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Proposal for the Algorithmic Use of the BKK-Number in the Algebraic Reduction of a 0-dimensional Polynomial System

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Abstract. For a regular 0-dimensional system P of polynomials with numerical coefficients, its BKK-number m equals the number of its zeros, counting multiplicities. In this paper, I analyze how the knowledge of m may be used for the computation of a Gröbner basis or more generally a border basis of P. It is also shown how numerical stability may be preserved in such an approach, and how near-singular systems are recognized and handled. There remain a number of open questions which should stimulate further research.

1. Introduction

When I prepared my invited lecture for SNC 2005, I had no intention of publishing it. Therefore, for the second part of my talk, I chose to put forward some preliminary ideas which I felt were worth being investigated further. After the presentation of the lecture in Xi'an, I was urged by a number of colleagues to publish its content, which — with a good deal of hesitation — I finally agreed to do. A later more thorough consideration of such a publication convinced me that there should actually be two separate papers: one should contain the first part of my talk which explained facts about the mathematical feasibility of extending significant parts of polynomial algebra into the realm of approximate data and approximate computation. The other one should put down the ideas in the second part of my talk in a more elaborate and formal fashion, as a stimulus for further research. This is that second paper.

It contains my ideas about a novel algorithmic approach to the numerical computation of a standard representation of the ideal or of the quotient ring resp. of a *regular 0-dimensional* system of polynomials with *numerical coefficients*. Here "regular" is used in analogy with its usage in Numerical Linear Algebra: no tiny change of coefficients can change the dimension of the quotient ring; cf. [11, Sect. 8].

The approach is based on the fact that the *exact number of zeros* of such a system (counting multiplicities) or — equivalently — the *exact vector dimension* of the quotient ring can be determined *a priori* by a symbolic algorithm with only the *supports* of the individual polynomials as input. I have wondered for many years why, by my observation, nobody in the large GB community has attempted to simplify the GB computation for regular 0-dimensional systems P by using its BKK-number as a helpful input, particularly after BKK computation has become a standard tool in the late 1990s; cf. e.g. [16, 10].

In the following, I present the steps of an algorithm which — in a good number of cases — computes the elements of a *border basis* of the ideal generated by P or, equivalently, the set of *multiplication matrices* of the associated quotient ring, both w.r.t. a particular *normal set* or monomial basis. The algorithm uses *no term order*; therefore it requires various choices and decisions which may determine its success; I have only been able to indicate potentially successful strategies. Besides these choices, the complete computation of the final result (border basis, multiplication matrices) consists only of substitutions and of the solution of blocks of *linear* vector equations. In particular, *no reductions to zero* are needed, which is an important feature for the numerical computation.

I have no proof that the algorithm will always succeed, even for the most clever choices in the preparatory part. This paper is rather an invitation to refine its ideas into a true algorithm, if only for some particular subclasses of regular 0-dimensional systems, or to establish that it must fail almost always. In any case, a good deal of insight should result from these investigations (which I am too old to tackle).

The paper begins with a summary of facts about border bases and multiplication matrices; it also introduces my terminology and notation. In Sect. 3, we show how the *syzygies* of a border basis, or the *consistency conditions* for the associated multiplication matrices, may be reduced to a minimal set which should — together with the system P — specify the basis uniquely. All this refers to a particular normal set whose number of elements must equal the BKK-number of P; its selection is discussed in Sect. 4. Then the constituent steps of the proposed algorithm are introduced and discussed in Sect. 5. In the following Sect. 6, the potential appearance of *ill-conditioning* is considered and algorithmic remedies are explained. Sect. 7 then deals with the possibility that the specified regular system P is actually *very close to singularity*: one type of singularity (BKK deficiency) may be dealt with algorithmically, and the other type (positive dimension) can only be diagnosed. The following Sect. 8 discusses various details of a potential implementation and points out potential sources of failure. Some conclusions complete the paper.

It should be mentioned that many parts of the material in this paper are also contained somewhere in my book on "Numerical Polynomial Algebra" [14]; but this paper presents them in a self-contained and systematic fashion which should help in bringing them to the attention of the community. I am also aware that some ideas in this paper are related to ideas in the more recent GB-algorithms of J.-C. Faugère (cf., e.g. [3]) and to ideas in papers by B. Mourrain and P. Trébuchet (cf., e.g., [11, 12]); but since their approaches and their notational framework differ from mine considerably, I have not pointed this out in detail. A more refined development of the algorithm proposed in this paper is contained in a forthcoming paper [7] of A. Kehrein and M. Kreuzer. Also I wish to thank SIAM Publ. for granting the permission to use figures which are adaptations of figures in [14].

2. Border Bases and Multiplication Tables

We denote by $\mathbb{P}^s$ the set (ideal) of all polynomials in s variables with coefficients in $\mathbb{C}$. Monomials in $\mathbb{P}^s$ are denoted by $x^k := x_1^{k_1} \dots x_s^{k_s}$, $k \in \mathbb{N}_0^s$. A set $\mathcal{N}$ of monomials in $\mathbb{P}^s$ is *closed* if $x^\mu \in \mathcal{N}$ implies $x^{\mu'} \in \mathcal{N}$ for all divisors $x^{\mu'}$ of x^μ. In $\mathbb{P}^s$, we consider a 0-dimensional *polynomial ideal* $\mathcal{I}$ with m zeros (counting multiplicities), with its *quotient ring* $\mathcal{R}[\mathcal{I}] := \mathbb{P}^s / \mathcal{I}$ of vector space dimension m. A closed m-element monomial set $\mathcal{N}$ is a *feasible normal set* for $\mathcal{R}$ if it is a basis of $\mathcal{R}$ as a vector space.

A regular polynomial system $P \subset \mathbb{P}^s$, with s equations, generates a *0-dimensional* ideal whose quotient ring dimension m *equals* the BKK-number of P. Note that the BKK-number of P depends only on the *supports* of the s polynomials in P, and that it may be computed via the mixed volume of the Newton polytope of P; cf., e.g., [15, 10].

Definition 2.1. For a closed monomial set $\mathcal{N} \subset \mathbb{P}^s$, the set of all monomials which are *not* in $\mathcal{N}$ but satisfy $x^k = x_\sigma x^j$ for some $x^j \in \mathcal{N}$ and some $x_\sigma, \sigma = 1(1)s$, is the *border set* $B[\mathcal{N}]$ of $\mathcal{N}$, with elements x^{k_ν}, $\nu = 1(1)N := |B[\mathcal{N}]|$.

It is well-known that the *multiplicative structure* of $\mathcal{R}$ is specified, w.r.t. the fixed basis $\mathcal{N}$, by the *multiplication matrices* $A_\sigma \in \mathbb{C}^{m \times m}$, $\sigma = 1(1)s$, which represent the residue classes $\bmod\, \mathcal{I}$ of the elements in $B[\mathcal{N}]$ in terms of the basis $\mathcal{N}$, with the elements x^{j_μ} of $\mathcal{N}$ arranged into a vector $\mathbf{b}$ in some arbitrary but *fixed* order:

$$x_\sigma \mathbf{b} \;\equiv\; A_\sigma \mathbf{b} \bmod \mathcal{I}, \quad \sigma = 1(1)s\,. \tag{1}$$

While many of the rows in the A_σ are simply unit rows because they refer to a normal set element "inside" $\mathcal{N}$ with the x_σ-shifted monomial also in $\mathcal{N}$, there must exist a particular *nontrivial* row $a_{k_\nu}^T$ in A_σ for each element x^{j_μ} of $\mathcal{N}$ with an x_σ-neighbor x^{k_ν} in $B[\mathcal{N}]$: $x^{k_\nu} = x_\sigma x^{j_\mu} \equiv a_{k_\nu}^T \mathbf{b}$. If an element x^{k_ν} in $B[\mathcal{N}]$ is an x_σ-neighbor of a normal set monomial for several distinct σ, then the same row $a_{k_\nu}^T$ will occur in all of the respective matrices A_σ.

