Chapter 2

Comparison Results

In this section we describe some comparison results for the Hessian and the Laplacian of the distance function and for the volume of geodesic balls under curvature conditions. In some cases, the results we are going to describe improve on classical results.

2.1 Hessian and Laplacian comparison

We begin by showing that a lower (resp. upper) bound on the radial sectional curvature of the form

$$\mathrm{Sect}_{rad} \geq -G(r(x)) \quad (\text{resp. } \mathrm{Sect}_{rad} \leq -G(r(x))) \tag{2.1}$$

implies an upper estimate for the Hessian, $\operatorname{Hess} r$, of the distance function $r(x)$ of the type

$$\operatorname{Hess}(r) \leq \frac{h'(r)}{h(r)}\big(\langle\, ,\rangle - dr \otimes dr\big) \quad \text{resp. } \operatorname{Hess}(r) \geq \frac{h'(r)}{h(r)}\big(\langle\, ,\rangle - dr \otimes dr\big) \tag{2.2}$$

for some appropriate function h. By taking traces, we will then obtain corresponding estimates for the Laplacian Δr. As we will see, an upper estimate for Δr requires only a lower bound for the radial Ricci curvature, while a lower estimate requires an upper bound for the radial sectional curvature.

To obtain these results we use an "analytic" approach inspired by P. Petersen, [128] avoiding, in this way, the "geometrical" Laplacian comparison theorem of R. Greene and H.H. Wu, [66].

We will need the following Sturm comparison result:

Lemma 2.1. *Let G be a continuous function on $[0,+\infty)$ and let ϕ, $\psi \in C^1\big([0,\infty)\big)$ with ϕ', $\psi' \in AC((0,+\infty))$ be solutions of the problems*

$$\begin{cases} \phi'' - G\phi \leq 0 \quad a.e. \ in \ (0,\infty), \\ \phi(0) = 0, \end{cases} \qquad \begin{cases} \psi'' - G\psi \geq 0 \qquad\qquad a.e. \ in \ (0,\infty), \\ \psi(0) = 0, \ \psi'(0) > 0. \end{cases}$$

If $\phi(r) > 0$ for $r \in (0,T)$ and $\psi'(0) \geq \phi'(0)$, then $\psi(r) > 0$ in $(0,T)$ and

$$\frac{\phi'}{\phi} \leq \frac{\psi'}{\psi} \quad and \quad \psi \geq \phi \quad on \ (0,T). \tag{2.3}$$

Proof. Since $\psi'(0) > 0$, $\psi > 0$ in a neighborhood of 0. We observe in passing that if G is assumed to be non-negative, then integrating the differential inequality satisfied by ψ we have

$$\psi'(r) = \psi'(0) + \int_0^r G(s)\psi(s)\,ds,$$

so that ψ' is positive in the interval where $\psi \geq 0$, and we conclude that, in fact, $\psi > 0$ on $(0, +\infty)$.

In the general case where no assumption is made on the signum of G, we let $\beta = \sup\{t : \psi > 0 \text{ in } (0,t)\}$ and $\tau = \min\{\beta, T\}$, so that ϕ and ψ are both positive in $(0, \tau)$. The function $\psi'\phi - \psi\phi'$ is continuous on $[0, +\infty)$ vanishes in $r = 0$, and satisfies

$$(\psi'\phi - \psi\phi')' = \psi''\phi - \psi\phi'' \geq 0,$$

a.e. in $(0, \tau)$. Thus $\psi'\phi - \psi\phi' \geq 0$ on $[0, \tau)$, and dividing through by $\phi\psi$ we deduce that

$$\frac{\psi'}{\psi} \geq \frac{\phi'}{\phi} \quad \text{in } (0, \tau).$$

Integrating between ϵ and r $(0 < \epsilon < r < \tau)$ yields

$$\phi(r) \leq \frac{\phi(\epsilon)}{\psi(\epsilon)}\psi(r)$$

and since

$$\lim_{\epsilon \to 0+} \frac{\phi(\epsilon)}{\psi(\epsilon)} = \frac{\phi'(0)}{\psi'(0)} \leq 1,$$

we conclude that in fact

$$\phi(r) \leq \psi(r) \quad \text{in } [0, \tau).$$

Since $\phi > 0$ in $(0, T)$ by assumption, this in turn forces $\tau = T$, for otherwise, $\tau = \beta < T$, and we would have, $\phi(\beta) > 0$, while by continuity, $\psi(\beta) = 0$, which is a contradiction. □

Using the above Sturm comparison result, we deduce a comparison result for solutions of Riccati (in)equalities of the form

$$\phi' + \phi^2 = G \quad (\geq G, \ \leq G)$$

on $(0, T)$ with appropriate asymptotic behavior as $r \to 0+$. Note in this respect that the substitution $g = \phi'/\phi$ transforms the Riccati inequality into the second-order linear inequality

$$g'' = Gg \quad (\geq Gg, \ \leq Gg)$$

and conversely.

Corollary 2.2. *Let G be a continuous function on $[0,+\infty)$ and let $g_i \in AC(0,T_i)$ be solutions of the Riccati differential inequalities*

$$g_1' + \frac{g_i^2}{\alpha} - \alpha G \le 0 \qquad g_2' + \frac{g_2^2}{\alpha} - \alpha G \ge 0$$

a.e. in $(0,T_i)$ satisfying the asymptotic condition

$$g_i(t) = \frac{\alpha}{t} + O(1) \quad \text{as } t \to 0+,$$

for some $\alpha > 0$. Then $T_1 \le T_2$ and $g_1(t) \le g_2(t)$ in $(0,T_1)$.

Proof. Since $\tilde{g}_i = \alpha^{-1} g_i$ satisfies the conditions in the statement with $\alpha = 1$, without loss of generality we may assume that $\alpha = 1$.

Observe that the function $g_i(s) - \frac{1}{s}$ is bounded and integrable in a neighborhood of $s = 0$, and let $\phi_i \in C^1([0,T_i))$ be the positive function on $[0,T_i)$ defined by

$$\phi_i(t) = t \exp\Big\{ \int_0^t \big(g_i(s) - \frac{1}{s}\big)\, ds \Big\}.$$

Then $\phi_i(0) = 0$, $\phi_i > 0$ on $(0,T_i)$, $\phi_i' \in AC(0,T_i))$ and straightforward computations show that

$$\phi_i'(t) = g_i \phi_i(t), \quad \phi_i'(0) = 1$$

and

$$\phi_1'' \le G\phi_1 \text{ on } (0,T_1), \quad \phi_2'' \ge G\phi_2 \text{ on } (0,T_2).$$

An application of Lemma 2.1 shows that $T_1 \le T_2$ and $g_1 = \frac{\phi_1'}{\phi_1} \le \frac{\phi_2'}{\phi_2} = g_2$ on $(0,T_1)$, as required. $\square$

After this preparation we are ready to state our comparison result for the Hessian.

Theorem 2.3. *Let $(M, \langle\, ,\rangle)$ be a complete manifold of dimension m. Having fixed a reference point $o \in M$, let $r(x) = \mathrm{dist}_M(x,o)$, and let $D_o = M \setminus cut(o)$ be the domain of the normal geodesic coordinates centered at o. Given a smooth even function G on $\mathbb{R}$, let h be the solution of the Cauchy problem*

$$\begin{cases} h'' - Gh = 0, \\ h(0) = 0,\ h'(0) = 1, \end{cases}$$

and let $I = [0,r_0) \subseteq [0,+\infty)$ be the maximal interval where h is positive. If the radial sectional curvature of M satisfies

$$\mathrm{Sect}_{rad} \ge -G(r(x)) \qquad \text{on } B_{r_0}(o) \tag{2.4}$$

on $B_{r_o}(o)$, then

$$\mathrm{Hess}(r) \le \frac{h'}{h}\{\langle\, ,\rangle - dr \otimes dr\} \tag{2.5}$$

on $(D_o \setminus \{o\}) \cap B_{r_0}(o)$, *in the sense of quadratic forms. On the other hand, if*

$$\mathrm{Sect}_{rad} \leq -G\left(r\left(x\right)\right) \qquad on \;\; B_{r_0}(o), \tag{2.6}$$

then

$$\mathrm{Hess}\,(r) \geq \frac{h'}{h}\left\{\langle\,,\rangle - dr \otimes dr\right\}. \tag{2.7}$$

Proof. We essentially follow the direct approach by P. Petersen, [128], thus avoiding the classical use of Jacobi fields.

Observe first of all that $\mathrm{Hess}\,(r)(\nabla r, X) = 0$ for every $X \in T_xM$, and $x \in D_o \setminus \{o\}$. Indeed, let γ be the geodesic parametrized by arc length issuing from o with $\gamma(s_o) = x$, then γ is an integral curve of ∇r, namely, $\dot\gamma(s) = \nabla r(\gamma(s))$ so that $D_{\nabla r}\nabla r(x) = D_{\gamma(s_o)}\dot\gamma = 0$.

Next, since $\mathrm{Hess}\,(r)$ is symmetric, T_xM has an orthonormal basis consisting of eigenvectors of $\mathrm{Hess}\,(r)$. Denoting by $\lambda_{max}(x)$ and $\lambda_{min}(x)$, respectively, the greatest and smallest eigenvalues of the $\mathrm{Hess}\,(r)$ in the orthogonal complement of $\nabla r(x)$, the theorem amounts to showing that on $(D_o \setminus \{o\}) \cap B_{r_0}(o)$,

(i) if (2.4) holds, then $\lambda_{max}(x) \leq \dfrac{h'}{h}(r(x))$,

(ii) if (2.6) holds, then $\lambda_{min}(x) \geq \dfrac{h'}{h}(r(x))$.

Let $x \in D_o \setminus \{o\}$, and let again γ be the minimizing geodesic joining o to x. We claim that, if (2.4) holds, then the Lipschitz function λ_{max} satisfies

$$\begin{cases} \frac{d}{ds}\left(\lambda_{\max}\circ\gamma\right) + \left(\lambda_{\max}\circ\gamma\right)^2 \leq G & \text{for a.e. } s > 0, \\ \lambda_{\max}\circ\gamma = \frac{1}{s} + o\left(1\right), \text{ as } s \to 0^+. & \end{cases} \tag{2.8}$$

Similarly, if (2.6) holds, then the Lipschitz function λ_{min} satisfies

$$\begin{cases} \frac{d}{ds}\left(\lambda_{\min}\circ\gamma\right) + \left(\lambda_{\min}\circ\gamma\right)^2 \geq G & \text{for a.e. } s > 0, \\ \lambda_{\min}\circ\gamma = \frac{1}{s} + o\left(1\right), \text{ as } s \to 0^+. & \end{cases} \tag{2.9}$$

Since $\phi = h'/h$ satisfies

$$\phi' + \phi^2 = G \,\text{on}\; (0, r_o), \quad \phi(s) = \frac{1}{s} + 0(s) \;\text{ as } s \to 0+,$$

the required conclusion follows at once from Corollary 2.2. It remains to prove that λ_{max} and λ_{min} satisfy the required differential inequalities. To this end, given a smooth real function u, denote by $\mathrm{hess}\,(u)$ the $(1,1)$ symmetric tensor field defined by

$$\mathrm{hess}\,(u)\,(X) = D_X \nabla u,$$

so that

$$\mathrm{Hess}\,(u)\,(X,Y) = \langle \mathrm{hess}\,(u)\,(X)\,, Y\rangle\,.$$

By definition of covariant derivative in $TM^* \otimes TM$,

$$D_X(\mathrm{hess}(u))(Y) = D_X\big[\mathrm{hess}(u)(Y)\big] - \mathrm{hess}(u)(D_X Y),$$

so that, recalling the definition of the curvature tensor, we deduce the Ricci commutation rule

$$D_X(\mathrm{hess}(u))(Y) - D_Y(\mathrm{hess}(u))(X) = R(X,Y)\nabla u.$$

Now, choose $u = r(x)$, $X = \nabla r$, and let γ be the minimizing geodesic joining o to $x \in D_o \setminus \{o\}$. For every unit vector $Y \in T_xM$ such that $Y \perp \dot\gamma(s_o)$, define a vector field $Y \perp \dot\gamma$, by parallel translation along γ. Since, as noted above, $\mathrm{hess}\,(r)(\nabla r) \equiv 0$, we compute

$$\begin{aligned}
D_{\dot\gamma(t_0)}\big[\mathrm{hess}\,(r)(Y)\big] &= D_{\dot\gamma(t_0)}(\mathrm{hess}\,(r))(Y) + \mathrm{hess}\,(r)(D_{\dot\gamma(t_0)}Y)\\
&= D_{\nabla(r)}(\mathrm{hess}\,(r))(Y)\\
&= D_Y(\mathrm{hess}\,(r))(\nabla r) + R(\nabla r, Y)\nabla r\\
&= D_Y[\mathrm{hess}\,(r)(\nabla r)] - \mathrm{hess}\,(r)(D_Y\nabla r) - R(Y,\nabla r)\nabla r\\
&= -\mathrm{hess}\,(r)(\mathrm{hess}\,(r)(Y)) - R(Y,\nabla r)\nabla r,
\end{aligned}$$

that is,

$$D_{\dot\gamma(t_o)}[\mathrm{hess}\,(r)(Y)] + \mathrm{hess}\,(r)(\mathrm{hess}\,(r)(Y)) - R(Y,\nabla r)\nabla r.$$

