Operator Theory:
Advances and Applications, Vol. 174, 173–186

Uniform and Smooth Benzaid-Lutz Type Theorems and Applications to Jacobi Matrices

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Abstract. Uniform and smooth asymptotics for the solutions of a parametric system of difference equations are obtained. These results are the uniform and smooth generalizations of the Benzaid-Lutz theorem (a Levinson type theorem for discrete linear systems) and are used to develop a technique for proving absence of accumulation points in the pure point spectrum of Jacobi matrices. The technique is illustrated by proving discreteness of the spectrum for a class of unbounded Jacobi operators.

Mathematics Subject Classification (2000). 39A11, 39A12, 47B36, 47A10.

Keywords. Difference equations, Asymptotics, Jacobi matrices, Discrete Spectrum.

1. Introduction

The asymptotic behavior of solutions of discrete linear systems can be obtained by means of discrete Levinson type theorems [4, 6]. Here we are mainly concerned with asymptotically diagonal linear systems to which the Benzaid-Lutz theorem can be applied.

Consider the system

$$\vec{x}_{n+1} = (\Lambda_n + R_n)\vec{x}_n\,, \qquad n \geq n_0\,, \tag{1}$$

where $\vec{x}_n$ is a d-dimensional vector, $\Lambda_n + R_n$ is an invertible $d \times d$ matrix, and $\Lambda_n = \operatorname{diag}\{\nu_n^{(k)}\}_{k=1}^d$. The Benzaid-Lutz theorem [2, 4, 6] asserts that, when the sequence $\{\Lambda_n\}_{n\geq n_0}$ satisfies the Levinson condition for $k = 1, \dots, d$ (see below Def. 2.1) and

$$\sum_{n=n_0}^{\infty} \frac{\|R_n\|}{|\nu_n^{(k)}|} < \infty\,, \qquad k = 1, \dots, d\,,$$

Partially supported by project PAPIIT IN 101902, DGAPA-UNAM.

then, there is a basis $\{\vec{x}_n^{(k)}\}_{n\ge n_0}$ ($k=1,\dots,d$) in the space of solutions of (1) such that

$$\left\| \frac{\vec{x}_n^{(k)}}{\prod_{i=n_0}^{n-1}\nu_i^{(k)}} - \vec{e}_k \right\| \to 0\,, \quad \text{as } n\to\infty\,, \quad \text{for } k=1,\dots,d\,,$$

where $\{\vec{e}_k\}_{k=1}^d$ is the canonical basis in $\mathbb{C}^d$. This result has its counterpart for linear systems of ordinary differential equations [3]. Loosely speaking, if the conditions of the Benzaid-Lutz theorem hold, the solutions $\{\vec{x}_n^{(k)}\}_{n\ge n_0}$ of (1) asymptotically behave as the solutions of the unperturbed system

$$\vec{x}_{n+1} = \Lambda_n \vec{x}_n\,, \qquad n\ge n_0\,.$$

Let us now consider the second order difference equation for the sequence $\{x_n\}_{n=1}^\infty$,

$$b_{n-1}x_{n-1} + q_n x_n + b_n x_{n+1} = \lambda x_n\,, \quad \lambda\in\mathbb{R}\,, \quad n\ge 2\,, \tag{2}$$

where $\{q_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are real sequences and $b_n\neq 0$ for any $n\in\mathbb{N}$. This equation can be written as follows

$$\vec{x}_{n+1} = B_n(\lambda)\vec{x}_n\,, \quad \lambda\in\mathbb{R}\,, \quad n\ge 2\,. \tag{3}$$

where, $\vec{x}_n := \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$ and $B_n(\lambda) := \begin{pmatrix} 0 & 1 \\ -\frac{b_{n-1}}{b_n} & \frac{\lambda-q_n}{b_n} \end{pmatrix}$. In general, difference equations of order d can be reduced to similar systems with $d\times d$ matrices.

It is well known that the spectral analysis of Jacobi operators having the matrix representation

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \qquad \begin{array}{c} 0\neq b_n\in\mathbb{R}\,, \forall n\in\mathbb{N}\,, \\ q_n\in\mathbb{R}\,, \forall n\in\mathbb{N}\,, \end{array}$$

with respect to the canonical basis in $l_2(\mathbb{N})$, can be carried out on the basis of the asymptotic behavior of the solutions of (2), for example using Subordinacy Theory [5, 8]. In its turn, in certain cases (see Sec. 5), the asymptotics of solutions of (3) (and therefore of (2)) can be obtained by the Benzaid-Lutz theorem applied point-wise with respect to $\lambda\in\mathbb{R}$.

In this paper we obtain sufficient conditions for a parametric Benzaid-Lutz system of the form

$$\vec{x}_{n+1}(\lambda) = (\Lambda_n(\lambda) + R_n(\lambda))\vec{x}_n(\lambda)\,, \qquad n\ge n_0\,,$$

to have solutions with certain smooth behavior with respect to λ (see Sec. 4). This result, together with a uniform (also with respect to λ) estimate of the asymptotic remainder of solutions of (2) obtained in [12], is used to develop a technique for excluding accumulation points in the pure point spectrum of difference operators. The technique is illustrated in a simple example.

