nabla\phi|^2,$$

while, since $\sigma - 2 < 0$, by monotone convergence, the RHS converges to

$$(\sigma - A - 1 - \epsilon)\int \phi^2\psi^{\sigma-2}|\nabla\psi|^2.$$

We may therefore conclude that

$$\int a_+(x)\,\phi^2\psi^\sigma + \frac{1}{\varepsilon}\int \psi^\sigma\,|\nabla\phi|^2 \geq (\sigma - A - 1 - \varepsilon)\int \varphi^2\psi^{\sigma-2}\,|\nabla\psi|^2\,. \tag{9.45}$$

According to Lemma 4.13, the function $\psi^{\sigma/2} \in W^{1,2}_{loc}$ and $\nabla\big(\psi^{\sigma/2}\big) = \frac{\sigma}{2}\psi^{(\frac{\sigma}{2}-1)}\nabla\psi$. By the Poincaré–Sobolev inequality, Young inequality and (9.45), we estimate

$$\begin{aligned}
S(\alpha)^{-1}\Big\{\int\big(\psi^{\frac{\sigma}{2}}\phi\big)^{\frac{2}{1-\alpha}}\Big\}^{1-\alpha} &\leq \int\big(|\nabla\big(\psi^{\frac{\sigma}{2}}\phi\big)|^2 + h\phi^2\psi^\sigma\big)\\
&\leq (1+\delta)\frac{\sigma^2}{4}\int\psi^{\sigma-2}\phi^2|\nabla\psi|^2 + (1+\frac{1}{\delta})\int\psi^\sigma|\nabla\phi|^2 + \int h\phi^2\psi^\sigma\\
&\leq C_{\delta,\epsilon}^{-1}\int\big(a_+ + C_{\delta,\epsilon}h\big)\phi^2\psi^\sigma + \Big(\frac{1}{\epsilon C_{\delta,\epsilon}} + 1 + \frac{1}{\delta}\Big)\int\psi^\sigma|\nabla\phi|^2,
\end{aligned}$$

where

$$C_{\delta,\epsilon} = \frac{4}{\sigma^2}\frac{\sigma - A - 1 - \epsilon}{1+\delta}.$$

Using Hölder's inequality in the first integral,

$$\int\big(a_+ + C_{\delta,\epsilon}h\big)\phi^2\psi^\sigma \leq ||a_+ + C_{\delta,\epsilon}h||_{L^{1/\alpha}}\Big\{\int\big(\psi^{\frac{\sigma}{2}}\phi\big)^{\frac{2}{1-\alpha}}\Big\}^{1-\alpha},$$

and rearranging, we finally obtain

$$\begin{aligned}
\Big\{S(\alpha)^{-1} - C_{\delta,\epsilon}^{-1}||a_+ + C_{\delta,\epsilon}h||_{L^{1/\alpha}}\Big\}\Big(\int\big(\psi^{\frac{\sigma}{2}}\phi\big)^{\frac{2}{1-\alpha}}\Big\}^{1-\alpha}&\\
\leq \Big\{\frac{1}{\epsilon C_{\delta,\epsilon}} + 1 + \frac{1}{\delta}\Big\}\frac{C}{R^2}\int_{B_{2r}}\psi^\sigma.&
\end{aligned}$$

To conclude, assume that (9.41) does not hold. Since, by dominated convergence,

$$C_{\delta,\epsilon}^{-1}||a_+ + C_{\delta,\epsilon}h||_{L^{1/\alpha}} \to \frac{\sigma^2}{4(\sigma - A - 1)}\left\|a_+(x) + \frac{4(\sigma - A - 1)}{\sigma^2}h\right\|_{L^{\frac{1}{\alpha}}(M)},$$

as $\epsilon, \delta \to 0+$, we may choose δ and ϵ small enough that the coefficient on the LHS is positive. Since ψ does not vanish identically, there exists R such that, for every $r > R$ the integral is strictly positive. The required contradiction follows by noting that, according to (9.40), the right-hand side tends to zero as $r \to +\infty$. □

