Chapter 4

Vanishing results

4.1 Formulation of the problem

As we mentioned in the introduction, the aim of this book is to present a unified approach to different geometrical questions such as the study of the constancy of harmonic maps, the topology at infinity of submanifolds, the L^2-cohomology, and the structure and rigidity of Riemannian and Kählerian manifolds (see Sections 6.1, 7.4, 7.5, 7.6, 8.1, and Appendix B).

The common feature of most of these problems lies in the fact that one identifies a suitable function ψ whose vanishing or, more generally, constancy, is the analytic counterpart of the desired geometric conclusion, and, using the peculiarities of the geometric data, one shows that the function ψ satisfies a differential inequality of the form

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geq 0 \tag{4.1}$$

weakly on M, as well as some suitable non-integrability condition. Typically, ψ represents the length of a section of a suitable vector bundle.

This is reminiscent of Bochner's original method: in the compact case, and under appropriate assumptions on the sign of the function ψ and of the coefficient $a(x)$, one concludes with the aid of the standard maximum principle.

In the non-compact case, one could conclude using a form of the maximum principle at infinity, see for instance [31] and the very recent [131], where, in some cases, one can also relax the boundedness conditions on ψ.

In the general case however, where no sign condition is imposed on $a(x)$ and/or the function ψ is not bounded, this method is not feasible.

The compactness of the ambient manifold is now replaced by the assumption that there exists a positive solution φ of a differential inequality suitably related to (4.1),

$$\Delta\varphi + Ha(x)\varphi \leq 0. \tag{4.2}$$

Combining the two inequalities enables us to rephrase the vanishing of ψ into an appropriate Liouville-type result.

We note that the existence of a positive solution to (4.2) is equivalent to the non-negativity of the bottom of the spectrum of the Schrödinger operator $-\Delta - Ha(x)$ (see Lemma 3.10 above), and one could interpret the condition on its spectrum as a sign condition on $a(x)$ in a suitably integrated sense. We also remark that a somewhat related approach has been used by other authors, see, e.g., P. Berard, [11]. However, he uses the condition on the spectral radius directly, and is therefore forced to restrict consideration to the L^2 case.

The main analytical tool used in proving our geometric results is a Liouville-type theorem for locally Lipschitz solutions of differential inequalities of the type

$$u\mathrm{div}\,(\varphi u) \geq 0$$

on M satisfying suitable non-integrability conditions (see Theorem 4.5 below).

Applying this result to a function u constructed in terms of ψ and φ yields the vanishing result for solutions of (4.1) alluded to above. The fairly weak regularity assumptions imposed on u are indeed necessary in order to treat the geometrical problems at hand.

4.2 Liouville and vanishing results

Theorem 4.1 below is a generalization of a Liouville-type result originally due to by H. Berestycki, L. Caffarelli and L. Nirenberg,[14], in a Euclidean setting. See also Proposition 2.1 in [6]. We remark that in the general case of a Riemannian manifold, the Euclidean technique works as well but does not yield the sharp result we are going to describe. We also note that when $\varphi \equiv 1$ we recover a classical result of Yau's, [168].

Theorem 4.1. *Let $(M, \langle\,,\,\rangle)$ be a complete manifold. Assume that $0 < \varphi \in L^\infty_{loc}(M)$ and $u \in L^\infty_{loc}(M) \cap W^{1,2}_{loc}(M)$ satisfy*

$$u\mathrm{div}\,(\varphi\nabla u) \geq 0, \textit{ weakly on } M. \tag{4.3}$$

If, for some $p > 1$,

$$\left(\int_{\partial B_r} |u|^p\,\varphi\right)^{-1} \notin L^1(+\infty), \tag{4.4}$$

then u is constant.

Proof. We begin by observing that assumption (4.3) means that, for every $0 \leq \sigma \in C^\infty_c(M)$, we have

$$0 \leq -\int \langle\nabla(\sigma u), \varphi\nabla u\rangle = -\int \big\{\langle\nabla\sigma, \varphi u\nabla u\rangle + \varphi\sigma|\nabla u|^2\big\}, \tag{4.5}$$

and it is therefore equivalent to the validity of the differential inequality

$$\mathrm{d}iv\,(\varphi u\nabla u) \geq \varphi|\nabla u|^2 \tag{4.6}$$

in the weak sense on M. Further, by a standard approximation argument, inequality (4.5) holds for every $0 \leq \sigma \in L^\infty(M) \cap W^{1,2}(M)$ compactly supported in M.

Next, let $a(t) \in C^1(\mathbb{R})$ and $b(t) \in C^0(\mathbb{R})$ satisfy

$$\text{(i) } a(u) \geq 0, \qquad \text{(ii) } a(u) + ua'(u) \geq b(u) > 0 \tag{4.7}$$

on M, and, for fixed $\epsilon, t > 0$, let ψ_ϵ be the Lipschitz function defined by

$$\psi_\epsilon(x) = \begin{cases} 1 & \text{if } r(x) \leq t, \\ \dfrac{t + \epsilon - r(x)}{\epsilon} & \text{if } t < r(x) < t + \epsilon, \\ 0 & \text{if } r(x) \geq t + \epsilon. \end{cases}$$

The idea of the proof is to apply the divergence theorem to the vector field $a(u)u\varphi\nabla u$. We use an integrated form of this idea in order to deal with the weak regularity of the functions involved.

For every non-negative compactly supported Lipschitz function ρ, we compute

$$\begin{aligned} -\int \langle \psi_\epsilon a(u)\nabla\rho, \varphi u \nabla u\rangle \\ &= -\int \langle \nabla(\rho\psi_\epsilon a(u)) - \rho\psi_\epsilon a'(u)\nabla u - \rho a(u)\nabla\psi_\epsilon, \varphi u\nabla u\rangle \\ &\geq \int \rho\psi_\epsilon\varphi|\nabla u|^2[a(u) + a'(u)u] + \rho a(u)\langle\nabla\psi_\epsilon, \varphi u\nabla u\rangle \\ &\geq \int \rho\psi_\epsilon\varphi b(u)|\nabla u|^2 - \frac{1}{\epsilon}\int_{B_{t+\epsilon}\setminus B_t} \rho a(u)\varphi|u||\nabla u|, \end{aligned}$$

where the first inequality follows from (4.6) using as test function $\rho\psi_\epsilon a(u)$, which is non-negative compactly supported and belongs to $L^\infty(M) \cap W^{1,2}(M)$ because of the assumptions imposed on a, u, φ, ψ_ϵ and ρ, while the second inequality is a consequence of (4.7) (ii) , and of the Cauchy–Schwarz inequality.

