Chapter XI

Manifolds and differential forms

In Chapter VIII, we learned about Pfaff forms and saw that differential forms of first degree are closely connected with the theory of line integrals. In this chapter, we will treat the higher-dimensional analogue of line integrals, in which differential forms of higher degree are integrated over certain submanifolds of $\mathbb{R}^n$. So this chapter will deal with the theory of differential forms.

In Section 1, we generalize what we know about manifolds. In particular, we explore the concept of a submanifold of a given manifold, and we introduce manifolds with boundary.

In Section 2, we compile the needed results from multilinear algebra. They form the algebraic foundation for the theory of differential forms: In Section 3, we treat differential forms on open subsets of $\mathbb{R}^n$. In Section 4, we make this theory global and then discuss the orientability of manifolds.

Because we always consider submanifolds of Euclidean spaces, we can naturally endow them with a Riemannian metric. In Section 5, we look more closely at this additional structure and explain several basic facts of Riemannian geometry. To accommodate the needs of physics, we also treat semi-Riemannian metrics; in the examples, we will always confine ourselves to Minkowski space.

Section 6, which concludes this chapter, makes the connection between the theory of differential forms and classical vector analysis. In particular, we study the operators gradient, divergence, and curl, and we derive their basic properties. We give their local coordinate representations and calculate these explicitly in several important examples.

In Section 2, which otherwise concerns linear algebra, we also introduce the Hodge star operator, which we will need in later sections to define the codifferential. Then we will be able unify the various operators of vector analysis into the language of the Hodge calculus. This material can be skipped on first reading: For this reason, we wait for the end of each section to discuss any material that uses Hodge theory.

In the entire book, we restrict to submanifolds of $\mathbb{R}^n$. However, apart from the definition of the tangent space, we structure all proofs so that they remain true or can be easily modified for abstract manifolds. Thus Chapters XI and XII give a first introduction to differential topology and differential geometry; though they sometimes lack the full elegance of the general theory, the many examples we consider do form a solid foundation for further study of the subject.

1 Submanifolds

In this section,

- M is an m-dimensional manifold and N is an n-dimensional manifold.

More precisely, this means M is an m-dimensional C^∞ submanifold of $\mathbb{R}^{\overline{m}}$ for some $\overline{m} \geq m$; a like statement holds for N.

For simplicity and to emphasize the essential, we restrict to the study of smooth maps. In particular, we always understand a **diffeomorphism** to be a C^∞ diffeomorphism, and we set

$$\mathrm{Diff}(M,N) := \mathrm{Diff}^\infty(M,N)\ .$$

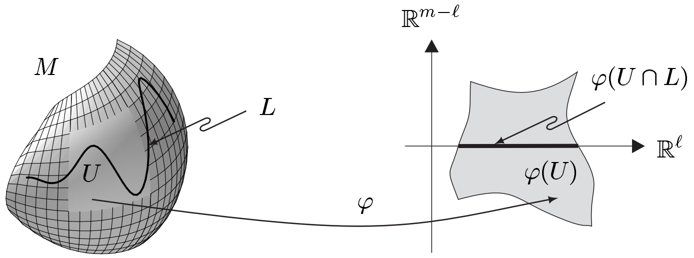
However, whenever anything is proved in the following, it will also hold for C^k manifolds and C^k maps, where, if necessary, $k \in \mathbb{N}^\times$ must be restricted appropriately. We will usually put these adjustments in remarks[1] and leave their verification to you.

Definitions and elementary properties

Let $0 \leq \ell \leq m$. A subset L of M is called an (***ℓ*-dimensional**) **submanifold of *M*** if for every $p \in L$ there is a chart (φ, U) of M around p such that[2]

$$\varphi(U \cap L) = \varphi(U) \cap \bigl(\mathbb{R}^\ell \times \{0\}\bigr)\ .$$

Every such chart is a **submanifold chart of *M* for *L***. The number $m - \ell$ is called the **codimension of *L* in *M***.



Clearly this definition directly generalizes of the idea of a submanifold of $\mathbb{R}^m$.

In the context of submanifolds, immersions play an important role. They will be introduced in analogy to the definition given Section VII.9.

Let $k \in \mathbb{N}^\times \cup \{\infty\}$. Then $f \in C^k(M,N)$ is a ***C^k* immersion** if $T_pf : T_pM \to T_{f(p)}N$ is injective for every $p \in M$. We call a C^k immersion f a ***C^k* embedding**

[1] In small print sections entitled "regularity".

[2] To avoid bothersome special cases, we interpret the empty set as a submanifold of dimension ℓ for every $\ell \in \{0, \ldots, m\}$ (see Section VII.9).

of M in N if f is a homeomorphism from M to $f(M)$ (where $f(M)$ is naturally provide with the relative topology of N). Instead of C^∞ immersion [or C^∞ embedding], we say for short **immersion** [or **embedding**].

1.1 Remarks **(a)** If L is an ℓ-dimensional submanifold of M and M is submanifold of N, then L is an ℓ-dimensional submanifold of N.

Proof Let $p \in L$, and let (φ, U) be a submanifold chart of M for L around p. Also let (ψ, V) be a submanifold chart of N for M around p. We can also assume $U = V \cap M$. Letting $X := \varphi(U) \subset \mathbb{R}^m$ and $Y := \mathrm{pr} \circ \psi(V) \subset \mathbb{R}^m$, where $\mathrm{pr} : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ denotes the canonical projection, we have

$$\chi := \mathrm{pr} \circ \psi \circ \varphi^{-1} \in \mathrm{Diff}(X, Y) .$$

Now we define $\Phi \in \mathrm{Diff}(Y \times \mathbb{R}^{n-m}, X \times \mathbb{R}^{n-m})$ by

$$\Phi(y, z) := \bigl(\chi^{-1}(y), z\bigr) \quad \text{for } (y, z) \in Y \times \mathbb{R}^{n-m} ,$$

and set $\Psi := \Phi \circ \psi$. Then $\Psi(V)$ is open in $\mathbb{R}^n$, and $\Psi \in \mathrm{Diff}(V, \Psi(V))$ with

$$\Psi(V \cap L) = \bigl(\varphi(U \cap L) \times \{0\}\bigr) \cap \bigl(\mathbb{R}^\ell \times \{0\}\bigr) = \Psi(V) \cap \bigl(\mathbb{R}^\ell \times \{0\}\bigr) \subset \mathbb{R}^n ,$$

as one can easily check. Therefore (Ψ, V) is a submanifold chart of N for L around p. ■

(b) Because the $\mathbb{R}^{\overline{m}} = \mathbb{R}^{\overline{m}} \times \{0\} \subset \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$ for $n \geq \overline{m}$, it follows from (a) that M is an m-dimensional submanifold of $\mathbb{R}^n$ for every $n \geq \overline{m}$. This shows that the "surrounding space" $\mathbb{R}^{\overline{m}}$ of M does not play an important role so long as we are only interested in the "inside properties" of M, that is, in properties that are described only with the help of charts and tangent spaces of M and which do not depend on how M is "situated" in the surrounding space.[3] However, how M is situated in $\mathbb{R}^{\overline{m}}$ does matter, for example, when defining the normal bundle $T^\perp M$.

(c) Let L be a submanifold of M. For the submanifold chart (φ, U) of M for L, we set

$$(\varphi_L, U_L) := (\varphi \,|\, U \cap L, U \cap L) .$$

Then (φ_L, U_L) is a chart for L, where $\varphi(U_L)$ is interpreted as an open subset of $\mathbb{R}^\ell$, that is, $\mathbb{R}^\ell \times \{0\} \subset \mathbb{R}^m$ is identified with $\mathbb{R}^\ell$.

