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An Algorithm of Real Root Isolation for Polynomial Systems with Applications to the Construction of Limit Cycles

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Abstract. By combining Wu's method, polynomial real root isolation and the evaluation of maximal and minimal polynomials, an algorithm for real root isolation of multivariate polynomials is proposed. Several examples from the literature are presented to illustrate the proposed algorithm.

Keywords. Wu's method, max-min polynomial, real root isolation.

1. Introduction

Solving systems of (symbolic or numerical) polynomial equations in several variables has been a very important topic of theoretical oriented research. Commonly used algorithmic methods for solving systems of polynomial equations include the characteristic set method introduced by Wu [20, 21], Gröbner basis method by Buchberger [1], and cylindrical algebraic decomposition method by Collins [4]. Numerical calculation methods include Newton method [18], homotopy method [9], and eigenvalue method [7].

A different approach for isolating the real roots of a univariate integral polynomial based on Sturm sequence, the derivative sequence and Descartes' rule of sign is widely used [8].

A natural questions is: Can the real root isolation algorithm be extended to the multivariate polynomial case?

In this paper, based on the characteristic set method, real root isolation and the evaluation for maximal and minimal polynomials, we propose an algorithm for isolating real roots of multivariate integral polynomial systems.

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This algorithm is an extension of a real root isolation algorithm for univariate integral polynomial. It results in a higher-dimensional isolated interval for each isolated real root [12, 13]. Based on this algorithm for multivariate polynomial systems, the stability analysis and the construction of small amplitude limit cycles for some differential polynomial systems are considered [12, 14, 15, 16].

Section 2 contains an introduction to Wu's method and the algorithm for real root isolation of multivariate polynomial systems. In Section 3, several examples from the literature are presented to illustrate the proposed algorithm. In Section 4, the construction of limit cycles for differential systems are given.

2. Wu's Method and Real Root Isolation

To prove the main result, we use the principle of the characteristic sets method which was introduced by Ritt [17] in the context of his work on differential algebra and has been considerably developed by Wu [20, 21]. The great success of theorem proving has stimulated the renewed interest in the characteristic sets method. To limit the space, we give here only the basic principle, i.e., the well ordering principle, and illustrate how this principle works.

Let $PS = \{f_1(x_1, \ldots, x_n), \ldots, f_s(x_1, \ldots, x_n)\}$ be any finite set of polynomials in n ordered variables $x_1 \prec \cdots \prec x_n$ with coefficients in certain basic field of characteristic 0, for instance, the field Q of rational numbers. We designate the complete set of zeros of the polynomials in PS by $\mathrm{Zero}(PS)$. If G is any other non-zero polynomial, the subset of $\mathrm{Zero}(PS)$ for which $G \neq 0$ will be denoted by $\mathrm{Zero}(PS/G)$. The following is the basic principle of the characteristic sets method [20, 21, 22].

Well Ordering Principle. Given a set PS of polynomials, one can compute by an algorithmic method another set CS of polynomials, called the *characteristic set* of PS, of the triangular form

$$CS: \quad \begin{cases} c_1(u_1, \ldots, u_d; y_1), \\ c_2(u_1, \ldots, u_d; y_1, y_2), \\ \qquad \cdots\cdots \\ c_r(u_1, \ldots, u_d; y_1, \ldots, y_r), \end{cases}$$

such that

$$\mathrm{Zero}(CS/J) \subset \mathrm{Zero}(PS) \subset \mathrm{Zero}(CS), \tag{2.1}$$

$$\mathrm{Zero}(PS) = \mathrm{Zero}(CS/J) \cup \bigcup_i \mathrm{Zero}(PS_i), \tag{2.2}$$

where $u_1, \ldots, u_d; y_1, \ldots, y_r$ $(d+r = n)$ is a rearrangement of $x_1, \ldots, x_n$, $J = \prod_i I_i$, I_i is the leading coefficient of c_i as polynomial in y_i, called the *initial* of c_i, and $PS_i = PS \cup \{I_i\}$.

The algorithm for triangularizing the polynomial set proceeds basically by successive pseudo-division of polynomials. From (2.1) and (2.2) the relation between the zeros of PS and CS is clear: any zero of PS is a zero of CS and, conversely, any zero of CS for which none of the initials vanishes is also a zero of PS. Therefore, under the condition that all the initials are not equal to 0, both PS and CS have the same zero set. For those zeros of PS making the vanishing of some initial I_i, we may consider further the enlarged polynomial set PS_i obtained from PS by adjoining I_i to it as required. Furthermore, in proceeding with each PS_i like PS by the same principle we would finally obtain a zero decomposition of the form

$$\text{Zero}(PS) = \bigcup_i \text{Zero}(CS_i/J_i), \tag{2.3}$$

in which CS_i is of triangular form as CS and J_i is the product of initials of the polynomials in CS_i for each i.

Equation (2.3) is called the Zero Decomposition Theorem [22]. Furthermore, by using any algorithm of polynomial factorization over algebraic extended fields, an ascending set CS can be decomposed into irreducible ascending sets. Hence, Wu obtains a variety decomposition theorem [22].

Variety Decomposition Theorem. For any given set PS of polynomials, there is an algorithm which computes a finite number of irreducible ascending sets IRR_k in a finite number of steps, such that

$$\text{Zero}(PS) = \bigcup_k \text{Zero}(IRR_k/J_k).$$

Moreover, redundant components can be removed by mere computations so that the union will become a non-contractible one.

Remark. Since the real roots of PS can be decomposed as the union of real roots of IRR_k, we only need to consider those components for which the irreducible ascending sets have isolated real roots. That is, the algebraic varieties associated to the irreducible ascending sets are of dimension zero. In such a case,

$$\text{Zero}(IRR_k/J_k) = \text{Zero}(IRR_k).$$

In the following discussions, we only consider the real root isolation algorithm in the positive cone R^n_+, since a transformation $x_i \to -x_i$ will change the negative real roots into the positive ones in R^n_+.

Now for any n-variate polynomial $f(x) = f(x_1, \ldots, x_n)$ in R_n^+, we denote the summation of the positive terms in $f(x)$ by $f^+(x_1, x_2, \ldots, x_n)$ and that of the negative terms in $f(x)$ by $f^-(x_1, x_2, \ldots, x_n)$. Clearly, $f = f^+ + f^-$.

The following theorems are direct results of the definition of f^+ and f^-.

