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On the Location of Zeros of an Interval Polynomial

Hiroshi Sekigawa and Kiyoshi Shirayanagi

Abstract. For an interval polynomial F, we provide a rigorous method for deciding whether there exists a polynomial in F that has a zero in a prescribed domain D. When D is real, we show that it is sufficient to examine a finite number of polynomials. When D is complex, we assume that the boundary C of D is a simple closed curve of finite length and C is represented by a piecewise rational function. The decision method uses the representation of C and the property that a polynomial in F is of degree one with respect to each coefficient regarded as a variable. Using the method, we can completely determine the set of real numbers that are zeros of a polynomial in F. For complex zeros, we can obtain a set X that contains the set $Z(F)$, which consists of all the complex numbers that are zeros of a polynomial in F, and the difference between X and $Z(F)$ can be as small as possible.

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1. Introduction

There are two premises for incorporating numeric or approximate computation in symbolic computation. One is that we know the exact values but use approximate computation for efficiency. An example is the theory of stabilizing algebraic algorithms [11, 12, 13]. The other is that inexact values are given.

In this article, we consider problems on the latter premise. That is, we treat the problems regarding zeros of real polynomials with perturbations. More precisely, let $[l_j, h_j] \subset \mathbb{R}$ be bounded closed intervals for $0 \leq j \leq d$. We consider the following types of problems.

- Does there exist a polynomial $f = a_d x^d + \cdots + a_0$ such that each a_j lies in the interval $[l_j, h_j]$ and f has a zero in the prescribed real (or complex) domain?
- What is the union of the sets of (real) zeros of polynomials $f = a_d x^d + \cdots + a_0$ such that each coefficient a_j lies in the interval $[l_j, h_j]$?

A similar but a slightly different problem on real zeros is treated in [5]. For a given real polynomial f that has no real zero, [5] provides an algorithm that finds the nearest real polynomial to f in the infinity norm among polynomials having a real zero. For complex domains, similar problems have been studied for complex polynomials (see, for example, [9, 4, 6, 2]). In such studies, complex perturbations are considered, and this is quite natural, since coefficients are complex. As described in [9, 7, 14], these studies can be viewed and understood in the common framework of fundamental observation from linear algebra.

Considering applications, we are very interested in the above types of problems since coefficients of polynomials may contain errors. For real polynomials, it is natural to consider only real perturbations, since in many practical examples real coefficients are obtained through measurements or observations and the errors are also real numbers. The methods in this article do not assume that perturbations are small except in several cases where small perturbations must be assumed so that the leading coefficient does not vanish.

It is also natural to consider only real zeros for many applications. Therefore, we treat real zeros in the first half and complex zeros in the second half. For real zeros, we provide a rigorous method for determining whether there exists a polynomial whose coefficients lie in the intervals $[l_j, h_j]$ and whose zero lies in the prescribed interval. We show that it is enough to examine only a finite number of polynomials. For complex zeros, we provide a rigorous method that first follows [9, 7, 14] but in the end differs from them because the perturbations are real. This type of research has already been carried out in control theory [3] and some results, such as Kharitonov's Theorem [8] and the Edge Theorem [1], have been obtained. The Edge Theorem is closely related to our main results on complex zeros; therefore, we will describe the relation between them.

This article is organized as follows. Section 2 introduces the notion of interval polynomials that can describe a set of polynomials with perturbations. Section 3 describes the decision method for real zeros. Section 4 describes the decision principle and computation methods for complex zeros and the relation with the Edge Theorem. Finally, Section 5 mentions future directions.

2. Definitions and Notations

In this section, we introduce interval polynomials to describe a set of polynomials with perturbations and pseudozeros to describe zeros of interval polynomials.

Definition 1 (Interval polynomials). For $1 \le j \le n$, let $e_j(x)$ be a monic polynomial in $\mathbb{R}[x]$ and $A_j = [l_j, h_j] \subset \mathbb{R}$ be a bounded closed interval. The set of polynomials

$$\left\{ \sum_{j=1}^{n} a_j e_j(x) \;\middle|\; a_j \in A_j \right\} \qquad (*)$$

is said to be an interval polynomial. A_j is said to be an interval coefficient.

For simplicity, the set described by $(*)$ may be denoted as follows:

$$A_1e_1(x) + A_2e_2(x) + \cdots + A_ne_n(x).$$

Note that an interval polynomial F is a convex set from the definition.

Definition 2 (Pseudozeros). Let F be an interval polynomial. We define a point $c \in \mathbb{C}$ as a pseudozero of F if and only if there exists $f \in F$ such that $f(c) = 0$. We write all pseudozeros of F as $Z(F)$. A pseudozero c of F is said to be a real pseudozero if c is real. We write all real pseudozeros of F as $Z_{\mathbb{R}}(F)$.

When computing, we restrict the real and the imaginary parts of numbers to rational numbers or real algebraic numbers and use exact computation unless mentioned otherwise.

3. Deciding the Set of Real Pseudozeros

In this section, for an interval polynomial F, we provide a method for determining real pseudozeros of F. The fundamental tool is a method for determining whether there exists a polynomial $f \in F$ such that f has a zero in a given closed interval $D = [d_1, d_2]$ in $\mathbb{R}$. When $d_1 = d_2$, this can be determined by using interval arithmetic with exact computation for endpoints. When $d_1 < d_2$, the following lemma is the fundamental tool.

Lemma 1. *Let F be an interval polynomial as described by $(*)$. Suppose that every e_j has no zero in the interior of the interval D. Then, each e_j is always positive or negative in the interior of D. We denote by P the set of all indices j such that $e_j > 0$ and by N the set of all indices j such that $e_j < 0$. We put*

$$f_l(x) = \sum_{j\in P} l_je_j(x) + \sum_{j\in N} h_je_j(x), \qquad f_h(x) = \sum_{j\in P} h_je_j(x) + \sum_{j\in N} l_je_j(x).$$

Then, two polynomials f_l and f_h belong to F.

1. *If at least one of $f_l(d_1)$, $f_l(d_2)$, $f_h(d_1)$ and $f_h(d_2)$ is 0, or there exists a pair with opposite signs, then there exists $f \in F$ such that f has a zero in D.*
2. *When $f_l(d_1)$, $f_l(d_2)$, $f_h(d_1)$, $f_h(d_2) > 0$, there exists $f \in F$ such that f has a zero in D if and only if f_l has a zero in D.*
3. *When $f_l(d_1)$, $f_l(d_2)$, $f_h(d_1)$, $f_h(d_2) < 0$, there exists $f \in F$ such that f has a zero in D if and only if f_h has a zero in D.*

Proof. Note that $f_l(c) \le f(c) \le f_h(c)$ hold for any $f \in F$ and any $c \in D$.

