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Results in f-algebras

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1. Introduction

We wrote a survey [18] on lattice ordered algebras five years ago. Why do we return to f-algebras once more? We hasten to say that there is only little overlap between the current paper and that previous survey. We have three purposes for the present paper. In our previous survey we remarked that one aspect that we did not discuss, while of some historical importance to the topic, is the theory of averaging operators. That theory has its roots in the nineteenth century and predates the rise of vector lattices. Positivity is a crucial tool in averaging, and positivity has been a fertile ground for the study of averaging-like operators. The fruits of positivity in averaging have recently (see [24]) started to appear in probability theory (to which averaging operators are close kin) and statistics. In the first section of our paper, we survey the literature for our selection of old theorems on averaging operators, at the same time providing some new perspectives and results as well.

Our second goal is to update the information from our previous survey on representation of disjointness preserving operators. Substantial new results have been obtained since and we intend to show that many of them can be understood from a generalized point of view, i.e., the structure theory of f-algebras. Indeed,

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in Section 6 we will prove the following new theorem that summarizes a rather large portion of the literature (e.g., [2, 3, 12, 13, 14, 32, 33]) on representation of order bounded disjointness preserving operators.

Theorem 1.1. *Let A be an n^{th}-root closed semiprime f-algebra and let B be a semiprime f-algebra. If $T : A \to B$ is an order bounded disjointness preserving operator then there exist an algebra and lattice homomorphism $S : \operatorname{Orth}(A) \to \operatorname{Orth}^{\infty}(\operatorname{Orth}(B^{\text{ru}}))$ and an element $w \in \operatorname{Orth}^{\infty}(\operatorname{Orth}(B^{\text{ru}}))$ such that*

$$T(f) = wS(f) \qquad \textit{for all} \quad f \in A.$$

Finally, to be able to present a proof of the latter theorem, we felt the need to lead the reader through the theory of various extended orthomorphisms and rings of quotients as available in the literature.

Last but not least, we have been involved in a study of the so-called square of a vector lattice [17], which in effect enables a systematic translation from the theory of order bounded bilinear maps that are separately disjointness preserving into the theory of order bounded disjointness preserving operators. The glue needed to achieve that translation is provided by so-called orthosymmetric bilinear maps introduced by Buskes and van Rooij in [16]. We need a brief appearance of orthosymmetric maps in the main result of the theorem above in our last section, and − as we said earlier − order bounded disjointness preserving operators have our interest in Section 5. The study of the geometric mean and square mean in f-algebras in our Section 3, apart from being interesting in its own right, provides exactly a foundation for a convexification procedure in vector lattices that leads to this square of vector lattices (see [5]).

2. Averaging operators

In his celebrated paper [48] written at the end of the 19^{th} century, Reynolds − a pioneer of theoretical fluid dynamics − introduced an operator that maps a function of time and space to its mean over some interval of time. For that operator, Reynolds was led to consider the algebraic identity

$$T\left(aT\left(b\right)+bT\left(a\right)\right)=T\left(a\right)T\left(b\right)+T\left(T\left(a\right)T\left(b\right)\right). \qquad (\mathcal{R})$$

An operator T with property $(\mathcal{R})$ is called a *Reynolds operator*. In his study, Reynolds also considered *averaging operators*, i.e., operators T that satisfy the identity

$$T(aT\left(b\right))=T\left(a\right)T\left(b\right). \qquad (\mathcal{A})$$

There now is an extensive literature on averaging operators, motivated to no small degree from their connection to conditional expectation in probability theory. Kampé de Fériet first recognized the importance of studying averaging and Reynolds operators in general, and substantially advanced the topic in [35]. A more algebraic study of these operators was initiated by Dubreil in [21], while the first study of averaging operators by means of functional analysis is due to

Birkhoff [11]. Interestingly, the averaging identity $(\mathcal{A})$ was being studied at about the same time as Kolmogorov's foundations of probability became known, whereas the connection with conditional expectation was made only many years later by Moy in [43].

Since those early beginnings of the history of averaging operators above, the identities $(\mathcal{R})$ and $(\mathcal{A})$ have been studied by many authors. Some were interested in the logical interdependence of the identities, others examined the relationship between $(\mathcal{R})$, $(\mathcal{A})$, and the differential equations describing the motion of fluids. Further research on the subject was motivated by the fact that both identities abundantly occur in probability theory, and, indeed, conditional expectation operators continue to be a source of inspiration for the general study of averaging and Reynolds operators [4, 10, 19, 22, 42, 49, 51, 53].

In the thirties of the previous century, Kampé de Fériet studied averaging operators on the set of real-valued functions that take only a finite number of values [35], while Birkhoff in [11] investigated them on spaces of real-valued continuous functions on a compact Hausdorff space. We remark that Sopka in [53] independently followed a similar path as Birkhoff, but the latter laced his study with a rather more algebraic point of view, setting the stage for our discussion. Our first proposition below appeared indeed in [11]. Following common terminology, a linear operator $T : A \to A$, where A is a real vector space, is called a *projection* whenever

$$T^2(a) = T(a) \qquad (\mathcal{P})$$

holds in A for all $a \in A$.

Proposition 2.1. (Birkhoff [11]) *Let A be an Archimedean f-algebra with unit element e and let $T : A \to A$ be an averaging operator such that $T(e) = e$. Then T is a projection and a Reynolds operator.*

By $C_0(X)$ we denote the (Archimedean and semiprime) f-algebra of all real-valued continuous functions on the locally compact Hausdorff space X that vanish at infinity. In [36], Kelley proved the following result which generalizes the case of compact X, established previously by Birkhoff in [11].

Theorem 2.2. (Kelley [36]) *A norm-one positive projection $T : C_0(X) \to C_0(X)$ is averaging if and only if the range of T is a subalgebra of $C_0(X)$.*

Kelley's proof of Theorem 2.2 is based on an integral representation for T and the fact that X may be decomposed into slices that render $T(a)$ to be the average of the value of a on each slice. Subsequently, Seever in [52] generalized Kelley's theorem as follows.

Theorem 2.3. (Seever [52]) *If $T : C_0(X) \to C_0(X)$ is a norm-one positive projection then*

$$T(aT(b)) = T(T(a)T(b)) \qquad (\mathcal{S})$$

holds for all $a, b \in C_0(X)$.

Let A be an (associative) algebra. Following the terminology by Huijsmans and de Pagter in [30], we call a linear operator $T : A \to A$ with property $(\mathcal{S})$ a *Seever operator*. Just like Kelley's proof of Kelley's theorem above, Seever's proof of his Theorem 2.3 uses the machinery of analysis. In [30], Huijsmans and de Pagter gave an f-algebra version of both of these theorems, crafting their proofs from the terrains of positivity and algebra. They restricted their results to semiprime f-algebras with the so-called *Stone condition*, which states that

$$a \wedge I \in A \quad \text{for all} \quad a \in A^+,$$

where I denotes the identity mapping on A and where A is considered as an f-subalgebra of the unital f-algebra Orth (A) of all orthomorphisms of A (see Section 4). Here is there theorem.

Theorem 2.4. (Huijsmans and de Pagter [30]) *Let A be an Archimedean semiprime f-algebra with the Stone condition and $T : A \to A$ be a positive contractive projection. Then T is a Seever operator.*

As a consequence of the previous theorem, Huijsmans and de Pagter also obtained a generalization of Kelley's theorem (see Theorem 2.2).

Theorem 2.5. (Huijsmans and de Pagter [30]) *Let A be an Archimedean semiprime f-algebra with the Stone condition and let $T : A \to A$ be a positive projection. Then the following are equivalent.*

(i) *T is averaging.*
(ii) *The range of T is a subalgebra of A and T is contractive.*

It turns out that the Stone condition in the preceding two theorems can be dropped. This was proved by Triki in [54] via extensions of positive projections. In addition, in his theorem below, A does not even need to be a vector lattice.

