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Abstract Model Theory as a Framework for Universal Logic

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Abstract. We suggest abstract model theory as a framework for universal logic. For this end we present basic concepts of abstract model theory in a general form which covers both classical and non-classical logics. This approach aims at unifying model-theoretic results covering as large a variety of examples as possible, in harmony with the general aim of universal logic.

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1. Introduction

Universal logic is a general theory of logical structures as they appear in classical logic, intuitionistic logic, modal logic, many-valued logic, relevant logic, paraconsistent logic, non-monotonic logic, topological logic, etc. The aim is to give general formulations of possible theorems and to determine the domain of validity of important theorems like the Completeness Theorem. Developing universal logic in a coherent uniform framework constitutes quite a challenge. The approach of this paper is to use a semantic approach as a unifying framework.

Virtually all logics considered by logicians permit a semantic approach. This is of course most obvious in classical logic and modal logic. In some cases, such as intuitionistic logic, there are philosophical reasons to prefer one semantics over another but the fact remains that a mathematical theory of meaning leads to new insights and clarifications.

Researchers may disagree about the merits of a semantic approach: whether it is merely illuminating or indeed primary and above everything else. It is quite reasonable to take the concept of a finite proof as the most fundamental concept in logic. From this predominantly philosophically motivated point of view semantics comes second and merely as a theoretical tool. It is also possible to take the mathematical concept of a structure as a starting point and use various logics merely as tools for the study of structures. From this predominantly mathematically motivated view, prevalent also in computer science logic, formal proofs are like certificates that we may acquire at will according to our needs. Finally, we may take the intermediate approach that formal languages are fundamental in logic, and there are many different ones depending on the purpose they are used for, but common to all is a mathematical theory of meaning. This is the approach of this paper.

Abstract model theory is the general study of model theoretic properties of extensions of first order logic (see [3]). The most famous result, and the starting point of the whole field, was Lindström's Theorem [22] (see Theorem 3.4 below) which characterized first order logic in terms of the Downward Löwenheim-Skolem Theorem and the Compactness Theorem. Subsequently many other characterizations of first order logic emerged. Although several characterizations of first order logic tailored for structures of a special form (e.g. topological [39]) were found, no similar characterizations were found for extensions of first order logic in the setting of ordinary structures. However, in [34] a family of new infinitary languages is introduced and these infinitary languages permit a purely model theoretic characterization in very much the spirit of the original Lindström's Theorem.

In Section 2 we present an approach to abstract model theory which is general enough to cover many non-classical logics. Even if few results exist in this generality, the approach suggests questions for further study.

Intertwined with the study of general properties of extensions of first order logic is naturally the study of the model theory of particular extensions. In this respect logics with generalized quantifiers and infinitary languages have been the main examples, but recently also fragments of first order logic such as the guarded fragment and finite variable fragments have been extensively studied. Chang [9] gives an early sketch of modal model theory, modern model theory of modal logic emphasizes the role of bisimulation (e.g. [26]).

It was noticed early on that particular properties of extensions of first order logic depended on set theoretical principles such as CH or $\diamondsuit$. A famous open problem is whether $L(Q_2)$ is axiomatizable (Q_2 is the quantifier "there exist at least $\aleph_2$ many"). In [35] the concept of a *logic frame* is introduced to overcome this dependence on metatheory. It becomes possible to prove results such as, if a logic of a particular form is axiomatizable, then it necessarily also satisfies the Compactness Theorem. We discuss logic frames in Section 5 and point out their particular suitability for the study of universal logic.

2. Abstract Model Theory

The basic concept of abstract model theory is that of an abstract logic. This concept is an abstraction of the concept of truth as a relation between structures of some sort or another and sentences of some sort or another. In its barest formulation, void of everything extra, when we abstract out all information about what kind of structures we have in mind, and also all information about what kind of sentences we have in mind, what is left is just the concept of a binary relation between two classes[1].

Definition 2.1. An *abstract logic* is a triple $L = (S, F, \models)$ where $\models\ \subseteq S \times F$. Elements of the class S are called the structures of L, elements of F are called the sentences of L, and the relation $\models$ is called the satisfaction relation of L.