First we prove Case 1. When one of $f_l(d_j)$ and $f_h(d_j)$ is 0, the statement is clear. If $f_l(d_1)f_l(d_2) < 0$ (resp. $f_h(d_1)f_h(d_2) < 0$), then the intermediate value theorem implies the statement. Suppose that $f_l(d_1)f_l(d_2) > 0$ and $f_h(d_1)f_h(d_2) > 0$ (see Fig. 1). Then, the sign of $f_l(d_1)$ and that of $f_h(d_1)$ should be opposite; otherwise all of the signs of $f_l(d_j)$ and $f_l(d_j)$ are the same. Put

$$t = \frac{f_h(d_1)}{f_h(d_1) - f_l(d_1)}.$$

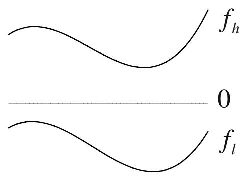

FIGURE 1. An example of the case $f_l(d_1)f_l(d_2),\ f_h(d_1)f_h(d_2) > 0$.

Then, the polynomial $g = (1-t)f_h + tf_l$ is in F and has a zero at d_1.

Since the proofs for Cases 2 and 3 are similar, we only show that of the former. Suppose that a polynomial $f \in F$ has a zero at c in the interval $[d_1, d_2]$. Then, $d_1 < c$ because $f_l(c) \le f(c) = 0$ and $f_l(d_1) > 0$. Therefore, the intermediate value theorem implies that f_l has a zero in the interval $[d_1, c]$. □

Remark 1. Arguments similar to the proof of Lemma 1 are valid for intervals $(-\infty, d_2]$, $[d_1, \infty)$, and $(-\infty, \infty)$ if no zero of e_j exists in the interior.

Remark 2. Under the same assumption of Lemma 1, every $f \in F$ has a zero in D if and only if both f_l and f_h have a zero in D.

Using Lemma 1, we can determine the real pseudozeros as follows.

Theorem 1. *Let F be an interval polynomial as described by $(*)$. Let all of the distinct real zeros of $\prod_{j=1}^{n} e_j$ be $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. We make intervals $D_0 = (-\infty, \alpha_1]$, $D_k = [\alpha_k, \alpha_{k+1}]$ $(1 \le k \le m-1)$ and $D_m = [\alpha_m, \infty)$.*

For the interval D_k, we denote the polynomials corresponding to $f_l(x)$ and $f_h(x)$ described in Lemma 1 by $f_{k,l}(x)$ and $f_{k,h}(x)$. Then, we have

$$Z_{\mathbb{R}}(F) = \bigcup_{k=0}^{m} \left\{ c \in D_k \mid f_{k,l}(c) \le 0 \le f_{k,h}(c) \right\}.$$

Proof. The inequalities $f_{k,l}(c) \le f(c) \le f_{k,h}(c)$ hold for any $c \in D_k$, $f \in F$ and $(1-t)f_{k,l} + tf_{k,h} \in F$ for any t $(0 \le t \le 1)$. These facts imply the statement. □

Corollary 1. *$Z_{\mathbb{R}}(F)$ is the union of a finite number (possibly zero) of closed intervals whose types are as follows:*

- *A closed interval $[\alpha, \beta]$, where α and β are zeros of $f_{k,l}$ or $f_{k,h}$.*
- *A closed interval $(-\infty, \alpha]$ or $[\alpha, \infty)$, where α is a zero of either $f_{k,l}$ or $f_{k,h}$.*
- *The whole real numbers $\mathbb{R}$.*

When the degrees of all polynomials in F are equal, only the first type appears.

Proof. Let m be the number of the distinct real zeros of $\prod_{j=1}^{n} e_j$. We take all of the distinct zeros of $f_{k,l}(x)$ and $f_{k,h}(x)$ in D_k described in Theorem 1 and denote them by $\beta_{k,1} < \beta_{k,2} < \cdots < \beta_{k,n(k)}$. Then, the signs of $f_{k,l}(x)$ and $f_{k,h}(x)$ do not change in the interval $(\beta_{k,p}, \beta_{k,p+1})$. The signs also do not change in the interval $(-\infty, \beta_{0,1})$ for $k = 0$ nor in the interval $(\beta_{m,n(m)}, \infty)$ for $k = m$.

Take every interval on which $f_{k,l}(x)$ is negative and $f_{k,h}(x)$ is positive and make it closed by adding the endpoints that are zeros of $f_{k,l}(x)$ or $f_{k,h}(x)$. $Z_{\mathbb{R}}(F)$ is the union of all such closed intervals. □

Example 1 (Lagrange interpolation). In the Lagrange interpolation for m points $a_1 < a_2 < \cdots < a_m$, each monic polynomial $e_j(x)$ is represented as follows:

$$e_j(x) = \prod_{k \neq j} (x - a_k).$$

We can easily compute real pseudozeros since $\{ x \in \mathbb{R} \mid e_j(x) = 0 \} = \{ a_k \mid k \neq j \}$.

Remark 3. The converse of the last part of Corollary 1 is not true. When the degrees are not constant, all types of intervals in Corollary 1 may appear. Consider the two monic polynomials $e_1(x)$ and $e_2(x)$:

$$e_1(x) = x^4 - 5, \qquad e_2(x) = x^4 - x^2.$$

We define three interval polynomials $F(x) \subset G(x) \subset H(x)$ as follows:

$$\begin{aligned} F(x) &= [1,\ 1.5]e_1(x) + [-1,\ -0.5]e_2(x), \\ G(x) &= [1,\ 1.5]e_1(x) + [-1.5,\ -0.5]e_2(x), \\ H(x) &= [0,\ 1.5]e_1(x) + [-1.5,\ 0]e_2(x). \end{aligned}$$

Note that both $e_1(x) - e_2(x)$ and $e_1(x) - e_2(x)/2$ belong to F $(\subset G \subset H)$, and $\deg(e_1(x) - e_2(x)) = 2$ and $\deg(e_1(x) - e_2(x)/2) = 4$. As described below, we have

$$\begin{aligned} Z_{\mathbb{R}}(F) &= \left[-\sqrt{5}, -\sqrt{5/2}\right] \cup \left[\sqrt{5/2}, \sqrt{5}\right], \\ Z_{\mathbb{R}}(G) &= \left(-\infty, -\sqrt{5/2}\right] \cup \left[\sqrt{5/2}, \infty\right), \\ Z_{\mathbb{R}}(H) &= \mathbb{R}. \end{aligned}$$

Real zeros of e_1 are $\pm\sqrt[4]{5}$ and real zeros of e_2 are 0 and ± 1. Since $f(-x) = f(x)$ for any $f \in H$, it is sufficient that we examine only in the interval $[0, \infty)$. We divide $[0, \infty)$ into three intervals: $[0, 1]$, $[1, \sqrt[4]{5}]$ and $[\sqrt[4]{5}, \infty)$. The polynomial e_1 is negative in $[0, \sqrt[4]{5})$ and positive in $(\sqrt[4]{5}, \infty)$. The polynomial e_2 is negative in $(0, 1)$ and positive in $(1, \infty)$.

