Chapter 4

Comultiplications. Exponentials. Deformations

Completely different notions are now expounded: first comultiplications and interior multiplications; then exponentials (defined without exponential series); finally deformations of Clifford algebras, which need both exponentials and interior products. Exterior algebras play an important role, because with a weak additional hypothesis (the existence of scalar products) we shall prove that Clifford algebras are isomorphic to them as K-modules (and even as comodules). The first two sections of Chapter **3** are sufficient prerequisites for almost all this chapter.

4.1 Coalgebras and comodules

Coalgebras

The category $\mathcal{A}lg(K)$ is a subcategory of $\mathcal{M}od(K)$; let us find out which particular properties an object A of $\mathcal{A}lg(K)$ does possess, with the requirement that these properties must be understandable inside the category $\mathcal{M}od(K)$. For this purpose, instead of defining the multiplication by a bilinear mapping $A\times A \to A$, we define it by the corresponding linear mapping $\pi_A : A \otimes A \to A$, and instead of mentioning the unit element 1_A, we mention the linear mapping $\varepsilon_A : K \to A$ such that $\varepsilon_A(1) = 1_A$. The associativity of the algebra A and the properties of its unit element are equivalent to the following properties, which only involve objects and morphisms of $\mathcal{M}od(K)$:

$$\pi_A \ (\pi_A \otimes \mathrm{id}_A) = \pi_A \ (\mathrm{id}_A \otimes \pi_A) \ ,$$
$$\pi_A \ (\mathrm{id}_A \otimes \varepsilon_A) = \text{canonical isomorphism } A \otimes K \longrightarrow A \ ,$$
$$\pi_A \ (\varepsilon_A \otimes \mathrm{id}_A) = \text{canonical isomorphism } K \otimes A \longrightarrow A.$$

By means of the automorphism $\top$ of $A \otimes A$ such that $\top(a \otimes b) = b \otimes a$, we can write the condition that means that the algebra A is commutative: $\pi_A \top = \pi_A$.

An object A of $\mathcal{M}od(K)$ is called a *coalgebra* if it satisfies the previous three conditions in the dual category of $\mathcal{M}od(K)$; this means the existence of a linear mapping $\pi'_A : A \to A \otimes A$ and a linear mapping $\varepsilon'_A : A \to K$ such that

$$\begin{aligned}
(\pi'_A \otimes \mathrm{id}_A)\ \pi'_A &= (\mathrm{id}_A \otimes \pi'_A)\ \pi'_A\ , \\
(\mathrm{id}_A \otimes \varepsilon'_A)\ \pi'_A &= \text{canonical isomorphism } A \longrightarrow A \otimes K\ , \\
(\varepsilon'_A \otimes \mathrm{id}_A)\ \pi'_A &= \text{canonical isomorphism } A \longrightarrow K \otimes A.
\end{aligned}$$

The mapping π'_A is called the *comultiplication* (or *coproduct*) of the coalgebra A, and ε'_A is called its *counit.* The coalgebra A is said to be *cocommutative* if moreover $\top \pi'_A = \pi'_A$.

The first motivation of these definitions is the following theorem.

(4.1.1) **Theorem.** *If A is a coalgebra and B an algebra, then* $\mathrm{Hom}_K(A, B)$ *is an algebra when it is provided with the following multiplication:*

$$(u, v) \longmapsto u * v = \pi_B\ (u \otimes v)\ \pi'_A \quad : \quad A \to A \otimes A \to B \otimes B \to B$$

for all u and v in $\mathrm{Hom}(A, B)$*; the unit element of this algebra* $\mathrm{Hom}(A, B)$ *is* $\varepsilon_B \varepsilon'_A$. *It is commutative whenever A is cocommutative and B commutative.*

Proof. A straightforward calculation shows that

$$(u * v) * w \ = \ \pi_B\ (\pi_B \otimes \mathrm{id}_B)\ (u \otimes v \otimes w)\ (\pi'_A \otimes \mathrm{id}_A)\ \pi'_A\ ,$$

and the analogous expression of $u*(v*w)$ shows that the associativity of $\mathrm{Hom}(A, B)$ is a consequence of the associativity of B and the coassociativity of A. Another calculation shows that

$$u * (\varepsilon_B \varepsilon'_A) \ = \ \pi_B\ (\mathrm{id}_B \otimes \varepsilon_B)\ (u \otimes \mathrm{id}_K)(\mathrm{id}_A \otimes \varepsilon'_A)\pi'_A \ = \ u\ ,$$

and in the same way $(\varepsilon_B \varepsilon'_A) * v = v$. The proof of the statement about commutativity is still easier. □

The basic ring K is both an algebra and a coalgebra; π_K is the canonical isomorphism $K \otimes K \to K$, and π'_K is the reciprocal isomorphism $K \to K \otimes K$; both ε_K and ε'_K are equal to id_K. The above theorem will often be used to state that the dual module $A^* = \mathrm{Hom}(A, K)$ is an algebra whenever A is a coalgebra.

Additional information. For interested readers we present the category of coalgebras (but hurried readers may go directly to comodules). When A and B are algebras, a linear mapping $f : A \to B$ is an algebra morphism if (and only if) the two following equalities are true:

$$f\pi_A \ = \ \pi_B\ (f \otimes f) \quad \text{and} \quad f\ \varepsilon_A \ = \ \varepsilon_B\ ;$$

therefore when A and B are coalgebras, we say that $f : A \to B$ is a *coalgebra morphism* if (by definition)

$$\pi'_B \, f \;=\; (f \otimes f)\, \pi'_A \quad \text{and} \quad \varepsilon'_B \, f \;=\; \varepsilon'_A .$$

If $f_1 : A' \to A$ is a coalgebra morphism and $f_2 : B \to B'$ an algebra morphism, then the mapping $\mathrm{Hom}(f_1, f_2)$ (defined in **1.5**) is an algebra morphism from $\mathrm{Hom}(A, B)$ into $\mathrm{Hom}(A', B')$.

When A and B are algebras, the algebra structure put on $C = A \otimes B$ (see **1.3**) corresponds to

$$\pi_C \;=\; (\pi_A \otimes \pi_B)\, \top_{2,3} \quad \text{and} \quad \varepsilon_C \;=\; (\varepsilon_A \otimes \varepsilon_B)\, \pi'_K \;;$$

here $\top_{2,3}$ means the reversion of the second and third factors, whatever the modules which they belong to may be: $\top_{2,3}(a \otimes b \otimes a' \otimes b') = a \otimes a' \otimes b \otimes b'$. Therefore when A and B are coalgebras, we make $C = A \otimes B$ become a coalgebra by setting

$$\pi'_C \;=\; \top_{2,3}\, (\pi'_A \otimes \pi'_B) \quad \text{and} \quad \varepsilon'_C \;=\; \pi_K\, (\varepsilon'_A \otimes \varepsilon'_B).$$

When A is both an algebra and a coalgebra, it is called a *bialgebra* when the four mappings π_A, ε_A, π'_A, ε'_A are related together by the four equalities

(a) $\pi'_A \pi_A \;=\; (\pi_A \otimes \pi_A)\, \top_{2,3}\, (\pi'_A \otimes \pi'_A)$,
(b) $\pi'_A \varepsilon_A \;=\; (\varepsilon_A \otimes \varepsilon_A)\, \pi'_K$,
(c) $\varepsilon'_A \pi_A \;=\; \pi_K\, (\varepsilon'_A \otimes \varepsilon'_A)$,
(d) $\varepsilon'_A \varepsilon_A \;=\; \mathrm{id}_K$;

these four conditions can be interpreted in two different ways; first we can observe that the conditions (a) and (b) mean that $\pi'_A : A \to A \otimes A$ is an algebra morphism, and that (c) and (d) mean that $\varepsilon'_A : A \to K$ is also an algebra morphism; but in a dual way we can also observe that (a) and (c) mean that $\pi_A : A \otimes A \to A$ is a coalgebra morphism, and that (b) and (d) mean that $\varepsilon_A : K \to A$ is also a coalgebra morphism. The ring K is a trivial example of a bialgebra. Later the symmetric algebra $\mathrm{S}(M)$ of a module M and its exterior algebra $\bigwedge(M)$ will receive comultiplications and counits that are algebra morphisms; consequently they will become bialgebras.

Comodules

The notion of *comodule* is derived from the notion of module by duality in an analogous way. First let A be a K-algebra, that is a K-module provided with two mappings π_A and ε_A as above; instead of describing the properties of a left A-module M by means of a bilinear mapping $A \times M \to M$, we will use the associated linear mapping; thus when M is a K-module, we can say that a linear mapping

$\pi_M : A \otimes M \to M$ makes it become a left A-module if these two conditions are satisfied:

$$\pi_M\ (\pi_A \otimes \mathrm{id}_M) = \pi_M\ (\mathrm{id}_A \otimes \pi_M)\ ,$$
$$\pi_M\ (\varepsilon_A \otimes \mathrm{id}_M) = \text{canonical isomorphism } K \otimes M \longrightarrow M.$$

When M is a right A-module, then π_M is a linear mapping from $M \otimes A$ into M satisfying the evident analogous conditions. Later we shall need right comodules, whence the following definition: when A is a K-coalgebra and M a K-module, we say that a linear mapping $\pi'_M : M \to M \otimes A$ makes M become a *right comodule over* A if the two following conditions are satisfied:

$$(\mathrm{id}_M \otimes \pi'_A)\ \pi'_M = (\pi'_M \otimes \mathrm{id}_A)\ \pi'_M\ ,$$
$$(\mathrm{id}_M \otimes \varepsilon'_A)\ \pi'_M = \text{canonical isomorphism } M \longrightarrow M \otimes K.$$

Comodules are interesting because they naturally become modules over suitable algebras. Indeed it is sensible to wonder whether M would be a module over the algebra $\mathrm{Hom}(A, B)$ defined in (4.1.1) when it is a comodule over the coalgebra A and a module over the algebra B. This statement is actually true provided that we use together a structure of comodule on one side and a structure of module on the other side, and require some compatibility between both structures; for instance we can suppose that M is a right A-comodule and a left B-module, and then we must require that the comultiplication π'_M and the multiplication π_M are compatible in the following sense:

$$\pi'_M\ \pi_M\ =\ (\pi_M \otimes \mathrm{id}_A)\ (\mathrm{id}_B \otimes \pi'_M) \qquad \begin{array}{ccc} B \otimes M & \longrightarrow & B \otimes M \otimes A \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \otimes A \end{array}\quad ;$$

this requirement may be interpreted in this way: the comultiplication π'_M must be B-linear. Before stating the announced theorem, let us recall other statements in which similar features appear. For instance a change of side also appears in the following statement: $\mathrm{Hom}(M, N)$ is a left B-module when M is a right B-module (and N merely a K-module); this change of side is easily explained by the contravariance of the functor $\mathrm{Hom}(\ldots, N)$. Besides, when we wish to make M become a left module over an algebra $B \otimes C$ (assuming that it is already a left module over B and C), we must also require that the structures of module over B and C are compatible: the operation in M of any element of B must commute with the operation of any element of C (see (1.3.3)); this means that the multiplication $C \otimes M \to M$ must be B-linear. These explanations should make the following theorem look quite natural.

(4.1.2) **Theorem.** *If M is a right comodule over the coalgebra A and a left module over the algebra B, and if the comultiplication π'_M is B-linear, then M is a* *left module over the algebra* $\mathrm{Hom}(A, B)$*; the operation in* M *of an element* u *of* $\mathrm{Hom}(A, B)$ *is this endomorphism of* M *:*

$$\pi_M\ (u \otimes \mathrm{id}_M) \top \pi'_M \quad : \quad M \to M \otimes A \to A \otimes M \to B \otimes M \to M.$$

Here $\top$ is the canonical isomorphism $M \otimes A \to A \otimes M$. Yet $\pi_M(u \otimes \mathrm{id}_M)\top\pi'_M$ is the same thing as $\pi_M\top(\mathrm{id}_M \otimes u)\pi'_M$ if $\top$ now means the canonical isomorphism $M \otimes B \to B \otimes M$.

Proof. We must prove that the mapping $u \longmapsto \pi_M(u \otimes \mathrm{id}_M)\top\pi'_M$ is an algebra morphism from $\mathrm{Hom}(A, B)$ into $\mathrm{End}(M)$. First the unit element $\varepsilon_B\varepsilon'_A$ of $\mathrm{Hom}(A, B)$ is mapped to the endomorphism

$$M \longrightarrow M \otimes A \longrightarrow M \otimes K \longrightarrow K \otimes M \longrightarrow B \otimes M \longrightarrow M$$

which is id_M because $(\mathrm{id}_M \otimes \varepsilon'_A)\pi'_M$ is the canonical isomorphism $M \to M \otimes K$, and $\pi_M(\varepsilon_B \otimes \mathrm{id}_M)$ is the canonical isomorphism $K \otimes M \to M$. Now let u and v be two elements of $\mathrm{Hom}(A, B)$; the following diagram contains the proof of the equality $u * (v * x) = (u * v) * x$ (for all $x \in M$); the endomorphism $x \longmapsto u * (v * x)$ appears if you go from M to M through the first column, whereas the endomorphism $x \longmapsto (u * v) * x$ appears if you go from M to M through the third column. The places where u and v are involved, are all indicated; the double arrows $\longleftrightarrow$ indicate canonical isomorphisms, that represent either a "commutativity" property of a tensor product, or an "associativity" property, according to the parentheses that are displayed; for instance the mapping $M \otimes (A \otimes A) \to (A \otimes A) \otimes M$ that appears in the third column is the canonical isomorphism $x \otimes a \otimes a' \longmapsto a \otimes a' \otimes x$; in all other arrows a multiplication or a comultiplication is involved:

$$\begin{tikzcd}
M \arrow[r] & M \otimes A \arrow[r] \arrow[d, leftrightarrow] & (M \otimes A) \otimes A \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & M \otimes (A \otimes A) \arrow[ddd, leftrightarrow] \\
& A \otimes M \arrow[r] \arrow[d, "v\otimes\mathrm{id}"] & A \otimes (M \otimes A) \arrow[d, "v\otimes\mathrm{id}\otimes\mathrm{id}"] & \\
& B \otimes M \arrow[r] \arrow[d] & B \otimes (M \otimes A) \arrow[dd, leftrightarrow] & \\
& M \arrow[d] & & (A \otimes A) \otimes M \arrow[ddd, "u\otimes v\otimes\mathrm{id}"] \\
& M \otimes A \arrow[d, leftrightarrow] & (B \otimes M) \otimes A \arrow[l] \arrow[d, leftrightarrow] & \\
& A \otimes M \arrow[d, "u\otimes\mathrm{id}"] & A \otimes (B \otimes M) \arrow[l] \arrow[d, "u\otimes\mathrm{id}\otimes\mathrm{id}"] & \\
M & B \otimes M \arrow[l] & B \otimes (B \otimes M) \arrow[l] \arrow[r, leftrightarrow] & (B \otimes B) \otimes M
\end{tikzcd}$$

The coassociativity hypothesis $(\mathrm{id}_M \otimes \pi'_A)\pi'_M = (\pi'_M \otimes \mathrm{id}_A)\pi'_M$ is involved in the first line, the associativity hypothesis $\pi_M(\pi_B \otimes \mathrm{id}_M) = \pi_M(\mathrm{id}_B \otimes \pi_M)$ is involved in the last line, and the compatibility hypothesis relating π'_M and π_M is involved in the middle of the first two columns; all other places of this diagram only require trivial verifications. $\square$

Of course the compatibility hypothesis is always fulfilled when $B = K$; thus every right A-comodule is a left A^*-module, and A itself is a A^*-module.

An example: the coalgebra $\mathrm{S}(M)$

The symmetric algebra $\mathrm{S}(M)$ of a K-module M is a cocommutative coalgebra, and the definition of its comultiplication is now explained because later we shall meet a similar but slightly more difficult comultiplication that makes every Clifford algebra become a comodule. The comultiplication $\pi' : \mathrm{S}(M) \to \mathrm{S}(M) \otimes \mathrm{S}(M)$ is the unique algebra morphism extending the linear mapping $M \to \mathrm{S}(M) \otimes \mathrm{S}(M)$ defined by $a \longmapsto a \otimes 1 + 1 \otimes a$; and the counit $\varepsilon' : \mathrm{S}(M) \to K$ is the unique algebra morphism extending the linear mapping $M \to K$ defined by $a \longmapsto 0$. Let us prove that $\mathrm{S}(M)$ is now a coalgebra.

Indeed, if we identify $\mathrm{S}(M) \otimes \mathrm{S}(M)$ with the algebra $\mathrm{S}(M \oplus M)$ (see (1.5.1)), then $a \otimes 1 + 1 \otimes a$ is identified with $(a, a) \in M \oplus M$, and thus π' becomes the mapping $\mathrm{S}(\delta)$ associated by the functor S with the linear mapping $\delta : M \to M \oplus M$ defined by $\delta(a) = (a, a)$. And if we identify K with the symmetric algebra $\mathrm{S}(0)$ of a zero module, then ε' becomes the mapping $\mathrm{S}(\zeta)$ associated by the functor S with the zero mapping $\zeta : M \to 0$. In the equality $(\delta \oplus \mathrm{id}_M)\,\delta = (\mathrm{id}_M \oplus \delta)\,\delta$ both members are equal to the mapping $a \longmapsto (a, a, a)$ from M into $M \oplus M \oplus M$, and it is not more difficult to verify that

$$(\mathrm{id}_M \oplus \zeta)\ \delta = \text{canonical isomorphism } M \to M \oplus 0\ ;$$
$$(\zeta \oplus \mathrm{id}_M)\ \delta = \text{canonical isomorphism } M \to 0 \oplus M\ ;$$

if we transform the previous three equalities by means of the functor S, we get the equalities that mean that $\mathrm{S}(M)$ is a coalgebra. Since δ is invariant by the automorphism $(a, b) \longmapsto (b, a)$ of $M \oplus M$, we can add that $\mathrm{S}(M)$ is a cocommutative coalgebra.

Therefore the dual module $\mathrm{S}^*(M) = \mathrm{Hom}(\mathrm{S}(M), K)$ is a commutative algebra. Let $\mathrm{S}^{*n}(M)$ be the set of all linear forms on $\mathrm{S}(M)$ that vanish on all $\mathrm{S}^j(M)$ such that $j \neq n$. Thus $\mathrm{S}^{*n}(M)$ is naturally isomorphic to $\mathrm{S}^n(M)^* = \mathrm{Hom}(\mathrm{S}^n(M), K)$, and $\mathrm{S}^*(M)$ is isomorphic to the direct product of its submodules $\mathrm{S}^{*n}(M)$. Nevertheless the direct sum of the submodules $\mathrm{S}^{*n}(M)$ is a subalgebra of $\mathrm{S}^*(M)$, because $f \vee g$ belongs to $\mathrm{S}^{*(i+j)}(M)$ for all $f \in \mathrm{S}^{*i}(M)$ and $g \in \mathrm{S}^{*j}(M)$.

The comultiplication of $\mathrm{S}(M)$ is involved in the Leibniz formula which gives the successive derivatives of a product; it is worth explaining this, because analogous Leibniz formulas will appear in the context of exterior and Clifford algebras. Let us assume that M is a vector space of finite dimension over $\mathbb{R}$, and let $\mathcal{C}^\infty(M)$ be the algebra of indefinitely differentiable real functions on M. With each vector

$a \in M$ is associated a derivation ∂_a; the value of a derivative $\partial_a f$ at any point $x \in M$ is

$$\partial_a f(x) \;=\; \lim_{t\to 0}\,(f(x+ta) - f(x))\;t^{-1}\;;$$

it is known that the derivations ∂_a are pairwise commuting, and because of the universal property of $\mathrm{S}(M)$ the mapping $a \longmapsto \partial_a$ extends to an algebra morphism from $\mathrm{S}(M)$ into $\mathrm{End}(\mathcal{C}^\infty(M))$; it maps every $w \in \mathrm{S}(M)$ to an operator ∂_w on $\mathcal{C}^\infty(M)$ which is called a partial differential operator with constant coefficients. The *Leibniz formula* tells how such an operator ∂_w operates on a product fg of two functions:

$$\partial_w(fg) \;=\; \pi_{\mathcal{C}}\;(\partial_{\pi'(w)}(f\otimes g)). \tag{4.1.3}$$

There are two interpretations of this formula: we can consider that each element of $\mathrm{S}(M)\otimes\mathrm{S}(M)$ operates in $\mathcal{C}^\infty(M)\otimes\mathcal{C}^\infty(M)$, so that $\pi'(w)$ operates on $f\otimes g$, and then $\pi_{\mathcal{C}}$ is the multiplication mapping associated with the algebra $\mathcal{C}^\infty(M)$; but we can also identify $\mathrm{S}(M)\otimes\mathrm{S}(M)$ with $\mathrm{S}(M\oplus M)$, and $f\otimes g$ with the function on $M\oplus M$ defined by $(x,y)\longmapsto f(x)g(y)$ (as it is usually done in functional analysis), and then $\pi_{\mathcal{C}}$ is the morphism from $\mathcal{C}^\infty(M\oplus M)$ into $\mathcal{C}^\infty(M)$ which maps each function $(x,y)\mapsto h(x,y)$ to the function $x\mapsto h(x,x)$; both interpretations are legitimate. To show that the above formula is the same thing as the ordinary Leibniz formula, it suffices to replace w with a symmetric power a^n of a vector $a\in M$; then

$$\pi'(a^n) = (\pi'(a))^n = (a\otimes 1 + 1\otimes a)^n \;=\; \sum_{k=0}^{n} \frac{n!}{k!\;(n-k)!}\;a^k\otimes a^{n-k}\;,$$

and thus we get the well-known formula

$$\partial_a^n(fg) \;=\; \sum_{k=0}^{n}\frac{n!}{k!\;(n-k)!}\;(\partial_a^k f)\;(\partial_a^{n-k} g).$$

4.2 Algebras and coalgebras graded by parities

Let G be an additive monoid with zero element (see **2.7**); a K-module M is said to be *graded over* G if it is the direct sum of submodules M_j indexed by the elements j of G. Such a decomposition into a direct sum is called a *grading* (or *gradation*) *over* G. An element $x\in M$ is said to be *homogeneous* if it belongs to some M_j, and j (well defined whenever $x\neq 0$) is called the *degree* of x and denoted by ∂x. Whenever ∂x is written, it is silently assumed that x is homogeneous. When $\gamma : G\to G'$ is a morphism between monoids with zero elements, every module M graded over G is also graded over G' : for all $j'\in G'$, $M_{j'}$ is the direct sum of all M_j such that $\gamma(j)=j'$. Here we shall especially use gradings over $\mathbb{Z}/2\mathbb{Z}$, which are called *parity gradings*. Often they come from gradings over $\mathbb{N}$ or $\mathbb{Z}$ by means of the evident monoid morphisms $\mathbb{N} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Nevertheless later in **4.5** we shall also use quite different gradings: when an algebra A is the direct sum of a subalgebra A^0 and an ideal A^+, such a decomposition means a grading over a monoid containing two elements, namely "zero" and "positive"; such a grading results from any grading over $\mathbb{N}$ by means of an evident morphism monoid.

When M and N are graded over G, $M \otimes N$ is also graded over it: $(M \otimes N)_j$ is the direct sum of all $M_{j'} \otimes N_{j''}$ such that $j' + j'' = j$. An element f of $\mathrm{Hom}(M, N)$ is said to be *homogeneous of degree* j if $f(x)$ is homogeneous of degree $\partial x + j$ whenever x is homogeneous in M; we write $j = \partial f$. A *graded morphism* is a homogeneous morphism of degree 0. In the category of G-graded K-modules we only accept morphisms which are finite sums of homogeneous morphisms, so that the module of morphisms between two G-graded modules M and N inherits a G-grading; when G is a finite monoid, this module coincides with $\mathrm{Hom}(M, N)$ if its grading is forgotten.

When G-gradings are involved, every module P that has not been given a particular grading, automatically receives the *trivial grading* such that $M_0 = M$ and $M_j = 0$ for all $j \neq 0$. The ring K is always trivially graded.

Let A be an algebra (resp. a coalgebra), the structure of which is defined by the linear mappings π and ε (resp. π' and ε'); A is said to be an *algebra graded over* G (resp. a *coalgebra graded over* G) when the module A is graded over G and when π and ε (resp. π' and ε') are graded morphisms. When A is an algebra, this means that the degree of 1_A is null, and that the degree of a product is the sum of the degrees of the factors. Many definitions impose this rule on other kinds of products too; for instance $\partial f(x) = \partial f + \partial x$ if f is a homogeneous morphism as above, and consequently $\partial(f' \circ f) = \partial f' + \partial f$. When A is a graded coalgebra, the coproduct $\pi'(a)$ of every homogeneous $a \in A$ can be written as a sum $\sum_i b_i \otimes c_i$ such that $\partial b_i + \partial c_i = \partial a$ for each term of this sum, and moreover $\varepsilon'(a) = 0$ if $\partial a \neq 0$.

Of course *graded modules or comodules* are defined in the same way by requiring that the corresponding products or coproducts are determined by graded morphisms.

For algebras and coalgebras graded over $\mathbb{Z}/2\mathbb{Z}$, besides many usual constructions, there are also twisted analogous ones; the twisted tensor products (defined in **3.2**) are typical examples. When we deal with exterior and Clifford algebras, twisting factors ± 1 appear very often, and in order to avoid any trouble with them, it is more convenient to introduce them everywhere according to the following rule.

(4.2.1) **Twisting rule.** Whenever in a product (of any kind) of homogeneous factors the order of two letters is reversed, this reversion must be compensated by a twisting factor that changes the sign if and only if both letters represent odd factors.