Theorem 2.6. (Triki [54]) *Let A be any majorizing subalgebra of the Archimedean semiprime f-algebra B and $T : A \to A$ be a positive contractive projection. Then T is a Seever operator. Moreover, T is averaging if and only if the range of T is a subalgebra of B.*

If in Theorem 2.6 B has a a Riesz norm $\|.\|$ (i.e., $\|a\| \leq \|b\|$ whenever $|a| \leq |b|$ in B) then the result holds without the extra condition 'A majorizes B'. More precisely, we have the following result.

Theorem 2.7. (Triki [54]) *Let A be a subalgebra of an Archimedean semiprime f-algebra B with a Riesz norm and let $T : A \to A$ be a positive contractive projection. Then T is a Seever operator. Moreover, T is averaging if and only if the range of T is a subalgebra of B.*

More recently, Triki (in [55]) also removed the semiprimeness assumption from the conditions of the theorem by Huijsmans and de Pagter above.

Theorem 2.8. (Triki [55]) *Let A be an Archimedean f-algebra and $T : A \to A$ be a positive contractive projection. Then T is a Seever operator and T is averaging if and only if the range of A is a subalgebra of A.*

It is not true that every positive projection onto a subalgebra is an averaging operator as can be seen from the following example due to Wulbert [59].

Example 2.9. (Wulbert [59]) *Put $X = [0,1] \cup \{2\}$ and let A be the subalgebra of $C(X)$ of all functions that vanish at the point 2. Let h be the function which is identically one on $[0,1]$, and vanishes at 2. Define the linear operator $T : C(X) \to C(X)$ by*

$$T(f)(x) = (f(x) + f(2))\,h(x) \quad \text{for all } f \in C(X), x \in X.$$

Then T is a positive projection on $C(X)$. However, if g is the constant function one on X, then $T(gT(g)) = 2h$ while $T(g)T(g) = 4h$.

Next we bring into focus various relationships between the algebraic identities $(\mathcal{A})$, $(\mathcal{P})$, $(\mathcal{R})$, and $(\mathcal{S})$ for a linear operator T on an f-algebra A. Consider first the properties $(\mathcal{A})$, $(\mathcal{P})$, $(\mathcal{R})$. Every operator T on an Archimedean semiprime f-algebra which satisfies two of these identities, also satisfies the third. This is the content of the following proposition.

Proposition 2.10. *Let A be an Archimedean semiprime f-algebra and let $T : A \to A$ be a linear operator. Then the following hold.*

(i) *If T is averaging and a projection then T is a Reynolds operator.*
(ii) *If T is averaging and a Reynolds operator then T is a projection.*
(iii) *If T is a Reynolds operator and a projection then T is averaging.*

It is easily verified that we can replace 'projection' by 'Seever operator' in the above result. So, if we consider the identities $(\mathcal{A})$, $(\mathcal{R})$, and $(\mathcal{S})$, then every operator T which satisfies two of those identities, also satisfies the third.

Proposition 2.11. *Let A be an Archimedean semiprime f-algebra and $T : A \to A$ be a linear operator. Then the following hold.*

(i) *If T is averaging and a Reynolds operator then T is a Seever operator.*
(ii) *If T is averaging and a Seever operator then T is a Reynolds operator.*
(iii) *If T is a Reynolds and Seever operator then T is averaging.*

Next we will deal with the relationship between the Reynolds identity $(\mathcal{R})$ and the averaging identity $(\mathcal{A})$ in connection with topological properties of certain function algebras. Before doing so, we present an example − due to Scheffold [51] − of a Reynolds operator that is not averaging.

Example 2.12. (Scheffold [51]) *Consider the operator $T : C([0,1]) \to C([0,1])$ defined by*

$$T(f)(x) = \int_0^1 f(tx)\,\mathrm{d}t \quad \text{for all } f \in C(X), x \in X.$$

It is easily verified that T is a Reynolds operator. At the same time, T is of course far from being averaging.

In [50], Rota considered Reynolds operators on the space $L_\infty(S,\Sigma,m)$ with closed range in the L_1-topology and showed that they are automatically averaging.

Theorem 2.13. (Rota [50]) *Let $L_\infty(S,\Sigma,m)$ and $L_1(S,\Sigma,m)$ denote bounded measurable and integrable functions on a σ-finite measure space, respectively. Let $T : L_\infty(S,\Sigma,m) \to L_\infty(S,\Sigma,m)$ be a Reynolds operator which is continuous with respect to the L_1-topology. Then R is averaging if and only if the range of T is closed.*

Rota conjectured that Theorem 2.13 remains valid for Reynolds operators on $C(X)$ with X compact Hausdorff. In his Ph.D. thesis [44], Neeb solved Rota's conjecture.

Theorem 2.14. (Neeb [44]) *Let $T : C_0(X) \to C_0(X)$ be a continuous Reynolds operator. Then the following statements are equivalent.*

(i) *T is averaging.*
(ii) *T is a projection.*
(iii) *The range of T is closed.*

However, the following problem remains open.

Problem 2.15. *Does* Theorem 2.14 *hold for an order bounded Reynolds operator on a semiprime Archimedean f-algebra under the relative uniform topology*?

Returning to Seever's identity $(\mathcal{S})$, we note that since the publication of Seever's paper [52], the identity $(\mathcal{S})$ has been studied by many authors in connection with contractive projections. Besides the results reviewed above, we present several theorems by Hadded that deserve more interest. We begin with the following.

Proposition 2.16. (Hadded [25]) *Let A be a f-algebra with unit element and $T : A \to A$ be a Seever operator. Then T^2 is a projection and a Seever operator.*

A Seever operator T need not be a projection (although T^2 is a projection). Indeed, consider $A = \mathbb{R}^3$ with the pointwise operations and $T : A \to A$ defined by $T(x,y,z) = (0,x,z)$ for all $(x,y,z) \in A$.

At this point, let X be a compact Hausdorff space. We denote the evaluation map at a point $x \in X$ by δ_x, and the restriction of δ_x to a vector subspace B of $C(X)$ is indicated by $\delta_{x,B}$. Recall from [59] that B is said to have a *weakly separating quotient* if for every two distinct points x and y in X and for each scalar $t \neq 1$ such that $\delta_{x,B} = t\delta_{y,B}$, we have that $\delta_{x,B}$ is not an extreme point of $\{\varphi \in B' : ||\varphi|| \leq 1\}$, where B' is the norm dual space of B. In particular, the range of a positive projection has weakly separating quotient. Wulbert improved Seever's theorem (for compact X) by introducing the condition that the range of the norm-one projection T has a weakly separating quotient as follows.

Theorem 2.17. (Wulbert [59]) *Let A denote a subalgebra of $C(X)$ and let $T : A \to A$ be a norm-one projection. If the range of T has a weakly separating quotient then T is A Seever operator.*

Later in [22], Friedman and Russo gave the following example showing that the range of a Seever operator which in addition is a norm-one projection need not have a weakly separating quotient.

Example 2.18. (Friedman and Russo [22]) *Write* $X = [-2,-1] \cup [1,2]$ *and let* $\chi = \chi_{[1,2]}$ *be the characteristic function of the interval* $[1,2]$*. Define a linear operator* $T : A \to A$ *by*

$$T(f)(x) = \frac{1}{2}(\chi(x)\, f(x) - \chi(-x)\, f(-x)) \quad \text{for all } f \in C(X), x \in X.$$

Then T *is a contractive projection and a Seever operator but the range of* T *does not have a weakly separating quotient.*

In [25], Hadded introduced the notion of an almost positive projection as follows. A projection $T : A \to A$, where A is an f-algebra, is said to be *almost positive* if there exists an order projection $\pi_T : A \to A$ such that

$$T(\pi_T(T(f))) = T(f) \quad \text{for all } f \in A$$

and

$$\pi_T(T(f)) \in A^+ \quad \text{for all } f \in A^+.$$

Of course, a positive projection is almost positive. The following proposition characterizes almost positive projections.