2. Preliminaries

Throughout this work, unless otherwise stated, $\mathfrak{I}$ denotes some real interval. Besides, we shall refer in multiple occasions to the sequence of matrices Λ defined as follows

$$\Lambda := \{\Lambda_n(\lambda)\}_{n=n_0}^{\infty}\,, \quad \text{where } \Lambda_n := \operatorname{diag}\{\nu_n^{(k)}(\lambda)\}_{k=1}^{d}\,, \quad \lambda \in \mathfrak{I}\,. \tag{4}$$

Definition 2.1. The sequence Λ, given by (4), is said to satisfy the Levinson condition for k (denoted $\Lambda \in \mathcal{L}(k)$) if there exist an $N \geq n_0$ such that $\nu_n^{(k)}(\lambda) \neq 0$, for any $n \geq N$ and $\lambda \in \mathfrak{I}$, and if for some constant number $M > 1$, with k being fixed, each j $(1 \leq j \leq d)$ falls into one and only one of the two classes I_1 or I_2, where

(a) $j \in I_1$ if $\forall \lambda \in \mathfrak{I}$

$$\frac{\left|\prod_{i=N}^{n} \nu_i^{(k)}(\lambda)\right|}{\left|\prod_{i=N}^{n} \nu_i^{(j)}(\lambda)\right|} \to \infty \quad \text{as } n \to \infty\,, \quad \text{and}$$

$$\frac{\left|\prod_{i=n}^{n'} \nu_i^{(k)}(\lambda)\right|}{\left|\prod_{i=n}^{n'} \nu_i^{(j)}(\lambda)\right|} > \frac{1}{M}\,, \quad \forall\, n'\,, n \text{ such that } n' \geq n \geq N\,.$$

(b) $j \in I_2$ if $\forall \lambda \in \mathfrak{I}$

$$\frac{\left|\prod_{i=n}^{n'} \nu_i^{(k)}(\lambda)\right|}{\left|\prod_{i=n}^{n'} \nu_i^{(j)}(\lambda)\right|} < M\,, \quad \forall\, n'\,, n \text{ such that } n' \geq n \geq N\,.$$

Definition 2.2. Fix the natural numbers k $(k \leq d)$ and n_1, and assume that $\nu_n^{(k)}(\lambda) \neq 0$, $\forall n \geq n_1$ and $\forall \lambda \in \mathfrak{I}$. Let $X_k(n_1)$ denote the normed space containing all sequences $\vec{\varphi} = \{\vec{\varphi}_n(\lambda)\}_{n=n_1+1}^{\infty}$ of functions defined on $\mathfrak{I}$ and with range in $\mathbb{C}^d$, such that

$$\sup_{n>n_1} \sup_{\lambda \in \mathfrak{I}} \left\{ \|\vec{\varphi}_n(\lambda)\|_{\mathbb{C}^d} \frac{1}{\left|\prod_{i=n_1}^{n-1} \nu_i^{(k)}(\lambda)\right|} \right\} < \infty$$

and where the norm is defined by

$$\|\vec{\varphi}\|_{X_k(n_1)} = \sup_{n>n_1} \sup_{\lambda \in \mathfrak{I}} \left\{ \|\vec{\varphi}_n(\lambda)\|_{\mathbb{C}^d} \frac{1}{\left|\prod_{i=n_1}^{n-1} \nu_i^{(k)}(\lambda)\right|} \right\}. \tag{5}$$

Clearly, $X_k(n_1)$ is complete. It will be also considered the subspace $X_k^0(n_1)$ which contains all functions of $X_k(n_1)$ such that

$$\sup_{\lambda \in \mathfrak{I}} \left\{ \|\vec{\varphi}_n(\lambda)\|_{\mathbb{C}^d} \frac{1}{\left|\prod_{i=n_1}^{n-1} \nu_i^{(k)}(\lambda)\right|} \right\} \to 0 \text{ as } n \to \infty\,. \tag{6}$$

In $\mathbb{C}^d$ consider the canonical orthonormal basis $\{\vec{e}_k\}_{k=1}^d$. The $d \times d$ diagonal matrix $\operatorname{diag}\{\delta_{kl}\}_{l=1}^d$, where δ_{kl} $(k, l = 1, \dots, d)$ is the Kronecker symbol, is a projector to the one dimensional space generated by $\vec{e}_k$.

Definition 2.3. Assuming $\Lambda \in \mathcal{L}(k)$ (for some $k = 1, \dots, d$), let $\mathcal{P}_i(\Lambda, k) = \mathcal{P}_i$ be defined by

$$\mathcal{P}_i = \sum_{j \in I_i} \operatorname{diag}\{\delta_{jl}\}_{l=1}^d \qquad i = 1, 2, \tag{7}$$

where I_1 and I_2 are the classes of Definition 2.1.

3. Uniform asymptotics of solutions

The following result has been proven in [12].

Lemma 3.1. *Let the sequence Λ be defined as in* (4). *For any $n \in \mathbb{N}$, $\lambda \in \mathfrak{I}$, let $R_n(\lambda)$ be a $d \times d$ complex matrix. Fix the natural number $k \le d$ and assume that the following conditions hold:*

(i) $\Lambda \in \mathcal{L}(k)$.

(ii) $\sup_{\lambda \in \mathfrak{I}} \sum_{n=N}^{\infty} \frac{\|R_n(\lambda)\|}{|\nu_n^{(k)}(\lambda)|} < \infty$ *(N is given by the previous condition, see Def.* 2.1).