Theorem 2.1. *If* $x_i \in [a, b]$ $(1 \le i \le n,\ 0 < a \le b)$, *then*

$$\begin{aligned} &f^+(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) + f^-(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) \\ \le\ &f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\ \le\ &f^+(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) + f^-(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n). \end{aligned}$$

Note that both f^+ and f^- are monotone in R^n_+. More general, we have:

Theorem 2.2. *If* $x_{i_j} \in [a_{i_j}, b_{i_j}]$, $1 \le i_j \le n$ $(j = 1, \ldots, k)$, $0 < a_{i_j} \le b_{i_j}$, *then*

$$\begin{aligned} & f^+(x_1, \ldots, a_{i_1}, \ldots, a_{i_k}, \ldots, x_n) + f^-(x_1, \ldots, b_{i_1}, \ldots, b_{i_k}, \ldots, x_n) \\ \le \ & f(x_1, \ldots, x_{i_1}, \ldots, x_{i_k}, \ldots, x_n) \\ \le \ & f^+(x_1, \ldots, b_{i_1}, \ldots, b_{i_k}, \ldots, x_n) + f^-(x_1, \ldots, a_{i_1}, \ldots, a_{i_k}, \ldots, x_n). \end{aligned}$$

In Theorem 2.1 (resp. Theorem 2.2) the identities hold if and only if $a = b$ (resp. $a_{i_j} = b_{i_j}$).

By Theorem 2.2, when considering all the variables $x_1, \ldots, x_n$ in R_n^+, we can get the following estimation:

Theorem 2.3. *For given constants* $0 < a_i \le b_i$ $(i = 1, \ldots, n)$,

(1) *if* $f^+(a_1, a_2, \ldots, a_n) + f^-(b_1, b_2, \ldots, b_n) > 0$, *then for* $x_i \in [a_i, b_i]$ $(i = 1, \ldots, n)$,

$$f(x_1, x_2, \ldots, x_n) > 0;$$

(2) *if* $f^+(b_1, b_2, \ldots, b_n) + f^-(a_1, a_2, \ldots, a_n) < 0$, *then for* $x_i \in [a_i, b_i]$ $(i = 1, \ldots, n)$,

$$f(x_1, x_2, \ldots, x_n) < 0.$$

The command *realroot* in Maple [2] can be used to isolate the real roots of a univariate polynomial with integral coefficients. To isolate the real roots of a given polynomial $g(x)$ with intervals of length less than or equal to ac, we can take *realroot*$(g(x), ac)$ which gives $[[a_1, b_1], \ldots, [a_k, b_k]]$. Here, k is the exact number of distinct real roots of $g(x)$. Namely, in each interval $[a_i, b_i]$, there is one and only one real root of $g(x)$ and $b_i < a_{i+1}$ for all i. Furthermore, a_i and b_i have the same sign. This sequence of disjoint intervals with rational endpoints is called the *real root isolation intervals* of $g(x)$.

Now, we illustrate how to obtain the real root isolation domains of a triangular polynomial system $\{g_1(x), g_2(x, y)\}$ by *realroot* algorithm and the max-min polynomials.

The following notations and theorems will be used in the algorithm.

Definition 2.4. Suppose that $[a_1, b_1]$ is one of the real root isolation intervals of $g_1(x)$. The maximal and minimal polynomials of $g_2(x, y)$ on $[a_1, b_1]$ is defined by $\overline{g}_2(y)$ and $\underline{g}_2(y)$, respectively:

$$\overline{g}_2(y) = g_2^+(b_1, y) + g_2^-(a_1, y), \quad \underline{g}_2(y) = g_2^+(a_1, y) + g_2^-(b_1, y).$$

Clearly, if $\overline{x} \in [a_1, b_1]$ is a real root of $g_1(x)$, then $\underline{g}_2(y) \le g_2(\overline{x}, y) \le \overline{g}_2(y)$.

Theorem 2.5. *For the above* $\overline{g}_2(y)$, $\underline{g}_2(y)$, *there exists an interval with width* ac_1 *such that when* $b_1 - a_1 < ac_1$, *we have:*

(1) *the coefficients of the the leading term of both* $\overline{g}_2(y)$ *and* $\underline{g}_2(y)$ *have the same sign;*

(2) *the number of the real roots for both* $\overline{g}_2(y)$ *and* $\underline{g}_2(y)$ *are the same.*

Proof. Note that $\underline{g}_2(y) \to g_2(\bar{x}, y)$ and $\overline{g}_2(y) \to g_2(\bar{x}, y)$, if $a_1 \to \bar{x}$ and $b_1 \to \bar{x}$. Furthermore, $g_2(\bar{x}, y)$ is a squarefree polynomial in y; therefore, a perturbation of $\bar{x}$ will not change the properties of $g_2(\bar{x}, y)$: the changed polynomial will keep the sign of the leading coefficient as well as the number of real roots. □

Theorem 2.6. *Suppose that* $a \neq b$*; then there is an* ac_2 *such that the realroot command with accuracy (the length of isolating interval)* ac_2 *will lead to the sequence of real root isolation intervals of* $\overline{g}_2(y)$ *and* $\underline{g}_2(y)$*, respectively, as follows:*

$$L_1 := [[c_{11}, d_{11}], \ldots, [c_{1m}, d_{1m}]],$$

$$L_2 := [[c_{21}, d_{21}], \ldots, [c_{2m}, d_{2m}]].$$

Here, $[c_{1i}, d_{1i}] \bigcap [c_{2j}, d_{2j}] = \emptyset$ $(i, j = 1, \ldots, m)$*. Namely, we can get a total list of all the intervals in* L_1 *and* L_2*.*

Proof. In fact, the minimum root separation of an integral polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is determined completely by its coefficients a_i $(i = 1, \ldots, n)$; see [5]. Suppose that $y_1, y_2, \ldots, y_k$ $(k \geq 2)$ are all the distinct real roots of $f(x)$; then

$$\min_{i,j \in \{1,\ldots,k\}} |y_i - y_j| > \frac{1}{n^{n+1}(1 + \sum_{l=0}^{n} |a_l|)}.$$

Now we consider the polynomial $g_{12}(y) = \overline{g}_2(y)\underline{g}_2(y)$ which is the product of $\overline{g}_2(y)$ and $\underline{g}_2(y)$. Take ac_2 as the minimum root separation of g_{12}; then Theorem 2.5 leads to this theorem. □

Remark 1. In a concrete manipulation, the accuracy ac_2 is much larger than the minimum root separation length.

Remark 2. When $a = b$, $\bar{x}$ is the rational solution of $g_(x)$. Hence, $\overline{g}_2(y) = \underline{g}_2(y) = g_2(\bar{x}, y)$. That is, $L_1 = L_2$ in Theorem 2.6.