Proposition 2.19. (Hadded [25]) *Let* A *be an* f*-algebra and let* $T : A \to A$ *be a projection. Then* T *is almost positive if and only if there exist linear operators* $T_1, T_2 : A \to A$ *such that* $T = T_1 + T_2$*,* T_1 *is a positive projection given by* $T_1 = \pi T$ *for some order projection* π*, and* $T_1 T_2 = T_2^2 = 0$*.*

Hadded additionally linked Seever operators to almost positive projections as follows.

Theorem 2.20. (Hadded [25]) *Let* A *be a* σ*-Dedekind complete* f*-algebra with unit element and let* $T : A \to A$ *be a* σ*-order continuous contractive projection. Then* T *is a Seever operator if and only if* T *is almost positive.*

The assumption that T is σ-order continuous in the above theorem cannot be dropped as the following example shows.

Example 2.21. (Hadded [25]) *Let* A *be the Dedekind completion of* $C([-1,1])$*. Note that the Dedekind completion of* $C([-1,1])$ *equals* $C(X)$ *where* X *is the Gleason projective cover of* $[-1,1]$ (*combine Theorems* 12.9 *and* 14.18 *in* [34] *with* 10.54 *in* [57])*. Then there exists a surjective map from* X *to* $[-1,1]$ *for which no proper subset of* X *maps onto* $[-1,1]$*. Hence* (*using the Axiom of Choice*) *there exists a map* $[-1,1] \to X$ *with dense range. Composition of the latter map with elements of* $C(X)$ *yields an algebra and lattice homomorphic embedding of* A *into* $\mathbb{R}^{[-1,1]}$*. Thus we consider* A *as an* f*-subalgebra of* $\mathbb{R}^{[-1,1]}$*. Let* $T : A \to A$ *be the operator defined by*

$$T(f) = f(1)g_1 - f(-1)g_2 \quad \text{for all } f \in A,$$

where

$$g_1(x) = \begin{cases} 0 & \textit{for} \quad -1 \le x \le 1/3 \\ \dfrac{3}{2}x - \dfrac{1}{2} & \textit{for} \quad 1/3 \le x \le 1 \end{cases}$$

and

$$g_2(x) = \begin{cases} \dfrac{4}{3}x + \dfrac{1}{3} & \textit{for} \quad -1 \le x \le 0 \\ \dfrac{-1}{3}x + \dfrac{1}{3} & \textit{for} \quad 0 \le x \le 1 \end{cases}$$

Then T is a contractive projection and it satisfies Seever's identity, but T is not almost positive.

To link Seever's identity to almost positive projections in another way, we have to recall that if A is a semiprime f-algebra then so is its order continuous bidual $(A')'_n$ with respect to the Arens multiplication [7, 28, 29]. The upward directed net $\{a_i : i \in I\}$ in A^+ is said to be an *approximate unit* if $\sup\{a_i b : i \in I\} = b$ for all $b \in A^+$. The approximate unit $[0, I] \cap A$ is said to be *$\sigma(A, A')$-bounded* if $M_f = \sup\{f(a) : a \in [0, I] \cap A\} < \infty$ for all $f \in (A')^+$.

Theorem 2.22. (Hadded [25]) *Let A be a semiprime f-algebra with separating order dual such that A has a $\sigma(A, A')$-bounded approximate unit and let $T : A \to A$ be an order bounded contractive projection. Then T is a Seever operator if and only if $T''_n : (A')'_n \to (A')'_n$ is almost positive, where T''_n is the restriction of the biadjoint T'' of T to $(A')'_n$.*

Let $T : C_0(X) \to C_0(X)$ be a contractive projection. In the proof of [22], Freedman and Russo took an order projection M on the order bidual $C_0(X)''$ which verifies $T''MT'' = T''$ and then proved that T is a Seever operator if and only if MT'' is positive (see [22]). Hence, they actually proved that T is a Seever operator if and only if T'' is almost positive. Since $C_0(X)$ satisfies the hypothesis of Theorem 2.22 and $C_0(X)'' = (C_0(X)')'_n$, the Freedman-Russo result is a consequence of Theorem 2.22.

3. Square-mean closed and geometric-mean closed f-algebras

A vector lattice E is said to be *square-mean closed* if the set

$$\mathfrak{S}(a, b) = \{(\cos x)\, a + (\sin x)\, b : x \in [0, 2\pi]\}$$

has a supremum $\mathfrak{s}(a, b)$ in E for every $a, b \in E$ [5]. Notice that if E is square-mean closed then

$$\mathfrak{s}(a, b) = \mathfrak{s}(|a|, |b|) \ge 0 \text{ for all } a, b \in E.$$

In 1968, Lotz [40] proved that any Banach lattice is square-mean closed. Three years later, Luxemburg and Zaanen [39] extended Lotz's theorem to uniformly complete vector lattices. An elementary proof of this result was obtained more than two decades ago by Beukers, Huijsmans, and de Pagter in [8]. However, a square-mean closed Archimedean vector lattice need not be uniformly complete.

For instance, the vector lattice of all step functions on the real interval $[0,1]$ – equipped with the pointwise operations and ordering – is square-mean closed and not uniformly complete. Obviously, the f-algebra $\mathbb{R}$ of all real numbers is square-mean closed. Moreover,

$$\mathfrak{s}(a,b)^2 = a^2 + b^2 \quad \text{for all } a,b \in \mathbb{R}.$$

The latter identity extends to uniformly complete semiprime f-algebra as was proved by Beukers, Huijsmans, and de Pagter in [8]. Interestingly, their proof actually shows that the identity holds for any square-mean closed Archimedean f-algebra.

Theorem 3.1. *Let A be a square-mean closed Archimedean f-algebra. Then*

$$\mathfrak{s}(a,b)^2 = a^2 + b^2 \textit{ for all } a,b \in A.$$

If A in Theorem 3.1 is semiprime then $\mathfrak{s}(a,b)$ is the unique positive element c in A such that $c^2 = a^2 + b^2$. In fact, we can say more. First, let $N(A)$ denotes the set of all nilpotent elements of the Archimedean f-algebra A. Recall from [60] that

$$N(A) = \{a \in A : a^2 = 0\} = \{a \in A : ab = 0 \text{ for all } b \in A\}.$$

Hence, if a and b are two positive elements in an Archimedean f-algebra A then $a^2 = b^2$ if and only if $a - b \in N(A)$. This observation together with Theorem 3.1 quickly leads to the following.

Corollary 3.2. *Let A be a square-mean closed Archimedean f-algebra and $a,b,c \in A$ with $c \geq 0$. Then $c^2 = a^2 + b^2$ if and only if $c - \mathfrak{s}(a,b) \in N(A)$.*

Now we turn our attention to so-called geometric-mean closed Archimedean f-algebras. A vector lattice E is said to be *geometric-mean closed* if the set

$$\mathfrak{G}(a,b) = \left\{\frac{x}{2}a + \frac{1}{2x}b : x \in (0,\infty)\right\}$$

has an infimum $\mathfrak{g}(a,b)$ in A for every $a,b \in A^+$ [5]. We noticed above that any uniformly complete vector lattice is square-mean closed. However, uniform completeness also implies geometric-mean closedness. Indeed, every $C(X)$ is geometric-mean closed, hence so is every uniformly complete vector lattice.

Theorem 3.3. *Any uniformly complete vector lattice is geometric-mean closed.*

In particular, the f-algbra $\mathbb{R}$ is geometric-mean closed, a fact that goes back to the lever of Archimedes, and

$$\mathfrak{g}(a,b)^2 = ab \text{ for all } a,b \in \mathbb{R}^+ = [0,\infty).$$

Next, we prove that this equality holds in any geometric-mean closed Archimedean f-algebra.

