Quantum p-adic Spaces and Quantum p-adic Groups

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To the memory of Sasha Reznikov

Abstract. We discuss examples of non-commutative spaces over non-archimedean fields. Those include non-commutative and quantum affinoid algebras, quantized K3 surfaces and quantized locally analytic p-adic groups.

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1. Introduction

The paper is devoted to examples of quantum spaces over non-archimedean fields and is, in a sense, a continuation of [So1] (part of the material is borrowed from the loc. cit). There are three classes of examples which I discuss in this paper: quantum affinoid spaces, quantum non-archimedean Calabi-Yau varieties and quantum p-adic groups. Let us recall the definitions and discuss the contents of the paper.

Quantum affinoid algebras are defined similarly to the "classical case" $q = 1$. It is a special case of a more general notion of non-commutative affinoid algebra introduced in [So1]. Let k be a Banach field and $k\langle\langle T_1, \ldots, T_n\rangle\rangle$ be the algebra of formal series in free variables $T_1, \ldots, T_n$. For each $r = (r_1, \ldots, r_n), r_i \geq 0, 1 \leq i \leq n$ we define a subspace $k\langle\langle T_1, \ldots, T_n\rangle\rangle_r$ consisting of series $f = \sum_{i_1,\ldots,i_m} a_{i_1,\ldots,i_m} T_{i_1} \cdots T_{i_m}$ such that $\sum_{i_1,\ldots,i_m} |a_{i_1,\ldots,i_m}| r_{i_1} \cdots r_{i_m} < +\infty$. Here the summation is taken over all sequences $(i_1, \ldots, i_m), m \geq 0$ and $|\bullet|$ denotes the norm in k. In this paper we consider the case when k is a valuation field (i.e., a Banach field with respect to a multiplicative non-archimedean norm). In the non-archimedean case the convergency condition is replaced by the following one: $max \, |a_{i_1,\ldots,i_m}| r_{i_1} \cdots r_{i_m} \to 0$ as $i_1 + \cdots + i_m \to \infty$. Clearly each $k\langle\langle x_1, \ldots, x_n\rangle\rangle_r$ is a Banach algebra called the algebra of analytic functions on a non-commutative k-polydisc $E_{NC}(0, r)$ centered at zero and having the (multi)radius $r = (r_1, \ldots, r_n)$.

The norm is given by $max |a_{i_1,\ldots,i_m}| r_{i_1} \cdots r_{i_m}$. A *non-commutative k-affinoid alge-bra* is an admissible noetherian quotient of this algebra (cf. [Be1], Definition 2.1.1). Let us fix $q \in k^*$ such that $|q| = 1$, and $r = (r_1,\ldots,r_n), r_i \geq 0$. A *quantum k-affinoid algebra* is a special case of the previous definition. It is defined as an admissible quotient of the algebra $k\{T\}_{q,r} := k\{T_1,\ldots,T_n\}_{q,r}$ of the series $f = \sum_{l=(l_1,\ldots,l_n)\in\mathbf{Z}_+^n} a_l T_1^{l_1} \cdots T_n^{l_n}$ such that $a_l \in k$, $T_i T_j = q T_j T_i, j < i$, and $max|a_l| r^{|l|} \to 0$ as $|l| := l_1 + \cdots + l_n \to \infty$. The latter is also called the algebra of analytic functions on the quantum polydisc $E_q(0,r)$. It is less useful notion than the one of non-commutative affinoid algebra since there are few two-sided closed ideals in the algebra $k\{T_1,\ldots,T_n\}_{q,r}$. Nevertheless quantum affinoid algebras appear in practice (e.g., in the case of quantum Calabi-Yau manifolds considered below). In the case when all $r_i = 1$ we speak about *strictly k-affinoid* non-commutative (resp. quantum) algebras, similarly to [Be1]. Any non-archimedean extension K of k gives rise to a non-commutative (resp. quantum) affinoid k-algebra, cf. loc.cit. There is a generalization of quantum affinoid algebras which we will also call quantum affinoid algebras. Namely, let $Q = ((q_{ij}))$ be an $n \times n$ matrix with entries from k such that $q_{ij}q_{ji} = 1, |q_{ij}| = 1$ for all i,j. Then we define the quantum affinoid algebra as an admissible quotient of the algebra $k\{T_1,\ldots,T_n\}_{Q,r}$. The latter defined similarly to $k\{T_1,\ldots,T_n\}_{q,r}$, but now we use polynomials in variables $T_i, 1 \leq i \leq n$ such that $T_i T_j = q_{ij} T_j T_i$. One can think of $k\{T_1,\ldots,T_n\}_{Q,r}$ as of the quotient of $k\langle\langle T_i, t_{ij}\rangle\rangle_{r,\mathbf{1}_{ij}}$, where $1 \leq i,j \leq n$ and $\mathbf{1}_{ij}$ is the unit $n \times n$ matrix, by the two-sided ideal generated by the relations

$$t_{ij}t_{ji} = 1, T_i T_j = t_{ij} T_j T_i, t_{ij}a = at_{ij},$$

for all indices i,j and all $a \in k\langle\langle T_i, t_{ij}\rangle\rangle_{r,\mathbf{1}_{ij}}$. In other words, we treat q_{ij} as variables which belong to the center of our algebra and have the norms equal to one. Having the above-discussed generalizations of affinoid algebras we can consider their Berkovich spectra (sets of multiplicative seminorms). Differently from the commutative case, the theory of non-commutative and quantum analytic spaces is not developed yet (see discussion in [So1]).

Quantum Calabi-Yau manifolds provide examples of topological spaces equipped with rings of non-commutative affinoid algebras, which are quantum affinoid outside of a "small" subspace. More precisely, let $k = \mathbf{C}((t))$ be the field of Laurent series, equipped with its standard valuation (order of the pole) and the corresponding non-archimedean norm. Quantum Calabi-Yau manifold of dimension n over $\mathbf{C}((t))$, is defined as a ringed space $(X, \mathcal{O}_{q,X})$ which consists of an analytic Calabi-Yau manifold X of dimension n over $\mathbf{C}((t))$ and a sheaf of $\mathbf{C}((t))$-algebras $\mathcal{O}_{q,X}$ on X such that $\mathcal{O}_{q,X}(U)$ is a non-commutative affinoid algebra for any affinoid $U \subset X$ and the following two conditions are satisfied:

1) X is a $\mathbf{C}((t))$-analytic manifold corresponding to a maximally degenerate algebraic Calabi-Yau manifold X^{alg} of dimension n (see [KoSo2] for the definitions).

2) Let $Sk(X)$ be the skeleton of X defined in [KoSo1], and let us choose a projection $\pi : X \to Sk(X)$ described in the loc.cit. Then the direct image $\pi_*(\mathcal{O}_{q,X})$ is locally isomorphic (outside of a topological subvariety of the codimension at least two) to the sheaf of $\mathbf{C}((t))$-algebras $\mathcal{O}_{q,\mathbf{R}^n}^{can}$ on $\mathbf{R}^n$ which is characterized by the property that for any open connected subset $U \subset \mathbf{R}^n$ we have $\mathcal{O}_{q,\mathbf{R}^n}^{can}(U) = \{\sum_{l \in \mathbf{Z}^n} a_l z^l\}$ such that $a_l \in \mathbf{C}((t))$ and $\sup_{l \in \mathbf{Z}^n}(\log |a_l| + \langle l, x \rangle) < \infty$ for any $x \in U$. Here $\langle (l_1, \ldots, l_n), (x_1, \ldots, x_n) \rangle = \sum_{1 \le i \le n} l_i x_i$.

For a motivation of this definition see [KoSo1-2] in the "commutative" case $q = 1$. Roughly speaking, in that case the above definition requires the Calabi-Yau manifold X to be locally isomorphic (outside of a "small" subspace) to an analytic torus fibration $\pi_{can} : (\mathbf{G}_m^{an})^n \to \mathbf{R}^n$, where on $\mathbf{C}((t))$-points the canonical projection π_{can} is the "tropical" map $(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|)$. This is a "rigid-analytic" implementation of the Strominger-Yau-Zaslow conjecture in Mirror Symmetry (see [KoSo1-2] for more on this topic). In present paper we discuss the case $n = 2$, essentially following [KoSo1], [So1]. Perhaps the higher-dimensional case can be studied by the technique developed in a recent paper [GroSie1] (which in some sense generalizes to the higher-dimensional case ideas of [KoSo1]). We plan to return to this problem in the future.

Finally, we discuss the notion of p-adic quantum group. Quantum groups over p-adic fields and their representations will be discussed in more detail in the forthcoming paper [So2]. We have borrowed some material from there. Recall that quantum groups are considered in the literature either in the framework of algebraic groups or in some special examples of locally compact groups over $\mathbf{R}$. In the case of groups over $\mathbf{R}$ or $\mathbf{C}$ there is the following problem: how to describe, say, smooth or analytic (or rapidly descreasing) functions on a complex or real Lie group in terms of the representation theory of its enveloping algebra? Finite-dimensional representations give rise to the algebra of regular functions (via Peter-Weyl theorem), but more general classes of functions are not so easy to handle. The case of p-adic fields is different for two reasons. First, choosing a good basis in the enveloping algebra, we can consider series with certain restrictions on the growth of norms of their coefficients. This allows us to describe a basis of compact open neighbourhoods of the unit of the corresponding p-adic group. Furthermore, combining the ideas of [ShT1] with the approach of [So3] one can define the algebra of *locally analytic* functions on a compact p-adic group as a certain completion of the coordinate ring of the group. Dualizing, one obtains the algebra of locally-analytic distributions. According to [ShT1] modules over the latter provides an interesting class of p-adic representations, which contains, e.g., principal series representations. The above considerations can be "quantized", giving rise to quantum locally-analytic groups.

The present paper contains a discussion of the above-mentioned three classes of examples of non-commutative spaces. The proofs are omitted and will appear in separate publications. We should warn the reader that the paper does not present

a piece of developed theory. This explains its sketchy character. My aim is to show interesting classes of non-archimedean non-commutative spaces which can be obtained as analytic non-commutative deformations of the corresponding classical spaces. They deserve further study (for the quantum groups case see [So2]).

When talking about rigid analytic spaces we use the approach of Berkovich, which seems to be more suitable in the non-commutative framework. For this reason our terminology is consistent with [Be1].

2. Quantum affinoid algebras

Let k be a valuation field.