(iii) *for any $\epsilon > 0$ there exists an N_ϵ (which depends only on ϵ) such that $\forall \lambda \in \mathfrak{I}$ we have*

$$\sum_{n=N_\epsilon}^{\infty} \frac{\|R_n(\lambda)\|}{|\nu_n^{(k)}(\lambda)|} < \epsilon .$$

Then, for some $N_0 \ge N$ and any bounded continuous function, denoted by $\varphi_{N_0}(\lambda)$ $(\lambda \in \mathfrak{I})$, the operator T_k defined on any $\vec{\varphi} = \{\vec{\varphi}_n(\lambda)\}_{n=N_0+1}^{\infty}$ in $X_k(N_0)$ by

$$\begin{aligned}(T_k\vec{\varphi})_n(\lambda) = \mathcal{P}_1 \prod_{i=N_0}^{n-1} \Lambda_i(\lambda) \sum_{m=N_0}^{n-1} \left(\prod_{i=N_0}^{m} \Lambda_i(\lambda) \right)^{-1} R_m(\lambda)\vec{\varphi}_m(\lambda) \\ - \mathcal{P}_2 \prod_{i=N_0}^{n-1} \Lambda_i(\lambda) \sum_{m=n}^{\infty} \left(\prod_{i=N_0}^{m} \Lambda_i(\lambda) \right)^{-1} R_m(\lambda)\vec{\varphi}_m(\lambda), \quad n > N_0,\end{aligned} \tag{8}$$

has the following properties

1. $\|T_k\| < 1$
2. $T_k X_k(N_0) \subset X_k^0(N_0)$

Assuming that Λ, defined by (4), and $\{R_n(\lambda)\}_{n=n_0}^{\infty}$ satisfy the conditions of Lemma 3.1, let the sequence $\vec{\varphi} = \{\vec{\varphi}_n^{(k)}(\lambda)\}_{n=N_0}^{\infty}$ in $X_k(N_0)$ be a solution of

$$\vec{\varphi} = \vec{\psi}^{(k)} + T_k\vec{\varphi}, \tag{9}$$

where $\vec{\psi}^{(k)} = \{\vec{\psi}_n^{(k)}(\lambda)\}_{n=N_0+1}^{\infty}$ is defined by

$$\vec{\psi}_n^{(k)} = \prod_{i=N_0}^{n-1} \Lambda_i(\lambda)\vec{e}_k = \prod_{i=N_0}^{n-1} \nu_i^{(k)}(\lambda)\vec{e}_k \,, \qquad n > N_0 \,.$$

It is straightforwardly verifiable that one obtains an identity if substitutes (9) into

$$\vec{\varphi}_{n+1}(\lambda) = (\Lambda_n(\lambda) + R_n(\lambda))\vec{\varphi}_n(\lambda) \,, \qquad n > N_0 \,, \tag{10}$$

and take into account (8). Thus, $\vec{\varphi} \in X_k(N_0)$, defined as a solution of (9), is a solution of (10) for each $k \le d$. Notice that T_k's property 2, stated in Lemma 3.1, implies

$$\vec{\varphi} - \vec{\psi}^{(k)} \in X_k^0(N_0) \,. \tag{11}$$

The following assertion is the uniform version of the Benzaid-Lutz theorem [2, 4].

Theorem 3.2. *Let the sequences* Λ*, given by* (4)*, and* $\{R_n(\lambda)\}_{n=n_0}^{\infty}$ *satisfy the conditions of Lemma* 3.1 *for all* $k = 1, \ldots, d$*. Then one can find an* $N_0 \in \mathbb{N}$ *such that there exists a basis* $\{\vec{\varphi}^{(k)}(\lambda)\}_{k=1}^{d}$*,* $\vec{\varphi}^{(k)} = \{\vec{\varphi}_n^{(k)}(\lambda)\}_{n=N_0+1}^{\infty}$*, in the space of solutions of* (10) *satisfying*

$$\sup_{\lambda \in \mathfrak{I}} \left\| \frac{\vec{\varphi}_n^{(k)}(\lambda)}{\prod_{i=N_0}^{n-1} \nu_i^{(k)}(\lambda)} - \vec{e}_k \right\| \to 0, \qquad \text{as } n \to \infty \,, \quad \text{for } k = 1, \ldots, d \,. \tag{12}$$

Proof. We have d solutions of (10) given by (9) for $k = 1, \ldots, d$. Equation (12) follows directly from (11). That $\{\vec{\varphi}^{(k)}(\lambda)\}_{k=1}^{d}$ is a basis is a consequence of (12). Indeed, let $\Phi(n, \lambda)$ be the $d \times d$ matrix whose columns are given by the d vectors $\vec{\varphi}_n^{(k)}(\lambda)$ $(k = 1, \ldots, d)$; then (12) implies that, for sufficiently big n,

$$\forall \lambda \in \mathfrak{I} \,, \quad \det \Phi(n, \lambda) \ne 0 \,. \qquad \square$$

It is worth remarking that the uniform Levinson theorem (in the continuous case, i.e., for a system of ordinary differential equations) has already been proven in [10], where this result is used in the spectral analysis of a self-adjoint fourth-order differential operator.

4. Smoothness of solutions

Here we show that if the matrices $R_n(\lambda)$ and $\Lambda_n(\lambda)$ enjoy certain smooth properties with respect to λ, then the solutions of (10) obtained through Theorem 3.2 are also smooth.