Theorem 3.4. *Let A be a geometric-mean closed Archimedean f-algebra. Then*

$$\mathfrak{g}(a,b)^2 = ab \textit{ for all } a,b \in A^+.$$

Proof. Let $a, b \in A^+$ and notice that, by Lemma 4.1 in [5],

$$\mathfrak{g}(a,b) = \mathfrak{g}(a \vee b, a \wedge b).$$

Moreover,

$$(a \vee b)(a \wedge b) = ab.$$

Hence we may assume that $a \geq b$. Observe now that

$$\mathfrak{g}(a,b)^2 = \frac{1}{4} \inf\left\{\left(xa + \frac{1}{x}b\right)^2 : x \in (0,\infty)\right\},$$

since the multiplication in A is order continuous. Thus

$$4\left(\mathfrak{g}(a,b)^2 - ab\right) = \inf\left\{\left(xa - \frac{1}{x}b\right)^2 : x \in (0,\infty)\right\} \geq 0.$$

For convenience, put

$$c := \inf\left\{\left(xa - \frac{1}{x}b\right)^2 : x \in (0,\infty)\right\}.$$

Take $n \in \{1, 2, \dots\}$ and $k \in \{1, 2, \dots, n\}$. We find that

$$0 \leq c \leq \left(\sqrt{\frac{n}{k}}a - \sqrt{\frac{k}{n}}b\right)^2 = \frac{n}{k}\left(a - \frac{k}{n}b\right)^2 \leq n\left(a - \frac{k}{n}b\right)^2.$$

It follows from Proposition 4.1 in [8] that

$$0 \leq c \leq n \inf\left\{\left(a - \frac{k}{n}b\right)^2 : k \in \{1, 2, \dots, n\}\right\} \leq \frac{1}{n}b^2.$$

But then $c = 0$ because A is Archimedean and the proof is complete. □

Recall that if a and b are two positive elements in an Archimedean f-algebra A then $a^2 = b^2$ if and only if $a - b \in N(A)$. This leads to the following 'geometric-mean' version of a similar 'square-mean' version above.

Corollary 3.5. *Let A be a geometric-mean closed Archimedean f-algebra and a, b, $c \in A^+$. Then $c^2 = ab$ if and only if $c - \mathfrak{g}(a,b) \in N(A)$.*

We arrive in particular at the fact that if a and b are two positive elements in a geometric-mean closed semiprime Archimedean f-algebra A then $\mathfrak{g}(a,b)$ can be defined as the unique (positive) square-root of ab (compare with Theorem 4.2 in [8]).

In view of Theorems 3.1 and 3.4, and the identity

$$a^2 + b^2 = (a+b)^2 - 2ab$$

which holds for all a, b in the Archimedean f-algebra A, we may also expect that any geometric-mean closed Archimedean f-algebra is square-mean closed. Indeed, this follows from Theorem 4.4 in [5].

Theorem 3.6. *A geometric-mean closed Archimedean f-algebra is square-mean closed.*

We observe here that the identity

$$\mathfrak{s}(a,b)^2 = \left(a+b+\frac{\sqrt{2}}{2}\mathfrak{g}(a,b)\right)\left(a+b-\frac{\sqrt{2}}{2}\mathfrak{g}(a,b)\right)$$

holds for all positive elements a, b in a geometric-mean closed Archimedean f-algebra. Reflecting on that formula, it is natural to ask whether the converse of Theorem 3.6 holds. The answer is no, i.e., there exists a square-mean closed Archimedean f-algebra which is not geometric-mean closed. To that end we give the following example from [5].

Example 3.7. *Let $C(\mathbb{R}^+)$ be the Archimedean f-algebra of all real-valued continuous functions on $\mathbb{R}^+ = [0,\infty)$ and P be the vector subspace of $C(\mathbb{R}^+)$ consisting of all polynomial functions. Define for each $n \in \mathbb{N} = \{1, 2, \dots\}$ a vector subspace A_n of $C(\mathbb{R}^+)$ by induction as follows. Let $A_1 = P$ and for each $n \in \mathbb{N}$ let A_{n+1} be the vector subspace of $C(\mathbb{R}^+)$ generated by*

$$A_n \cup \left\{ \left(a^2+b^2\right)^{\frac{1}{2}} : a, b \in A_n \right\}$$

We claim that A_n is a subalgebra of $C(\mathbb{R}^+)$ for all $n \in \mathbb{N}$. To this end, we argue by induction. The result being trivial for A_1, let $n \in \mathbb{N}$ and assume that A_n is a subalgebra of $C(\mathbb{R}^+)$. Clearly, to show that A_{n+1} is a subalgebra of $C(\mathbb{R}^+)$, it suffices to prove that

$$a\left(b^2+c^2\right)^{\frac{1}{2}} \in A_{n+1} \text{ and } \left(a^2+b^2\right)^{\frac{1}{2}}\left(c^2+d^2\right)^{\frac{1}{2}} \in A_{n+1} \text{ for all } a, b, c \in A_n$$

Let $a, b, c, d \in A_n$ and put $u = (a+\mathbf{1})^2$ and $v = u - a$. Since A_n is a subalgebra of $C(\mathbb{R}^+)$, we get $0 \le u, v \in A_n$ and

$$a\left(b^2+c^2\right)^{\frac{1}{2}} = (u-v)\left(b^2+c^2\right)^{\frac{1}{2}} = \left((ub)^2+(uc)^2\right)^{\frac{1}{2}} - \left((vb)^2+(vc)^2\right)^{\frac{1}{2}} \in A_{n+1}.$$

On the other hand,

$$\left(a^2+b^2\right)^{\frac{1}{2}}\left(c^2+d^2\right)^{\frac{1}{2}} = \left((ac+bd)^2+(ad-bc)^2\right)^{\frac{1}{2}} \in A_{n+1}.$$

Accordingly, the union

$$A = \bigcup_{n\in\mathbb{N}} A_n$$

is a subalgebra of $C(\mathbb{R}^+)$. Furthermore, if $a \in A$ then there exists $n \in \mathbb{N}$ such that $a \in A_n$. Hence,

$$|a| = \left(a^2+0^2\right)^{\frac{1}{2}} \in A_{n+1} \subset A$$

and A is a vector sublattice of $C(\mathbb{R}^+)$. In summary, A is an Archimedean f-algebra with respect to the pointwise operations and ordering.

To show that A is square-mean closed, let $a, b \in A^+$ and choose $n \in \mathbb{N}$ such that $a, b \in A_n$. Observe that $\left(a^2+b^2\right)^{\frac{1}{2}}$ is the supremum in $C(\mathbb{R}^+)$ of $\mathfrak{S}(a,b)$.

But then the equality

$$\mathfrak{s}(a,b) = \sup \mathfrak{S}(a,b) = \left(a^2+b^2\right)^{\frac{1}{2}}$$

holds in A *because* $\left(a^2+b^2\right)^{\frac{1}{2}} \in A_{n+1} \subset A$. *Thus* A *is square-mean closed.*

Now, we prove by induction that all functions in A *are differentiable at* 0. *Any element of* A_1 *is a polynomial, hence differentiable at* 0. *Let* $n \in \mathbb{N}$ *and assume that all functions in* A_n *are differentiable at* 0. *Pick* $a \in A_{n+1}$ *and write*

$$a = b + \sum_{k=1}^{m} \lambda_k \left(a_k^2 + b_k^2\right)^{\frac{1}{2}}$$

for some $b, a_1, b_1, \ldots, a_m, b_m \in A_n$ *and* $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. *By the induction hypothesis,* b *and all* a_k, b_k *are differentiable at* 0. *Then so is* $\left(a_k^2+b_k^2\right)^{\frac{1}{2}}$. *It follows that* a *is differentiable at* 0.