Theorem 9.12 allows us to obtain a quantitative improvement on the results obtained in Section 7.4

Recall that if $(M, \langle\,,\,\rangle)$ is isometrically immersed into a Cartan–Hadamard manifold N, the following L^2-type Sobolev inequality holds:

$$S_2(m,\varepsilon)^{-1}\left(\int_M |v|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \int_M |\nabla v|^2 + \frac{\varepsilon^2(m-2)^2}{4(m-1)^2}\int_M |H|^2 v^2, \tag{9.46}$$

where

$$S(m,\varepsilon)=\left(\frac{2\sqrt{1+\varepsilon^{-2}}\,(m-1)}{(m-2)\,S_1(m)^{-1}}\right)^2, \tag{9.47}$$

and H is the mean curvature vector field of the immersion. We also recall that if the immersion is minimal, then the best constant in (9.47) is achieved by choosing $\varepsilon=+\infty$, and in this situation, we write $S_2(m)=S_2(m,+\infty)$.

Applying Theorem 9.12 instead of Theorem 4.5, we obtain the following improvements of Theorem 7.36 and Theorem 7.35.

Theorem 9.13. *Let $f:(M^m,\langle\,,\rangle)\to\mathbb{R}^n$ be a complete, minimal, immersed submanifold of dimension $m\geq 3$ whose second fundamental tensor* II *satisfies*

$$\left(\int_M |\mathrm{II}|^m\right)^{\frac{2}{m}} < 4S_2(m)^{-1}\,\frac{\frac{2}{(m+2)(n-m)-2}+m-1}{\left(2-\frac{1}{n-m}\right)m^2}. \tag{9.48}$$

Then f is totally geodesic.

Theorem 9.14. *Let $f:(M^m,\langle\,,\rangle)\to(N^n,(\,,))$ be an isometric immersion of the complete manifold M of dimension $m\geq 3$ into the Cartan–Hadamard manifold N whose sectional curvature (along f) satisfies*

$$(0\geq)\ {}^N\mathrm{Sect}_{f(x)}\geq -\,{}^N R(x) \tag{9.49}$$

for some function ${}^N R\in C^0(M)$. Denote by H and II respectively the mean curvature vector field and the second fundamental tensor of f. Assume that, for some $\varepsilon>0$,

$$\left\|\frac{m(m-2)^2\varepsilon^2}{4(m-1)^3}H^2+(m-1)\ {}^N R(x)+|\mathrm{II}|\left(|\mathrm{II}|+m\,|H|\right)(x)\right\|_{L^{\frac{m}{2}}} < \frac{m}{m-1}S_2(m,\varepsilon)^{-1}$$

Then M has only one end.

Remark 9.15. For the sake of comparison, recall that Theorem 7.36 yields the same conclusion under the stronger assumption that

$$\left\|\frac{(m-2)^2\varepsilon^2}{(m-1)^2}|H|^2+(m-1)\ {}^N R(x)+|\mathrm{II}|\left(|\mathrm{II}|+m\,|H|\right)(x)\right\|_{L^{\frac{m}{2}}} \leq S_2(m,\varepsilon)^{-1}. \tag{9.50}$$

Observe that the proof of Theorem 7.36 relies on a vanishing result, Theorem 4.5, which depends on the assumption that the bottom of the spectrum of a suitable Schrödinger operator is non-negative. This in turn is obtained by combining the integral bound (9.50) and the Sobolev inequality. By contrast, the argument in Theorem 9.12 uses the Sobolev inequality in a more direct way, and allows us to improve the constant.

Our next result is a further application of Theorem 9.12, and it implies a vanishing result for harmonic forms in L^2.