Figure 1 lists some building blocks from which many examples of abstract logics can be constructed.

Some obvious conditions immediately suggest themselves, such as closure under conjunction (see Definition 2.5 below) and closure under permutation of symbols. However, the spectrum of different logics is so rich that it seems reasonable to start with as generous a definition as possible. Still, even this definition involves a commitment to truth as a central concept which limits its applicability as an abstraction of e.g. fuzzy logic.

The value of a general concept like the above depends upon whether we can actually say anything on this level of generality. Surprisingly, already this very primitive definition allows us to formulate such a central concept as compactness and prove some fundamental facts about compactness.

We say that a subset T of F *has a model* if there is $A \in S$ such that $A \models T$ i.e. $\forall\phi \in T(A \models \phi)$. An abstract logic $L = (S, F, \models)$ is said to satisfy the *Compactness Theorem* if every subset of F, every finite subset of which has a model, has itself a model. We can now demonstrate even in this quite general setup that compactness is inherited by sublogics, a technique frequently used in logic. First we define the sublogic-relation. The definition below would be clearer if we applied it only to logics which have the same structures. This seems an unnecessary limitation and we are quite naturally lead to allowing a translation of structures, too.

Definition 2.2. An abstract logic $L = (S, F, \models)$ is a *sublogic* of another abstract logic $L' = (S', F', \models')$, in symbols

$$L \leq L',$$

if there are a sentence $\theta \in S'$ and functions $\pi : S' \to S$ and $f : F \to F'$ such that

1. $\forall A \in S \exists A' \in S'(\pi(A') = A \text{ and } A' \models' \theta)$
2. $\forall\phi \in F \forall A' \in S'(A' \models \theta \to (A' \models' f(\phi) \iff \pi(A') \models \phi))$.

The idea is that the structures in S' are richer than the structures in S, and therefore we need the projection π. The role of θ is to cut out structures in S' that are meaningless from the point of view of L. Typical projections are in Figure 2.

Lemma 2.3. *If $L \leq L'$ and L' satisfies the Compactness Theorem, then so does L.*

[1]In lattice theory such relations are called Birkhoff polarities [5]. A recent study is [14]. We are indebted to Lauri Hella for pointing this out.

Structures	Sentences
Valuations -double valuations -three-valued	Propositional logic Relevance logic Paraconsistent logic
Relational structures -monadic -ordered -finite -pseudofinite -topological -Banach space -Borel -recursive -ω-models	Predicate logic -with two variables -guarded fragment -with infinitary operations -with generalized quantifiers -higher order -positive bounded, etc -logic of measure and category
Kripke structures -transitive reflexive -equivalence relation -etc	Intuitionistic logic Modal logic -S4,S5, etc
Many-valued structures	Many-valued logic, fuzzy logic
Games	Linear logic

FIGURE 1. Building blocks of abstract logics

Proof. We give the easy details in order to illustrate how the different components of Definition 2.2 come into play. Suppose $T \subseteq F$ and every finite subset of T has a model. Let $T' = \{\theta\} \cup \{f(\phi) : \phi \in T\}$. Suppose $T'_0 = \{\theta\} \cup \{f(\phi) : \phi \in T_0\} \subseteq T'$ is finite. There is $A \in S$ such that $A \models T_0$. Let $A' \in S'$ such that $\pi(A') = A$ and $A' \models' \theta$. Then $A' \models T'_0$. By the Compactness Theorem of L' there is $A' \in S'$ such that $A' \models' T'$. Thus $\pi(A') \models T$. □

Another property that is inherited by sublogics is decidability (and axiomatizability). The formulation of these properties in abstract terms requires sentences of the abstract logic to be encoded in such a way that concepts of effectiveness apply. It is not relevant which concept of effectiveness one uses.

Definition 2.4. An abstract logic $L = (S, F, \models)$ is said to be *recursive* if F is effectively given. A recursive abstract logic $L = (S, F, \models)$ is an *effective sublogic* of another recursive abstract logic $L' = (S', F', \models')$, in symbols $L \leq_{\text{eff}} L'$, if $L \leq L'$ via θ, π and f such that $f(\phi)$ can be effectively computed from ϕ. A recursive abstract logic $L = (S, F, \models)$ is said to be *decidable* if the set $\{\phi \in F : \phi \text{ has a model}\}$ can be effectively decided inside F. It is *co-r.e. (or r.e.) for satisfiability* if $\{\phi \in F : \phi \text{ has a model}\}$ is co-r.e. (respectively, r.e.).