Definition 2.7. The sequences of the real root isolation intervals of $\overline{g}_2(y)$ and $\underline{g}_2(y)$ are called *matchable* with respect to ac, if

(1) the sequences of the real root isolation intervals L_1 and L_2 take one of the following forms:

(i) 1, 2, 2, 1, 1, 2, 2, 1, …, 1, 2, 2, 1;
(ii) 1, 2, 2, 1, 1, 2, 2, 1, …, 1, 2, 2, 1, 1, 2;
(iii) 2, 1, 1, 2, 2, 1, 1, 2, …, 2, 1, 1, 2;
(iv) 2, 1, 1, 2, 2, 1, 1, 2, …, 2, 1, 1, 2, 2, 1

(here "1" denotes an interval belonging to L_1, and "2" to L_2),

(2) in each interval of L_1 and L_2, the considered integral polynomial $g_2(\bar{x}, y)$ is monotone.

Now we consider the triangular system$\{g_1(x),\ g_2(x, y)\}$.

Theorem 2.8. *Suppose that the real root $\bar{x}$ of $g_1(x)$ in $[a_1, b_1]$ makes the initial $J(\bar{x}) \neq 0$. If $g_2(\bar{x}, y)$ is squarefree with respect to y, then there must be a constant ac such that $\overline{g}_2(y)$ and $\underline{g}_2(y)$ are matchable with respect to ac.*

Proof. Since $g_2(\bar{x}, y)$ is squarefree with respect to y, it must pass through y-axes at each of its real roots. That is, $\overline{g}_2(y), \underline{g}_2(y) \to g_2(\bar{x}, y)$, as $a_1, b_1 \to \bar{x}$ $(ac \to 0)$. Therefore, $\overline{g}_2(y)$ and $\underline{g}_2(y)$ also pass through the y-axes in the neighborhood of $g_2(\bar{x}, y)$.

Now consider the partial derivative $h(\bar{x}, y)$ of $g_2(\bar{x}, y)$ with respect to y. From Definition 2.4, we have

$$\underline{h}(y) \leq h(\bar{x}, y) \leq \overline{h}(y).$$

Clearly, for sufficiently small $|a_1 - b_1|$ (the length of the interval $[a_1, b_1]$), $\underline{h}(y)$ and $\overline{h}(y)$ have the same sign.

Hence, by Theorems 2.5 and 2.6, we can choose ac_1 as small as enough (i.e., $b_1 - a_1$ small enough) and take $ac = \min\{ac_1, ac_2\}$ such that $\overline{g}_2(y)$ and $\underline{g}_2(y)$ are matchable with respect to ac. □

The mrealroot algorithm for isolating the real roots of a set PS of polynomials can be described as follows (here, a three polynomial system will be considered). To guarantee the matchable condition in Theorem 2.8, we use Wu's method to decompose the real roots of PS to the union of those of a set of irreducible ascending sets.

Step 0. Consider system $PS = \{f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)\}$. Using Wu's method (Variety Decomposition Theorem), we can get its irreducible components IRR_i, $i = 1, \ldots, k$. Take the components whose corresponding varieties are of dimension zero. Denote one IRR as $\{g_1(x), g_2(x, y), g_3(x, y, z)\}$.

Step 1. Using *realroot* command to get the sequence of isolation intervals of $g_1(x)$ (here, we just consider the positive interval), where the accuracy ac_1 is given by Theorem 2.5. Now we choose one interval, say $[a_1, b_1]$, to illustrate our method.

Step 2. Construct the maximal and minimal polynomials $\overline{g}_2(y)$ and $\underline{g}_2(y)$ of $g_2(x, y)$ with indeterminate y. By using the *realroot* algorithm, we can obtain the sequences of isolating real roots, L_1, L_2.

Step 3. Take ac and check if $\overline{g}_2(y)$ and $\underline{g}_2(y)$ are matchable with respect ac, i.e., if L_1 and L_2 take one of the forms (i), (ii), (iii) and (iv) in Definition 2.7. If the answer is positive, then go to step 4. Otherwise, divide ac by 2 and go back to step 1. By Theorem 2.8, after finitely many steps, L_1 and L_2 will be matchable.

Step 4. $\overline{g}_2(y)$ and $\underline{g}_2(y)$ are matchable with respect to ac. Now we match the interval as follows:

Take $a_{2i} = \min\{c_{1i}, c_{2i}\}$ and $b_{2i} = \max\{d_{1i}, d_{2i}\}$, $i = 1, \dots, m$; then $\{g_1(x), g_2(x,y)\}$ has just one real root in each $[a_1, b_1] \times [a_{2i}, b_{2i}]$ for $i = 1, \dots, m$.

Step 5. Suppose that $G = [a_1, b_1] \times [a_2, b_2]$ is a real root isolation interval of $\{g_1(x), g_2(x,y)\}$. We may construct the maximal and minimal polynomials $\overline{g}_3(z)$ and $\underline{g}_3(z)$ of $g_3(x,y,z)$ as follows:

$$\overline{g}_3(z) = g_3^+(b_1, b_2, z) + g_3^-(a_1, a_2, z),$$
$$\underline{g}_3(z) = g_3^+(a_1, a_2, z) + g_3^-(b_1, b_2, z).$$

Step 6. Similar to steps 2 and 4, we obtain the real root isolation interval $G = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ of $\{g_1(x), g_2(x,y), g_3(x,y,z)\}$.

Step 7. Since the initials of IRR are not zero at its real roots, we can have all the real root isolation intervals of

$$\{f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)\}.$$

For a general n-variate polynomial system, we may describe the above seven-step algorithm (except step 0) as follows:

```
INPUT     {g1(x1), g2(x1,x2), ..., gn(x1,x2,...,xn)};   J(x1,...,xn)
OUTPUT    the real root isolation intervals of
          {g1(x1), g2(x1,x2), ..., gn(x1,x2,...,xn)};
          the signs of J(x1,...,xn) at every isolating interval
BEGIN
  LABEL
  G1 := {realroot(g1(x1), ac) intervals for positive real roots}
  IF G1 = ∅ THEN return ∅ END IF
  k := 2                                                  /* remark 3 */
  WHILE k ≤ n DO
    Gk = ∅
    gk(x1,...,xk) = gk+(x1,...xk) + gk-(x1,...,xk)
    FOR p in Gk-1 DO
      _gk(xk),   ‾gk(xk)                                  /* remark 4 */
      _Gk = {realroot(_gk(xk), ac) intervals with positive endpoints}
      ‾Gk = {realroot(‾gk(xk), ac) intervals with positive endpoints}
      flag = Qk(_Gk, ‾Gk)                                 /* remark 5 */
      IF (flag == false) GOTO LABEL
      Gk = Gk union Qk(_Gk, ‾Gk)
    END FOR
  k := k + 1
  END WHILE                                               /* remark 6*/
END
```

Remark 3. k denotes the k-th unknown x_k for the k-th polynomial.

Remark 4. $\underline{g}_k(x_k)$ and $\overline{g}_k(x_k)$ are minimal and maximal polynomials of $g_k(x_1, \ldots, x_k)$, respectively.