Theorem 9.16. *Let* $(M, \langle\, , \rangle)$ *be a complete Riemannian manifold supporting the Poincaré–Sobolev inequality* (9.38) *for some* $0 \le \alpha < 1$*, and assume that*

$$\mathrm{Ric} \ge -\rho(x)$$

for some continuous function ρ *satisfying*

$$||\rho_+ + \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1}h||_{L^{1/\alpha}} < \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1}S(\alpha)^{-1}, \qquad (9.51)$$

for some $p > (m-2)/(m-1)$*. If* ω *is a closed and co-closed* 1*-form satisfying*

$$\int_{B_r} |\omega|^p = o\big(r^2\big),$$

then $\omega \equiv 0$.

The proof follows by noting that the norm of a closed and co-closed 1-form satisfies the differential inequality

$$\psi\Delta\psi + \rho\,(x)\,\psi^2 \ge \frac{1}{(m-1)}\,|\nabla\psi|^2$$

and applying Thoerem 9.12. Noting that a harmonic L^2 form is automatically closed and co-closed, we recover the vanishing results of P.Li and J. Wang, [103], Theorem 4.2, for manifolds with a positive spectral gap. In the case of L^p harmonic 1-forms, which are not necessarily closed and co-closed, and therefore do not satisfy a refined Kato inequality, one has a vanishing result provided the right-hand side of (9.51) is replaced by $4(p-1)S(\alpha)^{-1}/p^2$.

Similar results can be given for L^p harmonic forms of any degree, or for harmonic maps with L^p energy density, provided one uses the appropriate Weitzenböck formula. For instance, we have the following

Theorem 9.17. *Let* $(M, \langle\, , \rangle)$ *be an* m*-dimensional complete Riemannian manifold, supporting the Poincaré–Sobolev inequality* (9.38) *for some* $0 \le \alpha < 1$*, and assume that*

$$\mathit{Ricci} \ge -\rho(x).$$

Let $f : M \to N$ *be a harmonic map into an* n*-dimensional non-positively curved Riemannian manifold* N*. If* $|df| \in L^p(M)$*, for some* $p > \frac{m-2}{m-1}$ *and*

$$||\rho_+ + \frac{4}{p^2}\frac{(m-1)(p-1)+1}{(m-1)}h||_{L^{1/\alpha}} < \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1}S(\alpha)^{-1},$$

then f *is constant.*

9.3 Gap Theorems, continued

In this section we will employ Theorem 9.12 to obtain isolation phenomena for the Ricci tensor of a scalar flat, conformally flat manifold.

A Riemannian manifold $(M, \langle\,,\,\rangle)$ of dimension m is said to be locally conformally flat if a neighborhood of each point of M can be conformally immersed into the standard sphere $\mathbb{S}^m$. When $m \geq 4$ this is equivalent to the fact that the Weyl tensor vanishes identically. The category of locally conformally flat spaces contains the manifolds of constant sectional curvature, hence, in particular, the space-forms $\mathbb{R}^m$, $\mathbb{H}^m_{-k^2}$, $\mathbb{S}^m_{k^2}$, where the subscripts refer to the constant sectional curvature of the space. Note that, according to the orthogonal decomposition of the Riemann tensor into its irreducible components, a conformally flat manifold of dimension $m \geq 3$ has constant sectional curvature if and only if it is Einstein, i.e., the traceless part of its Ricci tensor is identically equal to zero. As a consequence, by the H. Hopf classification theorem, the space forms are (up to isometries) the only complete, simply connected, locally conformally flat, Einstein manifolds. In this section we investigate other possible characterizations of space forms from the conformally-flat viewpoint.

Let $(M, \langle\,,\,\rangle)$ be a conformally flat manifold of dimension $m \geq 3$ with constant scalar curvature S. We carry out the computations that follow assuming that $m \geq 4$, but note that the conclusions we obtain also hold when $m = 3$. Conformal flatness and the decomposition of the Riemann tensor into its irreducible components yield

$$R_{ijkl} = \frac{1}{m-2}\left(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}\right) - \frac{S}{(m-1)(m-2)}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right), \quad (9.52)$$

where we have denoted with R_{ij} the components of the Ricci tensor. Taking covariant derivatives, tracing, and using the fact that S is constant give

$$\sum_i R_{ijkl,i} = \frac{1}{m-2}\Big(\sum_i R_{ik,i}\delta_{jl} - \sum_i R_{il,i}\delta_{jk} + R_{jl,k} - R_{jk,l}\Big). \quad (9.53)$$