Choosing ρ in such a way that $\rho \equiv 1$ on $\overline{B}_{t+\epsilon}$ the integral on the leftmost side vanishes, and, applying the Cauchy–Schwarz inequality to the second integral on the right-most side and rearranging, we deduce that

$$\begin{aligned} \int_{B_t} \varphi b(u)|\nabla u|^2 \\ &\leq \left(\frac{1}{\epsilon}\int_{B_{t+\epsilon}\setminus B_t} \frac{a(u)^2}{b(u)}\varphi u^2\right)^{1/2} \left(\frac{1}{\epsilon}\int_{B_{t+\epsilon}\setminus B_t} b(u)\varphi|\nabla u|^2\right)^{1/2}. \end{aligned} \tag{4.8}$$

Setting

$$H(t) = \int_{B_t} \varphi b(u)|\nabla u|^2,$$

it follows by the co-area formula (see Theorem 3.2.12 in [51]) that

$$H'(t) = \lim_{\epsilon\to 0+} \frac{1}{\epsilon}\int_{B_{t+\epsilon}\setminus B_t} b(u)\varphi|\nabla u|^2 = \int_{\partial B_t} b(u)\varphi|\nabla u|^2\mathcal{H}^{m-1} \text{ for a.e. } t.$$

Here $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure on ∂B_t, which coincides with the Riemannian measure induced on the regular part of ∂B_t (the intersection of ∂B_t with the complement of the cut locus of o, see [51], 3.2.46, or [30], Proposition 3.4).

Since the same conclusion holds for the first integral on the right-hand side of (4.8), letting $\epsilon \to 0+$ in (4.8) and squaring, we conclude that

$$H(t)^2 \leq \left(\int_{\partial B_t} \frac{a(u)^2}{b(u)}\varphi u^2\right) H'(t) \quad \text{for a.e.} \quad t. \tag{4.9}$$

At this point the proof follows the lines of that of Lemma 1.1 in [138]: assume by contradiction that u is non-constant, so that there exists $R_0 > 0$ such that $|\nabla u|$ does not vanish a.e. in B_{R_o}. Then for each $t > R_0$, $H(t) > 0$, and therefore the RHS of (4.9) is also positive. Integrating the inequality between R and r ($R_0 \leq R < r$) we obtain

$$H(R)^{-1} \geq H\left(R\right)^{-1} - H(r)^{-1} \geq \int_R^r \left(\int_{\partial B_t} \varphi \frac{a(u)^2}{b(u)} u^2\right)^{-1}. \tag{4.10}$$

Now, we consider the sequence of functions defined by

$$a_n\left(t\right) = \left(t^2 + \frac{1}{n}\right)^{\frac{p-2}{2}}, \quad b_n(t) = \min\{p-1, 1\}\, a_n(t)\ , \forall n \in \mathbb{N}.$$

Since condition (4.7) holds for every n, so does (4.10), whence, letting $n \to +\infty$ and using the Lebesgue dominated convergence theorem and Fatou's lemma we deduce that there exists $C > 0$ which depends only on p such that

$$\left(\int_{B_R} \varphi |u|^{p-2} |\nabla u|^2\right)^{-1} \geq C \int_R^r \left(\int_{\partial B_t} \varphi |u|^p\right)^{-1} dt.$$

The required contradiction is now reached by letting $r \to +\infty$ and using assumption (4.4). □

As the above proof shows, the conclusion of the theorem holds if one assumes that $0 < \varphi \in L^2_{loc}(M)$ and $u \in Lip_{loc}(M)$.

We observe that condition (4.4) is implied by $u\varphi^{1/p} \in L^p(M)$. Indeed, if this is the case and we set $f = \int_{\partial B_r} |u|^p \varphi$, then the assumption and the co-area formula show that $f \in L^1(+\infty)$, and by Hölder inequality

$$\int_{r_0}^r f^{-1} \geq (r - r_0)^2 \left(\int_{r_0}^r f\right)^{-1} \to +\infty \text{ as } r \to +\infty.$$

We also note that the conclusion of Theorem 4.1 fails if we assume that $p = 1$ in (4.4). Indeed, taking $\varphi \equiv 1$, (4.3) reduces to $u\Delta u \geq 0$, and P. Li and R. Schoen have constructed in [95] an example of a non-constant, L^1, harmonic, function on a complete manifold. Indeed, let $(M, \langle\,,\rangle$ be a model manifold in the sense of Greene and Wu, namely $M = \mathbb{R}^m$ as a manifold, with the metric given in polar coordinates by

$$\langle\,,\rangle = dr^2 + \sigma(r)^2 d\theta^2,$$

where $d\theta^2$ denotes the standard metric on the unit sphere $\mathbb{S}^{m-1}$, and σ is a smooth odd function on $\mathbb{R}$ which is positive for $r > 0$ and such that $\sigma'(0) = 1$.

Choose a non-negative non-identically zero compactly supported smooth function $a(t)$, and define the non-negative function

$$u(x) = \int_0^{r(x)} \sigma(t)^{-(m-1)} \Big\{ \int_0^t a(s)\sigma(s)^{m-1}\, ds \Big\} dt, \qquad (4.11)$$

where $r(x)$ denotes the distance function from 0. It is easily verified that u is smooth, and satisfies

$$\Delta u = a(r(x))$$

on $(\mathbb{R}^m, \langle\,,\rangle)$, and it is therefore a non-constant non-negative subharmonic function. Since u is radial, for ease of notation we will write $u(r)$. If we specify σ to be $\sigma(t) = t$ for $t \in [0, 1]$, and such that

$$\sigma(t) = \big(t(\log t)^\epsilon\big)^{-1/(m-1)} \exp\big(-\frac{t^2(\log t)^\epsilon}{m-1}\big),$$

for $t \in [T_o, +\infty)$, for some $\epsilon > 0$, and $T_o > 1$ sufficiently large, then it is easy to check that

$$u(r) \sim C \exp\big(r^2(\log r)^\epsilon\big)$$

and

$$\int_{\partial B_r} u \sim \frac{C'}{r(\log r)^\epsilon}$$

as $r \to +\infty$. Thus, if $\epsilon > 1$, then u is a non-negative, integrable subharmonic function on M. We note that in this case, the manifold $(M, \langle\,,\rangle)$ has finite volume.

Finally, we remark that Theorem 4.1 generalizes [14] (see the proof of Proposition 2.1 therein) in two directions, even in the case where $M = \mathbb{R}^m$. First, in their case $p = 2$; secondly they replace (4.4) by the more stringent condition

$$\int_{B_r} u^2\varphi \le Cr^2$$

for some constant $C > 0$. To see that the latter implies (4.4) simply note that its validity forces

$$\frac{r}{\int_{B_r} u^2\varphi} \notin L^1(+\infty)$$

which in turn implies (4.4) (see, e.g., [138], Proposition 1.3). Furthermore, although the approach used in [14] is applicable also in the case of Riemannian manifolds, in this general context, it does not yield a sharp result.