First we examine G since it is clear that $Z_{\mathbb{R}}(H) = \mathbb{R}$. In the interval $[0, 1]$,

$$g_h(x) = e_1(x) - \frac{3}{2}e_2(x) = -\frac{1}{2}(x^4 - 3x^2 + 10) < 0.$$

In the interval $[1, \sqrt[4]{5}]$,

$$g_h(x) = e_1(x) - \frac{1}{2}e_2(x) = \frac{1}{2}(x^4 + x^2 - 10) < 0.$$

In the interval $[\sqrt[4]{5}, \infty)$,

$$\begin{aligned} g_l(x) &= e_1(x) - \frac{3}{2}e_2(x) = -\frac{1}{2}(x^4 - 3x^2 + 10) < 0, \\ g_h(x) &= \frac{3}{2}e_1(x) - \frac{1}{2}e_2(x) = x^4 + \frac{1}{2}x^2 - \frac{15}{2} = \frac{1}{2}(2x^2 - 5)(x^2 + 3). \end{aligned}$$

Therefore, the set of all real pseudozeros $Z_{\mathbb{R}}(G)$ is $(-\infty, -\sqrt{5/2}] \cup [\sqrt{5/2}, \infty)$, that is, the union of two unbounded closed intervals.

To determine $Z_{\mathbb{R}}(F)$, it is sufficient to examine only the interval $[\sqrt[4]{5}, \infty)$ since F is a subset of G. Now

$$\begin{aligned} f_l(x) &= e_1(x) - e_2(x) = x^2 - 5, \\ f_h(x) &= g_h(x) = \frac{1}{2}(2x^2 - 5)(x^2 + 3). \end{aligned}$$

Therefore, the set of all real pseudozeros $Z_{\mathbb{R}}(F)$ is $[-\sqrt{5}, -\sqrt{5/2}] \cup [\sqrt{5/2}, \sqrt{5}]$, that is, the union of two bounded closed intervals. Note that $\sqrt[4]{5} < \sqrt{5/2}$.

Next, we apply Theorem 1 to Wilkinson's famous example.

Example 2 (Wilkinson). Put $e_1(x) = \prod_{j=1}^{20}(x - j)$ and $e_2(x) = x^{19}$. We consider the following two interval polynomials $F(x) \subset G(x)$:

$$F(x) = e_1(x) + [-2^{-23}, 0]e_2(x), \qquad G(x) = e_1(x) + [-2^{-30}, 0]e_2(x).$$

The "endpoint" polynomial $e_1 - 2^{-23}e_2$ of F is Wilkinson's original example.

First, we consider $F(x)$. Since the signs of e_1 and e_2 do not change in the region $x < 0$, the polynomials f_l and f_h for $x \le 0$ are as follows:

$$f_l(x) = e_1(x), \qquad f_h(x) = e_1(x) - 2^{-23}e_2(x).$$

Since $0 < f_l(x)$ for $x \le 0$, there is no pseudozero in the region.

The interval coefficient of e_1 consists of one point and $0 \le e_2(x)$ for $0 \le x$. Therefore, for $0 \le x$,

$$f_l(x) = e_1(x) - 2^{-23}e_2(x), \qquad f_h(x) = e_1(x).$$

The number of real zeros of f_l is 10 and there is no multiple root. We denote by $\alpha_1 < \alpha_2 < \cdots < \alpha_{10}$ the real zeros. They lie in the intervals as described below:

$$\begin{aligned} &\alpha_1 \in (1 - 10^{-24},\ 1 - 10^{-25}), && \alpha_2 \in (2 + 10^{-18},\ 2 + 10^{-17}), \\ &\alpha_3 \in (3 - 10^{-12},\ 3 - 10^{-13}), && \alpha_4 \in (4 + 10^{-10},\ 4 + 10^{-9}), \\ &\alpha_5 \in (5 - 10^{-7},\ 5 - 10^{-8}), && \alpha_6 \in (6 + 10^{-6},\ 6 + 10^{-5}), \\ &\alpha_7 \in (6.999,\ 6.9999), && \alpha_8 \in (8.001,\ 8.01), \\ &\alpha_9 \in (8.9,\ 8.99), && \alpha_{10} \in (20.1,\ 21). \end{aligned}$$

From the above inequalities,

$$\begin{aligned} \{\, c \in \mathbb{R} \mid 0 \le c,\ 0 \le f_h(c) \,\} &= [0,1] \cup [2,3] \cup [4,5] \cup [6,7] \cup [8,9] \cup [10,11] \\ &\quad \cup [12,13] \cup [14,15] \cup [16,17] \cup [18,19] \cup [20,\infty), \\ \{\, c \in \mathbb{R} \mid 0 \le c,\ f_l(c) \le 0 \,\} &= [\alpha_1,\alpha_2] \cup [\alpha_3,\alpha_4] \cup [\alpha_5,\alpha_6] \cup [\alpha_7,\alpha_8] \cup [\alpha_9,\alpha_{10}]. \end{aligned}$$

Therefore, the set of all real pseudozeros for F is as follows:

$$\begin{aligned} Z_{\mathbb{R}}(F) &= [\alpha_1,1] \cup [2,\alpha_2] \cup [\alpha_3,3] \cup [4,\alpha_4] \cup [\alpha_5,5] \cup [6,\alpha_6] \cup [\alpha_7,7] \cup [8,\alpha_8] \\ &\quad \cup [\alpha_9,9] \cup [10,11] \cup [12,13] \cup [14,15] \cup [16,17] \cup [18,19] \cup [20,\alpha_{10}]. \end{aligned}$$

Similar arguments hold for G. It is clear that $Z_{\mathbb{R}}(G) \cap (-\infty, 0]$ is empty. For $0 \le x$,

$$g_l(x) = e_1(x) - 2^{-30} e_2(x), \qquad g_h(x) = e_1(x).$$

The number of real zeros of $g_l(x)$ is 14 and there is no multiple root. We denote by $\beta_1 < \beta_2 < \cdots < \beta_{14}$ the real zeros. They lie in the intervals as described below:

$$\begin{array}{ll}
\beta_1 \in (1 - 10^{-26},\ 1 - 10^{-27}), & \beta_2 \in (2 + 10^{-20},\ 2 + 10^{-19}),\\
\beta_3 \in (3 - 10^{-14},\ 3 - 10^{-15}), & \beta_4 \in (4 + 10^{-12},\ 4 + 10^{-11}),\\
\beta_5 \in (5 - 10^{-9},\ 5 - 10^{-10}), & \beta_6 \in (6 + 10^{-8},\ 6 + 10^{-7}),\\
\beta_7 \in (7 - 10^{-5},\ 7 - 10^{-6}), & \beta_8 \in (8 + 10^{-5},\ 8 + 10^{-4}),\\
\beta_9 \in (8.999,\ 8.9999), & \beta_{10} \in (10.001,\ 10.01),\\
\beta_{11} \in (10.9,\ 10.99), & \beta_{12} \in (12.3,\ 12.4),\\
\beta_{13} \in (12.4,\ 12.5), & \beta_{14} \in (20.01,\ 20.1).
\end{array}$$