Lemma 4.1. *Let the sequences* $\{R_n(\lambda)\}_{n=n_0}^{\infty}$ *and* Λ*, defined in* (4)*, satisfy the conditions of Lemma* 3.1*, and suppose that the entries of* $R_n(\lambda)$ *and* $\Lambda_n(\lambda)$*, seen as functions of* λ*, are continuous on* $\mathfrak{I}$ *for every* $n \ge N_0$*, where* N_0 *is given by Lemma* 3.1*. Then the solution* $\vec{\varphi} = \{\vec{\varphi}_n(\lambda)\}_{n=N_0+1}^{\infty}$ *of* (9) *is such that* $\vec{\varphi}_n(\lambda)$*, as a function of* λ*, is continuous on* $\mathfrak{I}$ *for each* $n > N_0$*.*

Proof. From the definition of T_k it follows that if the sequence $\vec{\varphi} = \{\vec{\varphi}_n(\lambda)\}_{n=N_0+1}^{\infty}$ is such that $\vec{\varphi}_n(\lambda)$ is a continuous function on $\mathfrak{I}$, $\forall n > N_0$, then $(T_k\vec{\varphi})_n(\lambda)$ is continuous on $\mathfrak{I}$, $\forall n > N_0$. Indeed, from (8) one has that $(T_k\vec{\varphi})_n(\lambda)$ is a uniform convergent series of continuous functions. The assertion of the lemma then follows from the fact that the unique solution of (9) can be found by the method of successive approximations. □

Lemma 4.2. *Suppose that the sequences* $\{R_n(\lambda)\}_{n=n_0}^{\infty}$ *and* $\{\Lambda_n(\lambda)\}_{n=n_0}^{\infty}$ *satisfy the conditions of Theorem* 3.2 *and* $\mathfrak{I}$ *is a closed interval. Let* $R_n(\lambda)$ *and* $\Lambda_n(\lambda)$ *be matrices whose entries are continuous functions of* λ *on* $\mathfrak{I}$ *for every* $n \geq n_0$ *and such that*

$$\det(\Lambda_n(\lambda) + R_n(\lambda)) \neq 0\,, \quad \lambda \in \mathfrak{I}\,, \quad n_0 \leq n \leq N_0\,. \tag{13}$$

Then, the solutions $\{\vec{\varphi}^{(k)}(\lambda)\}_{k=1}^{d}$, $\vec{\varphi}^{(k)} = \{\vec{\varphi}_n^{(k)}(\lambda)\}_{n=N_0+1}^{\infty}$, *of* (10) *given by Theorem* 3.2 *can be extended to solutions* $\vec{\varphi}^{(k)} = \{\vec{\varphi}_n^{(k)}(\lambda)\}_{n=n_0}^{\infty}$ *of the system*

$$\vec{\varphi}_{n+1}(\lambda) = (\Lambda_n(\lambda) + R_n(\lambda))\vec{\varphi}_n(\lambda) \qquad n \geq n_0\,,$$

having the property that, given $n \geq n_0$ *fixed, for any* $\epsilon > 0$ *there exists* δ *such that*

$$\forall \lambda_1, \lambda_2 \in \mathfrak{I}\,, \quad |\lambda_1 - \lambda_2| < \delta \;\Rightarrow\; \left\|\vec{\varphi}_n^{(k)}(\lambda_1) - \vec{\varphi}_n^{(k)}(\lambda_2)\right\| < \epsilon\,, \quad k = 1, \ldots, d\,. \tag{14}$$

Proof. The proof is again straightforward. By Theorem 3.2 there exists an $N_0 \in \mathbb{N}$ such that the basis $\{\vec{\varphi}^{(k)}(\lambda)\}_{k=1}^{d}$ in the space of solutions of (10) satisfies (12). $\vec{\varphi}_n^{(k)}(\lambda)$ is continuous on $\mathfrak{I}$ for all $n > N_0$ as a consequence of Lemma 4.1. Since $\mathfrak{I}$ is closed, each $\vec{\varphi}_n^{(k)}(\lambda)$ is actually uniform continuous. Therefore, we have (14) for $n > N_0$. Now, for $n_0 \leq p \leq N_0$, one has

$$\vec{\varphi}_p^{(k)}(\lambda) = Q(\lambda, p, N_0)\vec{\varphi}_{N_0+1}^{(k)}(\lambda)\,,$$

where

$$Q(\lambda, p, N_0) := (\Lambda_p(\lambda) + R_p(\lambda))^{-1} \ldots (\Lambda_{N_0}(\lambda) + R_{N_0}(\lambda))^{-1}.$$

Condition (13) implies that $Q(\lambda, p, N_0)$ is always well defined, and the smooth properties of $R_n(\lambda)$ and $\Lambda_n(\lambda)$ imply that the entries of $Q(\lambda, p, N_0)$ are uniform continuous on $\mathfrak{I}$ for all p. Thus, from

$$\begin{aligned}\left\|\vec{\varphi}_p^{(k)}(\lambda_1) - \vec{\varphi}_p^{(k)}(\lambda_2)\right\| &\leq \left\|(Q(\lambda_1, p, N_0) - Q(\lambda_2, p, N_0))\vec{\varphi}_{N_0}^{(k)}(\lambda_2)\right\| \\ &\quad + \left\|Q(\lambda_1, p, N_0)(\vec{\varphi}_{N_0}^{(k)}(\lambda_1) - \vec{\varphi}_{N_0}^{(k)}(\lambda_2))\right\|\,.\end{aligned}$$

it follows that

$$\left\|\vec{\varphi}_p^{(k)}(\lambda_1) - \vec{\varphi}_p^{(k)}(\lambda_2)\right\| \to 0 \quad \text{as } \lambda_1 \to \lambda_2$$