Since Y is parallel,

$$\frac{d}{dt}\langle \mathrm{hess}\,(r)(Y), Y\rangle = \langle D_{\dot\gamma}[\mathrm{hess}\,(r)(Y)], Y\rangle,$$

and we conclude that

$$\frac{d}{ds}\left(\mathrm{Hess}\,(r)\,(\gamma)\,(Y,Y)\right) + \langle \mathrm{hess}\,(r)(\gamma)(Y), \mathrm{hess}\,(r)(\gamma)(Y)\rangle = -\mathrm{Sect}_\gamma\,(Y\wedge\dot\gamma)\,. \tag{2.10}$$

Now assume that $\mathrm{Sect}_{rad} \geq -G\,(r\,(x))$. Note that, for any unit vector field $X \perp \nabla r$,

$$\mathrm{Hess}\,(r)\,(X,X) \leq \lambda_{\max}.$$

Thus, if Y is chosen so that, at s_0,

$$\mathrm{Hess}\,(r)\,(\gamma)\,(Y,Y) = \lambda_{\max}\,(\gamma\,(s_0))\,,$$

then the function

$$\mathrm{Hess}\,(r)\,(\gamma)\,(Y,Y) - \lambda_{\max}\circ\gamma$$

attains its maximum at $s = s_0$ and, if at this point λ_{max} is differentiable, then its derivative vanishes:

$$\left.\frac{d}{ds}\right|_{s_0}\mathrm{Hess}\,(r)\,(\gamma)\,(Y,Y) - \left.\frac{d}{ds}\right|_{s_0}\lambda_{\max}\circ\gamma = 0.$$

Whence, using (2.10), we obtain, at s_0,

$$\frac{d}{ds}(\lambda_{\max}\circ\gamma)+(\lambda_{\max}\circ\gamma)^2\leq G,$$

which is the desired inequality stated in (2.8). The asymptotic behavior of $\lambda_{\max}\circ\gamma$ near $s=0^+$ follows from the fact that

$$\operatorname{Hess}(r)=\frac{1}{r}(\langle\,,\rangle-dr\otimes dr)+o(1),\ r\to 0^+,$$

as one can verify by a standard computation in normal coordinates at $o\in M$. The argument in the case where $\mathrm{Sect}_{rad}\leq -G$ is completely similar. ☐

As mentioned above, by taking traces in Theorem 2.3 we immediately obtain corresponding estimates for Δr. In particular, if $\mathrm{Sect}_{rad}\leq -G(r(x))$ it follows that

$$\Delta r(x)\geq (m-1)\frac{h'(r(x))}{h(r(x))}$$

on $(D_o\backslash\{o\})\cap B_{r_0}(o)$. Clearly the corresponding upper estimate holds if we assume instead that the radial sectional curvature is bounded below by $-G$. In this case however, the conclusion holds under the weaker assumption that the radial Ricci curvature is bounded from below by $-(m-1)G(r(x))$. Indeed we have the following Laplacian comparison theorem,

Theorem 2.4. *Maintaining the notation of the previous theorem, assume that the radial Ricci curvature of M satisfy*

$$Ric_{(M,\langle\,,\rangle)}(\nabla r,\nabla r)\geq -(m-1)G(r) \tag{2.11}$$

for some function $G\in C^0([0,+\infty))$, and let $h\in C^2([0,+\infty))$ be a solution of the problem

$$\begin{cases} h''-Gh\geq 0,\\ h(0)=0,\ h'(0)=1.\end{cases} \tag{2.12}$$

Then the inequality

$$\Delta r(x)\leq (m-1)\frac{h'(r(x))}{h(r(x))} \tag{2.13}$$

holds pointwise on $M\setminus(cut(o)\cup\{o\})$, and weakly on all of M.

Proof. Let $[0,r_0)\subseteq[0,+\infty)$ be the maximal interval where h is positive. Note that comparing h with the solution of the differential equation associated to (2.12) and using the remark at the beginning of the proof of Lemma 2.1 shows that if G is non-negative, then $r_o=+\infty$.

As in the proof of Theorem 2.3, let $D_o=M\setminus cut(o)$ be the maximal star-shaped domain of the normal coordinates at o. Fix any $x\in D_o\cap(B_{r_o}(o)\setminus\{o\})$

and let $\gamma : [0,l] \to M$ be the minimizing geodesic from o to x parametrized by arc-length. Set

$$\varphi(s) = (\Delta r) \circ \gamma(s), \quad s \in (0,l].$$

We claim that φ satisfies

$$\begin{cases} \text{i)} \quad \varphi(s) = \frac{m-1}{s} + o(1), \qquad \text{as } s \to 0^+, \\ \text{ii)} \quad \varphi' + \frac{1}{m-1}\varphi^2 \le (m-1)G, \quad \text{on } (0,l]. \end{cases} \tag{2.14}$$

Indeed (2.14) i) follows from the well-known fact that

$$\Delta r = \frac{m-1}{r} + o(1), \quad \text{as } r \to 0^+.$$

As for (2.14) ii), note that by tracing in (2.10) we deduce that

$$\frac{d}{dt}(\Delta r \circ \gamma) + |\mathrm{Hess}\, r|^2(\gamma) = -\mathrm{Ric}(\nabla r, \nabla r)(\gamma).$$

Using the elementary inequality

$$\frac{(\Delta r)^2}{m-1} \le |\mathrm{Hess}(r)|^2,$$

which in turn follows easily from the Cauchy–Schwarz inequality, we deduce that

$$\frac{d}{dt}(\Delta r \circ \gamma) + \frac{(\Delta r \circ \gamma)^2}{m-1} \le -\mathrm{Ric}(\nabla r, \nabla r)(\gamma). \tag{2.15}$$

Inequality (2.14) ii) follows from the assumption on Ric. Arguing as in the proof of Theorem 2.3 shows that (2.13) holds pointwise on $D_o \cap (B_{r_o}(o) \setminus \{o\})$.

Note now that a computation in polar geodesic coordinates shows that

$$\Delta r \circ \gamma(t) = \frac{1}{\sqrt{g(t,\theta)}} \frac{\partial \sqrt{g(t,\theta)}}{\partial t}$$

where $\theta = \gamma'(0)$ and $g(r,\theta)$ is the determinant of the metric in geodesic polar coordinates. Thus (2.13) can be rewritten in the form

$$\frac{1}{\sqrt{g(t,\theta)}} \frac{\partial \sqrt{g(t,\theta)}}{\partial t} \le (m-1)\frac{h'(t)}{h(t)}$$

whence, integrating and using the asymptotic behavior of h and $\sqrt{g}$ as $t \to 0+$, show that for every unit length $\theta \in T_oM$,

$$\sqrt{g(t,\theta)} \le h(t) \qquad \forall t < \min\{r_o, c(\theta)\}$$

where $c(\theta)$ denotes the distance of o from $cut(o)$ along the geodesic γ_θ. Since $\sqrt{g(t,\theta)} > 0$ if (t,θ) belongs to the domain of the geodesic polar coordinates, while, if $r_o < +\infty$, then $h(r_o) = 0$, we deduce that for all θ $c(\theta) \leq r_o$, and therefore $D_o \subset B_{r_o}(o)$.

Thus (2.13) holds pointwise on $M \setminus (\{o\} \cup cut(o))$, and it remains to prove that the inequality holds weakly on all of M. This is guaranteed by the following lemma. $\square$

Lemma 2.5. *Set $D_o = M \backslash cut(o)$ and suppose that*

$$\Delta r \leq \alpha(r) \quad \textit{pointwise on } \Omega \backslash \{o\} \tag{2.16}$$

for some $\alpha \in C^0\big((0, +\infty)\big)$. Let $v \in C^2(\mathbb{R})$ be non-negative and set $u(x) = v(r(x))$ on M. Suppose either

$$\text{i) } v' \leq 0 \quad \textit{or} \quad \text{ii) } v' \geq 0. \tag{2.17}$$

Then we respectively have

$$\text{i) } \Delta u \geq v''(r) + \alpha(r)v'(r); \quad \text{ii) } \Delta u \leq v''(r) + \alpha(r)v'(r) \tag{2.18}$$

weakly on M.

Proof. Let E_o be the maximal star-shaped domain on which $\exp_o$ is a diffeomorphism onto its image, so that $D_o = \exp(E_o)$ and we have $cut(o) = \partial\left(\exp_o(E_o)\right)$. Since E_o is a star-shaped domain, we can exhaust E_o by a family $\{E_o^n\}$ of relatively compact, star-shaped domains with smooth boundary. We set $\Omega^n = \exp_o(E_o^n)$ so that

$$\overline{\Omega}^n \subset \Omega^{n+1} \text{ and } \cup_n \Omega^n = D_o.$$

The fact that each E_o^n is star-shaped implies

$$\frac{\partial r}{\partial \nu_n} > 0, \quad \text{on } \partial\Omega^n \tag{2.19}$$

where ν_n denotes the outward unit normal to $\partial\Omega^n$. Now, we assume the validity of (2.17) i). Since $r \in C^\infty(\Omega^n \backslash \{o\})$, computing we get

$$\Delta u \geq v'' + \alpha(r)v' \quad \text{pointwise on } \Omega^n \backslash \{o\}. \tag{2.20}$$

Let $0 \leq \varphi \in C_0^\infty(M)$. We claim that, $\forall n$,

$$\int_{\Omega^n} u\Delta\varphi \geq \int_{\Omega^n} \left(v'' + \alpha(r)v'\right)\varphi + \varepsilon_n$$

with $\varepsilon_n \to 0$ as $n \to +\infty$. Since $M = \Omega \cup cut(o)$ and $cut(o)$ has measure 0, inequality (2.18) i) will follow by letting $n \to +\infty$. To prove the claim we fix $\delta > 0$ small and we apply the second Green formula on $\overline{\Omega^n} \backslash B_\delta(o)$ to obtain

$$\int_{\Omega^n \backslash B_\delta(o)} u\Delta\varphi = \int_{\Omega^n \backslash B_\delta(o)} \varphi\Delta u - \int_{\partial\Omega^n \cup \partial B_\delta(o)} \left(\varphi\frac{\partial u}{\partial \nu_n} - u\frac{\partial \varphi}{\partial \nu_n}\right) \tag{2.21}$$

where ν_n is the outward unit normal to $\partial\Omega^n \cup \partial B_\delta(o)$. We note that, according to (2.17) i) and (2.19),

$$\frac{\partial u}{\partial \nu_n} = v'(r)\frac{\partial r}{\partial \nu_n} \leq 0 \quad \text{on } \partial\Omega_n.$$

Using this, (2.20) and (2.21), we obtain

$$\int_{\Omega^n} u\Delta\varphi \geq \int_{\Omega^n} \left(v'' + \alpha(r)v'\right)\varphi + \varepsilon_n + I_\delta$$

with

$$\varepsilon_n = \int_{\partial\Omega^n} u\frac{\partial\varphi}{\partial\nu_n},$$

$$I_\delta = \int_{B_\delta(o)} \left[u\Delta\varphi - \left(v'' + \alpha(r)v'\right)\varphi\right] - \int_{\partial B_\delta(o)} \left[u\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial u}{\partial r}\right].$$

Clearly, $I_\delta \to 0$ as $\delta \to 0^+$. On the other hand, since $\varphi \in C_0^\infty(M)$ and $cut(o)$ has measure 0, using the divergence and Lebesgue theorems we see that, as $n \to +\infty$,

$$\varepsilon_n = \int_{\Omega^n} \operatorname{div}(u\nabla\varphi) \to \int_\Omega \operatorname{div}(u\nabla\varphi) = \int_M \operatorname{div}(u\nabla\varphi) = 0.$$

This proves the claim and the validity of (2.18) i).