The above concept of co-r.e. for satisfiability could be appropriately called *effective axiomatisability* for logics closed under negation (see Definition 3.3 below).

Structure A	Projection $\pi(A)$	θ expresses
Ordered structure $(M, <)$	Structure M	Axioms of order
Zero-place relations	Valuation	
Binary structure	Kripke structure	Transitivity (e.g.)
Structure with unary predicates	Many-sorted structure	

FIGURE 2. Typical projections

The Independence Friendly logic IF (see [16]) is not closed under negation, it is r.e. for satisfiability, but neither effectively axiomatizable nor co-r.e. for satisfiability.

Definition 2.5. An abstract logic $L = (S, F, \models)$ is said to be *closed under conjunction* if for every $\phi \in F$ and $\psi \in F$ there is $\phi \wedge \psi \in F$ such that $\forall A \in S(A \models \phi \wedge \psi \iff A \models \phi \text{ and } A \models \phi)$. We say that a recursive abstract logic L is *effectively closed under conjunction* if $\phi \wedge \psi$ can be found effectively from ϕ and ψ inside F.

Lemma 2.6. *Suppose $L \leq_{\mathit{eff}} L'$ and L' is effectively closed under conjunction. If L' is decidable (or co-r.e. for satisfiability), then so is L.*

Proof. If $\phi \in F$ has a model A in S, then there is a model A' in S' such that $A' \models \theta$ and $\pi(A') = A$, whence $A' \models \theta \wedge f(\phi)$. Conversely, if $\theta \wedge f(\phi) \in F$ has a model A' in S', then ϕ has the model $\pi(A')$ in S. Since there is an effective algorithm for"$\theta \wedge f(\phi)$ *has a model in* S'", there is also one for"ϕ *has a model in* S". □

Example (Predicate logic). Let S be the set of all first order structures of various vocabularies, and F the set of all first order sentences (with identity) built from the atomic formulas and the usual logical symbols $\exists, \forall, \wedge, \vee, \neg, \rightarrow, \leftrightarrow$ and parentheses. The relation $A \models_L \phi$ is defined as usual for structures A and sentences ϕ of the same vocabulary. Predicate logic is a recursive abstract logic which satisfies the Compactness Theorem and is co-r.e. for satisfiability but not decidable. Predicate logic on finite structures does not satisfy Compactness Theorem. It is r.e. for satisfiability but not co-r.e. for satisfiability and hence not decidable. It is not a sublogic of predicate logic but it is an effective sublogic of the extension of predicate logic with the generalized quantifier Q_0 ("there exists infinitely many"). Predicate logic on ordered structures is important in computer science (especially on finite ordered structures). It is clearly an effective sublogic of predicate logic. Another variant of predicate logic is many-sorted logic [28]. It is an effective sublogic of the ordinary predicate logic.

Example (Two variable predicate logic). Let S be the set of all first order structures of various vocabularies, and F the set of all first order sentences (with identity) built from the atomic formulas with just the two variables x and y, and the usual logical symbols $\exists, \forall, \wedge, \vee, \neg, \rightarrow, \leftrightarrow$, and parentheses. The relation $A \models_L \phi$ is defined as usual for structures A and sentences ϕ of the same vocabulary. This is a decidable abstract logic [29], which is an effective sublogic of predicate logic.

Example (Guarded fragment of predicate logic [1]). Let S be the set of all first order structures of various vocabularies, and F the set of all guarded first order sentences i.e. first order formulas where all quantifiers are of the form $\exists\vec{x}(R(\vec{x},\vec{y})\wedge\phi(\vec{x},\vec{y}))$ or $\forall\vec{x}(R(\vec{x},\vec{y})\rightarrow\phi(\vec{x},\vec{y}))$, where $R(\vec{x},\vec{y})$ is atomic. The relation $A \models_L \phi$ is defined as usual for structures A and sentences ϕ of the same vocabulary. This is a decidable abstract logic [1], which is an effective sublogic of predicate logic.