□

5. An application to a class of Jacobi matrices

In the Hilbert space $l_2(\mathbb{N})$, let J be the operator whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is a semi-infinite Jacobi matrix of the form

$$\begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} . \tag{15}$$

The elements of the sequence $\{b_n\}_{n=1}^{\infty}$ are defined as follows

$$b_n := n^{\alpha}\Big(1 + \frac{c_n}{n}\Big) , \quad \forall n \in \mathbb{N} , \tag{16}$$

where $\alpha > 1$ and $c_n = c_{n+2L}$ ($L \in \mathbb{N}$). We assume that $1+\frac{c_n}{n} \neq 0$ for all n. Clearly, the Jacobi operator J is symmetric and unbounded. J is closed by definition since the unbounded symmetric operator J is said to have the matrix representation (15) with respect to the canonical basis in $l_2(\mathbb{N})$ if it is the minimal closed operator satisfying

$$(Je_k, e_{k+1}) = (Je_{k+1}, e_k) = b_k , \quad \forall k \in \mathbb{N} ,$$

where $\{e_k\}_{k=1}^{\infty}$ is the canonical basis in $l_2(\mathbb{N})$ (see [1]). The class of Jacobi matrices given by (15) and (16) is said to have rapidly growing weights. This class is based on an example suggested by A.G. Kostyuchenko and K.A. Mirzoev in [9].

On the basis of subordinacy theory [5, 8], the spectral properties of J have been studied in [6, 11, 12]. The theory of subordinacy reduces the spectral analysis of operators to the asymptotic analysis of the corresponding generalized eigenvectors. This approach has proved to be very useful in the spectral analysis of Jacobi operators. In [6] it is proven that if

$$\left| \sum_{k=1}^{2L} (-1)^k c_k \right| \geq L(\alpha - 1) , \tag{17}$$

then $J = J^*$ and it has pure point spectrum. However, within the framework of subordinacy theory, one cannot determine if the pure point spectrum has accumulation points in some finite interval.

Equation (2) for J takes the form

$$b_{n-1}u_{n-1} + b_n u_{n+1} = \lambda u_n , \quad n > 1 , \ \lambda \in \mathbb{R} , \tag{18}$$

with $\{b_n\}_{n=1}^{\infty}$ given by (16). As was mentioned before, the asymptotic behavior of the solutions of (18) gives information on the spectral properties of J. If a solution $u(\lambda) = \{u_n(\lambda)\}_{n=1}^{\infty}$ of (18) satisfies the "boundary condition"

$$b_1 u_2 = \lambda u_1 \tag{19}$$

and turns out to be in $l_2(\mathbb{N})$, then $u(\lambda)$ is an eigenvector of J^* corresponding to the eigenvalue λ.

Using the results of Sections 3 and 4, we shall develop a technique to prove that J with weights given by (16) and (17) has discrete spectrum.

It is worth remarking that there are simpler methods for proving that the spectrum of J is purely discrete. Indeed, one can use for instance the asymptotic behavior of the solutions of (18) to show that the resolvent of J is compact. This has been done for a class of Jacobi operators in [7] and the technique developed there can in fact be used to obtain estimates for the eigenvalues.

The method we develop below may, nevertheless, be advantageous in some cases since it uses and preserves more information inherent in system (18). For simplicity, operator J has been chosen to illustrate the technique, but one can easily adapt the reasoning for other Jacobi operators. Our technique seems to be especially useful for operators having simultaneously intervals of pure point and absolutely continuous spectrum [6, Th. 2.2].

We begin by deriving from (18) a system suitable for applying our previous results, but first we introduce the following notation. Given a sequence of matrices $\{M_s(\lambda)\}_{s=1}^{\infty}$ ($\lambda \in \mathfrak{I}$) and a sequence $\{f_s\}_{s=1}^{\infty}$ of real numbers, we shall say that

$$M_s(\lambda) = \widehat{O}_{\mathfrak{I}}(f_s) \quad \text{as } s \to \infty .$$

if there exists a constant $C > 0$ and $S \in \mathbb{N}$ such that

$$\sup_{\lambda \in \mathfrak{I}} \|M_s(\lambda)\| < C\,|f_s| \,, \quad \forall s > S \,.$$

Now suppose that $\mathfrak{I}$ is a finite interval and rewrite (18), with $\lambda \in \mathfrak{I}$, in the form of (3). We have

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{b_{n-1}}{b_n} & \frac{\lambda}{b_n} \end{pmatrix}, \quad n \geq 2\,, \quad \lambda \in \mathfrak{I}\,.$$

Define the sequence of matrices $\{A_m(\lambda)\}_{m=1}^{\infty}$ as follows

$$A_m(\lambda) := \prod_{s=1+L(m-1)}^{Lm} B_{2s+1}(\lambda) B_{2s}(\lambda)\,, \quad m \in \mathbb{N}\,. \tag{20}$$

Whenever we have products of non-diagonal matrices, as in (20), we take them in “chronological” order, that is,

$$A_m(\lambda) := B_{2Lm+1}(\lambda) B_{2Lm} \dots B_{2L(m-1)+3}(\lambda) B_{2L(m-1)+2}\,.$$