The case of (2.17) ii) and (2.18) ii) can be dealt with in a similar way. □

Remark 2.6. We note that, for the above proofs to work, it is not necessary that (2.11) holds on the entire M. Indeed, for instance, if (2.11) is valid on $B_R(o)$, then (2.13) holds on $B_R(o)\backslash(\{o\} \cup cut(o))$ and weakly on $B_R(o)$.

We also remark that in the course of the proof we have shown that if the solution h of (2.12) vanishes at r_o, then $D_o \subset B_{r_o}(o)$ and therefore $M \subset \overline{B}_{r_o}(o)$. This easily yields the classical Bonnet-Myers theorem, stating that if $\operatorname{Ric} \geq (m-1)B^2$, then M is compact with diameter at most $\sqrt{\pi}/B$.

Remark 2.7. We note for future use that a modification of the above argument shows that on $M \setminus \{o\}$ the singular part of the distribution Δr is negative, and therefore it is the opposite of a positive measure concentrated on the cut locus. Indeed, let ϕ be a smooth, non-negative test function with support contained in $M \setminus \{o\}$. Arguing as above we may write

$$(\phi, \Delta r) = \int_M r\Delta\phi = \lim_n \Big(\int_{\Omega^n} \phi\Delta r + \int_{\partial\Omega^n} r\langle\nabla\phi, \nu\rangle - \int_{\partial\Omega^n} \phi\langle\nabla r, \nu\rangle\Big).$$

As $n \to +\infty$, the first term on the right-hand side tends to $\int_{E_o} \phi\Delta r$, and, as noted in the above proof, the second term tends to zero. Thus the limit of the third term exists, and we have

$$(r, \Delta\phi) - \int_{E_o} \phi\Delta r = (\phi, (\Delta r)_{sing}) = -\lim_n \int_{\partial\Omega^n} \phi\langle\nabla r\nu\rangle,$$

and since $\langle\nabla r, \nu\rangle \geq 0$, the limit is non-negative, as claimed.

In order to apply Theorem 2.4 one needs to find solutions of (2.12). We begin with the following fairly general result.

Lemma 2.8. *Suppose that G is a positive C^1 function on $[0,+\infty)$ such that*

$$\inf_{[0,+\infty)} \frac{G'}{G^{3/2}} > -\infty. \tag{2.22}$$

Then, there exists $D > 0$ sufficiently large that the function h defined by

$$h(r) = \frac{1}{D\sqrt{G(0)}} \left\{ e^{D\int_0^r \sqrt{G(s)}ds} - 1 \right\} \tag{2.23}$$

is a solution of (2.12).

Proof. Indeed, it is a simple matter to check that if h is as in the statement, then $h(0) = 0$, $h'(0) = 1$ and furthermore

$$h'' - G\ h \geq \frac{G}{\sqrt{G(0)}} \left[\inf_{[0,+\infty)} \frac{G'}{2G^{3/2}} + D - \frac{1}{D} \right]$$

so that, by (2.22), (2.12) holds provided $D > 0$ is sufficiently large. □

Remark 2.9. Assumption (2.22) implies $G(r)^{1/2} \notin L^1(+\infty)$. It is then a simple matter to check that

$$\frac{h'(r)}{h(r)} \sim \begin{cases} \frac{1}{r} & \text{as } r \to 0, \\ D\ G(r)^{1/2} & \text{as } r \to +\infty. \end{cases} \tag{2.24}$$

Remark 2.10. When we can find an explicit solution of the problem

$$\begin{cases} h'' - h\ G = 0, & \text{on } [0, +\infty), \\ h(0) = 0,\ h'(0) = 1, \end{cases} \tag{2.25}$$

then (2.13) yields a better estimate. For instance if $G(r) \equiv B^2$, $B > 0$,

$$h(r) = B^{-1} \sinh(Br)$$

satisfies (2.25) and we obtain

$$\Delta r \leq (m-1)B \coth(Br) \quad \text{weakly on } M \tag{2.26}$$

and pointwise on $M \setminus (\{o\} \cup cut(o))$. This estimate will be repeatedly used in the sequel.

We next describe some upper and lower estimates for h'/h and for h obtained in [20] for the case where G has the form $G(r) = B^2(1+r^2)^{\alpha/2}$ for some constants $B > 0$ and $\alpha \geq -2$. We begin with upper bounds.

Proposition 2.11. *Assume h is a solution of*

$$\begin{cases} h'' - B^2(1+r^2)^{\alpha/2}h = 0, \\ h(0) = 0 \quad h'(0) = 1, \end{cases} \tag{2.27}$$

where $B > 0$ and $\alpha \geq -2$. Set

$$B' = \begin{cases} B & \text{if } \alpha > -2, \\ \frac{1+\sqrt{1+4B^2}}{2} & \text{if } \alpha = -2. \end{cases} \tag{2.28}$$

Then

$$\frac{h'}{h}(r) \leq B' r^{\alpha/2}(1 + o(1)) \qquad \text{as } r \to +\infty. \tag{2.29}$$

Moreover there exists a constant C such that, when $r > 1$,

$$h(r) \leq C \begin{cases} \exp\left(\dfrac{2B'}{2+\alpha}(1+r)^{1+\alpha/2}\right) & \text{if } \alpha \geq 0, \\ r^{-\alpha/4}\exp\left(\dfrac{2B'}{2+\alpha}r^{1+\alpha/2}\right) & \text{if } -2 < \alpha < 0, \\ r^{B'} & \text{if } \alpha = -2. \end{cases} \tag{2.30}$$

Proof. The case where $\alpha \geq 0$ was already treated in [134]. It is a simple matter to check that the function ψ defined by

$$\psi(r) = B^{-1}\sinh\left(\frac{2B}{2+\alpha}\left[(1+r)^{1+\alpha/2} - 1\right]\right)$$

satisfies $\psi''/\psi \geq B^2(1+r^2)^{\alpha/2}$, $\psi(0) = 0$ and $\psi'(0) = 1$. The estimates (2.29) and (2.30) follow at once.

Next assume that $-2 < \alpha < 0$. Denoting by I_ν the modified Bessel function of order ν, it may be verified that the function defined by

$$\psi(r) = r^{1/2} I_{\frac{1}{2+\alpha}}\left(\frac{2B}{2+\alpha}r^{1+\alpha/2}\right),$$

is a positive C^1 solution on $[0, +\infty)$ of the (singular) differential equation

$$\psi'' = B^2 r^\alpha \psi$$

satisfying $\psi(0) = 0$ ([91], page 106). Since $r^\alpha \geq (1+r^2)^{\alpha/2}$, the argument above shows that $h'/h \leq \psi'/\psi$. Using the recurrence relation

$$I_\nu = \frac{1}{2}\left(I_{\nu+1} + I_{\nu-1}\right)$$

and the asymptotic representation

$$I_\nu(r) = \frac{e^r}{\sqrt{2\pi r}}\left(1 + o(1)\right) \qquad \text{as } r \to +\infty$$

([91], page 110 and page 123, respectively), it is not hard to show that

$$\frac{\psi'}{\psi}(r) = Br^{\alpha/2} - \frac{\alpha}{4}r^{-1} + O\left(r^{-2-\alpha/2}\right) \qquad \text{as } r \to +\infty$$

and (2.29) follows. Since we also have $\psi'(0) = C$ where $C = C_{\alpha,B}$ is a positive constant depending on α and B, we have $h \le C^{-1}\psi$ and (2.30) is again a consequence of the asymptotic representation of I_ν.

Finally, if $\alpha = -2$, we consider the function defined by

$$\psi(r) = r^\beta.$$

A simple computation shows that

$$\psi''/\psi \ge B^2(1+r^2)^{-1}$$

holds provided $\beta = \left[1 + \sqrt{1+4B^2}\right]/2$, showing that (2.29) holds. The validity of (2.30) is then established by integrating h'/h over $[1, r]$. □

Next we consider lower bounds. To illustrate the method, assume that $h'' = G(r)h$ and that $G(r)$ is non-increasing. Then

$$h''(r) = G(r)h(r) \ge G(R)h(r) \qquad \forall\, r \in [0, R].$$

Thus, the comparison argument used in the proof of Lemma 2.1 applied to the function $\phi(r) = G(R)^{-1/2}\sinh\left(G(R)^{1/2}r\right)$ implies that

$$\frac{h'}{h}(r) \ge G(R)^{1/2}\tanh\left(G(R)^{1/2}r\right).$$

Suppose now that $G(r)$ has the form $G(r) = B^2(1+r^2)^{\alpha/2}$. The condition that $G(r)$ be decreasing requires that we restrict ourselves to the case where $-2 \le \alpha \le 0$.

The case where $\alpha = 0$ is trivial since then $G(r)$ is constant, and we have the equality $h(r) = B^{-1}\sinh(Br)$.

If $-2 < \alpha < 0$, the argument above shows that

$$\frac{h'}{h} \ge B(1+r^2)^{\alpha/4} \ge Br^{\alpha/4}\left[1 + \frac{\alpha}{4}r^{-2}\right] \qquad r \gg 1,$$

where the last inequality follows by expanding the function $(1+r^{-2})^{\alpha/4}$. Integrating over $[R, r]$, $R \gg 1$, we obtain

$$\begin{aligned} h(r) &\ge h(R)\exp\left(B\int_R^r s^{\alpha/4}[1 + \alpha/(4s^2)]\right)\,ds \\ &\ge C\exp\left(\frac{2B}{2+\alpha}r^{1+\alpha/2}\right). \end{aligned}$$

Finally, if $\alpha = -2$, we consider the function $\phi(r) = (1+r)^{B'}$, with B' as in (2.28). Then

$$\frac{\phi''}{\phi}(r) = \frac{B'(B'-1)}{(1+r)^2} \geq \frac{B^2}{1+r^2}.$$

Thus $(h'\phi - h\phi')'(r) > 0$, $r > 0$, and since $(h'\phi - h\phi')(0) = 1$, we conclude that

$$\frac{h'}{h}(r) \geq B'(1+r)^{-1},$$

and then integrating

$$h(r) \geq C(1+r)^{B'}.$$

The above considerations prove the following

Proposition 2.12. *Let h be the solution of* (2.27) *with $-2 \leq \alpha \leq 0$, and let B' be defined as in* (2.28)*. Then*

$$\frac{h'}{h}(r) \geq B' r^{\alpha/2}(1+o(1)) \qquad \text{as } r \to +\infty, \tag{2.31}$$

and there exists a constant C such that when $r > 1$

$$h(r) \geq C \begin{cases} \exp\left(\dfrac{2B'}{2+\alpha} r^{1+\alpha/2}\right) & \text{if } -2 < \alpha \leq 0, \\ r^{B'} & \text{if } \alpha = -2. \end{cases} \tag{2.32}$$

We conclude this section by considering the case where G is a non-negative non-increasing function satisfying the condition

$$\int_0^{+\infty} tG(t)\,dt < +\infty. \tag{2.33}$$

According to a terminology introduced by U. Abresch [1] for the sectional curvature, if a manifold $(M, \langle\,,\,\rangle)$ satisfies

$$\mathrm{Sect}(x) \geq -G(r(x)) \quad \text{resp.} \quad \mathrm{Ric}(\nabla r, \nabla r) \geq -(m-1)G(r(x))$$

with G as above, then one says that M has asymptotically non-negative sectional, resp. Ricci, curvature. Note that under these assumption, G decays faster than quadratically at infinity, and in particular, condition (2.22) does not hold. This case is dealt with in the following lemma (see also [125], Lemma 1, and [171], Lemma 2.1).

Lemma 2.13. *Assume that $h \in C^2([0, +\infty)$ is the solution of the problem*

$$\begin{cases} h'' - G(r)h = 0, \\ h(0) = 1, \; h'(0) = 1, \end{cases} \tag{2.34}$$

where G is a continuous non-negative function on $[0, +\infty)$. Let ψ be defined by the formula

$$\psi(r) = \int_0^r e^{\int_0^s uG(u)\,du} ds.$$

Then we have the estimates

$$\text{(i)}\ \ 1 \le h'(r) \le \psi'(r) \quad \text{(ii)}\ \ \frac{1}{r} \le \frac{h'}{h} \le \frac{\psi'}{\psi} \ \textit{and}\ \ \text{(iii)}\ r \le h(r) \le \psi(r). \tag{2.35}$$

In particular, if

$$\int_0^{+\infty} tG(t)\,dt = b_0 < +\infty, \tag{2.36}$$

then, for every $r \ge 0$,

$$1 \le h'(r) \le e^{b_0} \quad \textit{and} \quad r \le h(r) \le e^{b_0} r. \tag{2.37}$$

Proof. Since the function $\phi(t) = t$ satisfies

$$\begin{cases} \phi'' - G(r)\phi \le 0, \\ \phi(0) = 1,\ \phi'(0) = 1, \end{cases}$$

applying Lemma 2.1 we easily obtain the left-hand side inequalities in (2.35) (ii) and (iii), which in turn imply (i).