In general the letters represent factors with arbitrary parities; if there are n letters, there are 2^n possible distributions of parities; let us assume that for instance the letters $\ell_1, \ell_2, \ldots, \ell_k$ are odd factors, and that $\ell_{k+1}, \ldots, \ell_n$ are even; if the factors $\ell_1,\ldots,\ell_k$ first appear in this order, and after some calculations in a different order, the product of all the twisting factors is the signature of the permutation inflicted on these k letters; since the signature of a product of permutations is the product of their signatures, we can forget which reversions of factors have been committed, and in which order, because at any moment the relative places of these k letters enable us to determine the exact value of the product of all twisting factors. Thus we can behave with twisting factors in an unconcerned way, and replace them with $\pm$ all along the calculations, since we are sure easily to find the value of their product at the end.

The uncompromising observance of the twisting rule is the wonderful remedy that delivers us from wasting an awful lot of energy in the calculation of a tremendous number of factors all belonging to the set $\{+1, -1\}$. We shall observe this rule even when it causes discrepancies with common use. Anyhow, it is hard to find a set of rules that accounts for all the fancies of common use, because it has conceded twisting factors without planning, always under constraint; the usefulness of systematic rules has been acknowledged only recently. The discrepancies with common use will be mentioned here at each occurrence.

When we systematically write a conventional sign $\pm$ which we calculate only at the end according to the permutation inflicted on the letters, we must carefully write every sign that is not automatically implied by the twisting rule. For instance the equality $\tau(xy) = \tau(y)\tau(x)$ involving the reversion τ (see (3.1.4)) may now be written $\tau(xy) = \pm(-1)^{\partial x \partial y}\tau(y)\tau(x)$, because the conventional sign $\pm$ automatically involves a twisting sign $(-1)^{\partial x \partial y}$ that here must be compensated.

When f and g belong respectively to $\mathrm{Hom}(M, M')$ and $\mathrm{Hom}(N, N')$, the ambivalent notation $f \otimes g$ means either an element of $\mathrm{Hom}(M, M') \otimes \mathrm{Hom}(N, N')$ or an element of $\mathrm{Hom}(M \otimes N,\ M' \otimes N')$; when all the involved modules are graded by parities, the latter $f \otimes g$ is replaced with the element $f \hat{\otimes} g$ of $\mathrm{Hom}(M \otimes N,\ M' \otimes N')$ defined in this way:

$$(f \hat{\otimes} g)(x \otimes y) \;=\; (-1)^{\partial g \partial x}\; f(x) \otimes g(y) \;; \tag{4.2.2}$$

when g is even, $f \hat{\otimes} g$ coincides with $f \otimes g$. This definition implies, for all $f' \in \mathrm{Hom}(M', M'')$ and all $g' \in \mathrm{Hom}(N', N'')$,

$$(f' \hat{\otimes} g') \circ (f \hat{\otimes} g) \;=\; (-1)^{\partial g' \partial f}\; f'f \hat{\otimes} g'g \;. \tag{4.2.3}$$

A lot of formulas of the same kind might be added, for instance the definition of the twisted tensor product of three morphisms:

$$(f \hat{\otimes} g \hat{\otimes} h)(x \otimes y \otimes z) \;=\; (-1)^{\partial g \partial x + \partial h \partial x + \partial h \partial y}\; f(x) \otimes g(y) \otimes h(z) \;.$$

When A is a coalgebra and B an algebra, both graded over $\mathbb{Z}/2\mathbb{Z}$, besides the algebra $\mathrm{Hom}(A, B)$ defined in (4.1.1), there is the *twisted algebra of morphisms*

$\mathrm{Hom}^\wedge(A,B)$ in which the product of two elements is defined in the following way:

$$u * v \;=\; \pi_B\ (u \mathbin{\hat{\otimes}} v)\ \pi'_A\ . \tag{4.2.4}$$

Theorem (4.1.1) remains valid for this new multiplication (provided that commutativity is replaced with twisted commutativity).

When A is a graded coalgebra, $\mathrm{Hom}^\wedge(A,K)$ is a graded algebra which often is still denoted by A^*. Nevertheless if the parity grading of A comes from a grading $A = \bigoplus A^n$ over $\mathbb{Z}$, in general A^* is not graded over $\mathbb{Z}$, because the $\mathbb{Z}$-grading is only available on the direct sum of all submodules like A^{*n} (the natural image of $(A^n)^*$ in A^*); this direct sum is a subalgebra. Moreover the elements of A^{*n} must be given the degree $-n$.

When M is a graded right comodule over A and a graded left module over B, there is a graded version of Theorem (4.1.2) stating that M is a graded left module over $\mathrm{Hom}^\wedge(A,B)$, provided that the operation of $u \in \mathrm{Hom}^\wedge(A,B)$ on $x \in M$ is defined by means of the twisted reversion $\top^\wedge$:

$$u * x \;=\; \pi_M\ (u \otimes \mathrm{id}_M)\ \top^\wedge\ \pi'_M\ (x) \quad \text{with} \quad \top^\wedge(y \otimes a) \;=\; (-1)^{\partial y \partial a} a \otimes y\ . \tag{4.2.5}$$

When C is the twisted tensor product of the graded algebras A and B, then

$$\pi_C \;=\; (\pi_A \otimes \pi_B)\ \top^\wedge_{2,3} \quad \text{with} \quad \top^\wedge_{2,3}(a \otimes b \otimes a' \otimes b') \;=\; (-1)^{\partial b \partial a'} a \otimes a' \otimes b \otimes b'.$$

The twisted tensor product C' of two graded coalgebras A' and B' is defined in an analogous way:

$$\pi'_{C'} \;=\; \top^\wedge_{2,3}\ (\pi'_{A'} \otimes \pi'_{B'}) \quad \text{if}\ \ C' = A' \mathbin{\hat{\otimes}} B'. \tag{4.2.6}$$

All these definitions are involved in the following proposition.

(4.2.7) **Proposition.** *Let A and A' be graded coalgebras, and B and B' graded algebras; there is a graded algebra morphism* (*called canonical morphism*) *from*

$$\mathrm{Hom}^\wedge(A,B) \mathbin{\hat{\otimes}} \mathrm{Hom}^\wedge(A',B') \quad \textit{into} \quad \mathrm{Hom}^\wedge(A \mathbin{\hat{\otimes}} A',\ B \mathbin{\hat{\otimes}} B')$$

that maps every $f \otimes f'$ to $f \mathbin{\hat{\otimes}} f'$.

Proof. Let us consider homogeneous elements f and g in $\mathrm{Hom}(A,B)$, f' and g' in $\mathrm{Hom}(A',B')$, a in A and a' in A'. Let $\sum_i b_i \otimes c_i$ and $\sum_j b'_j \otimes c'_j$ be the coproducts of a and a'. We must prove that

$$(f * g) \mathbin{\hat{\otimes}} (f' * g') \quad \text{and} \quad (-1)^{\partial f' \partial g}\ (f \mathbin{\hat{\otimes}} f') * (g \mathbin{\hat{\otimes}} g')$$

both map $a \otimes a'$ to the same element of $B \otimes B'$. Straightforward applications of the definitions show that they both map it to the element

$$\sum_{i,j}\ \pm\ f(b_i) g(c_i) \otimes f'(b'_j) g'(c'_j)\ ;$$

as explained above, we need not worry about the sign $\pm$; here it is determined by the parity of

$$(\partial f' + \partial g')\partial a \ + \partial g \partial b_i \ + \partial g' \partial b'_j . \qquad \square$$

Remarks.

(a) The nongraded version of (4.2.7) has not been stated; as a matter of fact, it is included in the above graded version when A, A', B, B' are all trivially graded.

(b) When A, B, A', B' are finitely generated projective modules, all canonical morphisms like $B \otimes A^* \to \mathrm{Hom}(A, B)$ or $A^* \otimes A'^* \to (A \otimes A')^*$ are bijective; consequently the canonical morphism described in (4.2.7) is also an isomorphism.

(c) The canonical morphism in (4.2.7) explains why the notation $f \hat{\otimes} f'$ is often replaced with $f \otimes f'$, since the former is the canonical image of the latter. Anyhow, when the twisting rule (4.2.1) is strictly observed, no ambiguity may occur.

4.3 Exterior algebras

The study of exterior algebras has already begun in **3.1** and **3.2** where they are treated as Clifford algebras of null quadratic forms. The exterior algebra of a module M is provided with an $\mathbb{N}$-grading: $\bigwedge(M) = \bigoplus_n \bigwedge^n(M)$. Moreover $\bigwedge^0(M) = K$ and $\bigwedge^1(M) = M$. The even subalgebra $\bigwedge_0(M)$ is the direct sum of all $\bigwedge^{2k}(M)$, and $\bigwedge_1(M)$ the direct sum of all $\bigwedge^{2k+1}(M)$.

Here is the universal property of the algebra $\bigwedge(M)$: every linear mapping f from M into any algebra P such that $f(a)^2 = 0$ for all $a \in M$, extends in a unique way to an algebra morphism $\bigwedge(M) \to P$. Each subspace $\bigwedge^n(M)$ has its own universal property: every alternate n-linear mapping g from M^n into any K-module P determines a unique linear mapping $g'' : \bigwedge^k(M) \to P$ such that

$$g(a_1, a_2, \ldots, a_n) \ = \ g''(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \quad \text{for all } a_1, a_2, \ldots, a_n \in M \ ;$$

the proof of this statement is analogous to that of (1.4.3).

These universal properties lead to functors $\bigwedge$ and $\bigwedge^n$ (or $\bigwedge_K$ and $\bigwedge_K^n$ when the basic ring must be specified). Indeed any K-linear mapping $f : M \to N$ extends to an algebra morphism $\bigwedge(f)$ from $\bigwedge(M)$ into $\bigwedge(N)$, and determines linear mappings $\bigwedge^n(f) : \bigwedge^n(M) \to \bigwedge^n(N)$ for all $n \in \mathbb{N}$.

If M and N are K-modules, the algebra $\bigwedge(M \oplus N)$ is canonically isomorphic to the twisted tensor product $\bigwedge(M) \hat{\otimes} \bigwedge(N)$; this is a particular case of (3.2.4).

If $K \to L$ is an extension of the basic ring, then $\bigwedge_L(L \otimes M)$ is canonically isomorphic to $L \otimes \bigwedge_K(M)$; this is a particular case of (3.1.9).

Like every Clifford algebra, $\bigwedge(M)$ admits a grade automorphism σ and a reversion τ, for which (3.1.5) and (3.2.8) give precise information.

When M is a free module, then $\bigwedge(M)$ too is a free module; this is stated in (3.2.5) when the rank is finite, in (3.2.7) when it is infinite, and moreover we know how to derive a basis of $\bigwedge(M)$ from every basis of M. The case of a finitely generated projective module M is treated in (3.2.6). When M is merely projective, there exists a module M' such that $M \oplus M'$ is free, consequently $\bigwedge(M) \hat{\otimes} \bigwedge(M')$ is a free module, and $\bigwedge(M)$ is projective because it is a direct summand of this free module.

In **4.5** we shall need the following lemma.

(4.3.1) **Lemma.** *If N is a submodule of M, the algebra $\bigwedge(M/N)$ is canonically isomorphic to the quotient of $\bigwedge(M)$ by the ideal $N \wedge \bigwedge(M)$ generated by N.*

Proof. With the quotient mapping $M \to M/N$ the functor $\bigwedge$ associates an algebra morphism vanishing on the ideal J generated by N in $\bigwedge(M)$, whence an algebra morphism $\bigwedge(M)/J \to \bigwedge(M/N)$. Conversely the mapping $M \to \bigwedge(M) \to \bigwedge(M)/J$ vanishes on N, and gives a linear mapping defined on M/N, which extends to an algebra morphism $\bigwedge(M/N) \to \bigwedge(M)/J$. Thus we have got two reciprocal morphisms. □

The exterior algebra $\bigwedge(M)$ becomes a coalgebra in the same way as $\mathrm{S}(M)$. The comultiplication $\pi' : \bigwedge(M) \to \bigwedge(M) \otimes \bigwedge(M)$ is the algebra morphism from $\bigwedge(M)$ into $\bigwedge(M) \hat{\otimes} \bigwedge(M)$ such that $\pi'(a) = a \otimes 1 + 1 \otimes a$ for all $a \in M$, and the counit $\varepsilon' : \bigwedge(M) \to K$ is the algebra morphism such that $\varepsilon'(a) = 0$ for all $a \in M$. If we identify $\bigwedge(M) \hat{\otimes} \bigwedge(M)$ with $\bigwedge(M \oplus M)$ and K with $\bigwedge(0)$, we recognize that π' and ε' are the algebra morphisms associated by the functor $\bigwedge$ with the linear mappings $\delta : a \longmapsto (a, a)$ and $\zeta : a \longmapsto 0$. This allows us to prove that π' and ε' actually give $\bigwedge(M)$ a structure of coalgebra.

Let us calculate $\pi'(x)$ when x is the exterior product of n elements a_1, $a_2, \ldots, a_n$ of M :

$$\pi'(x) = \sum_{j=0}^{k} \sum_{s} \operatorname{sgn}(s) \; (a_{s(1)} \wedge \cdots \wedge a_{s(j)}) \otimes (a_{s(j+1)} \wedge \cdots \wedge a_{s(n)}) \; ;$$

the second summation runs over the subset of all permutations s such that

$$s(1) < s(2) < \cdots < s(j) \quad \text{and} \quad s(j+1) < s(j+2) < \cdots < s(n) \; ,$$

and $\operatorname{sgn}(s)$ is the signature of s.

The dual space $\bigwedge^*(M) = \mathrm{Hom}^\wedge(\bigwedge(M), K)$ is an algebra for the multiplication defined by (4.2.4), and the specific symbol $\wedge$ is still used for this multiplication. Let us observe that $\bigwedge^*(M)$ is naturally isomorphic to the direct product of the modules $\bigwedge^n(M)^* = \mathrm{Hom}(\bigwedge^n(M), K)$, and even to their direct sum when M is finitely generated. The image of $\bigwedge^n(M)^*$ in $\bigwedge^*(M)$ is denoted by $\bigwedge^{*n}(M)$; its elements are the linear forms vanishing on all $\bigwedge^j(M)$ such that $j \neq n$, and as $\mathbb{Z}$-homogeneous elements, they have the degree $-n$. In an analogous way, $\bigwedge^{*\leq n}(M)$

(resp. $\bigwedge^{*\geq n}(M)$) is the set of all linear forms vanishing on all $\bigwedge^j(M)$ such that $j > n$ (resp. $j < n$).

Let f and g be elements of $\bigwedge^{*j}(M)$ and $\bigwedge^{*k}(M)$ respectively; since $\pi' = \bigwedge(\delta)$ is a morphism of $\mathbb{N}$-graded algebras, $f \wedge g$ belongs to $\bigwedge^{*(j+k)}(M)$; here is its value on the product of $j+k$ elements of M:

$$(f \wedge g)(a_1 \wedge a_2 \wedge \cdots \wedge a_{j+k})$$
$$= \sum_s \operatorname{sgn}(s)\ (-1)^{jk}\ f(a_{s(1)} \wedge \cdots \wedge a_{s(j)})\ g(a_{s(j+1)} \wedge \cdots \wedge a_{s(j+k)})\ ;$$

the summation runs on all permutations s satisfying the conditions required above.

Because of the universal property of $\bigwedge^n(M)$, $\bigwedge^n(M)^*$ can be identified with the set of all alternate n-linear forms on M; thus the above definition of $f \wedge g$ allows us to define the exterior product of an alternate j-linear form and an alternate k-linear form; the definition of the exterior product of two alternate multilinear forms has been classical long before comultiplications were used to explain it; nevertheless the twisting factor $(-1)^{jk}$ which appears above as a consequence of (4.2.2), has not been introduced in this classical definition; consequently a discrepancy with common use appears here: the product here denoted by $f \wedge g$ is understood elsewhere as the exterior product of g and f in this order.

When the twisting rule (4.2.1) is strictly observed, for all $h_1,\dots,h_n$ in $\bigwedge^{*1}(M)$ and all $a_1,\dots,a_n$ in M, the determinant of the matrix $\big(h_j(a_k)\big)$ (in which $j,\ k = 1, 2, \dots, n$) is equal to the value of $h_n \wedge h_{n-1} \wedge \cdots \wedge h_1$ on $a_1 \wedge a_2 \wedge \cdots \wedge a_n$.

Since $h \wedge h = 0$ for all $h \in \bigwedge^{*1}(M)$, the natural bijection $M^* \to \bigwedge^{*1}(M)$ extends to a canonical algebra morphism from $\bigwedge(M^*)$ into $\bigwedge^*(M)$. It is obviously an isomorphism when M is a free module of finite rank; consequently it is still an isomorphism when M is a finitely generated projective module.

Since $\bigwedge(M)$ is obviously a module over $\bigwedge(M)$ on the left (resp. right) side, $\bigwedge^*(M)$ is a module over $\bigwedge(M)$ on the right (resp. left) side; thus there are interior products $f \lfloor x$ and $x \rfloor f$ for all $f \in \bigwedge^*(M)$ and all $x \in \bigwedge(M)$. Because of the relation $f \lfloor x = (-1)^{\partial f \partial x} x \rfloor f$, both multiplications are equally useful; but the interior multiplication by x on the left side involves the canonical mapping $M \times M^* \to K$ defined by $(a, h) \longmapsto -h(a)$ according to the twisting rule (4.2.1); to avoid unpleasant twisting signs, we prefer the interior multiplication by x on the right side. By definition of $f \lfloor x$ the following identity holds for all $y \in \bigwedge(M)$:

$$(4.3.2) \qquad (f \lfloor x)(y) \;=\; f(x \wedge y)\ ;$$

thus $\bigwedge^*(M)$ becomes a right $\bigwedge(M)$-module:

$$(4.3.3) \qquad (f \lfloor x) \lfloor y \;=\; f \lfloor (x \wedge y).$$

When f and x belong respectively to $\bigwedge^{*j}(M)$ and $\bigwedge^k(M)$, then $f \lfloor x$ belongs to $\bigwedge^{*(j-k)}(M)$; thus the equality $\partial(f \lfloor x) = \partial f + \partial x$ is valid for $\mathbb{Z}$-degrees since the degrees of f, x and $f \lfloor x$ are respectively $-j$, k and $-j+k$.

The interior multiplication by an element a of M is a twisted derivation of degree $+1$, but since the multiplication in $\bigwedge^*(M)$ has been defined in agreement with the twisting rule (4.2.1), we get a formula slightly different from the usual one:

$$(4.3.4) \qquad (f \wedge g) \lfloor a \;=\; f \wedge (g \lfloor a) + (f \lfloor a) \wedge \sigma(g) \;;$$

indeed, if we write $\pi'(y) = \sum_i y_i' \otimes y_i''$ for some $y \in \bigwedge(M)$, then

$$\begin{aligned}((f \wedge g) \lfloor a)(y) &= (f \hat{\otimes} g)\big((a \otimes 1 + 1 \otimes a) \wedge \pi'(y)\big) \\ &= \sum_i (-1)^{\partial g(1+\partial y_i')} f(a \wedge y_i') g(y_i'') + \sum_i (-1)^{(1+\partial g)\partial y_i'} f(y_i') g(a \wedge y_i'') \;;\end{aligned}$$

the former (resp. latter) summation is the value of $(f \lfloor a) \wedge \sigma(g)$ (resp. $f \wedge (g \lfloor a)$) on y. □

When f vanishes on $\bigwedge^{>n}(M)$ and x has no component of degree $< n$, then $f \lfloor x$ belongs to K :

$$(4.3.5) \qquad f \lfloor x \;=\; f(x) \quad \text{for all} \quad f \in \bigwedge^{*\leq n}(M) \quad \text{and all} \quad x \in \bigwedge^{\geq n}(M) \;;$$

for instance $h \lfloor a = h(a)$ for all $a \in M$ and all $h \in \bigwedge^{*1}(M)$.

We can interpret $\bigwedge^*$ as a contravariant functor, namely $\mathrm{Hom}(\bigwedge(\dots),\, K)$; any linear mapping $w : M \to N$ determines an algebra morphism $\bigwedge^*(w) : \bigwedge^*(N) \to \bigwedge^*(M)$. Let g be an element of $\bigwedge^*(N)$ and x an element of $\bigwedge(M)$; straightforward calculations show that

$$(4.3.6) \qquad \bigwedge^*(w)(g) \;\lfloor\; x \;=\; \bigwedge^*(w)\big(\, g \;\lfloor\; \bigwedge(w)(x)\big).$$

It is natural to define the interior product of the elements

$$f \otimes g \in \bigwedge^*(M) \,\hat{\otimes}\, \bigwedge^*(N) \quad \text{and} \quad x \otimes y \in \bigwedge(M) \,\hat{\otimes}\, \bigwedge(N)$$

by the following formula:

$$(4.3.7) \qquad (f \otimes g) \;\lfloor\; (x \otimes y) \;=\; (1)^{\partial g \partial x}\; (f \lfloor x) \otimes (g \lfloor y).$$

This definition is so much the more sensible as it is compatible with the canonical morphism $f \otimes g \longmapsto f \hat{\otimes} g$ that is defined according to (4.2.7), and that maps $f \otimes g$ to a linear form on $\bigwedge(M) \hat{\otimes} \bigwedge(N)$. Indeed, because of the canonical isomorphism $\bigwedge(M) \hat{\otimes} \bigwedge(N) \cong \bigwedge(M \oplus N)$, we can identify $f \hat{\otimes} g$ with an element of $\bigwedge^*(M \oplus N)$ and $x \otimes y$ with an element of $\bigwedge(M \oplus N)$; therefore the interior product $(f \hat{\otimes} g) \lfloor (x \otimes y)$ is meaningful, and after some calculations it becomes clear that

$$(f \otimes g) \;\lfloor\; (x \otimes y) \;\longmapsto\; (f \hat{\otimes} g) \;\lfloor\; (x \otimes y).$$

These considerations lead us to the Leibniz formula in exterior algebras, which implies the derivation formula (4.3.4) as a particular case. There are two versions of this formula in (4.3.8) below, which correspond to the two possible interpretations of (4.1.3). With the diagonal mapping δ (that is $a \longmapsto (a,a)$) the contravariant functor $\bigwedge^*$ associates an algebra morphism $\bigwedge^*(\delta) : \bigwedge^*(M \oplus M) \to \bigwedge^*(M)$, and from the definition of the exterior product of two elements f and g of $\bigwedge(M)$ it immediately follows that $f \wedge g = \bigwedge^*(\delta)(f \hat{\otimes} g)$. This shows a close relation between $\bigwedge^*(\delta)$ and the morphism $\pi_* : \bigwedge^*(M) \hat{\otimes} \bigwedge^*(M) \to \bigwedge^*(M)$ that represents the multiplication in $\bigwedge^*(M)$. Now we claim that for all f, $g \in \bigwedge^*(M)$ and all $x \in \bigwedge(M)$,

$$(4.3.8) \qquad \begin{aligned} (f \wedge g) \rfloor x &= \bigwedge^*(\delta)\big((f \hat{\otimes} g) \ \rfloor \ \pi'(x)\big) \\ &= \ \pi_*\big((f \otimes g) \ \rfloor \ \pi'(x)\big). \end{aligned}$$

Indeed the former right-hand member comes from a direct application of (4.3.6), since $\pi' = \bigwedge(\delta)$ and $f \wedge g = \bigwedge^*(\delta)(f \hat{\otimes} g)$. The latter right-hand member involves the definition (4.3.7) and its compatibility with the canonical morphism $f \otimes g \longmapsto f \hat{\otimes} g$. □

Because of its parity grading, $\bigwedge^*(M)$ has a grade automorphism σ, and the equalities $(\sigma(f))(x) = f(\sigma(x))$ and $\sigma(f \rfloor x) = \sigma(f) \rfloor \sigma(x)$ are obviously true for all $f \in \bigwedge^*(M)$ and $x \in \bigwedge(M)$.

Let us define the reversion τ in $\bigwedge^*(M)$ by the formula $(\tau(f))(x) = f(\tau(x))$. This definition immediately implies that there are formulas analogous to (3.1.5) in $\bigwedge^*(M)$, but more work is necessary to verify that we have got an involution of $\bigwedge^*(M)$, in other words, $\tau(f \wedge g) = \tau(g) \wedge \tau(f) = (-1)^{\partial f \partial g}\tau(f) \wedge \tau(g)$. After some calculations this follows from

$$\begin{aligned} \pi' \circ \tau(x) &= (\tau \otimes \tau) \circ \pi'(x) \qquad \text{for all } x \in \textstyle\bigwedge_1(M), \\ &= (\tau \otimes \sigma\tau) \circ \pi'(x) \quad \text{for all } x \in \textstyle\bigwedge_0(M). \end{aligned}$$

Besides, for all $f \in \bigwedge^*(M)$ and all $x \in \bigwedge(M)$,

$$(4.3.9) \qquad \tau(f \rfloor x) \ = \ (-1)^{\partial x(\partial f + \partial x)} \, \tau(f) \ \rfloor \ \tau(x) \ ;$$

indeed from the definitions (in particular (4.3.2)) it follows that

$$\tau(f \rfloor x)(y) \ = \ (-1)^{\partial x \partial y} \ (\tau(f) \rfloor \tau(x))(y) \ ,$$

and we can suppose that $\partial y = \partial f + \partial x$ since both members of this equality vanish if y has another parity.

This classical interior multiplication $\bigwedge^*(M) \times \bigwedge(M) \to \bigwedge^*(M)$ will serve as a model for the interior multiplication presented in the next section.

4.4 Interior products in Clifford algebras

Let M be a K-module provided with a quadratic form $q : M \to K$, $\mathrm{C}\ell(M,q)$ the associated Clifford algebra, and $\rho : M \to \mathrm{C}\ell(M,q)$ the canonical morphism. Since the canonical algebra morphism $K \to \mathrm{C}\ell(M,q)$ is not always injective, we must distinguish the unit elements 1 in K and 1_q in $\mathrm{C}\ell(M,q)$. To get convenient notation, we denote the identity mappings of $\bigwedge(M)$, $\bigwedge^*(M)$ and $\mathrm{C}\ell(M,q)$ by $\mathrm{id}_\wedge$, id_* and id_q, and we denote the linear mappings that determine the algebra and coalgebra structures of $\bigwedge(M)$ by π, ε, π', ε', whereas π_* and π_q correspond to the multiplications in $\bigwedge^*(M)$ and $\mathrm{C}\ell(M,q)$.