We apply Theorem 4.1 to prove a uniqueness result for harmonic maps which largely improves on previous work in the literature. We recall that a ball $B_R(q)$ in a Riemannian manifold $(N,(\,,))$ is said to be regular if it does not intersect the cut locus of q, and, having denoted by $B \geq 0$ an upper bound for the sectional curvature of N on $B_R(q)$, one has $\sqrt{B}R < \pi/2$. Let q_b be the function defined by the formula

$$q_B(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } B = 0, \\ \frac{1}{B}\left(1 - \cos\left(\sqrt{B}t\right)\right) & \text{if } B > 0. \end{cases}$$

Assume that $f, g : (M, \langle\,,\rangle) \to B_R(q) \subset N$ are harmonic maps taking values in the regular ball $B_R(q)$ and define functions $\Phi, \psi, \varphi, u : M \to \mathbb{R}$ by setting

$$\begin{aligned} &\Phi(x) = -\log\left(\cos(\sqrt{B}\mathrm{dist}_N(q, f(x)) \cos(\sqrt{B}\mathrm{dist}_N(q, g(x))\right), \\ &\varphi(x) = e^{-\Phi(x)} \quad \text{and} \quad u = \varphi(x)^{-1} q_B\left(\mathrm{dist}_N(f(x), g(x))\right). \end{aligned} \tag{4.12}$$

Clearly, $u \geq 0$ and, since f and g take values in the regular ball $B_R(q)$, there exists a constant $C \geq 1$ such that

$$\begin{aligned} &C^{-1} \leq \varphi \leq 1 \qquad \text{and} \\ &C^{-1}\mathrm{dist}_N(f(x), g(x))^2 \leq u(x) \leq C\mathrm{dist}_N(f(x), g(x))^2 \end{aligned} \tag{4.13}$$

on M. Further, a result of W. Jäger and H. Kaul [86] shows that

$$\mathrm{div}\,(\varphi \nabla u) \geq 0 \quad \text{on } M,$$

and therefore, "a fortiori",

$$u\mathrm{div}\,(\varphi \nabla u) \geq 0 \quad \text{weakly on } M. \tag{4.14}$$

With this preparation we have the following uniqueness result:

Theorem 4.2. *Maintaining the notation introduced above, let $f, g : M \to N$ be harmonic maps taking values in the regular ball $B_R(q) \subset N$, and assume that, for some $p \geq 1$,*

$$\mathrm{dist}_N(f, g)^{2p} \in L^1(M). \tag{4.15}$$

In case $p = 1$, assume also that

$$\int_{\partial B_r} \mathrm{dist}_N(f, g)^2 \leq \frac{C}{r \log^\beta r} \tag{4.16}$$

for some constants $C, \beta > 0$ and for $r(x) >> 1$. If $\mathrm{vol}(M) = +\infty$, then $f \equiv g$.

Proof. As noted above, the functions φ and u satisfy (4.14), and, according to (4.13), the integrability condition (4.15) implies that

$$\varphi u^p = \varphi^{1-p} q_B\left(\mathrm{dist}_N(f(x), g(x))\right)^p \in L^1(M).$$

In particular

$$\left\{ \int_{\partial B_r} \varphi u^p \right\}^{-1} \notin L^1(+\infty).$$

If $p > 1$, we can use Theorem 4.1 above to deduce that u is constant, that is, there exists a constant $C_1 \geq 0$, such that

$$q_B(\mathrm{dist}_N(f(x), g(x))) = C_1 \varphi(x).$$

Since $\mathrm{vol}(M) = +\infty$ and φ is bounded away from zero, the integrability condition (4.15) forces $C_1 = 0$ and therefore $\mathrm{dist}_N(f(x), g(x)) \equiv 0$, as required.

The case $p = 1$ is a consequence of the following version of Theorem 4.1 above. □

Theorem 4.3. *Let $(M, \langle\, ,\rangle)$ be a complete manifold. Assume that $0 < \varphi \in L^2_{loc}(M)$ and $u \in Lip_{loc}(M)$ satisfy*

$$\mathrm{div}(\varphi \nabla u) \geq 0 \quad \textit{weakly on } M.$$

If $u \geq 0$ and

$$\text{(i)} \int_{\partial B_r} \varphi u \leq \frac{C}{r \log^\beta r}, \qquad \text{(ii)}\ u(x) \leq C e^{r(x)^2} \tag{4.17}$$

for some constants $C, \beta > 0$ and $r(x) \gg 1$, then u is constant.

Proof. We suppose that u is non-constant to get a contradiction. Thus, we proceed as in the proof of Theorem 4.1 above to arrive at

$$\left\{ \int_{B_R} \varphi b(u) |\nabla u|^2 \right\}^{-1} \geq \int_R^r \left\{ \int_{\partial B_t} \varphi \frac{a(u)^2}{b(u)} \right\}^{-1} \tag{4.18}$$

for $r > R \geq R_0$ sufficiently large and where the functions $a \in C^1(\mathbb{R})$ and $b \in C^0(\mathbb{R})$ satisfy

$$\text{(i)}\ a(u) \geq 0; \quad \text{(ii)}\ a'(u) \geq b(u) > 0 \text{ on } M. \tag{4.19}$$

Now, for every fixed $n \geq 1$, and for every $t \geq 0$, we let

$$a_n(t) = \log^\beta \left(1 + \log\left(1 + t + \frac{1}{n}\right)\right),$$
$$b_n(t) = a_n'(t) = \frac{\beta \log^{\beta - 1}\left(1 + \log\left(1 + t + \frac{1}{n}\right)\right)}{\left(1 + \log\left(1 + t + \frac{1}{n}\right)\right)\left(1 + t + \frac{1}{n}\right)}.$$

It is easy to verify that there exists a constant $\gamma = \gamma(\beta) > 0$ such that, for every $s \geq 0$,

$$\log^{1+\beta}(1 + \log(1 + s)) \leq \gamma s \left(1 + \log^{1+\beta}(1 + \log(1 + s))\right)$$

and therefore

$$\frac{a_n(t)^2}{b_n(t)} \leq \frac{1}{\beta}\gamma\left(\frac{1}{n}+t\right)\left(1+\log\left(1+t+\frac{1}{n}\right)\right)\left(1+\log^{1+\beta}\left(1+\log\left(1+t+\frac{1}{n}\right)\right)\right).$$

We substitute a_n, b_n into (4.18), let n tend to infinity in the resulting inequality and use the dominated convergence theorem and Fatou's lemma to deduce the existence of a constant $C > 0$ such that

$$\left\{\int_{B_R} \frac{\varphi|\nabla u|^2}{(1+u)(1+\log(1+u))\log^{1-\beta}(1+\log(1+u))}\right\}^{-1} \geq C\int_R^r \left\{\int_{\partial B_t} \varphi u(1+\log(1+u))\left(1+\log^{1+\beta}(1+\log(1+u))\right)\right\}^{-1}. \tag{4.20}$$

On the other hand, by (4.17),

$$\int_{\partial B_t} \varphi u(1+\log(1+u))\left(1+\log^{1+\beta}(1+\log(1+u))\right) \leq C\int_{\partial B_t} \varphi u t^2 (\log t)^{1+\beta} \leq Ct\log t.$$

By letting $r \to +\infty$, this contradicts (4.20). □

When $(N, (,))$ is a Cartan–Hadamard manifold, namely, a complete, simply connected manifold of non-positive sectional curvature, the above proof yields the next

Theorem 4.4. *Let $(N, (,))$ be Cartan–Hadamard and let $f, g : M \to N$ be harmonic maps such that, for some $p \geq 1$,*

$$\mathrm{dist}_N(f(x), g(x))^{2p} \in L^1(M) \tag{4.21}$$

and, for $p = 1$, add the conditions

$$\text{(i)} \int_{\partial B_r} \mathrm{dist}_N(f(x), g(x))^2 \leq \frac{C}{r\log^\beta r}; \quad \text{(ii)}\ \mathrm{dist}_N(f(x), g(x))^2 \leq Ce^{r(x)^2} \tag{4.22}$$

for some constants $\beta, C > 0$ and for $r(x)$ large enough. If $\mathrm{vol}(M) = +\infty$, then $f = g$.