Finally, we show that A *is not geometric-mean closed. We argue by contradiction. Let* e *and* u *be the functions in* $C(\mathbb{R}^+)$ *defined respectively by* $e(x) = x$ *and* $u(x) = 1$ *for all* $x \in \mathbb{R}^+$. *Clearly,* $e, u \in A$. *Assume that* $\mathfrak{G}(e,u)$ *has an infimum* $\mathfrak{s}(e,u)$ *in* A. *But* $\mathfrak{G}(e,u)$ *has an infimum in* $C(\mathbb{R}^+)$. *Indeed,*

$$b = \inf \mathfrak{G}(e,u) \text{ in } C\left(\mathbb{R}^+\right),$$

where $b(t) = t^{\frac{1}{2}}$ *for all* $t \in \mathbb{R}^+$. *Since* A *is uniformly dense in* $C(\mathbb{R}^+)$, *we get* $b \leq \mathfrak{s}(e,u)$. *Let* $t \in \mathbb{R}^+$ *and observe that*

$$\mathfrak{s}(e,u)(t) \leq xt + x^{-1} \text{ for all } x \in (0,\infty).$$

That is,

$$\mathfrak{s}(e,u)(t) \leq t^{\frac{1}{2}} = b(t)$$

It follows that $\mathfrak{s}(e,u) \leq b$. *Consequently,* $b = \mathfrak{s}(e,u) \in A$. *This contradicts the fact that all functions in* A *are differentiable at* 0.

We derive that A *is an example of an Archimedean* f*-algebra which is square-mean closed but not geometric-mean closed (notice that* A *is even unital).*

Remark 3.8. Interesting as the previous example is, after completing this survey, van Rooij (private communication) pointed out the following much easier and more elegant example.

Example 3.9. *Let* A *be the the Archimedean* f*-algebra of all Lipschitz functions on* $[0,1]$. *We denote the constant function one, the unit in* A, *by* $\mathbf{1}$. *For* $a, b \in A$, *we consider the complex-valued function* $f = a + ib$. *Then* $|f| = (a^2+b^2)^{\frac{1}{2}}$. *Moreover, for* $s, t \in [0,1]$ *it follows that* $||f(s)| - |f(t)|| \leq |f(s) - f(t)| \leq |a(s) - a(t)| + |b(s) - b(t)|$, *hence,* $(a^2+b^2)^{\frac{1}{2}} \in A$. *Thus,* A *is square-mean closed. Of course,* A *is not geometric-mean closed, because* $\sqrt{\mathbf{1}.e} = \sqrt{e}$ *is not in* A, *where* e *is the identity function.*

At the end of this section we remark once more that the geometric mean as studied above gives rise to a concrete construction of what is called the square of a vector lattice. In turn, the square of a vector lattice plays a fundamental role in understanding bilinear maps that are order bounded and separately disjointness preserving. Finally, the construction plays a role in understanding orthosymmetric bilinear maps, i.e., bilinear maps $T : E \times E \to F$ for vector lattices E, F with the property $T(a,b) = 0$ when a and b are disjoint. For more information about squares of vector lattices, we refer the reader to the survey by Bu, Buskes, and Kusarev on page 97 in this volume.

4. Maximal rings of quotients and (extended) orthomorphisms

We will first discuss so-called extended orthomorphisms. Let L be an Archimedean vector lattice. Luxemburg and Schep in [38] defined an order bounded linear operator $\pi : D_\pi \to L$, where D_π is an order dense order ideal in L, to be an *extended orthomorphism* of L if $|a| \wedge |b| = 0$ in D_π implies $|\pi(a)| \wedge |b| = 0$ in L. An extended orthomorphism π of L is called an *orthomorphism* of L if $D_\pi = L$. A natural equivalence relation can be introduced in the set of all extended orthomorphisms of L as follows. Two extended orthomorphisms of L are equivalent whenever they agree on an order dense order ideal in L or, equivalently, they are equal on the intersection of their domains. The intersection of two order dense order ideals in L is of course again an order dense order ideal in L. The set of all equivalence classes of extended orthomorphisms of L is denoted by $\mathrm{Orth}^\infty(L)$. With respect to the pointwise addition, scalar multiplication, and ordering, $\mathrm{Orth}^\infty(L)$ is an Archimedean vector lattice. The lattice operations in the vector lattice $\mathrm{Orth}^\infty(L)$ are given pointwise. It turns out that the vector lattice $\mathrm{Orth}^\infty(L)$ is an f-algebra with respect to composition as multiplication. Moreover, since extended orthomorphisms (and hence orthomorphisms) are order continuous, the set $\mathrm{Orth}(L)$ of all orthomorphisms of L can be embedded naturally in $\mathrm{Orth}^\infty(L)$ as an f-subalgebra. Obviously, the identity operator I_L of L serves as unit element in $\mathrm{Orth}^\infty(L)$ and in $\mathrm{Orth}(L)$. We summarize these facts in the following result, due to Luxemburg and Schep in [38].

Theorem 4.1. (Luxemburg and Schep [38]) *Let L be an Archimedean vector lattice. Then the following hold.*

(i) $\mathrm{Orth}^\infty(L)$ *is an Archimedean f-algebra with I_L as a unit element.*
(ii) $\mathrm{Orth}(L)$ *is an f-subalgebra of* $\mathrm{Orth}^\infty(L)$ *with I_L as a unit element.*

The algebraic properties and order structure of orthomorphisms had also been investigated earlier by Bigard and Keimel in [9], and by Conrad and Diem in [20]. Observe now that the f-algebra $\mathrm{Orth}^\infty(L)$ is commutative since it is Archimedean. Furthermore, due to in de Pagter [46], if L is uniformly complete then $\mathrm{Orth}^\infty(L)$ is von Neumann regular. We remind the reader that a commutative ring R is said to be *von Neumann regular* if for every $r \in R$ there exists $s \in R$ such that $r = r^2 s$.

Theorem 4.2. (de Pagter [46]) *If L is a uniformly complete vector lattice then* $\mathrm{Orth}^\infty(L)$ *is von Neumann regular.*

Next we turn to the maximal ring of quotients of a commutative semiprime ring. Our principal reference on the subject is the classical monograph [37] by Lambek. Let R be a commutative ring and assume that R in addition is semiprime, that is, 0 is the only nilpotent element in R. A ring ideal D of R is said to be *dense* in R if $r = 0$ whenever $r \in R$ and $rd = 0$ for all $d \in D$. Observe that the intersection of two dense ring ideals in R is again a dense ring ideal in R. A mapping $\pi : D_\pi \to R$, where D_π is a dense ring ideal in R, is called *fraction* of R if π is R-*linear*, that is to say, $\pi(c+d) = \pi(c) + \pi(d)$, $\pi(c-d) = \pi(c) - \pi(d)$, and $\pi(rd) = r\pi(d)$ for all $r \in R, c, d \in D_\pi$. Two fractions of R are identified if they coincide on some dense ring ideal of R. An obvious equivalence relation is thus obtained on the set of all fractions of R. The set of all equivalence classes is denoted by $Q(R)$ and called *the maximal ring of quotients of* R. Clearly, $Q(R)$ may be given a ring structure by defining addition and multiplication pointwise on the intersections of domains. Furthermore, $Q(R)$ is commutative and, since R is semiprime, it is von Neumann regular [37]. There is a natural and canonical embedding of R into $Q(R)$, and we accordingly regard R as a subring of $Q(R)$. Moreover, if S is a ring of which the elements are fractions of R then there exists a one-to-one ring homomorphism of S into $Q(R)$ that is induced by the canonical embedding of R into $Q(R)$. Less formally, S can be considered as a subring of $Q(R)$. For this reason, Utumi in [56] has called $Q(R)$ the *maximal* (or *complete*) *ring of quotients* of R (see also [6] by Banaschewski and [41] by Martinez).