(4.4.1) **Theorem.** *There exists a unique algebra morphism*

$$\pi'_q : \mathrm{C}\ell(M,q) \to \mathrm{C}\ell(M,q) \hat{\otimes} \bigwedge(M)$$

such that $\pi'_q(\rho(a)) = \rho(a) \otimes 1 + 1_q \otimes a$ *for all* $a \in M$*; it makes* $\mathrm{C}\ell(M,q)$ *become a right comodule over the coalgebra* $\bigwedge(M)$*.*

Proof. The unicity of π'_q is evident. Let $\delta : M \to M \oplus M$ be defined as in **4.1** and **4.3**: $\delta(a) = (a,a)$. It is clear that δ is a morphism from the quadratic module (M,q) into the orthogonal sum $(M,q) \perp (M,0)$; consequently it induces an algebra morphism $\mathrm{C}\ell(\delta)$ between the associated Clifford algebras; we can identify the Clifford algebra of this orthogonal sum with the twisted tensor product of $\mathrm{C}\ell(M,q)$ and $\bigwedge(M)$, and thus $\mathrm{C}\ell(\delta)$ maps $\rho(a)$ to $\rho(a)\otimes 1 + 1_q \otimes a$; this proves the existence of π'_q. The comultiplication π' of $\bigwedge(M)$ can also be identified with the algebra morphism derived from δ when δ is understood as a morphism from the trivial quadratic module $(M,0)$ into $(M,0) \perp (M,0)$, and its counit ε' is the algebra morphism associated with the zero morphism $\zeta : (M,0) \to (0,0)$. Consequently the required equalities

$$(\pi'_q \otimes \mathrm{id}_\wedge)\ \pi'_q = (\mathrm{id}_q \otimes \pi')\ \pi'_q \ ,$$
$$(\mathrm{id}_q \otimes \varepsilon')\ \pi'_q = \text{canonical isomorphism } \mathrm{C}\ell(M,q) \to \mathrm{C}\ell(M,q) \otimes K \ ,$$

are consequences of these equalities:

$$(\delta \oplus \mathrm{id}_M)\ \delta = (\mathrm{id}_M \oplus \delta)\ \delta = \text{mapping } a \longmapsto (a,a,a) \ ,$$
$$(\mathrm{id}_M \oplus \zeta)\ \delta = \text{canonical isomorphism } M \to M \oplus 0.$$ □

Since $\mathrm{C}\ell(M,q)$ is a right comodule over $\bigwedge(M)$ and a module over K, it is a left module over the algebra $\bigwedge^*(M) = \mathrm{Hom}^\wedge(\bigwedge(M), K)$ according to the graded version of Theorem (4.1.2). Consequently there is a multiplication $\bigwedge^*(M) \times \mathrm{C}\ell(M,q) \to \mathrm{C}\ell(M,q)$ that we shall call an interior multiplication and denote by $(f,x) \longmapsto f \rfloor x$. By definition,

$$(4.4.2) \qquad f \rfloor x \ = \ \sum_i \ (-1)^{\partial x'_i \partial x''_i}\ f(x''_i)\ x'_i \quad \text{if} \quad \pi'_q(x) = \sum_i x'_i \otimes x''_i \ ;$$

the reversion of x'_i and x''_i with the subsequent twisting sign is due to the presence of the twisted reversion $\top^\wedge$ in (4.2.5). We already know that

$$f \rfloor (g \rfloor x) \;=\; (f \wedge g) \rfloor x. \tag{4.4.3}$$

Let us consider an element $x = \rho(a_1)\rho(a_2)\cdots\rho(a_n)$ which is the product in $C\ell(M,q)$ of n elements of M; the calculation of $\pi'_q(x)$ is exactly similar to that of $\pi'(a_1 \wedge a_2 \wedge \cdots \wedge a_n)$ in **4.3**, but we write the result in a slightly different way to compensate the reversion of x'_i and x''_i in (4.4.2):

$$\pi'_q(x) \;=\; \sum_{j=0}^{n}\sum_{s} (-1)^{j(n-j)}\ \mathrm{sgn}(s)\ \big(\rho(a_{s(j+1)})\cdots\rho(a_{s(n)})\big) \otimes \big(a_{s(1)}\wedge\cdots\wedge a_{s(j)}\big)\ ;$$

the second summation still runs on the permutations s such that

$$s(1) < s(2) < \cdots < s(j) \quad \text{and} \quad s(j+1) < s(j+2) < \cdots < s(n)\ ;$$

a straightforward application of the definition (4.4.2) shows that

$$\begin{aligned} &f \ \rfloor\ \rho(a_1)\rho(a_2)\cdots\rho(a_n) \\ &\quad = \sum_{j=0}^{n}\sum_{s}\ \mathrm{sgn}(s)\ f\big(a_{s(1)}\wedge\cdots\wedge a_{s(j)}\big)\ \rho(a_{s(j+1)})\cdots\rho(a_{s(n)})\ . \end{aligned}$$

From this calculation we derive:

$$h \rfloor (xy) \;=\; (h \rfloor x)y \;+ \sigma(x)(h \rfloor y) \qquad \text{for all } h \in \textstyle\bigwedge^{*1}(M)\ , \tag{4.4.4}$$

$$\begin{aligned} &f \ \rfloor\ \big(\rho(a_1)\rho(a_2)\cdots\rho(a_n)\big) \\ &\quad = f\big(a_1 \wedge a_2 \wedge \cdots \wedge a_n\big)\ 1_q \qquad \text{for all } f \in \textstyle\bigwedge^{*\geq n}(M)\ ; \end{aligned} \tag{4.4.5}$$

in Formula (4.4.5), f must vanish on all $\bigwedge^j(M)$ with $j < n$. As for (4.4.4), it means that interior multiplications by elements of $\bigwedge^{*1}(M)$ are twisted derivations of odd degree. It is also clear that $f \rfloor x$ belongs to $C\ell^{\leq k-j}(M,q)$ when f belongs to $\bigwedge^{*j}(M)$ and x to $C\ell^{\leq k}(M,q)$.

Let w be a morphism from (M,q) into $(N,\tilde q)$, that is a linear mapping such that $\tilde q(w(a)) = q(a)$ for all $a \in M$. The functors $C\ell$ and $\bigwedge^*$ associate with w two algebra morphisms $C\ell(w) : C\ell(M,q) \to C\ell(N,\tilde q)$ and $\bigwedge^*(w) : \bigwedge^*(N) \to \bigwedge^*(M)$. This situation leads to a formula analogous to (4.3.6); for all $x \in M$ and all $g \in \bigwedge^*(N)$,

$$g \ \rfloor\ C\ell(w)(x) \;=\; C\ell(w)\ \big(\textstyle\bigwedge^*(w)(g)\ \rfloor\ x\ \big). \tag{4.4.6}$$

For the reversion τ there is a formula analogous to (4.3.9):

$$\tau(f \rfloor x) \;=\; (-1)^{\partial f(\partial f + \partial x)}\ \tau(f)\ \rfloor\ \tau(x). \tag{4.4.7}$$

Now we come to the Leibniz formulas. Although $\bigwedge^*(M)$ is not always a coalgebra, we can define a linear mapping $\pi^* : \bigwedge^*(M) \longrightarrow (\bigwedge(M)\,\hat\otimes\,\bigwedge(M))^*$

that looks like a comultiplication; it is the morphism associated with

$$\pi : \bigwedge(M) \hat{\otimes} \bigwedge(M) \longrightarrow \bigwedge(M)$$

by the contravariant functor $\mathrm{Hom}(\dots, K)$; consequently

$$\pi^*(f)\ (x \otimes y) \ = \ f(x \wedge y) \quad \text{for all } x,\ y \in \bigwedge(M). \tag{4.4.8}$$

It is worth noticing that π is the algebra morphism $\bigwedge(M) \hat{\otimes} \bigwedge(M) \to \bigwedge(M)$ associated by the functor $\bigwedge$ with the morphism $(a, b) \longmapsto a + b$ from $M \oplus M$ onto M; consequently π^* is the algebra morphism associated by the functor $\bigwedge^*$ with this mapping $(a, b) \longmapsto a + b$.

Besides, from (4.2.7) we derive a canonical morphism from $\bigwedge^*(M) \hat{\otimes} \bigwedge^*(M)$ into $(\bigwedge(M) \hat{\otimes} \bigwedge(M))^*$; when it is an isomorphism (for instance when M is a finitely generated projective module), π^* determines a morphism π'_* from $\bigwedge^*(M)$ into $\bigwedge^*(M) \hat{\otimes} \bigwedge^*(M)$ which makes $\bigwedge^*(M)$ actually become a coalgebra.

Here is the first *Leibniz formula* that shows the effect of the interior multiplication by $f \in \bigwedge^*(M)$ on the product xy of two elements of $\mathrm{C}\ell(M, q)$; it is meaningful because $x \otimes y$ and $\pi^*(f)$ can be understood as elements of $\mathrm{C}\ell((M, q) \perp (M, q))$ and $\bigwedge^*(M \oplus M)$:

$$f \rfloor (xy) \ = \ \pi_q \left(\pi^*(f) \ \rfloor \ (x \otimes y)\right). \tag{4.4.9}$$

Proof of (4.4.9). In the diagram just below the morphism that goes from

$$\mathrm{C}\ell(M, q) \hat{\otimes} \mathrm{C}\ell(M, q) \quad \text{to} \quad \mathrm{C}\ell(M, q)$$

through the left-hand column, is the mapping $(x \otimes y) \longmapsto f \rfloor (xy)$; but you get the mapping $(x \otimes y) \longmapsto \pi_q(\pi^*(f) \rfloor (x \otimes y))$ if you follow the longer path through the right-hand column. To prove that both paths give the same result, an oblique arrow has been added, which divides the diagram into two parts:

$$\begin{array}{ccc}
\mathrm{C}\ell(M,q) \hat{\otimes} \mathrm{C}\ell(M,q) & \longrightarrow & (\mathrm{C}\ell(M,q) \hat{\otimes} \bigwedge(M)) \hat{\otimes} (\mathrm{C}\ell(M,q) \hat{\otimes} \bigwedge(M)) \\
\downarrow & & \updownarrow \\
\mathrm{C}\ell(M,q) & & (\mathrm{C}\ell(M,q) \hat{\otimes} \mathrm{C}\ell(M,q)) \hat{\otimes} (\bigwedge(M) \hat{\otimes} \bigwedge(M)) \\
\downarrow & \swarrow & \updownarrow \\
\mathrm{C}\ell(M,q) \hat{\otimes} \bigwedge(M) & & (\bigwedge(M) \hat{\otimes} \bigwedge(M)) \hat{\otimes} (\mathrm{C}\ell(M,q) \hat{\otimes} \mathrm{C}\ell(M,q)) \\
\updownarrow & & \downarrow \\
\bigwedge(M) \hat{\otimes} \mathrm{C}\ell(M,q) & & \bigwedge(M) \hat{\otimes} (\mathrm{C}\ell(M,q) \hat{\otimes} \mathrm{C}\ell(M,q)) \\
\downarrow & & \downarrow \\
K \otimes \mathrm{C}\ell(M,q) & & K \otimes (\mathrm{C}\ell(M,q) \hat{\otimes} \mathrm{C}\ell(M,q)) \\
\updownarrow & & \updownarrow \\
\mathrm{C}\ell(M,q) & \longleftarrow & \mathrm{C}\ell(M,q) \hat{\otimes} \mathrm{C}\ell(M,q))
\end{array}$$

The upper part of this diagram shows two paths from $\mathrm{C}\ell(M, q) \hat{\otimes} \mathrm{C}\ell(M, q)$ into $\mathrm{C}\ell(M, q) \hat{\otimes} \bigwedge(M)$; they give equal morphisms because π'_q is an algebra morphism: $\pi'_q(xy) = \pi'_q(x)\pi'_q(y)$. The lower part of this diagram only requires a trivial verification. □

Unlike the Leibniz formula (4.3.8) which is an immediate consequence of (4.3.6), here (4.4.9) is not a consequence of (4.4.6); indeed π_q is not an algebra morphism, it is not associated by the functor $\mathrm{C}\ell$ with the mapping $(a,b) \longmapsto a+b$ from $M \oplus M$ to M, unless $q = 0$.

Besides, there is a definition analogous to (4.3.7):

$$(f \otimes g) \;\rfloor\; (x \otimes y) \;=\; (-1)^{\partial g \partial x} \; (f \rfloor x) \otimes (g \rfloor y)$$

for $f \in \bigwedge^*(M),\; g \in \bigwedge^*(N),\; x \in \mathrm{C}\ell(M,q)$ and $y \in \mathrm{C}\ell(N,\tilde{q})$. This definition too is compatible with the canonical morphism $\bigwedge^*(M) \hat{\otimes} \bigwedge^*(N) \to (\bigwedge(M) \hat{\otimes} \bigwedge(N))^*$. It can be used in case of an application of (4.4.9) when $\pi^*(f)$ is the image of some element $\sum_i f'_i \otimes f''_i \in \bigwedge^*(M) \hat{\otimes} \bigwedge^*(M)$:

$$f \rfloor (xy) \;=\; \sum_i \; (-1)^{\partial x \partial f''_i} \; (f'_i \rfloor x) \; (f''_i \rfloor y).$$

For instance every $\pi^*(h)$ with $h \in \bigwedge^{*1}(M)$ is the image of $h \otimes 1 + 1 \otimes h$, and this allows us to deduce the derivation formula (4.4.4) from the Leibniz formula.

Besides the Leibniz formula (4.4.9), the product $f \rfloor (xy)$ gives rise to two other formulas which involve the coproduct of x or y, and which are called *composite Leibniz formulas* because they need both interior multiplications $\lfloor$ and $\rfloor$. If $\pi'_q(x) = \sum_i x'_i \otimes x''_i$ and $\pi'_q(y) = \sum_j y'_j \otimes y''_j$ (with homogeneous x'_i and y'_j in $\mathrm{C}\ell(M,q)$, and homogeneous x''_i and y''_j in $\bigwedge(M)$), then

(4.4.10)
$$f \rfloor (xy) = \sum_i \; (-1)^{\partial f \partial x'_i} \; x'_i \; ((f \lfloor x''_i) \rfloor y) \;,$$
$$f \rfloor (xy) = \sum_j \; (-1)^{(\partial x + \partial y'_j)\partial y''_j} \; ((f \lfloor y''_j) \rfloor x) \; y'_j.$$

When x or y is an element $\rho(a)$ with $a \in M$, we get the *composite derivation formulas*

(4.4.11)
$$f \rfloor (\rho(a)y) = (f \lfloor a) \rfloor y + \rho(a)(\sigma(f) \rfloor y) \;,$$
$$f \rfloor (x\rho(a)) = (f \rfloor x)\rho(a) + (f \lfloor a) \rfloor \sigma(x).$$

Proof of (4.4.10). These two formulas are immediate consequences of these easy calculations:

$$f \rfloor (xy) = \sum_i \sum_j \; \pm f(x''_i \wedge y''_j) \; x'_i y'_j \;,$$
$$(f \lfloor x''_i) \rfloor y = \sum_j \; \pm f(x''_i \wedge y''_j) \; y'_i \;,$$
$$(f \lfloor y''_i) \rfloor x = \sum_i \; \pm f(y''_i \wedge x''_j) \; x'_j.$$

The three signs $\pm$ are those resulting from the twisting rule (4.2.1); if you wish to calculate them, remember that in the summations you must pay attention only to terms such that $\partial f = \partial x''_i + \partial y''_j$. $\square$

When $q = 0$, we get an interior multiplication $\bigwedge^*(M) \times \bigwedge(M) \to \bigwedge(M)$ satisfying all the properties stated here, with the consequent little changes of notation. The grading of $\bigwedge(M)$ over $\mathbb{Z}$ is involved in the following assertion: when f belongs to $\bigwedge^{*j}(M)$ and x to $\bigwedge^k(M)$, then $f \rfloor x$ belongs to $\bigwedge^{k-j}(M)$. Remember that this f has degree $-j$.

This interior multiplication appears in the composite Leibniz formula that deals with the same product $(f \wedge g) \lfloor x$ as (4.3.8), but does not involve the coproduct of x. If $\pi^*(f)$ is the image of $\sum_i f'_i \otimes f''_i$ (an element of $\bigwedge^*(M) \hat{\otimes} \bigwedge^*(M)$), then

$$(f \wedge g) \lfloor x \;=\; \sum_i (-1)^{\partial f''_i \partial g}\; f'_i \wedge (g \lfloor (f''_i \rfloor x)).$$

Here this composite Leibniz formula is never needed, and its proof is proposed as an exercise. When f belongs to $\bigwedge^{*1}(M)$, it gives a composite derivation formula.

Interior multiplications involving two factors respectively in $\bigwedge^*(M)$ (or $\bigwedge(M^*)$) and $\bigwedge(M)$ appear very often in the literature, yet with systematic discrepancies in the treatment of the twisting signs, since the twisting rule (4.2.1) is not always uncompromisingly enforced as it is here. More fundamental discrepancies appear when the factor undergoing the operation belongs to a Clifford algebra $\mathrm{C}\ell(M, q)$, because the assailing factor does not always belong to $\bigwedge^*(M)$ nor to $\bigwedge(M^*)$; sometimes it belongs to $\bigwedge(M)$ or even to $\mathrm{C}\ell(M, q)$ (as in (4.ex.8)). All these versions can be derived from the present one because the operation of an assailing factor belonging to $\bigwedge(M^*)$ or to $\bigwedge(M)$ or to $\mathrm{C}\ell(M, q)$ is always the operation of its natural image in $\bigwedge^*(M)$ by these natural morphisms:

$$\mathrm{C}\ell(M, q) \longrightarrow \bigwedge(M) \longrightarrow \bigwedge(M^*) \longrightarrow \bigwedge^*(M).$$

The first arrow $\Phi_{-\beta} : \mathrm{C}\ell(M, q) \to \bigwedge(M)$ is not an algebra morphism but a comodule isomorphism; it is associated with the "canonical scalar product" $\beta = \mathrm{b}_q/2$ as it is later explained at the end of **4.8**. The second arrow is the algebra morphism associated by the functor $\bigwedge$ with $\mathrm{d}_q : M \to M^*$; it is an isomorphism when q is nondegenerate. The third arrow is the algebra morphism that extends the natural injection $M^* \to \bigwedge^*(M)$; it is an isomorphism when M is projective and finitely generated.

This section ends with an easy yet very important result. Remember that for all $a \in M$, $\mathrm{d}_q(a)$ is the element of M^* such that $\mathrm{d}_q(a)(b) = \mathrm{b}_q(a, b)$; here this $\mathrm{d}_q(a)$ is silently identified with its canonical image in $\bigwedge^{*1}(M)$. For all $a \in M$ and all $x \in \mathrm{C}\ell(M, q)$, it is stated that

$$\rho(a)\; x \;-\; \sigma(x)\; \rho(a) \;=\; \mathrm{d}_q(a) \rfloor x. \tag{4.4.12}$$

Proof. We consider a as fixed. Let D_1 and D_2 be the mappings $x \longmapsto \rho(a)x - \sigma(x)\rho(a)$ and $x \longmapsto \mathrm{d}_q(a) \rfloor x$. On one side, for all $b \in M$,

$$D_1(\rho(b)) \;=\; \rho(a)\rho(b) + \rho(b)\rho(a) \;=\; \mathrm{b}_q(a,b)\; 1_q \;=\; D_2(\rho(b)) \;.$$

On the other side, for all x, $y \in \mathrm{C}\ell(M,q)$, and for $i = 1, 2$,

$$D_i(xy) \;=\; D_i(x)\; y \;+ (-1)^{\partial x} x\; D_i(y) \;;$$

indeed, when $i = 2$, this is a consequence of (4.4.4); and when $i = 1$, it is easy to verify that every odd element z in a graded algebra determines a twisted derivation $x \longmapsto zx - \sigma(x)z$. Since the algebra $\mathrm{C}\ell(M,q)$ is generated by $\rho(M)$, these common properties of D_1 and D_2 imply their equality. □

4.5 Exponentials in even exterior subalgebras

Let M be a K-module and $\bigwedge(M)$ its exterior algebra; the even subalgebra $\bigwedge_0(M)$ is commutative; it is the direct sum of $K = \bigwedge^0(M)$ and the ideal $\bigwedge_0^+(M)$ that is the direct sum of all $\bigwedge^{2i}(M)$ with $i > 0$; all elements in this ideal are nilpotent.

An element of $\bigwedge(M)$ is said to be *decomposable* if it is an element of $K = \bigwedge^0(M)$ or an element of $M = \bigwedge^1(M)$ or an exterior product of elements of M.

(4.5.1) **Theorem.** *There is a unique mapping* Exp *from* $\bigwedge_0^+(M)$ *into* $\bigwedge_0(M)$ *such that*

$$\mathrm{Exp}(x+y) \;=\; \mathrm{Exp}(x) \wedge \mathrm{Exp}(y) \quad \textit{for all } x \textit{ and } y \textit{ in } \textstyle\bigwedge_0^+(M),$$
$$\mathrm{Exp}(x) \;=\; 1 + x \quad \textit{whenever } x \textit{ is decomposable with even positive degree.}$$

Proof. The unicity of the mapping Exp is evident, since every element of $\bigwedge_0^+(M)$ is a sum of decomposable elements; every decomposition of x as a sum of decomposable elements allows us to calculate $\mathrm{Exp}(x)$, and we must prove that they all give the same value to $\mathrm{Exp}(x)$. If x is decomposable, $1-x$ is the $\wedge$-inverse of $1+x$ because $x \wedge x = 0$; thus it is easy to realize that the existence of the mapping Exp is equivalent to the following statement:

(4.5.2) *If* $x_1, x_2, \ldots, x_r$ *are decomposable elements in* $\bigwedge_0^+(M)$ *and if their sum vanishes, then*

$$(1+x_1) \wedge (1+x_2) \wedge \cdots \wedge (1+x_r) \;=\; 1 \;.$$

The existence of the mapping Exp is evident when K contains a subring isomorphic to the field $\mathbb{Q}$ of rational numbers, because every $x \in \bigwedge_0^+(M)$ is nilpotent, and the exponential series gives the value of $\mathrm{Exp}(x)$ in such a way that the statement (4.5.1) is true; this fact suggests a proof in three steps.

First step. If (4.5.2) is true for every module over $\mathbb{Z}$, then it is true for every module over K. Indeed the K-module M is also a $\mathbb{Z}$-module, and its identity mapping extends to a ring morphism from $\bigwedge_{\mathbb{Z}}(M)$ into $\bigwedge(M) = \bigwedge_K(M)$; the restricted mapping $\bigwedge^+_{\mathbb{Z}}(M) \to \bigwedge^+(M)$ is surjective, and its kernel is the ideal of $\bigwedge_{\mathbb{Z}}(M)$ generated by all elements $\lambda a \wedge b - a \wedge \lambda b$ with $\lambda \in K$ and a and $b \in M$. The decomposable elements $x_1, \dots, x_n$ in $\bigwedge^+_0(M)$ are images of decomposable elements $y_1, \dots, y_n$ in $\bigwedge_{\mathbb{Z}}(M)$, and since the sum $\sum_i x_i$ vanishes, the sum $\sum_i y_i$ is equal to a sum of several terms like $u \wedge (\lambda a \wedge b - a \wedge \lambda b) \wedge v$ with arbitrary decomposable factors u and v in $\bigwedge_{\mathbb{Z}}(M)$, both even or odd. If we have proved that (4.5.2) is valid in $\bigwedge_{\mathbb{Z}}(M)$, we can assert that the product of the r factors $1 + y_i$ and several other factors like

$$(1 - u \wedge \lambda a \wedge b \wedge v) \wedge (1 + u \wedge a \wedge \lambda b \wedge v)$$

is equal to 1. Therefore the product of the images of all these factors in $\bigwedge(M)$ is also equal to 1. The n factors $1 + y_i$ give the n factors $1 + x_i$ in $\bigwedge(M)$, but the above two factors give a product in $\bigwedge(M)$ equal to 1. It follows that (4.5.2) is also valid in $\bigwedge(M)$.

Second step. (4.5.2) is valid for free additive groups. Indeed if M is a free additive group, it can be considered as a subgroup of $\mathbb{Q} \otimes_{\mathbb{Z}} M$ whereas $\bigwedge_{\mathbb{Z}}(M)$ is a subring of $\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}}(M)$, itself canonically isomorphic to $\bigwedge_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M)$. The statement (4.5.2) is true for the $\mathbb{Q}$-module $\mathbb{Q} \otimes_{\mathbb{Z}} M$, consequently it is also true for the free group M.

Third step. (4.5.2) is true for all additive groups. Indeed if M is not free, there exists a surjective group morphism $N \to M$ defined on a free additive group N. If N_0 is the kernel of this morphism, the kernel of the ring morphism $\bigwedge_{\mathbb{Z}}(N) \to \bigwedge_{\mathbb{Z}}(M)$ is the ideal generated by N_0 (see (4.3.1)). Let $x_1, \dots, x_n$ be decomposable elements in $\bigwedge^+_{\mathbb{Z},0}(M)$, the sum of which is 0; they are the images of decomposable elements $y_1, \dots, y_n$ in $\bigwedge^+_{\mathbb{Z},0}(N)$, the sum of which belongs to the ideal generated by N_0; consequently their sum is also a sum of decomposable elements $z_1, \dots, z_r$ all belonging to this ideal. Since the property (4.5.2) is valid for the free group N, the product of all factors $(1+y_1), \dots, (1+y_n)$, $(1-z_1), \dots, (1-z_r)$ is equal to 1. If we transport this result in $\bigwedge(M)$ by means of the morphism $\bigwedge(N) \to \bigwedge(M)$, we get the awaited equality: the product of the factors $(1+x_1), \dots, (1+x_n)$ is 1. $\square$

(4.5.3) **Example.** Let M be a free K-module of rank 4 with basis (a, b, c, d), and $x = a \wedge b + c \wedge d$; obviously $x \wedge x = 2\, a \wedge b \wedge c \wedge d$ and

$$\mathrm{Exp}(x) = (1 + a \wedge b) \wedge (1 + c \wedge d) = 1 \ + a \wedge b \ + c \wedge d \ + a \wedge b \wedge c \wedge d.$$

When K is the field $\mathbb{Z}/2\mathbb{Z}$, then $x \wedge x = 0$, but $\mathrm{Exp}(x) \neq 1 + x$.