If $\mathcal{A} := \bigl\{ (\varphi_\lambda, U_\lambda) \ ; \ \lambda \in \Lambda \bigr\}$ is a set of submanifold charts of M for L such that L is covered by the coordinate patches (charted territories) $\{ U_\lambda \ ; \ \lambda \in \Lambda \}$, then $\bigl\{ (\varphi_{\lambda,L}, U_{\lambda,L}) \ ; \ \lambda \in \Lambda \bigr\}$ is an atlas for L, the **atlas induced by** $\mathcal{A}$.

Proof We leave the simple verifications to you. ■

(d) Suppose L and K are respectively ℓ- and k-dimensional submanifolds of M and N. Then $L \times K$ is an $(\ell{+}k)$-dimensional submanifold of the manifold $M \times N$, which is $(m{+}n)$-dimensional.

[3]In Section 4, it will be clear that tangent spaces also have an "inside" characterization.

Proof This follows simply from the definitions. We again leave the proof to you.[4] ■

(e) Let L be a submanifold of M. Then

$$i : L \to M \ , \quad p \mapsto p$$

is an embedding, the **natural embedding** of L in M; we write it as $i : L \hookrightarrow M$. We identify T_pL for $p \in L$ with its image in T_pM under the injection T_pi, that is, we regard T_pL as a vector subspace of T_pM: $T_pL \subset T_pM$.

Proof Let (φ, U) be a submanifold chart of M for L. Then i has the local representation

$$\varphi \circ i \circ \varphi_L^{-1} : \varphi_L(U_L) \to \varphi(U) \ , \quad x \mapsto (x, 0) \ .$$

Now the claim is clear. ■

(f) If $f : M \to N$ is an immersion, then $m \leq n$.

(g) Let L be a submanifold of M of dimension ℓ, and suppose f belongs to $\mathrm{Diff}(M, N)$. Then $f(L)$ is an ℓ-dimensional submanifold of N.

Proof We leave the simple check to you. ■

(h) Every open subset of M is an m-dimensional submanifold of M.

(i) If (φ, U) is a chart of M, then $\varphi : U \to \mathbb{R}^m$ is an embedding, and φ is a diffeomorphism from U to $\varphi(U)$.

(j) Suppose L and K are respectively submanifolds of M and N, and $i_L : L \hookrightarrow M$ and $i_K : K \hookrightarrow N$ are their respective natural embeddings. Let $k \in \mathbb{N} \cup \{\infty\}$ and $f \in C^k(M, N)$ with $f(L) \subset K$. Then the restriction of f to L satisfies

$$f \,|\, L := f \circ i_L \in C^k(L, K) \ ,$$

and the diagrams

$$\begin{array}{ccc} L & \xrightarrow{\ i_L\ } & M \\ {\scriptstyle f \,|\, L}\downarrow & & \downarrow{\scriptstyle f} \\ K & \xrightarrow{\ i_K\ } & N \end{array} \qquad\qquad \begin{array}{ccc} T_pL & \xrightarrow{\ T_p i_L\ } & T_pM \\ {\scriptstyle T_p(f \,|\, L)}\downarrow & & \downarrow{\scriptstyle T_p f} \\ T_{f(p)}K & \xrightarrow{\ T_{f(p)} i_K\ } & T_{f(p)}N \end{array}$$

commute. Identifying T_pL with its image in T_pM under T_pi_L, that is, regarding T_pL in the canonical way as a vector subspace of T_pM, we have in particular $T_p(f \,|\, L) = (T_pf) \,|\, T_pL$.

Proof This follows from obvious changes to the proof of Example VII.10.10(b), which is generalized by this statement. ■

[4] See Exercise VII.9.4.

(k) (regularity) Analogous definitions and statements hold when M is a C^k manifold for $k \in \mathbb{N}^\times$. In this case L is also a C^k manifold, and the natural inclusion $i : L \hookrightarrow M$ belongs to the class C^k. ■

The next theorem, a generalization of Proposition VII.9.10, shows that we can generate submanifolds using embeddings.

1.2 Theorem

(i) *Suppose $f : M \to N$ is an immersion. Then f is* locally *an embedding, that is, for every p in M, there is a neighborhood U such that $f \,|\, U$ is an embedding.*

(ii) *If $f : M \to N$ is an embedding, then $f(M)$ is an m-dimensional submanifold of N, and f is a diffeomorphism from M to $f(M)$.*

Proof (i) Let $p \in M$, and suppose (φ, U_0) and (ψ, V) are respectively charts of M around p and of N around $f(p)$ such that $f(U_0) \subset V$. Then

$$f_{\varphi,\psi} := \psi \circ f \circ \varphi^{-1} : \varphi(U_0) \to \psi(V)$$

is an immersion by Remark 1.1(i). By the immersion theorem (Theorem VII.9.7), there is an open neighborhood X of $\varphi(p)$ in $\varphi(U_0)$ such that $f_{\varphi,\psi}(X)$ is an m-dimensional submanifold of $\mathbb{R}^n$. Then $\psi \in \mathrm{Diff}(V, \psi(V))$ and Remark 1.1(g) imply that $f(U)$, with $U := \varphi^{-1}(X)$, is an m-dimensional submanifold of N.

By appropriately shrinking X, Remark VII.9.9(d) shows that $f_{\varphi,\psi}$ is a diffeomorphism from $X = \varphi(U)$ to $f_{\varphi,\psi}(X) = \psi \circ f(U)$. Therefore f is a diffeomorphism from U to $f(U)$, where $f(U)$ is provided with the topology induced by N. Therefore $f \,|\, U$ is an embedding.

(ii) Suppose f is an embedding. For $q \in f(M)$, suppose (ψ, V) is a chart of N around q and (φ, U) is a chart of M around $p := f^{-1}(q)$ with $f(U) \subset V$. Because f is topological from M to $f(M)$, we know $f(U)$ is open in $f(M)$. Therefore we can assume that $f(U) = f(M) \cap V$. Now it follows from the proof of (i) that $f(M) \cap V$ is an m-dimensional submanifold of N. Because this is true for every $q \in f(M)$, we conclude $f(M)$ is an m-dimensional submanifold of N.

By (i), f is a local diffeomorphism from M to $f(M)$. Because f is topological, it follows that $f \in \mathrm{Diff}(M, f(M))$. ■

From Remark VII.9.9(c), we know that the image of an injective immersion is generally not a submanifold. The following theorem gives a simple sufficient condition which tells whether an injective immersion is an embedding.

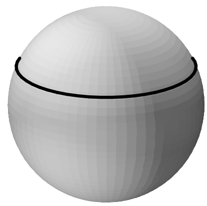
1.3 Theorem *Suppose M is compact and $f : M \to N$ is an injective immersion. Then f is an embedding, $f(M)$ is an m-dimensional submanifold of N, and $f \in \mathrm{Diff}(M, f(M))$.*

Proof Because M compact and $f(M)$ is a metric space, the bijective continuous map $f: M \to f(M)$ is topological (see Exercise III.3.3). Now the claim follows from Theorem 1.2. ■

1.4 Remark (regularity) Let $k \in \mathbb{N}^\times$. Then corresponding versions of Theorems 1.2 and 1.3 remain true when M and N are C^k manifolds and f belongs to the class C^k. ■

1.5 Examples **(a)** Suppose $1 \le \ell < m$, and let (x, y) denote a general point of $\mathbb{R}^{\ell+1} \times \mathbb{R}^{m-\ell} = \mathbb{R}^{m+1}$. Then

$$L_y := \sqrt{1-|y|^2}\, S^\ell \times \{y\}$$



is an ℓ-dimensional submanifold of the m-sphere S^m for every $y \in \mathbb{B}^{m-\ell}$. It is diffeomorphic to S^ℓ. The tangent space at the point $p \in L_y$ satisfies

$$T_pL_y = T_pS^m \cap \big(p, \mathbb{R}^{\ell+1} \times \{0\}\big) \subset T_p\mathbb{R}^{m+1} \ . \tag{1.1}$$