From the above inequalities,

$$\begin{aligned}
\{\, c \in \mathbb{R} \mid 0 \le c,\ 0 \le g_h(c) \,\} &= [0,1] \cup [2,3] \cup [4,5] \cup [6,7] \cup [8,9] \cup [10,11]\\
&\quad \cup [12,13] \cup [14,15] \cup [16,17] \cup [18,19] \cup [20,\infty),\\
\{\, c \in \mathbb{R} \mid 0 \le c,\ g_l(c) \le 0 \,\} &= [\beta_1,\beta_2] \cup [\beta_3,\beta_4] \cup [\beta_5,\beta_6] \cup [\beta_7,\beta_8] \cup [\beta_9,\beta_{10}]\\
&\quad \cup [\beta_{11},\beta_{12}] \cup [\beta_{13},\beta_{14}].
\end{aligned}$$

Therefore, the set of all real pseudozeros for G is as follows:

$$\begin{aligned}
Z_{\mathbb{R}}(G) &= [\beta_1, 1] \cup [2, \beta_2] \cup [\beta_3, 3] \cup [4, \beta_4] \cup [\beta_5, 5] \cup [6, \beta_6] \cup [\beta_7, 7]\\
&\quad \cup [8, \beta_8] \cup [\beta_9, 9] \cup [10, \beta_{10}] \cup [\beta_{11}, 11] \cup [12, \beta_{12}] \cup [\beta_{13}, 13]\\
&\quad \cup [14,15] \cup [16,17] \cup [18,19] \cup [20, \beta_{14}].
\end{aligned}$$

4. Deciding the Location of Pseudozeros

In this section, first we describe a principle for deciding the location of pseudozeros. Let F be an interval polynomial as described by $(*)$ and D be a domain in $\mathbb{C}$. We consider the following problem.

Problem 1. Does there exist a pseudozero of F in D?

Below, we assume that D is a closed domain in $\mathbb{C}$ whose boundary C is a simple curve. When D is not bounded, we further assume that the degree of $f \in F$ is constant. Since the domain D is not bounded when C is not a closed curve, from the above assumption on the degree we can construct a new closed domain $D' \subset D$ such that the following conditions are satisfied.

- The boundary of D' is a simple and closed curve.
- $Z(F) \cap D = Z(F) \cap D'$.

Therefore, we can assume that C is a simple and closed curve. Furthermore, we assume the following conditions.

Condition 1. C is of finite length and $C = \cup_{k=1}^{K} C_k$ ($K < \infty$), where each C_k is expressed by an injective function as

$$\varphi_k(s) + i\psi_k(s), \qquad s \in S_k.$$

Here $\varphi_k(s)$, $\psi_k(s) \in \mathbb{Q}(s)$ and S_k is either of type $[a, b]$, $[a, \infty)$, $(-\infty, b]$ or $\mathbb{R}$.

First, we reduce the problem to examine zeros on C. Second, we reduce it to examine polynomials whose coefficients are the endpoints of the interval coefficients with at most one exception.

4.1. Preliminaries

Take a polynomial $f_0 \in F$. We can determine whether f_0 has a zero on C using Sturm's algorithm, the sign variation method, or some other improved algorithm, and whether it has a zero in the interior of D using the argument principle when it has no zero on C. If f_0 has no zero in D, then Problem 1 is equivalent to asking whether there exists a pseudozero of F on C.

Proposition 1. *Suppose that a polynomial $f_0 \in F$ has no zero in D. When D is unbounded we assume that degrees of all polynomials in F are equal. Then, the following two conditions are equivalent.*

1. *There exists a polynomial $f \in F$ that has a zero in D.*
2. *There exists a polynomial $f \in F$ that has a zero on C.*

Proof. It is sufficient to prove that the first condition implies the second condition. Assume that f has a zero in D but no zero on C. Let g_t be $(1-t)f_0 + tf$. Then, $g_0 = f_0$, $g_1 = f$ and $g_t \in F$ for any t ($0 \le t \le 1$). We prove the statement by contradiction. Suppose that every g_t ($0 \le t \le 1$) has no zero on C.

When D is bounded, Rouché's theorem (see below) implies that the number of zeros of g_0 in D is equal to that of g_1. This contradicts the assumption.

When D is unbounded, the assumption that C is of finite length implies that the compliment D^c of D is bounded. Therefore, the number of zeros of g_0 in D^c is equal to that of g_1. Since $\deg g_0 = \deg g_1$, the number of zeros of g_0 in D is equal to that of g_1. This contradicts the assumption. □

The following is a version of Rouché's theorem.

Theorem 2 (Rouché's Theorem). *Let C be a simple closed curve of finite length in a domain $\Omega \subset \mathbb{C}$ and let the inside of C be in Ω. Suppose that $f(z)$ and $\varphi(z)$ are holomorphic on Ω and that $f(z) + t\varphi(z)$ has no zero on C for any t ($0 \le t \le 1$). Then, the number of zeros of $f(z)$ inside C is equal to that of $f(z) + \varphi(z)$.*

We provide a proof since this version is not described in standard textbooks.

Proof. The following inequality holds for $0 \le t_1 \le t_2 \le 1$:

$$|f(z) + t_2\varphi(z)| \le |f(z) + t_1\varphi(z)| + |t_2 - t_1||\varphi(z)|.$$

Let $m(t)$ be $\min_{z \in C}\{|f(z) + t\varphi(z)|\}$ and M be $\max_{z \in C}\{|\varphi(z)|\}$. Then,

$$m(t_2) \le |f(z) + t_1\varphi(z)| + |t_2 - t_1|M.$$

Therefore, the following inequality holds:

$$m(t_2) \leq m(t_1) + |t_2 - t_1|M.$$

When we interchange t_1 and t_2, the resulting inequality also holds. Therefore,

$$|m(t_2) - m(t_1)| \leq |t_2 - t_1|M.$$

This inequality implies that $m(t)$ is continuous in the interval $0 \leq t \leq 1$. From the hypothesis, $m(t) > 0$ holds for any t in $[0, 1]$. Therefore, $m = \min_{0 \leq t \leq 1}\{|m(t)|\}$ should be positive. Now, we denote the length of C by L and the maximum of $|f(z)\varphi'(z) - f'(z)\varphi(z))|$ on C by G. Let $N(t)$ be

$$\frac{1}{2\pi i}\int_C \frac{f'(z) + t\varphi'(z)}{f(z) + t\varphi(z)}dz.$$

Then, we have

$$|N(t_2) - N(t_1)| = \frac{1}{2\pi}\left|\int_C \frac{(t_1 - t_2)(f(z)\varphi'(z) - f'(z)\varphi(z))}{(f(z) + t_1\varphi(z))(f(z) + t_2\varphi(z))}dz\right| \leq \frac{|t_1 - t_2|GL}{2\pi m^2},$$

which implies that $N(t)$ is continuous on the interval $0 \leq t \leq 1$. Therefore, the equality $N(0) = N(1)$ holds since $N(t)$ is a nonnegative integer for any t. □

4.2. Main Theorem

In this subsection, we prove the main theorem.