Obviously, the collection of these nontrivial rows specifies the multiplicative structure of $\mathcal{R}[\mathcal{I}]$ and thus also the structure of $\mathcal{I}$:

Definition 2.2. The set of the $N := |B[\mathcal{N}]|$ polynomials

$$bb_\nu(x) \;:=\; x^{k_\nu} \;-\; a_{k_\nu}^T\, \mathbf{b}\,, \quad \nu = 1(1)N\,, \tag{2}$$

is a *border basis* $\mathcal{B}[\mathcal{I}]$ of $\mathcal{I}$; more specifically, it is the $\mathcal{N}$-*border basis* $\mathcal{B}_{\mathcal{N}[\mathcal{I}]}$ of $\mathcal{I}$. Cf. also [6, 8].

Except for $s = 1$, the number N of border basis elements bb_ν is $> s$, often $N \gg s$. Thus $\mathcal{B}_{\mathcal{N}[\mathcal{I}]}$ would be *overdetermined* if its elements bb_ν were considered as independent. $\mathcal{B}_{\mathcal{N}[\mathcal{I}]}$ can define a nontrivial ideal with m zeros *only* if the bb_ν satisfy the *system of syzygies* which arises in the following way:

In the border set $B[\mathcal{N}]$, two monomials x^{k_ν}, $x^{k_{\nu'}}$ are *neighbors* if they satisfy one of the following two relations:

$$\begin{aligned} &\text{(i) For some } \sigma, \quad x^{k'_\nu} \;=\; x_\sigma\, x^{k_\nu}\,; \\ &\text{(ii) For some } \sigma,\, \sigma',\; x_\sigma\, x^{k'_\nu} \;=\; x_{\sigma'}\, x^{k_\nu}\,. \end{aligned} \tag{3}$$

For the coefficients of the border basis elements bb_ν, $bb_{\nu'}$ associated with neighboring border monomials, this implies (cf. (1), (2), (3)) in case

$$\begin{aligned} \text{(i) } 0 &= \quad bb_{\nu'} - x_{\sigma'} bb_\nu + (a^T_{k_{\nu'}} - a^T_{k_\nu}\, x_{\sigma'})\,\mathbf{b} \quad \equiv\; (a^T_{k_{\nu'}} - a^T_{k_\nu}\, A_{\sigma'})\,\mathbf{b} \,\in\, \mathcal{I}\,; \\ \text{(ii) } 0 &= \; x_\sigma bb_{\nu'} - x_{\sigma'} bb_\nu + (a^T_{k_{\nu'}}\, x_\sigma - a^T_{k_\nu}\, x_{\sigma'})\,\mathbf{b} \;\equiv\; (a^T_{k_{\nu'}}\, A_\sigma - a^T_{k_\nu}\, A_{\sigma'})\,\mathbf{b} \,\in\, \mathcal{I}\,. \end{aligned} \tag{4}$$

Note that $a^T\,\mathbf{b} \in \mathcal{I}$ implies $a = 0$. Therefore we have

Theorem 2.1. *For a specified feasible normal set* $\mathcal{N}$*, the row vectors* $a^T_{k_\nu} \in \mathbf{C}^m$, $\nu = 1(1)N$ *whose components are the coefficients of the border basis* $\mathcal{B}_\mathcal{N}$ *of the 0-dimensional polynomial ideal* $\mathcal{I}$ (*cf.* (2)) *and the elements of the nontrivial rows of the multiplication matrices* A_σ *w.r.t.* $\mathcal{N}$ *of the quotient ring* $\mathcal{R}[\mathcal{I}]$ (*cf.* (1)) *must satisfy, for neighboring border set monomials* (*cf.* (3))*:*

$$\begin{aligned} \textit{in case (i)} \qquad a^T_{k_{\nu'}} &= a^T_{k_\nu}\, A_{\sigma'}\,; \\ \textit{in case (ii)} \quad a^T_{k_{\nu'}}\, A_\sigma &= a^T_{k_\nu}\, A_{\sigma'}\,. \end{aligned} \tag{5}$$

Theorem 2.1 gives necessary relations for the border basis polynomials of $\mathcal{I}$ as well as for the multiplication matrices A_σ of $\mathcal{R}[\mathcal{I}]$. For the latter, *commutativity* is known to be a necessary and sufficient condition, cf. [11]. But a closer analysis shows (cf. [14], Thm. 8.11) that the relations (5) are *identical* with the relations arising from the commutativity conditions $A_{\sigma'}\, A_\sigma \;=\; A_\sigma\, A_{\sigma'}$ for all pairs (σ, σ') when these are spelt out in terms of the rows $a^T_{k_\nu}$.

Corollary 2.2. *The conditions* (5) *of Theorem* 2.1 *are necessary and sufficient.*

Example 2.1. In $\mathbb{P}^3$, we consider a 0-dimensional ideal with $m = 6$ zeros and assume that the 6-point normal set $\mathcal{N}$ shown in Fig. 1 is feasible. (The figure depicts a set of monomials x^{j_μ} in $\mathbb{P}^3$ by the set of its exponents j_μ in $\mathbb{N}^3$; this is a commonly used visualization tool.) The associated border set $B[\mathcal{N}]$ is also indicated; each of its monomials is a "positive neighbor" of a monomial in $\mathcal{N}$. The total number N of border monomials turns out to be 9. The edges in the figure connect border monomials which are *neighbors* in the sense defined above. There are $\bar{N} = 14$ pairs of neighbors, four of which are of type (i) while the others are of type (ii).

As an example of a relation (4) of type (i), we consider the neighboring monomials $x^{(0,1,1)} = x_2x_3$ and $x^{(1,1,1)} = x_1x_2x_3$: they require the validity of

$$(a^T_{(1,1,1)} - a^T_{(0,1,1)}\, x_1)\, \mathbf{b} \in \mathcal{I}\,. \tag{6}$$

To use this relation computationally, we must replace those monomials x^k of $x_1\mathbf{b}$ which are in $B[\mathcal{N}]$ by $a_k^T\mathbf{b}$. When we now assume that — within an algorithm for the computation of the $a^T_{k_\nu}$ — the row vector $a^T_{(0,1,1)}$ has already been found, then the above relation constitutes a *linear* vector equation for the remaining unknown vectors $a^T_{k_\nu}$.

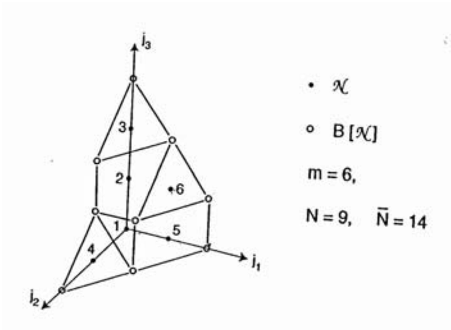

FIGURE 1

On the other hand, when we consider the type (ii) neighbors $x^{(1,1,1)} = x_1x_2x_3$ and $x^{(1,0,2)} = x_1x_3^2$ which require the validity of

$$(a^T_{(1,0,2)}\, x_2 - a^T_{(1,1,1)}\, x_3)\, \mathbf{b} \in \mathcal{I}\,,$$

and assume the row vector $a^T_{(1,1,1)}$ to be known, then we will obtain a linear vector equation only if $a^T_{(1,0,2)}$ is known, too.

3. Minimal Syzygy Bases

It is our goal to use the syzygy relations (4) for the computation of the nontrivial rows of the multiplication matrices for a specified feasible normal set, in the fashion indicated in Example 2.1. If we wanted to do this for an ideal generated by polynomials with integer or simple rational coefficients and in rational arithmetic, we might be willing to put up with the *overdetermination* which prevails in the complete system (4). For the *numerical treatment* of (4) which we have in mind, such an overdetermination is fatal: since the use of floating-point arithmetic excludes algorithmic decisions based on a strict equality of complex numbers, the interdependence of the relations (4) may remain undetected and lead to an *inconsistent* system.

At first, we consider the number of syzygy relations which should be operative in a minimal set. To determine N nontrivial vectors $a^T_{k_\nu}$ we should have N

independent vector equations. s of those will be furnished by the polynomial system which defines the ideal at hand. Thus, $N - s$ further independent equations should be necessary and sufficient to determine the $a^T_{k_\nu}$ uniquely. The following concept yields an overview of the complete set of relations (4):

Definition 3.1. For a specified normal set $\mathcal{N}$, the *border web* $BW_\mathcal{N}$ is the set of all pairs of monomials in $B[\mathcal{N}]$ which are neighbors in the sense of (3). In the visualization of monomials by their exponents, the border web $BW_\mathcal{N}$ is the set of all *edges* connecting the exponents of neighboring pairs. In this representation, $BW_\mathcal{N}$ is a *graph* in s-space.