Proof For $y \in \mathbb{B}^{m-\ell}$, the map

$$F_y : \mathbb{R}^{\ell+1} \to \mathbb{R}^{m+1} \ , \quad x \mapsto \big(\sqrt{1-|y|^2}\, x, y\big) \tag{1.2}$$

is a smooth immersion. Because S^ℓ and S^m are respectively submanifolds of $\mathbb{R}^{\ell+1}$ and $\mathbb{R}^{m+1}$ and because $F_y(S^\ell) \subset S^m$, Remark 1.1(j) with $i_\ell : S^\ell \hookrightarrow \mathbb{R}^{\ell+1}$ gives

$$f_y := F_y \,|\, S^\ell = F_y \circ i_\ell \in C^\infty(S^\ell, S^m) \ . \tag{1.3}$$

Clearly f_y is injective, and the chain rule of Remark VII.10.9(b) implies

$$T_pf_y = T_pF_y \circ T_pi_\ell \quad \text{for } p \in S^\ell \ .$$

Therefore T_pf_y is injective (see Exercise I.3.3), that is, f_y is an immersion. Because S^ℓ is compact, Theorem 1.3 shows that $L_y = f_y(S^\ell)$ is an ℓ-dimensional submanifold of S^m and is diffeomorphic to S^ℓ. Then (1.1) is a simple consequence of (1.2) and (1.3). ■

(b) (torus-like hypersurfaces of rotation) Let

$$\gamma : S^1 \to (0,\infty) \times \mathbb{R} \ , \quad t \mapsto \big(\rho(t), \sigma(t)\big)$$

be an injective immersion and therefore by Theorem 1.3 an embedding. Also let $i : S^m \hookrightarrow \mathbb{R}^{m+1}$, and define

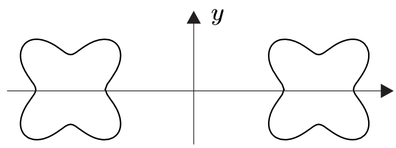
$$f : S^m \times S^1 \to \mathbb{R}^{m+1} \times \mathbb{R} \ , \quad (q,t) \mapsto \big(\rho(t)i(q), \sigma(t)\big) \ .$$

Then f is an embedding, and

$$T^{m+1} := f(S^m \times S^1)$$

is a hypersurface in $\mathbb{R}^{m+2}$, which is diffeomorphic to $S^m \times S^1$.

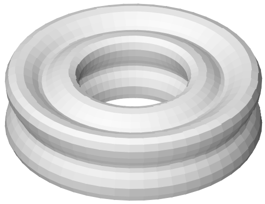
In the case $m = 0$, the set T^1 consists of two copies of the closed, smooth curve $\gamma(S^1)$, which has no points of self-intersection[5] and reflects symmetrically about the y-axis.



For $m = 1$, T^2 is the surface of rotation in $\mathbb{R}^3$ generated by rotating the meridional curve

$$\Gamma := \left\{ \big(\rho(t), 0, \sigma(t)\big) \ ; \ t \in S^1 \right\}$$

around the z-axis (see Example VII.9.11(e)). T^2 "is a 2-torus", that is, it is diffeomorphic to $\mathsf{T}^2 := S^1 \times S^1$. In particular, $\mathsf{T}^2_{a,r}$, the 2-torus from Example VII.9.11(f), is diffeomorphic to T^2.



In the general case, we call T^{m+1} a **torus-like hypersurface of rotation**.

Proof By Example VII.9.5(b), S^m and S^1 are m- and 1-dimensional manifolds, respectively. Therefore $S^m \times S^1$ is an $(m+1)$-dimensional manifold.

Suppose $(\varphi \times \psi, U \times V)$ is a product chart[6] of $S^m \times S^1$. Because γ is an immersion, its local representation with respect to ψ (and the trivial chart $\mathrm{id}_{\mathbb{R}^2}$ of $\mathbb{R}^2$), that is, $\gamma_\psi = (r, s)$ with $r := \rho \circ \psi^{-1}$ and $s := \sigma \circ \psi^{-1}$, satisfies

$$\big(\dot r(y), \dot s(y)\big) \neq (0,0) \quad \text{for } y \in \psi(V) \ . \tag{1.4}$$

Further, the local representation of f with respect to $\varphi \times \psi$ has the form

$$f_{\varphi\times\psi}(x,y) = \big(r(y)g(x), s(y)\big) \quad \text{for } (x,y) \in \varphi(U) \times \psi(V) \ ,$$

where $g := i \circ \varphi^{-1}$ is the parametrization of S^m belonging to φ. From this is follows that

$$\big[\partial f_{\varphi\times\psi}(x,y)\big] = \left[\begin{array}{c:c} r(y)\partial g(x) & \dot r(y) g(x) \\ \hdashline 0 & \dot s(y) \end{array}\right] \in \mathbb{R}^{(m+2)\times(m+1)} \ .$$

Because $r(y) > 0$ and because $\partial g(x)$ is injective, the first m columns of this matrix are linearly independent. If $\dot s(y) \neq 0$, then the matrix has rank $m+1$. If $\dot s(y) = 0$, then we have $\dot r(y) \neq 0$ by (1.4). From $|g(x)|^2 = (g(x) \,|\, g(x)) = 1$ for $x \in \varphi(U)$, it follows that $(g(x) \,|\, \partial_j g(x)) = 0$ for $1 \le j \le m$ and $x \in \varphi(U)$. This shows that the matrix has rank $m+1$ in this case as well. Therefore f is an immersion.

We now consider the equation $f(q,t) = (y,s)$ for some $(y,s) \in T^{m+1}$. From the relations $\rho(t)i(q) = y$ and $|i(q)| = 1$, it follows that $\rho(t) = |y|$. Because γ is injective, there is exactly one $t \in S^1$ such that $(\rho(t), \sigma(t)) = (|y|, s)$. Likewise, there is exactly one $q \in S^m$ with $i(q) = y/|y|$. Therefore the equation $(\rho(t)i(q), \sigma(t)) = (y,s)$, with (y,s) as above, has a unique solution (since $y = |y|\,(y/|y|)$). Hence f is an injective immersion

[5]Here and in the following, "curve" means a one-dimensional manifold (see Remark 1.19(a)).

[6]That is, (φ, U) and (ψ, V) are respectively charts of S^m and S^1, and $\varphi \times \psi(q,t) := \big(\varphi(q), \psi(t)\big)$.

of $S^m \times S^1$ in $\mathbb{R}^{m+2}$. Now all the claims follow from Theorem 1.3 because $S^m \times S^1$ is compact. ■

(c) Suppose L and M are submanifolds of N with $L \subset M$. Then L is a submanifold of M.

Proof Because $\mathrm{id}_N \in \mathrm{Diff}(N,N)$, we know $i := \mathrm{id}_N \,|\, L$ is an immersion of L in N with $i(L) \subset M$. Therefore it follows from Remark 1.1(j) that i is a bijective immersion of L in M. Because L and M carry the topology induced by N and because M induces the same topology on L, we know i, as a restriction of a diffeomorphism, is topological. Therefore i is an embedding, and the claim follows from Theorem 1.2. ■

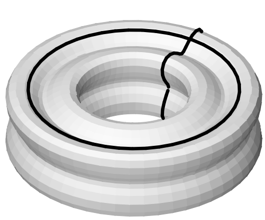
(d) Suppose the assumptions of (b) are satisfied with $m = 1$. Then for every $(q_0, t_0) \in S^1 \times S^1$, the images of

$$f(\,\cdot\,, t_0)\colon S^1 \to \mathbb{R}^3$$

and

$$f(q_0, \cdot\,)\colon S^1 \to \mathbb{R}^3$$

are one-dimensional submanifolds of T^2 and are diffeomorphic to S^1 (and therefore "circles").