A straightforward computation shows that

$$B_{2s+1}(\lambda)B_{2s}(\lambda) = -I + \begin{pmatrix} \frac{c_{2s}-c_{2s-1}+\alpha}{2s} & 0 \\ 0 & \frac{c_{2s+1}-c_{2s}+\alpha}{2s} \end{pmatrix} + \widehat{O}_{\mathfrak{I}}(s^{-1-\epsilon}), \quad \epsilon > 0\,.$$

Indeed, one can easily verify that

$$B_{2s+1}(\lambda)B_{2s}(\lambda) + I - \begin{pmatrix} \frac{c_{2s}-c_{2s-1}+\alpha}{2s} & 0 \\ 0 & \frac{c_{2s+1}-c_{2s}+\alpha}{2s} \end{pmatrix} = \begin{pmatrix} r_1(s) & r_2(s) \\ r_3(s)\lambda & r_4(s) + r_5(s)\lambda^2 \end{pmatrix}$$

where $r_l(s) = O(s^{-1-\epsilon})$ for $l = 1, \ldots, 5$. Clearly, up to the same asymptotic estimate, we may also write ($\epsilon > 0$)

$$B_{2s+1}(\lambda)B_{2s}(\lambda) = \begin{pmatrix} -e^{\frac{c_{2s-1}-c_{2s}-\alpha}{2s}} & 0 \\ 0 & -e^{\frac{c_{2s}-c_{2s+1}-\alpha}{2s}} \end{pmatrix} \left[I + \widehat{O}_{\mathfrak{I}}(s^{-1-\epsilon})\right].$$

Therefore,

$$A_m(\lambda) = \prod_{s=1+L(m-1)}^{Lm} \begin{pmatrix} -e^{\frac{c_{2s-1}-c_{2s}-\alpha}{2s}} & 0 \\ 0 & -e^{\frac{c_{2s}-c_{2s+1}-\alpha}{2s}} \end{pmatrix} \prod_{s=1+L(m-1)}^{Lm} \left[I + \widehat{O}_{\mathfrak{I}}(s^{-1-\epsilon})\right]$$

$$= (-1)^L \begin{pmatrix} \exp \sum_{s=1+L(m-1)}^{Lm} \frac{c_{2s-1}-c_{2s}-\alpha}{2s} & 0 \\ 0 & \exp \sum_{s=1+L(m-1)}^{Lm} \frac{c_{2s}-c_{2s+1}-\alpha}{2s} \end{pmatrix} \left[I + \widehat{O}_{\mathfrak{I}}(m^{-1-\epsilon})\right]$$

Let us define, for $m \in \mathbb{N}$, $\lambda \in \mathfrak{I}$, the matrices

$$\Lambda_m := \operatorname{diag}\{\nu_m^{(1)}, \nu_m^{(2)}\},$$

where

$$\begin{aligned} \nu_m^{(1)} &:= (-1)^L \exp \sum_{s=1+L(m-1)}^{Lm} \frac{c_{2s-1}-c_{2s}-\alpha}{2s} \\ \nu_m^{(2)} &:= (-1)^L \exp \sum_{s=1+L(m-1)}^{Lm} \frac{c_{2s}-c_{2s+1}-\alpha}{2s}, \end{aligned} \tag{21}$$

and

$$R_m(\lambda) := A_m(\lambda) - \Lambda_m$$

Observe that Λ_m does not depend on λ, and $R_m(\lambda) = \widehat{O}_{\mathfrak{I}}(m^{-1-\epsilon})$ as $m \to \infty$.

Lemma 5.1. *Let $\mathfrak{I}$ be a finite closed interval. There is a basis $\vec{x}^{(k)}(\lambda) = \{\vec{x}_n^{(k)}(\lambda)\}_{n=1}^{\infty}$ $(k = 1, 2)$ in the space of solutions of the system*

$$\vec{x}_{n+1}(\lambda) = A_n(\lambda)\vec{x}_n(\lambda), \qquad n \in \mathbb{N}, \quad \lambda \in \mathfrak{I}, \tag{22}$$

with $A_n(\lambda)$ given by (20), such that

$$\sup_{\lambda \in \mathfrak{I}} \left\| \frac{\vec{x}_n^{(k)}(\lambda)}{\prod_{i=1}^{n-1} \nu_i^{(k)}} - \vec{e}_k \right\| \to 0, \quad as\ n \to \infty, \quad for\ k = 1, 2,$$

where $\nu_i^{(k)}$ is defined in (21). Moreover, for any fixed $n \in \mathbb{N}$

$$\sup_{\substack{|\lambda'-\lambda|<\delta \\ \lambda', \lambda \in \mathfrak{I}}} \left\| \vec{x}_n^{(k)}(\lambda') - \vec{x}_n^{(k)}(\lambda) \right\| \to 0, \qquad as\ \delta \to 0, \quad k = 1, 2.$$

Proof. Write $A_n(\lambda) = \Lambda_n + R_n(\lambda)$ as was done before. We first show that the sequences $\{\Lambda_n\}_{n=1}^{\infty}$ and $\{R_n(\lambda)\}_{n=1}^{\infty}$ satisfy the conditions of Theorem 3.2. Let us prove that $\{\Lambda_n\}_{n=1}^{\infty} \in \mathcal{L}(k)$ for $k = 1, 2$. Define

$$\gamma := \frac{1}{2L} \sum_{s=1}^{L} c_{2s-1} - 2c_{2s} + c_{2s+1} \,.$$

It is not difficult to verify that for every $n \geq 2$ there is a constant K such that

$$\begin{aligned}\prod_{i=1}^{n} \frac{|\nu_i^{(1)}|}{|\nu_i^{(2)}|} &= \exp \sum_{i=1}^{n} \sum_{s=1+L(i-1)}^{Li} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} \\ &< K \exp\left\{ \gamma \sum_{s=1}^{n} \frac{1}{s} \right\}.\end{aligned}$$