The upper bounds are proved in a similar way. Indeed, since

$$\psi(r) \le r e^{\int_0^r sG(s)\,ds},$$

a straightforward computation shows that the function ψ satisfies

$$\begin{cases} \psi'' - G(r)\psi \ge 0, \\ \psi(0) = 1,\ \psi'(0) = 1, \end{cases}$$

and therefore we obtain the right-hand side inequalities in (2.35). It is clear that if (2.36) holds, then (2.35) yields (2.37) □

2.2 Volume comparison and volume growth

The following result is a somewhat generalized version of what is known in the literature as the Bishop-Gromov volume comparison theorem. In fact, on manifolds with Ricci curvature bounded from below, R. Bishop proved an upper estimate for the volume of balls not intersecting the cut locus of their centers, in terms of the volumes of corresponding balls in space-forms, [10]. Later, M. Gromov extended the estimate including the cut locus and improved the result by showing a monotonicity behavior.

Theorem 2.14. *Let $(M, \langle\,,\rangle)$ be a complete, m-dimensional Riemannian manifold satisfying*

$$\mathrm{Ric}\,(x) \geq -\,(m-1)\,G\,(r\,(x)) \;\; \text{on } M \tag{2.38}$$

for some non-negative function $G \in C^0\,([0,+\infty))$, where $r\,(x) = \mathrm{dist}\,(x,o)$ is the distance from a fixed reference origin $o \in M$. Let $h\,(t) \in C^2\,([0,+\infty))$ be the non-negative solution of the problem

$$\begin{cases} h''\,(t) - G\,(t)\,h\,(t) = 0, \\ h\,(0) = 0 \;\; h'\,(0) = 1. \end{cases} \tag{2.39}$$

Then, for almost every $R > 1$, the function

$$R \mapsto \frac{\mathrm{vol}\partial B_R\,(o)}{h(R)^{m-1}} \quad \text{is non-increasing} \tag{2.40}$$

and

$$\mathrm{vol}\partial B_R\,(o) \leq c_m h(R)^{m-1} \tag{2.41}$$

where c_m is the volume of the unit sphere in $\mathbb{R}^m$. Moreover,

$$R \mapsto \frac{\mathrm{vol}B_R\,(o)}{\int_0^R h\,(t)^{m-1}\,dt} \tag{2.42}$$

is a non-increasing function on $(0,+\infty)$.

Proof. In case o is a pole of M one simply integrates the radial vector field

$$X = h(r(x))^{-m+1}\nabla r$$

on concentric balls $B_R\,(o)$, and uses the divergence and the Laplacian comparison theorems. However, in general, objects are non-smooth and inequalities are intended in the sense of distributions. Therefore, we have to take some extra care.

The Laplacian comparison theorem asserts that

$$\Delta r\,(x) \leq (m-1)\,\frac{h'\,(r\,(x))}{h\,(r\,(x))} \tag{2.43}$$

pointwise on the open, star-shaped, full-measured set $M \setminus \mathrm{cut}\,(o)$ and weakly on all of M. Thus, for every $0 \leq \varphi \in Lip_c\,(M)$,

$$-\int \langle \nabla r, \nabla \varphi \rangle \leq (m-1) \int \frac{h'\,(r\,(x))}{h\,(r\,(x))}\varphi. \tag{2.44}$$

For any $\varepsilon > 0$, consider the radial cut-off function

$$\varphi_\varepsilon\,(x) = \rho_\varepsilon\,(r\,(x))\,h(r(x))^{-m+1} \tag{2.45}$$

where ρ_ε is the piecewise linear function

$$\rho_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in [0,r), \\ \frac{t-r}{\varepsilon} & \text{if } t \in [r, r+\varepsilon), \\ 1 & \text{if } t \in [r+\varepsilon, R-\varepsilon), \\ \frac{R-t}{\varepsilon} & \text{if } t \in [R-\varepsilon, R), \\ 0 & \text{if } t \in [R,\infty). \end{cases} \tag{2.46}$$

Note that

$$\nabla\varphi_\varepsilon = \left\{ -\frac{\chi_{R-\varepsilon,R}}{\varepsilon} + \frac{\chi_{r,r+\varepsilon}}{\varepsilon} - (m-1)\frac{h'(r(x))}{h(r(x))}\rho_\varepsilon \right\} h(r(x))^{-m+1} \nabla r,$$

for a.e. $x \in M$, where $\chi_{s,t}$ is the characteristic function of the annulus $B_t(o) \setminus B_s(o)$. Therefore, using φ_ε into (2.44) and simplifying, we get

$$\frac{1}{\varepsilon}\int_{B_R(o)\setminus B_{R-\varepsilon}(o)} h(r(x))^{-m+1} \le \frac{1}{\varepsilon}\int_{B_{r+\varepsilon}(o)\setminus B_r(o)} h(r(x))^{-m+1}.$$

Using the co-area formula we deduce that

$$\frac{1}{\varepsilon}\int_{R-\varepsilon}^{R} \mathrm{vol}\partial B_t(o)\, h(t)^{-m+1} \le \frac{1}{\varepsilon}\int_{r}^{r+\varepsilon} \mathrm{vol}\partial B_t(o)\, h(t)^{-m+1}$$

and, letting $\varepsilon \searrow 0$,

$$\frac{\mathrm{vol}\partial B_R(o)}{h(R)^{m-1}} \le \frac{\mathrm{vol}\partial B_r(o)}{h(r)^{m-1}} \tag{2.47}$$

for a.e. $0 < r < R$. Letting $r \to 0$, and recalling that $h(r) \sim r$ and $\mathrm{vol}\,\partial B_r \sim c_m r^{m-1}$ as $r \to 0$, we conclude that, for a.e. $R > 0$,

$$\mathrm{vol}\partial B_R(o) \le c_m h(R)^{m-1}, \text{ a.e. } R > 0.$$

To prove the second statement, we note that it was observed by M. Gromov, see [33], that for general real-valued functions $f(t) \ge 0$, $g(t) > 0$,

$$\text{if } t \to \frac{f(t)}{g(t)} \text{ is decreasing, then } t \to \frac{\int_0^t f}{\int_0^t g} \text{ is decreasing.}$$

Indeed, since f/g is decreasing, if $0 < r < R$,

$$\int_0^r f \int_r^R g = \int_0^r g\frac{f}{g}\int_r^R g \ge \frac{f(r)}{g(r)}\int_0^r g\int_r^R g \ge \int_0^r g\int_r^R g\frac{f}{g} = \int_0^r g\int_r^R f,$$

whence

$$\int_0^r f\int_0^R g = \int_0^r f\int_0^r g + \int_0^r f\int_r^R g \ge \int_0^r f\int_0^r g + \int_0^r g\int_r^R f = \int_0^r g\int_0^R f.$$

In particular, applying this observation to (2.47) and using the co-area formula we deduce that

$$r \to \frac{\mathrm{vol} B_r(o)}{\int_0^r h(t)^{m-1}\,dt} \quad \text{is decreasing,}$$

as required to conclude the proof. □

The following straightforward consequence of Theorem 2.14 is useful in applications.

Corollary 2.15. *Let $(M, \langle\, ,\rangle)$ be a complete, m-dimensional manifold satisfying* (2.38) *for some non-negative function $G(t) \in C^0([0,\infty))$.*

(i) *Having fixed $x_0 \in M$ and $0 < \bar{R} \le r(x_0)$, define*

$$sn_H(t) = \begin{cases} H^{-1}\sinh(Ht) & \text{if } H > 0, \\ t & \text{if } H = 0, \end{cases} \tag{2.48}$$

with

$$H^2 = \max_{B_{\bar{R}}(x_0)} G(r(x)) = \max_{[r(x_0)-\bar{R},\, r(x_0)+\bar{R}]} G(t). \tag{2.49}$$

Then, the function

$$R \mapsto \frac{\mathrm{vol} B_R(x_0)}{\int_0^R sn_H(t)^{m-1}\,dt}$$

is non-increasing on $[0, \bar{R}]$.

(ii) *If $G(t)$ is non-decreasing, then, for every $x_0 \in M$, the function*

$$R \mapsto \frac{\mathrm{vol} B_R(x_0)}{\int_0^R h_{x_0}(t)^{m-1}\,dt}$$

is non-increasing on $(0,+\infty)$. Here, h_{x_0} is the solution of the problem

$$\begin{cases} h''(t) - G_{x_0}(t)\,h(t) = 0, \\ h(0) = 0 \;\; h'(0) = 1, \end{cases}$$

where $G_{x_0}(t) = G(t + r(x_0))$.

Remark 2.16. Using the first part of the statement, J. Cheeger, M. Gromov and M. Taylor, [33], were able to deduce sharp volume growth estimates under point-inhomogeneous curvature conditions. See Theorem 2.26 and Corollary 2.27 below. See also Remark 2.23.

Proof. Part (i) is just a local version of Theorem 2.14. As for part (ii), let $G(t)$ be non-decreasing. Fix $x_0 \in M$, set $r_{x_0} = \mathrm{dist}(x_0, x)$, and observe that, by the triangle inequality,

$$\mathrm{Ric}(x) \ge -(m-1)\,G(r(x)) \ge -(m-1)\,G_{x_0}(r_{x_0}(x)).$$

Therefore, the assumptions of Theorem 2.14 are met with respect to the origin x_0. □

In the previous section, we made several choices of the lower Ricci bound $G(t)$ in such a way that the solutions $g(t)$ of the corresponding problem (2.39) can be explicitly estimated. In view of Theorem 2.14, these estimates can now be applied to get upper volume bounds. We limit ourselves to "polynomial situations" (see Proposition 2.11 and Lemma 2.13 above).

Corollary 2.17. *Let $(M, \langle\,,\rangle)$ be a complete, m-dimensional manifold satisfying*

$$\operatorname{Ric}(x) \geq -(m-1)\, G(r(x)) \quad \text{on } M$$

for some non-negative function $G \in C^0([0,+\infty))$, where $r(x) = \operatorname{dist}(x,o)$ is the distance from a fixed reference origin $o \in M$.

(i) *Assume $G(t) = B^2\left(1+t^2\right)^{-1}$, with $B>0$. Then, for every $R > r >> 1$,*

$$\frac{\operatorname{vol} B_R(o)}{\operatorname{vol} B_r(o)} \leq C\left(\frac{R}{r}\right)^{(m-1)B'+1}, \qquad \operatorname{vol}\partial B_r(o) \leq C r^{(m-1)B'},$$

where

$$B' = \frac{1+\sqrt{1+4B^2}}{2}$$

and $C>0$ is a suitable constant.

(ii) *Assume $\int_0^{+\infty} tG(t)\,dt = b_0 < +\infty$. Then, for every $R > r > 1$,*

$$\frac{\operatorname{vol} B_R(o)}{\operatorname{vol} B_r(o)} \leq e^{(m-1)b_0}\left(\frac{R}{r}\right)^{m}, \qquad \operatorname{vol}\partial B_r(o) \leq c_m e^{(m-1)b_0} r^{m-1},$$

where, as in (2.41), c_m is the $(m-1)$-volume of the unit sphere in $\mathbb{R}^m$.

The volume estimates described above depend on uniform bounds on the Ricci curvature. We next consider a situation in which the Ricci curvature satisfies some L^p-integrability conditions, and describe upper bounds for the volume growth of balls obtained by P. Petersen and G. Wei in [129], who consider the slightly less general case where the function G below is a non-negative constant and M is compact. Previous related results have been obtained by S. Gallot, [57], Li and Yau, [107] and D. Yang, [166].

As above, we assume that G is non-negative and continuous on $[0,+\infty)$ and that $h(t) \in C^2([0,+\infty))$ is the non-negative solution of the problem

$$\begin{cases} h''(t) - G(t)\,h(t) = 0, \\ h(0) = 0 \;\; h'(0) = 1. \end{cases}$$

Letting $r(x)$ be the distance function from the reference point o, we also define

$$\psi(x) = \max\{0, \Delta r(x) - (m-1)\frac{h'(r(x))}{h(r(x))}\} \tag{2.50}$$

in the domain D_o of the normal coordinates at o, and 0 on the cut locus, and let

$$\rho(x) = \max\{0, -\mathrm{Ric}(\nabla r, \nabla r) - (m-1)G(r(x))\}, \tag{2.51}$$

so that if $\rho \equiv 0$, that is $\mathrm{Ric}(\nabla r, \nabla r) \geq -(m-1)G(r(x))$, then, by the Laplacian comparison theorem, $\psi(x) \equiv 0$. For ease of notation, in the course of the arguments that follow we will denote by

$$A_G(r) = c_m h(r)^{m-1} \quad \text{and} \quad V_G(r) = c_m \int_0^r h(t)^{m-1} dt \tag{2.52}$$

the measures of the sphere and of the ball of radius r centered at the pole in the m-dimensional model manifold M_G with radial Ricci curvature equal to $-(m-1)G$, that is the manifold that is diffeomorphic to $\mathbb{R}^m$ and whose metric is given, in geodesic polar coordinates by

$$\langle\, , \rangle_G = dr^2 + h(r)^2 (\, ,)_{\mathbb{S}^{m-1}}.$$

We begin with the following lemma, which is a minor modification of Lemma 2.1 in [129].