We recall here the definition already given in the Introduction. Let us fix $r = (r_1, \ldots, r_n) \in \mathbf{R}_{\geq 0}^n$. We start with the algebra $k\langle T \rangle := k\langle T_1, \ldots, T_n \rangle$ of polynomials in n free variables and consider its completion $k\langle\langle T \rangle\rangle_r$ with respect to the norm $|\sum_{\lambda \in P(\mathbf{Z}_+^n)} a_\lambda T^\lambda| = \max_\lambda |a_\lambda| r^\lambda$. Here $P(\mathbf{Z}_+^n)$ is the set of finite paths in $\mathbf{Z}_+^n$ starting at the origin, and $T^\lambda = T_1^{\lambda_1} T_2^{\lambda_2} \cdots$ for the path which moves λ_1 steps in the direction $(1, 0, 0, \ldots)$ then λ_2 steps in the direction $(0, 1, 0, 0, \ldots)$, and so on (repetitions are allowed, so we can have a monomial like $T_1^{\lambda_1} T_2^{\lambda_2} T_1^{\lambda_3}$).

Definition 2.0.1. *We say that a noetherian Banach unital algebra A is non-commutative affinoid k-algebra if there is an admissible surjective homomorphism $k\langle\langle T \rangle\rangle_r \to A$ (admissibility means that the norm on the image is the quotient norm).*

In particular, affinoid algebras in the sense of [Be1] belong to this class (unfortunately the terminology is confusing in this case: commutative affinoid algebras give examples of non-commutative affinoid algebras!). Another class of examples is formed by quantum affinoid algebras defined in the Introduction.

Let us now recall the following definition (see [Be1]).

Definition 2.0.2. *Berkovich spectrum $M(A)$ of a unital Banach ring A consists of bounded multiplicative seminorms on A.*

If A is a k-algebra, we require that seminorms extend the norm on k. It is well-known (see [Be1], Th. 1.2.1) that if A is commutative then $M(A)$ is a non-empty compact Hausdorff topological space (in the weak topology). If $\nu \in M(A)$ then $Ker\, \nu$ is a two-sided closed prime ideal in A. Therefore it is not clear whether $M(A)$ is non-empty in the non-commutative case.

Algebras of analytic functions on the non-commutative and quantum poly-discs carry multiplicative "Gauss norms" (see Introduction), hence the Berkovich spectrum is non-empty in each of those cases. The following example can be found in [So3], [SoVo].

Let L be a free abelian group of finite rank d, $\varphi : L \times L \to \mathbf{Z}$ be a skew-symmetric bilinear form, $q \in K^*$ satisfies the condition $|q| = 1$. Then $|q^{\varphi(\lambda,\mu)}| = 1$ for any $\lambda, \mu \in L$. We denote by $A_q(T(L, \varphi))$ the *algebra of regular functions on the quantum torus* $T_q(L, \varphi)$. By definition, it is a k-algebra with generators $e(\lambda), \lambda \in L$, subject to the relation

$$e(\lambda)e(\mu) = q^{\varphi(\lambda,\mu)}e(\lambda + \mu).$$

The algebra of analytic functions on the analytic quantum torus $T_q^{an}(L, \varphi)$ consists by definition of series $\sum_{\lambda \in L} a(\lambda)e(\lambda), a(\lambda) \in k$ such that for all $r = (r_1, \dots, r_d), r_i > 0$ one has: $|a(\lambda)|r^\lambda \to 0$ as $|\lambda| \to \infty$ (here $|(\lambda_1, \dots, \lambda_d)| = \sum_i |\lambda_i|$).

Quantum affinoid algebra $k\{T\}_{q,r}$ discussed in the Introduction is the algebra of analytic functions on quantum polydisc of the (multi)radius $r = (r_1, \dots, r_n)$. It was shown in [So1] that $M(k\{T\}_{q,r})$ can be quite big as long as $|q - 1| < 1$. In particular, it contains "quantum" analogs of the norms $|f|_{E(a,\rho)}$ which is the "maximum norm" of an analytic function f on the polydisc centered at $a = (a_1, \dots, a_n)$ of the radius $\rho = (\rho_1, \dots, \rho_n)$, with the condition $a_i \leq \rho_i < r_i, 1 \leq i \leq n$. Similar result holds for the quantum analytic torus. This observation demonstrates an interesting phenomenon: differently from the formal deformation quantization, the non-archimedean analytic quantization "preserves" some part of the spectrum of the "classical" object.

The conventional definition of the quatization can be carried out to the analytic case with obvious changes. Indeed, the notion of Poisson algebra admits a straightforward generalization to the analytic case (Poisson bracket is required to be a bi-analytic map). Furthermore, for any commutative affinoid algebra A there is a notion of non-commutative A-affinoid algebra, which is a natural generalization of the notion of k-affinoid algebra (we use $A\langle\langle T_1, \dots, T_n \rangle\rangle_r$ instead of $k\langle\langle T_1, \dots, T_n \rangle\rangle_r$).

Let now $\mathcal{O}(E(0,r))$ be the algebra of analytic functions on a 1-dimensional polydisc $E(0,r) = M(k\{r^{-1}T\})$ of the radius r (the notation is from [Be1], Chapter 2). We say that a non-commutative $\mathcal{O}(E(0,r))$-affinoid algebra A is an *analytic quantization of a k-affinoid commutative Poisson algebra A_0 over the polydisc $E(0,r)$* if the following two conditions are satisfied:

   1) A is a topological $\mathcal{O}(E(0,r))$-algebra, free as a topological $\mathcal{O}(E(0,r))$-module.
   2) The quotient A/TA is isomorphic to A_0 as a k-affinoid Poisson algebra.

Then a quantization of a k-analytic space $(X, \mathcal{O}_X)$ iss a ringed space $(X, \mathcal{O}_{q,X})$ such that for any affinoid $U \subset X$ the algebra $\mathcal{O}_{q,X}(U)$ is an analytic quantization of $\mathcal{O}_X(U)$ over some polydisc $E(0,r)$.

Notice that the projection $A \to A_0$ induces an embedding of Berkovich spectra $M(A_0) \to M(A)$. Every element of A can be thought of as analytic

function on $E(0, r)$ with values in a non-commutative k-affinoid algebra. Suppose that $A \simeq A_0\{r^{-1}T\}$ as a $k\{r^{-1}T\}$-module. Then the topological vector space A is isomorphic to the space of analytic functions on $E(0, r)$ with values in A_0 (but the product is not a pointwise product of functions). Assume that $r \leq 1$ and consider the subspace A_1 of analytic functions $a(x)$ as above such that $|a(0)|_{A_0} \leq 1, |a(x) - a(0)|_{A_0} \leq |T(x)|, x \in E(0, r)$, where $\bullet|_{A_0}$ denotes the norm on A_0. Here x is interpreted as a seminorm on the Banach k-algebra $k\{r^{-1}T\}$, hence $|T(x)|$ is the norm of the generator T in the completition of the residue field $k\{r^{-1}T\}/Ker\,x$. It is clear that A_1 is in fact a Banach k-algebra. Hence the natural projection $a(x) \mapsto a(0)$ defines an embedding $M(A_0) \to M(A_1)$.

Suppose that X is an analytic spaces for which there is a notion of a skeleton $Sk(X)$ either in the sense of [KoSo1] (then X is assumed to be Calabi-Yau) or in the sense of [Be2,Be3]. Then in either of these cases there is a continuous retraction $\pi : X \to Sk(X)$. Suppose that the there is a quantization $(X, \mathcal{O}_{q,X})$ of $(X, \mathcal{O}_X)$ in the above sense.

Conjecture 2.0.3. *For any closed $V \subset X$ there is a natural embedding $i_V : V \subset M(\mathcal{O}_{q,X}(\pi^{-1}(V)))$ such that $\pi \circ i_V = id_V$. Moreover if $V_1 \subset V_2$ then the restriction of i_{V_1} to V_2 is equal to V_2.*

In other words, the skeleton survives an analytic quantization. The above conjecture is not very precise, because there is no general definition of a skeleton. The definition given in [KoSo1] is different from the one in [Be2,3] even for Calabi-Yau manifolds. Hence the conjecture is an "experimental fact" at this time.

3. Quantum Calabi-Yau varieties

Let X^{alg} be a maximally degenerate (in the sense of [KoSo1-2]) algebraic Calabi-Yau manifold over $\mathbf{C}((t))$ of dimension n and X be the corresponding $\mathbf{C}((t))$-analytic space. Then one can associate with X a PL-manifold $Sk(X)$ of real dimension n, called the skeleton of X (see [KoSo1-2]). A choice of Kähler structure on X^{alg} defines (conjecturally) a continuous retraction $\pi : X \to Sk(X)$. This map satisfies the condition 2) from the Introduction. In other words, it defines a (singular) analytic torus fibration over $Sk(X)$ with the generic fiber, isomorphic to the analytic space $M(k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/(|T_i| = c_i, 1 \leq i \leq n))$, where $c_i > 0, 1 \leq i \leq n$ are some numbers. Since the projection is Stein (see loc. cit), one can reconstruct X (as a ringed space) from the knowledge of $(Sk(X), \pi_*(\mathcal{O}_X))$, where $\mathcal{O}_X$ is the sheaf of analytic functions on X.

Let $B = Sk(X)$ and B^{sing} be the "singular subvariety" of real codimension two (see Introduction). It was observed in [KoSo1] that the norms of elements of the direct image sheaf $\pi_*(\mathcal{O}_X^\times)$ define an integral affine structure on $B^0 := B \backslash B^{\mathrm{sing}}$. Hence we would like to reconstruct the analytic space starting with a PL-manifold equipped with a (singular) integral affine structure. As we will explain in the next subsection the same data give rise to a sheaf of quantum affinoid algebras on B^0.

3.1. Integral affine structures and quantized canonical sheaf

Here we explain following [KoSo1] and [So1] how a manifold with integral affine structure defines a sheaf of (quantum) affinoid algebras.

Recall that an integral affine structure ($\mathbf{Z}$-affine structure for short) on an n-dimensional topological manifold Y is given by a maximal atlas of charts such that the change of coordinates between any two charts is described by the formula

$$x'_i = \sum_{1 \le j \le n} a_{ij} x_j + b_i,$$

where $(a_{ij}) \in GL(n, \mathbf{Z}), (b_i) \in \mathbf{R}^n$. In this case one can speak about the sheaf of $\mathbf{Z}$-affine functions, i.e., those which can be locally expressed in affine coordinates by the formula $f = \sum_{1 \le i \le n} a_i x_i + b, a_i \in \mathbf{Z}, b \in \mathbf{R}$. An equivalent description: $\mathbf{Z}$-affine structure is given by a covariant lattice $T^{\mathbf{Z}} \subset TY$ in the tangent bundle (recall that an affine structure on Y is the same as a torsion free flat connection on the tangent bundle TY).