Remark 5. Check if $\overline{g}_k(x_k)$ and $\underline{g}_k(x_k)$ are matchable with respect to ac. If the answer is positive, then finish the match to get the isolation intervals. Otherwise, choose $ac/2$ as new ac and return “false”.

Remark 6. The result G_n is just the set of real root isolation intervals of

$$\{g_1(x_1), g_2(x_1, x_2), \ldots, g_n(x_1, x_2, \ldots, x_n)\}.$$

Based on the above algorithm, we construct a command *mrealroot* in Maple:

$$> mrealroot([g_1(x_1), g_2(x_1, x_2), \ldots, g_n(x_1, x_2, \ldots, x_n)], [x_1, x_2, \ldots, x_n], \\ c, \quad [h_1(x_1, x_2, \ldots, x_n), \ldots, h_m(x_1, x_2, \ldots, x_n)]);$$

where n, m are positive integers and the positive number c is the upper bound of the width of the intervals of the real roots. Here c is used to control the accuracy of the interval solutions to its exact ones. If c is too wide to fulfil the matchable condition in Theorem 2.8, it will be substituted by a smaller one automated according to the algorithm. Even if c is omitted, the most convenient width is used for each interval returned.

The output is:

$$\Big[\big[x_{11}, x_{12}\big], \big[x_{21}, x_{22}\big], \ldots, \big[x_{n1}, x_{n2}\big], \underbrace{+, -, \widetilde{0}, \ldots, -}_{m}\Big].$$

Here, “+” (resp. “−”) denotes that a polynomial at the real root is positive (resp. negative), $\widetilde{0}$ means that the positiveness and negativeness are not determined (undecided). The above output means that there is a real root

$$(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) \in \big[[x_{11}, x_{12}\big] \times \big[x_{21}, x_{22}\big] \times \cdots \times \big[[x_{n1}, x_{n2}]\big]$$

such that $h_1(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) > 0, h_2(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) < 0, h_3(\overline{x}_1, \overline{x}_1, \ldots, \overline{x}_n)$ is undecided, ..., $h_m(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) < 0$.

3. Application to Polynomial Systems

The first example is from [6].

Example P1.

$$\begin{aligned} f_1 &= x^5 + y^2 + z^2 - 4, \\ f_2 &= x^2 + 2y^2 - 5, \\ f_3 &= xz - 1. \end{aligned} \tag{3.1}$$

By using step 0 in the algorithm, (3.1) is transformed to the triangular form:

$$\begin{aligned} g_1 &= 2x^7 + 2 - 3x^2 - x^4, \\ g_2 &= x^2 + 2y^2 - 5, \\ g_3 &= xz - 1. \end{aligned}$$

Take the *mrealroot* command

$$> mrealroot([g_1, g_2, g_3], [x, y, z], \frac{1}{10})$$

we get

$$\Big[\Big[1,1\Big],\Big[\tfrac{11}{8},\tfrac{23}{16}\Big],\Big[1,1\Big]\Big], \qquad \Big[\Big[\tfrac{13}{16},\tfrac{7}{8}\Big],\Big[\tfrac{23}{16},\tfrac{3}{2}\Big],\Big[\tfrac{9}{8},\tfrac{5}{4}\Big]\Big],$$

$$\Big[\Big[1,1\Big],\Big[-\tfrac{23}{16},-\tfrac{11}{8}\Big],\Big[1,1\Big]\Big], \qquad \Big[\Big[-\tfrac{3}{4},-\tfrac{11}{16}\Big],\Big[\tfrac{23}{16},\tfrac{25}{16}\Big],\Big[-\tfrac{3}{2},-\tfrac{21}{16}\Big]\Big],$$

$$\Big[\Big[\tfrac{13}{16},\tfrac{7}{8}\Big],\Big[-\tfrac{3}{2},-\tfrac{23}{16}\Big],\Big[\tfrac{9}{8},\tfrac{5}{4}\Big]\Big], \qquad \Big[\Big[-\tfrac{3}{4},-\tfrac{11}{16}\Big],\Big[-\tfrac{25}{16},-\tfrac{23}{16}\Big],\Big[-\tfrac{3}{2},-\tfrac{21}{16}\Big]\Big].$$

This means that the polynomial system (3.1) has six real roots.

The second example is also from [6].

Example P2.

$$\begin{aligned} f_1 &= x^2 - 2xz + 5, \\ f_2 &= xy^2 + yz + 1, \\ f_3 &= 3y^2 - 8xz. \end{aligned} \tag{3.2}$$

The step 0 gives us from (3.2) the triangular form

$$\begin{aligned} g_1 &= y(320 + 1600y^4 - 240y^5 - 471y^6 + 36y^7 - 48y^2 + 36y^8), \\ g_2 &= -40y^2 + 3y^3 + 6y^4 + 8x, \\ g_3 &= -8yz - 8 - 40y^4 + 3y^5 + 6y^6. \end{aligned}$$

Applying *mrealroot*:

$$> mrealroot([g_1, g_2, g_3], [x, y, z], \frac{1}{10^5})$$

we obtain two real root isolation intervals:

$$\Big[\Big[-\tfrac{188613}{65536},-\tfrac{377225}{131072}\Big],\Big[-\tfrac{72181}{65536},-\tfrac{36063}{32768}\Big],\Big[-\tfrac{185017}{65536},-\tfrac{369513}{131072}\Big]\Big],$$

$$\Big[\Big[-\tfrac{45}{16},-\tfrac{368639}{131072}\Big],\Big[\tfrac{126525}{131072},\tfrac{63315}{65536}\Big],\Big[\tfrac{50295}{16384},\tfrac{100711}{32768}\Big]\Big].$$

Example P3 [5].

$$\begin{aligned} f_1 &= x^2 + 2y^2 - y - 2z, \\ f_2 &= x^2 - 8y^2 + 10z - 1, \\ f_3 &= x^2 - 7yz. \end{aligned} \tag{3.3}$$

The triangular form of (3.3) is

$$\begin{aligned} g_1 &= 2450x^6 - 1241x^4 + 196x^2 - 49, \\ g_2 &= 86x^2 + 35yx^2 - 77y - 14, \\ g_3 &= 175x^4 + 70x^2z - 76x^2 - 154z + 21, \end{aligned}$$

with initial $J = 5x^2 - 11$.