Example (Propositional logic). Let us fix a set P of propositional symbols $p_0, p_1, \ldots$. Let S be the set of all functions $v : P \rightarrow \{0,1\}$, and F the set of all propositional sentences built from the symbols of P and the usual logical symbols $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ and parentheses. The relation $v \models_L \phi$ is defined to hold if $v(\phi) = 1$. Propositional logic is an effective sublogic of the (even two variable) predicate logic: We may treat p_n as a 0-place predicate symbol. A first order structure A gives rise to a valuation $\pi(A)$ which maps p_n to the truth value of P_n in A. Clearly, every valuation arises in this way from some structure.

Example (Modal logic). Let us fix a set P of propositional symbols $p_0, p_1, \ldots$. Let S be the set of all reflexive and transitive Kripke-structures, and F the set of all propositional modal sentences built from the symbols of P and the usual logical symbols of modal logic $\Box, \Diamond, \wedge, \vee, \neg, \rightarrow, \leftrightarrow$ and parentheses. The relation $\mathcal{K} \models_L \phi$ is defined as usual. Modal logic is an effective sublogic of predicate logic: A first order structure $A = (K, R, c, P_0, P_1, \ldots)$, where R is transitive and reflexive, gives rise to a Kripke-structure $\mathcal{K}$ in which (K, R) is the frame, c denotes the initial node and P_n indicates the nodes in which p_n is true. Clearly, every Kripke-structure arises in this way from some such structure A. Sentences are translated in the well-known way:

$$\begin{aligned}
g(p_n, x) &= P_n(x) \\
g(\neg\phi, x) &= \neg g(\phi, x) \\
g(\phi \wedge \psi, x) &= g(\phi, x) \wedge g(\psi, x) \\
g(\phi \vee \psi, x) &= g(\phi, x) \vee g(\psi, x) \\
g(\Box\phi, x) &= \forall y(R(x, y) \rightarrow g(\phi, y)) \\
g(\Diamond\phi, x) &= \exists y(R(x, y) \wedge g(\phi, y)) \\
f(\phi) &= g(\phi, c)
\end{aligned}$$

In view of Lemma 2.3 this sublogic relation gives immediately the Compactness Theorem, also for say, S4. In fact, basic modal logic is an effective sublogic of the two variable logic, and therefore by Lemma 2.6 decidable.

Example (Intuitionistic logic). We fix a set P of propositional symbols $p_0, p_1, \ldots$. Let S be the set of all transitive and reflexive Kripke-structures, and F the set of all propositional sentences built from the symbols of P and the usual logical symbols of intuitionistic logic $\supset, \wedge, \vee, \neg$ and parentheses. The relation $\mathcal{K} \models_L \phi$ is defined as usual. Intuitionistic logic is an effective sublogic of predicate logic, and satisfies therefore the Compactness Theorem.

3. Lindström Theorems

The most famous metatheorem about abstract logics is Lindström's Theorem characterizing first order logic among a large class of abstract logics [22]. This type of characterization results are generally called Lindström theorems even when the conditions may be quite different from the original result. The original Lindström's Theorem characterizes first order logic in a class abstract logics $L = (S, F, \models)$ satisfying a number of assumptions that we first review.

The most striking formulations of Lindström theorems assume negation. To discuss negation at any length we have to impose more structure onto the class S of structures of an abstract logic.

Definition 3.1. An *abstract logic with occurrence relation* is any quadruple $L = (S, F, \models, V)$, where $L = (S, F, \models)$ is an abstract logic and $V \subseteq S \times F$ is a relation (called *occurrence relation*) such that $\models \subseteq V$. An abstract logic $L = (S, F, \models, V)$ with occurrence relation is *classical* if S is a subclass of the class of all relational structures of various vocabularies, and L satisfies:

Isomorphism Axiom:	If $A \models \phi$ and $A \cong B \in S$ then $B \models \phi$.
Reduct Axiom:	If $V(B, \phi)$ and B is a reduct of $A \in S$, then $A \models \phi \iff B \models \phi$
Renaming Axiom:	Suppose every $A \in S$ is associated with $A' \in S$ obtained by renaming symbols in A. Then for every $\phi \in F$ there is $\phi' \in F$ such that for all $A \in S$, $V(A, \phi) \iff V(A', \phi')$ and $A \models \phi \iff A' \models \phi'$.