Lemma 2.18. *Maintaining the notation introduced above, for every fixed $p > 1/2$ we have*

$$\frac{d}{dR} \frac{\mathrm{vol}\, B_R(o)}{V_G(R)} \leq \frac{c_m R A_G(R)}{V_G(R)^{1+1/2p}} \Big(\frac{\mathrm{vol}\, B_R}{V_G(R)} \Big)^{1-1/2p} \Big(\int_{B_R} \psi^{2p} \Big)^{1/2p}. \tag{2.53}$$

Proof. According to Remark 2.7, for every non-negative Lipschitz function φ compactly supported in $M \setminus \{o\}$ we have

$$-\int_M \langle \nabla r, \nabla \varphi \rangle = (\varphi, \Delta r) = \int_{E_o} \varphi \Delta r + (\varphi, (\Delta r)_{sing}) \leq \int_{E_o} \varphi \Delta r,$$

where D_o is the domain of the normal geodesic coordinates at o. Applying this inequality to the function $\varphi_\varepsilon(x) = \rho_\varepsilon(r(x))\, h(r(x))^{-m+1}$ where ρ_ε is the Lipschitz cut-off function defined in (2.46), arguing as in the proof of Theorem 2.14, and using the fact that h is non-decreasing, and the definition of ψ, we deduce that for a.e. $0 < r < R$,

$$\begin{aligned} \frac{\mathrm{vol}\, \partial B_R}{A_G(R)} - \frac{\mathrm{vol}\, \partial B_r}{A_G(r)} &\leq \int_{B_R \setminus B_r} h(r(x))^{-m+1} \Big[\Delta r - (m-1) \frac{h'(r(x))}{h(r(x))} \Big] \\ &\leq h(r)^{-m+1} \int_{B_R \setminus B_r} \psi. \end{aligned}$$

Using Hölder inequality and the definition of A_G we obtain

$$\begin{aligned} A_G(r) \mathrm{vol}\, \partial B_R - A_G(R) \mathrm{vol}\, \partial B_r &\leq c_m A_G(R) \int_{B_R \setminus B_r} \psi(x) \\ &\leq c_m A_G(R) \big(\mathrm{vol}\, B_R \big)^{1-1/2p} \Big(\int_{B_R} \psi^{2p} \Big)^{1/2p}. \end{aligned}$$

Therefore, applying the co-area formula, and using the above inequality we obtain

$$\begin{aligned}\frac{d}{dR}\Big(\frac{\operatorname{vol} B_R}{V_G(R)}\Big) &= \frac{V_G(R)\operatorname{vol}\partial B_R - A_G(R)\operatorname{vol} B_R}{V_G(R)^2}\\ &= V_G(R)^{-2}\int_0^R \big(A_G(r)\operatorname{vol}\partial B_R - A_G(R)\operatorname{vol}\partial B_r\big)\,dr\\ &\leq V_G(R)^{-2}\int_0^R c_m A_G(R)\big(\operatorname{vol} B_R\big)^{1-1/2p}\Big(\int_{B_R}\psi^{2p}\Big)^{1/2p}\,dr\\ &= \frac{c_m R A_G(R)}{V_G(R)^{1+1/2p}}\Big(\frac{\operatorname{vol} B_R}{V_G(R)}\Big)^{1-1/2p}\Big(\int_{B_R}\psi^{2p}\Big)^{1/2p}\end{aligned}$$

as required. □

As noted before the statement of the lemma, if the Ricci tensor satisfies the inequality $\operatorname{Ric}(\nabla r, \nabla r) \geq -(m-1)G(r(x))$, then $\psi \equiv 0$, and we recover the fact that the function

$$r \mapsto \frac{\operatorname{vol}\partial B_r(o)}{A_G(r)}$$

is a decreasing function of r.

The following lemma (see Lemma 2.2 in [129]) allows to estimate the $2p$-norm of ψ over B_R in terms of the p-norm of ρ.

Lemma 2.19. *For every $p > m/2$ there exists a constant $C = C(m,p)$ such that for every R,*

$$\int_{B_R}\psi^{2p} \leq C\int_{B_R}\rho^p$$

with $\rho(x)$ as defined in (2.51).

Proof. Integrating in polar geodesic coordinates we have

$$\int_{B_R} f = \int_{S^{m-1}} d\theta \int_0^{\min\{R,c(\theta)\}} f(t\theta)\omega(t\theta)dt$$

where ω is the volume density with respect to Lebesgue measure $dtd\theta$, and $c(\theta)$ is the distance from o to the cut locus along the ray $t \to t\theta$. It follows that it suffices to prove that for every $\theta \in S^{m-1}$,

$$\int_0^{\min\{R,c(\theta)\}} \psi^{2p}(t\theta)\omega(t\theta)dt \leq C\int_0^{\min\{R,c(\theta)\}} \rho^p(t\theta)\omega(t\theta)dt. \tag{2.54}$$

An easy computation which uses (2.15) yields

$$\frac{\partial}{\partial t}\{\Delta r - (m-1)\frac{h'}{h}\} \leq -\frac{(\Delta r)^2}{m-1} - Ric(\nabla r, \nabla r) - (m-1)\Big\{\frac{h''}{h} - \Big(\frac{h'}{h}\Big)^2\Big\}.$$

Thus, recalling the definitions of ψ and ρ, we deduce that the locally Lipschitz function ψ satisfies the differential inequality

$$\psi' + \frac{\psi^2}{m-1} + 2\frac{h'}{h}\psi \leq \rho$$

on the set where $\rho > 0$ and a.e. on $(0, +\infty)$. Multiplying through by $\psi^{2p-2}\omega$ and integrating, we obtain

$$\int_0^r \psi'\psi^{2p-2}\omega + \frac{1}{m-1}\int_0^r \psi^{2p}\omega + 2\int_0^r \frac{h'}{h}\psi^{2p-1}\omega \leq \int_0^r \rho\psi^{2p-1}\omega. \tag{2.55}$$

On the other hand, integrating by parts, and recalling that $\omega^{-1}\partial\omega/\partial t = \Delta r \leq \psi + (m-1)\frac{h'}{h}$ and that $\psi(t\theta) = 0$ if $t \geq c(\theta)$, yield

$$\begin{aligned}\int_0^r \psi'\psi^{2p-2}\omega &= \frac{1}{2p-1}\psi(r)^{2p-1}\omega(r\theta) - \frac{1}{2p-1}\int_0^r \psi^{2p-1}\Delta r\omega \\ &\geq -\frac{1}{2p-1}\int_0^r \psi^{2p-1}\big(\psi + (m-1)\frac{h'}{h}\big)\omega.\end{aligned}$$

Substituting this into (2.55), and using Hölder inequality we obtain

$$\begin{aligned}\Big(\frac{1}{m-1} - \frac{1}{2p-1}\Big)\int_0^r \psi^{2p}\omega + \Big(2 - \frac{m-1}{2p-1}\Big)\int \psi^{2p-1}\omega \\ \leq \int_0^r \rho\psi^{2p-1}\omega \\ \leq \Big(\int_0^r \rho^p\omega\Big)^{1/p}\Big(\int_0^r \psi^{2p}\omega\Big)^{(p-1)/p},\end{aligned}$$

and, since the coefficient of the first integral on the left-hand side is positive, by the assumption on p, while the second summand is non-negative, rearranging and simplifying we conclude that (2.54) holds with

$$C(m,p) = \Big(\frac{1}{m-1} - \frac{1}{2p-1}\Big)^{-p}. \qquad \square$$

We are now ready to state the announced volume comparison theorem under L^p Ricci curvature assumptions.

Theorem 2.20. *Keeping the notation introduced above, if $p > m/2$, then for every $0 < r < R$,*

$$\frac{\operatorname{vol} B_R(o)}{V_G(R)} - \frac{\operatorname{vol} B_r(o)}{V_G(r)} \leq \Big(\frac{1}{2p}\int_r^R f(t)dt\Big)^{2p}, \tag{2.56}$$

where

$$f(t) = \frac{c_m t A_G(t)}{V_G(t)^{1+1/2p}}\Big(\int_{B_t} \rho^p\Big)^{1/2p}. \tag{2.57}$$

Moreover f is integrable in $r = 0$ and

$$\frac{\operatorname{vol} B_R(o)}{V_G(R)} \le \Big(1 + \frac{1}{2p}\int_0^R f(t)dt\Big)^{2p}, \tag{2.58}$$

and

$$\begin{aligned}\frac{\operatorname{vol} \partial B_R(o)}{A_G(R)} \le {} & \Big(1 + \frac{1}{2p}\int_0^R f(t)dt\Big)^{2p} \\ & + \frac{c_m R}{V_G(R)^{1/2p}}\Big(\int_{B_R}\rho^p\Big)^{1/2p}\Big(1 + \frac{1}{2p}\int_0^R f(t)dt\Big)^{2p-1}.\end{aligned} \tag{2.59}$$

Proof. Set

$$y(r) = \frac{\operatorname{vol} B_r(o)}{V_G(r)}.$$

According to (2.53) in Lemma 2.18, Lemma 2.19 and (2.57) we have

$$\begin{cases} y'(t) \le f(t)y(t)^{1-1/2p}, \\ y(t) \sim 1 \text{ as } t \to 0+, \quad y(t) > 0 \ \text{ if } t > 0, \end{cases}$$

whence, integrating between r and R we obtain

$$y(R)^{1/2p} - y(r)^{1/2p} \le \frac{1}{2p}\int_r^R f(t)\,dt,$$

that is, (2.56). Since

$$A_G(t) \sim c_m t^{m-1} \quad \text{and} \quad \int_{B_t}\rho^p = O(t^m) \ \text{ as } \ t \to 0,$$

f is integrable in $t = 0$, and letting $r \to 0$ we obtain (2.58).

On the other hand, according to (2.53) and Lemma 2.19,

$$\frac{\operatorname{vol} \partial B_R}{A_G(R)} \le \frac{\operatorname{vol} B_R}{V_G(R)} + \frac{c_m R}{V_G(R)^{1/2p}}\Big(\int_{B_t}\rho^p\Big)^{1/2p}\Big(\frac{\operatorname{vol} B_R}{V_G(R)}\Big)^{1-1/2p}$$

and the conclusion follows inserting (2.58). $\square$

Corollary 2.21. *Keeping the notation introduced above, we have:*

(i) *Let $G = 0$, so that $h(t) = t$, $A_G(t) = c_m t^{m-1}$ and*

$$\rho = \max\{0, -\operatorname{Ric}(\nabla r, \nabla r)\} = \operatorname{Ric}_-.$$

(1) *If* $\mathrm{Ric}_- \in L^p(M)$ *for some* $p > m/2$, *then there exist constants* C_1 *and* C_2, *depending only on* m *and* p, *such that*

$$\operatorname{vol} B_R \leq \frac{c_m}{m}\Big[1 + C_1 ||\mathrm{Ric}_-||^{1/2}_{L^p(M)} R^{1-m/2p}\Big]^{2p} R^m,$$

and

$$\operatorname{vol} \partial B_R \leq c_m \Big[1 + C_2 ||\mathrm{Ric}_-||^{1/2}_{L^p(M)} R^{1-m/2p}\Big]^{2p} R^{m-1}.$$

(2) *(cf.* [107], *Corollary* 1.2*) If*

$$\int_{B_R} \mathrm{Ric}_-^p = o\big(R^\gamma\big) \quad as \quad t \to +\infty, \tag{2.60}$$

for some $p > m/2$ *and* $\gamma > 0$, *then*

$$\operatorname{vol} B_R = o(R^{2p+\gamma}) \quad as \quad R \to +\infty,$$

and

$$\operatorname{vol} \partial B_R = o(R^{2p+\gamma-1}) \quad as \quad R \to +\infty.$$