Now the command

$$> mrealroot([g_1, g_2, g_3], [x, y, z], \frac{1}{10^5}, [J])$$

gives out two real root isolation intervals:

$$\left[\left[\frac{85823}{131072}, \frac{1341}{2048}\right], \left[\frac{48355}{131072}, \frac{24179}{65536}\right], \left[\frac{10879}{65536}, \frac{10881}{65536}\right], -\right],$$

$$\left[\left[-\frac{1341}{2048}, -\frac{85823}{131072}\right], \left[\frac{48355}{131072}, \frac{24179}{65536}\right], \left[\frac{10879}{65536}, \frac{10881}{65536}\right], -\right].$$

Example P4 [5].

$$\begin{aligned} f_1 &= x^4 + y^4 - 1, \\ f_2 &= x^5y^2 - 4x^3y^3 + x^2y^5 - 1. \end{aligned} \tag{3.4}$$

The triangular form of (3.4) is as follows:

$$\begin{aligned} g_1 = {} & 16y^5 - 36y^9 + 30y^{10} - y^8 + 144y^{15} - 64y^{11} + 758y^{16} - 251y^{12} + 28y^{13} \\ & - 126y^{14} + 4y^{21} - 12y^{17} - 128y^{22} + 192y^{18} + 48y^{23} - 112y^{19} - 757y^{20} \\ & + 32y^{26} - 16y^{27} + 249y^{24} + 2y^{28} + 1, \\ g_2 = {} & 4x+y-2y^5+2y^8x-2y^4x+y^2+16y^3-24y^7-xy^3+4y^{10}x-3y^{11}x+3y^7x+y^9 \\ & - 32xy^9 + 16y^{13}x - 4y^{14}x + y^{15}x - 8y^{15} + 16y^{11} + y^{16} - y^{12} + 16y^5x. \end{aligned}$$

The initial is

$$J = 4 + 2y^8 - 2y^4 - y^3 + 4y^{10} - 3y^{11} + 3y^7 - 32y^9 + 16y^{13} - 4y^{14} + y^{15} + 16y^5.$$

Taking the command

$$> mrealroot([g_1, g_2], [y, x], \frac{1}{10^5}, [J]);$$

we get four real roots:

$$\left[\left[-\frac{121139}{131072}, -\frac{60569}{65536}\right], \left[\frac{47213}{65536}, \frac{47299}{65536}\right], +\right], \quad \left[\left[\frac{94517}{131072}, \frac{47259}{65536}\right], \left[-\frac{60575}{65536}, -\frac{60563}{65536}\right], +\right],$$

$$\left[\left[-\frac{78307}{131072}, -\frac{39153}{65536}\right], \left[\frac{126675}{131072}, \frac{31673}{32768}\right], +\right], \quad \left[\left[\frac{126681}{131072}, \frac{63341}{65536}\right], \left[-\frac{19599}{32768}, -\frac{39111}{65536}\right], +\right].$$

In [4], Collins considered the following system of polynomials and found four solution points. By using the mrealroot algorithm, a PIII 550 computer gives out the four isolating real roots in less than 1.2 seconds.

Example P5.

$$\begin{aligned} f_1 &= -7xyz + 6yz - 14xz + 9z - 3xy - 12y - x + 1, \\ f_2 &= 2xyz - yz + 14z + 15xy + 14y - 15x, \\ f_3 &= -8xyz + 11yz - 12xz - 5z + 15xy + 2y + 10x - 14. \end{aligned} \tag{3.5}$$

The triangular form of (3.5) is as follows:

$$\begin{aligned}
g_1 = {} & -311308988x - 7694683176x^2 + 4541529810x^3 + 8951356045x^4 \\
& - 4587245014x^5 - 4919456115x^6 + 3307892784 + 198993030x^7 \\
& + 174040200x^8, \\
g_2 = {} & 34405x^2 + 7530x^4y - 46172x^3y - 65274x^2y + 22815x^3 - 13680x^4 \\
& - 17640x + 44624xy - 16296 + 54084y, \\
g_3 = {} & 92250x^5 + 46110x^4z + 541875x^4 - 692403x^3z + 143625x^3 - 844151x^2z \\
& - 886430x^2 + 639688xz - 319860x + 740880z + 27360x^5z + 228144,
\end{aligned}$$

and the initial is

$$\begin{aligned}
J = {} & -1860170521921098891 64x + 349875023661822013792x^2 \\
& - 186848402623007386128 + 337522017343330106770x^3 \\
& - 170675129780428490075x^4 - 155540642141382952602x^5 \\
& + 99439300041231374 85x^6 + 59968621217518218 90x^7.
\end{aligned}$$

Call the mrealroot command:

$$> mrealroot([g_1, g_2, g_3], [x, y, z], \frac{1}{10^{10}}, [J]);$$

the computer shows four isolating real roots:

$$[[\tfrac{33573910397}{34359738368}, \tfrac{16786955199}{17179869184}], [\tfrac{82505394681}{8589934592}, \tfrac{330021582735}{34359738368}], [-\tfrac{96628390617}{8589934592}, -\tfrac{386513559595}{34359738368}], -1],$$
$$[[-\tfrac{8880815695}{8589934592}, -\tfrac{35523262779}{34359738368}], [-\tfrac{16638954157}{17179869184}, -\tfrac{33277908053}{34359738368}], [-\tfrac{1073569409}{1073741824}, -\tfrac{34354220805}{34359738368}], -1],$$
$$[[-\tfrac{7659282837}{8589934592}, -\tfrac{30637131347}{34359738368}], [\tfrac{614939185923}{34359738368}, \tfrac{614939235161}{34359738368}], [\tfrac{11783002047}{17179869184}, \tfrac{11783002915}{17179869184}], 1],$$
$$[[\tfrac{173688058101}{34359738368}, \tfrac{86844029051}{17179869184}], [-\tfrac{36761640685}{17179869184}, -\tfrac{36761640679}{17179869184}], [-\tfrac{1676409165625}{34359738368}, -\tfrac{1676409164939}{34359738368}], 1].$$

In all the above examples, the systems are considered in R^n. In fact, we may consider systems in C^n. In this case a transformation $x \to x_1 + ix_2$ shall be made, where x_1 is the real part and x_2 the imaginary part of x.