Let Y be a manifold with $\mathbf{Z}$-affine structure. The sheaf of $\mathbf{Z}$-affine functions $Aff_{\mathbf{Z}} := Aff_{\mathbf{Z},Y}$ gives rise to an exact sequence of sheaves of abelian groups

$$0 \to \mathbf{R} \to Aff_{\mathbf{Z}} \to (T^*)^{\mathbf{Z}} \to 0,$$

where $(T^*)^{\mathbf{Z}}$ is the sheaf associated with the dual to the covariant lattice $T^{\mathbf{Z}} \subset TY$.

Let us recall the following notion introduced in [KoSo1], Section 7.1. Let k be a valuation field.

Definition 3.1.1. *A k-affine structure on Y compatible with the given $\mathbf{Z}$-affine structure is a sheaf Aff_k of abelian groups on Y, an exact sequence of sheaves*

$$1 \to k^{\times} \to Aff_k \to (T^*)^{\mathbf{Z}} \to 1,$$

together with a homomorphism Φ of this exact sequence to the exact sequence of sheaves of abelian groups

$$0 \to \mathbf{R} \to Aff_{\mathbf{Z}} \to (T^*)^{\mathbf{Z}} \to 0,$$

such that $\Phi = id$ on $(T^*)^{\mathbf{Z}}$ and $\Phi = $ val on $k^{\times}$, where val is the valuation map.

Since Y carries a $\mathbf{Z}$-affine structure, we have the corresponding $GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$-torsor on Y, whose fiber over a point x consists of all $\mathbf{Z}$-affine coordinate systems at x.

Then one has the following equivalent description of the notion of k-affine structure.

Definition 3.1.2. *A k-affine structure on Y compatible with the given $\mathbf{Z}$-affine structure is a $GL(n, \mathbf{Z}) \ltimes (k^{\times})^n$-torsor on Y such that the application of $\mathrm{val}^{\times n}$ to $(k^{\times})^n$ gives the initial $GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$-torsor.*

Assume that Y is oriented and carries a k-affine structure compatible with a given $\mathbf{Z}$-affine structure. Orientation allows us to reduce the structure group of the torsor defining the k-affine structure to $SL(n, \mathbf{Z}) \ltimes (k^{\times})^n$.

Let $q \in k, |q| = 1$, and $z_1, \ldots, z_n$ be invertible variables such that $z_i z_j = q z_j z_i$, for all $1 \leq i < j \leq n$. We define the sheaf of k-algebras $\mathcal{O}_q^{can}$ on $\mathbf{R}^n, n \geq 2$ by the formulas:

$$\mathcal{O}_q^{can}(U) = \left\{ \sum_{I=(I_1,\ldots,I_n)\in\mathbf{Z}^n} c_I z^I \; \right|$$

$$\left. \forall (x_1,\ldots,x_n) \in U \;\; \sup_I \left(\log(|c_I|) + \sum_{1 \leq m \leq n} I_m x_m \right) < \infty \right\},$$

where $z^I = z_1^{I_1} \ldots z_n^{I_n}$. Since $|q| = 1$ the convergency condition does not depend on the order of variables.

The sheaf $\mathcal{O}_q^{can}$ can be lifted to Y (we keep the same notation for the lifting). In order to do that it suffices to define the action of the group $SL(n,\mathbf{Z}) \ltimes (k^\times)^n$ on the canonical sheaf on $\mathbf{R}^n$. Namely, the inverse to an element $(A, \lambda_1, \ldots, \lambda_n) \in SL(n,\mathbf{Z}) \ltimes (k^\times)^n$ acts on monomials as

$$z^I = z_1^{I_1} \ldots z_n^{I_n} \mapsto \left(\prod_{i=1}^n \lambda_i^{I_i} \right) \; z^{A(I)} \; .$$

The action of the same element on $\mathbf{R}^n$ is given by a similar formula:

$$x = (x_1,\ldots,x_n) \mapsto A(x) - (val(\lambda_1),\ldots,val(\lambda_n)) \; .$$

Any n-dimensional manifold Y with integral affine structure admits a covering by charts with transition functions being integral affine transformations. This allows to define the sheaf $\mathcal{O}_{q,Y}^{can}$ as the one which is locally isomorphic to $\mathcal{O}_q^{can} = \mathcal{O}_{q,\mathbf{R}^n}^{can}$.

It is explained in [KoSo1] (see also [So1], Section 7.2) that for any open $U \subset \mathbf{R}^n$ the topological space $M(\mathcal{O}_q^{can}(U))$ for $q = 1$ is an analytic torus fibration in the sense of Introduction. Recall that an analytic torus fibration is a fiber bundle (X, Y, π) consisting of a commutative k-analytic space, a topological manifold Y and a continuous map $\pi : X \to Y$ such that it is locally isomorphic to the torus fibration $\mathbf{G}_m^n \to \mathbf{R}^n$ from Introduction. In that case π is a Stein map, and we have: $\pi^{-1}(U) = M(\mathcal{O}_{q=1}^{can}(U))$. Therefore we can think of the ringed space $(Y, \mathcal{O}_{q,Y}^{can})$ as of quantization of this torus fibration.

3.2. Model sheaf near a singular point

In the case of maximally degenerate K3 surfaces the skeleton is homeomorphic to $B = S^2$ (the two-dimensional sphere) equipped with an integral affine structure outside of the subset B^{sing} consisting of 24 points (see [KoSo1], Section 6.4, where the affine structure is described). The construction of the previous subsection gives rise to a sheaf of quantum $\mathbf{C}((t))$-affinoid algebras over $B^0 = B \setminus B^{sing}$. In order to complete the quantization procedure we need to extend the sheaf $\mathcal{O}_{q,B^0}^{can}$ to a neighbourhood of B^{sing}. It is explained in [KoSo1] (case $q = 1$) and in [So1] (case $|q| = 1$) that one has to modify this sheaf in order to extend it to singular points.

Summarizing, the quantization is achieved in two steps. First, we define a sheaf of non-commutative $\mathbf{C}((t))$-affinoid algebras in a neighbourhood of B^{sing} such that it is locally isomorphic to the canonical sheaf $\mathcal{O}_{q,B^0}^{can}$ outside of B^{sing} and gives a "local model" for the future sheaf $\pi_*(\mathcal{O}_{q,X})$ at the singularities. Second, we modify the sheaf $\mathcal{O}_{q,B^0}^{can}$ by applying (infinitely many times) automorphisms associated with edges of an infinite tree embedded in B, such that its external vertices belong to B^{sing}. Those modifications ensure that the resulting sheaf can be glued with the model sheaf at the singularities, and that it is indeed the direct image of the sheaf of analytic functions on a compact $\mathbf{C}((t))$-analytic K3 surface. More precisely, we do the following.

We start with an open covering of $\mathbf{R}^2$ by the following sets $U_i, 1 \le i \le 3$. Let us fix a number $0 < \varepsilon < 1$ and define

$$
\begin{aligned}
U_1 &= \{(x,y) \in \mathbf{R}^2 | x < \varepsilon|y|\} \\
U_2 &= \{(x,y) \in \mathbf{R}^2 | x > 0, y < \varepsilon x\} \\
U_3 &= \{(x,y) \in \mathbf{R}^2 | x > 0, y > 0\}
\end{aligned}
$$

Clearly $\mathbf{R}^2 \setminus \{(0,0)\} = U_1 \cup U_2 \cup U_3$. We will also need a slightly modified domain $U_2' \subset U_2$ defined as $\{(x,y) \in \mathbf{R}^2 | x > 0, y < \frac{\varepsilon}{1+\varepsilon}x\}$.

Let $\pi_{can} : (\mathbf{G}_m^{an})^2 \to \mathbf{R}^2$ be the canonical map defined in the Introduction (see also [KoSo1]). We define the following three open subsets of the two-dimensional analytic torus: $T_i := \pi_{can}^{-1}(U_i), i = 1, 3$ and $T_2 := \pi_{can}^{-1}(U_2')$. There are natural projections $\pi_i : T_i \to U_i$ given by the formulas

$$
\pi_i(| \bullet |) = \pi_{can}(| \bullet |) = (\log|\xi_i|, \log|\eta_i|), \ \ i = 1, 3
$$

$$
\pi_2(| \bullet |) = \begin{cases} (\log|\xi_2|, \log|\eta_2|) & \text{if } |\eta_2| < 1 \\ (\log|\xi_2| - \log|\eta_2|, \log|\eta_2|) & \text{if } |\eta_2| \ge 1 \end{cases}
$$

To each T_i we assign the algebra $\mathcal{O}_q(T_i)$ of series $\sum_{m,n} c_{mn}\xi_i^m\eta_i^n$ such that $\xi_i\eta_i = q\eta_i\xi_i$, $c_{mn} \in \mathbf{C}((t))$, and for the seminorm $|\bullet|$ corresponding to a point of T_i (which means that $(log|\xi_i|, log|\eta_i|) \in U_i$) one has: $\sup_{m,n}(m\,log|\xi_i| + n\,log|\eta_i|) < +\infty$. Similarly, we can define $\mathcal{O}_q(U)$ for any $U \subset U_i$. In this way we obtain a sheaf of quantum $\mathbf{C}((t))$-affinoid algebras on the set U_i. We will denote this sheaf by $\pi_{i*}(\mathcal{O}_{q,T_i})$.

We define the sheaf $\mathcal{O}_q^{can}$ on $\mathbf{R}^2 \setminus \{(0,0)\}$ as $\pi_{i*}(\mathcal{O}_{q,T_i})$ on each domain U_i, with identifications

$$
\begin{aligned}
(\xi_1, \eta_1) &= (\xi_2, \eta_2) & \text{on } U_1 \cap U_2 \\
(\xi_1, \eta_1) &= (\xi_3, \eta_3) & \text{on } U_1 \cap U_3 \\
(\xi_2, \eta_2) &= (\xi_3\eta_3, \eta_3) & \text{on } U_2 \cap U_3
\end{aligned}
$$

The notation for the sheaf is consistent with the previous subsection since $\mathcal{O}_q^{can}$ is locally isomorphic to the canonical sheaf associated with the standard integral affine structure.