As we pointed out after the proof of Theorem 4.1, an L^1-Liouville-type theorem for subharmonic functions is in general false if we do not require some extra assumptions. This explains the role of assumption (4.22) in Theorem 4.4 when $p = 1$.

We now come to the following consequence of Theorem 4.1, which will be the main ingredient in the geometric applications of Chapter 6 below.

Theorem 4.5. *Let $(M, \langle\, ,\rangle)$ be a complete manifold, $a(x) \in L^\infty_{loc}(M)$ and let $\psi \in Lip_{loc}(M)$ satisfy the differential inequality*

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geq 0 \qquad \textit{weakly on } M \tag{4.23}$$

for some $A \in \mathbb{R}$. Let also $\varphi \in Lip_{loc}(M)$ be a positive solution of

$$\Delta\varphi + Ha(x)\varphi \leq 0, \qquad \textit{weakly on } M, \tag{4.24}$$

for some H such that

$$H \geq A + 1, \ \ H > 0. \tag{4.25}$$

If

$$\left(\int_{\partial B_r} |\psi|^{2(\beta+1)}\right)^{-1} \notin L^1(+\infty) \tag{4.26}$$

for some β such that

$$A \leq \beta \leq H - 1, \ \ \beta > -1, \tag{4.27}$$

then there exists a constant $C \in \mathbb{R}$ such that

$$C\varphi = |\psi|^H \operatorname{sgn}\psi. \tag{4.28}$$

Further,

(i) *If $H - 1 > A$, then ψ is constant on M, and if in addition, $a(x)$ does not vanish identically, then ψ is identically zero;*

(ii) *If $H - 1 = A$, and ψ does not vanish identically, then φ and therefore $|\psi|^H$ satisfy (4.24) with equality sign.*

Proof. Set, for ease of notation, $\alpha = \frac{\beta+1}{H}$, and let u be the function defined by

$$u = \varphi^{-\alpha}|\psi|^\beta\psi,$$

so that the first assertion in the statement is that u is constant on M.

Noting that the restrictions imposed on β, and Lemma 4.12 in the Appendix at the end of this section, imply that $u \in C^0(M) \cap W^{1,2}_{loc}(M)$. Moreover,

$$\int \varphi^{2\alpha}|u|^2 = \int |\psi|^{2(\beta+1)},$$

so that (4.26) implies that (4.4) holds with $\varphi^{2\alpha}$ in place of φ, and $p = 2$, the constancy of u follows from Theorem 4.1 once we show that the differential inequality

$$u\mathrm{div}\big(\varphi^{2\alpha}\nabla u\big) \geq 0 \tag{4.29}$$

holds weakly on M. i.e. (see the beginning of the proof of Theorem 4.1), that for every non-negative, compactly supported function $\rho \in L^\infty(M) \cap W^{1,2}(M)$, we have

$$I = \int \big[\langle\varphi^{2\alpha}u\nabla u, \nabla\rho\rangle + \varphi^{2\alpha}|\nabla u|^2\rho\big] \leq 0.$$

Using the definition of u, and Lemma 4.13, we compute

$$\nabla u = -\alpha\varphi^{-\alpha-1}|\psi|^\beta\psi\nabla\varphi + (\beta+1)\varphi^{-\alpha}|\psi|^\beta\nabla\psi$$

whence

$$\begin{aligned} I = (\beta+1)\int\langle\nabla\psi,\psi|\psi|^{2\beta}\nabla\rho\rangle - \alpha\int\varphi^{-1}|\psi|^{2\beta+2}\langle\nabla\varphi,\nabla\rho\rangle \\ + \int\Big[(\beta+1)^2|\psi|^{2\beta}|\nabla\psi|^2\rho + \alpha^2|\psi|^{2\beta+2}\frac{|\nabla\varphi|^2}{\varphi^2}\rho\Big] \\ - 2\alpha(\beta+1)\int|\psi|^{2\beta}\psi\langle\frac{\nabla\varphi}{\varphi},\nabla\psi\rangle. \end{aligned} \tag{4.30}$$

We first consider the first integral on the right-hand side, and assume that $\beta<0$, the other case being easier. Since

$$\big|(\psi^2+\epsilon)^\beta\psi\nabla\psi\big| \le |\psi|^{2\beta+1}|\nabla\psi| = |\psi|^{1+\beta}|\psi|^\beta|\nabla\psi| \in L^1_{loc}$$

by Lemma 4.13, by the dominated convergence theorem,

$$\begin{aligned} &\int|\psi|^{2\beta}\psi\langle\nabla\psi,\nabla\rho\rangle = \lim_{\epsilon\to0+}\int(\psi^2+\epsilon)^\beta\psi\langle\nabla\psi,\nabla\rho\rangle \\ &= \lim_{\epsilon\to0+}\Big\{\int\langle\nabla\psi,\nabla\big[\psi(\psi^2+\epsilon)^\beta\rho\big]\rangle - (\psi^2+\epsilon)^\beta\frac{(2\beta+1)\psi^2+\epsilon}{\psi^2+\epsilon}|\nabla\psi|^2\rho\Big\}. \end{aligned} \tag{4.31}$$

According to (4.23), for every non-negative, compactly supported function $\sigma\in W^{1,2}(M)$,

$$\int\langle\nabla\psi,\nabla(\sigma\psi)\rangle \le \int\big(a(x)\psi^2 + A|\nabla\psi|^2\big)\sigma.$$

Applying the above inequality with $\sigma=\rho(\psi^2+\epsilon)^\beta$, and applying the dominated convergence theorem, we deduce that

$$\lim_{\epsilon\to0+}\int(\psi^2+\epsilon)^\beta\frac{(2\beta+1)\psi^2+\epsilon}{\psi^2+\epsilon}|\nabla\psi|^2\rho = (2\beta+1)\int|\psi|^{2\beta}|\nabla\psi|^2\rho,$$

and

$$\begin{aligned} \lim_{\epsilon\to0+}\int\langle\nabla\psi,\nabla\big[\psi(\psi^2+\epsilon)^\beta\rho\big]\rangle &\le \lim_{\epsilon\to0+}\int\big[a(x)\psi^2+A|\nabla\psi|^2\big](\psi^2+\epsilon)^\beta\rho \\ &= \int\big[a(x)|\psi|^{2\beta+2}+A|\psi|^{2\beta}|\nabla\psi|^2\big]\rho. \end{aligned}$$

Inserting these expressions into (4.31) we conclude that

$$\int|\psi|^{2\beta}\psi\langle\nabla\psi,\nabla\rho\rangle \le \int\big[a(x)|\psi|^{2\beta+2} + (A-2\beta-1)|\psi|^{2\beta}|\nabla\psi|^2\big]\rho. \tag{4.32}$$