A *classical abstract logic with vocabulary function* is the special case of an abstract logic with occurrence relation where we have a *vocabulary function* τ mapping F into S and the occurrence relation is defined by $V(A, \phi) \iff A$ is a $\tau(\phi)$-structure. We denote such an abstract logic by $(S, F, \models, \tau)$.

Intuitively, $V(A, \phi)$ means that the non-logical symbols occurring in ϕ have an interpretation in A. In classical abstract logics this means the vocabulary of the structure A includes the vocabulary of ϕ, whence the concept of a vocabulary function.

We say that a classical $L = (S, F, \models, \tau)$ is a *classical sublogic* of another classical $L' = (S', F', \models', \tau')$, $L \leq_c L'$, if $L \leq L'$ via θ, π and f such that $\tau'(f(\phi)) = \tau(\phi)$ for all $\phi \in F$, and $\pi(A)$ and A have the same universe for all $A \in S'$. A classical abstract logic L satisfies the *Downward Löwenheim-Skolem Theorem* if whenever $\phi \in F$ has a model, ϕ has a model with a countable universe.

Lemma 3.2. *If $L \leq_c L'$, L' is closed under conjunction, and L' satisfies the Downward Löwenheim-Skolem Theorem, then so does L.*

Definition 3.3. An abstract logic $L = (S, F, \models, V)$ with occurrence relation is said to be *closed under negation* if for every $\phi \in F$ there is $\neg\phi \in F$ such that

$$\forall A(V(A, \phi) \iff V(A, \neg\phi) \text{ and } A \models \neg\phi \iff A \not\models \phi). \tag{3.1}$$

The Independence Friendly logic IF [16] is not closed under negation (one way to see this is Corollary 4.2) if we define $V(A, \phi)$ to mean that the non-logical symbols occurring in ϕ have an interpretation in A. However, if we use a different definition letting $V(A, \phi)$ mean that the semantic game of ϕ is determined on A, then IF *is* closed under negation. But then IF is not what we called *classical* above.

A classical abstract logic $(S, F, \models, \tau)$ with vocabulary function is *fully classical* if S is the *whole* class of all relational structures. Among fully classical abstract logics we assume the sublogic relation $L \leq L'$ satisfies always the natural assumptions that every model A satisfies $\pi(A) = A$ and $A \models \theta$. In such a case we say that the abstract logic L' *extends* L. Two fully classical abstract logics are *equivalent* if they are sublogics of each other. Now we are ready to state:

Theorem 3.4 (Lindström's Theorem [22]). *Suppose L is a fully classical abstract logic closed under conjunction and negation extending first order logic. Then L is equivalent to first order logic if and only if L satisfies the Compactness Theorem and the Downward Löwenheim-Skolem Theorem.*

This important result gives a purely model-theoretic syntax-free characterization of first order logic. It has lead to attempts to find similar characterizations for other logics, also for non-classical logics. Indeed, de Rijke [26] has obtained a characterization for basic modal logic in terms of a notion of "finite rank". Other characterizations can be found in Figure 3. Many of them are very close in spirit to Theorem 3.4.

Another result of Lindström [22] tells us that a recursive fully classical abstract logic satisfying the closure conditions of Theorem 3.4 (effectively), which satisfies the Downward Löwenheim-Skolem Theorem and is effectively axiomatizable, is an effective sublogic of first order logic. This result has an extra limitation on definability of the set F of formulas of the abstract logic, typically satisfied by, but not limited to, the extensions of first order logic by finitely many generalized quantifiers.