Let us modify the canonical sheaf $\mathcal{O}_q^{can}$ in the following way. On the sets U_1 and $U_2 \cup U_3$ the new sheaf $\mathcal{O}_q^{mod}$ is isomorphic to $\mathcal{O}_q^{can}$ (by identifying of coordinates (ξ_1, η_1) and glued coordinates (ξ_2, η_2) and (ξ_3, η_3) respectively). On

the intersection $U_1 \cap (U_2 \cup U_3)$ we identify two copies of the canonical sheaf by an automorphism φ of $\mathcal{O}_q^{can}$ given (we skip the index of the coordinates) by

$$\varphi(\xi, \eta) = \begin{cases} (\xi(1+\eta), \eta) & \text{on} \quad U_1 \cap U_2 \\ (\xi(1+\eta^{-1}), \eta) & \text{on} \quad U_1 \cap U_3 \end{cases}$$

Finally we are going to introduce a sheaf of $\mathbf{C}((t))$-algebras $\mathcal{O}_q^{\text{sing}}$ on a small open disc $W \subset \mathbf{R}^2, \{(0,0)\} \in W$ such that $\mathcal{O}_q^{\text{sing}}|_{W \setminus \{(0,0)\}}$ is isomorphic to $\mathcal{O}_q^{\text{mod}}|_{W \setminus \{(0,0)\}}$. The sheaf $\mathcal{O}_q^{\text{sing}}$ provides a non-commutative deformation of the "local model sheaf" near a singular point (see [KoSo1], Section 8 about the latter).

Let us consider a non-commutative $\mathbf{C}((t))$-algebra $A_q(S)$ generated by α, β, γ subject to the following relations:

$$\alpha\gamma = q\gamma\alpha, q\beta\gamma = \gamma\beta,$$
$$\beta\alpha - q\alpha\beta = 1 - q,$$
$$(\alpha\beta - 1)\gamma = 1.$$

For $q = 1$ this algebra coincides with the algebra of regular functions on the surface $S \subset \mathbf{A}_{\mathbf{C}((t))}^3$ given by the equation $(\alpha\beta - 1)\gamma = 1$ and moreover, it is a flat deformation of the latter with respect to the parameter $q - 1$. It is explained in [KoSo1], Section 8, that there is a natural map $p : S^{an} \to \mathbf{R}^2$ of the corresponding analytic surface such that $p_*(\mathcal{O}_{S^{an}})$ is a local model near a singularity of the sheaf $\pi_*(\mathcal{O}_X)$, where X is the maximally degenerate K3 surface and π is the projection to the skeleton $Sk(X)$.

Let us denote by $\mathcal{O}_{q,r_1,r_2,r_3}(S^{an})$ the non-commutative affinoid algebra which is the quotient of $\mathbf{C}((t))\langle\langle \alpha, \beta, \gamma \rangle\rangle_{r_1,r_2,r_3}$ by the closed two-sided ideal generated by the above three relations for $A_q(S)$. Here $r_i, i = 1, 2, 3$ are arbitrary non-negative numbers. We denote by $\mathcal{O}_q(S^{an})$ the intersection of all algebras $\mathcal{O}_{q,r_1,r_2,r_3}(S^{an})$.

We define homomorphisms of non-commutative algebras $g_i : A_q(S) \to \mathcal{O}_q(T_i)$, $1 \le i \le 3$ by the following formulas (the notation is obvious):

$$\begin{aligned} g_1(\alpha, \beta, \gamma) &= (\xi_1^{-1}, \xi_1(1+\eta_1), \eta_1^{-1}) \\ g_2(\alpha, \beta, \gamma) &= ((1+\eta_2)\xi_2^{-1}, \xi_2, \eta_2^{-1}) \\ g_3(\alpha, \beta, \gamma) &= ((1+\eta_3)(\xi_3\eta_3)^{-1}, \xi_3\eta_3, (\eta_3)^{-1}) \end{aligned}$$

These homomorphisms correspond to the natural embeddings $T_i \hookrightarrow S^{an}, i = 1, 2, 3$. One can use these homomorphisms in order to show an existence of non-trivial multiplicative seminorms on $A_q(S)$ and construct explicitly some of the corresponding representations of $A_q(S)$ in a k-Banach vector space.

For example, let us consider a Banach vector space V_r consisting of series $\sum_{i \in \mathbf{Z}} a_i T^i, a_i \in k$ such that $|a_i| r^i \to 0$ as $|i| \to \infty$, where $r > 0$ is some number. Let $\tau : V_r \to V_r$ be the shift operator: $\tau(f)(T) = f(qT)$. Define $\alpha = T$ (operator of multiplication by T), $\gamma = -\tau^{-1}$ and $\beta = T^{-1} \circ (1 - \tau)$. One checks that all the relations of $A_q(S)$ are satisfied, and moreover, the seminorm on $A_q(S)$ induced by the operator norm is multiplicative. (Similar considerations apply to the analytic quantum torus derived from $\xi\eta = q\eta\xi$. Then the element $\sum_{n,m \in \mathbf{Z}} a_{nm} \xi^n \eta^m$ transforms

the series $f = \sum_{i \in \mathbf{Z}} c_i T^i$ into $\sum_{n,m \in \mathbf{Z}} a_{nm} q^{nm} T^m f(q^n T)$.) Rescaling the action of α and γ by arbitrary non-zero numbers one can adjust the action of β in such a way that the norms of operators α, β, γ "cover" an open neighborhood of the point $(1, 1, 1)$. More precisely, let us consider the map $f : M(\mathcal{O}_q(S^{an})) \to \mathbf{R}^3$ defined by the formula $f(x) = (a, b, c)$ where $a = \max(0, \log |\alpha|_x), b = \max(0, \log |\beta|_x), c = \log |\gamma|_x = -\log |\alpha\beta - 1|_x$. Here $|\cdot|_x = \exp(-val_x(\cdot))$ denotes the mulitplicative seminorm corresponding to the point $x \in M(\mathcal{O}_q(S^{an}))$. Then the image of f is homeomorphic to $\mathbf{R}^2$, similarly to the case $q = 1$ considered in [KoSo1]. More precisely, let us decompose $M(\mathcal{O}_q(S^{an})) = S_- \cup S_0 \cup S_+$ according to the sign of $\log |\gamma|_x$ where $x \in M(\mathcal{O}_q(S^{an}))$. Then

$$
\begin{array}{rcl}
f(S_-) &=& \{\, (a, b, c) \in \mathbf{R}^3 \,|\, c < 0, a \geq 0, b \geq 0, \, ab(a + b + c) = 0 \,\} \\
f(S_0) &=& \{\, (a, b, c) \in \mathbf{R}^3 \,|\, c = 0, a \geq 0, b \geq 0, \, ab = 0 \,\} \\
f(S_+) &=& \{\, (a, b, c) \in \mathbf{R}^3 \,|\, c > 0, a \geq 0, b \geq 0, \, ab = 0 \,\}
\end{array}
$$

In fact the image of the map f coincides with the image of the embedding $j : \mathbf{R}^2 \to \mathbf{R}^3$ given by formula

$$
j(x, y) = \left\{
\begin{array}{ll}
(-x, \, \max(x + y, 0), \, -y) & \text{if} \quad x \leq 0 \\
(0, \, x + \max(y, 0), \, -y) & \text{if} \quad x \geq 0
\end{array}
\right.
$$

Proofs of the above observations are different from the case $q = 1$. Indeed, there are no one-dimensional modules over $A_q(S)$ corresponding to the points of the surface S. Therefore it is not obvious that there are multiplicative seminorms x on $A_q(S)$ with the prescribed value of $f(x)$. Seminorms on $A_q(S)$ arise from representations of this algebra in k-Banach vector spaces: if $\rho : A_q(S) \to \operatorname{End}_k(V)$ is such a representaion then we can define $|a|_\rho = ||\rho(a)||$, where $||\rho(a)||$ is the operator norm in the Banach algebra $\operatorname{End}_k(V)$ of bounded operators on V. Such seminorms are, in general, submultiplicative: $|ab|_\rho \leq |a|_\rho |b|_\rho$. We are interested in those which are multiplicative. This can be achieved, e.g., by mapping of $A_q(S)$ into an algebra which admits multiplicative seminorms. We discussed above the homomorphisms g_i of $A_q(S)$ into analytic quantum tori. Let us consider a different example of such homomorphism. Let $\delta = (\alpha\beta - 1)\gamma$. One checks that δ is a central element in the quantum affinoid algebra $B_q(S)$ generated by the first three relations for $A_q(S)$ (i.e., we drop the relation $\delta = 1$). Let us consider the quantum affinoid algebra B generated by $\beta^{\pm 1}, \gamma^{\pm 1}, \delta$ subject to the relations:

$$
\beta\delta = \delta\beta, \gamma\delta = \delta\gamma, \gamma\beta = q\beta\gamma,
$$

and such that β^{-1} is inverse to β and γ^{-1} is inverse to γ. There is an embedding of algebras $A_q(S) \to B/(\delta - 1)B$ induced by the linear map $A_q(S) \to B$ such that β and γ are mapped into the corresponding elements of B and $\alpha \mapsto (1 + \delta\gamma^{-1})\beta^{-1}$. Notice that for any $r_1 > 0, r_2 > 0, r_3 \geq 0$ one can define a multiplicative norm $|\bullet|_{r_1, r_2, r_3}$ on B such that $|\sum_{n \in \mathbf{Z}, m \in \mathbf{Z}, l \in \mathbf{Z}_+} c_{nml} \beta^n \gamma^m \delta^l|_{r_1, r_2, r_3} = \max_{n,m,l} |c_{nml}| r_1^n r_2^m r_3^l$. Moreover, we can complete B with respect to this multiplicative norm and obtain a quantum affinoid algebra B_{r_1, r_2, r_3}. We can also invert

δ and do the same construction. In this way we obtain the quantum affinoid algebra denoted by $B^{(1)}_{r_1,r_2,r_3}$. Since $A_q(S)$ is embedded into the quotient of any of these algebras by the central ideal, we obtain plenty multiplicative norms on $A_q(S)$ and on its completions.

We denote by p the composition $j^{-1} \circ f$. In the case $q = 1$ it is an analytic torus fibration over the set $\mathbf{R}^2 \setminus \{(0,0)\}$. Let now W be a small disc in $\mathbf{R}^2$ centered at the origin. We need to define the non-commutative affinoid algebra $\mathcal{O}_q^{\mathrm{sing}}(W)$. In the commutative case $q = 1$ it is defined as $\mathcal{O}_{S^{an}}^{\mathrm{sing}}(p_*^{-1}(W)) = p_*(\mathcal{O}_{S^{an}})(W)$.