Analogously for some constant $\widetilde{K}$

$$\begin{aligned}\prod_{i=1}^{n} \frac{|\nu_i^{(1)}|}{|\nu_i^{(2)}|} &= \exp \sum_{i=1}^{n} \sum_{s=1+L(i-1)}^{Li} \frac{c_{2s-1} - 2c_{2s} + c_{2s+1}}{2s} \\ &> \widetilde{K} \exp\left\{ \gamma \sum_{s=1}^{n} \frac{1}{s} \right\}.\end{aligned}$$

Clearly, one obtains similar estimates interchanging $k = 1, 2$. Thus i holds. Conditions ii and iii follow from the fact that $\nu_n^{(k)} \to 1$ as $n \to \infty$ and $R_n = \widehat{O}_{\mathfrak{I}}(n^{-1-\epsilon})$.

Now observe that (13) holds for the system (22), and for $n \in \mathbb{N}$ the entries of $R_n(\lambda)$ and Λ_n are continuous functions of $\lambda \in \mathfrak{I}$. Therefore, the conditions of Lemma 4.2 are satisfied. □

Lemma 5.2. *Let $\mathfrak{I}$ be any closed finite interval of $\mathbb{R}$. Then, there exists a solution $u(\lambda) = \{u_n(\lambda)\}_{n=1}^{\infty}$ of* (18)*, with $\{b_n\}_{n=1}^{\infty}$ given by* (16) *and satisfying* (17)*, such that*

$$\sum_{n=1}^{\infty} \sup_{\lambda \in \mathfrak{I}} |u_n(\lambda)|^2 < \infty \,.$$

Moreover, for any fixed $n \in \mathbb{N}$,

$$\sup_{\substack{|\lambda' - \lambda| < \delta \\ \lambda', \lambda \in \mathfrak{I}}} |u_n(\lambda') - u_n(\lambda)| \to 0 \,, \qquad \text{as } \delta \to 0 \,.$$

Proof. By (20) and (22), it is clear that

$$\vec{x}_{n+1}^{(k)}(\lambda) = \begin{pmatrix} u_{2Ln+1}^{(k)}(\lambda) \\ u_{2Ln+2}^{(k)}(\lambda) \end{pmatrix} . \tag{23}$$

Thus, Lemma 5.1 yields that for $n \in \mathbb{N}$ and some constants $C, C' > 0$

$$\begin{aligned}\sup_{\lambda\in\Im}\left|u^{(1)}_{2Ln+2}(\lambda)\right| &\le \sup_{\lambda\in\Im}\left\|\vec{x}^{(1)}_{n+1}(\lambda)\right\| \\ &\le C\left|\prod_{i=1}^{n}\nu_i^{(1)}\right| \\ &= C\exp\sum_{i=1}^{n}\sum_{s=1+L(i-1)}^{Li}\frac{c_{2s-1}-c_{2s}-\alpha}{2s} \\ &\le C'\exp\left\{\sum_{s=1}^{L}\frac{c_{2s-1}-c_{2s}-\alpha}{2L}\sum_{s=1}^{n}\frac{1}{s}\right\},\end{aligned}$$

where we have use the periodicity of the sequence $\{c_k\}_{k=1}^{\infty}$. Thus for some constant C'' we have

$$\sup_{\lambda\in\Im}\left|u^{(1)}_{2Ln+2}(\lambda)\right| \le C'' n^{\beta}, \quad \beta := \frac{1}{2L}\sum_{s=1}^{2L}(-1)^{s+1}c_s - \frac{\alpha}{2}.$$

Analogously, there is a $\widetilde{C} > 0$ such that

$$\sup_{\lambda\in\Im}\left|u^{(2)}_{2Ln+2}(\lambda)\right| \le \widetilde{C} n^{\tilde\beta}, \quad \tilde\beta := \frac{1}{2L}\sum_{s=1}^{2L}(-1)^{s}c_s - \frac{\alpha}{2}.$$

Since $\alpha > 1$, (17) implies that either for $k = 1$ or $k = 2$

$$\sum_{n=1}^{\infty}\sup_{\lambda\in\Im}\left|u^{(k)}_{2Ln+2}(\lambda)\right|^2 < \infty \tag{24}$$

The first assertion of the lemma follows from (24) and the fact that there is a constant C such that

$$\sup_{\lambda\in\Im}\Big\|\prod_{j=2}^{s}B_{2Ln+j}(\lambda)\Big\| < C \qquad s = 2,3,\dots,2L, \qquad n\in\mathbb{N}.$$

Now, Lemma 5.1 and (23) yield, for $n \in \mathbb{N}$ and $k = 1, 2$,

$$\sup_{\substack{|\lambda'-\lambda|<\delta \\ \lambda',\,\lambda\in\Im}}\left|u^{(k)}_{2Ln+2}(\lambda') - u^{(k)}_{2Ln+2}(\lambda)\right| \to 0, \quad \text{as } \delta\to 0. \tag{25}$$