Now, note that tracing with respect to the indices i, m in the second Bianchi identities

$$R_{ijkl,m} + R_{ijmk,l} + R_{ijlm,k} = 0$$

yields

$$\sum_i R_{ijkl,i} = -\sum_i R_{ijik,l} - \sum_i R_{ijli,k} = -R_{jk,l} + R_{jl,k}, \quad (9.54)$$

and tracing again with respect to j, l and using the fact that S is constant we deduce that the Ricci tensor satisfies

$$R_{ik,k} = 0.$$

Inserting this and (9.54) in (9.53) we conclude that

$$\frac{m-3}{m-2}(R_{jl,k} - R_{jk,l}) = 0,$$

showing that Ric is a Codazzi tensor, namely, the following commutation relations for the coefficients of the covariant derivative of Ric hold,

$$R_{ij,k} = R_{ik,j}. \tag{9.55}$$

As a consequence, the traceless part

$$T = \mathrm{Ric} - \frac{S}{m}\langle\, , \,\rangle$$

is again Codazzi so that, as first observed by J.P. Bourguignon, [18], we have the validity of the refined Kato-type inequality

$$|DT|^2 \geq \frac{m+2}{m} \; |\nabla |T||^2 . \tag{9.56}$$

Further covariant differentiation of Ric yields the commutation formulas

$$R_{ij,kl} - R_{ij,lk} = R_{it}R_{tjkl} + R_{tj}R_{tikl}. \tag{9.57}$$

Using (9.52), (9.55), (9.57), and the fact that S is constant, we compute

$$\begin{aligned} \frac{1}{2}\Delta |Ric|^2 = |DRic|^2 + \frac{m}{m-2} tr\left(\mathrm{ric}^{(3)}\right) \\ - \frac{(2m-1)\, S}{(m-2)\,(m-1)} |\mathrm{Ric}|^2 + \frac{S^3}{(m-1)\,(m-2)} \end{aligned} \tag{9.58}$$

where $\mathrm{ric}^{(3)}$ is the third composition power of the Ricci endomorphism. Using the identity

$$|T|^2 = |\mathrm{Ric}|^2 - \frac{S^2}{m},$$

and expressing $tr\left(\mathrm{ric}^{(3)}\right)$ in terms of T we obtain, with the obvious meaning of the symbols,

$$\frac{1}{2}\Delta |T|^2 = |DT|^2 + \frac{m}{m-2} tr\left(T^{(3)}\right) + \frac{S}{m-1} |T|^2 .$$

A simple algebraic lemma due to M. Okumura, [122], shows that

$$tr\left(T^{(3)}\right) \geq -\frac{m-2}{\sqrt{m\,(m-1)}} |T|^3 . \tag{9.59}$$

Inserting (9.59) into (9.58) gives

$$\frac{1}{2}\Delta |T|^2 \geq |DT|^2 - \frac{m}{\sqrt{m(m-1)}}|T|^3 + \frac{S}{m-1}|T|^2 .$$

Setting $u = |T|$ and using (9.56), we rewrite the above inequality in the form

$$u\Delta u + \left(\sqrt{\frac{m}{m-1}}u - \frac{S}{m-1}\right)u^2 \geq \frac{2}{m}|\nabla u|^2 , \tag{9.60}$$

pointwise on $\{x \in M : |T|(x) \neq 0\}$ and weakly on all of M.