For each $i = 1, 2, 3$ we define the $\mathbf{C}((t))$-affinoid algebras $\mathcal{O}_q^{\mathrm{sing}}(p^{-1}(U_i))$ such that for every $x \in M(\mathcal{O}_q^{\mathrm{sing}}(p^{-1}(U_i)))$ one has $p(x) \in U_i$ (it coincides with the intersection of all completions of $A_q(S)$ with respect to multiplicative seminorms x such that $p(x) \in U_i, i = 1, 2, 3$). Similarly we define algebras $\mathcal{O}_q^{\mathrm{sing}}(p^{-1}(W))$ and $\mathcal{O}_q^{\mathrm{sing}}(p^{-1}(W^0))$, where $W^0 = W \setminus \{(0,0)\}$ or, more generally, any $\mathcal{O}_q^{\mathrm{sing}}(p^{-1}(U))$ for U being an open subset of W. Using homomorphisms $g_i, i = 1, 2, 3$ one proves that if $U \subset U_i$ then $\mathcal{O}_q^{\mathrm{sing}}(p^{-1}(U))$ is isomorphic to $\mathcal{O}_q^{\mathrm{mod}}(\pi_i^{-1}(U))$. The latter is defined as the set of series $\sum_{m,n \in \mathbf{Z}} c_{mn} \xi_i^m \eta_i^n$ such that $\pi_i(| \bullet |) \in U$ for any multiplicative seminorm $|\bullet|$ such that $\sup_{m,n}(\log|c_{mn}| + m \log|\xi_i| + n \log|\eta_i| < \infty)$, if $(x, y) \in U$. The isomorphism of sheaves $\mathcal{O}_q^{\mathrm{sing}}|_{W \setminus \{(0,0)\}} \simeq \mathcal{O}_q^{\mathrm{mod}}|_{W \setminus \{(0,0)\}}$ follows. Details of this construction will be explained elsewhere.

3.3. Trees, automorphisms and gluing

As was explained in [KoSo1] in the case of $q = 1$ and in [So1] in the case $|q| = 1$, one has to modify the canonical sheaf in order to glue it with $\mathcal{O}_q^{\mathrm{sing}}$. Here we explain the construction following [KoSo1], [So1], leaving the details to a separate publication. The starting point for the construction is a subset $\mathcal{L} \subset B$ which is an infinite tree. We called it *lines* in [KoSo1].

The definition is quite general. Here we discuss the 2-dimensional case, while a much more complicated higher-dimensional case was considered in the recent paper [GroSie1]. For a manifold Y which carries a $\mathbf{Z}$-affine structure a *line* l is defined by a continuous map $f_l : (0, +\infty) \to Y$ and a covariantly constant (with respect to the connection which gives the affine structure) nowhere vanishing integer-valued 1-form $\alpha_l \in \Gamma((0, +\infty), f_l^*((T^*)^{\mathbf{Z}})$. A set $\mathcal{L}$ of lines is required to be decomposed into a disjoint union $\mathcal{L} = \mathcal{L}_{in} \cup \mathcal{L}_{com}$ of *initial* and *composite* lines. Each composite line is obtained as a result of a finite number of "collisions" of initial lines. A collision is described by a Y-shape figure, where the bottom leg of Y is a composite line, while two other segments are "parents" of the leg, so that the leg is obtained as a result of the collision. A construction of the set $\mathcal{L}$ satisfying the axioms from [KoSo1] was proposed in [KoSo1], Section 9.3. Generalization to the higher-dimensional case can be found in [GroSie1]. In the two-dimensional case the lines form an infinite tree embedded into B. The edges have rational slopes with respect to the integral affine structure. The tree is dense in B^0.

With each line l (i.e., edge of the tree) we associate a continuous family of automorphisms of stalks of sheaves of algebras $\varphi_l(t) : (\mathcal{O}_q^{can})_{Y,f_l(t)} \to (\mathcal{O}_q^{can})_{Y,f_l(t)}$.

Automorphisms φ_l can be defined in the following way (see [KoSo1], Section 10.4).

First we choose affine coordinates in a neighborhhod of a point $b \in B \setminus B^{\text{sing}}$, identifyin b with the point $(0,0) \in \mathbf{R}^2$. Let $l = l_+ \in \mathcal{L}_{in}$ be (in the standard affine coordinates) a line in the half-plane $y > 0$ emerging from $(0,0)$ (there is another such line l_- in the half-plane $y < 0$, see [KoSo1] for the details). Assume that t is sufficiently small. Then we define $\varphi_l(t)$ on topological generators ξ, η by the formula

$$\varphi_l(t)(\xi, \eta) = (\xi(1 + \eta^{-1}), \eta).$$

In order to extend $\varphi_l(t)$ to the interval $(0, t_0)$, where t_0 is not small, we cover the corresponding segment of l by open charts. Then a change of affine coordinates transforms η into a monomial multiplied by a constant from $(\mathbf{C}((t)))^\times$. Moreover, one can choose the change of coordinates in such a way that $\eta \mapsto C\eta$ where $C \in (\mathbf{C}((t)))^\times, |C| < 1$ (such change of coordinates preserve the 1-form dy. Constant C is equal to $exp(-L)$, where L is the length of the segment of l between two points in different coordinate charts). Therefore η extends analytically in a unique way to an element of $\Gamma((0, +\infty), f_l^*((\mathcal{O}_q^{can})^\times))$. Moreover the norm $|\eta|$ strictly decreases as t increases, and remains strictly smaller than 1. Similarly to [KoSo1], Section 10.4 one deduces that $\varphi_l(t)$ can be extended for all $t > 0$. This defines $\varphi_l(t)$ for $l \in \mathcal{L}_{in}$.

Next step is to extend $\varphi_l(t)$ to the case when $l \in \mathcal{L}_{com}$, i.e., to the case when the line is obtained as a result of a collision of two lines belonging to $\mathcal{L}_{in}$. Following [KoSo1], Section 10, we introduce a group G which contains all the automorphisms $\varphi_l(t)$, and then prove the factorization theorem (see [KoSo1], Theorem 6) which allows us to define $\varphi_l(0)$ in the case when l is obtained as a result of a collision of two lines l_1 and l_2. Then we extend $\varphi_l(t)$ analytically for all $t > 0$ similarly to the case $l \in \mathcal{L}_{in}$.

More precisely, the construction of G goes such as follows. Let $(x_0, y_0) \in \mathbf{R}^2$ be a point, $\alpha_1, \alpha_2 \in (\mathbf{Z}^2)^*$ be 1-covectors such that $\alpha_1 \wedge \alpha_2 > 0$. Denote by $V = V_{(x_0,y_0),\alpha_1,\alpha_2}$ the closed angle

$$\{(x, y) \in \mathbf{R}^2 | \langle \alpha_i, (x, y) - (x_0, y_0) \rangle \geq 0, i = 1, 2\}$$

Let $\mathcal{O}_q(V)$ be a $\mathbf{C}((t))$-algebra consisting of series $f = \sum_{n,m \in \mathbf{Z}} c_{n,m} \xi^n \eta^m$, such that $\xi\eta = q\eta\xi$ and $c_{n,m} \in \mathbf{C}((t))$ satisfy the condition that for all $(x, y) \in V$ we have:

1. if $c_{n,m} \neq 0$ then $\langle (n, m), (x, y) - (x_0, y_0) \rangle \leq 0$, where we identified $(n, m) \in \mathbf{Z}^2$ with a covector in $(T_p^* Y)^{\mathbf{Z}}$;
2. $\log |c_{n,m}| + nx + my \to -\infty$ as long as $|n| + |m| \to +\infty$.

For an integer covector $\mu = adx + bdy \in (\mathbf{Z}^2)^*$ we denote by $R_\mu := R_{(a,b)}$ the monomial $\xi^a \eta^b$. Then $R_{(a,b)} R_{(c,d)} = q^{ad-bc} R_{(c,d)} R_{(a,b)} = q^{-bc} R_{(a+c,b+d)}$. We define a prounipotent group $G := G(q, \alpha_1, \alpha_2, V)$ which consists of automorphisms

of $\mathcal{O}_q(V)$ having the form $f \mapsto e^g f e^{-g}$ where

$$g = \sum_{n_1,n_2 \geq 0, n_1+n_2 > 0} c_{n_1,n_2} R_{\alpha_1}^{-n_1} R_{\alpha_2}^{-n_2}$$

where $c_{n_1,n_2} \in \mathbf{C}((t))$ and

$$\log |c_{n,m}| - n_1 \langle \alpha_1, (x,y) \rangle - n_2 \langle \alpha_2, (x,y) \rangle \leq 0 \quad \forall (x,y) \in V.$$

The latter condition is equivalent to $\log |c_{n,m}| - \langle n_1\alpha_1 + n_2\alpha_2, (x_0, y_0) \rangle \leq 0$. The assumption $|q| = 1$ ensures that the product is well defined.

Let us consider automorphisms as above such that in the series for g the ratio $\lambda = n_2/n_1 \in [0, +\infty]_\mathbf{Q} := \mathbf{Q}_{\geq 0} \cup \infty$ is fixed. Such automorphism form a commutative subgroup $G_\lambda := G_\lambda(q, \alpha_1, \alpha_2, V) \subset G$. There is a natural map $\prod_\lambda G_\lambda \to G$, defined as in [KoSo1], Section 10.2. The factorization theorem proved in the loc. cit states that this map is a bijection of sets.

Example 3.3.1. *Let us consider the automorphism, discussed above:*

$$\varphi(\xi, \eta) = (\xi(1 + \eta^{-1}), \eta).$$

One can check that the transformation $\xi \mapsto \xi(1 + \eta^{-1})$ has the form

$$exp(Li_{2,q}(\eta^{-1})/(q-1))\xi exp(-Li_{2,q}(\eta^{-1})/(q-1)),$$

where $Li_{2,q}(x)$ is the quantum dilogarithm function (see, e.g., [BR]). It satisfies the property $(x; q)_\infty = exp(Li_{2,q}(-x)/(q-1))$, where $(a; q)_N = \prod_{0 \leq n \leq N}(1 - aq^n)$ for $1 \leq N \leq \infty$. Using the formula $(x; q)_\infty = \sum_{n \geq 0} \frac{(-1)^n q^{n(n-1)/2} x^n}{(q;q)_n}$ one can show that $\lim_{q \to 1} Li_{2,q}(x) = Li_2(x) = \sum_{n \geq 1}(-1)^n x^n/n^2$, which is the ordinary dilogarithm function (the latter appeared in [\bar{K}oSo1], Section 10.4 in the reconstruction problem of rigid analytic K3 surfaces).