In a similar, but easier way, using (4.24) one verifies that

$$
\begin{aligned}
&-\int \varphi^{-1}|\psi|^{2\beta+2}\langle\nabla\varphi,\nabla\rho\rangle\\
&\quad\leq \int \big[-Ha(x)|\psi|^{2\beta+2}-|\psi|^{2\beta+2}\frac{|\nabla\varphi|^2}{\varphi^2}+2(\beta+1)|\psi|^{2\beta}\psi\langle\frac{\nabla\varphi}{\varphi},\nabla\psi\rangle\big]\rho. \qquad (4.33)
\end{aligned}
$$

Substituting (4.32) and (4.33) into (4.30), and recalling the value of α and the condition satisfied by β, we conclude that

$$
I \;\leq\; (\beta+1)\int\left[(A-\beta)|\psi|^{2\beta}|\nabla\psi|^2\rho+\frac{\beta+1-H}{H^2}|\psi|^{2\beta+2}\frac{|\nabla\varphi|^2}{\varphi^2}\rho\right]\;\leq\; 0,
$$

as required to show that (4.29) holds.

In particular, ψ has constant sign, and if we assume that $\psi \not\equiv 0$, multiplying ψ by a suitable constant we may assume that ψ is strictly positive, and

$$\varphi = \psi^H.$$

Inserting this equality into (4.24) we have

$$H\psi^{H-2}\big[\psi\Delta\psi+(H-1)|\nabla\psi|^2+a(x)\psi^2\big]\leq 0, \qquad (4.34)$$

whence, multiplying (4.23) by $H\psi^{H-2}$, and subtracting the resulting inequality from (4.34) we obtain

$$H\big[(H-1)-A\big]\psi^{H-2}|\nabla\psi|^2\leq 0. \qquad (4.35)$$

Thus, if $H-1>A$, $|\nabla\psi|^2\equiv 0$, and ψ and therefore φ are constant. It follows from (4.24) that

$$\Delta\varphi+Ha(x)\varphi=Ha(x)\varphi\leq 0, \text{ so that } a(x)\leq 0,$$

while, (4.23) implies that

$$\psi\Delta\psi+a(x)\psi^2+A|\nabla\psi|^2=a(x)\psi^2\geq 0 \text{ so that } a(x)\geq 0,$$

and we conclude that $a(x)\equiv 0$. In particular, if $a(x)\not\equiv 0$, then ψ must vanish identically.

Finally, assume that $A=H-1$, and that ψ does not vanish identically, so that, as noted above, we may assume that ψ is strictly positive, and that $\varphi=\psi^H$. On the other hand, it follows from (4.24) and Lemma 3.10 that there exists a positive C^1 function v satisfying

$$\Delta v+Ha(x)v=0 \quad\text{weakly on } M. \qquad (4.36)$$

Repeating the argument with v in place of φ, we deduce that there exists $\tilde c\neq 0$ such that

$$\tilde c v=\psi^H=\varphi.$$

Thus φ is a positive multiple of v and we conclude that it also satisfies (4.36). □

We remark that Theorem 4.5 fails if the exponent $2(\beta+1)$ in the integrability condition (4.28) is replaced by $p(\beta+1)$ for some $p > 2$. Indeed, it was shown in [16] that if $a(x)$ and $b(x)$ are non-negative continuous functions on $\mathbb{R}^m$ satisfying

$$a(x) \leq \frac{(m-2)^2}{4}|x|^{-2} \quad , \quad a(x) = \frac{(m-2)^2}{4}|x|^{-2} \text{ if } |x| \gg 1$$

and

$$b(x) = \frac{|x|^{(m-2)(\sigma-1)/2}}{(\log|x|)^{\sigma+1}(\log\log|x|)(\log\log\log|x|)^2} \text{ if } |x| \gg 1$$

for some $\sigma > 1$, then the equation

$$\Delta u + a(x)u - b(x)u^{\sigma} = 0 \tag{4.37}$$

has a family of positive solutions u_α ($\alpha > 0$) satisfying

$$u_\alpha(0) = \alpha \quad \text{and} \quad u_\alpha(x) \sim |x|^{-(m-2)/2}\log|x| \quad \text{as } |x| \to +\infty.$$

In particular, u_α is a solution of (4.23) with $A = 0$, and

$$\int_{\partial B_r} |u_\alpha|^q \asymp r^{1+(m-2)(2-q)/2}(\log r)^q,$$

so that

$$\left(\int_{\partial B_r} |u_\alpha|^q\right)^{-1} \notin L^1(+\infty)$$

for every $q > 2$.

On the other hand, it is well known that in this case $\lambda_1\big([-\Delta - a(x)]_{\mathbb{R}^m}\big) = 0$, so there exists a positive solution φ of

$$\Delta\varphi + a(x)\varphi = 0 \quad \text{on } \mathbb{R}^m \tag{4.38}$$

(see, e.g., [19], Lemma 3 and subsequent Remark 4). Since in this case $H = 1$, applying Theorem 4.5 we would conclude that

$$c\varphi = u_\alpha$$

for some constant c which is necessarily positive, since both u, $\varphi > 0$. But then u_α would be a solution of (4.38) and this is impossible since it satisfies (4.37) and b is non-zero.

We also note that a minor modification of the above proof yields the following

Theorem 4.6. *Let $a(x)$, $b(x) \in C^0(M)$ and assume that $b(x) \geq 0$. Let $H > 0$, $K > -1$ and $A \in \mathbb{R}$ be constants satisfying*

$$A \leq H(K+1) - 1, \tag{4.39}$$

and suppose that there exists a positive $Lip_{loc}(M)$ solution of the differential inequality

$$\Delta\varphi + Ha(x)\varphi \le -K\frac{|\nabla\varphi|^2}{\varphi} \qquad on \;\; M. \tag{4.40}$$

Then the differential inequality

$$u\Delta u + a(x)u^2 - b(x)u^{\sigma+1} \ge -A|\nabla u|^2, \qquad \sigma \ge 1, \tag{4.41}$$

has no non-negative $Lip_{loc}(M)$ solutions on M satisfying

$$\operatorname{supp} u \cap \{x \in M \,:\, b(x) > 0\} \neq \emptyset \tag{4.42}$$

and

$$\Big(\int_{\partial B_r} u^{2(\beta+1)}\Big)^{-1} \notin L^1(+\infty), \tag{4.43}$$

for some β satisfying $\beta > -1$, $A \le \beta \le H(K+1) - 1$.

As an immediate corollary of Theorem 4.5 we have

Corollary 4.7. *Let $a(x) \in L^\infty_{loc}(M)$, $A \in \mathbb{R}$, $H \ge A+1, H > 0$, and set ${}^HL = -\Delta - Ha(x)$. Assume that $\psi \in Lip_{loc}(M)$ is a changing sign solution of* (4.23) *satisfying* (4.26) *for some β such that $\beta > -1$, $A \le \beta \le H-1$. Then $\lambda_1({}^HL_M) < 0$.*

Proof. Assume by contradiction that $\lambda_1({}^HL_M) \ge 0$. By Lemma 3.10 there exists $0 < \varphi \in C^1(M)$ satisfying $\Delta\varphi + Ha(x)\varphi = 0$ on M. By Theorem 4.5, there exists a constant C such that $C\varphi = |\psi|^{H-1}\psi$, and since ψ changes sign, while φ is strictly positive, this yields the required contradiction. □

In the case of Euclidean space, the integrability condition (4.26) follows assuming a suitable upper estimate for ψ, and yields the following (slight) improvement of [14] Theorem 1.7.