In the late 1960s M. Tani, [157], showed that the universal cover of a compact, orientable, m-dimensional, locally conformally flat Riemannian manifold $(M, \langle\,,\,\rangle)$ with positive Ricci curvature and constant scalar curvature S is isometrically a sphere. This result has been generalized by S.I. Goldberg, [61], in the complete (non-necessarily orientable) case under the additional assumption that, for some $\varepsilon > 0$,

$$\frac{S^2}{m-1} - |\mathrm{Ric}|^2 \geq \varepsilon > 0, \qquad \text{on } M \tag{9.61}$$

(see also [79]). In fact, combining a classification theorem by S. Zhu, [170], with a PDEs global symmetry result due to L. Caffarelli, B. Gidas and J. Spruck, [22], we prove that the above characterization holds by merely assuming that the left-hand side of (9.61) is strictly positive at one point.

Theorem 9.18. *Let $(M, \langle\,,\,\rangle)$ be a complete, locally conformally flat Riemannian manifold of dimension $m \geq 3$ and with constant scalar curvature $S > 0$. If*

$$\frac{S^2}{m-1} - |\mathrm{Ric}|^2 \geq 0, \qquad \text{on } M \tag{9.62}$$

and the strict inequality holds at some point, then the universal cover of $(M, \langle\,,\,\rangle)$ is isometrically a sphere.

Proof. We note that, by a lemma of Okumura, [123], inequality (9.62) implies that $\mathrm{Ric} \geq 0$ on M. Therefore, according to [170], the universal cover $\tilde{M}$ of M is either isometric to $\mathbb{R} \times \mathbb{S}^{m-1}_{S/(m-1)(m-2)}$ or conformally equivalent to $\mathbb{R}^m$ or $\mathbb{S}^m_1$. Since, by assumption, inequality (9.62) is strict somewhere, the first case is excluded. On the other hand, $\tilde{M}$ cannot be conformally diffeomorphic to $\mathbb{R}^m$. In fact, from Theorem 8.1 in [22] we know that a Riemannian metric on $\mathbb{R}^m$ which is of constant, positive scalar curvature and conformally related to the canonical metric must be a spherical metric, hence incomplete. It follows that $\tilde{M}$ is conformally a sphere, hence M is compact. The conclusion now follows from (the easy case of) Goldberg's argument. Namely, keeping the notation introduced above, we have only to show that $u \equiv 0$, i.e., that M is Einstein. From (9.60) we obtain

$$\Delta u^2 \geq b(x)\,u^2 \qquad \text{on } M \tag{9.63}$$

with

$$b(x) = \frac{2}{m-1}\left(S - \sqrt{m(m-1)}u(x)\right).$$

On the other hand, according to (9.62) and the fact that the strict inequality holds somewhere, we have

$$0 \not\equiv b(x) \geq C\left\{S^2 - (m-1)|\mathrm{Ric}|^2(x)\right\} \geq 0,$$

for some absolute constant $C > 0$. Since M is compact, applying the usual maximum principle we conclude that $b(x)$ is a positive constant and $u \equiv 0$, as desired. $\square$

Analogous characterizations hold in the case where the scalar curvature is non-positive.

Our first result in this direction deals with Euclidean space and can be thought of as an extension of Theorem 9.18 to the scalar flat case.

Theorem 9.19. *Let $(M, \langle\, , \rangle)$ be a complete, simply connected, locally conformally flat Riemannian manifold of dimension $m \geq 3$ and zero scalar curvature. Assume that*

$$\|\mathrm{Ric}\|_{L^{\frac{m}{2}}(M)} < \frac{2\omega_m^{\frac{2}{m}}(m-2)^3\sqrt{m-1}}{m^2\sqrt{m}} \tag{9.64}$$

where ω_m denotes the volume of the standard sphere $\mathbb{S}_1^m$. Then $(M, \langle\, , \rangle)$ is isometric to Euclidean space.