(4.5.4) **Corollary.** *For all $x \in \bigwedge^+_0(M)$ and all $h \in \bigwedge^{*1}(M)$,*

$$h \rfloor \mathrm{Exp}(x) \ = \ (h \rfloor x) \wedge \ \mathrm{Exp}(x).$$

Proof. Formula (4.4.4) shows that $(h \rfloor x) \wedge x = 0$ when x is decomposable with degree ≥ 2; consequently the equality in (4.5.4) is true when x is decomposable. When this equality is true for x and y, it is still true for $x + y$ because (4.4.4) allows us to write

$$\begin{aligned} h \rfloor \operatorname{Exp}(x+y) &= (h \rfloor \operatorname{Exp}(x)) \wedge \operatorname{Exp}(y) \; + \; \operatorname{Exp}(x) \wedge (h \rfloor \operatorname{Exp}(y)) \\ &= (h \rfloor x) \wedge \operatorname{Exp}(x) \wedge \operatorname{Exp}(y) \; + \; \operatorname{Exp}(x) \wedge (h \rfloor y) \wedge \operatorname{Exp}(y) \\ &= (h \rfloor (x+y)) \wedge \operatorname{Exp}(x+y) \; ; \end{aligned}$$

the conclusion follows. □

(4.5.5) **Corollary.** *If $w : M \to N$ is a linear mapping, for all $x \in \bigwedge_0^+(M)$,*

$$\operatorname{Exp}(\bigwedge(w)(x)) \;=\; \bigwedge(w)(\operatorname{Exp}(x)).$$

This corollary is evident, and also the following identity involving the reversion in $\bigwedge(M)$:

$$\operatorname{Exp}(\tau(x)) \;=\; \tau(\operatorname{Exp}(x)).$$

Now in the algebra $\bigwedge^*(M)$ we consider the subalgebra $\bigwedge_0^*(M)$ of all elements vanishing on $\bigwedge_1(M)$, and in this subalgebra, the ideal $\bigwedge_0^{*+}(M)$ of all elements also vanishing on $K = \bigwedge^0(M)$. The elements of this ideal are all nilpotent when M is finitely generated, but nonnilpotent elements exist in $\bigwedge^{*2}(M)$ when M is free with infinite bases.

(4.5.6) **Theorem.** *There exists a unique mapping* Exp *from $\bigwedge_0^{*+}(M)$ into $\bigwedge_0^*(M)$ such that the following equalities hold for all $f \in \bigwedge_0^{*+}(M)$ and all $a \in M$:*

$$(\operatorname{Exp}(f))(1) = 1 \quad \textit{and} \quad \operatorname{Exp}(f) \lfloor a \;=\; \operatorname{Exp}(f) \wedge (f \lfloor a) \; .$$

Proof. Let g_k be the restriction of $\operatorname{Exp}(f)$ to the sum $\bigwedge^{\leq k}(M)$ of all components of degree $\leq k$. First g_0 must be the identity mapping of K, and the other g_k are determined by induction on k by the following requirement, for all $a \in M$ and all $x \in \bigwedge^k(M)$ (see (4.3.2)):

$$g_{k+1}(a \wedge x) \;=\; (\operatorname{Exp}(f) \lfloor a)(x) \;=\; (g_k \wedge (f \lfloor a))(x) \; ;$$

this proves the unicity of $\operatorname{Exp}(f)$. Moreover since f vanishes on $\bigwedge_1(M)$, $f \lfloor a$ vanishes on $\bigwedge_0(M)$, and consequently, by induction on k, g_k vanishes on $\bigwedge_1(M) \cap \bigwedge^{\leq k}(M)$; thus the induction that determines the restrictions g_k, implies that $\operatorname{Exp}(f)$ vanishes on $\bigwedge_1(M)$ as required.

We prove the existence of g_k by induction on k. The existence of g_0 and g_1 is evident, and g_1 vanishes on $M = \bigwedge^1(M)$. We assume the existence of g_k for some $k \geq 1$, and we consider the following $(k+1)$-linear form g' on M^{k+1} :

$$g'(a_1, a_2, \ldots, a_{k+1}) \;=\; (g_k \wedge (f \lfloor a_1))(a_2 \wedge a_3 \wedge \cdots \wedge a_{k+1}) \; ;$$

if g' is alternate in all variables, it determines a linear form on $\bigwedge^{k+1}(M)$ that allows us to extend g_k to a linear form g_{k+1} on $\bigwedge^{\leq k+1}(M)$. Obviously g' is alternate in $a_2, a_3, \dots, a_{k+1}$; thus it suffices to prove that it vanishes if $a_1 = a_2$; in other words it suffices to prove the following equality for all $a \in M$ and all $x \in \bigwedge^{\leq k-1}(M)$:

$$(g_k \wedge (f \lfloor a))(a \wedge x) \ = \ 0 \ .$$

By means of (4.3.2), (4.3.4) and (4.3.3) we obtain

$$(g_k \wedge (f \lfloor a))(a \wedge x) \ = \ \big((g_k \wedge (f \lfloor a)) \lfloor a\big)(x) \ = \ -\big((g_k \lfloor a) \wedge (f \lfloor a)\big)(x) \ ;$$

because of the induction hypothesis, $g_k \lfloor a$ is equal to $g_{k-1} \wedge (f \lfloor a)$; and since $f \lfloor a$ is odd, its exterior square vanishes; all this proves the desired equality and completes the proof. □

(4.5.7) **Corollary.** *For all f and g in $\bigwedge_0^{*+}(M)$,*

$$\mathrm{Exp}(f+g) \ = \ \mathrm{Exp}(f) \wedge \mathrm{Exp}(g) \ .$$

Proof. Indeed by (4.3.4) we can write (for all $a \in M$)

$$\begin{aligned}(\mathrm{Exp}(f) \wedge \mathrm{Exp}(g)) \lfloor a &= \mathrm{Exp}(f) \wedge \mathrm{Exp}(g) \wedge (g \lfloor a) \ + \ \mathrm{Exp}(f) \wedge (f \lfloor a) \wedge \mathrm{Exp}(g) \\ &= \mathrm{Exp}(f) \wedge \mathrm{Exp}(g) \wedge ((f+g) \lfloor a).\end{aligned}$$ □

(4.5.8) **Corollary.** *If $w : M \longrightarrow N$ is a linear mapping, for all $g \in \bigwedge_0^{*+}(N)$*

$$\bigwedge^*(w)(\mathrm{Exp}(g)) \ = \ \mathrm{Exp}(\bigwedge^*(w)(g)) \ .$$

This is an easy consequence of (4.3.6); and the following equality involving the reversion in $\bigwedge^*(M)$ is also evident: $\mathrm{Exp}(\tau(g)) = \tau(\mathrm{Exp}(g))$.

Here is a last technical lemma involving a Clifford algebra.

(4.5.9) **Lemma.** *The equality* $\mathrm{Exp}(f) \rfloor x = x$ *holds if x belongs to $\mathrm{C}\ell^{\leq k}(M,q)$ and $f(\bigwedge^{\leq k}(M)) = 0$. The same equality holds if x belongs to the subalgebra of $\mathrm{C}\ell(M,q)$ generated by a direct summand N of M and $f(\bigwedge(N)) = 0$.*

Proof. From (4.5.6) it is easy to deduce that $\mathrm{Exp}(f)(\bigwedge^j(M)) = 0$ if $f(\bigwedge^{\leq k}(M)) = 0$ and $1 \leq j \leq k$, or that $\mathrm{Exp}(f)(\bigwedge^+(N)) = 0$ if $f(\bigwedge(N)) = 0$. Then $\pi'_q(x)$ can be written as a sum $\sum_i x'_i \otimes x''_i$ in which $(x'_1, x''_1) = (x, 1)$ whereas $x''_i \in \bigwedge^+(M)$ for all $i \neq 1$; moreover each x''_i belongs to $\bigwedge^{\leq k}(M)$ (resp. $\bigwedge(N)$) if x belongs to $\mathrm{C}\ell^{\leq k}(M,q)$ (resp. to the subalgebra generated by N). Now the conclusion $\mathrm{Exp}(f) \rfloor x = x$ follows from the definition (4.4.2). □

4.6 Systems of divided powers

Section **4.6** is just an appendix to **4.5** and hurried readers are advised to skip it. Although the system of divided powers in $\bigwedge_0(M)$ becomes superfluous if we use the exponentials presented in **4.5**, it is worth knowing that divided powers also account for the existence of these exponentials. Besides, systems of divided powers appear in many other places and lead to the universal algebras $\Gamma(M)$ mentioned below.

Let A be a commutative K-algebra such that the canonical morphism $K \to A$ is injective, and A is the direct sum of the image of K and some ideal A^+; we identify K with its image in A, and write $A = K \oplus A^+$. This situation may also be described in this manner: there are two algebra morphisms $\varepsilon_A : K \to A$ and $\varepsilon'_A : A \to K$ such that $\varepsilon'_A \varepsilon_A = \mathrm{id}_K$; the ideal A^+ is then the kernel of ε'_A. A *system of divided powers* on A is a sequence of mappings $x \longmapsto x^{[n]}$ from A^+ into A, such that the following six conditions are satisfied whenever x and y are in A^+, m and n in $\mathbb{N}$, and λ in K :

$$x^{[0]} = 1,\ x^{[1]} = x \quad \text{and} \quad x^{[n]} \in A^+ \text{ for all } n > 0\ ; \tag{4.6.1}$$

$$(\lambda x)^{[n]} \ = \ \lambda^n x^{[n]}\ ; \tag{4.6.2}$$

$$(x+y)^{[n]} \ = \ \sum_{k=0}^{n} x^{[k]}\, y^{[n-k]}\ ; \tag{4.6.3}$$

$$(xy)^{[n]} \ = \ x^n y^{[n]} \ = \ x^{[n]} y^n\ ; \tag{4.6.4}$$

$$x^{[m]} x^{[n]} \ = \ \frac{(m+n)!}{m!\, n!}\ x^{[m+n]}\ ; \tag{4.6.5}$$

$$\text{if } n > 0, \text{ then } (x^{[n]})^{[m]} \ = \ \frac{(mn)!}{m!\,(n!)^m}\ x^{[mn]}. \tag{4.6.6}$$

It is known that the rational number that appears in (4.6.5) is an integer. The rational number that appears in (4.6.6) is also an integer (provided that $n > 0$); this can be proved by induction on m with the help of the equality

$$\frac{(mn)!}{m!\,(n!)^m} \ = \ \frac{((m-1)n)!}{(m-1)!\,(n!)^{m-1}}\ \frac{(mn-1)!}{(mn-n)!\,(n-1)!}.$$

It is possible to prove that (4.6.4) is a consequence of the five other conditions; the proof begins with the identity $xy = (x+y)^{[2]} - x^{[2]} - y^{[2]}$ which follows from (4.6.3) and (4.6.1).

By using (4.6.1) and (4.6.5) and by induction on n it is easy to prove, for all $x \in A^+$ and $n \in \mathbb{N}$,

$$x^n \ = \ n!\ x^{[n]}\ ; \tag{4.6.7}$$

this explains why $x^{[n]}$ is called the nth divided power of x (divided by $n!$).

Let us suppose that the canonical morphism $\mathbb{Z} \to K$ extends to a ring morphism $\mathbb{Q} \to K$; then (4.6.7) proves the existence and unicity of a system of divided powers on every algebra A that is decomposable as $A = K \oplus A^+$. Indeed it is easy to prove that the six conditions (4.6.1) to (4.6.6) are consequences of (4.6.7) when all integers $n!$ are invertible in K. Therefore a system of divided powers is interesting only when some integers are *not* invertible in K.

The even exterior algebra $\bigwedge_0(M)$ is provided with a system of divided powers that can be deduced from Theorem (4.5.1) in this way: by means of the polynomial extension $K \to K[t]$ it is possible to define $\mathrm{Exp}(tx)$ in $K[t] \otimes \bigwedge_0(M)$, and then $x^{[m]}$ is the factor multiplied by t^m in the development of $\mathrm{Exp}(tx)$. The ideas underlying the proof of (4.5.1) can also show that a system of divided powers has actually been defined in this way.

The extension $K \to K[t]$ can also serve to define divided powers in $\bigwedge_0^*(M)$. When M is finitely generated, $\bigwedge_{K[t]}^*(K[t] \otimes M)$ can be identified with $K[t] \otimes \bigwedge^*(M)$ which is the direct sum of the submodules $t^n \otimes \bigwedge^*(M)$; in all cases $\bigwedge_{K[t]}^*(K[t] \otimes M)$ can be identified with a subalgebra of the direct product of the submodules $t^n \otimes \bigwedge^*(M)$; this fact gives sense to this definition: $f^{[m]}$ is the factor multiplied by t^m in the development of $\mathrm{Exp}(tf)$. Consequently

$$f^{[m+1]} \lfloor a \;=\; f^{[m]} \wedge (f \lfloor a) \quad \text{for all } a \in M;$$

this allows us to prove that a system of divided powers has been obtained.

Here is another nontrivial example of a system of divided powers. Let M be a K-module. The group $\mathcal{S}_n$ of all permutations of $\{1, 2, 3, \ldots, n\}$ acts in the nth tensor power $\mathrm{T}^n(M)$ of M; for all $s \in \mathcal{S}_n$, the action of s^{-1} in $\mathrm{T}^n(M)$ is the following one:

$$s^{-1}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \;=\; a_{s(1)} \otimes a_{s(2)} \otimes \cdots \otimes a_{s(n)}.$$

The elements of $\mathrm{T}^n(M)$ that are invariant under the action of $\mathcal{S}_n$ make up the submodule $\mathrm{ST}^n(M)$ of all symmetric n-tensors. Of course $\mathrm{ST}^0(M) = K$ and $\mathrm{ST}^1(M) = M$. The direct sum $\mathrm{ST}(M)$ of all $\mathrm{ST}^n(M)$ becomes a commutative algebra when it is provided with the following multiplication; if $y \in \mathrm{ST}^j(M)$ and $z \in \mathrm{ST}^k(M)$, their symmetric product is the symmetrized tensor

$$y \vee z \;=\; \sum_s \; s^{-1}(y \otimes z) \;,$$

where the summation runs only on those $s \in \mathcal{S}_{j+k}$ such that

$$s(1) < s(2) < s(3) < \cdots < s(j) \quad \text{and} \quad s(j+1) < s(j+2) < \cdots < s(j+k).$$

By induction on n it is easy to prove that the symmetric product of n elements of M is given by

$$a_1 \vee a_2 \vee \cdots \vee a_n \;=\; \sum_s \; s^{-1}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \;,$$

with a summation running over all $s \in \mathcal{S}_n$. In particular the nth symmetric power of a (element of M) and its nth tensor power are related by the equality

$$a \vee a \vee \cdots \vee a \;=\; n!\; a \otimes a \otimes \cdots \otimes a.$$

Of course much more work is necessary to prove that $\mathrm{ST}(M)$ is a commutative and associative algebra provided with a system of divided powers such that $a^{[n]}$ is the nth tensor power of a for all $a \in M$ and all $n \in \mathbb{N}$. Yet the unicity of this system of divided power is obvious.

It is known that the algebra $\mathrm{S}^*(M)$ dual to the coalgebra $\mathrm{S}(M)$ is also provided with a system of divided powers (see (4.ex.2)). When M is a finitely generated projective module, each dual space $(\mathrm{S}^n(M))^*$ is canonically isomorphic to $\mathrm{ST}^n(M^*)$.

The algebras provided with a system of divided powers constitute a subcategory $\mathcal{D}iv(K)$ of $\mathcal{C}om(K)$; a morphism in this category is an algebra morphism $f : K \oplus A^+ \longrightarrow K \oplus B^+$ such that $f(A^+) \subset B^+$ and $f(x^{[n]}) = f(x)^{[n]}$ for all $x \in A^+$ and all $n \in \mathbb{N}$. It is sensible to ask wether a module M may freely generate an algebra in this category, in the same way as it generates the algebras $\mathrm{T}(M)$ and $\mathrm{S}(M)$ in the categories $\mathcal{A}lg(K)$ and $\mathcal{C}om(K)$. The answer is positive: with M is associated a universal algebra $\Gamma(M)$ provided with a system of divided powers; it is an $\mathbb{N}$-graded algebra such that $\Gamma^0(M) = K$ and $\Gamma^1(M) = M$. Its universal property says that every linear mapping f from M into an object $K \oplus A^+$ of $\mathcal{D}iv(K)$ such that $f(M) \subset A^+$, extends in a unique way to a morphism $f' : \Gamma(M) \to K \oplus A^+$ in the category $\mathcal{D}iv(K)$.

In particular there is a unique morphism from $\Gamma(M)$ into the previous algebra $\mathrm{ST}(M)$ that maps every $a \in M$ to itself; it is known that it is an isomorphism whenever M is a projective module. Besides, in the category $\mathcal{C}om(K)$ there is a unique algebra morphism from $\mathrm{S}(M)$ into $\Gamma(M)$ that maps every $a \in M$ to itself; it is an isomorphism whenever there is a ring morphism $\mathbb{Q} \to K$.

It remains to report that the submodule $\Gamma^2(M)$ of this algebra $\Gamma(M)$ is canonically isomorphic to the module defined in **2.1** and already denoted by $\Gamma^2(M)$. In other words, for every quadratic mapping $q : M \to N$ there exists a unique linear mapping $\tilde{q} : \Gamma^2(M) \to N$ such that $q(a) = \tilde{q}(a^{[2]})$ for all $a \in M$.

4.7 Deformations of Clifford algebras

This is the main section in Chapter **4**. The notations are those of **4.3** and **4.4**; we still consider the same three algebras, and all the notations referring to each one are here recalled without comment:

$\mathrm{C}\ell(M,q) \;:\; \rho\,,\; 1_q\,,\; \mathrm{id}_q\,,\; \pi_q\,,\; \pi'_q\;;$

$\bigwedge(M) \;:\; \mathrm{id}_\wedge\,,\; \pi\,,\; \pi'\,,\; \varepsilon\,,\; \varepsilon'\;;$

$\bigwedge^*(M) \;:\; \mathrm{id}_*\,,\; \pi_*\,,\; \pi^*\;.$

Although the unit element of $\bigwedge^*(M)$ is ε', it is rather denoted by 1 wherever this notation causes no ambiguity. Each of these three algebras is provided with a grade automorphism σ and a reversion τ.

Now a new figure β appears; it is any bilinear form $\beta : M \times M \to K$. All the notations referring to β are presented here together. With q and β we associate the quadratic form q' such that $q'(a) = q(a) + \beta(a,a)$ for all $a \in M$, and ρ' is the canonical mapping $M \to C\ell(M,q')$. According to the twisting rule (4.2.1), with β we associate the *twisted opposite bilinear form* β^{to} defined by $\beta^{to}(a,b) = -\beta(b,a)$. The linear mappings $M \to M^*$ determined by β and β^{to} are denoted by d_β and d_β^{to} :

$$\mathrm{d}_\beta(a)(b) = \beta(a,b) \quad \text{and} \quad \mathrm{d}_\beta^{to}(a)(b) = -\beta(b,a) \ .$$

The *opposite bilinear form* $(a,b) \longmapsto \beta(b,a)$ appears only later in (4.7.15) when the reversion τ gets involved; it is denoted by $-\beta^{to}$ (rather than β^o). In $\bigwedge^{*2}(M)$ there is an element $[\beta]$ such that

$$[\beta](a \wedge b) \ = \ \beta(a,b) - \beta(b,a) \quad \text{for all } a,\ b \in M.$$

Moreover, since $\bigwedge^2(M \oplus M)$ is canonically isomorphic to the direct sum of $\bigwedge^2(M) \otimes 1$, $1 \otimes \bigwedge^2(M)$ and $M \otimes M$, in $\bigwedge^{*2}(M \oplus M)$ there is a submodule canonically isomorphic to $(M \otimes M)^*$; and since the bilinear forms on M are in bijection with the elements of $(M \otimes M)^*$, β has a canonical image $\beta_{\prime\prime}$ in $\bigwedge^{*2}(M \oplus M)$. Thus $\beta_{\prime\prime}$ is the element of $\bigwedge^{*2}(M \oplus M)$ such that

$$\beta_{\prime\prime}((a_1,b_1) \wedge (a_2,b_2)) \ = \ \beta(a_1,b_2) - \beta(a_2,b_1) \ .$$

Since we can identify $(a_1,b_1) \wedge (a_2,b_2) \in \bigwedge(M \oplus M)$ with

$$(a_1 \wedge a_2) \otimes 1 \ + 1 \otimes (b_1 \wedge b_2) \ + (a_1 \otimes b_2) - (a_2 \otimes b_1) \ \in \bigwedge(M) \hat{\otimes} \bigwedge(M),$$

we can also say that $\beta_{\prime\prime}$ is the linear form on $\bigwedge(M) \hat{\otimes} \bigwedge(M)$ that vanishes on $\bigwedge^i(M) \otimes \bigwedge^j(M)$ whenever $(i,j) \neq (1,1)$, and such that $\beta_{\prime\prime}(a \otimes b) = \beta(a,b)$ for all $a,\ b \in M$.

This is not yet sufficient, since we shall also use the three images of β in $\bigwedge^{*2}(M \oplus M \oplus M)$. A notation like $\beta_{\prime\prime\cdot}$ or $\beta_{\prime\cdot\prime}$ or $\beta_{\cdot\prime\prime}$ should clearly enough indicate which of these three images we consider: $\beta_{\prime\prime\cdot}$ (resp. $\beta_{\prime\cdot\prime}$) (resp. $\beta_{\cdot\prime\prime}$) is the linear form on $\bigwedge(M) \hat{\otimes} \bigwedge(M) \hat{\otimes} \bigwedge(M)$ that vanishes on $\bigwedge^i(M) \otimes \bigwedge^j(M) \otimes \bigwedge^k(M)$ whenever (i,j,k) is not equal to $(1,1,0)$ (resp. $(1,0,1)$) (resp. $(0,1,1)$) and such that

$$\beta_{\prime\prime\cdot}(a \otimes b \otimes 1) = \beta(a,b) \ , \quad \text{resp.} \ \ \beta_{\prime\cdot\prime}(a \otimes 1 \otimes b) = \beta(a,b) \ ,$$
$$\text{resp.} \ \ \beta_{\cdot\prime\prime}(1 \otimes a \otimes b) = \beta(a,b).$$

Later we shall even use $\beta_{\prime\cdot\prime\cdot}$ which is one of the six images of β in $\bigwedge^{*2}(M \oplus M \oplus M \oplus M)$, and we rely on the reader to guess its definition.

(4.7.1) **Definition.** The *deformation* of the algebra $\mathrm{C}\ell(M,q)$ by the bilinear form β is the K-module $\mathrm{C}\ell(M,q)$ provided with the following multiplication:

$$(x,y) \longmapsto x \star y = \pi_q\big(\mathrm{Exp}(\beta_{\prime\prime}) \rfloor (x \otimes y)\big) \ ;$$

this deformation is denoted by $\mathrm{C}\ell(M,q;\beta)$.

Usually the word "deformation" (when it does not mean an infinitesimal deformation) refers to a family of multiplications depending on a parameter t in such a way that the initial multiplication is obtained for $t = 0$. Nonetheless there is no impassable gap between this usual concept of deformation and Definition (4.7.1); indeed if we consider the polynomial extension $K \to K[t]$, the deformation of $K[t] \otimes \mathrm{C}\ell(M,q)$ by the bilinear form $t \otimes \beta$ gives the initial multiplication when t is replaced with 0, and the new one when t is replaced with 1.

The notation $\bigwedge(M;\beta)$ means $\mathrm{C}\ell(M,0;\beta)$; nonetheless the modified multiplication of $\bigwedge(M;\beta)$ is simply denoted by $(x,y) \longmapsto xy$ (instead of $x \star y$), since the initial multiplication in $\bigwedge(M)$ is already denoted by the proper symbol $\wedge$.

Since $\mathrm{Exp}(\beta_{\prime\prime})$ is even, it is clear that the deformed algebra $\mathrm{C}\ell(M,q;\beta)$ is graded by the same subspaces $\mathrm{C}\ell_0(M,q)$ and $\mathrm{C}\ell_1(M,q)$ as $\mathrm{C}\ell(M,q)$.

More than the half of this section is devoted to the proof of the following five theorems, and near the end, a sixth theorem shall be added.

(4.7.2) **Theorem.** *The deformation $\mathrm{C}\ell(M,q;\beta)$ is an associative algebra with the same unit element 1_q.*

(4.7.3) **Theorem.** *These two equalities are true for all $a \in M$ and all $x \in \mathrm{C}\ell(M,q)$:*

(a) $\rho(a) \star x = \rho(a)\, x + \mathrm{d}_\beta(a) \rfloor x$;

(b) $x \star \rho(a) = x\, \rho(a) + \mathrm{d}_\beta^{to}(a) \rfloor \sigma(x)$.

Here $\mathrm{d}_\beta(a)$ and $\mathrm{d}_\beta^{to}(a)$ must be understood as elements of $\bigwedge^{*1}(M)$, that are linear forms on $\bigwedge(M)$ vanishing on all $\bigwedge^j(M)$ such that $j \neq 1$; such identifications of elements of M^* with their image in $\bigwedge^{*1}(M)$ will be silently committed when the context obviously requires them.