This concept of a border web permits us to use graph theoretic considerations for the reduction of the system (4) to a subset of $N - s$ independent relations. At first, we realize that the graph $BW_\mathcal{N}$ contains *closed loops.* But a relation (4) represented by an edge "closing a loop" holds if the relations for the remaining edges of the loop hold; cf. [14, Propositions 8.14 and 8.15].

By a well-known theorem of graph theory , the "breaking of all loops" in a connected graph with N nodes results in a graph with exactly $N - 1$ edges. While this has brought us closer to our goal, we still have to get rid of $s - 1$ further relations or edges resp.

We note that our restriction to syzygies arising from neighboring border monomials is not compulsory. Like with Gröbner bases, syzygies between border basis polynomials whose leading terms are *relatively prime* are satisfied automatically; cf. [2]. In our visualization, relatively prime border monomials become nodes in the border web which lie in disjoint subspaces (e.g. $x^{(1,0,0)}$ and $x^{(0,0,3)}$). Therefore we are permitted to augment our border web graph by *"virtual"* edges between such nodes.

The introduction of these further edges makes it possible to delete some further "real" edges of $BW_\mathcal{N}$ which close a loop with a virtual edge. Altogether, $s-1$ real edges may be deleted because $s-1$ virtual edges may be introduced without creating loops between virtual edges; cf. [14, Proposition 8.14 (b)]. This reduces the total number of remaining edges and thus of operative equations (4) to the desired $N-s$. Such a *minimal border web* will be denoted by $\overline{BW}_\mathcal{N}$. Naturally, throughout the deletion process, we must take care that the graph remains connected and that each one of the final support leading monomials of the autoreduced system retains some *real* edge(s) issuing from it.

Example 3.1. We continue with the situation of Example 2.1; cf. the visualized border web of Fig. 1. When we consider the node $(2, 0, 0)$ as the root of a tree, it becomes obvious that the 3 edges in the $2, 3$-plane close loops. After their deletion, we may still delete the two upward edges issuing form $(1, 1, 0)$ and the edge between $(1, 1, 1)$ and $(1, 0, 2)$. This leaves $N - 1 = 8$ edges.

Now we introduce the two virtual edges from $(2, 0, 0)$ to $(0, 2, 0)$ and $(0, 0, 3)$ resp. which are obviously in the disjoint subspaces 1-axis, 2-axis, 3-axis; cf. Fig. 2. This enables us to delete the real edges ending in $(0, 2, 0)$ and $(0, 0, 3)$ without

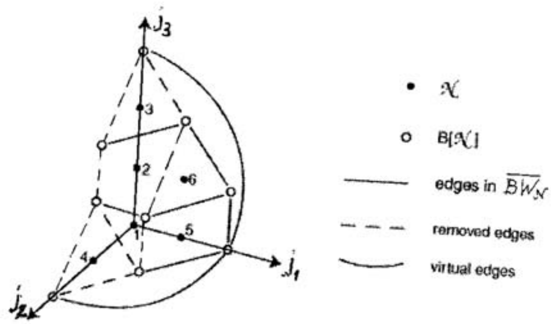

FIGURE 2

disconnecting these nodes from the graph. Thus, we have arrived at $N - s = 6$ conditions (4) which, together with the 3 generating polynomials of $\mathcal{I}$, should suffice to specify the complete border basis.

The minimal border web $BW_{\mathcal{N}}$ is not at all uniquely determined. E.g., we could have retained the type (i) edges $[(1,1,0),(1,1,1)]$ and $[(0,1,1),(0,1,2)]$ and deleted the type (ii) edges $[(2,0,1),(1,1,1)]$ and $[(1,0,2),(0,1,2)]$ instead.

Actually, in the algorithmic computation of a border basis, the minimal border web is not selected *a priori* but formed recursively during the computation; cf. Sects. 5 and 8.

4. Selection of a Tentative Normal Set

As explained in the Introduction, we assume that we *know* the number m of zeros (counting multiplicities) of the 0-dimensional ideal $\mathcal{I}$ defined by the *regular* polynomial system $P = \{p_\nu,\ \nu = 1(1)s\}$. This number is also the expected vector space dimension of the quotient ring $\mathcal{R}[\mathcal{I}]$ and hence the number of elements in a monomial basis $\mathcal{N}$ of $\mathcal{R}$. Remember that the number m computed from the mixed volume of the Newton polytope of P (cf. e.g. [4]) depends only on the *supports* of the polynomials p_ν and not on their specified coefficients. Thus, a closed set $\mathcal{N}$ with the correct number m of monomials and a structure compatible with the supports of the p_ν may still not be *feasible* for the actual coefficients of P. This can only be discovered during the border basis computation; the appropriate measures will be discussed in later sections.

As a first step towards the selection of a normal set for $\mathcal{R}$ we attempt to simplify the system P by *autoreduction*:

In Gröbner basis computation, autoreduction is controlled by term order: we check whether the leading monomial of some p_ν divides terms in another polynomial $p_{\nu'}$. If yes, we eliminate these terms, in the sequence of term order, by the subtraction of suitable multiples of p_ν. The possibly new leading monomial of the reduced $p_{\nu'}$ may now divide terms in another polynomial etc. Clearly, this simplification procedure does not change the ideal $\mathcal{I}[P]$. It must stop because term orders can only decrease in this well-known procedure.

Border bases do not refer to a term order but to a normal set $\mathcal{N}$, with a partial ordering given *a-posteriori* by the "distance" to $\mathcal{N}$; cf. [14, Definition 8.3]. But autoreduction must take place *before* the specification of the normal set. Therefore, we proceed as follows:

We consider the monomial sets S_ν of the *supports* of the p_ν and their *internal borders* consisting of those monomials in the S_ν with no multiples in S_ν. For each p_ν, we select a monomial in the internal border of its support S_ν as *support-leading* monomial which we use like the (term order)-leading monomial above. In the absence of term order, we must avoid eliminations in a $p_{\nu'}$ which would introduce new border terms into $S_{\nu'}$; this requires the *simultaneous* elimination in sets of polynomials where the support-leading terms of each polynomial also occur in the other polynomials. This is easily achieved by the solution of a linear system for the support-leading terms. Naturally, each elimination in a $p_{\nu'}$ redefines $S_{\nu'}$ and may require the selection of a new support-leading monomial which — possibly — permits further elimination. Because the supports can only shrink, the procedure must come to a stop. However, depending on the choice of the support-leading monomials, the results may differ considerably; cf. Example 4.1 below. The support sets of the final autoreduced system, *without* the respective support-leading monomials, will be denoted by $\bar{S}_\nu$.

Now we are ready to select a normal set $\mathcal{N}$ of proper magnitude m. For this purpose we take the *union* of the s truncated final support sets $\bar{S}_\nu$. Then we complement this union into a *closed convex* monomial set, i.e. a set which contains all divisors of one of its elements. If the resulting set has exactly m elements, it is a (tentative) normal set for $\mathcal{R}[\mathcal{I}]$.

Otherwise, we have to adjoin further monomials such that the set remains convex and that no multiples of the final support-leading monomials are appended. For an arbitrary choice of support-leading monomials, it may not be possible to reach a set of m elements. But we know that m member normal sets for $\mathcal{R}[\mathcal{I}]$ must exist, with one monomial of each autoreduced support not in $\mathcal{N}$; therefore it must be possible to reach such a normal set, perhaps with some additional sophistication in the approach.

Example 4.1. To avoid confusing complications, we take a very simple situation: $s = 2$, and dense polynomials p_1, p_2 with degrees 2 and 3 resp.; this makes $m = 6$. A less trivial situation will appear in Sect. 7; cf. Example 7.2 and Fig. 5 (i).

(i) At first, we follow the Gröbner basis pattern. Use of the term order $\texttt{tdeg}[x_2, x_1]$ makes x_2^2 and x_2^3 the leading monomials in p_1, p_2 resp. With suitable multiples of p_1, we may eliminate the x_2^3, $x_1x_2^2$ and x_2^2 terms in p_2; the leading monomial of the reduced p_2 is $x_1^2x_2$, and the union of the truncated supports is

$\{1, x_1, x_2, x_1^2, x_1x_2, x_1^3\}$ (cf. Fig. 3); it has exactly 6 elements. This normal set also arises in the course of a Gröbner basis algorithm; but its *a priori* knowledge cuts the basis computation short; cf. Example 5.1 (i).