Theorem 4.3. (Anderson [1]) *Let A be an Archimedean f-algebra with unit element e. Then the following hold.*

(i) *$Q(A)$ is an Archimedean von Neumann regular f-algebra with e as a unit element.*
(ii) *A is an f-subalgebra of $Q(A)$.*

Now, let A be an Archimedean semiprime f-algebra and consider the linear operator $\iota : A \to \mathrm{Orth}(A)$ defined by

$$\iota(a)(x) = ax \quad \text{for all } a, x \in A.$$

Obviously, ι is a one-to-one lattice and ring homomorphism. Furthermore, it is not hard to see that the range of ι is a ring ideal in $\mathrm{Orth}(A)$. In summary, the elements of $\mathrm{Orth}(A)$ can be considered as fractions of A. It follows that $\mathrm{Orth}(A)$ is (after suitable identifications) an f-subalgebra of $Q(A)$. But then $Q(\mathrm{Orth}(A))$ is contained in $Q(A)$ since elements in $Q(\mathrm{Orth}(A))$ are clearly fractions of A. We derive that $Q(A) = Q(\mathrm{Orth}(A))$. The latter equality together with Theorem 4.3 leads to the following.

Corollary 4.4. *Let A be an Archimedean semiprime f-algebra. Then the following hold.*

(i) *$Q(A)$ is an Archimedean von Neumann f-algebra with unit element.*
(ii) $\mathrm{Orth}(A)$ (*and then A*) *is an f-subalgebra of $Q(A)$.*

The definition of $\mathrm{Orth}^{\infty}(L)$ for an Archimedean vector lattice L is of course somewhat analogous to the definition of $Q(R)$ for a commutative semiprime ring R. When we add to this the many properties that $\mathrm{Orth}^{\infty}(A)$ and $Q(A)$ share when A is an Archimedean semiprime f-algebra A, one suspects that the two objects are isomorphic. Unfortunately, this is not true in general. An example in this direction is provided by de Pagter in [46].

Example 4.5. (de Pagter [46]) *Let A be the set of all real-valued continuous functions on the real interval $[0,1]$ which are piecewise polynomial. Clearly, A is an Archimedean unital* (*and then semiprime*) *f-algebra with respect to the pointwise operation and ordering. Define $a \in A$ by*

$$a(t) = 1 + t \quad \textit{for all}\, t \in [0,1]$$

and $\pi : A \to A$ by

$$\pi(x)(t) = (ax)(t) = a(t)\, x(t) \quad \textit{for all}\, x \in A, t \in [0,1]\,.$$

Clearly, $\pi \in \mathrm{Orth}^{\infty}(A)$ and $\pi \in Q(A)$. The principal ring ideal $aA = \{ax : x \in A\}$ is dense in A. Consider the fraction $\sigma : aA \to A$ defined by

$$\sigma(ax) = x \quad \textit{for all}\, x \in A,$$

that is, σ is the multiplication by the function $1/a$. Obviously, σ is the inverse of π in $Q(A)$. However, one can prove by contradiction that π does not have an inverse in $\mathrm{Orth}^{\infty}(A)$.

In spite of de Pagter's example, $\mathrm{Orth}^{\infty}(A)$ can be embedded in $Q(A)$ as an f-subalgebra. This result was proved by de Pagter [46] in case that A has a unit and Wickstead [58] extended that to Archimedean semiprime f-algebras.

Theorem 4.6. (Wickstead [58]) *Let A be an Archimedean semiprime f-algebra. Then $\mathrm{Orth}^{\infty}(A)$ is an f-subalgebra of $Q(A)$.*

Though we know from de Pagter's example that the converse of Theorem 4.6 fails, Wickstead in [58] proved that the maximal ring of quotients can, in fact, be viewed as consisting of some kind of orthomorphisms. Indeed, an order bounded linear operator $\pi : D_{\pi} \to L$, where D_{π} is an order dense vector sublattice of L, is called a *weak orthomorphism* of L if $|a| \wedge |b| = 0$ in D_{π} implies $|\pi(a)| \wedge |b| = 0$ in L. Hence, a weak orthomorphism of L is an extended orthomorphism of L if and only if D_{π} is an order dense order ideal in L. Unlike extended orthomorphisms, weak orthomorphisms do not, in general, have an additive structure (see [58]). Fortunately, this 'bad' behavior is absent in the case of an Archimedean semiprime f-algebra A. Indeed, amongst those extensions of weak orthomorphisms on A, which are again weak orthomorphisms of A, there is one which has a largest domain. The set of all weak orthomorphisms of A which have maximal domain is denoted by $\mathrm{Orth}^{w}(A)$. It turns out that pointwise operations and ordering make $\mathrm{Orth}^{w}(A)$ into an Archimedean f-algebra with unit element. Actually, we have more.

Theorem 4.7. (Wickstead [58]) *Let A be an Archimedean semiprime f-algebra. Then the following hold.*

(i) $\mathrm{Orth}^{w}(A)$ *is an Archimedean von Neumann regular f-algebra with I_A as a unit element.*

(ii) $\mathrm{Orth}^{\infty}(A)$ (*and hence* $\mathrm{Orth}(A)$) *is an f-subalgebra of* $\mathrm{Orth}^{w}(A)$.

In particular, $\mathrm{Orth}^{w}(A)$ is commutative and has positive squares. The upshot of it all is that $Q(A)$ can indeed be identified with $\mathrm{Orth}^{w}(A)$.

Theorem 4.8. (Wickstead [58]) If *A is an Archimedean semiprime f-algebra then* $Q(A) = \mathrm{Orth}^{w}(A)$.

Under the extra condition of uniform completeness, the extended orthomorphisms and the maximal ring of quotients coincide as well. This result is also due to Wickstead in [58] and, in the unital case, to de Pagter in [46]. In summary, we have the following theorem, the last result of this section.

Theorem 4.9. (Wickstead [58]) *Let A be a uniformly complete semiprime f-algebra A. Then $Q(A) = \mathrm{Orth}^{\infty}(A) = \mathrm{Orth}^{w}(A)$.*

5. Order bounded disjointness preserving operators

Let L and M be vector lattices. A (linear) operator $T : A \to B$ is said to be *disjointness preserving* if $|T(a)| \wedge |T(b)| = 0$ for all $a, b \in A$ with $|a| \wedge |b| = 0$. If A and B are Archimedean semiprime f-algebras, then the operator $T : A \to B$ is disjointness preserving if and only if T is *separating*, meaning that, $T(a)T(b) = 0$ in B whenever $ab = 0$ in A.

In 1983, Arendt proved in [2] that if X and Y are compact Hausdorff spaces and $T : C(X) \to C(Y)$ is an order bounded disjointness preserving operator (a *Lamperti operator* in Arendt's terminology) then T is a weighted composition operator. First, let $\mathrm{coz}(w)$ denote the cozero-set of a real-valued function w on Y, i.e.,

$$\mathrm{coz}(w) = \{y \in Y : w(y) \neq 0\},$$

and denote by $\mathbf{1}$ the function identically equal to one on X.

Theorem 5.1. (Arendt [2]) *Let X and Y be compact Hausdorff spaces. An order bounded operator $T : C(X) \to C(Y)$ is disjointness preserving if and only if there exists a map $h : Y \to X$ such that*

$$T(a)(y) = T(\mathbf{1})(y)\, a(h(y)) \quad \text{for all } a \in C(X), y \in Y.$$

Furthermore, h is continuous and uniquely determined on $\mathrm{coz}(T(\mathbf{1}))$.