(ii) *Let* $G = H^2 > 0$ *be constant, so that* $h(t) = H^{-1} \sinh Ht$, *and*

$$V_G(t) \sim \frac{c_m}{(m-1)2^{m-1}H^m} e^{(m-1)Ht} \quad as \quad t \to +\infty.$$

If $\rho \in L^p(M)$ *for some* $p > m/2$, *then there exist constants* C_3 *and* C_4 *depending only on* p, m *and* H *such that*

$$\operatorname{vol} B_R \leq \big(1 + C_3 ||\rho||^{1/2}_{L^p(M)}\big)^{2p} V_G(R),$$

and

$$\operatorname{vol} \partial B_R \leq \big(1 + C_4 ||\rho||^{1/2}_{L^p(M)}\big)^{2p} A_G(R),$$

Proof. Assume that $G = 0$ and that $\mathrm{Ric}_- \in L^p$. Then we may estimate

$$f(t) = \frac{c_m t A_G(t)}{V_G(t)^{1+1/2p}} \Big(\int_{B_t} \mathrm{Ric}_-^p\Big)^{1/2p} \leq C ||\mathrm{Ric}_-||^{1/2}_{L^p(M)} t^{-m/2p}$$

and (i) (1) follows at once from (2.58) and (2.59). Similarly, if (2.60) holds, then

$$f(t) = o\big(t^{(\gamma-m)/2p}\big) \quad \text{as} \quad t \to +\infty,$$

and (i) (2) follows as before from (2.58) and (2.59). Finally, if $G = H^2 > 0$ and $\rho \in L^p(M)$, we may estimate

$$f(t) \leq C ||\rho||^{1/2}_{L^p(M)} \max\{1, t\} V_G(t)^{-1/2p}$$

and since the right-hand side is integrable on $[0, +\infty)$, the required conclusion follows once again from (2.58) and (2.59). $\square$

The Bishop-Gromov comparison theorem enables one to get an upper estimate of the volumes of geodesic balls, assuming a lower control on the Ricci tensor. However, there are special situations where lower volume estimates can be deduced from the same curvature conditions. To see this, let us first consider the next important consequences of Theorem 2.14, as was pointed out in [33].

Corollary 2.22. *Let* $(M, \langle\, ,\rangle)$ *and* h *be as in Theorem* 2.14.

(i) *For every* $r_1 \le r_2 \le r_3$,

$$\frac{\mathrm{vol}B_{r_3}(o) - \mathrm{vol}B_{r_2}(o)}{\int_{r_2}^{r_3} h(t)^{m-1}\,dt} \le \frac{\mathrm{vol}B_{r_1}(o)}{\int_0^{r_1} h(t)^{m-1}\,dt}. \tag{2.61}$$

(ii) *Assume that the lower Ricci curvature bound is a constant function* $G(t) \equiv H^2 \ge 0$. *Let* $x, y \in M$, *set* $d = \mathrm{dist}(x,y)$ *and take* $r + R < d$. *Then*

$$\mathrm{vol}B_R(x)\,\frac{\int_0^r sn_H(t)^{m-1}\,dt}{\int_{d-R}^{d+R} sn_H(t)^{m-1}\,dt} \le \mathrm{vol}B_r(y), \tag{2.62}$$

where sn_H *is defined in* (2.48).

Remark 2.23. As will be clear from the proof, property (2.61) can be localized according to the first part of Corollary 2.22. In this situation, condition (2.61) becomes: for every $r_1 \le r_2 \le r_3 \le \bar{R}$,

$$\frac{\mathrm{vol}B_{r_3}(x_0) - \mathrm{vol}B_{r_2}(x_0)}{\int_{r_2}^{r_3} sn_H(t)^{m-1}\,dt} \le \frac{\mathrm{vol}B_{r_1}(x_0)}{\int_0^{r_1} sn_H(t)^{m-1}\,dt}$$

with H defined in (2.49).

Proof. We first consider case (i) . A repeated use of Theorem 2.14 gives

$$\begin{aligned}
\mathrm{vol}B_{r_3}(o) - \mathrm{vol}B_{r_2}(o) &= \mathrm{vol}B_{r_3}(o) - \frac{\mathrm{vol}B_{r_2}(o)}{\int_0^{r_2} h(t)^{m-1}\,dt}\int_0^{r_2} h(t)^{m-1}\,dt \\
&\le \mathrm{vol}B_{r_3}(o) - \frac{\mathrm{vol}B_{r_3}(o)}{\int_0^{r_3} h(t)^{m-1}\,dt}\int_0^{r_2} h(t)^{m-1}\,dt \\
&= \frac{\mathrm{vol}B_{r_3}(o)}{\int_0^{r_3} h(t)^{m-1}\,dt}\int_{r_2}^{r_3} h(t)^{m-1}\,dt \\
&\le \frac{\mathrm{vol}B_{r_1}(o)}{\int_0^{r_1} h(t)^{m-1}\,dt}\int_{r_2}^{r_3} h(t)^{m-1}\,dt,
\end{aligned}$$

proving (2.61).

We now consider case (ii) . The desired inequality easily follows from (2.61). Indeed, observe that

$$B_R(x) \subset B_{R+d}(y) \setminus B_{d-R}(y).$$

Therefore, recalling that $d-R \geq r$ and using (2.61), we conclude that

$$\begin{aligned}\mathrm{vol}B_R(x) &\leq \mathrm{vol}B_{d+R}(y) - \mathrm{vol}B_{d-R}(y) \\ &\leq \int_{d-R}^{d+R} sn_H(t)^{m-1}\,dt \frac{\mathrm{vol}B_r(y)}{\int_0^r sn_H(t)^{m-1}\,dt}.\end{aligned}$$

□

Now, by way of example, assume that $(M, \langle\,,\rangle)$ is a complete manifold of non-negative Ricci curvature. Using (2.62) of Corollary 2.22 with $H=0$, so that $\mathrm{sn}_H(t)=t$, and radii $d=s-1$, $r=s-2$, and $R=1$, we obtain

$$\mathrm{vol}B_s(y) \geq \frac{(s-2)^m}{s^m-(s-2)^m}\mathrm{vol}B_1(x) \geq Cs,$$

for every $s >> 1$ and for some constant $C>0$. Namely, a complete manifold of non-negative Ricci curvature has at least a linear volume growth. This result was originally due to E. Calabi and S.T. Yau. Similar conclusions can be reached in a slightly more general situation. The starting point is that, in the above Ricci curvature assumption,

$$\frac{\mathrm{vol}B_R(x)}{\mathrm{vol}B_r(x)} \leq \left(\frac{R}{r}\right)^m$$

for every $x \in M$, $R \geq r > 0$. Thus, in particular, $(M, \langle\,,\rangle)$ enjoys the *doubling property*. This means that, for some (hence any) $\alpha > 1$, there is a constant $D_\alpha > 1$, such that

$$\mathrm{vol}B_{\alpha R}(x) \leq D_\alpha \mathrm{vol}B_R(x)$$

regardless of $x \in M$ and $R > 0$. We recall the following characterization of manifolds enjoying the doubling property.

Lemma 2.24. *A complete manifold $(M, \langle\,,\rangle)$ has the doubling property if and only if for some (hence any) $\alpha > 1$ there exists a constant $D_\alpha > 1$ such that the following holds. Let $A \subset M$ be a bounded set. Then, for every ball $B_r(x)$ centered at $x \in A$ and of radius r satisfying*

$$r < \delta_x(A) = \sup_{y\in A} \mathrm{dist}(x,y),$$

we have

$$\frac{\mathrm{vol}(A)}{\mathrm{vol}B_r(x)} \leq D_\alpha \left(\frac{\delta_x(A)}{r}\right)^{\log_\alpha D_\alpha}. \tag{2.63}$$

In particular, for every $0 < r \leq R$ we have

$$\frac{\mathrm{vol}B_R(x)}{\mathrm{vol}B_r(x)} \leq D_\alpha \left(\frac{R}{r}\right)^{\log_\alpha D_\alpha}.$$

Proof. We only need to show that if M has the doubling property, then the property holds. Assume therefore that $(M, \langle\, , \rangle)$ satisfies the doubling property with constants $\alpha, D_\alpha > 1$. Fix $B_r(x)$ as in the statement and consider $j \in \mathbb{N}$ satisfying

$$\alpha^{j-1} \leq \frac{\delta_x(A)}{r} \leq \alpha^j.$$

Note that

$$A \subset B_{\delta_x(A)}(x) \subset B_{r\alpha^j}(x).$$

Therefore, using the doubling property,

$$\mathrm{vol}(A) \leq \mathrm{vol}B_{r\alpha^j}(x) \leq D_\alpha^j \mathrm{vol}B_r(x).$$

To conclude, note that

$$D_\alpha^{j-1} = \left(\alpha^{j-1}\right)^{\log_\alpha D_\alpha} \leq \left(\frac{\delta_x(A)}{r}\right)^{\log_\alpha D_\alpha}. \qquad \square$$

Direct use of this Lemma gives a (quantitative) volume growth result which is true regardless of any monotonicity property.

Proposition 2.25. *Let $(M, \langle\, , \rangle)$ be a complete manifold with the doubling property and let $x \in M$. Then, there exist explicit constants $C = C(x, \alpha) > 0$ and $k = k(D_\alpha, \alpha) > 0$ such that, for every $R > 1$,*

$$\mathrm{vol}B_R(x) \geq CR^k. \tag{2.64}$$

Proof. Fix $x \in M$ and, for every j, choose $y_j \in \partial B_{\frac{\alpha^{j-1}+\alpha^j}{2}}(x)$. Then, we have the inclusions

$$B_{\alpha^j}(x) \setminus B_{\alpha^{j-1}}(x) \supset B_{\frac{\alpha^j - \alpha^{j-1}}{2}}(y_j); \qquad B_{\frac{\alpha^j+3\alpha^{j-1}}{2}}(y_j) \supset B_{\alpha^{j-1}}(x).$$

Using Lemma 2.24, we deduce

$$\begin{aligned}\mathrm{vol}B_{\alpha^j}(x) - \mathrm{vol}B_{\alpha^{j-1}}(x) &\geq \mathrm{vol}B_{\frac{\alpha^j-\alpha^{j-1}}{2}}(y_j) \\ &\geq D_\alpha^{-1}\left(\frac{\alpha^j - \alpha^{j-1}}{\alpha^j + 3\alpha^{j-1}}\right)^{\log_\alpha D_\alpha} \mathrm{vol}B_{\frac{\alpha^j+3\alpha^{j-1}}{2}}(y_j) \\ &\geq D_\alpha^{-1}\left(\frac{\alpha-1}{\alpha+3}\right)^{\log_\alpha D_\alpha} \mathrm{vol}B_{\alpha^{j-1}}(x).\end{aligned}$$

Therefore

$$\mathrm{vol}B_{\alpha^j}(x) \geq E\mathrm{vol}B_{\alpha^{j-1}}(x),$$

where we have set

$$E = 1 + \left(\frac{\alpha-1}{\alpha^2+3\alpha}\right)^{\log_\alpha D_\alpha} > 1.$$

Iterating this inequality j-times gives

$$\mathrm{vol}B_{\alpha^j}(x) \geq E^j \mathrm{vol}B_1(x),$$

and choosing $k = \log_\alpha E > 0$ we finally obtain

$$\mathrm{vol}B_{\alpha^j}(x) \geq \alpha^{jk}\mathrm{vol}B_1(x) = \left(\alpha^{j+1}\right)^k \frac{\mathrm{vol}B_1(x)}{\alpha^k}. \tag{2.65}$$

Now, for any given $R > 1$, let j satisfy $\alpha^j < R < \alpha^{j+1}$. From (2.65) we conclude the validity of (2.64) with $C = \mathrm{vol}B_1(x)/\alpha^k$. □

The last result of the section is the following impressive volume growth property of complete manifolds with almost non-negative Ricci curvature (in the sense specified in (2.66), (2.67) see also Corollary 2.27 below). It is a contribution of P. Li and R. Schoen, [95], and P. Li and M. Ramachandran, [98]. Its proof relies heavily on the work by Cheeger-Gromov-Taylor [33].

Theorem 2.26. *For every $m \geq 2$ there exists a constant $\varepsilon = \varepsilon(m) > 0$ such that the following holds. Let $(M, \langle\,,\rangle)$ be a complete manifold satisfying*

$$\mathrm{Ric} \geq -(m-1)\,G(r(x)) \quad \text{on } M \tag{2.66}$$

where $G(t) \in C^0([0,+\infty))$ is a positive function such that

$$G(t) = \frac{\varepsilon^2}{t^2} \text{ for } t >> 1, \tag{2.67}$$

and $r(x) = \mathrm{dist}(x, o)$, for some reference origin $o \in M$. Then, for every $d > 1$,

$$\mathrm{vol}\left(B_{\frac{r(x)}{d}}(x)\right) \to +\infty, \text{ as } r(x) \to +\infty. \tag{2.68}$$

Proof. We divide the quite involved proof in several steps.