Theorem 3. *Let F be an interval polynomial as described by $(*)$ and D be a closed domain whose boundary C satisfies Condition 1. When D is unbounded we assume that degrees of all polynomials in F are equal. Suppose the following conditions.*

- *There exists a polynomial $f_0 \in F$ that does not have a zero in D.*
- *There exists a point $\alpha_0 \in C$ that is not a pseudozero of F.*

Then, the following two conditions are equivalent.

1. *There exists a polynomial $f \in F$ having a zero in D.*
2. *There exists a polynomial $f \in F$ such that f has a zero on C and the number of coefficients a_j of f that are not l_j or h_j is at most one.*

We prove the following lemma for the proof of Theorem 3.

Lemma 2. *Consider the following simultaneous equations with a parameter $z \in \mathbb{C}$,*

$$\begin{cases} a_1(z)x + b_1(z)y = c_1(z), \\ a_2(z)x + b_2(z)y = c_2(z), \end{cases} \tag{1}$$

where $a_j(z)$, $b_j(z)$, $c_j(z)$ are continuous with respect to z. Let Γ be a simple curve of finite length whose two endpoints are z_0 and z_1 $(z_0 \neq z_1)$. Let $d(z)$ be the determinant

$$\begin{vmatrix} a_1(z) & b_1(z) \\ a_2(z) & b_2(z) \end{vmatrix}.$$

Suppose the following conditions:

- $d(z_0) = 0$ *and* $d(z) \neq 0$ *for* $z \in \Gamma \setminus \{z_0\}$.
- *The solutions of* (1) *are bounded for* $z \in \Gamma \setminus \{z_0\}$.

Then, the simultaneous equations (1) *are indeterminate at* $z = z_0$.

Proof. For $z \in \Gamma \setminus \{z_0\}$ we put

$$n_x(z) = \begin{vmatrix} c_1(z) & b_1(z) \\ c_2(z) & b_2(z) \end{vmatrix}, \qquad n_y(z) = \begin{vmatrix} a_1(z) & c_1(z) \\ a_2(z) & c_2(z) \end{vmatrix}.$$

Then, the solution x and y of (1) can be represented as functions of z as follows:

$$x(z) = \frac{n_x(z)}{d(z)}, \qquad y(z) = \frac{n_y(z)}{d(z)}.$$

The functions $n_x(z)$ and $n_y(z)$ converge to 0 since $d(z)$ converges to 0 as z tends to z_0 on Γ and the solution of (1) is bounded in $\Gamma \setminus \{z_0\}$. Therefore,

$$n_x(z_0) = n_y(z_0) = 0, \tag{2}$$

since $n_x(z)$ and $n_y(z)$ are continuous. Furthermore, if $a_j(z_0) = b_j(z_0) = 0$ hold, then $c_j(z_0) = 0$. The reason is as follows. Since there exists a positive number M such that $|x(z)| \leq M$ and $|y(z)| \leq M$ hold for $z \in \Gamma \setminus \{z_0\}$, the inequality

$$|c_j(z)| = |a_j(z)x(z) + b_j(z)y(z)| \leq (|a_j(z)| + |b_j(z)|)M$$

holds for $z \in \Gamma \setminus \{z_0\}$. Since $(|a_j(z)| + |b_j(z)|)M$ converges to 0 when z tends to z_0, $c_j(z)$ converges to 0 and $c_j(z_0) = 0$ follows from the fact that $c_j(z)$ is continuous.

We prove the lemma by dividing it into three cases.

First, we prove the case $a_1(z_0) = a_2(z_0) = 0$. If $b_1(z_0) \neq 0$, then the second equation of (1) is equal to the first equation multiplied by $b_2(z_0)/b_1(z_0)$. Again, $b_1(z_0) \neq 0$ implies the conclusion. If $b_1(z_0) = 0$, then $c_1(z_0) = 0$ and the equations of (1) become the second equation only. Furthermore, if $b_2(z_0) = 0$, the second equation also vanishes.

Second, we prove the case $a_1(z_0) = 0$ and $a_2(z_0) \neq 0$. (The case $a_1(z_0) \neq 0$ and $a_2(z_0) = 0$ is similar.) The assumption $d(z_0) = 0$ implies that $b_1(z_0) = 0$. Therefore, $c_1(z_0) = 0$ and the second equation of (1) vanishes. Then, the assumption $a_2(z_0) \neq 0$ implies the conclusion.

The last case is that when both $a_1(z_0)$ and $a_2(z_0)$ are not 0. The assumption $d(z_0) = 0$ and (2) imply that the second equation of (1) is equal to the first equation multiplied by $a_2(z_0)/a_1(z_0)$ and $a_2(z_0) \neq 0$ implies the conclusion. □

The proof of the main theorem is as follows.

Proof. It is sufficient to prove that condition (1) implies condition (2) under the assumptions of the theorem.

From Proposition 1, there exists a polynomial $g \in F$ having a zero α on C.

If the number of coefficients of g that are not the endpoints of the interval coefficients is less than two, the proof is done.

Otherwise, we take two of them and write them as t_1 and t_2. Then, we can write the real part of $g(z)$ and the imaginary part of $g(z)$ as follows:

$$\operatorname{Re} g(z) = a_1(z)t_1 + b_1(z)t_2 + c_1(z), \qquad \operatorname{Im} g(z) = a_2(z)t_1 + b_2(z)t_2 + c_2(z),$$

where $a_j(z)$, $b_j(z)$, $c_j(z)$ are continuous functions with respect to z. The equation $g(\alpha) = 0$ is equivalent to the simultaneous equations

$$\begin{cases} a_1(z)t_1 + b_1(z)t_2 + c_1(z) = 0, \\ a_2(z)t_1 + b_2(z)t_2 + c_2(z) = 0. \end{cases} \tag{3}$$

We consider these to be the simultaneous equations of t_1 and t_2 with parameter z. If the determinant

$$\begin{vmatrix} a_1(z) & b_1(z) \\ a_2(z) & b_2(z) \end{vmatrix} \tag{4}$$

is 0 at $z = \alpha$, then we can move t_1 and t_2 as α is a zero, until either t_1 or t_2 reaches one of the end points of the interval coefficients.