Proof Because $f(\,\cdot\,, t_0)$ and $f(q_0, \cdot\,)$ as restrictions of embeddings are themselves embeddings, $f(S^1, t_0)$ and $f(q_0, S^1)$ are submanifolds of $\mathbb{R}^3$ diffeomorphic to S^1, and they lie in T^2. The claim now follows from (c). ■

Submersions

Suppose $f \in C^1(M,N)$. Then we say $p \in M$ is a **regular point** of f if T_pf is surjective. Otherwise p is a **singular point**. A point $q \in N$ is said to be a **regular value** of f if every $p \in f^{-1}(q)$ is a regular point. If every point of M is regular, we say f is a **regular map** or a **submersion**.

These definitions generalize concepts introduced in Section VII.8.

1.6 Remarks **(a)** If p is a regular point of f, then $m \geq n$. Every $q \in N \backslash f(M)$ is a regular value of f.

(b) The point $p \in M$ is a regular point of $f = (f^1, \ldots, f^n) \in C^1(M, \mathbb{R}^n)$ if and only if the cotangent vectors[7]

$$df^j(p) := d_pf^j = \mathrm{pr}_2 \circ T_pf^j \in T_p^*M \quad \text{for } 1 \leq j \leq n$$

are linearly independent.

(c) A singular point of $f \in C^1(M, \mathbb{R})$ is also called a **critical point**. Therefore $p \in M$ is a critical point of f if and only if $df(p) = 0$.[8] ■

[7]See Section VIII.3.
[8]See Remark VII.3.14(a).

The following theorem generalizes the regular value theorem to the case of maps between manifolds.

1.7 Theorem (regular value) *Suppose $q \in N$ is a regular value of the map $f \in C^\infty(M,N)$. Then $L := f^{-1}(q)$ is a submanifold of M of codimension n. For $p \in L$, the kernel of T_pf is T_pL.*

Proof Let $p_0 \in f^{-1}(q)$. Let (φ, U) be a chart of M around p_0, and let (ψ, V) be a chart of N around q with $f(U) \subset V$. Then it follows from the chain rule that for every $p \in U \cap f^{-1}(q)$, the point $\varphi(p)$ is a regular point of the local representation

$$f_{\varphi,\psi} := \psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n) \ .$$

In other words, $y := \psi(q)$ is a regular value of $f_{\varphi,\psi}$. Therefore Theorem VII.9.3 guarantees that $(f_{\varphi,\psi})^{-1}(y)$ is an $(m-n)$-dimensional submanifold of $\mathbb{R}^m$. Hence there are open sets X and Y of $\mathbb{R}^m$ and a $\Phi \in \mathrm{Diff}(X,Y)$ such that

$$\Phi(X \cap (f_{\varphi,\psi})^{-1}(y)) = Y \cap (\mathbb{R}^{m-n} \times \{0\}) \ .$$

By replacing $\varphi(U)$ and X with their intersection, we can assume that $\varphi(U) = X$. But then $\varphi_1 := \Phi \circ \varphi$ is a chart of M around p with

$$\begin{aligned} \varphi_1\bigl(f^{-1}(q) \cap U\bigr) &= \Phi \circ \varphi\bigl(f^{-1} \circ \psi^{-1}(y) \cap U\bigr) \\ &= \Phi\bigl((f_{\varphi,\psi})^{-1}(y) \cap X\bigr) = Y \cap \bigl(\mathbb{R}^{m-n} \times \{0\}\bigr) \end{aligned}$$

and is therefore a submanifold chart of M for $f^{-1}(q)$. The second claim now follows from an obvious modification of the proof of Theorem VII.10.7. ■

1.8 Remarks **(a)** Theorem 1.7 has a converse that says that every submanifold of M can be represented locally as the fiber of a regular map. More precisely, it says that if L is an ℓ-dimensional submanifold of M, then for every $p \in L$ there are a neighborhood U in M and an $f \in C^\infty(U, \mathbb{R}^{m-\ell})$ such that $f^{-1}(0) = U \cap L$, and 0 is a regular value of f.

Proof Suppose (φ, U) is a submanifold chart of M around p for L. Then the function defined by $f(q) := (\varphi^{\ell+1}(q), \ldots, \varphi^m(q))$ for $q \in U$ belongs to $C^\infty(U, \mathbb{R}^{m-\ell})$ and satisfies $f^{-1}(0) = U \cap L$. Because φ is a diffeomorphism, 0 is a regular value f. ■

(b) (regularity) If q is a regular value of $f \in C^k(M,N)$ for some $k \in \mathbb{N}^\times$, then $f^{-1}(q)$ is a C^k submanifold of M. In this case it suffices to assume that M is itself a C^k manifold. ■

1.9 Examples **(a)** Suppose X is open in $\mathbb{R}^m \times \mathbb{R}^n$ and $q \in \mathbb{R}^n$ is a regular value of $f \in C^\infty(X, \mathbb{R}^n)$ with $M := f^{-1}(q) \neq \emptyset$. Then M is an m-dimensional submanifold of X. For

$$\pi := \mathrm{pr} \mid M : M \to \mathbb{R}^m$$

with

$$\mathrm{pr} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \ , \quad (x,y) \mapsto x \ ,$$

we have $\pi \in C^\infty(M, \mathbb{R}^m)$. Finally let $p \in M$, and suppose $D_1 f(p) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ is surjective.[9] Then p is regular point of π if and only if $D_2 f(p)$ is bijective.

Proof The regular value theorem guarantees that M is an m-dimensional submanifold of X with $T_pM = \ker(T_p f)$ for $p \in M$. Because π is the restriction of a linear and therefore smooth map, it follows from Remark 1.1(j) that $\pi \in C^\infty(M, \mathbb{R}^m)$ and $T_p\pi = T_p \operatorname{pr} \,|\, T_pM$.

It follows from $T_p \operatorname{pr} = \bigl(p, \partial \operatorname{pr}(p)\bigr)$ and $\partial \operatorname{pr}(p)(h,k) = h$ for $(h,k) \in \mathbb{R}^m \times \mathbb{R}^n$ that $T_p\pi$ is surjective if and only if for every $y \in \mathbb{R}^m$ there is an $(h,k) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\partial f(p)(h,k) = D_1 f(p)h + D_2 f(p)k = 0$$

and $h = y$. This is because $D_1 f(p)$ is surjective if and only if for every $z \in \mathbb{R}^n$ there is a $k \in \mathbb{R}^n$ such that $D_2 f(p)k = z$ or, equivalently, if and only if $D_2 f(p)$ itself is surjective. Because $D_2 f(p) \in \mathcal{L}(\mathbb{R}^n)$, this finishes the proof. ■

(b) ("cusp catastrophe") For

$$f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} , \quad \bigl((u,v),x\bigr) \mapsto u + vx + x^3 ,$$

we have

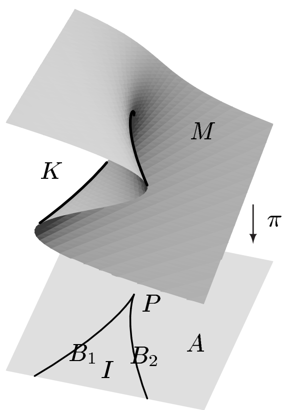
$$\bigl[D_1 f(w,x)\bigr] = [1, x] \in \mathbb{R}^{1\times 2} , \quad \text{where} \quad w := (u,v) .$$

Therefore 0 is a regular value of f, and $M := f^{-1}(0)$ is a surface in $\mathbb{R}^3$. Because $D_2 f(w,x) = v + 3x^2$, we know by (a) that

$$K := \bigl\{ \bigl((u,v),x\bigr) \in M \ ;\ v + 3x^2 = 0 \bigr\}$$

is the set of singular points of the projection $\pi : M \to \mathbb{R}^2$. It satisfies

$$K = \gamma(\mathbb{R}) \quad \text{with} \quad \gamma : \mathbb{R} \to \mathbb{R}^3 , \quad t \mapsto (2t^3, -3t^2, t) . \tag{1.5}$$



In particular, K is a 1-dimensional submanifold of M, a smoothly embedded curve. Its projection $B := \pi(K)$ is the image of

$$\sigma : \mathbb{R} \to \mathbb{R}^2 , \quad t \mapsto (2t^3, -3t^2) ,$$

a Neil parabola.[10] It is the union of the 0-dimensional manifold $P := \{(0,0)\} \in \mathbb{R}^2$, the "cusp", and the two one-dimensional manifolds $B_1 := \sigma((-\infty, 0))$ and $B_2 := \sigma((0,\infty))$.