(4.7.4) **Theorem.** *Among all the associative multiplications on the K-module $\mathrm{C}\ell(M,q)$ that admit 1_q as a unit element, the multiplication defined by* (4.7.1) *is the only one satisfying the equality* (a) *in* (4.7.3) (*for all $a \in M$ and all $x \in \mathrm{C}\ell(M,q)$*). *It is also the only one satisfying the equality* (b).

(4.7.5) **Theorem.** *Let us set $q'(a) = q(a) + \beta(a,a)$. There is a unique algebra morphism Φ_β from $\mathrm{C}\ell(M,q')$ into $\mathrm{C}\ell(M,q;\beta)$ such that $\Phi_\beta(\rho'(a)) = \rho(a)$ for all $a \in M$; it is a morphism of graded algebras. It is also a morphism of right comodules over $\bigwedge(M)$, and consequently a morphism of left modules over $\bigwedge^*(M)$.*

(4.7.6) **Theorem.** *The algebra morphisms $\Phi_\beta : \mathrm{C}\ell(M,q') \to \mathrm{C}\ell(M,q;\beta)$ and $\Phi_{-\beta} : \mathrm{C}\ell(M,q) \to \mathrm{C}\ell(M,q';-\beta)$ are reciprocal bijections.*

Proof of the five theorems, and corollaries

Proof of (4.7.2). Since $\beta_{\prime\prime}$ vanishes on $\bigwedge(M)\otimes 1$ and $1\otimes\bigwedge(M)$, the equalities $\mathrm{Exp}(\beta_{\prime\prime})\rfloor(x\otimes 1_q)=x\otimes 1_q$ and $\mathrm{Exp}(\beta_{\prime\prime})\rfloor(1_q\otimes x)=1_q\otimes x$ follow from (4.5.9) and imply $x\star 1_q = 1_q\star x = x$ for all $x\in \mathrm{C}\ell(M,q)$. The associativity of the $\star$-multiplication is the most difficult stage in this section because it involves two Leibniz formulas slightly more sophisticated than the simple formula (4.4.9). Let C be a graded right comodule over a graded coalgebra A; later C and A will be $\mathrm{C}\ell(M,q)$ and $\bigwedge(M)$. Thus $C\otimes\mathrm{C}\ell(M,q)$ (resp. $\mathrm{C}\ell(M,q)\otimes C$) is a left module over the algebra $A^*\hat{\otimes}\bigwedge^*(M)$ (resp. $\bigwedge^*(M)\hat{\otimes}A^*$). If f (resp. g) is an element of this algebra, if x, y, z are elements of $\mathrm{C}\ell(M,q)$, and ξ, ζ elements of C, then

$$f\rfloor(\xi\otimes yz)=(\mathrm{id}_C\otimes\pi_q)\ \big((\mathrm{id}_A\otimes\pi)^*(f)\ \rfloor\ (\xi\otimes y\otimes z)\big)\ ,$$
$$g\rfloor(xy\otimes\zeta)=(\pi_q\otimes\mathrm{id}_C)\ \big((\pi\otimes\mathrm{id}_A)^*(g)\ \rfloor\ (x\otimes y\otimes\zeta)\big)\ ;$$

these formulas are proved exactly like (4.4.9); the mappings $(\mathrm{id}_A\otimes\pi)^*$ and $(\pi\otimes\mathrm{id}_A)^*$ are associated by the functor $\mathrm{Hom}(\dots,K)$ with $\mathrm{id}_A\otimes\pi$ and $\pi\otimes\mathrm{id}_A$.

Now let us calculate $x\star(y\star z)$. By means of Definition (4.7.1), the Leibniz formula devoted to $f\rfloor(\xi\otimes yz)$, and also (4.4.3), (4.5.7) and (4.5.8), we obtain

$$\begin{aligned}x\star(y\star z)&=\pi_q\ \big(\mathrm{Exp}(\beta_{\prime\prime})\ \rfloor\ \big((\mathrm{id}_q\otimes\pi_q)\ ((1\otimes\mathrm{Exp}(\beta_{\prime\prime}))\rfloor(x\otimes y\otimes z))\big)\big)\\ &=\pi_q\ (\mathrm{id}_q\otimes\pi_q)\ \big(\mathrm{Exp}\big((\mathrm{id}_\wedge\otimes\pi)^*(\beta_{\prime\prime})\ +1\otimes\beta_{\prime\prime}\big)\ \rfloor\ (x\otimes y\otimes z)\big).\end{aligned}$$

Obviously $1\otimes\beta_{\prime\prime}=\beta_{\cdot\prime\prime}$. Let us verify that $(\mathrm{id}_\wedge\otimes\pi)^*(\beta_{\prime\prime})=\beta_{\prime\prime\cdot}+\beta_{\prime\cdot\prime}$; indeed

$$\begin{aligned}&(\mathrm{id}_\wedge\otimes\pi)^*(\beta_{\prime\prime})\ (a_1\otimes b_1\otimes 1+a_2\otimes 1\otimes b_2+1\otimes a_3\otimes b_3)\\ &\quad=\beta_{\prime\prime}(a_1\otimes b_1+a_2\otimes b_2+1\otimes(a_3\wedge b_3))\ =\ \beta(a_1,b_1)+\beta(a_2,b_2).\end{aligned}$$

All this shows that

$$x\star(y\star z)\ =\ \pi_q\ (\mathrm{id}_q\otimes\pi_q)\ \big(\mathrm{Exp}\big(\beta_{\prime\prime\cdot}+\beta_{\prime\cdot\prime}+\beta_{\cdot\prime\prime}\big)\ \rfloor\ (x\otimes y\otimes z)\big).$$

In the same way we can calculate that

$$(x\star y)\star z\ =\ \pi_q\ (\pi_q\otimes\mathrm{id}_q)\ \big(\mathrm{Exp}\big(\beta_{\prime\prime\cdot}+\beta_{\prime\cdot\prime}+\beta_{\cdot\prime\prime}\big)\ \rfloor\ (x\otimes y\otimes z)\big).$$

We remember that $\pi_q(\pi_q\otimes\mathrm{id}_q)=\pi_q(\mathrm{id}_q\otimes\pi_q)$ because the algebra $\mathrm{C}\ell(M,q)$ is associative, and the proof is complete. □

If we calculated the product of four factors in the algebra $\mathrm{C}\ell(M,q;\beta)$, we should find a similar result involving the six images of β in $\bigwedge^{*2}(M\oplus M\oplus M\oplus M)$, and so forth....

Proof of (4.7.3). By means of (4.3.2) we get (for all $a\in M$)

$$\beta_{\prime\prime}\lfloor(a\otimes 1)=1\otimes\mathrm{d}_\beta(a)\quad\text{and}\quad\beta_{\prime\prime}\lfloor(1\otimes a)=\mathrm{d}_\beta^{to}(a)\otimes 1\ ;$$

here $\mathrm{d}_\beta(a)$ and $\mathrm{d}_\beta^{to}(a)$ must be understood as elements of $\bigwedge^{*1}(M)$. Now the formulas (a) and (b) are immediate consequences of the equalities

$$\begin{aligned}\mathrm{Exp}(\beta_{//}) \ \rfloor \ (\rho(a)\otimes x) &= \rho(a)\otimes x \ + \ 1_q\otimes(\mathrm{d}_\beta(a)\rfloor x)\ ,\\ \mathrm{Exp}(\beta_{//}) \ \rfloor \ (x\otimes\rho(a)) &= x\otimes\rho(a) \ + \ (\mathrm{d}_\beta^{to}(a)\rfloor\sigma(x))\otimes 1_q\ ,\end{aligned}$$

which themselves can be easily proved by means of the composite derivations formulas (4.4.11). Indeed $\rho(a)\otimes x$ (for instance) is the product of $\rho(a)\otimes 1_q$ and $1_q\otimes x$, and in the proof of (4.7.2) we have already noticed that the second factor is invariant by the interior multiplication by $\mathrm{Exp}(\beta_{//})$; thus (4.4.11) implies

$$\mathrm{Exp}(\beta_{//})\rfloor(\rho(a)\otimes x) \ = \ \rho(a)\otimes x \ + \big(\mathrm{Exp}(\beta_{//})\lfloor(a\otimes 1)\big)\rfloor(1_q\otimes x)\ ;$$

then (4.5.6) implies

$$\mathrm{Exp}(\beta_{//})\lfloor(a\otimes 1) \ = \ \mathrm{Exp}(\beta_{//})\wedge(1\otimes\mathrm{d}_\beta(a))\ ;$$

this allows us to complete the proof of (a) with the help of (4.3.7) and (4.5.9):

$$\begin{aligned}\big(\mathrm{Exp}(\beta_{//})\lfloor(\rho(a)\otimes 1)\big)\rfloor(1_q\otimes x) &= \mathrm{Exp}(\beta_{//})\rfloor\big((1\otimes\mathrm{d}_\beta(a))\rfloor(1_q\otimes x)\big)\\ &= \ 1_q\otimes(\mathrm{d}_\beta(a)\rfloor x).\end{aligned}$$

The proof of the formula (b) is similar. □

(4.7.7) **Examples** of applications of (4.7.3). For all a, b, $c\in M$ we can write

$$\begin{aligned}\rho(a)\star\rho(b) &= \rho(a)\,\rho(b) \ +\beta(a,b)\,1_q\ ,\\ \rho(a)\star\rho(a) &= (q(a)+\beta(a,a))\,1_q\ ,\\ \rho(a)\star\rho(b)\star\rho(c) &= \rho(a)\,\rho(b)\,\rho(c) \ +\beta(b,c)\,\rho(a)-\beta(a,c)\,\rho(b)+\beta(a,b)\,\rho(c)\ .\end{aligned}$$

In the proof of (4.7.4) we shall use the filtration of the algebra $\mathrm{C}\ell(M,q)$ by the submodules $\mathrm{C}\ell^{\le k}(M,q)$ defined in **3.1**; when x belongs to $\mathrm{C}\ell^{\le k}(M,q)$, then $f\rfloor x$ also belongs to it for all $f\in\bigwedge^*(M)$.

Proof of (4.7.4). Let us forget the algebra $\mathrm{C}\ell(M,q;\beta)$ and assume that there is an associative multiplication on the K-module $\mathrm{C}\ell(M,q)$ admitting 1_q as a unit element and satisfying (4.7.3)(b) (for instance) for all $a\in M$ and all $x\in\mathrm{C}\ell(M,q)$; we denote it by $(x,y)\longmapsto x\star y$. Obviously the product $x\star y$ is uniquely determined for all y in $\mathrm{C}\ell^{\le 1}(M,q)$. Let us assume that it is uniquely determined for all y in $\mathrm{C}\ell^{\le k}(M,q)$, and let us prove that it is still uniquely determined for all y in $\mathrm{C}\ell^{\le k+1}(M,q)$. We can suppose that $y = z\,\rho(a)$ for some z in $\mathrm{C}\ell^{\le k}(M,q)$. The condition (b) still determines the value of $x\star y$:

$$x\star(z\,\rho(a)) \ = \ x\star(z\star\rho(a)-\mathrm{d}_\beta^{to}(a)\rfloor\sigma(z)) \ = \ (x\star z)\star\rho(a) \ - x\star(\mathrm{d}_\beta^{to}(a)\rfloor\sigma(z))\ ;$$

it suffices to remember that $f\rfloor\sigma(z)$ belongs to $\mathrm{C}\ell^{\le k}(M,q)$ for all $f\in\bigwedge^*(M)$. With the condition (4.7.3)(a) the proof is similar. □

Here is a corollary of (4.7.4); it involves the twisted opposite algebra $\mathrm{C}\ell(M,q)^{to}$ provided with the multiplication $x^{to}y^{to} = (-1)^{\partial x \partial y}(yx)^{to}$ (see **3.2**).

(4.7.8) **Corollary.** *The mapping $x \longmapsto x^{to}$ is an algebra isomorphism from*

$$\mathrm{C}\ell(M,q;-\mathrm{b}_q) \quad \textit{onto} \quad \mathrm{C}\ell(M,q)^{to}.$$

Proof. For all $a \in M$ and all $x \in \mathrm{C}\ell(M,q)$, the product of $\rho(a)^{to}$ and x^{to} in $\mathrm{C}\ell(M,q)^{to}$ is equal to $(\sigma(x)\rho(a))^{to}$; from (4.4.12) we deduce that

$$\sigma(x)\rho(a) \;=\; \rho(a)x - \mathrm{d}_q(a) \rfloor x \;;$$

the right-hand member is the product of $\rho(a)$ and x in $\mathrm{C}\ell(M,q;-\mathrm{b}_q)$, and this suffices to conclude. □

Proof of (4.7.5). The equality $\rho(a) \star \rho(a) = q'(a,a)1_q$ (see (4.7.7)) proves the existence of the algebra morphism Φ_β from $\mathrm{C}\ell(M,q')$ into $\mathrm{C}\ell(M,q;\beta)$. Since the algebra $\mathrm{C}\ell(M,q;\beta)$ admits the same parity grading as $\mathrm{C}\ell(M,q)$, Φ_β is a graded morphism. The main assertion in (4.7.5) is that Φ_β is a morphism of right comodules over $\bigwedge(M)$, in other words,

$$\pi'_q \circ \Phi_\beta \;=\; (\Phi_\beta \otimes \mathrm{id}_\wedge) \circ \pi'_{q'}.$$

The right-hand member of this equality is the algebra morphism from $\mathrm{C}\ell(M,q')$ into $\mathrm{C}\ell(M,q;\beta) \hat{\otimes} \bigwedge(M)$ that maps every $\rho'(a)$ to $\rho(a) \otimes 1 + 1_q \otimes a$; therefore it suffices to prove that π'_q is also an algebra morphism from $\mathrm{C}\ell(M,q;\beta)$ into $\mathrm{C}\ell(M,q;\beta) \hat{\otimes} \bigwedge(M)$. Let us denote by Π the linear mapping representing the multiplication in $\mathrm{C}\ell(M,q) \hat{\otimes} \bigwedge(M)$; the product of two elements ξ and ζ in $\mathrm{C}\ell(M,q;\beta) \hat{\otimes} \bigwedge(M)$ is

$$\xi \star \zeta \;=\; \Pi\,(\mathrm{Exp}(\beta_{\prime.\prime.}) \;\rfloor\; (\xi \otimes \zeta)) \;;$$

consequently it suffices to prove the following equality for $x,\, y \in \mathrm{C}\ell(M,q)$:

$$\Pi\big(\, \mathrm{Exp}(\beta_{\prime.\prime.}) \;\rfloor\; (\pi'_q(x) \otimes \pi'_q(y))\big) \;=\; \pi'_q \circ \pi_q\big(\, \mathrm{Exp}(\beta_{\prime\prime}) \;\rfloor\; (x \otimes y)\big).$$

Since π'_q is an algebra morphism, $\pi'_q \circ \pi_q = \Pi \circ (\pi'_q \otimes \pi'_q)$, and thus it suffices to prove that

$$\mathrm{Exp}(\beta_{\prime.\prime.}) \;\rfloor\; (\pi'_q(x) \otimes \pi'_q(y)) \;=\; (\pi'_q \otimes \pi'_q)\big(\mathrm{Exp}(\beta_{\prime\prime}) \;\rfloor\; (x \otimes y)\big)\;.$$

This last equality is an immediate consequence of (4.4.6), when the morphism $w : (M,q) \to (N,\tilde{q})$ appearing there is here replaced with the morphism

$$\Delta : (M,q) \perp (M,q) \longrightarrow (M,q) \perp (M,0) \perp (M,q) \perp (M,0)$$

defined by $\Delta(a,b) = (a,a,b,b)$; indeed all this implies that

$$\mathrm{C}\ell(\Delta) = \pi'_q \otimes \pi'_q \quad \text{and} \quad {\bigwedge}^*(\Delta)(\beta_{\prime.\prime.}) = \beta_{\prime\prime}\;. \qquad \square$$

Since Φ_β is a morphism of comodules, the equality $\Phi_\beta(f \rfloor x) = f \rfloor \Phi_\beta(x)$ holds for all $x \in \mathrm{C}\ell(M,q')$ and all $f \in \bigwedge^*(M)$. Besides, it is clear that the objects presented here behave nicely in the case of a direct sum; if (M,q) is the direct sum of two orthogonal submodules (M_1,q_1) and (M_2,q_2), and if β is the direct sum of two bilinear forms β_1 and β_2 respectively on M_1 and M_2, then $\mathrm{C}\ell(M,q;\beta)$ is canonically isomorphic to the twisted tensor product of $\mathrm{C}\ell(M_1,q_1;\beta_1)$ and $\mathrm{C}\ell(M_2,q_2;\beta_2)$, and by this isomorphism Φ_β becomes $\Phi_{\beta_1} \otimes \Phi_{\beta_2}$. Whence the following consequence of (4.7.5).

(4.7.9) **Corollary.** *For every bilinear form $\beta' : M \times M \to K$, the algebra morphism Φ_β from $\mathrm{C}\ell(M,q')$ into $\mathrm{C}\ell(M,q;\beta)$ is also an algebra morphism from $\mathrm{C}\ell(M,q';\beta')$ into $\mathrm{C}\ell(M,q;\,\beta+\beta')$.*

Proof. We must prove this equality for all $\xi \in \mathrm{C}\ell(M,q') \otimes \mathrm{C}\ell(M,q')$:

$$\Phi_\beta \circ \pi_{q'}(\ \mathrm{Exp}(\beta'_{''})\ \rfloor\ \xi\ ) \ = \ \pi_q \circ (\ \mathrm{Exp}(\beta_{''} + \beta'_{''})\ \rfloor\ (\Phi_\beta \otimes \Phi_\beta)(\xi)\ )\ ;$$

since Φ_β is a morphism from $\mathrm{C}\ell(M,q')$ into $\mathrm{C}\ell(M,q;\beta)$, we know that

$$\Phi_\beta \circ \pi_{q'}(\ \mathrm{Exp}(\beta'_{''})\ \rfloor\ \xi\ ) \ = \ \pi_q \circ (\ \mathrm{Exp}(\beta_{''})\ \rfloor\ (\Phi_\beta \otimes \Phi_\beta)(\mathrm{Exp}(\beta'_{''})\ \rfloor\ \xi)\ )\ ;$$

since the interior multiplication by $\mathrm{Exp}(\beta_{''}+\beta'_{''})$ is equivalent to successive interior multiplications by $\mathrm{Exp}(\beta'_{''})$ and $\mathrm{Exp}(\beta_{''})$, it suffices to verify that

$$(\Phi_\beta \otimes \Phi_\beta)(\mathrm{Exp}(\beta'_{''})\ \rfloor\ \xi) \ = \ \mathrm{Exp}(\beta'_{''})\ \rfloor\ (\Phi_\beta \otimes \Phi_\beta)(\xi)\ ;$$

since $\Phi_\beta \otimes \Phi_\beta$ can be identified with the algebra morphism

$$\Phi_{\beta\perp\beta}\ :\ \mathrm{C}\ell((M,q') \perp (M,q')) \longrightarrow \mathrm{C}\ell((M,q) \perp (M,q)\ ;\ \beta \perp \beta)\ ,$$

the conclusion follows from the fact that $\Phi_{\beta\perp\beta}$ is a morphism of comodules. □

Proof of (4.7.6). Because of (4.7.9), Φ_β is also an algebra morphism from

$$\mathrm{C}\ell(M,q';-\beta) \quad \text{into} \quad \mathrm{C}\ell(M,q).$$

Consequently $\Phi_\beta \circ \Phi_{-\beta}$ is an algebra morphism from $\mathrm{C}\ell(M,q)$ into itself which maps every $\rho(a)$ to itself; this proves that $\Phi_\beta \circ \Phi_{-\beta}$ is the identity mapping. And the same for $\Phi_{-\beta} \circ \Phi_\beta$. □

More generally, if we set $q''(a) = q'(a)+\beta'(a,a)$ and consider the isomorphism $\Phi_{\beta'}$ from $\mathrm{C}\ell(M,q'')$ onto $\mathrm{C}\ell(M,q';\beta')$, we can deduce from (4.7.9) that

$$\Phi_\beta \circ \Phi_{\beta'} \ = \ \Phi_{\beta+\beta'}. \tag{4.7.10}$$

Here are other corollaries of the previous results.

(4.7.11) **Corollary.** *For all $k \in \mathbb{N}$, $\Phi_\beta(\mathrm{C}\ell^{\leq k}(M,q')) = \mathrm{C}\ell^{\leq k}(M,q)$; moreover Φ_β induces an algebra isomorphism between the graded algebras $\mathrm{Gr}(\mathrm{C}\ell(M,q'))$ and $\mathrm{Gr}(\mathrm{C}\ell(M,q))$ defined in* **3.1**.

Proof. Since the interior multiplication by $d_\beta(a)$ maps $C\ell^{\leq k}(M,q)$ into itself, the equality (4.7.3)(a) shows (by induction on k) that $\Phi_\beta(C\ell^{\leq k}(M,q')) \subset C\ell^{\leq k}(M,q)$. The opposite inclusion is proved by means of $\Phi_{-\beta}$. Thus it is clear that Φ_β induces bijections

$$C\ell^{\leq k}(M,q') \,/\, C\ell^{\leq k-1}(M,q') \;\longrightarrow\; C\ell^{\leq k}(M,q) \,/\, C\ell^{\leq k-1}(M,q) \;,$$

resulting in a bijection $\mathrm{Gr}(\Phi_\beta) : \mathrm{Gr}(C\ell(M,q')) \to \mathrm{Gr}(C\ell(M,q))$.

For every $x \in C\ell^{\leq k}(M,q)$, $\rho(a) \star x$ and $\rho(a)x$ are congruent modulo $C\ell^{\leq k}(M,q)$; consequently, for all $b \in \mathrm{Gr}^1(C\ell(M,q'))$ and all $y \in \mathrm{Gr}^k(C\ell(M,q'))$,

$$\mathrm{Gr}(\Phi_\beta)(by) \;=\; \mathrm{Gr}(\Phi_\beta)(b)\ \mathrm{Gr}(\Phi_\beta)(y) \quad \text{in} \quad \mathrm{Gr}^{k+1}(C\ell(M,q')) \;;$$

since the algebra $\mathrm{Gr}(C\ell(M,q'))$ is generated by $\mathrm{Gr}^1(C\ell(M,q'))$, this suffices to conclude that $\mathrm{Gr}(\Phi_\beta)$ is an algebra morphism. □

(4.7.12) **Corollary.** *The mapping $x \longmapsto x^{to}$ is an algebra isomorphism from*

$$C\ell(M,q;\, \beta^{to} - \mathrm{b}_q) \quad \textit{onto} \quad C\ell(M,q;\beta)^{to}.$$

Proof. Let F be this mapping, and F' the analogous mapping $C\ell(M,q') \to C\ell(M,q')^{to}$. Since Φ_β and $\Phi_{-\beta}$ are reciprocal bijections, we can write $F = \Phi_\beta^{to} \circ F' \circ \Phi_{-\beta}$. Moreover an easy calculation shows that $\beta^{to} - \mathrm{b}_q$ and $\beta - \mathrm{b}_{q'}$ are the same thing. Now $\Phi_{-\beta}$ is an isomorphism from $C\ell(M,q;\ \beta - \mathrm{b}_{q'})$ onto $C\ell(M,q';-\mathrm{b}_{q'})$ because of (4.7.9); then F' is an isomorphism from $C\ell(M,q';-\mathrm{b}_{q'})$ onto $C\ell(M;q')^{to}$ because of (4.7.8); and finally Φ_β^{to} is an isomorphism $C\ell(M;q')^{to} \to C\ell(M,q;\beta)^{to}$. □

Additional information

A sixth theorem is now added to the five previous ones.

(4.7.13) **Theorem.** *Let β and β' be two bilinear forms $M \times M \to K$ such that $\beta(a,a) = \beta'(a,a)$ for all $a \in M$, and let f be the element of $\bigwedge^{*2}(M)$ such that $f(a \wedge b) = \beta'(a,b) - \beta(a,b)$ for all a, $b \in M$. Then $\Phi_{\beta'-\beta}$ (that is the unique algebra isomorphism $C\ell(M,q;\beta) \to C\ell(M,q;\beta')$ leaving all elements of $\rho(M)$ invariant) is the mapping $x \longmapsto \mathrm{Exp}(f) \rfloor x$.*

Proof. The bilinear form $(a,b) \longmapsto \beta'(a,b) - \beta(a,b)$ is alternate and defines an element $f \in \bigwedge^{*2}(M)$. From (4.7.9) we deduce that $\Phi_{\beta'-\beta}$ is an algebra isomorphism $C\ell(M,q;\beta) \to C\ell(M,q;\beta')$. As a linear endomorphism of $C\ell(M,q)$ it is characterized by these two properties: it leaves invariant all elements of $C\ell^{\leq 1}(M,q)$, and for every $a \in M$ and $x \in C\ell(M,q)$ it maps $\rho(a) \star x$ (product in $C\ell(M,q;\beta)$) to the product of $\rho(a)$ and $\Phi_{\beta'-\beta}(x)$ in $C\ell(M,q;\beta')$. Let us verify that the mapping $x \longmapsto \mathrm{Exp}(f) \rfloor x$ satisfies these two properties. First it leaves invariant all elements of $C\ell^{\leq 1}(M,q)$ because of (4.5.9). Secondly we must verify that, for every $a \in M$ and $x \in C\ell(M,q)$,

$$\mathrm{Exp}(f) \rfloor (\rho(a)x) + (\mathrm{Exp}(f) \wedge \mathrm{d}_\beta(a)) \rfloor x \;=\; \rho(a)(\mathrm{Exp}(f) \rfloor x) + (\mathrm{d}_{\beta'}(a) \wedge \mathrm{Exp}(f)) \rfloor x.$$

From the definitions (in particular (4.3.2)) it follows immediately that $\mathrm{d}_{\beta'}(a) - \mathrm{d}_\beta(a) = f \lfloor a$; consequently the previous equality is equivalent to

$$\mathrm{Exp}(f) \rfloor (\rho(a)x) \;=\; (\mathrm{Exp}(f) \wedge (f \lfloor a)) \rfloor x + \rho(a)(\mathrm{Exp}(f) \rfloor x) \;;$$

since $\mathrm{Exp}(f) \wedge (f \lfloor a)$ is the same thing as $\mathrm{Exp}(f) \lfloor a$ (see (4.5.6)), this is an example of a composite derivation formula like (4.4.11). □

Like $C\ell(M,q')$, the algebra $C\ell(M,q;\beta)$ admits a reversion that we shall now calculate as a corollary of (4.7.13); the definition of $[\beta] \in \bigwedge^{*2}(M)$ has been given at the beginning.