Let us now assume that lines l_1 and l_2 collide at $p = f_{l_1}(t_1) = f_{l_2}(t_2)$, generating the line $l \in \mathcal{L}_{com}$. Then $\varphi_l(0)$ is defined with the help of factorization theorem. More precisely, we set $\alpha_i := \alpha_{l_i}(t_i)$, $i = 1, 2$ and the angle V is the intersection of certain half-planes $P_{l_1,t_1} \cap P_{l_2,t_2}$ defined in [KoSo1], Section 10.3. The half-plane $P_{l,t}$ is contained in the region of convergence of $\varphi_l(t)$. By construction, the elements $g_0 := \varphi_{l_1}(t_1)$ and $g_{+\infty} := \varphi_{l_2}(t_2)$ belong respectively to G_0 and $G_{+\infty}$. The we have:

$$g_{+\infty} g_0 = \prod_\to \left((g_\lambda)_{\lambda \in [0,+\infty]_\mathbf{Q}} \right) = g_0 \ldots g_{1/2} \ldots g_1 \ldots g_{+\infty}.$$

Each term g_λ with $0 < \lambda = n_1/n_2 < +\infty$ corresponds to the newborn line l with the direction covector $n_1\alpha_{l_1}(t_1) + n_2\alpha_{l_2}(t_2)$. Then we set $\varphi_l(0) := g_\lambda$. This transformation is defined by a series which is convergent in a neighborhood of p, and using the analytic continuation we obtain $\varphi_l(t)$ for $t > 0$, as we said above. Recall that every line carries an integer 1-form $\alpha_l = adx + bdy$. By construction, $\varphi_l(t) \in G_\lambda$, where λ is the slope of α_l.

Having automorphisms φ_l assigned to lines $l \in \mathcal{L}$ we proceed as in [KoSo1], Section 11, modifying the sheaf $\mathcal{O}_q^{can}$ along each line. We denote the resulting

sheaf by $\mathcal{O}_q^{\mathrm{mod}}$. It is isomorphic to the previously constructed sheaf $\mathcal{O}_q^{\mathrm{mod}}$ in a neighborhood of the point $(0,0)$.

Remark 3.3.2. *The appearance of the dilogarithm function in the above example can be illustrated in the picture of collision of two lines (say, $(x,0)$ and $(0,y)$ $x,y \geq 0$) which leads to the appearance of the new line, which is the diagonal $(x,x), x \geq 0$. Then the factorization theorem gives rise to the five-term identity $g_\infty g_0 = g_0 g_1 g_\infty$, which is the quantum version of the famous five-term identity for the dilogarithm function.*

4. p-adic quantum groups

4.1. How to quantize p-adic groups

Let L be a finite algebraic extension of the field $\mathbf{Q}_p$ of p-adic numbers, and K be a discretely valued subfield of the field of complex p-adic numbers $\mathbf{C}_p$ containing L. All the fields carry non-archimedean norms, which we will denote simply by $|\bullet|$ (sometimes we will be more specific, using the notation like $|x|_K$ in order to specify which field we consider). We denote by $\mathcal{O}_L$ the ring of integers of L and by m_L the maximal ideal of $\mathcal{O}_L$. Let G be a locally L-analytic group, which is the group of L-points of a split reductive algebraic group $\mathbf{G}$ over L. Let $H \subset G$ be an open maximal compact subgroup. We would like to define quantum analogs of the algebras $C^{la}(G,K), C^{la}(H,K)$ of locally analytic functions on G and H, as well as their strong duals $D^{la}(G,K)$, $D^{la}(H,K)$, which are the algebras of locally analytic distributions on G and H respectively (see [SchT1], [Em1]). Modules over the algebras of locally analytic distributions were used in [SchT1-5], [Em1] for a description of locally analytic admissible representations of locally L-analytic groups. Our aim is to derive "quantum" analogs of those results. In this paper we will discuss definitions of the algebras only.

First of all we are going to define locally analytic functions and locally analytic distributions on the quantized compact group H. Let us explain our approach in the case of the "classical" (i.e., non-quantum) group H. We are going to present definitions of $C^{la}(H,K)$ and $D^{la}(H,K)$ in such a way that they can be generalized to the case of quantum groups. The difficulty which one needs to overcome is to define everything using only two Hopf algebras: the universal enveloping algebra $U(g)$, $g = \mathrm{Lie}(G)$ and the algebra $K[G]$ of *regular* functions on the algebraic group G. Our construction consists of three steps.

1) For a "sufficiently small" open compact subgroup $H_r \subset H$ we define the algebra $C^{an}(H_r, K)$ of *analytic* functions on H_r. Here $0 < r \leq 1$ is a parameter, such that $H_1 = H$, and if $r_1 < r_2$ then $H_{r_1} \subset H_{r_2}$. The strong dual to $C^{an}(H_r, K)$ is denoted by $D^{an}(H_r, K)$. It is, by definition, the algebra of *analytic* distributions on H_r. It can be described (see [Em1], Section 5.2) as a certain completion of the universal enveloping algebra $U(g)$ of the Lie algebra $g = \mathrm{Lie}(G)$.

2) For any $r \leq 1$ we define a norm on the algebra of regular functions $K[G]$ such that the completion with respect to this norm is the algebra of *continuous* functions $C(H_r, K)$ on the group H_r.

3) In order to define *locally analytic* functions on H we consider a family of seminorms $|f|_l$ on $K[G]$, where $f \in K[G]$ and l runs through the set $D^{an}(H_r, K)$. More precisely, for every $l \in D^{an}(H_r, K)$ we define a seminorm $|f|_l = ||(id \otimes l)(\delta(f))||$, where δ is the coproduct on the Hopf algebra $K[G]$ and $|| \bullet ||$ is the norm defined above on the step 2). The completion of $K[G]$ with respect to the topology defined by the family of seminorms $| \bullet |_l$ is the algebra $C^{la}(H, K)$ of locally analytic functions on H defined in [SchT1]. The strong dual $D^{la}(H, K) := C^{la}(H, K)'_b$ is the algebra of locally analytic distributions introduced in [SchT1].

We recall that the locally analytic representation theory of G developed in [SchT1-6] is based on the notion of coadmissible module over the algebra $D^{la}(H, K)$, where $H \subset G$ is an open compact subgroup. Therefore, from the point of view of representation theory, it suffices to quantize $D^{la}(H, K)$.

4.2. Quantization of "small" compact subgroups

We would like to quantize $D^{la}(H, K)$ following the above considerations. We will do that for the class of algebraic quantum groups introduced by Lusztig (see [Lu1], [Lu2]). Let us fix $q \in L^\times$ such that $|q| = 1$ (this restriction is not necessary for algebraic quantization, but it will be important when we discuss convergent series). We will assume that there is $h \in \mathcal{O}_L$ such that $|h| < 1, exp(h) = q$. Let $\mathbf{G}$ be a semisimple simply-connected algebraic group over $\mathbf{Z}$, associated with a Cartan matrix $((a_{ij}))$ (more precisely, in order to be consistent with the terminology of [Lu1] we start with a *root datum* of finite type associated with a Cartan datum, see [Lu1], Chapter 2. These data give rise to the Cartan matrix in the ordinary sense). The algebraic group $\mathbf{G}(\mathbf{C})$ of $\mathbf{C}$-points of $\mathbf{G}$ was quantized by Drinfeld (see, e.g., [KorSo] Chapters 1,2). We will need a $\mathbf{Z}$-form of the quantized algebraic group $\mathbf{G}$ introduced by Lusztig (see [Lu1]). It allows us to define the quantized group over an arbitrary field. We need to be more specific when speaking about "quantized" group. More precisely, following [Lu1] one can define Hopf L-algebras $U_q(g_L)$ and $L[G]_q$, which are the quantized enveloping algebra of the Lie algebra g_L of $\mathbf{G}(L)$ and the algebra of *regular* functions on the algebraic quantum group $\mathbf{G}(L)$ respectively. Extending scalars to K we obtain Hopf K-algebras $U_q(g_K) = K \otimes_L U_q(g_L)$ and $K[G]_q = K \otimes_L L[G]_q$. We will also need $\mathbf{Z}$-forms of the above Hopf algebras, which will be denoted by $U := U_A$ and $A[G]_v$ respectively. The latter are Hopf algebras over the ring $A = \mathbf{Z}[v, v^{-1}]$, where v is a variable. The algebras U_L and $L[G]_q$ are obtained by tensoring of U and $A[G]_v$ respectively with L in such a way that v acts on L by multiplication by q.

As an A-module the algebra U is isomorphic to the tensor product $U \simeq U^+ \otimes U^0 \otimes U^-$ where $U^\pm$ are the quantized Borel subalgebras and U^0 is the quantized Cartan subalgebra (see [Lu2]). Recall that U^+ (resp. U^-) is an A-algebra generated

by the divided powers $E_i^{(N)}$ (resp $F_i^{(N)}$) of the Chevalley generators E_i (resp. F_i) of the quantized enveloping $\mathbf{Q}(v)$-algebra $\mathbf{U}$, where $1 \leq i \leq n := \operatorname{rank} g_L$ (see [Lu2]). The algebra U^0 is generated over A by the generators $K_i^{s_i} \binom{K_i}{n_i}_v$, where $\binom{K_i}{n_i}_v = \prod_{1 \leq m \leq n_i} \frac{K_i v^{d_i(-m+1)} - K_i^{-1} v^{d_i(m-1)}}{v^{d_i m} - v^{-d_i m}}$. Here $n_i \in \mathbf{Z}_+$, $s_i \in \{0,1\}$, and $K_i, 1 \leq i \leq n$ are the standard Chevalley generators of $\mathbf{U}$. The integer numbers $d_i \in \{1,2,3\}$ satisfy the condition that $((d_i a_{ij}))$ is a symmetric positive definite matrix with $a_{ii} = 2$, and $a_{ij} \leq 0$ if $i \neq j$. Recall that there is a canonical reduction of U_A at $v = 1$, which is a Hopf $\mathbf{Z}$-algebra $U_{\mathbf{Z}}$. It is the universal enveloping of the integer Lie algebra $g_{\mathbf{Z}}$ of the corresponding Chevalley group. We will denote by $t_i \in g_L$ the generators at $v = 1$ corresponding to K_i, keeping the same notation $E_i^{(N)}, F_i^{(N)}$ for the rest of the generators of $g_{\mathbf{Z}}$. Thus we have the standard decompostion $g_{\mathbf{Z}} = g_{\mathbf{Z}}^+ \oplus h_{\mathbf{Z}} \oplus g_{\mathbf{Z}}^-$, where the Lie algebra $g_{\mathbf{Z}}^+$ is generated (as a Lie algebra over $\mathbf{Z}$) by $E_i^{(N)}$, the Lie algebra $g_{\mathbf{Z}}^-$ is generated by $F_i^{(N)}$ and the commutative Lie algebra $h_{\mathbf{Z}}$ is generated by $\binom{t_i}{N} := t_i(t_i - 1) \cdots (t_i - N + 1)/N!$, $1 \leq i \leq n$ (see [St], Theorem 2). Lie algebra g_L is generated by the standard Chevalley generators $E_i = E_i^{(1)}, F_i = F_i^{(1)}, t_i, 1 \leq i \leq n$. The Hopf algebra $U/(v-1)U$ is the universal enveloping algebra $U(g_L)$ of g_L.