There are two traditions in abstract model theory. One based on back-and-forth systems, particularly suitable for infinitary logic and interpolation theorems. The work on bisimulation shows that it is a similarly suitable setup for modal logics. The other tradition is the method of identities associated with generalized quantifiers and compact logics [27, 32]. It seems difficult to combine these two traditions. This culminates in the open question whether there is an extension of first order logic satisfying both the Compactness Theorem and the *Interpolation Theorem*: If every model of ϕ is a model of ψ, then there is a sentence θ such that every model of ϕ is a model of θ, every model of θ is a model of ψ, and

Abstract logic L	Reference
Monadic logic	Tharp [37]
- with Q_1	Caicedo [7]
Predicate logic	Lindström [22]
- with a monotone generalized quantifier	Flum [10]
- on pseudofinite structures	Väänänen [38]
Infinitary logic	
- $L_{\kappa\omega}$	Barwise [2]
- L^1_κ	Shelah,Väänänen [34]
Modal logic	de Rijke [26]
Topological logic	Sgro [30]
Various topological and related logics	Ziegler [39]
Banach space logic	Iovino [18]

FIGURE 3. Some examples of Lindström theorems

θ contains (in the obvious sense) only non-logical symbols common to both ϕ and ψ. For logics closed under negation (see Definition 3.3) this is equivalent to the *Separation Theorem*: If ϕ and ψ have no models in common, then there is a sentence θ such that every model of ϕ is a model of θ, θ and ψ have no models in common, and θ contains only non-logical symbols common to both ϕ and ψ.

Where did the two traditions reach an impasse? With back-and-forth systems the problem arose that uncountable partially isomorphic structures need not be isomorphic. With identities and compact logics the problem is the existence of a fundamental function for the relevant identity, and that is a difficult partition theoretic question.

The difficulties of the study of extensions of first order logic raise (among others) the question, is there a logic of "many" that we could somehow understand, e.g. axiomatize. A recent result of Shelah [33] shows that it is consistent that $L(Q_1, Q_2)$ is non-compact. On the other hand, logics with cofinality quantifiers are axiomatizable and compact [32]. It seems that the cofinality quantifiers behave much better than the "many"-type quantifiers.

With the new infinitary languages of [34] one can express "*there is an uncountable sequence*" in a way which does not allow one to say "*there is an infinite sequence*". This proves to be crucial. Note that the generalized quantifier "*there exists uncountably many*" is axiomatizable but the quantifier "*there exists infinitely many*" is not. The new infinitary logics of [34] transform this phenomenon from generalized quantifiers to infinitary logic. Thereby also a new Lindström theorem arises.

Structures	Sentences
Relational structures	Existential sentences
	Existential universal, FI logic [24]
	Transfinite game formulas [17]
	Existential second order, IF logic [16]
Banach space structures	Positive bounded formulas [18]
Kripke structures	Intuitionistic logic

Figure 4. Examples of lack of negation.

4. Abstract logic without negation

There are many examples of abstract logics $L = (S, F, \models, V)$ with a natural occurrence relation which are not closed under negation, see Figure 4. Several concepts of abstract model theory have definitions which are equivalent if we have negation but otherwise different. It is not immediately obvious which of these definitions are the most natural ones when we do not have negation. For the Interpolation Theorem it seems that the Separation Theorem is the right formulation in the absence of negation. This question is further studied in [11]. A formulation of Lindström's Theorem for logics not closed under negation states (as pointed out in [10]):

Theorem 4.1 (Lindström's Theorem without negation). *Suppose L is a fully classical abstract logic extending first order logic and closed under conjunction and disjunction, which satisfies the Compactness Theorem and the Downward Löwenheim-Skolem Theorem. If $\phi \in F$ and $\psi \in F$ have no models in common, then there is a first order sentence θ such that every model of ϕ is a model of θ but θ has no models in common with ψ.*

Corollary 4.2 (Total lack of negation). *Suppose L is a fully classical abstract logic extending first order logic and closed under conjunction and disjunction, which satisfies the Compactness Theorem and the Downward Löwenheim-Skolem Theorem. Then only the first order sentences in F have a negation (in the sense of (*3.1*).*

The Independence Friendly logic IF is an example of a logic which satisfies the assumptions of the above corollary [16]. Consequently, only the first order sentences in IF have a negation. As an illustration of an abstract logic with partial negation we consider the following example:

Definition 4.3 ([11]). Let us consider predicate logic as built from atomic and negated atomic formulas by means of $\forall, \exists, \wedge, \vee$. Let $L(m, n)$ be the extension of this predicate logic obtained by adding the generalized quantifiers

$$Q_m x\phi(x) \iff \text{there are at least } \aleph_m \text{ elements } x \text{ satisfying } \phi(x)$$

and

$$\check{Q}_n x\phi(x) \iff \text{all but fewer than } \aleph_n \text{ elements } x \text{ satisfy } \phi(x).$$

In other words, $L(m,n)$ is built from atomic and negated atomic formulas by means of $Q_m, \check{Q}_n, \forall, \exists, \wedge, \vee$.

Theorem 4.4 ([11]). *The abstract logic $L(m,n)$ satisfies the Compactness Theorem if $m < n$, and also (by [31]) if $m \geq n$ and $\aleph_m^\omega = \aleph_m$. It satisfies total lack of negation if $n \neq m$, and it satisfies the Separation Theorem if and only if $n < m$.*

We can define a kind of partial negation $\sim\phi$ in $L(m,n)$ as follows:

$$
\begin{aligned}
\sim\phi &= \neg\phi \text{ if } \phi \text{ atomic} \\
\sim\phi &= \phi \text{ if } \phi \text{ negated atomic} \\
\sim(\phi \wedge \psi) &= \sim\phi \vee \sim\psi \\
\sim(\phi \vee \psi) &= \sim\phi \wedge \sim\psi \\
\sim\exists x\phi &= \forall x \sim\phi \\
\sim\forall x\phi &= \exists x \sim\phi \\
\sim Q_m x\phi &= \check{Q}_n x \sim\phi \\
\sim\check{Q}_n x\phi &= Q_m x \sim\phi
\end{aligned}
$$

What can we say about $Q_m x\phi \wedge \sim Q_m x\phi$ for first order ϕ? The meaning of this sentence is that ϕ is satisfied by at least $\aleph_m$ elements but still all but fewer than $\aleph_n$ elements satisfy $\neg\phi$. If $m < n$ this is perfectly possible. So in this case $\sim\psi$ acts as a weak negation which is not even in contradiction with ψ unless ψ is first order. In a sense, $L(m,n)$ does not satisfy the Law of Contradiction if $\sim$ is interpreted as its negation. If $m > n$ then $Q_m x\phi \wedge \sim Q_m x\phi$ cannot hold but now $Q_m x\phi \vee \sim Q_m x\phi$ may fail. Thus in this case $\sim\psi$ acts as a strong negation which does not cover the complement of ψ unless ψ is first order. In a sense, $L(m,n)$ does not satisfy the Law of Excluded Middle if $\sim$ is interpreted as its negation. Finally, if $m = n$, $\sim$ acts as a perfect negation for $L(m,n)$, i.e. $A \models \sim\phi \iff A \not\models \phi$.

Open Question. Is there a model theoretic characterization of any abstract logic $L = (S, F, V)$ with occurrence relation which is not closed under negation?

5. Logic frames

Having totally neglected the aspect of syntax we now bring it back with the concept of a logic frame. Without specifying the axioms and rules of proof, for example, the question of the *Completeness Theorem* has to be reduced to the question whether the set of valid sentences is recursively enumerable. From such knowledge one could in principle devise axioms and rules that yield a Completeness Theorem. However, it is often relevant to know whether particular axioms and rules constitute a complete set. This is all the more important in the case of logics which are originally defined via axioms and rules, such as constructive logic. The below concept of a

Structures	Sentences	axioms and rules by
Valuations	Propositional logic	Post [25]
Relational structures	Predicate logic	Gödel [12]
-monadic		Löwenheim
-ω-models		Orey [23]
	-with two variables	Scott [29]
	-infinitary	Karp [19]
	-with Q_1	Keisler [20]
	-higher order	Henkin [15]
-topological	-invariant	Ziegler [39]
-analytic	-existential bounded	Iovino [18]
-Borel	-Borel logic	Friedman[36]
Kripke structures	Intuitionistic logic	Kripke [21]
	Modal logic, S4	Kripke [21]
Many-valued structures	Many-valued logic	Belluce-Chang [4]
	Fuzzy logic	Hajek [13]
Games	Linear logic, additive	Blass [6]

FIGURE 5. Examples of complete logic frames

logic frame captures in abstract form the combination of syntax, semantics and proof theory of a logic. An *axiom* of an abstract logic L is simply a sentence of L, i.e. an element of F. An *inference rule* is any collection of functions defined in the set F.