(4.7.14) **Proposition.** *The reversion τ_β in $C\ell(M,q;\beta)$ maps every $x \in C\ell(M,q;\beta)$ to*

$$\tau_\beta(x) \;=\; \mathrm{Exp}([\beta]) \;\rfloor\; \tau(x) \;=\; \tau(\;\mathrm{Exp}(-[\beta]) \;\rfloor\; x) \;.$$

Proof. Because of (4.7.13), the mapping $x \longmapsto \mathrm{Exp}([\beta]) \rfloor x$ is the isomorphism of $C\ell(M,q;-\beta^{to})$ onto $C\ell(M,q;\beta)$ that leaves invariant all elements of $\rho(M)$. Thus the proof of (4.7.14) is completed by the following lemma.

(4.7.15) **Lemma.** *The mapping $x \longmapsto \tau(x)^o$ is an isomorphism from $C\ell(M,q;\beta)$ onto the opposite algebra $C\ell(M,q;-\beta^{to})^o$.*

Proof. We must prove that

$$\tau\big(\;\mathrm{Exp}(\beta_{''}) \;\rfloor\; (\tau(x) \otimes \tau(y))\;\big) \;=\; \mathrm{Exp}(-\beta_{''}^{to}) \;\rfloor\; (y \otimes x) \;;$$

we observe that $\tau(\beta_{''}) = -\beta_{''}$ (see (3.1.5)), and because of (3.2.8) and (4.4.7) the previous equality is equivalent to

$$(-1)^{\partial x \partial y}\; \mathrm{Exp}(-\beta_{''}) \;\rfloor\; (x \otimes y) \;=\; \mathrm{Exp}(-\beta_{''}^{to}) \;\rfloor\; (y \otimes x) \;;$$

this is an immediate consequence of (4.4.6) when w is the reversion mapping $M \oplus M \to M \oplus M$ defined by $w(a,b) = (b,a)$; indeed

$$C\ell(w)(x \otimes y) = (-1)^{\partial x \partial y} y \otimes x \;, \quad \text{and} \quad \bigwedge^*(w)(\beta_{''}) = \beta_{''}^{to} \;. \qquad \square$$

Examples. It is clear that $[\beta] = 0$ if and only if β is symmetric. When 2 is invertible in K, for every pair (q,q') of quadratic forms on M, there exists a unique symmetric bilinear form β such that $q'(a) = q(a) + \beta(a,a)$ for all $a \in M$. On the contrary, when the equality $2 = 0$ holds in K, then $[\beta]$ is strictly determined by $q' - q$ because

$$[\beta](a \wedge b) \;=\; \beta(a,b) + \beta(b,a) \;=\; \mathrm{b}_{q'}(a,b) - \mathrm{b}_q(a,b).$$

This sections ends with some routine information.

(4.7.16) **Proposition.** *Let (M, q) and (M', q') be two quadratic modules, β and β' bilinear forms respectively on M and M', and $f : M \to M'$ a linear mapping such that*

$$\forall a, b \in M, \quad q(a) = q'(f(a)) \quad \text{and} \quad \beta(a, b) = \beta'(f(a), f(b)) \ ;$$

then the algebra morphism $\mathrm{C}\ell(f) : \mathrm{C}\ell(M, q) \to \mathrm{C}\ell(M', q')$ is also an algebra morphism from $\mathrm{C}\ell(M, q; \beta)$ into $\mathrm{C}\ell(M', q'; \beta')$.

Proof. For every x, $y \in \mathrm{C}\ell(M, q)$ we must verify that

$$\mathrm{C}\ell(f) \circ \pi_q\big(\ \mathrm{Exp}(\beta_{\prime\prime}) \ \rfloor \ (x \otimes y)\big) \ = \ \pi_{q'}\big(\ \mathrm{Exp}(\beta'_{\prime\prime}) \ \rfloor \ (\mathrm{C}\ell(f)(x) \otimes \mathrm{C}\ell(f)(y))\ \big).$$

If f_2 is the morphism from $(M, q) \perp (M, q)$ into $(M', q') \perp (M', q')$ such that $f_2(a, b) = (f(a), f(b))$, it is obvious that

$$\mathrm{C}\ell(f)(x) \otimes \mathrm{C}\ell(f)(y) = \mathrm{C}\ell(f_2)(x \otimes y) \quad \text{and} \quad \mathrm{C}\ell(f) \circ \pi_q = \pi_{q'} \circ \mathrm{C}\ell(f_2)\ ,$$

and it is easy to verify that $\beta_{\prime\prime} = \bigwedge^*(f_2)(\beta'_{\prime\prime})$. Thus the conclusion follows from (4.4.6). $\square$

Interior products and exponentials have been presented in **4.4** and **4.5** without mentioning their behaviour in case of an extension $K \to K'$ of the basic ring; indeed it is clear that they behave as expected. Since later we shall again use localizations in a systematic way, we just add the following evident statement.

(4.7.17) **Lemma.** *Let $K \to K'$ be a ring morphism, q' and β' the quadratic form and the bilinear form on $K' \otimes M$ derived from q and β. The algebra $\mathrm{C}\ell_{K'}(K' \otimes M,\ q'; \beta')$ is canonically isomorphic to $K' \otimes \mathrm{C}\ell(M, q; \beta)$.*

4.8 Applications of deformations

A Clifford algebra $\mathrm{C}\ell(M, q)$ is not at all convenient when the canonical mappings $K \to \mathrm{C}\ell_0(M, q)$ and $\rho : M \to \mathrm{C}\ell_1(M, q)$ are not both injective; whence the following definition.

(4.8.1) **Definition.** A quadratic module (M, q), or the quadratic form q itself, is said to be *cliffordian* if the canonical morphisms from K and M into $\mathrm{C}\ell(M, q)$ are both injective, and allow us to identify K and M with submodules of $\mathrm{C}\ell(M, q)$. It is said to be *strongly cliffordian* if moreover K is a direct summand of $\mathrm{C}\ell_0(M, q)$, and M a direct summand of $\mathrm{C}\ell_1(M, q)$.

When (M, q) is cliffordian, K and M are silently identified with their canonical images in $\mathrm{C}\ell(M, q)$ unless it is otherwise specified (for instance when M itself is an algebra already containing K as a subalgebra). Almost everywhere in the literature additional hypotheses ensure the quadratic modules to be strongly cliffordian, and let the definition (4.8.1) become useless; nevertheless an example of a non-cliffordian quadratic form has been given in (3.1.3).

(4.8.2) **Proposition.** *Let us suppose that (M, q) is not a cliffordian quadratic module; let K' be the image of K in $\mathrm{C}\ell(M, q)$, and $M' = \rho(M)$ the image of M, considered as a module over K'. We get a cliffordian quadratic form q' on M' if we set $q'(a') = a'^2$ for all $a' \in M'$. Moreover the identity mapping of M' extends to an isomorphism $\mathrm{C}\ell_{K'}(M', q') \to \mathrm{C}\ell_K(M, q)$ of algebras over K or K'.*

Proof. It is clear that q' is a K'-quadratic form $M' \to K'$; since $\mathrm{C}\ell(M, q)$ is also a K'-algebra, the universal property of $\mathrm{C}\ell_{K'}(M', q')$ associates with $\mathrm{id}_{M'}$ an algebra morphism from $\mathrm{C}\ell_{K'}(M', q')$ into $\mathrm{C}\ell_K(M, q)$. This already proves that K' and M' are mapped injectively into $\mathrm{C}\ell_{K'}(M', q')$. Conversely for all $a' = \rho(a) \in M'$ we can write $q'(a')1_{q'} = (q(a)1_q)1_{q'} = q(a)1_{q'}$ and consequently with $\rho : M \to M'$ the universal property of $\mathrm{C}\ell_K(M, q)$ associates an algebra morphism from $\mathrm{C}\ell_K(M, q)$ into $\mathrm{C}\ell_{K'}(M', q')$. Obviously these algebra morphisms are reciprocal isomorphisms. □

Proposition (4.8.2) shows that the Clifford algebra of a non-cliffordian quadratic module (M, q) is also the Clifford algebra of a cliffordian one (M', q') canonically derived from it, and that the properties of (M, q) are observable in its Clifford algebra only as far as they are inherited by (M', q'). Up to now, nothing seems to be known about what is inherited and what is lost.

Here is an immediate consequence of the definition (4.8.1).

(4.8.3) **Proposition.** *Let $q : M \to K$ be a cliffordian quadratic form; for every $\lambda \in K$, λq is also a cliffordian quadratic form. When λ is invertible, and q strongly cliffordian, then λq too is strongly cliffordian.*

Proof. On the module $\mathrm{C}\ell(M, q)$ we define the following multiplication:

$$\begin{aligned} (x, y) \longmapsto x * y &= xy \quad \text{if } x \text{ or } y \text{ is even,} \\ &= \lambda xy \text{ if } x \text{ and } y \text{ are odd.} \end{aligned}$$

It is easy to prove that this new multiplication is still associative, with the same unit element 1_q. Since $\rho(a) * \rho(a) = \lambda q(a)$ for all $a \in M$, the mapping ρ induces an algebra morphism g from $\mathrm{C}\ell(M, \lambda q)$ into the new algebra $\mathrm{C}\ell(M, q)$. Since K and M are mapped injectively into the module $\mathrm{C}\ell(M, q)$, they are already mapped injectively into $\mathrm{C}\ell(M, \lambda q)$.

When λ is invertible, the stronger conclusion follows from the bijectiveness of g. It is bijective because similarly there is an algebra morphism g' from $\mathrm{C}\ell(M, q)$ into $\mathrm{C}\ell(M, \lambda q)$ provided with a new multiplication such that $x * y = \lambda^{-1} xy$ when x and y are odd; thus gg' and $g'g$ are algebra endomorphisms of respectively $\mathrm{C}\ell(M, q)$ and $\mathrm{C}\ell(M, \lambda q)$. Since gg' and $g'g$ leave invariant the elements of M, they are the identity automorphisms. There is another proof using (3.8.7), because $(M, \lambda q)$

is isomorphic to the tensor product of (M,q) and the free discriminant module generated by an element d such that $d^2 = \lambda$. $\square$

The following statement is an immediate corollary of the five theorems at the beginning of **4.7**.

(4.8.4) **Proposition.** *Let q and q' be two quadratic forms on M; if there exists a bilinear form $\beta : M \times M \to K$ such that $q'(a) = q(a) + \beta(a,a)$ for all $a \in M$, then q' is cliffordian (resp. strongly cliffordian) if and only if q is cliffordian (resp. strongly cliffordian).*

Here is a less trivial application of the results of **4.7**.

(4.8.5) **Theorem.** *Let us suppose that M is the direct sum of the submodules M' and M''; let q be a quadratic form on M, and q' and q'' its restrictions to M' and M''; and let $f' : \mathrm{C}\ell(M',q') \to \mathrm{C}\ell(M,q)$ and $f'' : \mathrm{C}\ell(M'',q'') \to \mathrm{C}\ell(M,q)$ be the algebra morphisms derived from the canonical injections $M' \to M$ and $M'' \to M$.*

(a) *The following multiplication mapping is bijective:*

$$\mathrm{C}\ell(M',q') \otimes \mathrm{C}\ell(M'',q'') \longrightarrow \mathrm{C}\ell(M,q)\ , \quad x' \otimes x'' \longmapsto f'(x')\ f''(x'').$$

(b) *When q'' is strongly cliffordian, then f' is injective and allows us to identify $\mathrm{C}\ell(M',q')$ with a subalgebra of $\mathrm{C}\ell(M,q)$.*

(c) *When q' and q'' are both strongly cliffordian, then q too is strongly cliffordian.*

Proof. Let β be the bilinear form on M such that $\beta(a,b)$ vanishes whenever a belongs to M', and also whenever b belongs to M'', but is equal to $-\mathrm{b}_q(a,b)$ when a and b belong respectively to M'' and M'; in other words, β is the bilinear form such that $\beta(M',M) = \beta(M,M'') = 0$ and such that M' and M'' are orthogonal for the quadratic form $a \longmapsto q(a) + \beta(a,a)$. Therefore the isomorphism Φ_β can be identified with an isomorphism from $\mathrm{C}\ell(M',q') \mathbin{\hat{\otimes}} \mathrm{C}\ell(M'',q'')$ onto $\mathrm{C}\ell(M,q;\beta)$. Since the restrictions of β to M' and M'' vanish, f' and f'' are also algebra morphisms into $\mathrm{C}\ell(M,q;\beta)$. Since $\beta(M',M'') = 0$, for all $x' \in M'$ and all $x'' \in M''$ the product of $f'(x')$ and $f''(x'')$ in $Cl(M,q;\beta)$ is equal to their product in $\mathrm{C}\ell(M,q)$; indeed this can be proved with the help of the formula (4.7.3)(a) and by induction on k for every $x' \in \mathrm{C}\ell^{\leq k}(M',q')$. Consequently Φ_β maps every $x' \otimes x''$ to $f'(x')f''(x'')$. This proves the statement (a).

When q'' is strongly cliffordian, K is a direct summand of $\mathrm{C}\ell(M'',q'')$, and $\mathrm{C}\ell(M',q')$ is isomorphic to a direct summand of $\mathrm{C}\ell(M',q') \mathbin{\hat{\otimes}} \mathrm{C}\ell(M'',q'')$ by the mapping $x' \longmapsto x' \otimes 1_{q''}$; thus the injectiveness of f' follows from that of Φ_β. When q' and q'' are both strongly cliffordian, then $K \otimes K$ is a direct summand of $\mathrm{C}\ell(M',q') \mathbin{\hat{\otimes}} \mathrm{C}\ell(M'',q'')$, and $M' \otimes K$ and $K \otimes M''$ too; consequently K and $M' \oplus M''$ are direct summands of $\mathrm{C}\ell(M,q)$. $\square$

Admissible scalar products

The null quadratic form on M is obviously strongly cliffordian because $\mathrm{C}\ell(M,0) = \bigwedge(M)$; this observation leads to the following definition.

(4.8.6) **Definition.** A bilinear form $\beta : M \times M \to K$ is called an *admissible scalar product* (or simply a *scalar product*) for the quadratic form $q : M \to K$ if $q(a) = \beta(a,a)$ for all $a \in M$.

When β is an admissible scalar product for q, then $\mathrm{b}_q = \beta - \beta^{to}$.

Here is an immediate consequence of several results of **4.7**, especially (4.7.11).

(4.8.7) **Theorem.** *When q admits a scalar product β, then q is strongly cliffordian and there is a comodule isomorphism $\Phi_{-\beta} : \bigwedge(M) \to \mathrm{C}\ell(M,q)$ such that*

$$\mathrm{C}\ell^{\leq k}(M,q) \;=\; \mathrm{C}\ell^{\leq k-1}(M,q) \oplus \Phi_{-\beta}\left(\bigwedge^{k}(M)\right) \quad \textit{for all } k > 0 \;.$$

Besides, the canonical morphism $\bigwedge(M) \to \mathrm{Gr}(\mathrm{C}\ell(M,q))$ defined in (3.1.8) *is an isomorphism.*

Now we state sufficient conditions ensuring the existence of scalar products.

(4.8.8) **Theorem.** *Let (M,q) be a quadratic module.*

(a) *When M is a projective module, there are always admissible scalar products for q, and $\mathrm{C}\ell(M,q)$ too is a projective module.*

(b) *When the mapping $a \longmapsto 2a$ is bijective from M onto M, there is a unique* symmetric *scalar product β admissible for q, which is defined by the equality $\beta(a,b) = 2\,\mathrm{b}_q(a/2,\, b/2)$; this symmetric scalar product is called the* canonical scalar product *derived from q.*

Proof. When M is projective, the existence of admissible scalar products has already been proved for a quite different purpose: see Lemma (2.5.3). It is already known that $\bigwedge(M)$ too is a projective module (see (3.2.6), and (3.2.7) if M is not finitely generated); because of the comodule isomorphism $\Phi_{-\beta} : \bigwedge(M) \to \mathrm{C}\ell(M,q)$, the same is true for $\mathrm{C}\ell(M,q)$.

When the mapping $a \longmapsto 2a$ is bijective, there is a reciprocal mapping $a \longmapsto a/2$. If β is an admissible *symmetric* scalar product, then $\mathrm{b}_q = 2\beta$ and consequently $2\,\mathrm{b}_q(a/2,\, b/2) = \beta(a,b)$. Conversely if β is defined by this equality, it is an admissible scalar product because $\mathrm{b}_q(a,a) = 2q(a)$ for all $a \in M$. □

When an admissible scalar product β has been chosen, the algebra $\mathrm{C}\ell(M,q)$ is often replaced with $\bigwedge(M;\beta)$ which is then treated as a module provided with two multiplications: the Clifford multiplication $(x,y) \longmapsto xy$ and the exterior multiplication $(x,y) \longmapsto x \wedge y$. The equalities (a) and (b) in (4.7.3) are now written in this way (for all $a \in M$ and $x \in \bigwedge(M)$):

(4.8.9) $$ax \;=\; a \wedge x + \mathrm{d}_\beta(a) \rfloor x \quad \text{and} \quad xa \;=\; x \wedge a + \mathrm{d}_\beta^{to}(a) \rfloor \sigma(x) \;;$$

here is more information about the relations between both multiplications.

(4.8.10) **Proposition.** *Let x and y be elements of $\bigwedge^j(M)$ and $\bigwedge^k(M)$ respectively, and xy their product in the Clifford algebra $\bigwedge(M;\beta)$. If $j \leq k$ (resp. $j \geq k$), xy belongs to*

$$\bigwedge^{j+k}(M)\oplus\bigwedge^{j+k-2}(M)\oplus\bigwedge^{j+k-4}(M)\oplus\cdots\oplus\bigwedge^{k-j}(M) \ \Big(\textit{resp.}\cdots\oplus\bigwedge^{j-k}(M)\Big).$$

Its component of degree $j+k$ is always $x \wedge y$. Its component of degree $|j-k|$ is

$$\bigwedge(\mathrm{d}_\beta)(x) \ \rfloor\, y \quad (\textit{resp.}\ (-1)^{\partial x \partial y} \ \bigwedge(\mathrm{d}_\beta^{to})(y) \ \rfloor\, x\ ).$$

Here the notation $\bigwedge(\mathrm{d}_\beta)$ means the algebra morphism $\bigwedge(M) \to \bigwedge(M^*)$ associated by the functor $\bigwedge$ with the mapping d_β; because of the canonical morphism $\bigwedge(M^*) \to \bigwedge^*(M)$, the interior multiplication by $\bigwedge(\mathrm{d}_\beta)(x)$ is meaningful. The strict observance of the twisting rule (4.2.1) gives immediately the correct sign for the component in $\bigwedge^{j-k}(M)$ when $j \geq k$; nevertheless we must remember that this rule has given $\bigwedge^*(M)$ a multiplication that for some people is that of $(\bigwedge^*(M))^o$.

Proof of (4.8.10). Everything is trivial when $j=0$ or $k=0$; and when $j=1$ or $k=1$, we have just to use the equalities (4.8.9), that here can be written in this way (for a and b in M):

$$ay \ = \ a \wedge y \ + \ \mathrm{d}_\beta(a) \,\rfloor\, y\ , \quad \text{and} \quad xb \ = \ x \wedge b \ + \ (-1)^{\partial x} \ \mathrm{d}_\beta^{to}(b) \,\rfloor\, x\ .$$

Then we proceed by induction. Let us treat for instance the case $j \geq k$ which requires an induction on k. Our induction hypothesis is that (4.8.10) is true for (j,k) and $(j,\ k-1)$, and we want to prove that it is true for $(j,\ k+1)$ if $j > k$. Consequently we replace (x,y) with $(x,\ y \wedge b)$:

$$\begin{aligned} x\ (y \wedge b) &= xyb \ - \ x\ (\mathrm{d}_\beta^{to}(b) \,\rfloor\, \sigma(y)) \\ &= (xy) \wedge b \ + \ \mathrm{d}_\beta^{to}(b) \,\rfloor\, \sigma(xy) \ - \ x\ (\mathrm{d}_\beta^{to}(b) \,\rfloor\, \sigma(y)) \\ &= (xy) \wedge b \ + \ (\mathrm{d}_\beta^{to}(b) \,\rfloor\, \sigma(x))\ \sigma(y); \end{aligned}$$

the component of $x(y \wedge b)$ in $\bigwedge^{j+k+1}(M)$ comes from the first term $(xy) \wedge b$ and is equal to $x \wedge y \wedge b$; its component in $\bigwedge^{j-k-1}(M)$ comes from the second term and is equal to

$$(-1)^{(1+\partial x)\partial y} \bigwedge(\mathrm{d}_\beta^{to})(\sigma(y)) \,\rfloor\ (\mathrm{d}_\beta^{to}(b) \,\rfloor\, \sigma(x)) = (-1)^{\partial x(1+\partial y)} \bigwedge(\mathrm{d}_\beta^{to})(y \wedge b) \,\rfloor\, x.$$

□

Now we prove that the Clifford algebra of a free module is a free module.

(4.8.11) **Proposition.** *Let M be a free module with a basis $(e_j)_{j\in J}$ indexed by a totally ordered set J. The linear mapping $\Phi : \bigwedge(M) \to \mathrm{C}\ell(M,q)$ such that*

$$\Phi(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}) \;=\; e_{j_1}\; e_{j_2} \cdots e_{j_k} \quad \textit{whenever} \quad j_1 < j_2 < \cdots < j_k \ ,$$

is an isomorphism of K-modules, and even an isomorphism of comodules over $\bigwedge(M)$.

Proof. Let β be the bilinear form on M defined in this way: $\beta(e_i, e_j)$ is equal to 0 when $i < j$, equal to $q(e_j)$ when $i = j$, and equal to $\mathrm{b}_q(e_i, e_j)$ when $i > j$. In $\bigwedge(M;\beta)$ the following equality can be proved by induction on k:

$$e_{j_1}e_{j_2}\cdots e_{j_k} \;=\; e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} \quad \text{whenever} \quad j_1 < j_2 < \cdots < j_k;$$

this proves that $\Phi = \Phi_\beta$. □

In the situation of (4.8.5) the identification of $\mathrm{C}\ell(M', q')$ with a subalgebra of $\mathrm{C}\ell(M,q)$ is legitimate whenever M is a projective module (and M' a direct summand of M); this result must now be improved.

(4.8.12) **Lemma.** *Let M be a finitely generated projective module, M' a direct summand of M, q a quadratic form on M, and q' its restriction to M'. Thus $\mathrm{C}\ell(M', q')$ can be identified with a subalgebra of $\mathrm{C}\ell(M,q)$. An element $x \in \mathrm{C}\ell(M,q)$ belongs to $\mathrm{C}\ell(M', q')$ if and only if $h \rfloor x = 0$ for every linear form $h \in M^*$ such that $h(M') = 0$. If q is nondegenerate, or more generally if d_q induces a surjective mapping $M'^{\perp} \to (M/M')^*$, this condition is equivalent to $\mathrm{d}_q(a) \rfloor x = 0$ for every $a \in M'^{\perp}$.*

Proof. It is clear that $h \rfloor x = 0$ if $x \in \mathrm{C}\ell(M', q')$ and $h(M') = 0$. Conversely let us suppose that $h \rfloor x = 0$ whenever $h(M') = 0$. By means of localizations we can reduce the problem to the case of free modules M and M'; let $(e_1, e_2, \dots, e_m)$ be a basis of M such that $(e_1, e_2, \dots, e_n)$ is a basis of M'; thus the products $e_{j_1}e_{j_2}\cdots e_{j_k}$ with $j_1 < j_2 < \cdots < j_k$ constitute a basis of $\mathrm{C}\ell(M,q)$, and if we moreover require $j_k \le n$, we get a basis of $\mathrm{C}\ell(M', q')$. Suppose that x does not belong to $\mathrm{C}\ell(M', q')$; by writing x in the above basis of $\mathrm{C}\ell(M,q)$, we would find some $k \in \{n{+}1,\ n{+}2, \dots, m\}$ such that $x = ye_k{+}z$ with some y and z both in the subalgebra generated by $(e_1, e_2, \dots, e_{k-1})$, and with $y \neq 0$; let h be the linear form such that $h(e_j) = 0$ whenever $j \neq k$, but $h(e_k) = 1$; now $h \rfloor x = (-1)^{\partial y} y \neq 0$; this contradicts the assumption that $h \rfloor x = 0$ whenever $h(M') = 0$.

If the mapping $M'^{\perp} \to (M/M')^*$ induced by d_q is surjective, the submodule of all $h \in M^*$ such that $h(M') = 0$ is equal to the submodule of all $\mathrm{d}_q(a)$ with $a \in M'^{\perp}$. And from (2.3.7) we know that this mapping is bijective if q is nondegenerate. □

Canonical scalar products

Canonical scalar products, which exist when the mapping $a \longmapsto 2a$ is bijective (see (4.8.8)), have special properties that deserve a separate exposition. The bijectiveness of this mapping does not require 2 to be invertible in K; but when M is finitely generated, it implies that $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$ containing the image of 2 in K; indeed, because of Nakayama's lemma (1.12.1), the equality $M_{\mathfrak{p}} = 2M_{\mathfrak{p}}$ implies $M_{\mathfrak{p}} = 0$ if the image of 2 falls in $\mathfrak{p}$. The notation $\mathrm{b}_q/2$ means the canonical scalar product even if 2 is not invertible in K.