Corollary 4.8. *Let $a(x) \in L^\infty_{loc}(\mathbb{R}^m)$, and let $\psi \in Lip_{loc}(\mathbb{R}^m)$ be a changing sign solution of*

$$\psi\Delta\psi + a(x)\psi^2 \ge 0 \qquad on \;\; \mathbb{R}^m,$$

such that, for some $H \ge 1$,

$$|\psi(x)| = \mathcal{O}\Big(r(x)^{-(m-2)/2H}(\log r(x))^{1/2H}\Big), \qquad as \;\; r(x) \to +\infty.$$

If ${}^HL = -\Delta - Ha(x)$, then $\lambda_1\big({}^HL_{\mathbb{R}^m}\big) < 0$.

Similar results can be obtained on Riemannian manifolds where $\operatorname{vol}\partial B_r$ satisfies a suitable upper bound. This in turn follows, by the volume comparison theorem, from appropriate lower bounds on the Ricci curvature (see, Section 2.2 and [20], Appendix). We leave the details to the interested reader.

Theorem 4.5 yields also the following generalization of Theorem 2 (and Corollary 2) in [54].

Corollary 4.9. *Let $(M, \langle\, ,\rangle)$ be a complete manifold, and let $a(x) \in L^{\infty}_{loc}(M)$ and $A < 0$. Suppose that $\psi \in Lip_{loc}$ is a non-constant weak solution of the differential inequality*

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geq 0,$$

satisfying

$$\Big(\int_{\partial B_r} \psi^2\Big)^{-1} \notin L^1(+\infty). \tag{4.44}$$

Then, there exists $H_o \in [0, 1)$ such that, for every $H > H_o$, the differential inequality

$$\Delta\varphi + Ha(x)\varphi \leq 0 \tag{4.45}$$

has no positive, locally Lipschitz weak solution on M, while if $0 \leq H \leq H_o$, such a solution of (4.45) *exists.*

Proof. Recall that, according to Lemma 3.10, the existence of a positive, locally Lipschitz weak solution of (4.45) is equivalent to

$$\lambda_1\big({}^{H}L\big) \geq 0,$$

where ${}^{H}L = -\Delta - Ha(x)$.

Observe next that if $0 < H_1 \leq H_2$, then, by the variational characterization of the bottom of $\lambda_1\big({}^{H}L\big)$, we have

$$\lambda_1\big({}^{H_1}L\big) \geq \frac{H_1}{H_2}\lambda_1\big({}^{H_2}L\big). \tag{4.46}$$

(see the argument in the proof of Theorem 2 in [54]). Thus, if we denote by S the set of $H \geq 0$ such that (4.45) holds, S is not empty, since $\lambda_1(-\Delta) \geq 0$, and if H_2 is in S, then so is H_1.

An application of Theorem 4.5 with $A < \max\{A, 0\} = 0 = \beta = H - 1$ implies that if $H = 1$, then (4.45) has no positive locally Lipschitz solution, for otherwise ψ would necessarily be constant, against the assumption. Thus $1 \notin S$, and $H_o = \sup S \leq 1$

Now one concludes as in Corollary 2 in [54] showing, by an approximation argument, that S is closed, so that $1 > H_o \in S$. □

To see that Corollary 4.9 implies Theorem 2 and Corollary 2 in [54], it suffices to observe that if $ds^2 = \mu(z)|dz|^2$ is a complete metric on the unit disk D, with Gaussian curvature K, then $\psi = \mu^{-1/2}$ is a non-constant solution of

$$\psi\Delta\psi - K\psi^2 = |\nabla\psi|^2$$

and

$$\int \psi^2 d\mathrm{vol}_{ds^2} = \int \mu^{-1}\mu\, dxdy = \mathrm{vol}_{Eucl}(D) < +\infty.$$

According to the remark after the proof of Theorem 4.1, condition (4.44) holds, and Corollary4.9 implies that there exists $H_o \in [0,1)$ such that equation

$$\Delta\varphi - HK(x)\varphi = 0$$

has no positive solution if $H > H_o$ and has a positive solution if $0 \leq H \leq H_o$.

Following the above line of investigation, we are naturally led to the next result, which extends some known facts in minimal surfaces theory to minimal hypersurfaces of Euclidean space; see Corollary 4.11 below.

We recall that a minimal hypersurface $f : (M^m, \langle\,,\rangle) \to \mathbb{R}^{m+1}$ is stable if it (locally) minimizes area up to second order or, equivalently, if the bottom of the spectrum $\lambda_1 L_M)$ of the operator $L = -\Delta - |\mathrm{II}|^2$ is non-negative. Here $|\mathrm{II}|$ denotes the length of the second fundamental tensor of the immersion.

We also recall that a Riemannian metric $\widetilde{\langle\,,\rangle}$ on a (generic) manifold M is said to be a pointwise conformal deformation of a metric $\langle\,,\rangle$ if there exists a positive function $\rho \in C^\infty(M)$ such that $\widetilde{\langle\,,\rangle}_x(v,w) = \rho^2(x)\langle\,,\rangle_x(v,w)$, for every $x \in M$ and $v, w \in T_xM$.

Theorem 4.10. *Let $f : (M^m, \langle\,,\rangle) \to \mathbb{R}^{m+1}$ be a complete, stable, minimal hypersurface of dimension $m \geq 2$. Then $\langle\,,\rangle$ cannot be pointwise conformally deformed to a Riemannian metric $\widetilde{\langle\,,\rangle}$ of scalar curvature $\tilde{S}(x) \leq 0$ and finite volume.*

Proof. We first consider the case where $m \geq 3$. By contradiction, we assume that there exists a conformal metric $\widetilde{\langle\,,\rangle}$ on M with scalar curvature $\tilde{S}(x) \leq 0$ and finite volume $\widetilde{\mathrm{vol}}(M) < +\infty$. Denoting by $S(x)$ the scalar curvature of the original metric, minimality and the Gauss equations imply

$$S(x) = -|\mathrm{II}(x)|^2. \tag{4.47}$$

According to Lemma 3.10, the stability of f is then equivalent to the existence of a positive solution $\varphi \in C^\infty(M)$ of

$$\Delta\varphi - S(x)\varphi = 0 \qquad \text{on } M. \tag{4.48}$$

Setting

$$H = \frac{4(m-1)}{m-2} > 1; \qquad a(x) = -\frac{1}{H}S(x),$$

we can rewrite (4.48) in the form

$$\Delta\varphi + Ha(x)\varphi = 0 \qquad \text{on } M.$$

Now, let

$$\widetilde{\langle\,,\rangle} = \psi^{\frac{4}{m-2}}\langle\,,\rangle. \tag{4.49}$$