Proof The point $(u,v,x) \in \mathbb{R}^3$ belongs to K if and only if it satisfies the equations

$$u + vx + x^3 = 0 \quad \text{and} \quad v + 3x^2 = 0 . \tag{1.6}$$

[9]We apply the notations of Section VII.8.

[10]See Remark VII.9.9(a).

By eliminating v from the first equation, we see that (1.6) is equivalent to

$$2x^3 = u \quad \text{and} \quad 3x^2 = -v \ .$$

This proves (1.5). For the derivative of the map

$$g : \mathbb{R}^3 \to \mathbb{R}^2 \ , \quad (u,v,x) \mapsto (u - 2x^3, v + 3x^2) \ ,$$

we find

$$\bigl[\partial g(u,v,x)\bigr] = \begin{bmatrix} 1 & 0 & -6x^2 \\ 0 & 1 & 6x \end{bmatrix} \in \mathbb{R}^{2\times 3} \ .$$

This matrix has rank 2, which shows that 0 is a regular value of g. Therefore by the regular value theorem, $K = g^{-1}(0)$ is a 1-dimensional submanifold of $\mathbb{R}^3$. Because $K \subset M$ it follows from Remark 1.5(c) that K is a submanifold of M. The rest is obvious. ■

1.10 Remark (catastrophe theory) We consider now a point particle of mass 1 moving along the real axis with potential energy U and total energy

$$E(\dot{x}, x) = \frac{\dot{x}^2}{2} + U(x) \quad \text{for } x \in \mathbb{R} \ .$$

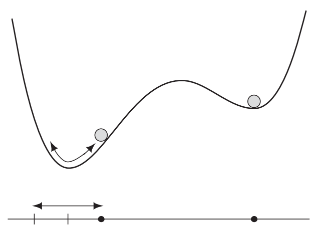
According to Example VII.6.14(a), Newton's equation of motion is

$$\ddot{x} = -U'(x) \ .$$

From Examples VII.8.17(b) and (c), we know that the critical points of the energy E are exactly the points $(0, x_0)$ with $U'(x_0) = 0$. Because the Hessian matrix of E has the form

$$\begin{bmatrix} 1 & 0 \\ 0 & U''(x_0) \end{bmatrix}$$

at $(0, x_0)$, it is positive definite if and only if $U''(x_0) > 0$. Hence it follows from Theorem VII.5.14 that $(0, x_0)$ is an isolated minimum of the total energy if and only if x_0 is an isolated minimum of the potential energy.[11] It is graphically clear that an isolated minimum of the total energy is "stable" in the sense that $\bigl(\dot{x}(t), x(t)\bigr)$ stays in "the neighborhood" of $(0, x_0)$ for all $t \in \mathbb{R}^+$ if this is true as its motion begins, that is, at $t = 0$.

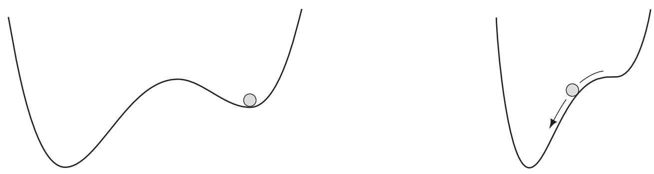


Intuitively, one can understand how x will move along the axis $\mathbb{R}$ by imagining a small ball rolling without friction along the graph of U while experiencing the force of gravity. If it lies on the "bottom of a potential well", that is, at a local minimum, then it will not move because $\dot{x}(t) = U'(x_0) = 0$. If the ball is released near a local minimum then ball will roll downhill past the minimum and up the other "slope of the valley" until

[11] We consider only the "generic" case in which $U''(x_0) \neq 0$ is satisfied if $U'(x_0) = 0$.

it "runs out" of kinetic energy at its original height. Then it will reverse course and roll until it again comes to instantaneous rest where it was initially released. Thus the ball will execute a periodic oscillation about x_0.[12]

Now we assume that U depends continuously on additional "control parameters" $u, v, \ldots$. By varying these parameters, we can vary the graph of U continuously. In this way, it can happen that a local minimum merges first into a saddle point and then ceases to be a critical point. A ball that had previously been confined to the neighborhood of the local minimum would then leave this neighborhood and oscillate about another resting point.



Now consider an observer who can see the ball move but is unaware of the mechanism underlying the process. She would see that the ball, which had before rested peacefully at a certain place, would suddenly, "for no apparent reason", begin to roll and oscillate periodically about another (fictitious) center. It would seem to be a sudden and drastic change of the situation, a "catastrophe".

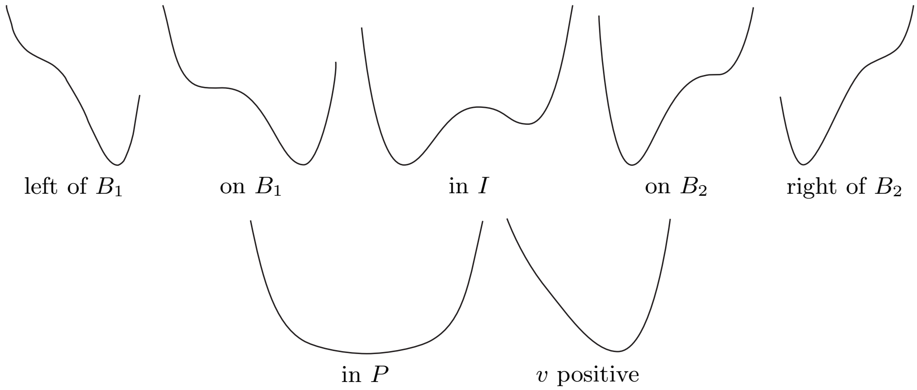
In order to understand such catastrophes (and avoid them if necessary), one must understand the mechanism by which they occur. In the situation described above, this boils down to understanding how the critical points of the potential (and in particular the relative minima) depend on the control parameters.

To illustrate, we consider the potential

$$U_{(u,v)} : \mathbb{R} \to \mathbb{R} , \quad x \mapsto ux + vx^2/2 + x^4/4$$

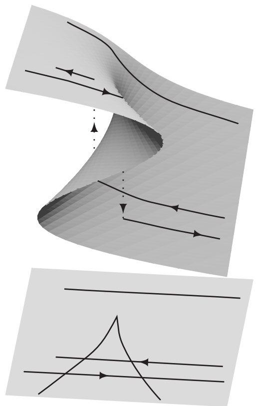
for $(u, v) \in \mathbb{R}^2$. The critical points of $U_{(u,v)}$ are just the zeros of the function f from Example 1.9(b). Therefore the manifold M, the *catastrophe manifold*, describes all critical points of the two parameter set $\{ U_{(u,v)} \ ; \ (u, v) \in \mathbb{R}^2 \}$ of potentials. Of particular interest is that subset of M, the *catastrophe set* K, consisting of all singular points of the projection π from M to the parameter space. In our example, K is a curve smoothly embedded in M, the *fold curve*, because the catastrophe manifold is "folded" along K. The image of K under π, that is, the projection of the fold curve onto the parameter plane, is the *bifurcation set* B. Every point of $\mathbb{R}^2 \backslash B$ is a regular point of π. The fiber $\pi^{-1}(u, v)$ consists of exactly one point for $(u, v) \in A \cup P$, exactly two points for $(u, v) \in B_1 \cup B_2$, and exactly three points for $(u, v) \in I$, where A and I are depicted in the illustration to Example 1.9(b). The following pictures show the qualitative form of the potential $U_{(u,v)}$ when (u, v) belongs to these sets.