In what follows, while keeping the above notation, we will assume for simplicity that $L = \mathbf{Q}_p$.

We will need the following extension of $U_q(g_L)$. Let us fix a basis $\{\alpha_i\}_{1 \leq i \leq n}$ of simple roots of g_L, as well as invariant bilinear form on this Lie algebra such that $(\alpha_i, \alpha_j) = d_i a_{ij}$.

Let $h_{\mathcal{O}_L} = \oplus_{1 \leq i \leq n} \mathbf{Z}_p t_i$. We fix a global chart $\psi : h_{\mathcal{O}_L} \to T^0$, where $T^0 = \mathbf{T}(\mathcal{O}_L)$ is the maximal compact torus. Then any element $a \in T^0$ can be written as an analytic function $t = \psi(\sum_{1 \leq i \leq n} x_i t_i) := t(x_1, \ldots, x_n)$, where $x_i \in \mathbf{Z}_p$. Let us introduce a unital topological Hopf L-algebra $U_q^{an}(g_L)$ which is a Hopf L-algebra generated by $E_i^{(N)}, F_i^{(N)}, 1 \leq i \leq n, N \geq 1$ and the elements $t(x) = t(x_1, \ldots, x_n) \in T^0$ as above, such that the relations between $E_i^{(N)}, F_i^{(N)}$ are the same as in $U_q(g_L)$, and $t(x)E_i = E_i t(x + v_i), t(x)F_i = F_i t(x - v_i)$, where $v_i = (a_{1i}, a_{2i}, \ldots, a_{ni})$. The elements $K_i^{\pm 1} = \exp(\pm d_i h)$ (recall that $\exp(h) = q$) belong to this algebra and together with $E_i^{(N)}, F_i^{(N)}, 1 \leq i \leq n, N \geq 1$ generate the Hopf algebra isomorphic to $U_q(g_L)$.

There is a natural non-degenerate pairing $U_q^{an}(g_L) \otimes L[G]_q \to L$ which extends the natural non-degenerate pairing $U_A \otimes A[G]_v \to A$ defined in [Lu2]. Extending scalars we obtain the algebra $U_q^{an}(g_K)$ and the pairing $U_q^{an}(g_K) \otimes K[G]_q \to K$.

For the rest of this subsection we will assume that $d_i = 1$, i.e., $((a_{ij}))$ is symmetric, and $L = \mathbf{Q}_p$. These conditions can be relaxed. We make them in order to simplify formulas.

There is a natural action $U_q^{an}(g_K) \otimes K[G]_q \to K[G]_q$ (right action) given by the formula $l(f) = (id \otimes l)(\delta(f))$, where $l \in U_q^{an}(g_K), f \in K[G]_q$ and $\delta : K[G]_q \to K[G]_q \otimes K[G]_q$ is the coproduct.

Recall that $K[G]_q \simeq \oplus_\Lambda m_\Lambda V(\Lambda)$ which is the sum of irreducible finite-dimensional highest weight $U_q(g_K)$-modules $V(\Lambda)$ with multiplicities m_Λ. This is also an isomorphism of $U_q^{an}(g_K)$-modules. Each element $E_i^{(N)}, F_i^{(N)}$ acts locally nilpotently on $K[G]_q$, while each $t(x)$ acts as a semi-simple linear map.

Let $R = \oplus_{1 \le i \le n} \mathbf{Z}\alpha_i$ be the set of roots of $g_{\mathbf{Z}}$. We denote by R^+ (resp R^-) the set of positive (resp. negative) roots. We will often write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha \in R^+$ (resp. $\alpha \in R^-$). Following [KorSo], Chapter 4, or [Lu1], Chapter 3, 41, one can construct quantum root vectors $E_\alpha^{(N)}, F_\alpha^{(N)} \in U_q(g_K), \alpha > 0, N \ge 1$, such that $E_{\alpha_i}^{(N)} = E_i^{(N)}, F_{\alpha_i}^{(N)} = F_i^{(N)}$ (in order to keep track of integrality of the coefficients we are going to use the formulas from [Lu1]). Let us fix a convex linear order on the set of roots, such that all negative roots preceed all positive roots (convexity means that $\alpha < \alpha + \beta < \beta$ for positive roots and the oppoiste inequalities for negative roots) .

For every $0 < r \le 1$ we define $U_q(g_K)(r)$ as a K-vector space consisting of series

$$\xi = \sum_{m \in \mathbf{Z}^n, \alpha > 0, s_\alpha, p_\alpha \ge 0} c_{m,s,p} t^m / m! \prod_{\alpha > 0} F_\alpha^{(p_\alpha)} E_\alpha^{(s_\alpha)},$$

such that $c_{m,s,p} \in K, t^m/m! = t_1^{m_1}/m_1! \cdots t_n^{m_n}/m_n!, |c_{m,s,p}|r^{-(|s|+|p|+|m|)} \to 0$ as $|m| + |s| + |p| \to \infty$. Here and below m, s, p denote multi-indices. We define $|\xi|_r = \sup_{m,s,p} |c_{m,s,p}|r^{-(|s|+|p|+|m|)}$. Let $t_{\alpha_i}(x), 1 \le i \le n$ be an ordered basis of T^0 (see [SchT1], Section 4). Then, as a topological K-vector space (with the topology defined by the norm $| \bullet |_r$) the space $U_q(g_K)(r)$ is isomorphic to the K-vector space of infinite series

$$\eta = \sum_{N_i, M_i \ge 0, l_i \in \mathbf{Z}} b_{M,N} \prod_{1 \le i \le n} (t_{\alpha_i}(x) - 1)^{l_i} F_i^{(M_i)} E_i^{(N_i)},$$

such that $M = (M_i), N = (N_i)$, and $|b_{M,N,l}||t_{\alpha_i}(x) - 1|_r r^{-(|M|+|N|)} \to 0$ as $|M| + |N| + |l| \to \infty$, equipped with the norm defined by

$$|\eta|_r' = \sup_{M,N,l} |b_{M,N,l}||t_{\alpha_i}(x) - 1|_r r^{-(|M|+|N|)}.$$

It is easy to see that $U_q(g_K)(r)$ is a K-Banach vector space. It contains Banach vector subspaces $U_q^+(g_K)(r)$ (resp. $U_q^-(g_K)(r)$) which are closures of vector subspaces generated by all the elements $E_\alpha^{(N)}$ (resp. $F_\alpha^{(N)}$). It also contains an analytic neighborhood of $1 \in T^0$, which is an analytic group isomorphic to the ball of radius r in the Lie algebra $h_{\mathcal{O}_L}$. The latter is an analytic Lie group via Campbell-Hausdorff formula. We can always assume that r belongs to the algebraic closure of L, thus the corresponding analytic groups are in fact affinoid.

Proposition 4.2.1. *The norm $|\xi|_r$ (equivalently the norm $|\eta|_r'$) gives rise to a Banach K-algebra structure on $U_q(g_K)(r)$.*

Similarly to the case $q = 1$ (see [Em1], Section 5.2) one can ask whether the algebra $U_q(g_K)(r)$ corresponds to a "good" analytic group. Let us consider

the completion of the tensor product $U_q(g_K)(r) \otimes U_q(g_K)(r)$ with respect to the minimal Banach norm. Then we have the following result.

Proposition 4.2.2. *The Hopf algebra structure on $U_q^{an}(g_K)$ admits a continuous extension to $U_q(g_K)(r)$, making it into a topological Hopf algebra.*

Let us consider the topological K-algebra $U_q^{(1)}(g_K)$ which is the projective limit of $U_q(g_K)(r)$ for all $0 < r < 1$. Then we have the following result, which is a corollary of the previous proposition.

Proposition 4.2.3. *The Hopf algebra structure on $U_q^{an}(g_K)$ admits a continuous extension to $U_q^{(1)}(g_K)$, making it into a topological Hopf algebra.*

Since the elements E_α, F_α act locally nilpotently on $K[G]_q$, there is a well-defined action of $U_q(g_K)(r)$ on $K[G]_q$, which extends to the action of $U_q^{(1)}(g_K)$ on $K[G]_q$. Notice that the pairing $U_q(g_K)(r) \otimes K[G]_q \to K, (l, f) \mapsto l(f)$ is non-degenerate. In particular we can define the norm on $K[G]_q$ by the formula $||f||_r = \sup_{l \neq 0} \frac{|l(f)|}{|l|_r}, l \in U_q(g_K)(r)$.

Let now $H_r, r = p^{-N}$ be a "small" compact open subgroup of G. This means that the exponential map $exp : \mathbf{Z}^d = h_{\mathbf{Z}} \oplus g_{\mathbf{Z}}^+ \oplus g_{\mathbf{Z}}^- \to G$ defines an analytic isomorphism $B(0, r) \to H_r$, where $B(0, r) \subset \mathbf{Z}_p^d$ is the ball consisting of points $(x_i, x_\alpha, y_\alpha) \in \mathbf{Z}_p^d, 1 \leq i \leq n, \alpha > 0$ such that $x_\alpha, y_\alpha \in p^N \mathbf{Z}_p, x_i \in p^N \mathbf{Z}_p$, for all $\alpha > 0, 1 \leq i \leq n$.