(ii) Without a term order, we may choose the monomials x_2^2 and x_1^3 as support-leading monomials of p_1, p_2 resp.; note that the internal border of the two supports consists of the monomials of degree 2 and 3 resp. The reduction proceeds as before; but now the support-leading monomial of p_2 remains unchanged throughout; the union $\{1, x_1, x_2, x_1^2, x_1x_2, x_1^2x_2\}$ of the final truncated supports has $m = 6$ elements and is a nice normal set for a border basis of our $\{p_1, p_2\}$; cf. Example 5.1 (ii).

(iii) Another normal set arises when we take x_1x_2 as the support-leading monomial of p_1, which we are free to do. Now we can eliminate the $x_1x_2^2$ and $x_1^2x_2$ terms in p_2 and choose either x_2^3 or x_1^3 as support leading monomial in the reduced p_2. For the former choice, we obtain $\{1, x_1, x_2, x_1^2, x_2^2, x_1^3\}$ as normal set for a border basis. It looks a little strange but functions alright as we shall see in Example 5.1 (iii).

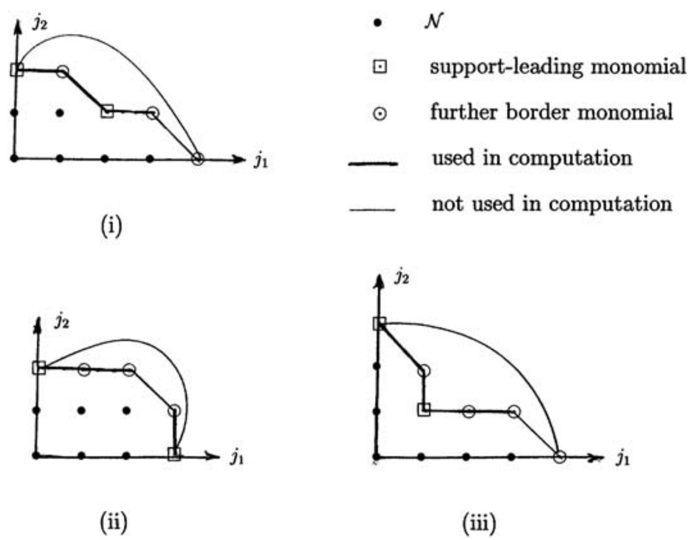

FIGURE 3

5. Algorithmic Determination of Border Bases

We assume that we have selected a tentative normal set $\mathcal{N}$, with its border web $BW_{\mathcal{N}}$, for a specified 0-dimensional polynomial system $P \subset \mathbb{P}^s$. Now we want to compute the coefficients of the associated border basis $\mathcal{B}_{\mathcal{N}} = \{bb_\nu, \nu = 1(1)n\}$ for the ideal generated by P.

By the selection procedure of Sect. 4, s of the bb_ν are specified by the polynomials of the autoreduced system P (we keep the notation p_ν for these polynomials): the support-leading monomials x^{ℓ_ν} of these p_ν are in the border set $B[\mathcal{N}]$. Thus it is natural to start the computation with syzygy relations (4) along edges of $BW_\mathcal{N}$ issuing from these monomials x^{ℓ_ν}, $\nu = 1(1)s$.

Assume at first that there is an edge of type (i) issuing from a particular one of these support-leading x^{ℓ_ν} in the σ'-direction and that the σ'-shifted truncated support set $x_{\sigma'}\bar{S}_\nu$ of $p_\nu = bb_\nu = x^{\ell_\nu} - a_{\ell_\nu}^T \mathbf{b}$ remains *completely within* $\mathcal{N} \cup \{x^{\ell_{\nu'}}, \nu' = 1(1)s\}$. (This is most likely to happen for a p_ν of lowest degree.) Then, after the substitution of the occurring $x^{\ell_{\nu'}}$, $a_{\ell_\nu}^T x_{\sigma'}\mathbf{b}$ is a polynomial in $\mathcal{N}$ and must *equal* the polynomial $a_{k_{\nu'}}\mathbf{b}$ for $x^{k_{\nu'}} = x_{\sigma'}x^{\ell_\nu}$; cf. (4)(i). Thus, the border basis polynomial $bb_{k_{\nu'}}$ is obtained directly as $bb_{k_{\nu'}} := x^{k_{\nu'}} - a_{\ell_\nu}^T x_{\sigma'}\mathbf{b}$. Obviously, as a first step in our computation, we should attempt to utilize all possibilities of this kind.

After this initial phase, which may not be present at all, we must utilize the relations (4) along the other edges issuing from the x^{ℓ_ν}, $\nu = 1(1)s$. Each edge to a type (i) neighbor (cf. (3)) will generate a linear vector equation. If some pair of support-leading monomials is connected by a type (ii) edge, the associated relation (4)(ii) also leads to a linear vector equation. Generally, these equations will involve further vectors $a_{k_\nu}^T$ associated with border monomials x^{k_ν} introduced by the shifts of the x_{ℓ_ν}. Hopefully, there will be as many vector equations as unknown vectors. Note that edges issuing from border web nodes whose vectors have been found in the initial phase may be employed in the same fashion.

After more vectors $a_{k_\nu}^T$ have been found from this block of linear vector equations, the procedure may be further continued. But in choosing further edges we must watch that the emerging web of used edges becomes a *minimal* border web. This requests mainly that we do not use edges which close a loop; otherwise, we would introduce an equation which is dependent on equations which have been used already. With floating-point computation, this may not be realized and spoil the computation.

Hopefully, we may thus recursively generate *blocks* of linear vector equations, with as many equations as unknown vectors each, until all $N - s$ unknown vectors $a_{k_\nu}^T$ have been determined. In principle, this should always be possible, at least for a cleverly chosen normal set $\mathcal{N}$:

From the theory of Gröbner bases [2] we know that, for a regular 0-dimensional system P, the Gröbner basis w.r.t. any term order may be computed by a finite number of *rational* operations. It is clear that this holds also for the border basis associated with the normal set of a Gröbner basis. For a fixed system P, the change from the border basis polynomials for one normal set to those for another one also requires the solution of linear systems only; cf. [14, Sect. 8.1.1]. Therefore, the coefficients of *any* border basis for a regular 0-dimensional system P are *rational functions* of the coefficients of P. This raises some hope that border bases may generally be computed along the lines indicated, possibly with a more refined strategy. Note that — like in the F_5 algorithm of [3] — no *reductions to zero* will ever arise in our algorithm. These have been exposed as the main obstacles for a stable floating-point implementation of GB-algorithms; cf. [9].

Before we turn to the discussion of numerical difficulties which may arise even when we have recursive blocks of the correct number of linear equations, we continue our simple-minded example of Sect. 4 to explain the described algorithmic procedure in more detail. For a less trivial example, we refer to Example 7.2 and Fig. 5 (i).

Example 5.1. The original polynomials p_1, p_2 are dense in 2 variables, of degrees 2 and 3 resp., and $m = 6$; cf. Example 4.1.