Proof. Maintaining the notation introduced above, we observe that in the present setting $u = |\mathrm{Ric}|$ while (9.60) becomes

$$u\Delta u + a(x)u^2 \geq \frac{2}{m}|\nabla u|^2 \tag{9.65}$$

with

$$a(x) = \sqrt{\frac{m}{m-1}}u(x).$$

Again, we have to show that $u \equiv 0$. To this end we reason by contradiction. Since $(M, \langle\, , \rangle)$ is simply connected and locally conformally flat, by a result of N. Kuiper, [90], there is a (global) conformal immersion (in fact an embedding) of M into the standard sphere $\mathbb{S}_1^m$. It follows by a result of R. Schoen and S.T. Yau, [147], that the Yamabe invariant $Q(M)$ of M defined by

$$Q(M) = \inf_{\phi \in C_c^\infty(M)\setminus\{0\}} \frac{\int\left(|\nabla\varphi|^2 + \frac{m-2}{4(m-1)}S\phi^2\right)}{\left(\int|\phi|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}}$$

satisfies

$$Q(M) = Q(\mathbb{S}_1^m) = \frac{m(m-2)\omega_m^{\frac{2}{m}}}{4}. \tag{9.66}$$

Thus, since $S = 0$, we have

$$\int |\nabla\varphi|^2 \geq Q\left(\mathbb{S}_1^m\right)\left(\int \varphi^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}, \text{ for each } \varphi \in C_c^\infty(M)\setminus\{0\}$$

which is the Sobolev inequality (9.38) with $h \equiv 0$, $\alpha = \frac{2}{m}$ and $S(\alpha) = Q\left(\mathbb{S}_1^m\right)^{-1}$. According to (9.65), conditions (9.39) and (9.40) are satisfied with $A = -\frac{2}{m}$ and $\sigma = \frac{m}{2}$. By Theorem 9.12 and (9.66) we conclude that

$$\|\mathrm{Ric}\|_{L^{\frac{m}{2}}(M)} \geq \frac{2\omega_m^{\frac{2}{m}}(m-2)^3\sqrt{m-1}}{m^2\sqrt{m}},$$

contradicting (9.64). ☐

In the case of negative scalar curvature, we establish the following result:

Theorem 9.20. *Let $(M, \langle\, ,\,\rangle)$ be a complete, locally conformally flat Riemannian manifold of dimension $m \geq 4$ and constant scalar curvature $S < 0$. Assume that, for some fixed $\varepsilon \geq 0$ and p, satisfying $m-1 < mp < (m-1)(m-2)$, the following conditions hold:*

$$\text{i) } \left|\mathrm{Ric} - \frac{S}{m}\langle\, ,\,\rangle\right| \leq -\varepsilon S, \qquad \text{ii) } \left|\mathrm{Ric} - \frac{S}{m}\langle\, ,\,\rangle\right| \in L^p(M) \tag{9.67}$$

and furthermore

$$\lambda_1(-\Delta) > \frac{p^2}{4}\frac{m}{m(p-1)+2}(-S)\left\{\frac{\varepsilon\sqrt{(m-1)\,m}+1}{(m-1)}\right\} \tag{9.68}$$

where $\lambda_1(-\Delta)$ denotes the bottom of the spectrum of the (positive definite) Laplace operator $-\Delta$. Then, the universal cover of $(M, \langle\, ,\,\rangle)$ is isometric to m-dimensional Hyperbolic space.

We remark that the restrictions on m and p follow from substituting into (9.68) the values of the scalar curvature and the bottom of the spectrum of m-dimensional hyperbolic space, for which (9.67) i) and ii) hold with $\varepsilon = 0$ and every $p > 0$.

Proof. As in the previous arguments the key point is to show that u, i.e., the length of the traceless Ricci tensor, vanishes identically.

Using (9.67) i) in (9.60) we see that u satisfies

$$u\Delta u - S\left(\varepsilon\sqrt{\frac{m}{m-1}} + \frac{1}{m-1}\right)u^2 \geq \frac{2}{m}|\nabla u|^2,$$

and $u \in L^p(M)$ by assumption (9.67) ii). If u were not identically zero, an application of Theorem 9.12, with the choices $A = -\frac{2}{m}$ and $a(x) = -S\left(\varepsilon\sqrt{\frac{m}{m-1}} + \frac{1}{m-1}\right)$, would contradict (9.68). Thus, $u \equiv 0$ as required. ☐