Example P6.

$$\begin{aligned}
f_1 &= 2x^2 - xy + 4, \\
f_2 &= xy - 2y^2 + 4.
\end{aligned} \tag{3.6}$$

After the transformation $x \to x_1 + ix_2$, $y \to y_1 + iy_2$, we have

$$\begin{aligned}
f_1 &= p_{11} + ip_{12}, \\
f_2 &= p_{21} + ip_{22},
\end{aligned}$$

where

$$\begin{aligned}
p_{11} &= 2x_1^2 + 4 - 2x_2^2 - x_1y_1 + x_2y_2, \\
p_{12} &= 4x_1x_2 - x_1y_2 - x_2y_1, \\
p_{21} &= x_1y_1 + 4 - x_2y_2 - 2y_1^2 + 2y_2^2, \\
p_{22} &= x_1y_2 + x_2y_1 - 4y_1y_2.
\end{aligned} \tag{3.7}$$

By using Wu's method, (3.7) can be transformed to

$$\begin{aligned} g_1 &= 3y_1^4 - 6y_1^2 - 1, \\ g_2 &= 2y_1^3 - x_1y_1^2 - 4y_1 + x_1, \\ g_3 &= -y_1^2 + 2 + y_2^2, \\ g_4 &= y_1^2x_2 - 2y_1^2y_2 - x_2. \end{aligned} \tag{3.8}$$

The initial is $J = (y_1 - 1)(y_1 + 1)$. Taking the *mrealroot* command for (3.8):

$$> mrealroot([g_1, g_2, g_3, g_4], [y_1, x_1, y_2, x_2], \frac{1}{10^2}, [J]);$$

we obtain four pairs of real roots

$$\left[\left[\tfrac{187}{128}, \tfrac{47}{32}\right], \left[\tfrac{39}{128}, \tfrac{7}{16}\right], \left[\tfrac{23}{64}, \tfrac{51}{128}\right], \left[\tfrac{169}{128}, \tfrac{97}{64}\right], +\right],$$

$$\left[\left[\tfrac{187}{128}, \tfrac{47}{32}\right], \left[\tfrac{39}{128}, \tfrac{7}{16}\right], \left[-\tfrac{51}{128}, -\tfrac{23}{64}\right], \left[-\tfrac{97}{64}, -\tfrac{199}{128}\right], +\right],$$

$$\left[\left[-\tfrac{47}{32}, -\tfrac{187}{128}\right], \left[-\tfrac{7}{16}, -\tfrac{39}{128}\right], \left[\tfrac{23}{64}, \tfrac{51}{128}\right], \left[\tfrac{169}{128}, \tfrac{97}{64}\right], +\right],$$

$$\left[\left[-\tfrac{47}{32}, -\tfrac{187}{128}\right], \left[-\tfrac{7}{16}, -\tfrac{39}{128}\right], \left[-\tfrac{51}{128}, -\tfrac{23}{64}\right], \left[-\tfrac{97}{64}, -\tfrac{169}{128}\right], +\right].$$

Therefore, the corresponding four complex roots of (3.6) (with $x = (x_1, x_2)$, $y = (y_1, y_2)$) are

$$\left[\left[\tfrac{39}{128}, \tfrac{7}{16}\right] \times \left[\tfrac{169}{128}, \tfrac{97}{64}\right], \left[\tfrac{187}{128}, \tfrac{47}{32}\right] \times \left[\tfrac{23}{64}, \tfrac{51}{128}\right]\right],$$

$$\left[\left[\tfrac{39}{128}, \tfrac{7}{16}\right] \times \left[-\tfrac{97}{64}, -\tfrac{199}{128}\right], \left[\tfrac{187}{128}, \tfrac{47}{32}\right] \times \left[-\tfrac{51}{128}, -\tfrac{23}{64}\right]\right],$$

$$\left[\left[-\tfrac{7}{16}, -\tfrac{39}{128}\right] \times \left[\tfrac{169}{128}, \tfrac{97}{64}\right], \left[-\tfrac{47}{32}, -\tfrac{187}{128}\right] \times \left[\tfrac{23}{64}, \tfrac{51}{128}\right]\right],$$

$$\left[\left[-\tfrac{7}{16}, -\tfrac{39}{128}\right] \times \left[-\tfrac{97}{64}, -\tfrac{169}{128}\right], \left[-\tfrac{47}{32}, -\tfrac{187}{128}\right] \times \left[-\tfrac{51}{128}, -\tfrac{23}{64}\right]\right].$$

4. Application to Differential Systems

The problem of distinguishing between centers and foci and the construction of small amplitude limit cycles for polynomial differential systems has a tight connection with Hilbert's 16th problem. Based on the real root isolation algorithm, the construction of small amplitude limit cycles for differential polynomial systems was proposed in [11]. After the Liapunov constants are obtained for a system, the question for the estimation of the number of small amplitude limit cycles bifurcated from a fine focus becomes the following: can we isolate the real roots for the polynomial system of the first k Liapunov constants?

In this section, based on the method in [14] and the *mrealroot* algorithm, we give three examples for cubic systems.

In what follows, $L_j(i)$ denotes the j-th Liapunov constant (focal value) for the i-th example.

Example D1. In 1980, Coleman [3] proposed the conjecture that a Kolmogorov prey-predator system may have more than two stable limit cycles. In [13], this conjecture is confirmed. In that paper, the idea of *mrealroot* algorithm is proposed.

The constructed system is as follows:

$$\begin{aligned}\dot{x} &= x(-2 - a_0 + a_1 + a_0 x - 2a_1 x + y + a_1 x^2 + xy),\\ \dot{y} &= y(2 + a_2 - x - y - 2a_2 y + a_2 y^2).\end{aligned}$$

Here, $a_1 > 0, a_2 > 0$. Clearly, $(1, 1)$ is a positive equilibrium of the system. By using the transformation

$$\overline{x} = x - 1, \qquad \overline{y} = y - 1,$$

the original system (we use x, y instead of $\overline{x}, \overline{y}$) is changed to

$$\begin{aligned}\dot{x} &= (x + 1)(x + a_0 x + 2y + a_1 x^2 + xy),\\ \dot{y} &= (y + 1)(-x - y + a_2 y^2).\end{aligned} \tag{4.1}$$

When $a_0 = 0$, $(0, 0)$ is a focus. The first three focal values of (4.1) are

$$\begin{aligned}L_1(1) &= 3a_2 - a_2^2 + 2a_1 - 1 + 2a_1^2,\\ L_2(1) &= 451a_2^2 + 546a_2a_1 + 427a_1^2 - 392a_2^2a_1 - 328a_2^3 - 472a_1^3 + 46a_2^3a_1\\ &\quad - 92a_2a_1^3 + 176a_2^2a_1^2 - 876a_2a_1^2 + 74a_2^4 - 648a_1^4 - 238a_2 - 182a_1 + 44,\\ L_3(1) &= -889784a_2^4 - 28184 + 3433278a_2a_1^2 + 944669a_2^3 - 1652639a_2^3a_1\\ &\quad + 1918142a_1^3 - 5270694a_2a_1^3 - 5028128a_2^2a_1^2 - 3163812a_1^4\\ &\quad - 579519a_2^2 + 221080a_1 + 196690a_2 - 1112353a_2a_1 + 2063089a_2^2a_1\\ &\quad + 499258a_1a_2^4 + 135384a_1^6 + 763936a_2a_1^5 - 338976a_2^3a_1^3 + 308.\end{aligned}$$