(i) With the term order $\texttt{tdeg}[x_2, x_1]$. we have obtained the normal set and border web of Fig. 3 (i), with x_2^2 and $x_1^2x_2$ as final support-leading border monomials with specified coefficient vectors, and the further 3 border monomials $x_1x_2^2$, $x_1^3x_2$, and x_1^4 whose associated coefficient vectors have to be determined. For a clear notation, we set $\mathbf{b} := (1, x_1, x_2, x_1^2, x_1x_2, x_1^3)^T$; then, with $a_k^T := (\alpha_{k,1}, \alpha_{k,2}, \alpha_{k,3},$
$\alpha_{k,4}, \alpha_{k,5}, \alpha_{k,6},)$, $a_k^T\,\mathbf{b}$ denotes the polynomial $\alpha_{k,1} + \alpha_{k,2}\,x_1 + \alpha_{k,3}\,x_2 + \alpha_{k,4}\,x_1^2$ $+\alpha_{k,5}\,x_1x_2 + \alpha_{k,6}\,x_1^3$.
In the initial phase of the algorithm, we see that the x_1-shift of the truncated support set $\bar{S}_1$ generates only normal set monomials and the border monomial $x_1^2x_2$ whose coefficient vector $a_{(2,1)}^T$ is known from the autoreduced p_2. Hence, the substitution of $a_{(2,1)}^T$ into $a_{(0,2)}^T\,x_1\mathbf{b}$ yields $a_{(1,2)}^T$ explicitly:

$$a_{(1,2)}^T := (0,\ \alpha_{(0,2),1}, 0,\ \alpha_{(0,2),2},\ \alpha_{(0,2),3},\ \alpha_{(0,2),4}) + \alpha_{(0,2),5}\,a_{(2,1)}^T\,.$$

For the simultaneous computation of $a_{(3,1)}^T$ and $a_{(4,0)}^T$, we may use the type (i) relation from an x_1-shift of $x_1^2x_2$ together with the type (ii) relation between $x_1x_2^2$ and $x_1^2x_2$, both of which introduce the vectors $a_{(3,1)}^T$ and $a_{(4,0)}^T$ linearly, with known scalar coefficients:

$$\mathbf{a_{(3,1)}^T} = (0,\ \alpha_{(2,1),1},\ 0,\ \alpha_{(2,1),2},\ \alpha_{(2,1),3},\ \alpha_{(2,1),4}) + \alpha_{(2,1),5},\ a_{(2,1)}^T + \alpha_{(2,1),6},\ \mathbf{a_{(4,0)}}^T\ ,$$
$$(0,\ \alpha_{(1,2),1}, 0,\ \alpha_{(1,2),2},\ \alpha_{(1,2),3},\ \alpha_{(1,2),4}) + \alpha_{(1,2),5}\,a_{(2,1)}^T + \alpha_{(1,2),6}\,\mathbf{a_{(4,0)}}^T =$$
$$(0, 0, \alpha_{(2,1),1}, 0, \alpha_{(2,1),2}, 0) + \alpha_{(2,1),3}a_{(0,2)}^T + \alpha_{(2,1),4}a_{(2,1)}^T + \alpha_{(2,1),5}a_{(1,2)}^T$$
$$+\alpha_{(2,1),6}\mathbf{a_{(3,1)}}^T.$$

Thus, the complete computation of the Gröbner basis for $\texttt{tdeg}[x_2, x_1]$ consists of substitutions and the solution of a 2×2 linear system with the matrix $\begin{pmatrix} 1 & -\alpha_{(2,1),6} \\ \alpha_{(2,1),6} & -\alpha_{(1,2),6} \end{pmatrix}$. It is hard to believe that the GB could be computed otherwise with fewer operations. (The reduced border web underlying this computation is shown in Fig. 3 (i). The case when the matrix is (near-) singular will be treated in the next section.)

(ii) With the rectangular normal set of Fig. 3 (ii), we have 3 type (i) edges and one type (ii) edge; thus, it is possible to use only type (i) relations for the computation of the 3 unknown vectors $a^T_{(1,2)}$, $a^T_{(2,2)}$ and $a^T_{(3,1)}$. The computation is very similar to the one for the preceeding normal set:
Again, the x_1-shift of the truncated support of p_1 introduces only normal set monomials and x_1^3 with its known vector $a^T_{(3,0)}$ into the expression for $a^T_{(1,2)}$. A simultaneous x_1-shift of $x_1x_2^2$ and x_2-shift of x_1^3 yields the two linear vector equations for $a^T_{(2,2)}$ and $a^T_{(3,1)}$. The manipulations are slightly simpler than in the previous case. The associated reduced border web (cf. Fig. 3 (ii)) remains connected by the virtual edge from x_2^2 to x_1^3.

(iii) With the unusual normal set of Fig. 3 (iii), the initial phase of the algorithm is empty. With simultaneous x_1- and x_2-shifts of the support-leading monomial x_1x_2, we obtain the system of linear equations for $a^T_{(2,1)}$ and $a^T_{(1,2)}$. Then we may use the type (ii) relation for $x_1x_2^2$ and x_2^3 together with the x_1-shift of $x_1^2x_2$ to obtain the two vector equations for $a^T_{(4,0)}$ and $a^T_{(3,1)}$. The lonely node x_1^4 is connected to the web by the virtual edge from x_1^4 to x_2^3; cf. Fig. 3 (iii).

6. Ill-Conditioned Situations

As explained in the Introduction, we assume throughout that the coefficients in the regular 0-dimensional polynomial system $P = \{p_\nu,\ \nu = 1(1)s\}$ are of *limited accuracy* and that the computation of a border basis for $\mathcal{I}[P]$ proceeds in *floating-point arithmetic*. Therefore, we must take special care to avoid *ill-conditioned* situations where small perturbations may be amplified excessively.

At first, the specified system P itself may be ill-conditioned, i.e. it may specify the ideal $\mathcal{I}[P]$ very poorly, with a high sensitivity to tiny changes in P. In a linear system, this happens when the (linear) polynomials are nearly linearly dependent or — equivalently — the system is close to singularity. The corresponding situation with polynomial systems will be discussed in the next section.

Now assume that P, with m joint zeros, is *not* an ill-conditioned system so that well-conditioned representations of $\mathcal{R}[\mathcal{I}[P]]$ must exist. Yet there may be closed convex sets of m monomials which represent $\mathcal{R}$ in an extremely sensitive fashion and thus are ill-suited as a basis for $\mathcal{R}$. How this may happen is easily seen:

For a specified normal set $\mathcal{N}$ and normal set vector $\mathbf{b}(x)$, we have (cf. (2))

$$bb_\nu \;=\; x^{k_\nu} - a^T_{k_\nu}\,\mathbf{b}(x) \;\in\; \mathcal{I}[P]\,, \quad \nu = 1(1)N\,,$$

which implies, for each ν,

$$z_\mu^{k_\nu} \;=\; a^T_{k_\nu}\,\mathbf{b}(z_\mu)\,, \quad \mu = 1(1)m\,, \tag{7}$$

where the z_μ are the m zeros of P. The extension to multiple zeros is straightforward but quite technical; we refer the reader to [14, Sect. 8.5]. By (7), the coefficient vectors $a_{k_\nu}^T$ can only be well-defined if the matrix

$$\mathbf{b}(z) := \begin{pmatrix} | & & & | \\ \mathbf{b}(z_1) & \dots & \dots & \mathbf{b}(z_m) \\ | & & & | \end{pmatrix} \in \mathbb{C}^{m\times m} \tag{8}$$

is well-conditioned. Clearly, the condition of $\mathbf{b}(z)$ depends strongly on the relation between the zero set $\{z_\mu,\ \mu = 1(1)m\}$ and the chosen normal set $\mathcal{N}$; it may differ considerably for different m-element normal sets.

Since the zeros z_μ are unknown at the time of the computation of the $a_{k_\nu}^T$, we cannot form $\mathbf{b}(z)$ explicitly. But we must take an ill-conditioning which appears in the computation of some $a_{k_\nu}^T$ as indication of an ill-conditioned $\mathbf{b}(z)$ and hence of an ill-suited normal set $\mathcal{N}$.

An ill-conditioning can already appear during the *autoreduction* phase through the choice of the support-leading monomials which strongly determine the normal sets which are admissible; cf. Sect. 4. Due to the dominant role which they play in the subsequent computation of the remaining $a_{k_\nu}^T$, the coefficient of the final support-leading monomial x^{ℓ_ν} of each autoreduced polynomial p_ν, $\nu = 1(1)s$, must not be very small relative to the coefficients in that p_ν.

If one or several of these coefficients should be tiny, we should change the (possibly recursive) selection of the support-leading monomials. Often it will be discernible which choice has introduced the ill-conditioning. In any case, the number of possible selections is generally quite limited.

Example 6.1. Assume the situation of Examples 4.1/5.1. We now specify the highest order terms of p_1 and p_2:

$$p_1(x_1,x_2) = -.75\,x_1^2 + 2.1\,x_1x_2 + x_2^2 + \dots,$$
$$p_2(x_1,x_2) = x_1^3 - .3\,x_1^2x_2 + 4.1\,x_1x_2 + 2.6\,x_2^3 + \dots;$$

we aim for the final support-leading monomials of version (ii) of these examples which lead to the normal set of Fig. 3 (ii). However, reduction of p_2 by p_1 as in Example 4.1 (ii) leads to a p_2 with the 3rd order terms $-.02\,x_1^3 - 1.206\,x_1^2x_2$, where the coefficient of the proposed support-leading monomial x_1^3 is quite small. Here, the relief is straightforward: for the same final p_2, we use $x_1^2x_2$ as support-leading monomial which leads to the normal set of Fig. 3 (i).