If the determinant is not 0 at $z = \alpha$, the solutions t_1 and t_2 are continuous with respect to z whenever the determinant is not 0. Therefore, when we move z from α to α_0 on C, one of the following occurs.

(a) The determinant (4) is not 0, and either t_1 or t_2 reaches one of the endpoints of the interval coefficients.

(b) The determinant (4) is 0 at a point β.

If the determinant (4) is not 0, and t_1 and t_2 are in the interval coefficients as z tends to α_0, then from Lemma 2, the simultaneous equations (3) are indeterminate at $z = \alpha_0$. This contradicts the assumption that α_0 is not a pseudozero of F.

When case (a) occurs, we find that there exists $h \in F$ such that h has a zero on C and the number of coefficients of h that are not equal to the endpoints of the interval coefficients is less than that of g. If case (b) occurs, we can move t_1 and t_2 as (3) holds at $z = \beta$, until either t_1 or t_2 reaches the endpoints of the interval coefficients. That is, also in this case, we can find a polynomial $h \in F$ such that h has a zero on C and the number of coefficients of h that are not equal to the endpoints of the interval coefficients is less than that of g.

We apply this procedure repeatedly until condition (2) is satisfied. □

4.3. Edge Theorem

Here, we describe the relation between Theorem 3 and the Edge Theorem [1].

In the Edge Theorem, the notion of a polynomial polytope appears, which is an extension of an interval polynomial. The set of polynomials represented as a convex combination of a given finite number of polynomials is said to be a polynomial polytope. For more details see textbooks on control theory.

Theorem 4 (Edge Theorem). *Suppose that a domain $D \subset \mathbb{C}$ satisfies the condition "any point in the compliment of D is on a path to infinity." Let F be a polynomial polytope. All zeros of any polynomial in F are contained in D if and only if all zeros of any exposed edge of F are contained in D.*

Remark 4. For an interval polynomial, an exposed edge is a subset of the set of polynomials whose coefficients are the endpoints of the interval coefficients except at most one coefficient. However, we should examine all such polynomials since there is no efficient way to find all exposed edges.

Suppose that Theorem 3 can be applied to a domain D and the Edge Theorem can be applied to the compliment D^c of D. Then, as described above, we can solve Problem 1 using Theorem 3. We can also solve Problem 1 by applying the Edge Theorem to D^c because the negation of the statement "there exists a polynomial $f \in F$ such that at least one zero of f belongs to D" is "all zeros of any polynomial in F belong to D^c." The computational cost when we use Theorem 3 is slightly high; determining whether there exists a zero in D for a given polynomial in F and whether there exists a polynomial f in F such that f has a zero at a given point on C are added.

However, the strong point of Theorem 3 over the Edge Theorem is that there exists a domain D such that Theorem 3 can be applied to D but the Edge Theorem cannot be applied to D^c. Closed disks and closed rectangles are such examples.

4.4. Computation Method

In this section, we describe the computation method using Theorem 3.

First, we show the method for deciding whether a given point on C is a pseudozero of an interval polynomial.

4.4.1. Polynomials Having a Zero at a Given Point of the Boundary. Let F be an interval polynomial as described by $(*)$. Then, we can write

$$F(x) = \left\{ \sum_{j=1}^{n} \{(h_j - l_j)t_j + l_j\}\, e_j(x) \,\middle|\, 0 \le t_j \le 1 \right\}.$$

Therefore, there exists a polynomial $f \in F$ such that f has a zero at a complex number α if and only if the equation

$$\sum_{j=1}^{n} \{(h_j - l_j)t_j + l_j\}\, e_j(\alpha) = 0$$

has a solution $0 \le t_j \le 1$ for all j. If (1) $\mathrm{Re}(h_j - l_j)e_j(\alpha) < 0$ or (2) $\mathrm{Re}(h_j - l_j)e_j(\alpha) = 0$ and $\mathrm{Im}(h_j - l_j)e_j(\alpha) < 0$, we replace t_j by $1 - t_j$ and we write the resulting equation as follows:

$$\sum_{j=1}^{n} a_j t_j = b. \tag{5}$$

Here, we take $\arg z$ for $z \in \mathbb{C}$ in the range $-\pi < \arg z \le \pi$. Therefore, the above substitution implies the inequalities $-\pi/2 < \arg a_j \le \pi/2$ for $a_j \ne 0$. We consider the problem that (5) has a solution $0 \le t_j \le 1$ for all j.

Lemma 3. *The set* $\{ \sum_{j=1}^{n} a_j t_j \mid 0 \le t_j \le 1 \}$ *is equal to the convex hull of the set* $\{ \sum_{j=1}^{n} \varepsilon_j a_j \mid \varepsilon_j = 0, 1 \}$.

Proof. It is clear that the set $\{\sum_{j=1}^{n} a_j t_j \mid 0 \le t_j \le 1\}$ contains the convex hull of the set $\{\sum_{j=1}^{n} \varepsilon_j a_j \mid \varepsilon_j = 0,\, 1\}$.

To show the latter contains the former, take any element $\beta = \sum_{j=1}^{n} a_j t_j$ in $\{\sum_{j=1}^{n} a_j t_j \mid 0 \le t_j \le 1\}$ and sort t_j in increasing order $0 \le t_{J(1)} \le t_{J(2)} \le \cdots \le t_{J(n)} \le 1$. Then,

$$\beta = \sum_{j=1}^{n} a_j t_j = \left(1 - t_{J(n)}\right) \cdot 0 + t_{J(1)} \sum_{j=1}^{n} a_{J(j)} + \sum_{k=2}^{n} \left\{ \left(t_{J(k)} - t_{J(k-1)}\right) \sum_{j=k}^{n} a_{J(j)} \right\}$$

and the equalities show that β is represented as a convex combinations of 0 and $\sum_{j=k}^{n} a_{J(j)}$ $(1 \le k \le n)$. □

Hence, we construct the convex hull of the set $V = \{\sum_{j=1}^{n} \varepsilon_j a_j \mid \varepsilon_j = 0 \text{ or } 1\}$ and determine whether b is in the convex hull. There are 2^n points in V in general, but the convex hull can be constructed efficiently: We can construct it by examining at most n points $a_1, a_2, \ldots, a_n$.

Theorem 5. *Let the set V be as above. First, sort $a_j \ne 0$ as the arguments in increasing order. Note that $-\pi/2 < \arg a_j \le \pi/2$ hold. If two or more points, say a_j, a_k, a_l, have the same argument, then we add them up together and replace a_j, a_k, a_l with the sum, and write the results as $p_1, p_2, \ldots, p_m$. Then, the vertices of the convex hull are, in counterclockwise order, $0, v_1, \ldots, v_{2m-1}$, where*

$$v_j = \begin{cases} \displaystyle\sum_{k=1}^{j} p_k & (1 \le j \le m), \\ \displaystyle\sum_{k=j-m+1}^{m} p_k & (m+1 \le j \le 2m-1). \end{cases}$$

We need a lemma for the proof. For $z_1, z_2 \in \mathbb{C}$, we define

$$d(z_1, z_2) = \begin{vmatrix} \operatorname{Re} z_1 & \operatorname{Re} z_2 \\ \operatorname{Im} z_1 & \operatorname{Im} z_2 \end{vmatrix}.$$

Then, the following lemma is clear.