[12]This plausible scenario can be proved using the theory of ordinary differential equations; see for example [Ama95].



Now consider a continuous curve C in the parameter space that begins in A and ends in I (or the reverse), while staying in $B_1 \cup B_2$. While moving continuously along this curve, the number of points in the inverse image of π will change suddenly from 1 to 3 (or from 3 to 1). As illustrated at right, one such curve C is obtained by projecting a curve Γ on the catastrophe manifold M that "jumps" when crossing the fold curve. In short, the value of x experiences a "catastrophe".

These facts have led to many interpretations of "catastrophe theory" which — not least because of its name — have been leveraged to great popularity and, especially in the popularized science literature, have kindled exaggerated hopes that the subject will somehow explain or help prevent real-world catastrophes. We refer to [Arn84] for a critical, nontechnical introduction to catastrophe theory, and we recommend [PS78] for a detailed presentation and several applications of the mathematical theory of singularities, of which catastrophe theory is a part. ■

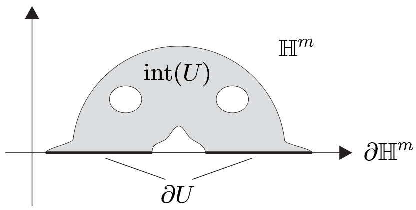


Submanifolds with boundary

We know that the open unit ball $\mathbb{B}^m$ and its boundary, the $(m-1)$-sphere S^{m-1}, are respectively m- and $(m-1)$-dimensional submanifolds of $\mathbb{R}^m$. However, the closed ball $\overline{\mathbb{B}}^m = \mathbb{B}^m \cup S^{m-1}$ is not a manifold, because a point $p \in \partial\mathbb{B}^m = S^{m-1}$ has no neighborhood U *in* $\overline{\mathbb{B}}^m$ that is mapped topologically onto an open set V of $\mathbb{R}^m$; such a neighborhood U, as the homeomorphic image of an open set V, would likewise need to be open in $\mathbb{R}^m$, which is not true. In the neighborhood of p, that is, "by viewing it with a very strong microscope", $\overline{\mathbb{B}}^m$ does not look like $\mathbb{R}^m$, but rather like a half-space. To capture such situations also, we must generalize the idea of a manifold by allowing subsets of half-spaces to be parameter sets.

In the following, $m \in \mathbb{N}^\times$, and

$$\mathbb{H}^m := \mathbb{R}^{m-1} \times (0,\infty)$$



is the **open upper half-space** of $\mathbb{R}^m$. We identify its boundary $\partial\mathbb{H}^m = \mathbb{R}^{m-1}\times\{0\}$ with $\mathbb{R}^{m-1}$ if there is no fear of misunderstanding. If U is an open subset of $\overline{\mathbb{H}}^m := \overline{\mathbb{H}^m} = \mathbb{R}^{m-1}\times\mathbb{R}^+$, we call $\mathrm{int}(U) := U\cap\mathbb{H}^m$ the **interior** and $\partial U := U\cap\partial\mathbb{H}^m$ the **boundary** of U. Note that the boundary ∂U is not the topological boundary[13] of U either in $\overline{\mathbb{H}}^m$ or in $\mathbb{R}^m$ (unless $U = \mathbb{H}^m$ in the latter case).

Suppose X is open in $\overline{\mathbb{H}}^m$ and E is a Banach space. Then $f : X \to E$ is said to be **differentiable** at the boundary point $x_0 \in \partial X$ if there is a neighborhood U of x_0 *in* $\mathbb{R}^m$ and a differentiable function $f_U : U \to E$ that agrees with f in $U \cap X$. Then it follows from Proposition VII.2.5 that

$$\begin{aligned}\partial_j f_U(x_0) &= \lim_{t\to 0+}\bigl(f_U(x_0+te_j) - f_U(x_0)\bigr)/t\\ &= \lim_{t\to 0+}\bigl(f(x_0+te_j) - f(x_0)\bigr)/t\end{aligned}$$

for $1 \le j \le m$, where $(e_1,\ldots,e_m)$ is the standard basis of $\mathbb{R}^m$. This and Proposition VII.2.8 show that $\partial f_U(x_0)$ is already determined by f. Therefore the **derivative**

$$\partial f(x_0) := \partial f_U(x_0) \in \mathcal{L}(\mathbb{R}^m, E)$$

of f is well defined at x_0, that is, independent of the choice of the local continuation f_U of f.

A map $f : X \to E$ is said to be **continuously differentiable** if f is differentiable at every point of X and if the map

$$\partial f : X \to \mathcal{L}(\mathbb{R}^m, E)\ , \quad x \mapsto \partial f(x)$$

is continuous.[14]

The higher derivatives of f are defined analogously, and these are also independent of the particular local continuation. For $k \in \mathbb{N}^\times \cup \{\infty\}$, the C^k maps of X to E form a vector space, which, as in the case of open subsets of $\mathbb{R}^m$, we denote by $C^k(X,E)$.

Suppose Y is open in $\overline{\mathbb{H}}^m$. Then $f : X \to Y$ is also called a C^k **diffeomorphism**, and we write $f \in \mathrm{Diff}^k(X,Y)$, if f is bijective and if f and f^{-1} belong to the class C^k. In particular, $\mathrm{Diff}(X,Y) := \mathrm{Diff}^\infty(X,Y)$ is the set of all smooth, that is, C^∞, diffeomorphisms from X to Y.

[13]From this point on, we use the symbol ∂M exclusively for boundaries, and, for clarity, we write $\mathrm{Rd}(M)$ for the topological boundary of a subset M of a topological space, that is, we put $\mathrm{Rd}(M) := \overline{M}\backslash\mathring{M}$.

[14]Naturally, we say f is differentiable at $x_0 \in \mathrm{int}(X)$ if $f \mid \mathrm{int}(X)$ is differentiable at x_0.

1.11 Remarks Suppose X and Y are open in $\overline{\mathbb{H}}^m$ and $f: X \to Y$ is a C^k diffeomorphism for some $k \in \mathbb{N}^\times \cup \{\infty\}$.

(a) If ∂X is not empty, then $\partial Y \neq \emptyset$, and $f \,|\, \partial X$ is a C^k diffeomorphism from ∂X to ∂Y. Also, $f \,|\, \mathrm{int}(X)$ belongs to $\mathrm{Diff}^k\bigl(\mathrm{int}(X), \mathrm{int}(Y)\bigr)$.

Proof Suppose $p \in \partial X$ and $q := f(p)$ belongs to $\mathrm{int}(Y)$. Then it follows from the inverse function theorem, Theorem VII.7.3, (applied to a local extension of f) that $\partial f^{-1}(q)$ is an automorphism of $\mathbb{R}^m$. It therefore follows, again from Theorem VII.7.3, that f^{-1} maps a suitable neighborhood V of q in $\mathrm{int}(Y)$ to an open neighborhood U of p *in* $\mathbb{R}^m$. But because $f^{-1}(V) \subset X \subset \overline{\mathbb{H}}^m$ and $p = f^{-1}(q) \in \partial X$, this is not possible. Therefore $f(\partial X) \subset \partial Y$. Analogously we find $f^{-1}(\partial Y) \subset \partial X$. This shows $f(\partial X) = \partial Y$.