During the computation proper, ill-conditioning may appear in the numerical solution of the blocks of linear equations for groups of $a_{k_\nu}^T$; cf. Example 5.1. This means that the relations (4) entering into that linear system are near-dependent. If they have arisen from edges of a *minimal* border web, this indicates that the selected normal set $\mathcal{N}$ furnishes an ill-conditioned basis of the quotient ring $\mathcal{R}[\mathcal{I}[P]]$ and that we should switch to another normal set. Naturally, we want to preserve as much as possible of the previous computation in that switch. This may generally be achieved in the following way:

We append one of the current border monomials to the normal set in exchange for one of the current normal set monomials. As candidates for this "degradation" we consider, at first, those border monomials which have figured in a relation (4) of the ill-conditioned block of equations. But the switch requires that the inclusion of that border monomial does not violate the *closedness* of $\mathcal{N}$; this means that it must not be a multiple of another border element. Usually, this decreases the number of candidates considerably; in some cases, the candidate set must even be extended to border monomials involved in an earlier step of the algorithm.

Similarly, the normal set monomial to be "upgraded to border" must not possess a multiple in the new normal set $\mathcal{N}'$. Of course, the switch may also generate further new border elements whose coefficient vectors, in the new basis $\mathcal{N}'$, have to be computed. But before we proceed to do this, we must *rewrite* the previously computed coefficient vectors into the new basis $\mathcal{N}'$, with normal set vector $\mathbf{b}'$.

Instead of an explanation of this rewriting procedure in general terms, we explain it in the context of our previous simple example. This will make it clear how to proceed in more realistic cases.

Example 6.2. Take the situation of Example 4.1/5.1 version (i) and consider the second step which yields the 2 equations for the coefficient vectors of $x_1^3x_2$ and x_1^4. Assume that this 2×2 system is ill-conditioned, i.e. that $\alpha_{(1,2),6} \approx \alpha^2_{(2,1),6}$. A scrutiny of the normal set of Fig. 3 (i) shows that only one of the 3 natural candidates $x_1x_2^2$, $x_1^2x_2$ and $x_1^3x_2$ is not a multiple of another border monomial, viz. $x_1^2x_2$. Also, the only normal set element without a multiple in the new normal set $\mathcal{N}'$ is x_1^3 which therefore becomes a border element. $\mathcal{N}'$ is now the normal set of Fig. 3 (ii) with the normal set vector $\mathbf{b}' = (1, x_1, x_2, x_1^2, x_1x_2, x_1^2x_2)^T$ which differs from $\mathbf{b} = (1, x_1, x_2, x_1^2, x_1x_2, x_1^3)^T$ only in the last (6th) component.

Therefore we introduce the truncated vector $\hat{\mathbf{b}} := (1, x_1, x_2, x_1^2, x_1x_2, 0)^T$ and write the polynomials $a_j^T\,\mathbf{b}$ as $aj^T\,\hat{\mathbf{b}} + \alpha_{j,6}x_1^3$. Thus, all we need is a representation of the new border monomial x_1^3 in terms of $\hat{\mathbf{b}}$ and the new normal set monomial $x_1^2x_2$. It is obtained by the inversion of

$$x_1^2x_2 \;=\; a^T_{(2,1)}\,\mathbf{b} \;=\; a^T_{(2,1)}\,\hat{\mathbf{b}} + \alpha_{(2,1),6}\,x_1^3$$

$$\text{into} \qquad x_1^3 \;=\; \frac{1}{\alpha_{(2,1),6}}\,[-a^T_{(2,1)}\,\hat{\mathbf{b}} + x_1^2x_2\,]\,.$$

Thus the original representation of the border element $x_1x_2^2$

$$x_1x_2^2 \;\equiv\; a^T_{(1,2)}\,\hat{\mathbf{b}} + \alpha_{(1,2),6}\,x_1^3 \qquad \text{becomes}$$

$$x_1x_2^2 \;\equiv\; (\,a^T_{(1,2)} - \frac{1}{\alpha_{(2,1),6}}\,a^T_{(2,1)}\,)\,\hat{\mathbf{b}} + \frac{\alpha_{(1,2),6}}{\alpha_{(2,1),6}}\,x_1^2x_2 \;=:\; (a'_{(1,2)})^T\,\mathbf{b}'\,.$$

Of course, in a more realistic situation, further adaptations of the normal set may be necessary to reach a well-conditioned basis for $\mathcal{R}[\mathcal{I}[P]]$ which must exist if P is well-conditioned. Also, if *no* ill-conditioned blocks arise during a border basis computation, we may conclude that the computed border basis $\mathcal{B}_{\mathcal{N}}$ is a well-conditioned basis for $\mathcal{I}[P]$.

7. Near-Singular Situations

A polynomial system with as many polynomials as variables is called *singular* if it
- either has (only isolated) but *fewer zeros* than its BKK-number m indicates,
- or it possesses a *zero manifold*.

This agrees with the usage for *linear* systems where the BKK-number is always 1; there it is also obvious that both cases are only the two sides of the same coin (singular matrix). With exact polynomials, either situation can only arise for very special precise values of the coefficients. Numerically, like in numerical linear algebra, one will generally meet only near-singular systems which are *very close* to an exact singular system.

Treated as exact systems, such near-singular systems are extremely ill-conditioned. The rapid movements of some of the m disjoint zeros during an assumed transition into an exactly singular system have been analyzed in [14, Sects. 9.4 and 9.5]. With *empirical systems* whose coefficients have a limited accuracy, it is generally more meaningful to assume that a very-nearly-singular systems stands for a nearby strictly singular one. With this taken into account, the ill-conditioning disappears and an algorithmic treatment becomes feasible. This is in analogy with the situation for dense clusters and multiple zeros: if a dense cluster of $\bar{m}$ zeros is treated as an $\bar{m}$-fold zero of a nearby system, the determination of this zero and of its structure becomes well-conditioned, cf. [5] and [14, Sect. 9.3].

Let us first consider the case of fewer than m zeros where some zeros have "diverged" to infinity, as was already remarked by D. Bernstein [1]. Here, for *any* m-element normal set, the attempt to compute a border basis must meet with a *singular, inconsistent* block of linear vector equations. If this fact is computationally well established, we may assume that we have a BKK-deficient system and treat it as such:

We consider a particular m-element normal set and the associated singular block which has arisen; we assume at first that its numerical rank deficiency is 1. Then there exists one linear combination of the linear vector equations which annihilates the terms with the unknown vectors and thus furnishes a *linear relation between the monomials of* $\mathcal{N}$. We select an element x^{j_μ} of $\mathcal{N}$ whose upgrading into border is feasible and detach it from $\mathcal{N}$ to generate an $(m-1)$-member normal set $\mathcal{N}'$; then we obtain the coefficient vector $(a')^T_{j_\mu}$ of x^{j_μ} w.r.t. $\mathcal{N}'$ by solving the linear relation in $\mathcal{N}$ for x^{j_μ}. Previously specified or computed coefficient vectors of border elements w.r.t. $\mathcal{N}$ are converted by the substitution of $(a')^T_{j_\mu}\,\mathbf{b}'$ for x^{j_μ} in their normal forms.

The computation may then be continued and — hopefully — completed, resulting in a border basis $\mathcal{B}_{\mathcal{N}'}$ w.r.t. $\mathcal{N}'$, whose $(m-1)\times(m-1)$ multiplication matrices furnish the $m-1$ zeros of a strictly BKK-deficient system P' very close to our specified system P. These zeros will generally be excellent approximations of the $m-1$ exact zeros of P with moderate modulus while the remaining exact zero lies extremely far from the origin and may therefore be considered as *diverged to* ∞ for practical purposes; cf. [14, Sect. 9.5].

A higher BKK-deficiency will become apparent either through a higher rank deficiency of some singular block or through a sequence of rank deficiencies 1 as above.