First Step. Fix

$$0 < \xi < 1, \qquad \beta > \frac{2}{\left(2^{1/m} - 1\right)} > 1.$$

If $\varepsilon = \varepsilon(m, \xi, \beta) > 0$ is sufficiently small, then the following holds: For every $x \in M$ choose a geodesic γ parametrized by arc length which realizes the distance between $o = \gamma(0)$ and $x = \gamma(r(x))$. Define a set of values $t_0 < t_1 < \cdots < t_k \leq r(x)$ by

$$t_0 = 0; \qquad t_i = (\beta+1)\sum_{j=0}^{i-1}\beta^j, \tag{2.69}$$

t_k being the largest value so that $t_k < r(x)$. Define $x_j = \gamma(t_j)$. Then, if k is large enough,

$$\mathrm{vol}B_{\beta^k}(x_k) \geq C_1\left(\frac{\beta^m}{(\beta+2)^m - \beta^m}\right)^{k(1-\xi)} \mathrm{vol}B_1(o), \tag{2.70}$$

for some constant $C = C(\beta) > 0$.

Indeed, observe that $\{x_i\}$ form a set of points along γ with the property that

$$\begin{cases} \text{(i)} & r(x_i) = t_i = (\beta+1)(\beta-1)^{-1}\left(\beta^i - 1\right), \\ \text{(ii)} & \operatorname{dist}(x_i, x_{i+1}) = t_{i+1} - t_i = \beta^i + \beta^{i+1}, \\ \text{(iii)} & \operatorname{dist}(x_k, x) = r(x) - t_k < t_{k+1} - t_k = \beta^k + \beta^{k+1}. \end{cases} \tag{2.71}$$

Thus, the geodesic balls $\left\{B_{\beta^i}(x_i)\right\}$ are disjoint and their closures cover the set $\gamma\left([0, t_k + \beta^k)\right)$. Furthermore

$$B_{\beta^{i-1}}(x_{i-1}) \subset B_{\beta^i + 2\beta^{i-1}}(x_i) \setminus B_{\beta^i}(x_i). \tag{2.72}$$

Setting

$$H_i^2 = \max_{B_{\beta^i+2\beta^{i-1}}(x_i)} G(r(y)),$$

from Corollary 2.22 (i) and (2.72), we deduce that

$$\begin{aligned} \operatorname{vol} B_{\beta^i}(x_i) &\geq T_i \left\{\operatorname{vol} B_{\beta^i+2\beta^{i-1}}(x_i) - \operatorname{vol} B_{\beta^i}(x_i)\right\} \\ &\geq T_i \operatorname{vol} B_{\beta^{i-1}}(x_{i-1}) \end{aligned}$$

where

$$\begin{aligned} T_i &= \frac{\displaystyle\int_0^{\beta^i} H_i^{-1} \sinh^{m-1}(H_i t)\, dt}{\displaystyle\int_{\beta^i}^{\left(\beta^i+2\beta^{i-1}\right)} H_i^{-1} \sinh^{m-1}(H_i t)\, dt} \\ &= \frac{\displaystyle\int_0^{\beta^i H_i} \sinh^{m-1} t\, dt}{\displaystyle\int_{\beta^i H_i}^{\left(\beta^i+2\beta^{i-1}\right)H_i} \sinh^{m-1} t\, dt}. \end{aligned} \tag{2.73}$$

Iterating this inequality we conclude that

$$\operatorname{vol} B_{\beta^k}(x_k) \geq \prod_{i=1}^{k} T_i \operatorname{vol} B_1(o). \tag{2.74}$$

The validity of (2.70) will follow once we show that, for every i, hence k, large enough,

$$T_i \geq \left(\frac{\beta^m}{(\beta+2)^m - \beta^m}\right)^{1-\xi}. \tag{2.75}$$

Note that, according to (2.66) and (2.67), for sufficiently large $i \geq i_0 = i_0(\beta, m)$, we have

$$\operatorname{Ric} \geq -(m-1)H_i^2 \quad \text{on} \quad B_{\beta^i+2\beta^{i-1}}(x_i)$$

with

$$H_i = \frac{\varepsilon}{\operatorname{dist}\left(o, \partial B_{\beta^i + 2\beta^{i-1}}\left(x_i\right)\right)} = \varepsilon \frac{(\beta - 1)}{2\beta^{i-1} - \beta - 1}.$$

In particular, for every $i \geq i_0$,

$$\left(\beta^i + 2\beta^{i-1}\right) H_i = \varepsilon \frac{(\beta + 2)(\beta - 1)}{2 - \frac{\beta+1}{\beta^{i+1}}} \leq \varepsilon A,$$

where

$$A = A\left(\beta, i_0\right) = \frac{(\beta - 1)\left(\beta + 2\right)}{2 - \frac{\beta+1}{\beta^{i_0 - 1}}}. \tag{2.76}$$

Let $0 < \delta = \delta\left(m, \beta, \xi\right) < 1$ be defined by the equation

$$\left(\frac{1}{1+\delta}\right)^{m-1} = \left\{\frac{(\beta+2)^m - \beta^m}{\beta^m}\right\}^{\xi}.$$

The definition is consistent because our assumption that $\beta > 2/\left(2^{1/m} - 1\right)$ guarantees that the right-hand side is < 1. If $\varepsilon = \varepsilon\left(m, \delta\right) > 0$ is chosen small enough, we can approximate

$$t \leq \sinh t \leq t\left(1 + \delta\right) \text{ on } [0, \varepsilon A]$$

which in turn, used in (2.73), gives

$$\begin{aligned} T_i &\geq \left(\frac{1}{1+\delta}\right)^{m-1} \frac{(\beta^i H_i)^m}{[(\beta^i + 2\beta^{i-1})H_i]^m - (\beta^i H_i)^m} \\ &= \left(\frac{1}{1+\delta}\right)^{m-1} \frac{\beta^m}{(\beta+2)^m - \beta^m}, \end{aligned}$$

proving (2.75).

Second Step. There exists a constant $0 < \alpha = \alpha\left(\beta\right) < 1$ such that, for $r\left(x\right) >> 1$, hence $k >> 1$,

$$\operatorname{vol}\left(B_{\alpha r(x)}\left(x\right)\right) \geq \operatorname{vol} B_{\beta^k}\left(x_k\right). \tag{2.77}$$

To see this, first observe that, by the triangle inequality,

$$B_{\operatorname{dist}(x_k, x) + \beta^k}\left(x\right) \supset B_{\beta^k}\left(x_k\right). \tag{2.78}$$

On the other hand, x and x_k lie on the minimizing geodesic γ from o to x, so that

$$\frac{\operatorname{dist}\left(x_k, x\right) + \beta^k}{r\left(x\right)} = \frac{\operatorname{dist}\left(x_k, x\right) + \beta^k}{\operatorname{dist}\left(x_k, x\right) + r\left(x_k\right)}.$$

Since, by (2.71) (i), $r\left(x_k\right) \geq \beta^k$, the function

$$t \mapsto \frac{t + \beta^k}{t + r\left(x_k\right)}$$

is monotone increasing. Whence, using (2.71) (iii),

$$\begin{aligned}\frac{\operatorname{dist}(x_k,x)+\beta^k}{r(x)} &= \frac{\operatorname{dist}(x_k,x)+\beta^k}{\operatorname{dist}(x_k,x)+r(x_k)}\\ &\le \frac{2\beta^k+\beta^{k+1}}{\beta^k+\beta^{k+1}+\frac{(\beta+1)(\beta^k-1)}{\beta-1}}\\ &= \frac{2+\beta}{1+\beta+\frac{\beta+1}{\beta-1}\left(1-\beta^{-k}\right)}.\end{aligned}$$

Note that

$$\lim_{k\to+\infty}\frac{2+\beta}{1+\beta+\frac{\beta+1}{\beta-1}\left(1-\beta^{-k}\right)} = \frac{2+\beta}{1+\beta+\frac{\beta+1}{\beta-1}} < 1.$$

Therefore, up to choosing k (hence $r(x)$) large enough, say $k \ge k_0 = k_0(\beta) > 0$, we have

$$\frac{\operatorname{dist}(x_k,x)+\beta^k}{r(x)} \le \alpha,$$

with

$$\alpha = \frac{2+\beta}{1+\beta+\frac{\beta+1}{\beta-1}\left(1-\beta^{-k_0}\right)} < 1.$$

It follows that

$$B_{\alpha r(x)}(x) \supset B_{\operatorname{dist}(x_k,x)+\beta^k}(x)$$

which in turn, combined with (2.78), proves (2.77).

Third Step. Let $d > 1$ be fixed. There exists a constant $C_2 = C_2(\alpha, d, \varepsilon) > 0$ such that, for $r(x) >> 1$,

$$\operatorname{vol} B_{\frac{r(x)}{d}}(x) \ge C_2 \operatorname{vol}\left(B_{\alpha r(x)}(x)\right). \tag{2.79}$$

Indeed, suppose $d > 1/\alpha$, for otherwise there would be nothing to prove. Note that, for every $y \in B_{\alpha r(x)}(x)$,

$$\operatorname{Ric}(y) \ge -(m-1)\frac{\varepsilon^2}{r(x)^2(1-\alpha)^2},$$

provided $r(x) >> 1$. Therefore, by Corollary 2.15 with $H = \varepsilon / r(x)(1-\alpha)$ we deduce

$$\frac{\operatorname{vol} B_{\frac{r(x)}{d}}(x)}{\operatorname{vol}\left(B_{\alpha r(x)}(x)\right)} \ge C_2 := \frac{\int_0^{\frac{\varepsilon}{d(1-\alpha)}} \sinh^{m-1}(t)\,dt}{\int_0^{\frac{\alpha\varepsilon}{(1-\alpha)}} \sinh^{m-1}(t)\,dt}.$$

Fourth Step. Putting together (2.70), (2.77) and (2.79) gives

$$\operatorname{vol} B_{\frac{r(x)}{d}}(x) \ge C_3\left(\frac{\beta^m}{(\beta+2)^m-\beta^m}\right)^{k(1-\xi)} \operatorname{vol} B_1(o)$$

with $C_3 = C_1 C_2 > 0$. Since, by our choice of $\beta > 2/\left(2^{1/m}-1\right)$,

$$\left(\frac{\beta^m}{(\beta+2)^m - \beta^m}\right) > 1,$$

the validity of (2.68) follows by letting $r(x)$, hence k, tend to $+\infty$. □

We point out the following consequence of the above proof.

Corollary 2.27. *Suppose we are given*

$$m \geq 2, \qquad 0 < \xi < 1, \qquad \beta > \frac{2}{\left(2^{1/m}-1\right)} > 1, \qquad d > 1.$$

Then, there exists $\varepsilon = \varepsilon(m, \xi, \beta) > 0$ sufficiently small such that the following holds. Let $(M, \langle\, ,\rangle)$ be a complete manifold satisfying the curvature conditions (2.66) *and* (2.67) *above. Then*

$$\mathrm{vol} B_{\frac{r(x)}{d}}(x) \geq C r(x)^{\log_\beta E}, \ \text{ as } r(x) \to +\infty,$$

where

$$E = \left(\frac{\beta^m}{(\beta+2)^m - \beta^m}\right)^{(1-\xi)} > 1$$

and $C = C(m, \beta, \xi, d) > 0$ is a suitable constant.

Proof. Let $x \in M$ be such that $r(x) >> 1$. Recall that, by definitions of k and $\{t_j\}$ in the First Step, we have

$$t_{k+1} = \frac{(\beta+1)\left(\beta^{k+1}-1\right)}{\beta - 1} \geq r(x).$$

Therefore

$$k \geq \log_\beta r(x) + \log_\beta\left(\frac{\beta-1}{\beta+1}\right) - 1.$$

On the other hand, in the Fourth Step we obtained, for $r(x) >> 1$,

$$\mathrm{vol} B_{\frac{r(x)}{d}}(x) \geq C_3 \left(\frac{\beta^m}{(\beta+2)^m - \beta^m}\right)^{k(1-\xi)} \mathrm{vol} B_1(o).$$

It follows that

$$\mathrm{vol} B_{\frac{r(x)}{d}}(x) \geq C E^{\log_\beta r(x)},$$

where we have set

$$C = C_3 \mathrm{vol} B_1(o) \left(\frac{\beta^m}{(\beta+2)^m - \beta^m}\right)^{(1-\xi)\left\{\log\left(\frac{\beta-1}{\beta+1}\right)-1\right\}}$$

and

$$E = \left(\frac{\beta^m}{(\beta+2)^m - \beta^m}\right)^{(1-\xi)}.$$

□

2.3 A monotonicity formula for volumes

In the previous sections we obtained volume growth estimates under curvature conditions. In the present section we replace the curvature conditions with the assumption that the manifold M is isometrically immersed in an appropriate ambient space N.