Definition 5.1 ([35]). A *logic frame* is a quadruple $L = (S, F, \models, A)$ where $(S, F, \models)$ is an abstract logic and A is a class of axioms and inference rules of L.

We write $T \vdash \phi$ if ϕ is derivable from T in the usual sense using the axioms and rules of L. A logic frame $L = (S, F, \models, A)$ satisfies: the *Soundness Theorem* if $T \vdash \phi$ implies every model of T is a model of ϕ, the *Completeness Theorem* if $T \vdash \phi$ holds exactly if every model of T is a model of ϕ, and the *Recursive Compactness Theorem* if every L-theory which is recursive in the set of axioms and rules, every finite subset of which which has a model, itself has a model.

Literature of logic has numerous examples of logic frames satisfying the Completeness Theorem (see Figure 5). In the case of extensions of first order logic the question of completeness depends in many cases on set theory. One of the oldest open problems concerning extensions of first order logic is the question whether the extension $L_{\omega\omega}(Q_2)$ of first order logic by the quantifier "*there exist at least* $\aleph_2$ *many*" is effectively axiomatizable or satisfies the Compactness Theorem (restricted to countable vocabularies). The answer is "yes" if the Generalized Continuum Hypothesis is assumed [8] but remains otherwise open (see however [33]). The concept of logic frames helps us here.

Theorem 5.2. [35] *The fully classical logic frame* $L = (S, L_{\omega\omega}(Q_2), \models, A_K)$*, where* A_K *is the set of axioms and rules of Keisler* [20]*, satisfies the Completeness Theorem if and only if L satisfies the Compactness Theorem (for countable vocabularies).*

There is a more complicated axiomatization A (the details of A are omitted) of an arbitrary $L_{\omega\omega}(Q_{\alpha_1}, \dots, Q_{\alpha_n})$ with the same property as A_K above.

Theorem 5.3. [35] *The extension* $L(\vec{Q}) = L_{\omega\omega}(Q_{\alpha_1}, \dots, Q_{\alpha_n})$ *of first order logic has a canonically defined set A of axioms and rules such that the fully classical logic frame* $L = (S, L(\vec{Q}), \models, A)$ *satisfies: If L satisfies the Completeness Theorem, then it satisfies the Compactness Theorem (and vice versa).*

Why is this interesting? The point is that we cannot decide on the basis of ZFC whether L satisfies the Compactness Theorem or not, but we can decide on the basis of ZFC alone that all we need to care about is the Completeness Theorem. We can also prove a general result about logics of the form

$$L(\vec{Q}) = L_{\omega\omega}(Q_{\alpha_1}, Q_{\alpha_2}, \dots)$$

as long as no α_n is a limit of some of the other ordinals α_i.

Theorem 5.4. [35] *The extension* $L(\vec{Q}) = L_{\omega\omega}(Q_{\alpha_1}, Q_{\alpha_2}, \dots)$ *of first order logic has a canonically defined set A of axioms and rules such that the fully classical logic frame* $L = (S, L(\vec{Q}), \models, A)$ *satisfies: If L satisfies the Recursive Compactness Theorem, then it satisfies the Compactness Theorem.*

What is interesting in the above theorem is that we cannot decide on the basis of ZFC whether L satisfies the Compactness Theorem or not, but we can decide on the basis of ZFC alone that if there is a counter-example to compactness, it is recursive in the axioms.

The results about logic frames up to now have been about connections between completeness and compactness. But we can also ask, are there Lindström theorems for logic frames. In particular, no answer to the following question is known even in the case of first order logic:

Open Question. Is there a characterization of any of the known complete logic frames $L = (S, F, \models, A)$ in terms of natural conditions on $S, F,$ and A?

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