Because X and Y in $\overline{\mathbb{H}}^m$ are open, both ∂X and ∂Y are open in $\partial \mathbb{H}^m = \mathbb{R}^{m-1}$, and $f \,|\, \partial X$ is a bijection from ∂X to ∂Y. Because $f \,|\, \partial X$ and $f^{-1} \,|\, \partial Y$ obviously belongs to the class C^k, we know $f \,|\, \partial X$ is a C^k diffeomorphism from ∂X to ∂Y. The last statement is now clear. ■

(b) For $p \in \partial X$, we have $\partial f(p)(\partial \mathbb{H}^m) \subset \partial \mathbb{H}^m$ and $\partial f(p)(\pm \mathbb{H}^m) \subset \pm \mathbb{H}^m$.

Proof From $f(\partial X) = \partial Y$, it follows that $f^m \,|\, \partial X = 0$ for the m-th coordinate function f^m of f. From this we get $\partial_j f^m(p) = 0$ for $1 \le j \le m-1$. Therefore the Jacobi matrix of f has at p the form

$$\bigl[\partial f(p)\bigr] = \left[\begin{array}{ccc:c} & & & \partial_m f^1(p) \\ & \partial(f \,|\, \partial \mathbb{H}^m)(p) & & \vdots \\ & & & \partial_m f^{m-1}(p) \\ \hdashline 0 & \cdots & 0 & \partial_m f^m(p) \end{array}\right] . \tag{1.7}$$

Because $f(X) \subset Y \subset \overline{\mathbb{H}}^m$, the inequality $f^m(q) \ge 0$ holds for $q \in X$. Hence we find

$$\partial_m f^m(p) = \lim_{t \to 0+} t^{-1}\bigl(f^m(p + te_m) - f^m(p)\bigr) = \lim_{t \to 0+} t^{-1} f^m(p + te_m) \ge 0 .$$

Since $\partial f(p) \in \mathcal{L}\mathrm{aut}(\mathbb{R}^m)$ (see Remark VII.7.4(d)) and since $\partial_m f^m(p) \ge 0$, we have $\partial_m f^m(p) > 0$. From (1.7), we read off

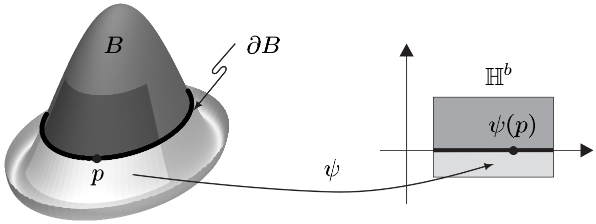
$$\bigl(\partial f(p)x\bigr)^m = \partial_m f^m(p)t \quad \text{for } x := (y,t) \in \mathbb{R}^{m-1} \times \mathbb{R} .$$

Therefore the sign of the m-th coordinate of $\partial f(p)x$ agrees with $\mathrm{sign}(t)$, and we are done. ■

We can now define the concept of submanifold with boundary. A subset B of the n-dimensional manifold N is said to be a ***b*-dimensional submanifold of *N* with boundary** if for every $p \in B$ there is a chart (ψ, V) of N around p, a **submanifold chart** of N around p for B, such that

$$\psi(V \cap B) = \psi(V) \cap \bigl(\overline{\mathbb{H}}^b \times \{0\}\bigr) \subset \mathbb{R}^n . \tag{1.8}$$

Here we say p is a **boundary point** of B if $\psi(p)$ lies in $\partial\mathbb{H}^b := \partial\mathbb{H}^b \times \{0\}$.



The set of all boundary points forms the **boundary**[15] ∂B of B. The set $\mathrm{int}(B) := B\backslash\partial B$ is called the **interior** of the submanifold B with boundary. Finally B is a **hypersurface in** N **with boundary** if $b = n - 1$.

1.12 Remarks

(a) Every submanifold M of N, in the sense given in the beginning of this section, is a submanifold with boundary, but with an empty boundary. We call such objects (**sub**)**manifolds without boundary**.

(b) The boundary ∂B and the interior $\mathrm{int}(B)$ are well defined, that is, independent of charts.

Proof Suppose (χ, W) is another submanifold chart of N around p for B. Also let f be the restriction of the transition function $\chi \circ \psi^{-1}$ to $\psi(V \cap W) \cap (\overline{\mathbb{H}}^b \times \{0\})$, understood as an open subset of $\overline{\mathbb{H}}^b$. Then it follows from Remark 1.11(a) that $\chi(p)$ belongs to $\partial\mathbb{H}^b$ if and only if $\psi(p)$ does. ■

(c) Suppose $p \in \mathrm{int}(B)$. Then (1.8) implies

$$\psi\big(V \cap \mathrm{int}(B)\big) = \psi(V) \cap \big(\mathbb{H}^b \times \{0\}\big) \ .$$

Because $\mathbb{H}^b$ is diffeomorphic to $\mathbb{R}^b$, this shows that $\mathrm{int}(B)$ is a b-dimensional submanifold of N without boundary.

(d) In the case $p \in \partial B$, it follows from (1.8) that

$$\psi(V \cap \partial B) = \psi(V) \cap \big(\mathbb{R}^{b-1} \times \{0\}\big) \ .$$

Therefore ∂B is a $(b-1)$-dimensional submanifold of N without boundary.

(e) Every b-dimensional submanifold of N with boundary is a b-dimensional submanifold of $\mathbb{R}^{\overline{n}}$ with boundary.

Proof This follows in analogy to the proof of Remark 1.1(a). ■

(f) (regularity) It is clear how C^k submanifolds with boundary are defined for $k \in \mathbb{N}^\times$, and that the analogues of (a)–(c) remain true. ■

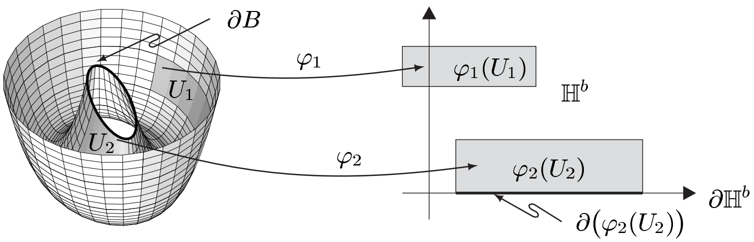
[15]Note that the boundary ∂B and the interior $\mathrm{int}(B)$ are generally different from the topological boundary $\mathrm{Rd}(B)$ and the topological interior $\mathring{B}$ of B. In the context of statements about manifolds, we will understand "boundary" and "interior" in the sense of the definitions above.

Local charts

Suppose B is a b-dimensional submanifold of N with boundary. We call the map φ a (b-dimensional local) **chart** of (or for) B around p if

- $U := \mathrm{dom}(\varphi)$ is open in B, where B carries the topology induced by N (and therefore by $\mathbb{R}^{\overline{n}}$).
- φ is a homeomorphism from U to an open subset X of $\overline{\mathbb{H}}^b$.
- $i_B \circ \varphi^{-1} : X \to N$ is an immersion, where $i_B : B \to N,\ \ p \mapsto p$ denotes the injection.

Note that except for the fact that $\varphi(U)$ is open *in* $\overline{\mathbb{H}}^b$ and $\mathbb{R}^n$ is replaced by N, this definition agrees literally with the definition of a C^∞ chart of a submanifold of $\mathbb{R}^n$ (see Section VII.9).



1.13 Remarks **(a)** If (ψ, V) is a submanifold chart of N for B, the intersected chart $(\varphi, U) := (\psi \,|\, V \cap B, V \cap B)$ is a b-dimensional chart for B.

(b) If (φ_1, U_1) and (φ_2, U_2) are charts of B around $p \in B$, then $\varphi_j(U_1 \cap U_2)$ is open in $\overline{\mathbb{H}}^b$ for $j = 1, 2$, and **transition function** $\varphi_2 \circ \varphi_1^{-1}$ satisfies

$$\varphi_2 \circ \varphi_1^{-1} \in \mathrm{Diff}\big(\varphi_1(U_1 \cap U_2), \varphi_2(U_1 \cap U_2)\big) \ .$$

(c) Suppose (φ, U) is a chart for B around $p \in \partial B$. Then

$$(\varphi_{\partial B}, U_{\partial B}) := (\varphi \,|\, U \cap \partial B, U \cap \partial B)$$

is a chart for ∂B, a $(b-1)$-dimensional submanifold of N without boundary.