When β is a symmetric scalar product, from (4.7.14) we deduce that the Clifford algebra $\bigwedge(M;\beta)$ and the exterior algebra $\bigwedge(M)$ have the same reversion τ, which is described by (3.1.5). The equalities (4.8.9) now look like this:

$$ax \;=\; a \wedge x + \mathrm{d}_\beta(a) \rfloor x \quad \text{and} \quad xa \;=\; a \wedge \sigma(x) - \mathrm{d}_\beta(a) \rfloor \sigma(x) \;;$$

they are equivalent to the following equalities (with $a \in M$), sometimes attributed to Riesz:

$$(4.8.13) \qquad 2\; a \wedge x \;=\; ax + \sigma(x)a \quad \text{and} \quad \mathrm{d}_q(a) \rfloor x \;=\; ax \;-\sigma(x)a.$$

Whereas the latter equality in (4.8.13) is the same thing as (4.4.12), the former equality is a new result; it leads to paying some attention to formulas of the following kind, in which E is some subset of the group $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$:

$$(4.8.14) \qquad \mathrm{card}(E)\; a_1 \wedge a_2 \wedge \cdots \wedge a_n \;=\; \sum_{s \in E} \mathrm{sgn}(s)\; a_{s(1)}a_{s(2)} \cdots a_{s(n)} \;;$$

indeed from (4.8.13) we can derive by induction on n the existence of a subset E of cardinal 2^{n-1} for which the equality (4.8.14) holds. Nevertheless when $n > 2$, it is well known that this equality already holds with a smaller subset, because of the following equality which is an easy consequence of (4.7.7):

$$2\; a_1 \wedge a_2 \wedge a_3 \;=\; a_1a_2a_3 - a_3a_2a_1.$$

It is still an open question to know whether (4.8.14) holds with a subset E of cardinal 2^k if k is the greatest integer such that $2k \leq n$. For $n = 5$ the answer is already known since

$$\begin{aligned} &4\; a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \\ &\quad = a_1a_2a_3a_4a_5 + a_1a_5a_4a_3a_2 \; - a_2a_5a_4a_3a_1 - a_3a_4a_5a_2a_1 \;. \end{aligned}$$

It is worth adding that (4.8.14) also holds when E is the whole group $\mathcal{S}_n$; often this assertion has been proved with the assumption of an orthogonal basis in M; for a quite general proof see (4.ex.14).

The bijection $\Phi_{-\beta}$ (with $\beta = \mathrm{b}_q/2$) allows us to carry onto $\mathrm{C}\ell(M, q)$ the $\mathbb{N}$-grading of $\bigwedge(M)$; therefore we set $\mathrm{C}\ell^n(M, q) = \Phi_{-\beta}(\bigwedge^n(M))$ for every $n \in \mathbb{N}$. The notation $\mathrm{C}\ell^n(M,q)$ can also be understood as an abbreviation of $\mathrm{C}\ell^n(M,q;-\beta)$ since the algebra $\mathrm{C}\ell(M,q;-\beta)$ is canonically isomorphic to the $\mathbb{N}$-graded algebra $\bigwedge(M)$. The subspaces $\mathrm{C}\ell^n(M,q)$ determine a grading of the module $\mathrm{C}\ell(M,q)$ that is compatible with the natural filtration of the algebra $\mathrm{C}\ell(M,q)$ (see (4.8.7)). Besides, every automorphism g of (M,q) determines an automorphism $\mathrm{C}\ell(g)$ of the algebra $\mathrm{C}\ell(M,q)$ which leaves every subspace $\mathrm{C}\ell^n(M,q)$ invariant; indeed (4.8.13) implies that $\mathrm{C}\ell(g)$ is also an automorphism of the exterior algebra $\mathrm{C}\ell(M,q;-\beta)$.

The usefulness of these subspaces $\mathrm{C}\ell^n(M,q)$ appears for instance in the research of the quadratic extension $\mathrm{QZ}(M,q)$ mentioned in (3.7.6).

(4.8.15) **Proposition.** *When 2 is invertible in K and (M,q) is a quadratic space of constant rank r, then $\mathrm{QZ}(M,q)$ is equal to $\mathrm{C}\ell^0(M,q)\oplus\mathrm{C}\ell^r(M,q)$, and its discriminant module is $\mathrm{C}\ell^r(M,q)$. Besides, for $k=0,1,2,\dots,r$, the multiplication mapping π_q induces an isomorphism*

$$\mathrm{C}\ell^r(M,q)\otimes\mathrm{C}\ell^k(M,q)\longrightarrow\mathrm{C}\ell^{r-k}(M,q)\ ,\quad z\otimes x\longmapsto zx\ =(-1)^{k(r-1)}xz.$$

Proof. We know that $\mathrm{QZ}(M,q)$ is the direct sum of $K=\mathrm{C}\ell^0(M,q)$ and its discriminant module, which is the submodule of all $z\in\mathrm{QZ}(M,q)$ mapped to $-z$ by its standard involution φ. Because of (3.5.13), this means that $az=-\sigma(z)a$ for all $a\in M$; because of (4.8.13), this is equivalent to the equality $a\wedge z=0$ which characterizes the elements z of $\bigwedge^r(M)$ in $\bigwedge(M)$ (see (3.2.6)), and the elements of $\mathrm{C}\ell^r(M,q)$ in $\mathrm{C}\ell(M,q)$.

To prove the last assertion of (4.8.15), we can suppose that M admits an orthogonal basis $(e_1,e_2,\dots,e_r)$ such that $q(e_1)$, $q(e_2)$,...,$q(e_r)$ are all invertible, since localizations allow us to reduce the general case to that one. For every subset F of $B=\{1,2,\dots,r\}$, we denote by e_F the product of all elements e_j such that $j\in F$, when the order of the factors is the order of their indices; their product in $\mathrm{C}\ell(M,q)$ is also their exterior product in $\mathrm{C}\ell(M,q;-\beta)$ since every $\beta(e_i,e_j)$ vanishes if $i\neq j$. Moreover $\mathrm{C}\ell^r(M,q)$ is the submodule generated by e_B. Now it suffices to observe that

$$e_Be_F\ =\ \pm\,(\prod_{j\in F}\ q(e_j))\ e_{B\setminus F}\ ,$$

and that the products e_F (resp. $e_{B\setminus F}$), with F a subset of cardinal k, constitute a basis of $\mathrm{C}\ell^k(M,q)$ (resp. $\mathrm{C}\ell^{r-k}(M,q)$). □

For all $x\in\mathrm{C}\ell(M,q)$, the parallel projection of x in $\mathrm{C}\ell^0(M,q)=K$ with respect to $\mathrm{C}\ell^{>0}(M,q)$ is called the *scalar component* of x and denoted by $\mathrm{Scal}(x)$.

(4.8.16) **Proposition.** *The bilinear form $(x,y)\longmapsto\mathrm{Scal}(xy)$ is symmetric and the submodules $\mathrm{C}\ell^n(M,q)$ are pairwise orthogonal for it. It is nondegenerate whenever (M,q) is a quadratic space.*

Proof. Let x and y be elements of $\mathrm{C}\ell^j(M,q)$ and $\mathrm{C}\ell^k(M,q)$ respectively; from (4.8.10) we deduce that $\mathrm{Scal}(xy)=0$ whenever $j\neq k$. Consequently we only have to compare Scal(xy) and Scal(yx) when $j = k$. Since the formulas (3.1.5) are also valid for the reversion τ of $C\ell(M,q)$, we can write (when $j = k$)

$$\mathrm{Scal}(xy) \;=\; \mathrm{Scal}(\tau(xy)) \;=\; \mathrm{Scal}(\tau(y)\tau(x)) \;=\; \mathrm{Scal}(yx).$$

When (M,q) is a quadratic space, we can suppose that (M,q) admits an orthogonal basis $(e_1,e_2,\dots,e_r)$ as in the last part of the proof of (4.8.15); then the products e_F defined above constitute a basis of $C\ell(M,q)$. Since $e_1,\dots,e_r$ are pairwise anticommuting, it is easy to prove that it is an orthogonal basis of $(C\ell(M,q),\ \mathrm{Scal})$. The nondegeneracy of Scal follows from the fact that $(e_F)^2$ is always an invertible element of K, since it is $\pm\prod_{j\in F} q(e_j)$. □

When M is a finitely generated projective module, more precise properties of the linear form $\mathrm{Scal} : C\ell(M,q) \to K$ are stated in (4.8.17); they imply that it is invariant by all automorphisms of $C\ell(M,q)$, and not only by the automorphisms $C\ell(g)$ derived from an automorphism g of (M,q). The trace of an endomorphism of a finitely generated projective module has been defined by elementary means at the beginning of **3.6**; for instance if M is a finitely generated projective module, if its rank takes the values $r_1,\ r_2,\ \dots,\ r_k$ and if $e_1,\ e_2,\ \dots,\ e_k$ are the corresponding idempotents of K (see (1.12.8)), than the trace of the identity mapping of $C\ell(M,q)$ is

$$\mathrm{tr}(\mathrm{id}_q) \;=\; 2^{r_1}e_1 \;+2^{r_2}e_2 \;+\cdots+2^{r_n}e_n\ ;$$

the bijectiveness of the mapping $a \longmapsto 2a$ implies that $r_i = 0$ whenever $2e_i$ is not invertible in Ke_i; therefore $\mathrm{tr}(\mathrm{id}_q)$ is invertible in K.

(4.8.17) **Proposition.** *When M is a finitely generated projective module (such that the mapping $a \longmapsto 2a$ is bijective), for all $x \in C\ell(M,q)$ the traces of the multiplications $y \longmapsto xy$ and $y \longmapsto yx$ are both equal to* $\mathrm{tr}(\mathrm{id}_q)\,\mathrm{Scal}(x)$.

Proof. This is obviously true when x belongs to $K = C\ell^0(M,q)$; therefore it suffices to prove that the traces of the multiplications by x both vanish when x belongs to $C\ell^{>0}(M,q)$. This is clear when x is odd, because these multiplications permute $C\ell_0(M,q)$ and $C\ell_1(M,q)$. When x is even, it is a sum of elements like $2a\wedge z$ with $a \in M$ and $z \in C\ell_1(M,q)$; because of (4.8.13), $2a \wedge z$ is a Lie bracket $az - za$ (in the ordinary nongraded sense); the multiplication by a Lie bracket (on either side) is a Lie bracket of multiplications, and the trace of a Lie bracket of endomorphisms is always 0. □

Comment. The invertibility of $\mathrm{tr}(\mathrm{id}_q)$ ensures the interest of the property of Scal(x) stated in (4.8.17), and allows us to compare it with the reduced trace $\mathrm{tr}(x)$ when $C\ell(M,q)$ is a graded Azumaya algebra. Reduced traces are defined in (3.6.6) and (3.6.7); when A is a graded Azumaya algebra of constant rank n^2 or $2n^2$, the traces of the multiplications by an element $x \in A$ are both equal to $n\,\mathrm{tr}(x)$. Therefore if (M,q) is a quadratic space of constant rank $2k$ or $2k-1$, the equality $\mathrm{tr}(x) = 2^k\mathrm{Scal}(x)$ holds for all $x \in C\ell(M,q)$.

Exercises

(4.ex.1) Let A be an algebra (associative with unit 1_A); suppose that the underlying K-module A is graded over an additive group G (therefore $A = \bigoplus_{j\in G} A_j$) in such a way that $A_iA_j \subset A_{i+j}$ for all $(i,j) \in G^2$. Prove that 1_A belongs to A_0 (and consequently A is a graded algebra).

Give a counterexample when G is merely an additive monoid (for instance $G = \{$"zero","positive"$\}$).

(4.ex.2)* Let M be a module. Since $\mathrm{S}(M)$ is a coalgebra, the dual module $\mathrm{S}^*(M)$ is an algebra; it is the direct sum of $\mathrm{S}^{*0}(M)$ which is isomorphic to K, and the ideal $\mathrm{S}^{*+}(M)$ of all linear forms vanishing on $\mathrm{S}^0(M) = K$.

(a) Define an interior multiplication $\mathrm{S}^*(M) \times \mathrm{S}(M) \to \mathrm{S}^*(M)$ that makes $\mathrm{S}^*(M)$ become a module over $\mathrm{S}(M)$, and give its elementary properties. The notation $f \rfloor x$ is still suitable, but the twisting rule (4.2.1) is not relevant for symmetric algebras.

(b) Prove that there exists a unique mapping Exp from $\mathrm{S}^{*+}(M)$ into $\mathrm{S}^*(M)$ such that the following equalities hold for all $f \in \mathrm{S}^{*+}(M)$ and all $a \in M$:

$$\mathrm{Exp}(f)(1) = 1 \quad \text{and} \quad \mathrm{Exp}(f) \rfloor a \;=\; \mathrm{Exp}(f) \vee (f \rfloor a)\ .$$

(c) Let $\mathrm{ST}^k(M^*)$ be the submodule of symmetric tensors in $\mathrm{T}^k(M^*)$. Define a canonical mapping $\mathrm{ST}^k(M^*) \to \mathrm{S}^{*k}(M)$ and prove that it is bijective when M is a finitely generated projective module.

(4.ex.3) Consider the exterior algebra $\bigwedge(M)$, and prove that the four mappings π, ε, π', ε' (defined in **4.3**) let $\bigwedge(M)$ become a bialgebra according to the definition in **4.1**, provided that this definition is adapted to the twisting rule (4.2.1). Prove that the automorphism σ (that is $x \longmapsto (-1)^{\partial x}x$) is the inverse of $\mathrm{id}_\wedge$ in the algebra $\mathrm{Hom}^\wedge(\bigwedge(M), \bigwedge(M))$ defined by (4.2.4):

$$\pi \circ (\mathrm{id}_\wedge \otimes \sigma) \circ \pi' \;=\; \varepsilon \circ \varepsilon' \;=\; \pi \circ (\sigma \otimes \mathrm{id}_\wedge) \circ \pi'\ .$$

Such an inverse of the identity mapping is called an *antipode*, and a bialgebra with an antipode is called a *Hopf algebra*.

(4.ex.4) Let (M,q) be a quadratic module; is it possible to make the exterior powers $\bigwedge^k(M)$ become quadratic modules in a natural way?

(a) First suppose that 2 is invertible in K. By means of the algebra morphism $\bigwedge(\mathrm{d}_q) : \bigwedge(M) \longrightarrow \bigwedge(M^*)$, prove the existence of a unique quadratic form $\hat{q}$ on $\bigwedge(M)$ such that the submodules $\bigwedge^k(M)$ are pairwise orthogonal, and such that this equality holds for every sequence $(a_1, b_1, a_2, b_2, \dots, a_k, b_k)$ of elements of M :

$$\mathrm{b}_{\hat{q}}(a_1 \wedge a_2 \wedge \cdots \wedge a_k\ ,\ b_k \wedge \cdots b_2 \wedge b_1) \;=\; \det(\mathrm{b}_q(a_i, b_j))_{1\le i,j\le k}.$$

Prove that $(\bigwedge(M),\ \hat{q})$ is a quadratic space whenever (M,q) is one.

(b)* Without any hypothesis on K, deduce from the concept of "half-determinant" (see (2.ex.13)) that $\bigwedge^k(M)$ is still a quadratic module at least for every *odd* exponent k.
Hint. First consider free modules.

(4.ex.5) Assume that q and q' are strongly cliffordian quadratic forms on the module M, according to Definition (4.8.1); prove that $q + q'$ is still strongly cliffordian.
Hint. Use the morphism $a \longmapsto (a, a)$ from $(M, q + q')$ into $(M, q) \perp (M, q')$; observe that $M \oplus M$ is the direct sum of $M \oplus 0$ and the image Δ of this morphism; deduce from (4.8.5) that the subalgebra generated by Δ in $C\ell((M, q) \perp (M, q'))$ is isomorphic to $C\ell(M,\ q + q')$.

(4.ex.6) Let (M, q) be a quadratic module such that M is a finitely generated projective module of nonzero constant rank r. Because of (4.8.7) there is a surjective mapping $p : C\ell(M, q) \to \bigwedge^r(M)$ with kernel $C\ell^{<r}(M, q)$. We also consider the dual module $C\ell^*(M, q) = \mathrm{Hom}(C\ell(M, q), K)$ and its parity grading: for $i = 0, 1$, $C\ell_i^*(M, q)$ is the submodule of all linear forms vanishing on $C\ell_{1-i}(M, q)$.

(a) For every $\omega^* \in \bigwedge^{*r}(M)$, let F_{ω^*} be the linear form on $C\ell(M, q)$ defined by $F_{\omega^*}(x) = \omega^*(p(x))$. Prove that the linear forms F_{ω^*} make up a direct summand of $C\ell^*(M, q)$ of constant rank 1, that they have the same parity as r, and deduce from (3.2.1) the equality

$$\forall x,\ y \in C\ell(M, q), \quad F_{\omega^*}(xy) \ = \ (-1)^{\partial x \partial y}\, F_{\omega^*}(yx)\ .$$

(b) Suppose that (M, q) is a quadratic space, and consider the discriminant module D of $\mathrm{QZ}(M, q)$ (the centralizer of $C\ell_0(M, q)$ in $C\ell(M, q)$). With every $w \in \bigwedge^{*r}(M) \otimes D$ we associate a linear form G_w on $C\ell(M, q)$ in this way:

$$\forall \omega^* \in \bigwedge^{*r}(M)\ ,\ \forall d \in D\ ,\ \forall x \in C\ell(M, q)\ , \quad G_{\omega^* \otimes d}(x) \ = \ \omega^*(p(dx)).$$

Prove that the linear forms G_w make up a direct summand of $C\ell^*(M, q)$ of constant rank 1, that they are all even, and satisfy the equality

$$\forall x,\ y \in C\ell(M, q), \quad G_w(xy) \ = \ G_w(yx)\ .$$

Comment. When A is a graded Azumaya algebra, in (6.ex.11) it is proved that the submodule of all $h \in A_0^*$ such that $h(xy) = h(yx)$ for all $x,\ y \in A$, is a free direct summand of constant rank 1, and that it is generated by the reduced trace $x \longmapsto \mathrm{tr}(x)$ defined in **3.6**; consequently when $A = C\ell(M, q)$, it is the submodule just found above; since it is free, $D \otimes \bigwedge^r(M)$ too is free, whence $D \cong \bigwedge^r(M)$.

(c) Suppose that (M, q) is a quadratic space such that the mapping $a \longmapsto 2a$ is bijective from M onto M; in this case D is equal to $C\ell^r(M, q)$ (see (4.8.15)), which p maps bijectively onto $\bigwedge^r(M)$. Verify that

$$G_{\omega^* \otimes d}(x) \ = \ \omega^*(p(d))\ \mathrm{Scal}(x)\ .$$

Scalar products β and algebras $\bigwedge(M;\beta)$

(4.ex.7) Let (M,q) be a quadratic module with M a finitely generated projective module, and β an admissible scalar product for q (see (4.8.6)). Here we shall prove the existence of an associative multiplication on $\bigwedge(M)$ admitting 1 as a unit element and satisfying the conditions (4.8.9), without the help of (4.7.1) and the subsequent theorems.

(a) From the universal properties of $C\ell(M,q)$ and $C\ell(M,-q)$ deduce the existence of an algebra morphism Ψ from $C\ell(M,q)\hat{\otimes} C\ell(M,q)^{to}$ into $\mathrm{End}(\bigwedge(M))$ such that, for all $a\in M$ and all $x\in\bigwedge(M)$,

$$\begin{aligned} &\Psi(\rho(a)\otimes 1_q^{to})(x) = a\wedge x + \mathrm{d}_\beta(a)\rfloor x \\ \text{and}\qquad &\Psi(1_q\otimes\rho(a)^{to})(x) = a\wedge x + \mathrm{d}_\beta^{to}(a)\rfloor x. \end{aligned}$$

(b) For every $z\in C\ell(M,q)$ we set $f(z)=\Psi(z\otimes 1_q^{to})(1)$ and $g(z)=\Psi(1_q\otimes z^{to})(1)$; prove that f and g are bijections from $C\ell(M,q)$ onto $\bigwedge(M)$ such that $f(1_q)=g(1_q)=1$ and $f(\rho(a))=g(\rho(a))=a$ for all $a\in M$.
Hint. Localizations, (3.1.7) and perhaps (3.ex.4).

(c) Let $\bigwedge(M;\beta)$ be the module $\bigwedge(M)$ provided with the following multiplication:

$$\begin{aligned} (x,y)\longmapsto\ xy &= \Psi(f^{-1}(x)\otimes g^{-1}(y)^{to})(1) \\ &= \Psi(f^{-1}(x)\otimes 1_q^{to})(y) \ = \ (-1)^{\partial x\partial y}\ \Psi(1_q\otimes g^{-1}(y)^{to})(x)\ . \end{aligned}$$

Verify that $xy = f(f^{-1}(x)\,f^{-1}(y)) = g(g^{-1}(x)\,g^{-1}(y))$. Prove that $\bigwedge(M;\beta)$ is an associative algebra with unit element 1, in which both equalities (4.8.9) are valid. Moreover $f=g$.

Comment. This construction of $\bigwedge(M;\beta)$ comes from [Chevalley 1954] ; there is no doubt that Chevalley knew both equalities (4.8.9); but he thought (in accordance with (4.7.4)) that the first one was sufficient to characterize the multiplication in $\bigwedge(M;\beta)$, and consequently he only defined the algebra morphism $z\longmapsto\Psi(z\otimes 1_q^{to})$ from $C\ell(M,q)$ into $\mathrm{End}(\bigwedge(M))$.

(d) Verify (without (4.7.5)) that the interior multiplication by any $h\in M^*$ is also a twisted derivation of $\bigwedge(M;\beta)$; it suffices to verify that $h\rfloor(ax) = h(a)x - a(h\rfloor x)$ for all $a\in M$.

(4.ex.8) Let (M,q) be a quadratic module, β an admissible scalar product for q, and $\bigwedge(M;\beta)$ the derived algebra. When x and y are elements of respectively $\bigwedge^j(M)$ and $\bigwedge^k(M)$, by definition the interior product $x\rfloor y$ (resp. $x\lfloor y$) is the component of the Clifford product xy in $\bigwedge^{k-j}(M)$ (resp. $\bigwedge^{j-k}(M)$). Consequently $x\rfloor y$ vanishes whenever $j>k$, whereas $x\lfloor y$ vanishes whenever $j<k$ (see (4.8.10)). Moreover $x\rfloor y$ and $x\lfloor y$ are the same element of K when $j=k$; for instance $a\rfloor b = a\lfloor b = \beta(a,b)$ for all a and $b\in M$. By bilinearity the interior products $x\rfloor y$ and $x\lfloor y$ are defined for all x and y in $\bigwedge(M)$.

(a) Prove these equalities for all x, y, z in $\bigwedge(M)$:

$$(x \wedge y) \rfloor z \;=\; x \rfloor (y \rfloor z) \quad \text{and} \quad (x \lfloor y) \lfloor z \;=\; x \lfloor (y \wedge z)\ .$$

(b) Prove these equalities for all x, $y \in \bigwedge(M)$ and all $a \in M$:

$$\begin{aligned} a \rfloor (x \wedge y) &= (a \rfloor x) \wedge y + \sigma(x) \wedge (a \rfloor y)\ , \\ a \rfloor (xy) &= (a \rfloor x)\ y + \sigma(x)\ (a \rfloor y)\ , \\ (x \wedge y) \lfloor a &= \ (x \lfloor a)\sigma(y) + x \wedge (y \lfloor a)\ , \\ (xy) \lfloor a &= \ (x \lfloor a)\ \sigma(y) + x\ (y \lfloor a)\ . \end{aligned}$$

(c) Suppose that β is symmetric and that x and y are homogeneous for the parity grading of $\bigwedge(M)$; prove that $\;y \lfloor x \;=\; (-1)^{\partial x(1+\partial y)}\ x \rfloor y$.

Comment. Such a concept of interior multiplication is only advisable in elementary presentations of Clifford algebras, when K is a field of characteristic $\neq 2$, and β is nondegenerate and symmetric; thus the algebra $\bigwedge(M;\beta)$ can be constructed in an elementary way (without quotient of $\mathrm{T}(M)$) by means of an orthogonal basis of M. For many applications of Clifford algebras, this may be sufficient.

(4.ex.9) Let (M, q) be a quadratic module, β an admissible scalar product for q, and $\bigwedge(M;\beta)$ the derived algebra; besides, let g be an automorphism of (M, q). The functors $\bigwedge$ and $\mathrm{C}\ell$ associate with g an automorphism $\bigwedge(g)$ of $\bigwedge(M)$ and an automorphism $\mathrm{C}\ell(g)$ of $\mathrm{C}\ell(M, q)$. If we replace $\mathrm{C}\ell(M, q)$ with $\bigwedge(M;\beta)$, we get an automorphism of $\bigwedge(M;\beta)$ also denoted by $\mathrm{C}\ell(g)$; here we are interested in a comparison between $\bigwedge(g)$ and $\mathrm{C}\ell(g)$, both considered as linear automorphisms of the module $\bigwedge(M)$.