We point out that Jarosz in [32] independently obtained Arendt's result. We now look at Theorem 5.1 from a more algebraic point of view. For every $a \in C(X)$, the function $S(a)$ defined by

$$S(a)(y) = 0 \text{ if } y \notin \mathrm{coz}(T(\mathbf{1})) \quad \text{and} \quad S(a)(y) = a(h(y)) \text{ if } y \in \mathrm{coz}(T(\mathbf{1}))$$

need not be a member of $C(Y)$. But $S(a)$ naturally is an element of the maximal ring of quotients $Q(C(Y))$ of $C(Y)$ [26]. Another version of Arendt's result thus arises as follows. If $T : C(X) \to C(Y)$ is an order bounded disjointness preserving operator, then there exists a lattice and ring homomorphism $S : C(X) \to Q(C(Y))$ such that $T(a) = T(\mathbf{1})S(a)$ for all $a \in C(X)$. Recently Boulabiar proved in [13] that the latter version is true for arbitrary Archimedean unital f-algebras.

Theorem 5.2. (Boulabiar [13]) *Let A and B be Archimedean f-algebras with unit elements. A ordered bounded operator $T : A \to B$ is disjointness preserving if and only if there exists a lattice and ring homomorphism $S : A \to Q(A)$ such that*

$$T(a) = T(e)S(a) \quad \textit{for all } a \in A,$$

where e indicates the unit element of A.

The following $C_0(X)$-version of Theorem 5.1 was proved by Jeang and Wong in [33].

Theorem 5.3. (Jeang-Wong [33]) *Let X and Y be locally compact Hausdorff spaces. An order bounded operator $T : C_0(X) \to C_0(Y)$ is a disjointness preserving if and only if there exist a function $w : Y \to \mathbb{R}$, which is continuous on* $\operatorname{coz}(w)$, *and a function $h : Y \to X$ such that*

$$T(a)(y) = w(y)\,a(h(y)) \quad \textit{for all } a \in C_0(X), y \in Y$$

Moreover, h is continuous and uniquely determined on $\operatorname{coz}(w)$.

One might hope that Theorem 5.3 can be obtained from Theorem 5.1 by extending an order bounded disjointness preserving operator $T : C_0(X) \to C_0(Y)$ to an order bounded disjointness preserving operator $T^{\alpha} : C(\alpha X) \to C(\alpha Y)$, where αX denotes the one-point compactification of X. However, Jeang and Wong [33] provided the following example of an order bounded disjointness preserving operator T which does not have any such extensions.

Example 5.4. (Jeang-Wong [33]) *Let $X = \mathbb{R}^{+}$ and $Y = \mathbb{R}$ with the usual topology and define $w, h : \mathbb{R} \to \mathbb{R}$ by*

$$w(y) = \begin{cases} 1 & \textit{if } \ y > 2 \\ y-1 & \textit{if } \ 0 \le y \le 2 \\ -1 & \textit{if } \ y < 0 \end{cases} \quad \textit{and} \quad h(y) = \begin{cases} y & \textit{if } \ y \ge 0 \\ -y & \textit{if } \ y < 0. \end{cases}$$

The weighted composition operator $T : C_0(X) \to C_0(Y)$ defined by

$$T(a)(y) = w(y)\,a(h(y)) \quad \textit{for all } a \in C(X), y \in Y$$

is an order bounded disjointness preserving operator. But no order bounded linear extension $T^{a} : C(\alpha X) \to C(\alpha Y)$ of T can be disjointness preserving.

Now, let X and Y be completely regular spaces. In [3], Araujo, Beckenstein, and Narici proved that if $T : C(X) \to C(Y)$ is a bijective disjointness preserving operator and if the inverse operator T^{-1} of T preserves disjointness as well (such an operator T is said to be *biseparating* in [3]), then T is a weighted composition operator and the realcompactification υX of X is homeomorphic to the realcompactification υY of Y (see the classical book [23] for realcompactification of completely regular spaces).

Theorem 5.5. (Araujo-Beckenstein-Narici [3]) *Let X and Y be completely regular topological spaces and $T : C(X) \to C(Y)$ be a bijective disjointness preserving operator such that T^{-1} also preserves disjointness. Then there exist an homeomorphism $h : \upsilon Y \to \upsilon X$ such that*

$$T(a)(y) = T(\mathbf{1})(y)\, a(h(y)) \quad \textit{for all } a \in C(X), y \in Y.$$

In Theorem 5.5, the composition operator $S : C(X) \to C(Y)$ defined by $S(f) = a \circ h$ for all $a \in C(Y)$ is obviously a lattice and ring isomorphism. Hence, Theorem 5.5 can be stated more algebraically as follows. If $T : C(X) \to C(Y)$ is a bijective disjointness preserving operator with T^{-1} disjointness preserving then there exist a lattice and ring isomorphism $S : C(X) \to C(Y)$ such that

$$T(a) = T(\mathbf{1})\, S(a) \quad \text{for all } a \in C(X).$$

This algebraic version of Theorem 5.5 was obtained by Boulabiar, Buskes, Henriksen in [12] for the more general setting of unital Archimedean f-algebras.

Theorem 5.6. (Boulabiar-Buskes-Henriksen [12]) *Let A and B be Archimedean f-algebras with unit elements. If T is an order bounded disjointness preserving operator $T : A \to B$ with T^{-1} disjointness preserving, then there exists a lattice and ring isomorphism $S : A \to B$ such that*

$$T(a) = T(e)\, S(a) \quad \textit{for all } a \in A,$$

where e denotes the unit element of A.

Bijective disjointness preserving operators on $C_0(X)$-algebras have been studied by Jeang and Wong in [33]. They obtained the following.

Theorem 5.7. (Jeang-Wong [33]) *Let X and Y be locally compact Hausdorff spaces and let $T : C_0(X) \to C_0(Y)$ be a bijective disjointness preserving operator. Then there exist $w \in C_b(Y)$ and an homeomorphism $h : Y \to X$ such that*

$$T(a)(y) = w(y)\, a(h(y)) \quad \textit{for all } a \in C_0(X), y \in Y.$$

Notice that in Theorem 5.7, the operator under consideration is not assumed to be order bounded. Actually, the hypotheses imply automatic order boundedness. This is a particular case of a result by Huijsmans and de Pagter to the effect that any invertible disjointness preserving operator between two Banach lattices is bounded. In [14], Boulabiar and Buskes gave alternative proofs of Theorems 5.5 and 5.7 based on the following theorem by Hart [27]. The vector sublattice of a vector lattice M generated by a subset E of M is denoted by $\mathfrak{R}(E)$.

Theorem 5.8. (Hart [27]) *Let L and M be Archimedean vector lattices and T be an order bounded disjointness preserving operator $T : L \to M$. Then there exists a lattice and ring homomorphism $\widetilde{T} : \mathrm{Orth}(L) \to \mathrm{Orth}(\mathfrak{R}(T(L)))$ such that*

$$\widetilde{T}(\pi)(T(a)) = T(\pi(a)) \quad \textit{for all } \pi \in \mathrm{Orth}(L), a \in L.$$

Theorem 5.8 leads to the following nice application in [27] to f-algebras. Once more we recall to the reader that if A is an Archimedean semiprime f-algebra then A can be embedded in the unital f-algebra $\mathrm{Orth}(A)$ of all orthomorphisms of A as an f-subalgebra and a ring ideal. This identification is taken into consideration below without further ado.

Corollary 5.9. *Let A and B be Archimedean semiprime f-algebras and let T be a bijective order bounded disjointness preserving operator from A onto B. Then there exists a unique algebra and lattice isomorphism $\widetilde{T}$ from $\mathrm{Orth}(A)$ onto $\mathrm{Orth}(B)$ such that*

$$T(fg) = Tf\widetilde{T}g \qquad (f, g \in A).$$

6. A new representation theorem

Let A be an Archimedean semiprime f-algebra. For $n > 1$ we say that A *is n^{th}-root closed* if for every $f \in A^+$ there exists an element $f^{\frac{1}{n}} \in A^+$ such that $\left(f^{\frac{1}{n}}\right)^n = f$.