Definition 4.2.4. *The space of analytic functions on the quantum group H_r (notation $C^{an}(H_r, K)_q$) is the completion of $K[G]_q$ with respect to the norm $|| \bullet ||_r$.*

Proposition 4.2.5. *The space $C^{an}(H_r, K)_q$ is a Banach Hopf K-algebra.*

Definition 4.2.6. *The algebra of analytic distributions on the quantum group H_r (notation $D^{an}(H_r, K)_q$) is the strong dual to $C^{an}(H_r, K)_q$.*

One can define a norm $|| \bullet ||$ on $K[G]_q$ such that the completion with respect to this norm is by definition the algebra $C(H, K)_q$ of continuous functions on the open maximal compact subgroup $H = H_1$ (in the case of $q = 1$ this is a theorem, not a definition). Then we proceed as follows.

Any linear functional $l \in D^{an}(H_r, K)_q$ defines a seminorm $| \bullet |_l$ on $K[G]_q$ such that

$$|f|_l = ||(id \otimes l)\delta(f)||.$$

The collection of seminorms $| \bullet |_l, l \in D^{an}(H_r, K)_q$ gives rise to a locally convex topology on $K[G]_q$.

Definition 4.2.7. *The space $C^{an}(H, H_r, K)_q$ of functions on the quantum group H which are locally analytic with respect to the quantum group H_r is the completion of $K[G]_q$ in the topology defined by the collection of seminorms $| \bullet |_l$.*

Definition 4.2.8.

a) *The space $C^{la}(H, K)_q$ of locally analytic functions on the quantum group H is the inductive limit $\varinjlim_{r \leq 1} \mathcal{O}^{an}(H, H_r, K)$ (i.e., it consists of functions on quantum group H which are locally analytic with respect to some H_r, $r < 1$).*

b) *The space $D^{la}(H, K)_q$ of locally analytic distributions on the quantum group H is the strong dual to $C^{la}(H, K)_q$.*

Since some details related to the proof of the following results are not finished, I formulate it as a conjecture.

Conjecture 4.2.9. *Both spaces $C^{la}(H, K)_q$ and $D^{la}(H, K)_q$ are topological Hopf K-algebras. Furthermore, $D^{la}(H, K)_q$ is a Frechét-Stein algebra in the sense of [SchT1].*

In the next subsection we will explain the definition of the norm $|| \bullet ||$ in the case of the group $SL_2(\mathbf{Z}_p)$. The general case is similar, but requires more details. It will be considered in [So2].

4.3. The $GL_2(\mathbf{Z}_p)$-case

We will use the notation $K[GL_2(\mathbf{Q}_p)]_q$ for the algebra of regular K-valued functions on the algebraic quantum group $GL_2(\mathbf{Q}_p)$. It is known (see [KorSo], Chapter 3) that $K[GL_2(\mathbf{Q}_p)]_q$ is generated by generators $t_{ij}, 1 \leq i, j \leq 2$ subject to the relations

$$t_{11}t_{12} = q^{-1}t_{12}t_{11}, \qquad\qquad t_{11}t_{21} = q^{-1}t_{21}t_{11}, \tag{1}$$

$$t_{12}t_{22} = q^{-1}t_{22}t_{12}, \qquad\qquad t_{21}t_{22} = q^{-1}t_{22}t_{21}, \tag{2}$$

$$t_{12}t_{21} = t_{21}t_{12}, \quad t_{11}t_{22} - t_{22}t_{11} = \left(q^{-1} - q\right)t_{12}t_{21}, \tag{3}$$

The element $det_q = t_{11}t_{22} - q^{-1}t_{12}t_{21}$ generates the center of the above algebra. As a result, the algebra $K[SL_2(\mathbf{Q}_p)]_q$ of regular functions on quantum group $SL_2(\mathbf{Q}_p)$ is obtained from the above algebra by adding one more equation

$$t_{11}t_{22} - q^{-1}t_{12}t_{21} = 1. \tag{4}$$

We are going to use the ideas of the representation theory of quantized algebras of functions (see [KorSo]).

Let V be a separable K-Banach vector space. This means that V contains a dense K-vector subspace spanned by the orthonormal basis $e_m, m \geq 0$ (orthonormal means that $||e_m|| = 1$ for all m). Let us consider the following representations $V_c, c \in K$ of $K[GL_2(\mathbf{Q}_p)]_q$ in V (cf. [KorSo], Chapter 4, Section 4.1):

$$t_{11}(e_m) = a_{11}(m)e_{m-1}, t_{21}(e_m) = a_{21}(m)e_m,$$

$$t_{12}(e_m) = a_{12}(m)e_m, t_{22}(e_m) = a_{22}(m)e_{m+1},$$

$$det_q = c.$$

Here $a_{ij}(m) \in K$ and $e_m = 0$ for $m < 0$. In particular the line Ke_0 is invariant with respect to the subalgebra A_+ generated by t_{11}, t_{21}. Let us assume that not

all $a_{11}(m)$ are equal to zero. Then it is easy to see from the commutation relations between t_{ij} that

$$a_{21}(m) = a_{21}(0)q^{-m}, a_{12}(m) = a_{12}(0)q^{-m}, m \geq 0.$$

Moreover

$$a_{11}(m+1)a_{22}(m) - a_{11}(m)a_{22}(m-1) = (q^{-1} - q)q^{-2m}h_0,$$

where $h_0 = a_{21}(0)a_{12}(0)$. Let $s(m) = a_{11}(m)a_{22}(m-1), m \geq 1$. Then we have

$$s(m+1) - s(m) = (q^{-1} - q)q^{-2m}h_0, s(1) = (q^{-1} - q)h_0.$$

It follows that

$$s(m) = (q^{-1} - q)(1 + q^{-2} + \cdots + q^{-2(m-1)})h_0 = q(q^{-2m} - 1)h_0,$$

for all $m \geq 1$. Since the quantum determinant is equal to c, we have

$$s(m+1) = a_{11}(m+1)a_{22}(m) = c + q^{-2m-1}h_0.$$

Comparing two formulas for $s(m)$ we see that

$$h_0 = a_{21}(0)a_{12}(0) = -cq^{-1}.$$

From now on we will assume that $|1 - q| < 1$.

Then the operators t_{12} and t_{21} are bounded. We also have $|s(m)| = |c(q^{-m} - 1)| = |c|, m \geq 1$.

Assume that $a_{21}(0) \neq 0$. Then the above representations (which are algebraically irreducible as long as q is not a root of 1) depend on the parameters $a_{21}(0), a_{11}(m), a_{22}(m), m \geq 0$ subject to the relations $a_{11}(m)a_{22}(m-1) = c(1 - q^{-2m})$. We will further specify restrictions on these parameters. The idea is the same as in [KorSo], Chapter 3, where in order to define continuous functions on the quantum group $SU(2)$ we singled out irreducible representations of $\mathbf{C}[SL_2(\mathbf{C})]_q$ corresponding to the intersection of the group $SU(2)$ with the big Bruhat cell for $SL_2(\mathbf{C})$. This intersection is the union of symplectic leaves of the Poisson-Lie group $SU(2)$. Kernel of an irreducible representation defines a symplectic leaf ("orbit method") which explains the relationship of representation theory and symplectic geometry. Notice that in the case $q = 1$ one can define the algebra of continuous function $C(SU(2))$ in the following way. For any function $f \in \mathbf{C}[SL_2(\mathbf{C})]$ one takes its restriction to the above-mentioned union of symplectic leaves. Since the latter is dense in $SU(2)$, the completion of the algebra $C[SL_2(\mathbf{C})]$ with respect to the sup-norm taken over all irreducible representations corresponding to the symplectic leaves is exactly $C(SU(2))$. Now we observe that symplectic leaves in $SL_2(\mathbf{C})$ are algebraic subvarieties, therefore they exist over any field. We will use the same formulas in the case of any p-adic field L (in this section we take $L = \mathbf{Q}_p$). In order to specify a symplectic leaf in $GL_2(\mathbf{C})$ we need in addition to fix the value of the determinant (it belongs to the center of the Poisson algebra $\mathbf{C}[GL_2(\mathbf{C})]$).

Let us recall (see [KorSo], Chapter 3) that to every element $t, c \in K^{\times}$ one can assign a 1-dimensional representation $W_{c,t} = \mathbf{Q}_p e_0$ of $K[GL_2(L)]_q$ such that

$t_{11}(e_0) = te_0, t_{22}(e_0) = ct^{-1}e_0$, and the rest of generators act on e_0 trivially. Recall (see [KorSo], Chapter 1) that complex 2-dimensional symplectic leaves of $GL_2(\mathbf{C})$ are algebraic subvarieties $S_{c,t}$ given by the equations:

$$t_{11}t_{22} - t_{21}t_{12} = c, t_{12} = t^2 t_{21},$$

where c, t are non-zero complex numbers. We define symplectic leaves over $\mathbf{Q}_p$ by the same formulas, taking $c, t \in \mathbf{Q}_p$. In order to define the norm of the restriction of a regular function $f \in \mathbf{C}[GL_2(\mathbf{Q}_p)]$ on $GL_2(\mathbf{Z}_p)$ we can choose a subset in the set of symplectic leaves $S_{c,t}$ such that the union of their intersection with $GL_2(\mathbf{Z}_p)$ is dense in the latter group. It suffices to take those leaves $S_{c,t}$ for which $|c| \leq 1, t \in \mathbf{Z}_p^\times$, and both t_{12} and t_{21} are non-zero.

Let us consider infinite-dimensional representation $V_{c,t}$ as above for which $a_{12}(0) = t^2 a_{21}(0), |c| \leq 1$ for a fixed $i \geq 0$, and $a_{21}(0) \neq 0$. We will also assume that the norm of the operators corresponding to t_{11} and t_{22} is less or equal than 1. It follows from the equality $t^2 a_{21}^2(0) = -cq^{-1}$ that $|a_{21}(0)| = |c|$, hence the norm of the operators corresponding to t_{12} and t_{21} is less or equal than $|c| \leq 1$. It follows that the norm of the operator $\pi_{c,t}(f)$ corresponding to an element $f \in K[GL_2(\mathbf{Q}_p)]_q$ acting in $V_{c,t}$ is bounded from above as $V_{c,t}$ run through the set of irreducible representations with the above restrictions on c, t. In addition, we are going to consider only those $c \in K^\times$ for which $-cq^{-1}$ is a square in K. We define the norm $||f||_{GL_2(\mathbf{Z}_p),q}, f \in K[GL_2(\mathbf{Q}_p)]_q$ as the supremum of norms of the operators $\pi_{c,t}(f)$ corresponding an element f in all representations $V_{c,t}$ as above. This is the desired sup-norm which we used in our definition of the algebra of locally analytic functions.

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