In this context, we recall that if M is an m-dimensional manifold minimally immersed into $\mathbb{R}^n$, denoting by $B_R^{\mathbb{R}^n}(p)$ the Euclidean ball of radius R centered at $p \in \mathbb{R}^n$, then the monotonicity formula states that

$$\frac{\operatorname{vol}\left(M \cap B_R^{\mathbb{R}^n}(x_o)\right)}{R^m}$$

is a non-decreasing function of R (see, e.g., [38]), and therefore

$$\operatorname{vol}\left(M \cap B_R^{\mathbb{R}^n}(x_o)\right) \geq c_m R^m,$$

where c_m is the volume of the unit sphere in $\mathbb{R}^m$. Such estimates have proven to be important in a number of applications, including the regularity theory of harmonic maps. A variation of the argument shows that similar monotonicity formulas hold replacing the "exterior ball" $B_R^{\mathbb{R}^n}(x_o)$ with the intrinsic ball $B_R(p)$, $p \in M$. As a matter of fact, we have the following slightly more general result concerning the volume growth of manifolds which admit a bi-Lipschitz, harmonic immersion into a Cartan–Hadamard space. Before stating the theorem, we recall that a geodesic ball $B_\rho^N(q)$ in N is said to be regular if it does not intersect the cut locus of q, and, having denoted by k an upper bound for the sectional curvature of N on $B_\rho^N(q)$, one has $\sqrt{k}\rho < \pi/2$.

Theorem 2.28. *Let $(M, \langle\,,\rangle)$ be a complete, non-compact Riemannian manifold of dimension $\dim M = m$, immersed into a complete manifold $(N, (\,,))$ via $f : M \to N$. Assume that either N is Cartan–Hadamard, or that $f(M)$ is contained in a regular geodesic ball. Furthermore, suppose that the immersion f is harmonic and bi-Lipschitz so that there exist positive constants A and B such that*

$$A\langle X, X\rangle \leq \big(df(X), df(X)\big) \leq B\langle X, X\rangle \tag{2.80}$$

for every $X \in TM$. Fix an origin $o \in M$ and denote $\rho(y) = \operatorname{dist}_N(y, f(o))$. Next, set

$$k = \sup_{f(M)} {}^N\mathrm{Sect}$$

and define sn_k to be the unique solution of the Cauchy problem

$$\begin{cases} \ddot{\mathrm{sn}}_k + k\,\mathrm{sn}_k = 0, \\ \mathrm{sn}_k(0) = 0;\ \dot{\mathrm{sn}}_k(0) = 1. \end{cases}$$

Set also

$$\mathrm{cn}_k(t) = \dot{\mathrm{sn}}_k(t).$$

Then, there exists a constant B_1,

$$B_1 \begin{cases} = 1 & \textit{if } f \textit{ is isometric,} \\ \leq \sqrt{mB} & \textit{otherwise,} \end{cases} \tag{2.81}$$

such that

$$R \longmapsto \frac{\int_{B_R(o)} \mathrm{cn}_k(\rho \circ f)}{\left\{\mathrm{sn}_k\left(\sqrt{B}R\right)\right\}^{\frac{mA}{B_1\sqrt{B}}}} \tag{2.82}$$

is a non-decreasing function. In particular, in case N *is Cartan–Hadamard, for every* $R \geq R_0 > 0$ *it holds that*

$$vol B_R(o) \geq C \begin{cases} R^{\frac{mA}{B_1\sqrt{B}}} & \textit{if } k = 0, \\ \sinh^{\frac{mA}{B_1\sqrt{B}}-1}\left(\sqrt{-kB}R\right)\tanh\left(\sqrt{-kB}R\right) & \textit{if } k < 0, \end{cases} \tag{2.83}$$

for a suitable constant $C = C(R_0, k, A, B, m) > 0$.

Proof. Consider the function

$$F(x) = \mathrm{in}_k \circ \rho \circ f(x)$$

where we have set

$$\mathrm{in}_k(t) = \int_0^t \mathrm{sn}_k(s)\, ds.$$

The chain rule for the Hessian yields

$$^M\Delta F(x) = \sum_{i=1}^m {}^N\mathrm{Hess}(\mathrm{in}_k \circ \rho)(df(e_i), df(e_i)) + d(\mathrm{in}_k \circ \rho)(\tau(f))$$

where $\{e_i\}$ is any local orthonormal frame filed in $(M, \langle\, ,\rangle)$. Since f is harmonic, we have

$$\tau(f) = 0,$$

and therefore

$$^M\Delta F(x) = \sum_{i=1}^m {}^N\mathrm{Hess}(\mathrm{in}_k \circ \rho)(df(e_i), df(e_i)). \tag{2.84}$$

Note that, by Hessian comparison,

$$\begin{aligned} {}^N\mathrm{Hess}(\mathrm{in}_k \circ \rho) &= \ddot{\mathrm{in}}_k(\rho)\, d\rho \otimes d\rho + \dot{\mathrm{in}}_k(\rho)\ {}^N\mathrm{Hess}(\rho) \\ &= \mathrm{cn}_k(\rho)\, d\rho \otimes d\rho + \mathrm{sn}_k(\rho)\ {}^N\mathrm{Hess}(\rho) \\ &\geq \mathrm{cn}_k(\rho)\, d\rho \otimes d\rho + \mathrm{sn}_k(\rho)\ \frac{\mathrm{cn}_k(\rho)}{\mathrm{sn}_k(\rho)}\{(\, ,\,) - d\rho \otimes d\rho\} \\ &= \mathrm{cn}_k(\rho)(\, ,\,). \end{aligned}$$

Therefore, using the fact that f is bi-Lipschitz and $\mathrm{cn}_k(\rho) > 0$,

$$
\begin{aligned}
RHS(2.84) &\geq \sum_{i=1}^{m} \mathrm{cn}_k(\rho(f)) \big(df(e_i), df(e_i)\big) \\
&\geq mA\ \mathrm{cn}_k(\rho(f)).
\end{aligned}
$$

It follows that

$$
{}^{M}\Delta F \geq mA\ \mathrm{cn}_k(\rho(f)). \tag{2.85}
$$

Starting from (2.85), we now apply the divergence theorem on the (intrinsic) geodesic ball $B_R(o)$ of M and use Schwarz inequality and Gauss' lemma to obtain

$$
\begin{aligned}
mA \int_{B_R(o)} \mathrm{cn}_k(\rho(f)) &\leq \int_{B_R(o)} {}^{M}\Delta F \\
&= \int_{\partial B_R(o)} \left\langle {}^{M}\nabla F, \frac{\partial}{\partial r} \right\rangle \\
&\leq \int_{\partial B_R(o)} \left| {}^{M}\nabla F \right|_M .
\end{aligned} \tag{2.86}
$$

It is readily seen that

$$
\left| {}^{M}\nabla F \right|_M \leq B_1\ \mathrm{sn}_k(\rho(f)) \tag{2.87}
$$

with B_1 satisfying (2.81). Indeed,

$$
\begin{aligned}
\left| {}^{M}\nabla F \right|_M &= \dot{\mathrm{in}}_k(\rho(f)) \left| {}^{M}\nabla(\rho \circ f) \right|_M \\
&= \mathrm{sn}_k(\rho(f)) \left| {}^{M}\nabla(\rho \circ f) \right|_M .
\end{aligned}
$$

Moreover, with respect to local orthonormal frames $\{e_i\}$ on M, $\{E_A\}$ on N with dual frames $\{\theta^i\}$,$\{\Theta^A\}$, we have

$$
d\rho = \rho_A \Theta^A,\ df = f_i^A \theta^i \otimes E_A
$$

so that

$$
{}^{M}\nabla(\rho \circ f) = \rho_A f_i^A e_i.
$$

This latter implies

$$
\left| {}^{M}\nabla(\rho \circ f) \right|_M = |D\mathbf{y}|_{\mathbb{R}^m}
$$

where

$$
D = \left(f_i^A\right) \in M_{n,m}(\mathbb{R}), \qquad \mathbf{y} = (\rho_A)^t \in \mathbb{R}^n.
$$

Clearly, in case f is isometric, $D = \left(\delta_i^A\right)$, while, in the general bi-Lipschitz case,

$$
|D| = \sqrt{\sum \langle df(e_i), df(e_i)\rangle_N} = |df| \leq \sqrt{mB}.
$$

Now recall that, by Gauss' lemma,

$$|\mathbf{y}|_{\mathbb{R}^n} = \left|{}^N\nabla\rho\right|_N = 1,$$

and therefore

$$|D\mathbf{y}|_{\mathbb{R}^m} \leq \sup_{|\mathbf{x}|=1} |D\mathbf{x}|_{\mathbb{R}^m} \begin{cases} = 1 & f \text{ isometric,} \\ \leq \sqrt{mB} & \text{otherwise.} \end{cases}$$

This proves the claimed inequality (2.87) with

$$B_1 = \sup_M \sup_{|\mathbf{x}|=1} |D\mathbf{x}|_{\mathbb{R}^m}.$$

Inserting (2.87) into (2.86) gives

$$mA \int_{B_R(o)} \mathrm{cn}_k\left(\rho\left(f\right)\right) \leq B_1 \int_{\partial B_R(o)} \mathrm{sn}_k\left(\rho\left(f\right)\right). \tag{2.88}$$

Let us elaborate the RHS of this latter. Observe that the positive function

$$t \longmapsto \frac{\mathrm{sn}_k\left(t\right)}{\mathrm{cn}_k\left(t\right)} \text{ is increasing.} \tag{2.89}$$

Furthermore, from the very definitions of dist_N and dist_M, we see that

$$\rho\left(f\left(x\right)\right) \leq \sqrt{B}\,\mathrm{dist}_M\left(x, o\right). \tag{2.90}$$

Indeed,

$$\begin{aligned} \rho\left(f\left(x\right)\right) &= \mathrm{dist}_N\left(f\left(x\right), f\left(o\right)\right) \\ &= \inf_{\substack{\Gamma\subset N \\ \Gamma(0)=f(o),\ \Gamma(1)=f(x)}} \int_0^1 \left|\dot{\Gamma}\left(t\right)\right|_N \\ &\leq \inf_{\substack{\gamma\subset M \\ \gamma(0)=o,\ \gamma(1)=x}} \int_0^1 \left|\frac{d}{dt} f\circ\gamma\left(t\right)\right|_N \\ &\leq \sqrt{B} \inf_{\substack{\gamma\subset M \\ \gamma(0)=o,\ \gamma(1)=x}} \int_0^1 \left|\dot{\gamma}\left(t\right)\right|_M \\ &= \sqrt{B}\,\mathrm{dist}_M\left(x, o\right). \end{aligned}$$

Combining (2.90) with (2.89) we deduce

$$\begin{aligned} RHS\,(2.88) &= \int_{\partial B_R(o)} \frac{\mathrm{sn}_k\left(\rho\left(f\right)\right)}{\mathrm{cn}_k\left(\rho\left(f\right)\right)} \mathrm{cn}_k\left(\rho\left(f\right)\right) \\ &\leq B_1 \frac{\mathrm{sn}_k\left(\sqrt{B}R\right)}{\mathrm{cn}_k\left(\sqrt{B}R\right)} \int_{\partial B_R(o)} \mathrm{cn}_k\left(\rho\left(f\right)\right) \end{aligned}$$

so that

$$mA\int_{B_R(o)} \mathrm{cn}_k\left(\rho\left(f\right)\right) \leq B_1 \frac{\mathrm{sn}_k\left(\sqrt{B}R\right)}{\mathrm{cn}_k\left(\sqrt{B}R\right)} \int_{\partial B_R(o)} \mathrm{cn}_k\left(\rho\left(f\right)\right). \tag{2.91}$$

Using the co-area formula, (2.91) can be written in the form

$$\frac{d}{dR}\log\left(\frac{\int_{B_R(o)} \mathrm{cn}_k\left(\rho\left(f\right)\right)}{\mathrm{sn}_k^{\frac{mA}{B_1\sqrt{B}}}\left(\sqrt{B}R\right)}\right) \geq 0 \tag{2.92}$$

which implies the validity of (2.82). □