Lemma 4. 1. *For any z, z_1 and $z_2 \in \mathbb{C}$ and $a \in \mathbb{R}$,*

$$d(z, z) = 0, \qquad d(z_1, z_2) = -d(z_2, z_1), \qquad d(az_1, z_2) = d(z_1, az_2) = a \cdot d(z_1, z_2),$$
$$d(z_1 + z_2, z) = d(z_1, z) + d(z_2, z), \qquad d(z, z_1 + z_2) = d(z, z_1) + d(z, z_2).$$

2. *When z_1 and z_2 are not 0 and $-\pi/2 < \arg z_1, \arg z_2 \le \pi/2$ hold, $d(z_1, z_2) > 0$ holds if and only if $\arg z_1 < \arg z_2$ holds, and $d(z_1, z_2) = 0$ holds if and only if $\arg z_1 = \arg z_2$ holds.*

Now, we prove Theorem 5.

Proof. For any j $(1 \le j \le 2m)$, it is sufficient to prove that an arbitrary point $\sum_{k=1}^{n} \varepsilon_k a_k$ is sitting at the left of or on the straight line from v_{j-1} to v_j (we put $v_0 = v_{2m} = 0$). To prove this, we introduce the following two statements.

- For any j and $a = \sum_{k=1}^{n} \varepsilon_k a_k$, the inequality $d(p_j, a - v_{j-1}) \geq 0$ holds.
- Any $a = \sum_{k=1}^{n} \varepsilon_k a_k$ satisfying $d(p_j, a - v_{j-1}) = 0$ lies between v_{j-1} and v_j.

First, we prove the first statement when $1 \leq j \leq m$. We divide $\sum_{k=1}^{n} \varepsilon_k a_k$ into three parts: S_1 consisting of a_k's whose arguments are less than $\arg p_j$, S_2 consisting of a_k's whose arguments are equal to $\arg p_j$, and S_3 consisting of a_k's whose arguments are greater than $\arg p_j$. Then, we have

$$d(p_j, a - v_{j-1}) = d(p_j, a - \sum_{k=1}^{j-1} p_k) = d(p_j, S_1 - \sum_{k=1}^{j-1} p_k) + d(p_j, S_2) + d(p_j, S_3).$$

From the definitions of S_2 and S_3, we have $d(p_j, S_2) = 0$ and $d(p_j, S_3) \geq 0$. Furthermore, the definition of S_1 implies

$$d(p_j, S_1) = d(p_j, \sum_{k} \varepsilon_k a_k) \geq d(p_j, \sum_{\arg a_k < \arg p_j} a_k) = d(p_j, \sum_{k=1}^{j-1} p_k).$$

Hence, the first statement is proved.

Next, we prove the second statement. Lemma 4 implies that the equality $d(p_j, a_k) = 0$ holds if and only if $a_k = 0$ or $\arg a_k = \arg p_j$ (The construction of p_j implies that $p_j \neq 0$). For the sum $a = \sum_{k=1}^{n} \varepsilon_k a_k$, we only add $a_k \neq 0$. The proof of the first statement implies that the equality $d(v_j - v_{j-1}, a) = 0$ holds if and only if the equality $\varepsilon_k = 1$ holds for k such that $\arg a_k < \arg p_j$ and the equality $\varepsilon_k = 0$ holds for k such that $\arg a_k > \arg p_j$. Therefore, the equality

$$a = v_{j-1} + \sum_{\arg a_k = \arg p_j} \varepsilon_k a_k$$

holds, and the equalities $v_j = v_{j-1} + p_j$ and

$$p_j = \sum_{\arg a_k = \arg p_j} a_k$$

imply the statement.

Similar arguments hold for $m + 1 \leq j \leq 2m$, considering that $v_j - v_{j-1} = -p_{j-m}$, by dividing $\sum_{k=1}^{n} \varepsilon_k a_k$ into S_1 consisting of a_k's whose arguments are less than $\arg p_{j-m}$, into S_2 consisting of a_k's whose arguments are equal to $\arg p_{j-m}$, and into S_3 consisting of a_k's whose arguments are greater than $\arg p_{j-m}$. □

4.4.2. Polynomials Having Zeros on the Boundary. Let F be an interval polynomial as described by $(*)$. For a polynomial $f = \sum_{j=1}^{n} a_j e_j \in F$, suppose that a_j is either l_j or h_j for $j \neq \lambda$. Here, we describe a method for determining whether we can make f have a zero on the segment $C_k \subset C$ by moving a_λ.

We write the representation of C_k as $\varphi_k(s) + i\psi_k(s)$, $s \in S_k$. For simplicity, writing a_λ as t and $[l_\lambda, h_\lambda]$ as $[l, h]$, we have

$$f(x) = t e_\lambda(x) + \sum_{j \neq \lambda} a_j e_j(x), \qquad t \in [l, h].$$

Substituting $\varphi_k(s) + i\psi_k(s)$ for x, we have

$$f(\varphi_k(s) + i\psi_k(s)) = u(s,t) + iv(s,t),$$

where u, $v \in \mathbb{Q}(s,t)$. Therefore, f has a zero on C_k for some $a_\lambda \in [l_\lambda, h_\lambda]$ if and only if the following simultaneous equations have a solution $s \in S_k$ and $l \leq t \leq h$.

$$\begin{cases} u(s,t) = 0, \\ v(s,t) = 0. \end{cases} \tag{6}$$

These equations are of degree one with respect to t. Therefore, solving $u = 0$ and $v = 0$, we write $t = T_1(s)$ and $t = T_2(s)$, where T_1, $T_2 \in \mathbb{Q}(s)$ (see Remark 5 below). Moreover, we put

$$T_1(s) - T_2(s) = \frac{P(s)}{Q(s)}, \tag{7}$$

where P, $Q \in \mathbb{Q}[s]$ and $\gcd(P,Q) = 1$.

Therefore, the problem is whether there exists a zero $\alpha \in S_k$ of P that satisfies $l \leq T(\alpha) \leq h$ by putting $T = T_1$ or T_2 (we can take either).

When $l = h$, we set

$$T(s) - l = \frac{P_l(s)}{Q_l(s)}, \tag{8}$$

where P_l, $Q_l \in \mathbb{Q}[s]$. Put $G_l = \gcd(P, P_l)$, we can solve Problem 1 by examining whether there exists a zero $\alpha \in S_k$ of G_l since the equations $T(\alpha) = l$ and $P(\alpha) = 0$ hold for any zero α of G_l, and $T(\beta) \neq l$ for any zero β of P/G_l.