The following new theorem implies all of the results about representation of order bounded disjointness preserving operators cited in the previous section. We remind the reader that B^{ru} stands for the uniform completion of B as defined by Quinn in [47].

Theorem 6.1. *Let $n > 1$. Let A be an n^{th}-root closed semiprime f-algebra and let B be a semiprime f-algebra. If $T : A \to B$ is an order bounded disjointness preserving operator then there exist an algebra and lattice homomorphism $S : \mathrm{Orth}(A) \to \mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$ and an element $w \in \mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$ such that*

$$T(f) = wS(f) \qquad \textit{for all } f \in A.$$

The proof consists of three ingredients. It heavily relies on the beautiful theorem by Hart above. Secondly, we need the following result by Buskes and van Rooij about orthosymmetric maps (for the definition of which we refer back to the end of Section 2), introduced in [16].

Theorem 6.2. (Buskes-Van Rooij [16]) *Every orthosymmetric map is symmetric.*

And thirdly, we need the following extension theorem by Buskes and van Rooij in [15].

Theorem 6.3. (Buskes-Van Rooij [15]) *An orthomorphism on a majorizing vector sublattice extends uniquely to an orthomorphism on the whole space.*

We will give an example to show that not for all semiprime f-algebras A a representation like the one above is valid, even when B equals the real numbers and T is a lattice homomorphism. The proof of Theorem 6.1 is now in order.

We start with the following lemma which easily follows from Theorem 6.2 above.

Lemma 6.4. *Let A and B be Archimedean semiprime f-algebras. If $p : A \times A \to B$ is an orthosymmetric map and a, b, and c are elements of A then $p(ab, c) = p(a, bc)$.*

Before we give the proof of our Theorem 6.1, we remark that, by Theorem 4.9, we could alternatively employ (as is also evident from our proof below) the maximal ring of quotients $Q(\mathrm{Orth}(B^{\mathrm{ru}}))$ instead of $\mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$. Now the proof of the main theorem.

Proof of Theorem 6.1. *Step* 1. We first construct the lattice and algebra homomorphism $S : \mathrm{Orth}(A) \to \mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$. By Hart's theorem 5.8, for every $\pi \in \mathrm{Orth}(A)$ there exists a unique $\widetilde{\pi}$ in $\mathrm{Orth}(\mathcal{R}(T(A)))$ such that

$$\widetilde{\pi}T = T\pi. \tag{1}$$

We denote by $\mathcal{I}(T(A))$ the order ideal generated by $\mathcal{R}(T(A))$ in B^{ru}.

By the Buskes-van Rooij Theorem 6.3, $\widetilde{\pi}$ extends uniquely to an element of $\mathrm{Orth}(\mathcal{I}(T(A)))$. This extension is again called $\widetilde{\pi}$. We extend $\widetilde{\pi}$ once more to an element $S(\pi)$ of $\mathrm{Orth}\left(\mathcal{I}(T(A)) \oplus \mathcal{I}(T(A))^{\mathrm{d}}\right)$ defined by

$$S(\pi)(f) = 0 \qquad \text{for all } f \in \mathcal{I}(T(A))^{\mathrm{d}},$$

where $\mathcal{I}(T(A))^{\mathrm{d}}$ denotes the disjoint complement of $\mathcal{I}(T(A))$ in B^{ru}. We consider $S(\pi)$ as an element of $\mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$. The map S that sends $\pi \in \mathrm{Orth}(A)$ to $S(\pi) \in \mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$ clearly is a lattice and algebra homomorphism.

Step 2. We now show that the equality

$$T(fg) = T(f)\,S(g)$$

holds in $\mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$ for all $f, g \in A$. Take $f, g \in A$ and consider $\pi_f \in \mathrm{Orth}(A)$, the multiplication by f. According to (1), an identity which we henceforth consider in $\mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$, and using the identifications made in the previous section, we obtain

$$S(f)\,T(g) = S(\pi_f)(T(g)) = \widetilde{\pi_f}T(g) = T\pi_f(g) = T(fg).$$

Step 3. In this step we construct the weight w. Let $f \in A^{+}$ and consider

$$w_f = \left(\frac{\left(T\left(f^{\frac{1}{n}}\right)\right)^{n}}{T(f)}\right)^{\frac{1}{n-1}}$$

as an element of the formal ring of quotients $q(\mathrm{Orth}(B^{\mathrm{ru}}))$ of $\mathrm{Orth}(B^{\mathrm{ru}})$. This w_f naturally is an element of the maximal ring of quotients $Q(\mathrm{Orth}(B^{\mathrm{ru}}))$ of $\mathrm{Orth}(B^{\mathrm{ru}})$. By Theorem 4.9, w_f is an element of $\mathrm{Orth}^{\infty}(\mathrm{Orth}(B^{\mathrm{ru}}))$.

For $g \in A^{+}$ we claim that

$$T\left(f^{\frac{1}{n}}\right)^{n} T(g) = T\left(g^{\frac{1}{n}}\right)^{n} T(f).$$

To this end, we define for a given $u \in A^{+}$ the positive bilinear map $\varphi : A \times A \to \operatorname{Orth}^{\infty}(\operatorname{Orth}(B^{\mathrm{ru}}))$ by

$$\varphi(x,y) = T(xu)\, T(y) \qquad \text{for all } x, y \in A.$$

Let $x, y \in A$ such that $x \wedge y = 0$ and observe that $(xu) \wedge y = 0$ so

$$T(xu) \wedge T(y) = 0.$$

Thus

$$\varphi(x,y) = T(xu)\, T(y) = 0$$

and φ is orthosymmetric. It follows by Theorem 6.2 that φ is symmetric and from Lemma 6.4 that

$$T(xu)\, T(y) = T(x)\, T(yu).$$

Therefore,

$$T\left(f^{\frac{1}{n}}\right)^{n} T(g) = T\left(f^{\frac{1}{n}}\right) \cdots T\left(f^{\frac{1}{n}}\right) T\left(g^{\frac{1}{n}} \cdots g^{\frac{1}{n}}\right) = T\left(g^{\frac{1}{n}}\right)^{n} T(f),$$

and $w_f = w_g$. Putting $w = w_f$, we have now proved that

$$T(f) = wS(f) \qquad \text{for all } f \in A^{+}$$

and hence also

$$T(f) = wS(f) \qquad \text{for all } f \in A.$$

Corollary 6.5. *If T is in addition surjective then S maps $\operatorname{Orth}(A)$ to $\operatorname{Orth}(B)$ and $w \in \operatorname{Orth}(B)$. If T is bijective then S maps A to B and w is invertible.* □

Proof. Assume that T is surjective. That S maps $\operatorname{Orth}(A)$ to $\operatorname{Orth}(B)$ is obvious. Remark that

$$wT(f^2) = w^2 S(f^2) = w^2 S(f)^2 = T(f)^2 \text{ for all } f \in A.$$

Therefore, $wg \in B$ for all $g \in B$, i.e., w is in $\operatorname{Orth}(B)$. If T is bijective then so is S (see [27]) and then $w = T \circ S^{-1}$ is invertible as well. Consequently, $S = w^{-1}T$ maps A to B. □

We now observe that all seven Theorems 5.1 through 5.7 immediately follow as consequences from our main result. The condition that A is n^{th}-root closed can not be deleted from the main theorem as the following example shows.

Example 6.6. *Let A be the f-algebra of the piecewise polynomial functions on $[0,1]$ that are 0 at 0. Then the lattice homomorphism $T : A \to \mathbb{R}$ that assigns to a function its right derivative at 0 is not representable as in the main theorem above. Indeed, denote the idenitity function on $[0,1]$ by f. Suppose that T has a representation as above with S an algebra and lattice homomorphism $A \to \mathbb{R}$ and α a nonzero real number such that $T = \alpha S$. Then $S(f) \neq 0$, hence $S(f^2) \neq 0$, but $T(f^2) = 0$, a contradiction.*

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