Then the smooth positive function ψ is a solution of the Yamabe equation, and, since $\tilde{S}\,(x) \leq 0$, we deduce that

$$\Delta\psi + a(x)\psi = -\frac{1}{H}\tilde{S}(x)\psi^{\frac{m+2}{m-2}} \geq 0, \text{ on } M. \tag{4.50}$$

Since

$$\int_M \psi^{\frac{2m}{m-2}} d\mathrm{vol} = \widetilde{\mathrm{vol}}(M) < +\infty$$

we have

$$\frac{1}{\int_{\partial B_r(o)} \psi^{2(\beta+1)}} \notin L^1\,(+\infty)$$

where

$$\beta = \frac{2}{m-2}$$

satisfies

$$0 < \beta < H - 1.$$

Applying Theorem 4.5, case 1, with $A = 0$ we therefore conclude that ψ, and therefore φ, is a positive constant and $S\,(x) \equiv 0$. According to (4.47) and (4.49) we deduce that $f\,(M)$ is an affine hyperplane and hence $\left(M, \widetilde{\langle\,,\rangle}\right)$ is homothetic to $(\mathbb{R}^m, can)$. But this clearly contradicts the assumption that $\widetilde{\mathrm{vol}}\,(M) < +\infty$.

The case $m = 2$ is completely similar. This time, we replace (4.49) with

$$\widetilde{\langle\,,\rangle} = \psi^2\langle\,,\rangle$$

and, instead of (4.50), we use the corresponding Yamabe equation

$$\psi\Delta\psi - S\,(x)\psi^2 = -\tilde{S}\,(x)\psi^4 + |\nabla\psi|^2\,.$$

Thus, ψ satisfies

$$\psi\Delta\psi - S\,(x)\,\psi^2 \geq |\nabla\psi|^2\,.$$

Since

$$\int_M \psi^2 d\mathrm{vol} = \widetilde{\mathrm{vol}}\,(M) < +\infty$$

we have

$$\frac{1}{\int_{\partial B_r(o)} \psi^2} \notin L^1\,(+\infty)\,.$$

On the other hand, the stability assumption implies the existence of a positive, smooth solution φ of (4.48). Therefore we can apply Theorem 4.5, case 1, with the choices $\beta = 0$, $a(x) = -S(x)$, $H = 1$, $A = -1$. Reasoning as above, we reach the desired contradiction. □

Using a classical universal covering argument, together with the Riemann-Köbe uniformization theorem, we easily recover Corollary 4 in [54]:

Corollary 4.11. *Let $f : (M, \langle\, ,\rangle) \to \mathbb{R}^3$ be a 2-dimensional, complete, stable, minimal surface. Then $f(M)$ is parabolic, and hence is an affine plane.*

Proof. Let $\pi : \left(\bar{M}, \bar{\langle\, ,\rangle}\right) \to (M, \langle\, ,\rangle)$ be the Riemannian universal covering of M. Then, $\bar{f} = f \circ \pi : \left(\bar{M}, \bar{\langle\, ,\rangle}\right) \to \mathbb{R}^3$ defines a complete, minimal surface. Moreover $\bar{f}$ is stable because any positive solution φ of (4.48) on M lifts to a positive solution $\bar{\varphi} = \varphi \circ \pi$ of $\bar{\Delta}\bar{\varphi} - \bar{S}(y)\bar{\varphi} = 0$ on $\bar{M}$. Here the bar-quantities refer to the covering metric $\bar{\langle\, ,\rangle}$. Since there are no compact minimal surfaces in the Euclidean space, the Uniformization Theorem implies that $\left(\bar{M}, \bar{\langle\, ,\rangle}\right)$ is conformally diffeomorphic to either $\mathbb{R}^2$ or the open unit disc $D_1 \subset \mathbb{R}^2$. In view of Theorem 4.10 the second possibility cannot occur so that M must be parabolic. To conclude that f is totally geodesic, simply note that, by (4.48), φ is a positive superharmonic function. Therefore φ must be constant and $S(x) = -\left|\mathrm{II}\right|^2 \equiv 0$. □

4.3 Appendix: Chain rule under weak regularity

This section provides the technical justification for the distributional computations needed in the proofs of Theorems 4.5 and 5.16 and Lemma 5.17 in the next section. First, we present a regularity result.

Lemma 4.12. *Let $a(x) \in L^{\infty}_{loc}(M)$ and $A \in \mathbb{R}$. Let $\psi \in Lip_{loc}(M)$ be a weak solution of*

$$\psi\Delta\psi + a(x)\,\psi^2 + A\left|\nabla\psi\right|^2 \geq 0 \text{ on } M.$$

Then

$$|\psi|^{p-1}\psi \in W^{1,2}_{loc}(M) \tag{4.51}$$

provided

$$\begin{cases} p \geq 1 & \text{if } A \geq 1, \\ p > \max\left\{0, \frac{A+1}{2}\right\} & \text{if } A < 1 \end{cases}$$

and, furthermore,

$$\nabla\left(\left(\psi^2+\varepsilon\right)^{(p-1)/2}\psi\right) \overset{L^2}{\rightharpoonup} \nabla\left(|\psi|^{p-1}\psi\right) \quad \text{as } \varepsilon \to 0+\,. \tag{4.52}$$

Proof. We treat only the case $p < 1$, the other case being easier. Consider the family of functions $\left(\psi^2+\varepsilon\right)^{(p-1)/2}\psi$ and note that, as $\varepsilon \to 0+$,

$$\left(\psi^2+\varepsilon\right)^{(p-1)/2}\psi \to |\psi|^{p-1}\psi \quad \text{in } L^2_{loc}.$$

We are going to use the fact that if a sequence $\{f_n\}$ is uniformly bounded in $W^{1,2}_{loc}$ and converges to f strongly in L^2_{loc}, then the limit function f is in $W^{1,2}_{loc}$ and ∇f_n converges to ∇f weakly in L^2_{loc} (see [55], Lemma 6.2, page 16). Since

$$|\nabla\big((\psi^2+\varepsilon)^{(p-1)/2}\psi\big)| = \left(\psi^2+\varepsilon\right)^{(p-1)/2}\frac{p\psi^2+\varepsilon}{\psi^2+\varepsilon}|\nabla\psi| \leq \left(\psi^2+\varepsilon\right)^{(p-1)/2}|\nabla\psi|$$

it suffices to show that the right-hand side is uniformly bounded in L^2_{loc} as $\varepsilon \to 0+$.