When $l < h$, we compute (8) and

$$T(s) - h = \frac{P_h(s)}{Q_h(s)}, \tag{9}$$

where P_h, $Q_h \in \mathbb{Q}[s]$, $\gcd(P_h, Q_h) = 1$. Put $G_l = \gcd(P, P_l)$ and $G_h = \gcd(P, P_h)$. Then, G_l and G_h are relatively prime because $T(\alpha) = l$ for any zero α of G_l and $T(\beta) = h$ for any zero β of G_h. Therefore, we can divide zeros of P into three groups: the zeros of G_l, the zeros of G_h, and the zeros of $P/(G_l G_h)$.

We only need to examine real zeros since $S_k \subset \mathbb{R}$. For a real zero α of G_l, we examine whether $\alpha \in S_k$, and we can solve this using, for example, Sturm's algorithm (or some other efficient algorithm). We carry out a similar procedure for a real zero of G_h.

For a real zero α of $P/(G_l G_h)$, we examine whether $\alpha \in S_k$ and $l \leq T(\alpha) \leq h$. The former can be carried out using, for example, Sturm's algorithm. The latter can be carried out using approximate computation with error analysis, for example, by interval computation, under the assumption that we can raise the precision as high as desired since $T(\alpha)$ is not equal to l or h.

To summarize, for the rational function $\varphi_k(s) + i\psi_k(s)$ $(s \in S_k)$ that represents the segment C_k of C, the simultaneous equations in (6) determined by f and the range $[l, h]$ of t, we carry out the following computations.

- Noting that the equations in (6) are of degree one with respect to t, solving $u = 0$ and $v = 0$, we write $t = T_1(s)$ and $t = T_2(s)$, where T_1, $T_2 \in \mathbb{Q}(s)$ (see Remark 5 below).
- Compute (7).
- Put $T(s)$ as either $T_1(s)$ or $T_2(s)$ (we can use either).
- When $l = h$, compute (8). If $\gcd(P, P_l)$ has a zero in S_k, the answer to the question posed in Problem 1 is “Yes.”
- When $l < h$, compute (8) and (9), and put $G_l = \gcd(P, P_l)$, $G_h = \gcd(P, P_h)$.
  If G_l or G_h has a zero in S_k, the answer is “Yes.”
  If both G_l and G_h have no zero in S_k, when $P/(G_l G_h)$ has a zero α that is in S_k and $l < T(\alpha) < h$, the answer is “Yes.”

The number of polynomials f to be examined is at most $n2^{n-1}$. For each polynomial f, we examine each segment C_k of C. If we obtain “Yes” for Problem 1 during the examination, the rest of the procedure is not needed. If we do not obtain “Yes” after the whole examination is done, then the answer is “No.”

Remark 5. When $e_\lambda(x)$ is constant, $v(s, t)$ in (6) is a rational function only in s. In this case, when computing P and Q, we put $t = T(s)$ by solving t from $u = 0$ and set the left-hand side of (7) to v.

The order of the computational steps in the determination of whether f has a zero in D for a given polynomial $f \in F$ and whether α is a pseudozero of F for a given point $\alpha \in C$ is a polynomial in n. On the other hand, the order of the computational steps in the determination of whether there exists a pseudozero of F on C is 2^n times a polynomial in n and K, since the number of polynomials to be examined is of order 2^n as described above and the number of C_k is K. Therefore, the order of the total computational steps in the determination of whether there exists a pseudozero of F in D is 2^n times a polynomial in n and K.

4.4.3. Experiments. We carried out experimental computations for the following examples. We used the experimental computer algebra system Risa/Asir [10] on a computer with an Intel (R) Xeon$^{\mathrm{TM}}$ processor (3.2 GHz) and 4 GB of memory.

Example 3. Solve Problem 1 for the interval polynomial

$$F = [0.9995, 1.0005]x^2 + [-0.6185, -0.6175]x + [0.9995, 1.0005]$$

and the domain $D = \{\, z \in \mathbb{C} \mid |z - (0.3096 + 0.9526 \cdot i)| \le 0.0004 \,\}$.

Example 4. Solve Problem 1 for the interval polynomial F in Example 3 and the domain D that is the rectangle whose vertexes are $0.3092 + i \cdot 0.95$, $0.31 + i \cdot 0.95$, $0.31 + i \cdot 0.953$ and $0.3092 + i \cdot 0.953$.

We obtained “No” for Example 3 and “Yes” for Example 4 within 0.1 s in both cases.

4.5. Rough Shape of Pseudozeros

Since, unlike real pseudozeros, the shape of pseudozeros is complicated, we should be content with a rough shape that is almost equal to the exact shape in general. For example, for an interval polynomial F, we should be content with a set $X \supset Z(F)$ that is a union of congruent closed rectangles intersecting $Z(F)$. We say that a set X is a rough shape for $Z(F)$ with precision ε, which is a positive real number, if the longest edge of the congruent rectangles is less than or equal to ε.

If the set $Z(F)$ is bounded and an initial rectangle containing $Z(F)$ is given, we can obtain a rough shape with arbitrary precision using the above computation methods for Theorem 3. If all polynomials in F have the same degree, we can compute an initial rectangle using, for example, the Cauchy bound for an algebraic equation. Once the initial rectangle is obtained, we divide it into four congruent rectangles and examine whether each of them intersects $Z(F)$. Similar computations are performed recursively for the rectangles that intersect $Z(F)$.

Note that we cannot use the Edge Theorem for determining a rough shape of pseudozeros of an interval polynomial since the outside of a rectangle does not satisfy the precondition for the Edge Theorem.

For efficient computation, several techniques are needed in order to avoid redundant computations. These remain for future study.

5. Conclusion

We have proposed a method for determining whether there exist a polynomial in a given interval polynomial that has a zero in a prescribed domain. The method is rigorous but is not efficient for a complex domain. Avoiding redundant computations, especially when computing a rough shape of pseudozeros, is one of our future directions. Another direction is to consider the following type of problem: For a given interval polynomial F and a given domain D, does every polynomial in F have a zero in D?

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Hiroshi Sekigawa and Kiyoshi Shirayanagi[1]
NTT Communication Science Laboratories
Nippon Telegraph and Telephone Corporation
3-1, Morinosato Wakamiya, Atsugi-shi,
Kanagawa, 243-0198 Japan
e-mail: `sekigawa@theory.brl.ntt.co.jp`
`shirayan@theory.brl.ntt.co.jp`[2]

[1]The current address is: Department of Mathematical Sciences, Tokai University, 1117 Kitakaname, Hiratsuka-shi, Kanagawa, 259-1292 Japan
[2]The current mail address is: `shirayan@ss.u-tokai.ac.jp`