Since for any $s = 2, 3, \dots, 2L$

$$\begin{pmatrix} u^{(k)}_{2Ln+s}(\lambda) \\ u^{(k)}_{2Ln+s+1}(\lambda)\end{pmatrix} = \prod_{j=2}^{s}B_{2Ln+j}(\lambda)\begin{pmatrix} u^{(k)}_{2Ln+1}(\lambda) \\ u^{(k)}_{2Ln+2}(\lambda)\end{pmatrix},$$

the following inequality holds for $n \in \mathbb{N}$ and $s = 2, 3, \ldots, 2L$

$$\left|u^{(k)}_{2Ln+s+1}(\lambda') - u^{(k)}_{2Ln+s+1}(\lambda)\right| \leq \left\| \left(\prod_{j=2}^{s} B_{2Ln+j}(\lambda') - \prod_{j=2}^{s} B_{2Ln+j}(\lambda)\right) \vec{x}^{(k)}_{n+1}(\lambda)\right\| + \left\| \prod_{j=2}^{s} B_{2Ln+j}(\lambda') \left(\vec{x}^{(k)}_{n+1}(\lambda') - \vec{x}^{(k)}_{n+1}(\lambda)\right)\right\| .$$

Taking into account the smooth properties of the finite product $\prod_{j=2}^{s} B_{2Ln+j}(\lambda)$ and Lemma 5.1, one obtains from the last inequality and (25) the second assertion of the lemma for any $n \geq 2L + 2$. To complete the proof use the invertibility and smoothness of the matrices $B_n(\lambda)$ for $n < 2L + 2$. □

Remark 1. Let $u(\lambda)$ ($\lambda \in \mathfrak{I}$) be the solution mentioned in the previous lemma. If $J = J^*$ and $\lambda_0 \in \mathfrak{I}$ is such that (19) is satisfied, then λ_0 is in the point spectrum of J and $u(\lambda_0)$ is the corresponding eigenvector.

Theorem 5.3. *Let J be the Jacobi operator defined by* (15), (16) *and* (17). *Then the spectrum of J is discrete.*

Proof. It is already known that the spectrum of J, denoted $\sigma(J)$, is pure point [6]. Suppose that $\sigma(J)$ has a point of accumulation μ in some finite closed interval $\mathfrak{I}$. Let λ and λ' ($\lambda \neq \lambda'$) be arbitrarily chosen from $\sigma(J) \cap \mathfrak{I} \cap V_{\frac{\delta}{2}}(\mu)$, where $V_{\frac{\delta}{2}}(\mu)$ is a $\frac{\delta}{2}$-neighborhood of μ. Consider

$$\begin{aligned} \left|(u(\lambda), u(\lambda'))_{l_2(\mathbb{N})}\right| &= \left|\sum_{n=1}^{\infty} u_n(\lambda)\overline{u_n(\lambda')}\right| \\ &\geq \left|\sum_{n=1}^{N_1} u_n(\lambda)\overline{u_n(\lambda')}\right| - \left|\sum_{n>N_1} u_n(\lambda)\overline{u_n(\lambda')}\right| . \end{aligned} \tag{26}$$

As a consequence of Lemmas 5.1 and 5.2, one can choose N_1, δ and $n_0 \leq N_1$ so that

$$\left|\sum_{n>N_1} u_n(\lambda)\overline{u_n(\lambda')}\right| < \frac{1}{4}\left|u_{n_0}(\mu)\right|^2 < \frac{1}{2}\left|u_{n_0}(\lambda)\right|^2 . \tag{27}$$

Now, consider the first term in the right-hand side of (26)

$$\begin{aligned} \left|\sum_{n=1}^{N_1} u_n(\lambda)\overline{u_n(\lambda')}\right| &\geq \sum_{n=1}^{N_1} \left|u_n(\lambda)\right|^2 - \left|\sum_{n=1}^{N_1} u_n(\lambda)(\overline{u_n(\lambda') - u_n(\lambda)})\right| \\ &\geq \left|u_1(\lambda)\right|^2 - \left|\sum_{n=1}^{N_1} u_n(\lambda)(\overline{u_n(\lambda') - u_n(\lambda)})\right| . \end{aligned}$$

Since $|\lambda' - \lambda| < \delta$, we have

$$\left|\sum_{n=1}^{N_1} u_n(\lambda)(\overline{u_n(\lambda') - u_n(\lambda)})\right| \le \max_{1\le n\le N_1} \omega_n(\delta) \sum_{n=1}^{N_1} |u_n(\lambda)| \,,$$

where

$$\omega_n(\delta) = \sup_{\substack{|\lambda'-\lambda|<\delta \\ \lambda',\, \lambda\in\mathfrak{I}}} |u_n(\lambda') - u_n(\lambda)|$$

is the modulus of continuity of $u_n(\lambda)$ on $\mathfrak{I}$. By the second assertion of Lemma 5.2, taking δ sufficiently small, one obtains

$$\left|\sum_{n=1}^{N_1} u_n(\lambda)(\overline{u_n(\lambda') - u_n(\lambda)})\right| < \frac{1}{2} |u_{n_0}(\lambda)|^2 \,. \tag{28}$$

From (26), (27), and (28)

$$(u(\lambda), u(\lambda'))_{l^2(\mathbb{N})} > |u_{n_0}(\lambda)|^2 - \frac{1}{2} |u_{n_0}(\lambda)|^2 - \frac{1}{2} |u_{n_0}(\lambda)|^2 = 0 \,.$$

But this cannot be true since $J = J^*$ and it must be that $u(\lambda) \perp u(\lambda')$.

□

Acknowledgment

I express my deep gratitude to the referee for very useful remarks and a hint on the literature.

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