As previously, we explain details of the algorithmic procedure by demonstration with a simple example:

Example 7.1. Again we take $s = 2$ and polynomials of degrees 2 and 3 resp., but here we have to specify the coefficients: $P = \{p_1, p_2\}$, with

$$p_1(x_1, x_2) := 100\,x_1^2 - 15\,x_1x_2 - 76\,x_2^2 + 25\,x_1 - 15\,x_2 + 1\,,$$
$$p_2(x_1, x_2) := x_1^2x_2 - .95\,x_1x_2^2 - 2\,x_1^2 + x_1x_2 - .5\,x_2^2 + 4\,x_1 - 5\,x_2 + 3\,.$$

The BKK-number of P is 6. The following computations have been performed with Maple 9.5, with `Digits:=10`.

We autoreduce p_2 and choose $\mathcal{N}$ as in Example 4.1 (ii); then we proceed as in Example 5.1 (ii). But in the joint computation of $a_{(3,1)}^T$ and $a_{(2,2)}^T$ we meet a numerically singular matrix, with a determinant of $\mathrm{O}(10^{-10})$. The attempt to overcome the extreme ill-conditioning by a switch of monomials between $\mathcal{N}$ and $B[\mathcal{N}]$ as described in Sect. 6 brings no relief; the singularity perseveres. Thus we (correctly) assume that P is BKK-deficient, with deficiency 1.

The only detachable monomial of $\mathcal{N}$ is $x_1^2x_2$ for which we obtain a representation w.r.t $\mathcal{N}' := \mathcal{N} \setminus \{x_1^2x_2\}$ from the linear relation between the right-hand sides of the singular system. Thus, instead of determining the vectors $a_{(3,1)}^T$ and $a_{(2,2)}^T$, the singular 2-block has determined the *one* vector $(a')_{(2,1)}^T$ only. It remains to substitute $(a')_{(2,1)}^T\,\mathbf{b}'$ into the representation of the other $\mathcal{N}'$-border monomials. This completes the task.

The 5 zeros computed from the eigenvectors of the associated multiplication matrices are very good approximations of the exact zeros of the autoreduced system P. With the use of high accuracy, the missing 6th zero of this very-near-singular system is discovered at approx. $(1.1\cdot10^{10}, 1.1\cdot10^{10})$ which may be regarded as ∞.

Let us now consider the case where an apparently regular system, with BKK-number m, possesses a *zero manifold* of dimension $d > 0$ while systems arbitrarily close to it have only the m isolated zeros (counting multiplicities). The analysis of what happens when the coefficients of such a system move from regular to singular values is very interesting (cf. [14, Sect. 9.4]); but we will not discuss it here. In the context of this paper, we want to understand how this type of singularity manifests itself in the computation of a border basis so that we may recognize a very-near singular system of this kind. In applications, such systems which are extremely ill-conditioned should be identified with a nearby strictly singular system. Often, the fact that a situation admits a manifold of solutions may be the most important result of the analysis of the model.

For a positive-dimensional polynomial system P, the associated quotient ring has vector dimension *infinity* so that a monomial basis must have infinitely many elements. In our visualization of normal sets by the exponents of the monomials, it means that these exponents most cover a complete subspace of dimension d. Fig. 4 shows a typical normal set for a 1-dimensional system with 2 variables; the 4 monomials *above* the infinite sequences reveal the existence of 4 isolated zeros besides the manifold.

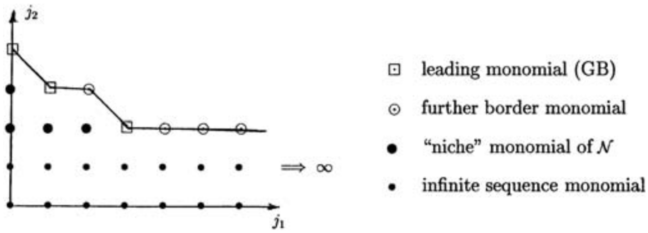

FIGURE 4

The Gröbner basis for this normal set consists of 3 polynomials, with leading monomials $x_2^4, x_1x_2^2, x_1^2x_2$; the absence of a leading monomial on the x_1-axis indicates the 1-dimensionality. Multiplication matrices for this basis formed in the usual manner would have infinitely many rows and columns; but it can be shown that a finite rectangular section of the matrices contains the complete information about the zeros; cf. [14, Chapter 11]. It is not yet clear how border bases could be defined in a meaningful way in this situation; cf. conclusions in [6].

In such a system, we should hit upon a block of linear vector equations which is *singular but consistent.* This means that the coefficient vectors $a_{k_\nu}^T$ of one of the border monomials x^{k_ν} figuring in that block may be moved into the normal set *without* simultaneous conversion of a normal set monomial into a border monomial. For a 0-dimensional system, there cannot exist *more* than m normal set monomials; hence the system must actually be positive-dimensional!

With the availability of further elements in the normal set, it should now be possible to extend $\mathcal{N}$ further and further in a coordinate direction. Very soon, this computation will become recursive and need not be continued. The coefficient vectors for other remaining border monomials should be computable in the normal fashion. A complete analysis of this situation is still in the future. We will again restrict ourselves to the discussion of an example.

Example 7.2. We consider the system

$$\begin{aligned} p_1(x_1,x_2,x_3) &:= x_1^2 + x_1x_2 - x_1x_3 - x_1 - x_2 + x_3\,, \\ p_2(x_1,x_2,x_3) &:= x_1x_2 + c\,x_2^2 - x_2x_3 - x_1 - c\,x_2 + x_3\,, \\ p_3(x_1,x_2,x_3) &:= x_1x_3 + x_2x_3 - x_3^2 - x_1 - x_2 + x_3\,; \end{aligned}$$

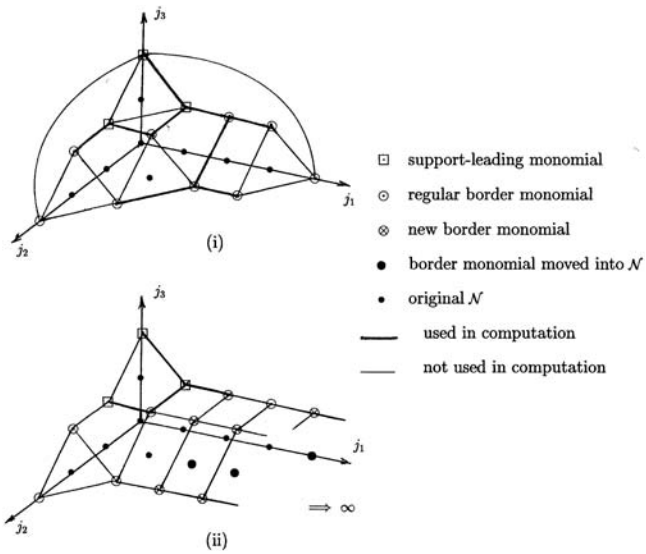

FIGURE 5

the parameter c remains indeterminate at first. We select the support-leading monomials according to the `tdeg`(x_3, x_2, x_1) order as $x_1x_3,\ x_2x_3,\ x_3^2$ resp. With $m{=}8$, we obtain the (apparent) normal set $\mathcal{N} := \{1,\ x_1,\ x_2,\ x_3,\ x_1^2,\ x_1x_2,\ x_2^2,\ x_1^3\}$, with a border set of 12 monomials. The complete border web has 21 edges, only $12-3=9$ of which should figure in a minimal web; cf. Fig. 5 (i). Until now, there is now immediate sign of the singularity.

Starting with the support-leading nodes, we obtain the following 4 by 4-block for $a^T_{(2,1,0)}, a^T_{(1,2,0)}, a^T_{(2,0,1)}, a^T_{(1,1,1)}$ from the type (ii) relation (4) for the edge $[x_3^2, x_1x_3]$ and the type (i) relations along the edges $[x_1x_3, x_1^2x_3]$, $[x_1x_3, x_1x_2x_3]$ and $[x_2x_3, x_1x_2x_3]$:

$$\begin{pmatrix} -2 & -c & 1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ -1 & -c & 0 & 1 \end{pmatrix} \begin{pmatrix} a^T_{(2,1,0)} \\ a^T_{(1,2,0)} \\ a^T_{(2,0,1)} \\ a^T_{(1,1,1)} \end{pmatrix} \mathbf{b}$$

$$= \begin{pmatrix} -2 & -2 & 2 & 0 & 1-c & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -c & 1 & 0 & 0 & c-1 & 0 \\ -1 & -1 & 1 & 0 & 1-c & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1^3 \end{pmatrix}.$$

It is easily seen that both sides of the equation are annihilated by pre-multiplication with $(-1,\ 1,\ 0,\ 1)$, for any value of the parameter c; therefore, we actually have a

1-dimensional singular system. (In this simple example, it is also not difficult to see that there are two lines of zeros: $x_1 = x_3$, $x_2 = 0$ and $x_1 = 1 - x_3$, $x_2 = 1$ for all values of c, with special situations for $c = 0$ and $c = 1$.)