By assumption, for any $0 \leq \rho \in Lip_c(M)$, we have

$$-\int \langle \nabla\psi, \nabla(\rho\psi)\rangle \geq -\int a(x)\,\psi^2\rho - A\int |\nabla\psi|^2\rho,$$

that is,

$$-\int \psi\,\langle\nabla\psi, \nabla\rho\rangle \geq -\int a(x)\,\psi^2\rho + (-A+1)\int |\nabla\psi|^2\rho. \tag{4.53}$$

Fix $\varepsilon > 0$ and choose

$$\rho = \left(\psi^2+\varepsilon\right)^{p-1}\phi^2$$

where $0 \leq \phi \in C^\infty_c(M)$. Then,

$$\nabla\rho = 2(p-1)\phi^2\left(\psi^2+\varepsilon\right)^{p-2}\psi\nabla\psi + 2\phi\left(\psi^2+\varepsilon\right)^{p-1}\nabla\phi,$$

so that, using the Cauchy–Schwarz and Young inequalities and the fact that $p-1 < 0$, we estimate

$$\begin{aligned}
&LHS \text{ of } (4.53)\\
&= -2\int \phi\left(\psi^2+\varepsilon\right)^{p-1}\psi\langle\nabla\psi,\nabla\phi\rangle - 2(p-1)\int\phi^2\left(\psi^2+\varepsilon\right)^{p-2}\psi^2|\nabla\psi|^2\\
&\leq 2\int\phi\left(\psi^2+\varepsilon\right)^{p-1/2}|\nabla\psi|\,|\nabla\phi| - 2(p-1)\int\phi^2\left(\psi^2+\varepsilon\right)^{p-1}|\nabla\psi|^2\\
&\leq \frac{4}{\eta}\int\left(\psi^2+\varepsilon\right)^p|\nabla\phi|^2 - (2p-2-\eta)\int\phi^2\left(\psi^2+\varepsilon\right)^{p-1}|\nabla\psi|^2.
\end{aligned}$$

Moreover

$$\begin{aligned}
RHS \text{ of } (4.53) &= -\int a(x)\psi^2\left(\psi^2+\varepsilon\right)^{p-1}\phi^2 + (-A+1)\int\phi^2\left(\psi^2+\varepsilon\right)^{p-1}|\nabla\psi|^2\\
&\geq -\int |a(x)|\left(\psi^2+\varepsilon\right)^p\phi^2 + (-A+1)\int\phi^2\left(\psi^2+\varepsilon\right)^{p-1}|\nabla\psi|^2,
\end{aligned}$$

for $\eta > 0$. Combining the two inequalities and rearranging we obtain

$$\begin{aligned}
&(2p-A-1-\eta)\int\phi^2\left(\psi^2+\varepsilon\right)^{p-1}|\nabla\psi|^2\\
&\qquad\leq \frac{4}{\eta}\int\left(\psi^2+\varepsilon\right)^p|\nabla\phi|^2 + \int|a(x)|\left(\psi^2+\varepsilon\right)^p\phi^2.\\
&\qquad\qquad\leq \int\max\{1,|\psi|^{2p}\}\left(\frac{4}{\eta}|\nabla\phi|^2 + |a(x)|\phi^2\right).
\end{aligned}$$

Since $2p-A-1 > 0$, we may choose $\eta > 0$ small enough that the $(2p-A-1-\eta) > 0$, and conclude that the left-hand side is uniformly bounded in L^2_{loc} as $\varepsilon \to 0+$, as required to conclude the proof. □

Next we prove that, in the above assumptions, one can use the ordinary chain rule to compute the weak gradient of $|\psi^{p-1}|\psi$ even if $p < 1$. Note that, in this situation, the function $x \longmapsto |x|^{p-1}x$ is not Lipschitz so that standard results in the literature do not apply directly.

Lemma 4.13. *Let* $0 < p_o$ (< 1) *and assume that* $\psi \in Lip_{loc}(M)$ *satisfies* (4.51) *and* (4.52), *for every* $p > p_o$. *Then for every such* p,

$$|\psi|^{p-1}\nabla\psi \in L^2_{loc}(M) \tag{4.54}$$

and

$$\nabla\left(|\psi|^{p-1}\psi\right) = p|\psi|^{p-1}\nabla\psi, \text{ a.e. on } M, \tag{4.55}$$

where the LHS is understood in the sense of distribution and the RHS is defined almost everywhere, and is equal to 0 *where the* $\nabla\psi$ *vanishes.*

Proof. Let $p_o < p'$ (< 1) be any real number, and $\Omega \subset\subset M$ a fixed domain. Using $\nabla\psi \in L^2(\Omega)$ as a test function in (4.52) we have

$$\int_\Omega \langle\nabla\big((\psi^2+\varepsilon)^{(p'-1)/2}\psi\big), \nabla\psi\rangle \to \int_\Omega \langle\nabla(|\psi|^{p'-1}\psi), \nabla\psi\rangle, \text{ as } \varepsilon \to 0+. \tag{4.56}$$

Since

$$p'\left(\psi^2+\varepsilon\right)^{(p'-1)/2}|\nabla\psi|^2 \le \langle\nabla\big((\psi^2+\varepsilon)^{(p'-1)/2}\psi\big), \nabla\psi\rangle,$$

it follows from (4.56) and the monotone convergence theorem that

$$\lim_{\varepsilon\to 0}\left(\psi^2+\varepsilon\right)^{(p'-1)/2}|\nabla\psi|^2 = \begin{cases} 0 & \text{if } \nabla\psi = 0, \\ |\psi|^{p'-1}|\nabla\psi|^2 & \text{if } \nabla\psi \neq 0,\ \psi \neq 0, \\ +\infty & \text{if } \nabla\psi \neq 0,\ \psi = 0 \end{cases}$$

is integrable on Ω. In particular, the set where $\psi = 0$ and $\nabla\psi \neq 0$ has measure zero, showing that the vector field $|\psi|^{(p'-1)/2}\nabla\psi$ is defined almost everywhere and in $L^2(\Omega)$. Therefore we may use this vector field in (4.52) and, arguing as above, show that $|\psi|^{3(p'-1)/4}\nabla\psi \in L^2(\Omega)$. Iterating, we deduce that, for every n,

$$|\psi|^{(2^n-1)(p'-1)/2^n}\nabla\psi \in L^2(\Omega). \tag{4.57}$$

Now, given $p > p_o$, let $p' = 2^n(p-1)/(2^n-1)+1$ so that $(2^n-1)(p'-1)/2^n = p-1$. Choosing n large enough that $p' > p_o$, shows that (4.54) holds.

Finally, to prove (4.55), let $\rho \in C_c^\infty(M)$. By (4.52),

$$\int \langle\nabla\big((\psi^2+\varepsilon)^{(p-1)/2}\psi\big), \nabla\rho\rangle \to \int \langle\nabla(|\psi|^{p-1}\psi), \nabla\rho\rangle, \text{ as } \varepsilon \to 0+. \tag{4.58}$$

On the other hand,

$$\nabla\big((\psi^2+\varepsilon)^{(p-1)/2}\psi\big) = (\psi^2+\varepsilon)^{(p-1)/2}\frac{p\psi^2+\varepsilon}{\psi^2+\varepsilon}\nabla\psi \to p|\psi|^{p-1}\nabla\psi$$

pointwise a.e., and its absolute value is bounded above by $p|\psi|^{p-1}|\nabla\psi|$ which is in L^2_{loc} by (4.54). Therefore we may apply the dominated convergence theorem to the left-hand side of (4.58) to obtain

$$\int \langle p|\psi|^{p-1}\nabla\psi, \nabla\rho\rangle = \int \langle \nabla(|\psi|^{p-1}\psi), \nabla\rho\rangle$$

as required. □