We can obtain simpler and equivalent $\bar{L}_1, \bar{L}_2, \bar{L}_3$ with $\bar{L}_1 = L_1(1)$, $\bar{L}_3 = L_3(1)$ and

$$\bar{L}_2 = 638a_2^3 - 709a_2^4 - 30a_2 - 139a_2^2 + 120a_2^5 + 8a_2^6 + 3,$$

provided that the initial $I = 8a_2^3 - 51a_2^2 - 12a_2^2a_1 + 32a_2a_1 + 55a_2 - 19 - 14a_1 \neq 0$. Now, taking the commend

$$mrealroot([\bar{L}_1, \bar{L}_2], [a_2, a_1], 1/10^{10}, [\bar{L}_3, I]),$$

we obtain eight solutions:

$$\left[\left[\frac{688939015}{8589934592}, \frac{1377878031}{17179869184}\right], \quad \left[\frac{5077638577}{17179869184}, \frac{5077638579}{17179869184}\right], \quad -, -\right],$$

$$\left[\left[\frac{688939015}{8589934592}, \frac{1377878031}{17179869184}\right], \quad \left[-\frac{22257507763}{17179869184}, -\frac{22257507761}{17179869184}\right], \quad -, \widetilde{0}\right],$$

$$\left[\left[-\frac{2985891391}{17179869184}, -\frac{1492945695}{8589934592}\right], \quad \left[\frac{8810195803}{17179869184}, \frac{8810195805}{17179869184}\right], \quad +, -\right],$$

$$\left[\left[-\frac{84633102747}{4294967296}, -\frac{338532410987}{17179869184}\right], \quad \left[-\frac{265975581777}{17179869184}, -\frac{265975581775}{17179869184}\right], \quad -, \widetilde{0}\right],$$

$$\left[\left[\frac{62789278049}{17179869184}, \frac{31394639025}{8589934592}\right], \quad \left[\frac{7689817571}{8589934592}, \frac{15379635145}{17179869184}\right], \quad +, -\right],$$

$$\left[\left[-\frac{84633102747}{4294967296}, -\frac{338532410987}{17179869184}\right], \quad \left[\frac{248795712591}{17179869184}, \frac{248795712593}{17179869184}\right], \quad +, -\right],$$

$$\left[\left[\frac{62789278049}{17179869184}, \frac{31394639025}{8589934592}\right], \quad \left[-\frac{32559504329}{17179869184}, -\frac{16279752163}{8589934592}\right], \quad -, \widetilde{0}\right],$$

$$\left[\left[-\frac{2985891391}{17179869184}, -\frac{1492945695}{8589934592}\right], \quad \left[-\frac{25990064989}{17179869184}, -\frac{25990064987}{17179869184}\right], \quad -, \widetilde{0}\right].$$

The first solution (with $a_1 > 0, a_2 > 0$) is what we need. In this case, the system can have three small amplitude limit cycles among which two are stable.

Example D2. In [10], the following cubic Kolmogorov system is considered:

$$\begin{aligned} \dot{x} &= x(x - 2y + 2)(Ax + y + B), \\ \dot{y} &= y(2x - y - 2)(Dx + y + C). \end{aligned} \tag{4.2}$$

Here $(2, 2)$ is a positive fixed point. With the transformation $\overline{x} = x - 2, \overline{y} = y - 2$, system (4.2) takes the form (here we use x, y instead of $\overline{x}, \overline{y}$)

$$\begin{aligned} \dot{x} &= (x + 2)(x - 2y)(A(x + 2) + y + 2 + B), \\ \dot{y} &= (y + 2)(2x - y)(D(x + 2) + y + 2 + C). \end{aligned} \tag{4.3}$$

To ensure $(0, 0)$ to be a center-focus form, we need $D = A + B/2 - C/2$. Substituting it into (4.3), we have

$$\begin{aligned} \dot{x} &= (x + 2)(x - 2y)(A(x + 2) + y + 2 + B), \\ \dot{y} &= (y + 2)(2x - y)((A + B/2 - C/2)(x + 2) + y + 2 + C). \end{aligned} \tag{4.4}$$

At $(0, 0)$, the first four focal values of (4.4) are $L_1(2), L_2(2), L_3(2), L_4(2)$. When $C = -4$, $I_{21} = 2 + B + 2A \neq 0$, $I_{22} \neq 0$ and $b + 4 \neq 0$, we can have reduced focal values: $\bar{L}_1 = L_1(2) = 0$, $\bar{L}_4 = L_4(2)$ and

$$\begin{aligned} \bar{L}_2 &= -3B^2 + 4AB - 22B + 28A^2 + 56A, \\ \bar{L}_3 &= 5639949B^8 - 158890977B^7 - 9906667659B^6 - 118203801471B^5 \\ &\quad - 407483203554B^4 + 118362692448B^3 + 417384231264B^2, \end{aligned}$$

with

$$\begin{aligned} I_2 = {} & 18252402AB^3 + 309080730B^2A + 1053193848AB - 3997665B^4 \\ & - 132974583B^3 - 759844470B^2 + 403771368B. \end{aligned}$$

Taking the commend

$$mrealroot([\bar{L}_2, \bar{L}_3], [B, A], 1/10^{20}, [\bar{L}_4, I_2, B + 4]),$$

we have four solutions

$$\begin{aligned}
&\Big[\Big[-\frac{2573675109802661915841}{1475739525896764129 28}, -\frac{402136735906665924 35}{2305843009213693952}\Big], \\
&\quad \Big[-\frac{1183730637891661183}{288230376151711744}, -\frac{3030350433002652628 47}{7378697629483820646 4}\Big], \quad +, +, -, -\Big], \\
&\Big[\Big[-\frac{2906285907974342590 35}{3689348814741910323 2}, -\frac{1162514363189737036139}{1475739525896764129 28}\Big], \\
&\quad \Big[\frac{1454501429562568981 21}{7378697629483820646 4}, \frac{7272507147812844906 3}{3689348814741910323 2}\Big], \quad -, +, -, -\Big], \\
&\Big[\Big[-\frac{1278947606251465505551}{1475739525896764129 28}, -\frac{6394738031257327527 75}{7378697629483820646 4}\Big], \\
&\quad \Big[\frac{8015183426052381884 1}{3689348814741910323 2}, \frac{2003795856513095471 1}{9223372036854775808}\Big], \quad -, +, -, -\Big], \\
&\Big[\Big[-\frac{1278947606251465505551}{1475739525896764129 28}, -\frac{6394738031257327527 75}{7378697629483820646 4}\Big], \\
&\quad \Big[\frac{1173301997102876793 15}{1475739525896764129 28}, \frac{5866509985514383966 1}{7378697629483820646 4}\Big], \quad -, +, -, -\Big].
\end{aligned}$$

Clearly at all the four zeros, $\bar{L}_1$, $\bar{L}_2$, and $\bar{L}_3$ are independent with respect to $\bar{L}_4$; therefore there are four classes of values for A, B, C, D such that system (4.2) has four small amplitude limit cycles at $(0, 0)$.