Of the border monomials in the above equation, only $x_1^2 x_2$ and $x_1 x_2^2$ are eligible for transfer into the normal set. We choose $x_1^2 x_2$ and can now determine the coefficient vectors $a_{(1,2,0)}^T$, $a_{(2,0,1)}^T$ and $a_{(1,1,1)}^T$ uniquely (except for $c = 1$) in terms of the enlarged normal set. Also, we have two new border monomials: $x_1^2 x_2^2$ and $x_1^2 x_2 x_3$. It turns out that $a_{(2,1,1)}^T$ can only be computed if the potential border monomial $x_1^3 x_2$ is also included into the normal set, cf. Fig. 5 (ii). Thus, the situation becomes recursive and will continue to infinity.

Note that a border basis for the same system, with generically perturbed coefficients, can readily be computed for the normal set chosen above; cf. Fig. 5 (i).

Numerically, the *very-near-singular* case will be more important. It is now clear that it may be recognized by a block which is a tiny perturbation of a singular consistent set of vector equations. An exact solution would be extremely sensitive and therefore ill-determined in this case; actually, except in the case of confluence, some of the exact isolated zeros of a very-near-singular system of this kind may approach *any* point on the singular manifold in the transition to strict singularity (cf. [14, Sect. 4])! Therefore, the determination of that manifold is generally far more relevant than the computation of "exact" zeros.

8. Implementation

It is a main purpose of this publication to stimulate attempts towards an implementation of the algorithmic procedure described in the previous sections. This will permit large-scale experimentation on non-trivial problems which, in turn, will lead to a preliminary assessment of the potential efficiency of our approach and of its numerical stability.

An implementation will have to proceed in two stages:

In the first stage, exact data and exact (rational) computation will be assumed and used. Thus, the emphasis will lie on the automatization of the decisions which have to be made to determine the algorithmic flow; numerical stability will not be a topic in this stage. In the second stage, floating-point data and floating-point computation will be assumed so that the numerical condition of the individual steps has to be checked; this will introduce further restrictions on the choices and also potential changes in the algorithmic flow.

It is the overall strategic goal of the organizational phase of our algorithm to generate a normal set whose border web permits the recursive selection of sets of syzygies which lead to blocks of linear vector equations for as many coefficient vectors $a_{j_\nu}^T$ as there are equations, until the representations of all border monomials have been found. From Sect. 2, we remember that *linear* equations are generated

- either by type (i) edges $[x^{j_\nu}, x^{j_\nu+e_\sigma}]$, with $a_{j_\nu}^T$ known[1],

- or by type (ii) edges $[x^{k_\mu+e_{\sigma_1}}, x^{k_\mu+e_{\sigma_2}}]$, $x^{k_\mu} \in \mathcal{N}$, with $a_{k_\mu+e_{\sigma_1}}^T$ *and* $a_{k_\mu+e_{\sigma_2}}^T$ known.

Thus, our computation can only get started, if there are type (i) edges issuing from the support-leading nodes and/or type (ii) edges connecting two support-leading nodes. Also, the overall occurrence of many type (i) edges in the web should help in the continuation of the computation. It will require some ingenuity to build a strategy for an optimal satisfaction of these requirements into the initial phase of the algorithm which chooses the support-leading monomials for the autoreduction, and then selects an m-element closed normal set consistent with the supports and the final support-leading monomials.

Besides, it is an open question whether there do exist normal sets for *arbitrary* regular polynomial systems which support the computation of the complete border basis by our approach; efforts towards an implementation may help to answer that question. Since we *know* from GB theory that there exist normal sets with border bases whose coefficients are rational functions of the data, a positive answer does not appear impossible.

Naturally, the unfamiliar choices which have to be made stem from the fact that we work *without a term order*. Actually, this is one of the major attractions of our approach, particularly when numerical stability is also an issue. As remarked in Sect. 4, when we choose the support-leading monomials as leading monomials w.r.t. a term order, we will generally obtain the reduced GB for that term order as the "corner subset" of the border basis, perhaps after enforced changes in the initial normal set. With this restricted set of normal sets, an answer to the question whether our approach will always succeed, or for which classes of systems, would be of particular interest. In Example 7.2, we have employed the term order $\texttt{tdeg}(x_3, x_2, x_1)$; our algorithm has worked in the singular as well as in the regular case (with perturbed coefficients) and generated the GB.

While the selection of syzygies for the beginning of the computation is determined by the support-leading monomials with their specified normal forms, the further continuation of the computation is not always clear. Probably, for an algorithm, one should simply form all linear syzygies whose edges do not close loops, and then look for blocks with as many equations as unknown vectors. If there is no such block, the algorithm cannot be continued. In this case, a feasible exchange between a normal set and a border monomial may often open the deadlock so that a full restart with a different normal set is not necessary. How to check that algorithmically, with a meaningful strategy, is not clear.

In the second stage of the design of an implementation of our approach, an algorithm which complies with all the considerations so far must now be adapted to the use of floating-point arithmetic. To my knowledge, none of the presently available GB-packages can handle that situation (or only with a ridiculous digit

[1]Here, e_σ is the σ-th unit vector.

swell, when floating-point data are converted to rational data and exact computation is then employed). But models of real-life situations have floating-point data, with a limited accuracy, in almost all cases!

Now, the considerations in Sects. 6 and 7 must also be implemented. As usual, the threshold for what is called ill-conditioned is arbitrary to some extent: with a 10 digit computation, e.g., relative condition numbers should supposedly not exceed 10^6 or so. If a computation shows signs of numerical instability, the floating-point accuracy should be raised with care (the algorithm must provide for that). When the results do not agree for higher accuracies, the system itself probably defines its ideal in such an ill-conditioned fashion that a determination is not meaningful.

The algorithmic checking of the relative sizes of data must begin with the selection of the support-leading terms; their coefficients must not be much smaller in modulus than those of the remaining terms. If there is only one highest degree term with a very small coefficient, one must attempt to get rid of it during autoreduction; if this is not possible the specified system itself is ill-conditioned.

During the further computation, the condition of the linear equation blocks must be checked, and the exchange mechanism of Sect. 6 must be used if a condition number is too high relative to the floating-point accuracy. Now there is generally a very small set of candidates for both parts in the exchange; if there is still a choice, the generation of fewer or more computable edges may be used as a criterion.

This holds also for the cases where the ill-conditioning is not due to an ill-chosen normal set but to a near-singularity in the specified system. Now, the ill-conditioning will not disappear with an exchange, or the situation does not permit an exchange. One should then proceed as discussed in Sect. 7.

Once more it is an open question whether there may exist situations where the stabilized algorithm runs into a dead end while the instable computation would have succeeded. Note also that the exchange mechanism in Sect. 6 generally proceeds just like the "extension" mechanism which had been proposed by myself many years ago (cf. [13]); but now we are not violating the term order as with the "extended GB" approach because there is no term order which restricts us. This shows once more that numerical stability in the computation of an ideal basis can only be realized without a pre-specified term order.

9. Conclusions

I have tried to show how the use of the BKK-number opens the way for a new algorithmic approach to the numerical computation of a border basis for a regular polynomial system. This approach employs *no term order* and *no reductions to zero*. The first feature permits more flexibility in the representation of the quotient ring while the second feature permits a stable implementation in floating-point arithmetic. Both features should make the approach attractive.

It should be interesting to compare this approach with the one of [12] and to analyze the similarities and differences. In the approach of [12], one begins with a tentative normal set which is generally too large and has to be successively reduced during the computation of normal forms for border elements.

No serious implementation of my approach exists so far; it is hoped that this publication may lead to implementations in the near future. Only then, the potential of the approach may seriously be assessed. In particular, it should then become clear whether the approach can be made to work for arbitrary regular systems or only for certain subclasses of such systems and how its efficiency compares with other recent approaches.

In any case, on the way to an implementation new insights about border bases should come to light, with potential applications in other parts of numerical polynomial algebra.

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