(a) Prove the existence of $\delta \in \bigwedge^{*2}(M)$ such that

$$\forall a,\ b \in M, \qquad \delta(a \wedge b) \;=\; \beta(g(a), g(b)) - \beta(a, b).$$

(b) Prove the following equalities, for all $x \in \bigwedge(M)$:

$$\mathrm{C}\ell(g)(x) \;=\; \bigwedge(g)(\,\mathrm{Exp}(\delta)\ \rfloor\ x\,) \;=\; \mathrm{Exp}\left(\bigwedge^{*}(g^{-1})(\delta)\right)\ \rfloor\ \bigwedge(g)(x).$$

Hint. Let us set $\theta(x) = \bigwedge(g)(\mathrm{Exp}(\delta) \rfloor x)$; the main difficulty is to prove that $\theta(xy) = \theta(x)\theta(y)$; here is the beginning of the calculations:

$$\begin{aligned} \theta(xy) &= \bigwedge(g)(\ \mathrm{Exp}(\delta)\ \rfloor\ \pi(\mathrm{Exp}(\beta_{\prime\prime}) \rfloor (x \otimes y))) \\ &= \bigwedge(g) \circ \pi(\ \mathrm{Exp}(\pi^*(\delta) + \beta_{\prime\prime})\ \rfloor\ (x \otimes y))\ ; \\ \theta(x)\ \theta(y) &= \pi\left(\ \mathrm{Exp}(\beta_{\prime\prime})\ \rfloor\ \bigwedge(g, g)\big((\mathrm{Exp}(\delta) \rfloor x) \otimes (\mathrm{Exp}(\delta) \rfloor y)\big)\right) \\ &= \bigwedge(g) \circ \pi\left(\ \mathrm{Exp}\left(\bigwedge^{*}(g, g)(\beta_{\prime\prime}) + \delta \otimes 1 + 1 \otimes \delta\right)\ \rfloor\ (x \otimes y)\right)\ . \end{aligned}$$

(c) Suppose that 2 is invertible in K, and prove that the above equalities are equivalent to this one, in which $[\beta]$ is defined as in **4.7**:

$$\mathrm{C}\ell(g)(x) \;=\; \mathrm{Exp}\left(\frac{1}{2}[\beta]\right) \;\rfloor\; \bigwedge(g)\left(\;\mathrm{Exp}\left(\frac{-1}{2}[\beta]\right) \;\rfloor\; x \;\right).$$

(4.ex.10)* With every bilinear form β on the module M is associated an algebra $\bigwedge(M;\beta)$ that is isomorphic to the Clifford algebra of the quadratic form $a \longmapsto \beta(a,a)$. In a dual way some people have associated a "Clifford coalgebra" with any element γ of $M \otimes M$. Let $\gamma_{\prime\prime}$ be the natural image of γ in $\bigwedge(M) \hat{\otimes} \bigwedge(M)$; the comultiplication $\pi'_\gamma : \bigwedge(M) \to \bigwedge(M) \otimes \bigwedge(M)$ is defined by

$$\pi'_\gamma(x) \;=\; \mathrm{Exp}(\gamma_{\prime\prime}) \wedge \pi'(x)\ .$$

Prove that π'_γ and ε' (that is the projection $\bigwedge(M) \longrightarrow \bigwedge^0(M) = K$) make $\bigwedge(M)$ become a coalgebra.

Now suppose that M is a finitely generated projective module, so that the algebras $\bigwedge(M^*)$ and $\bigwedge^*(M)$ can be identified; since γ induces a bilinear form on M^*, a deformation $\bigwedge(M^*;\gamma)$ can be defined; let $\pi_\gamma : \bigwedge(M^*) \otimes \bigwedge(M^*) \to \bigwedge(M^*)$ be the corresponding multiplication mapping. Prove the following equality for all $x \in \bigwedge(M)$ and all f and $g \in \bigwedge(M^*)$:

$$(\pi_\gamma(f \otimes g))(x) \;=\; (f \hat{\otimes} g)(\pi'_\gamma(x))\ .$$

Canonical scalar products

(4.ex.11) Let (M,q) be a quadratic module with M a finitely generated module of rank ≤ 4 at every prime ideal, and such that the mapping $a \longmapsto 2a$ is bijective from M onto M.

(a) Prove that $\mathrm{C}\ell^0(M,q) \oplus \mathrm{C}\ell^4(M,q)$ is a subalgebra contained in the center of $\mathrm{C}\ell_0(M,q)$, and that it is the submodule of all $x \in \mathrm{C}\ell_0(M,q)$ such that $\tau(x) = x$.
(b) Prove that M is the submodule of all $x \in \mathrm{C}\ell_1(M,q)$ such that $\tau(x) = x$. This has been proved in (3.ex.19) with less hypotheses but with more difficulty.
(c) When (M,q) is a quadratic space of constant rank 4, prove that $\mathrm{C}\ell^2(M,q)$ is a projective module of constant rank 3 over $\mathrm{QZ}(M,q) = \mathrm{C}\ell^0(M,q) \oplus \mathrm{C}\ell^4(M,q)$ (see (4.8.15)).

(4.ex.12) Let (M,q) be again a quadratic module with M a finitely generated module of rank ≤ 4, and such that the mapping $a \longmapsto 2a$ is bijective from M onto M. The main purpose of this exercise is to prove that the square of every element of $\mathrm{C}\ell^3(M,q)$ or $\mathrm{C}\ell^4(M,q)$ belongs to $K = \mathrm{C}\ell^0(M,q)$.

(a) Prove this when (M,q) is a quadratic space of constant rank 4, by means of (4.8.15) or (2.6.2).

In the following parts, where (M,q) is merely a quadratic module of rank ≤ 4, the proofs are not so simple, and $\mathrm{C}\ell(M,q)$ is replaced with the algebra $\bigwedge(M;\beta)$ (where $\beta = \mathrm{b}_q/2$) provided with a Clifford multiplication and an exterior one.

(b) Take y and y' in $\bigwedge^3(M)$ and prove that the Clifford product yy' has no component in $\bigwedge^4(M)$; consequently $yy' + y'y$ belongs to K. Besides, when h is an element of M^*, verify that $(h \rfloor y)\,(h \rfloor y')$ has no component in $\bigwedge^4(M)$.
Hint. After localization, you can suppose that M is generated by (a,b,c,d), and that $y = a \wedge b \wedge c$ and $y' = a \wedge b \wedge d$; deduce from (4.7.7) that

$$a \wedge b \wedge c = -cba + \beta(b,c)a - \beta(a,c)b + \beta(a,b)c\ ,$$
$$a \wedge b \wedge d = abd - \beta(b,d)a + \beta(a,d)b - \beta(a,b)d\ ,$$

and remember (4.8.10); finally observe that $yy' + y'y$ is invariant by τ. A direct calculation proves that

$$(h \rfloor (a \wedge b \wedge c)) \wedge (h \rfloor (a \wedge b \wedge d)) \ = \ 0.$$

(c) Prove that zz' belongs to K for all z and z' in $\bigwedge^4(M)$.
Hint. After localization, you can suppose that $z = z' = a \wedge y$ for some $a \in M$ and some $y \in \bigwedge^3(M)$; from (b) above you know that $y^2 \in K$; set $h = \mathrm{d}_q(a)$ and use (4.8.13) in this way:

$$4\ (a \wedge y)^2 \ = \ (ay - ya)^2 \ = \ h \rfloor (yay) \ - ay^2a - ya^2y$$
$$\text{with} \quad yay \ = \ (h \rfloor y)\ y \ - ay^2\ .$$

For another point of view, see (4.ex.16).

(4.ex.13) Let (M,q) be a quadratic space of constant rank r, such that the mapping $a \longmapsto 2a$ is bijective from M onto M, and let N be a direct summand of M of constant rank s. The restrictions of q to N and $N^\perp$ are denoted by q' and q'', and $\mathrm{C}\ell(N,q')$ and $\mathrm{C}\ell(N^\perp,q'')$ are treated as subalgebras of $\mathrm{C}\ell(M,q)$. Prove that the multiplication

$$\mathrm{C}\ell^r(M,q) \otimes \mathrm{C}\ell^s(M,q) \longrightarrow \mathrm{C}\ell^{r-s}(M,q) \qquad \text{(see (4.8.15))}$$

induces a bijection

$$\mathrm{C}\ell^r(M,q) \otimes \mathrm{C}\ell^s(N,q') \longrightarrow \mathrm{C}\ell^{r-s}(N^\perp,q'').$$

Hint. By means of (4.8.13) calculate $2b \wedge (zx)$ when b, z and x belong respectively to $N^\perp$, $\mathrm{C}\ell^r(M,q)$ and $\mathrm{C}\ell^s(N,q')$; you must discover that $b \wedge (zx) = 0$; this implies that zx lies in $\mathrm{C}\ell^{r-s}(N^\perp,q'')$.

(4.ex.14)* We suppose that 2 is invertible in K and we look for a method allowing us to prove formulas like (4.8.14) for suitable subsets E of the group $\mathcal{S}_n$ of permutations of $\{1,2,\ldots,n\}$. Such formulas involve algebras $\bigwedge(M;\beta)$ with $\beta = \mathrm{b}_q/2$, which are provided with exterior and Clifford multiplications. First, for any subset E, we define the following n-multilinear mapping P_E from M^n into $\bigwedge(M;\beta)$:

$$P_E(a_1, a_2, \ldots, a_n) \;=\; \sum_{s\in E} \operatorname{sgn}(s)\; a_{s(1)}a_{s(2)}\cdots a_{s(n)}\ .$$

Besides let Π_n be the set of all sets $\{B_1, B_2, \ldots, B_k\}$ such that $0 \le 2k \le n$ and $B_1,\ B_2,\ \ldots,\ B_k$ are pairwise disjoint subsets of $\{1, 2, \ldots, n\}$ all of cardinal 2. When $k > 0$, they constitute a partition of a subset of even cardinal $2k$. With every $\varpi \in \Pi_n$ we associate another n-multilinear mapping P_ϖ defined in this way: $P_\varpi(a_1, a_2, \ldots, a_n)$ is the exterior product of all elements a_i such that i does not belong to $B_1 \cup B_2 \cup \cdots \cup B_k$, still multiplied by all $\beta(a_i, a_j)$ such that $\{i, j\}$ is one of the sets $B_1,\ B_2,\ \ldots,\ B_k$; of course the order of the factors in the exterior product is the order of their indices.

(a) Explain that there exist integers N_ϖ independent of K, M and q, which allow you to write

$$P_E(a_1, a_2, \ldots, a_n) \;=\; \sum_{\varpi} N_\varpi\; P_\varpi(a_1, a_2, \ldots, a_n).$$

Moreover $N_\varpi = \operatorname{card}(E)$ when ϖ is the element of Π_n such that $k = 0$.

(b) Suppose that for all quadratic modules (M, q) over the field $\mathbb{Q}$ the n-linear mapping P_E vanishes whenever the variables a_i and a_j (with $i \ne j$) are equal. Prove that $N_\varpi = 0$ if i or j or both belong to $B_1 \cup B_2 \cup \cdots \cup B_k$.
What happens when P_E is always an alternate n-multilinear mapping?

(c) *Example.* Prove that the 5-linear mapping

$$(a_1, a_2, a_3, a_4, a_5) \longmapsto a_1a_2a_3a_4a_5 + a_1a_5a_4a_3a_2 - a_2a_5a_4a_3a_1 - a_3a_4a_5a_2a_1$$

is always alternate.

(4.ex.15) We suppose that (M, q) is a quadratic module such that the mapping $a \longmapsto 2a$ is bijective from M onto M. Let J be the kernel of the symmetric bilinear form $(x, y) \longmapsto \operatorname{Scal}(xy)$ mentioned in (4.8.16). Prove that J is an ideal of $C\ell(M, q)$, that it is the direct sum of all the intersections $J \cap C\ell^k(M, q)$, and that $J \cap C\ell^0(M, q) = 0$.
Comment. When (M, q) is a quadratic space, $C\ell(M, q)$ is a graded Azumaya algebra, and Proposition (6.7.4) implies $J = 0$ when J is a graded ideal of $C\ell(M, q)$ such that $J \cap K = 0$; the equality $J = 0$ only means that the bilinear form $(x, y) \longmapsto \operatorname{Scal}(xy)$ is weakly nondegenerate; compare with (4.8.16).

(4.ex.16) Let (M, q) be a quadratic module such that the mapping $a \longmapsto 2a$ is bijective from M onto M. We suppose that M is a finitely generated module, and consequently there exists an integer r such that the rank of M at every prime ideal is $\le r$. Let x and y be elements of $C\ell^j(M, q)$ and $C\ell^k(M, q)$ respectively; for every integer m between 0 and r, let $\Gamma_m(x, y)$ be the component of xy in $C\ell^m(M, q)$.

Prove that $\Gamma_m(y,x)$ vanishes when these two conditions are not both satisfied: first $|j-k| \le m \le \inf(j+k,\ 2r-j-k)$, secondly the parity of m must be that of $j+k$. Moreover, the number of values of m satisfying both conditions is $1+\inf(j,\ k,\ r-j,\ r-k)$.

Hint. The first result follows from $yx = \tau(\tau(x)\tau(y))$; then the inequalities $|j-k| \le m \le j+k$ follow from (4.8.10); but the inequality $m \le 2r-j-k$ (only valid for a symmetric β) requires more work. By localizations reduce the problem to the case of a module M generated by r elements $a_1, a_2, \dots, a_r$; you can suppose that x and y are exterior products of some of these r elements; thus xy is a sum of terms of this kind: exterior products of elements a_i multiplied by some factors $\beta(a_h, a_i)$ and some *universal* integers (independent of M, q and K); to prove that these integers vanish when $m > 2r-j-k$, you can assume that $K = \mathbb{Q}$, and use an orthogonal basis $(b_1, b_2, \dots, b_r)$ of M besides the basis $(a_1, a_2, \dots, a_r)$.

A characterization of Clifford algebras

(4.ex.17) According to Theorem (4.8.7) the canonical morphism $\bigwedge(M) \to \mathrm{Gr}(C\ell(M,q))$ is often an isomorphism. Conversely, if the graded algebra $\mathrm{Gr}(A)$ derived from some filtered algebra A is isomorphic to an exterior algebra, in some cases A must be isomorphic to a Clifford algebra. Observe that a graded algebra isomorphism $\bigwedge(N) \to \mathrm{Gr}(A)$ already implies $A^{\le -1} = 0$ and $A^{\le 0} = K$. Following [Roy 1964], we will prove the following statement: if 2 is invertible in K, and if $\mathrm{Gr}(A)$ is isomorphic (as a graded algebra) to the exterior algebra of a free module N, then there exists a unique submodule M of A such that $A^{\le 1} = K \oplus M$ and a^2 belongs to K for all $a \in M$; moreover the mapping $a \longmapsto a^2$ is a quadratic form q on M, and id_M extends to an algebra isomorphism from $C\ell(M,q)$ onto A.

For every $k \in \mathbb{N}$ the notation g_k means the canonical mapping $A^{\le k} \to \mathrm{Gr}^k(A)$. The proof will be achieved in four steps.

(a) Since N is free, there is a family $(b_j)_{j\in J}$ constituting a basis of a submodule of $A^{\le 1}$ supplementary to K. Since $g_1(b_j)^2$ vanishes, b_j^2 belongs to $A^{\le 1}$; and since b_j and b_j^2 commute, there are scalars λ_j and μ_j such that $b_j^2 = \lambda_j b_j + \mu_j$. Replace b_j with $e_j = b_j - \lambda_j/2$, so that $e_j^2 \in K$. Let M be the submodule generated by all the e_j.

(b) If i and j are distinct elements of J, for the same reasons $(e_i+e_j)^2$ can be written $\lambda_{i,j}(e_i+e_j) + \mu_{i,j}$ for some scalars $\lambda_{i,j}$ and $\mu_{i,j}$. Observe that $(e_i+e_j)^2$ commutes with e_j, whence $\lambda_{i,j} = 0$. Conclude that $a^2 \in K$ for every $a \in M$.

(c) Prove that M is the only submodule satisfying the above stated properties.

(d) Prove the bijectiveness of $C\ell(M,q) \to A$.

(e) The assumption about the invertibility of 2 has been used several times above; the following counter-example shows that it is probably indispensable. Let K be the field $\mathbb{Z}/2\mathbb{Z}$, and A the quotient of the polynomial ring $K[x]$ by the ideal generated by $x^2 - x - 1$, which inherits the natural increasing filtration of $K[x]$. Let b be the image of x in A; since $g_1(b)^2 = g_2(b^2) = 0$, $\mathrm{Gr}(A)$ is isomorphic to an exterior algebra. Prove that A is not isomorphic to a Clifford algebra.

Weyl algebras (for interested readers)

(4.ex.18) Let M be a K-module, and ψ an alternate bilinear form on M; the Weyl algebra $\mathrm{W}(M,\psi)$ (or $\mathrm{W}_K(M,\psi)$) is the quotient of the tensor algebra $\mathrm{T}(M)$ by the ideal generated by all elements

$$a \otimes b - b \otimes a - \psi(a,b) \quad \text{with} \quad a, b \in M.$$

The natural morphism $M \to \mathrm{T}(M) \to \mathrm{W}(M,\psi)$ is denoted by ρ, and 1_ψ is the unit element of $\mathrm{W}(M,\psi)$. This first exercise about $\mathrm{W}(M,\psi)$ only requires the knowledge expounded in **3.1** and **3.2**.

State the universal property directly derived from this definition. Develop an elementary theory for this algebra $\mathrm{W}(M,\psi)$ by following the ideas presented in **3.1** and **3.2**. In particular, you must explain that $\mathrm{W}(M,\psi)$ is provided with a *twisted reversion* τ, such that $\tau(\rho(a)) = \rho(a)$ for all $a \in M$, and $\tau(xy) = (-1)^{\partial x \partial y}\tau(y)\tau(x)$ for all (homogeneous) x and $y \in \mathrm{W}(M,\psi)$.

Prove the theorem analogous to (3.2.4): the Weyl algebra of $(M,\psi) \perp (M',\psi')$ is isomorphic to the ordinary tensor product $\mathrm{W}(M,\psi) \otimes \mathrm{W}(M',\psi')$.

(4.ex.19) The notation is the same as in (4.ex.18). Explain why $\mathrm{W}(M,\psi)$ is a comodule over the coalgebra $\mathrm{S}(M)$. Define the interior product $f \rfloor x$ of an element f of $\mathrm{S}^*(M) = \mathrm{Hom}(\mathrm{S}(M), K)$ and an element x of $\mathrm{W}(M,\psi)$, and state the elementary properties of this operation.

(4.ex.20) The notation is the same as in (4.ex.18); we also consider the symmetric algebra $\mathrm{S}(M)$. An admissible scalar product for ψ is a bilinear form β on M such that $\psi(a,b) = \beta(a,b) - \beta(b,a)$ for all a and $b \in M$. Of course, when 2 is invertible in K, there is a canonical scalar product $\beta = \psi/2$. Assuming that M is a finitely generated projective module, prove the existence of scalar products β for any alternate bilinear form ψ on M. Then, following (4.ex.7), define an algebra morphism Ψ from $\mathrm{W}(M,\psi) \otimes \mathrm{W}(M,\psi)^o$ into $\mathrm{End}(\mathrm{S}(M))$, and bijections f and g from $\mathrm{W}(M,\psi)$ onto $\mathrm{S}(M)$ that enable you to define an algebra $\mathrm{S}(M;\beta)$ isomorphic to $\mathrm{W}(M,\psi)$. Besides, K and M can be identified with their images in $\mathrm{W}(M,\psi)$, and the notations 1_ψ and $\rho(a)$ can be replaced with 1 and a.

(4.ex.21) The notation is the same as in (4.ex.18) and (4.ex.19). Let β be any bilinear form on M, and ψ' the alternate bilinear form defined by $\psi'(a,b) = \psi(a,b) + \beta(a,b) - \beta(b,a)$. Here $\beta_{\prime\prime}$ is the element of $S^{*2}(M \oplus M)$ naturally derived from β. To define $\mathrm{Exp}(\beta_{\prime\prime})$ in $S^*(M \oplus M)$, you may either assume that the natural ring morphism $\mathbb{Z} \to K$ extends to a ring morphism $\mathbb{Q} \to K$, or use the results of (4.ex.2) if you are more courageous. On $\mathrm{W}(M,\psi)$ a new multiplication is defined in this way:

$$(x,y) \longmapsto x \star y = \pi_\psi\big(\mathrm{Exp}(\beta_{\prime\prime}) \rfloor (x \otimes y)\big) .$$

Prove that this multiplication admits 1_ψ as a unit element, that it satisfies equalities analogous to (a) and (b) in (4.7.3), that it is associative, and that the resulting algebra $\mathrm{W}(M,\psi;\beta)$ is isomorphic to $\mathrm{W}(M,\psi')$ through a bijection Φ_β that is also an isomorphism of comodules over $\mathrm{S}(M)$.

(4.ex.22)* Let β be a bilinear form on M, and ψ the alternate bilinear form defined by $\psi(a,b) = \beta(a,b) - \beta(b,a)$. Moreover let $L = K[[t]]$ be the ring of formal series with coefficients in K. We identify $\mathrm{S}_L(L \otimes M)$ with $L \otimes \mathrm{S}(M)$, and we embed it into the algebra $\bar{\mathrm{S}}_L(L \otimes M)$ which is by definition the direct product of all submodules $t^j \otimes S^k(M)$. For every $n \in \mathbb{N}$ let $F^{\geq n}$ be the direct product of all submodules $t^j \otimes S^k(M)$ such that $2j + k \geq n$; these ideals $F^{\geq n}$ determine a decreasing filtration of $\bar{\mathrm{S}}_L(L \otimes M)$: $F^{\geq m} \vee F^{\geq n} \subset F^{\geq m+n}$; mind that $t \otimes 1$ has degree 2. We provide $\bar{\mathrm{S}}_L(L \otimes M)$ with the topology for which these ideals $F^{\geq n}$ constitute a basic family of neighbourhoods of 0; thus the subalgebra $K[t] \otimes \mathrm{S}(M)$ is dense in $\bar{S}_L(L \otimes M)$.

According to (4.ex.21), or to (4.ex.20) (when M is finitely generated and projective), we can define an algebra $\mathrm{S}_L(L \otimes M;\ t \otimes \beta)$, that is the module $L \otimes \mathrm{S}(M)$ provided with a new multiplication which lets it become isomorphic to $\mathrm{W}_L(L \otimes M,\ t \otimes \psi)$:

$$\forall a,\ b \in M, \quad (1 \otimes a)(1 \otimes b) - (1 \otimes b)(1 \otimes a) = t\ \psi(a,b).$$

Prove that this new multiplication extends by continuity to $\bar{\mathrm{S}}_L(L \otimes M)$. Moreover the submodules $F^{\geq n}$ also determine a filtration for this new multiplication: $F^{\geq m}\ F^{\geq n} \subset F^{\geq m+n}$.

Comment. Thus we get a "formal enlargement" $\bar{\mathrm{S}}_L(L \otimes M;\ t \otimes \beta)$ of the algebra $\mathrm{W}_L(L \otimes M,\ t \otimes \psi)$. When x is any element of $L \otimes \mathrm{S}^{\geq 1}(M)$, any power series in x is formally convergent in this enlargement.

(4.ex.23)* Let M be a vector space of finite dimension r over $\mathbb{R}$, β an element of $M \otimes M$, and ψ the element of $M \otimes M$ derived from β by the skew symmetrization $a \otimes b \longmapsto a \otimes b - b \otimes a$. We treat ψ as an alternate bilinear form on the dual space M^*, and β as an admissible scalar product. From ψ we can derive a Weyl algebra $\mathrm{W}(M^*,\psi)$, but here we are rather concerned with the Weyl algebra $\mathrm{W}_{\mathbb{C}}(\mathbb{C} \otimes M^*,\ i \otimes \psi)$ (where $i = \sqrt{-1}$), and we are going to construct an "enlargement" of this complex Weyl algebra. We use Fourier transformation according to this definition: the letters x and y represent variables running respectively through M and M^*, and the Fourier transform of a regular enough function f on M is defined in this way:

$$\mathcal{F}(f)(y) = (2\pi)^{-r/2} \int_M \exp(iy(x))\ f(x)\ dx\ ;$$

let $\mathrm{A}(M)$ be the space of functions $f : M \to \mathbb{C}$ such that $\mathcal{F}(f)$ is a distribution on M^* with compact support; $\mathrm{A}(M)$ contains the algebra $\mathrm{S}(M^*)$ identified with the algebra of polynomial functions on M. If f and g are elements of $\mathrm{A}(M)$, it is known that their ordinary product fg satisfies this equality, in which φ is any "test function" on M^* (that is an infinitely derivable function, the derivatives of which are all "rapidly vanishing" at infinity):

$$\int_{M^*} \varphi(y)\ \mathcal{F}(fg)(y)\ dy \ = \ (2\pi)^{-r/2} \int_{M^*\oplus M^*} \varphi(y_1+y_2)\ \mathcal{F}(f)(y_1)\mathcal{F}(g)(y_2)\ dy_1 dy_2.$$

Their $\star$-product is the element $f \star g$ of $\mathrm{A}(M)$ defined by this equality, which, according to the principles of Fourier analysis, is the natural translation of the definition proposed in (4.ex.21):

$$\begin{aligned} &\int_{M^*} \varphi(y)\ \mathcal{F}(f\star g)(y)\ dy \\ &\quad = (2\pi)^{-r/2} \int_{M^*\oplus M^*} \varphi(y_1+y_2)\ \exp(-i\beta(y_1,y_2))\ \mathcal{F}(f)(y_1)\mathcal{F}(g)(y_2)\ dy_1 dy_2. \end{aligned}$$

Prove that this multiplication on $\mathrm{A}(M)$ is associative, admits the constant function 1 as a unit element, and satisfies this equality:

$$\forall f,\ g \in M^*, \quad f\star g - g \star f \ = \ i\ \psi(f,g)\ .$$

You can even write formulas analogous to (4.8.9) for a $\star$-product $f \star g$ in which f or g belongs to M^*; for instance if f belongs to M^*, and if ∂_a is the partial derivation along the vector $a \in M$ such that $h(a) = \beta(f,h)$ for all $h \in M^*$, then $f\star g = fg + i\partial_a(g)$.
Comment. Unfortunately when neither $\mathcal{F}(f)$ nor $\mathcal{F}(g)$ has a compact support in M^*, the above definition in general fails to define a $\star$-product $f\star g$; existence theorems for this $\star$-product (with various additional hypotheses) require sophisticated functional analysis, and are outside the scope of this book. Of course when the supports of $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are not compact, it becomes important that φ and all its derivatives rapidly vanish at infinity; and above all, the factor i always present beside β plays a capital role, because the function $\exp(-i\beta(y_1,y_2))$ is bounded on $M^* \oplus M^*$.