Example D3. Consider the following system [19]

$$\begin{aligned} \dot{x} &= y + Bx^3 + (C - G)x^2y + (3D - H)xy^2 + Ey^3, \\ \dot{y} &= -x - Ax^3 - (3B + F)x^2y - (C + G)xy^2 - Dy^3. \end{aligned} \tag{4.5}$$

Clearly, $(0, 0)$ is an equilibrium. The first five focal values are $L_1(3), \ldots, L_5(3)$. Well ordering the focal values in the order of $H \prec F \prec A \prec G$, we obtain

$$\begin{aligned}
L_1 &= -F - H, \\
L_2 &= 4DG + 2FG - FE + FA + 4BG, \\
L_3 &= Gp_{53}q_{51}I_{51}^{-1}, \\
L_4 &= -G^2p_{54}q_{51}I_{51}^{-1}, \\
L_5 &= (32EC + 48E^2 - 9D^2 + 30DB - 9B^2)(-9B^3 - 9B^2D + 32BC^2 + 144BCE \\
&\quad + 144BE^2 + 9BD^2 - 48DCE - 144DE^2 + 9D^3),
\end{aligned}$$

where

$$
\begin{aligned}
p_{53} &= 5A - 5E + 2G,\\
p_{54} &= 3G - 15E - 5C,\\
q_{51} &= (A^2D - ADC + BEA - BGA - BCA + 3DAG - DAE + 2B^3\\
&\quad - DGE + DCE - 2DCG + 2DG^2 - 2D^3 - 2BD^2 - BE^2 - 2BG^2\\
&\quad + BCE + 2B^2D - 2BCG + 3BGE),\\
I_{51} &= (2G - E + A)^2.
\end{aligned}
$$

Let $B = 1, C = 1, D = 1$ and $G = -1$, and take the function

$$mrealroot([p_{54}, p_{53}, L_2, L_1], [E, A, F, H], 1/10^{10}, [L_5, q_{51}I_{51}]);$$

then we obtain

$$
\Big[\Big[-\tfrac{9162596899}{17179869184}, -\tfrac{4581298449}{8589934592}\Big], \Big[-\tfrac{1145324613}{8589934592}, -\tfrac{286331153}{2147483648}\Big],
$$

$$
\Big[-\tfrac{85899345925}{17179869184}, -\tfrac{85899345915}{17179869184}\Big]\Big], \Big[\tfrac{85899345915}{17179869184}, \tfrac{85899345925}{17179869184}\Big], \quad -, -\Big].
$$

This real root makes the first six focal values to be independent. Therefore, system (4.5) has six small amplitude limit cycles.

References

[1] B. Buchberger. An algorithm for finding a basis for the residue class of zero-dimensional polynomial ideal. *Aequationes Math.* 4/3, 374–383 (1970).

[2] B.W. Char, K.O. Geddes, G.H. Gonnet et al. *Maple V — Language Reference Manual.* New York: Springer (1991).

[3] C.S. Coleman. Hilbert's 16th problem: How many cycles? In: *Differential Equation Models* (M. Braun et al., eds.), pp. 279–297. Berlin: Springer (1982).

[4] G.E. Collins. Quantifier elimination by cylindrical algebraic decomposition — Twenty years of progress. In: *Quantifier Elimination and Cylindrical Algebraic Decomposition* (B. Caviness and J.R. Johnson, eds.), pp. 8–23. Wien, New York: Springer (1998).

[5] D. Cox, J. Little, D. O'Shea. *Ideals, Varieties, and Algorithms* (2nd edn.). New York, Berlin: Springer (1996).

[6] D. Cox, J. Little, D. O'Shea. *Using Algebraic Geometry.* New York, Berlin: Springer (1996).

[7] G. Feng, S. Zhang. Reducing the multivariate polynomial system to eigentvalue problem. In: *Proc. Int. Workshop Math. Mech.* (W.-T. Wu and M. Cheng, eds.), pp. 19–27. Beijing: Int. Academic Publ. (1992).

[8] J.R. Johnson. Algorithm for polynomial real root isolation. In: *Quantifier Elimination and Cylindrical Algebraic Decomposition* (B. Caviness and J.R. Johnson, eds.), pp. 269–299. New York: Springer (1998).

[9] T.Y. Li, T. Sauer, J.A. Yorke. Numerically determining solutions of systems of polynomial equations. *Bull. Amer. Math. Soc.* 18, 173–177 (1988).

[10] N.G. Lloyd, J.M. Pearson, E. Saez et al. Limit cycles of a cubic Kolmogorov system. *Appl. Math. Lett.* 9, 15–18 (1996).

[11] Z. Lu, B. He. Multiple stable limit cycles for a cubic Kolmogorov prey-predator system. *Chinese J. Engin. Math.* 18, 115–117 (2001).

[12] Z. Lu, B. He, Y. Luo. *An Algorithm of Real Root Isolation for Polynomial Systems with Applications*. Bejing: Science Press (2004).

[13] Z. Lu, B. He, Y. Luo, L. Pan. An algorithm of real root isolation for polynomial systems. *MM Research Preprints* (Academia Sinica) 20, 187–198 (2001).

[14] Z. Lu, B. He, Y. Luo, L. Pan. The construction of small amplitude limit cycles for differential polynomial systems. Preprint (2000).

[15] Z. Lu, Y. Luo. Two limit cycles in three-dimensional Lotka-Volterra systems. *Comput. Math. Appl.* 44, 51–66 (2002).

[16] Z. Lu, Y. Luo. Three limit cycles for a three-dimensional Lotka-Volterra competitive system with a heteroclinic cycle. *Comput. Math. Appl.* 46, 281–288 (2003).

[17] J.F. Ritt. *Differential Algebra*. New York: Amer. Math. Soc. (1950).

[18] S. Smale. The fundamental theorem of algebra and complexity theory. *Bull. Amer. Math. Soc.* 4, 1–36 (1981).

[19] D. Wang. A class of cubic differential systems with 6-tuple focus. *J. Diff. Eqns.* 87, 305–315 (1990).

[20] W.-T. Wu. On the decision problem and the mechanization of theorem proving in elementary geometry. *Sci. Sinica* 21, 150–172 (1978).

[21] W.-T. Wu. *Mechanical Theorem Proving in Geometries: Basic Principle*. Wien, New York: Springer (1994).

[22] W.-T. Wu. *Mathematics Mechanization*. Beijing: Science Press/Kluwer (2000).

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