(d) All concepts and definitions, for example, differentiability of maps and local representations, that can be described using charts of manifolds, carry over straightforwardly to submanifolds with boundary. In particular, $i_B : B \hookrightarrow N$, that is, the natural embedding $p \mapsto p$ of B in N, is a smooth map.

(e) If C is a submanifold of M with boundary and $f \in \mathrm{Diff}(B, C)$, then $f(\partial B) = \partial C$, and $f \,|\, \partial B$ is a diffeomorphism from ∂B to ∂C.

Proof This follows from Remark 1.11(a). ∎

(f) Suppose B is a b-dimensional submanifold of N with boundary, and $f \in C^\infty(B, M)$ is an **embedding**, that is, f is a bijective immersion and a homeomorphism from B to $f(B)$. Then $f(B)$ is a b-dimensional submanifold M with boundary satisfying $\partial f(B) = f(\partial B)$, and f is a diffeomorphism from B to $f(B)$.

Proof The proof of Theorem 1.2(ii) also applies here. ■

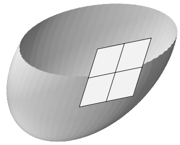
(g) (regularity) All previous statements transfer literally to C^k submanifolds with boundary. ■

Naturally, we again say a family $\{ (\varphi_\alpha, U_\alpha) \ ; \ \alpha \in \mathsf{A} \}$ of charts of B with $B = \bigcup_\alpha U_\alpha$ is an **atlas** of B.

Tangents and normals

Suppose B is a submanifold of N with boundary, and let $p \in \partial B$. Also suppose (φ, U) is a chart of B around p. Then we define the **tangent space** T_pB of B at the point p by

$$T_pB := T_{\varphi(p)}(i_B \circ \varphi^{-1})(T_{\varphi(p)}\mathbb{R}^b) \ ,$$



where $b := \dim(B)$. Therefore T_pB is a ("full") b-dimensional vector subspace of the tangent space T_pN of N at p (and not, say, a half-space). An obvious modification of the proof of Remark VII.10.3(a) shows that T_pB is well defined, that is, independent of which chart is used. In this case, we define the **tangent bundle** TB of B by $TB := \bigcup_{p \in B} T_pB$.

1.14 Remarks **(a)** For $p \in \partial B$, $T_p\partial B$ is a $(b{-}1)$-dimensional vector subspace of T_pB.

Proof This is a simple consequence of Remarks 1.12(d) and 1.13(c). ■

(b) Suppose $p \in \partial B$ and (φ, U) is chart of B around p. Letting

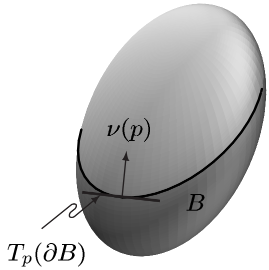
$$T_p^{\pm}B := T_{\varphi(p)}(i_B \circ \varphi^{-1})\big(\varphi(p), \pm\overline{\mathbb{H}}^b\big) \ ,$$

we have $T_pB = T_p^+B \cup T_p^-B$ and $T_p^+B \cap T_p^-B = T_p(\partial B)$. The vector v is an **inward pointing** [or an **outward pointing**] tangent vector if and only if v belongs to the set $T_p^+B \backslash T_p(\partial B)$ [or $T_p^-B \backslash T_p(\partial B)$]. This is the case if and only if the b-th component of $(T_p\varphi)v$ is positive [or negative].

Proof From Remarks 1.11(b) and 1.13(b), it follows easily that $T_p^{\pm}B$ is defined in a coordinate-independent way. ■

(c) Let C be a submanifold of M with or without boundary. For $f \in C^1(C, N)$, the **tangential** T_pf of f at $p \in C$ is defined as in the case of manifolds without boundary. Then the analogues of Remarks VII.10.9 remain true. ■

Suppose $p \in \partial B$. Then $T_p(\partial B)$ is a $(b-1)$-dimensional vector subspace of the b-dimensional vector space T_pB. As a vector subspace of T_pN (and therefore of $T_p\mathbb{R}^{\overline{n}}$), T_pB is an inner product space with the inner product $(\cdot\,|\,\cdot)_p$ induced by the Euclidean scalar product on $\mathbb{R}^{\overline{n}}$. Hence there is exactly one unit vector $\nu(p)$ in T_p^-B that is orthogonal to $T_p(\partial B)$, and we call it the **outward** (**unit**) **normal vector** of ∂B at p. Clearly $-\nu(p) \in T_p^+B$ is the unique inward pointing vector of T_pB that is orthogonal to $T_p(\partial B)$, and we call it the **inward** (**unit**) **normal** vector ∂B at p.



The regular value theorem

We have already seen that submanifolds without boundary can be represented in many cases (actually always, locally) as fibers of regular maps. We will now extend this important and simple criterion to the case of submanifolds with boundary.

1.15 Theorem (regular value) *Suppose c is a regular value of $f \in C^\infty(N,\mathbb{R})$. Then*

$$B := f^{-1}\big((-\infty, c]\big) = \big\{\, p \in N\ ;\ f(p) \le c \,\big\}$$

is an n-dimensional submanifold of N with boundary with $\partial B = f^{-1}(c)$ and $\operatorname{int}(B) = f^{-1}\big((-\infty,c)\big)$. For $p \in \partial B$, we have $T_p(\partial B) = \ker(d_pf)$, and the outward unit normal $\nu(p)$ on ∂B is given by $\nabla_p f(p)/|\nabla_p f|_p$.

Proof Because $f^{-1}((-\infty,c))$ is open in N and is therefore an n-dimensional submanifold of N, it suffices to consider $p \in f^{-1}(c)$.

Therefore let $p \in f^{-1}(c)$, and let (ψ, V) be a chart of N around p such that $\psi(p) = 0$. Then $g := c - f \circ \psi^{-1}$ belongs to $C^\infty(\psi(V),\mathbb{R})$ and satisfies $g(0) = 0$ and $g(x) \ge 0$ if and only if x lies in $\psi(V \cap B)$. Also 0 is a regular point of g. By renaming the coordinates (that is, by composing ψ with a permutation), we can assume that $\partial_n g(0) \ne 0$ and therefore $\partial_n g(0) > 0$.

Consider the map $\varphi \in C^\infty(\psi(V),\mathbb{R}^n)$ defined by $\varphi(x) := (x^1,\ldots,x^{n-1},g(x))$. It satisfies $\varphi(0) = 0$ and

$$\partial\varphi = \left[\begin{array}{ccc:c} & & & 0 \\ & 1_{n-1} & & \vdots \\ & & & 0 \\ \hdashline \partial_1 g & \cdots & \partial_{n-1} g & \partial_n g \end{array}\right] .$$

Therefore $\partial\varphi(0)$ is an automorphism of $\mathbb{R}^n$, and Theorem VII.7.3 (the inverse function theorem) guarantees the existence of open neighborhoods U and W of 0 in $\psi(V)$ such that $\varphi \,|\, U$ is a diffeomorphism from U to W.

Letting $V_0 := \psi^{-1}(U)$ and $\chi := \varphi \circ \psi \,|\, V_0$, we see that (χ, V_0) is a chart of N around p with $\chi(p) = 0$ and $\chi(B \cap V_0) = \chi(V_0) \cap \overline{\mathbb{H}}^n$. This shows that B is a submanifold of N with boundary with $\partial B = f^{-1}(c)$ and $\operatorname{int}(B) = f^{-1}((-\infty, c))$. Thus we get from Theorem 1.7 that

$$T_p(\partial B) = \ker(T_p f) = \ker(d_p f) \quad \text{for } p \in \partial B \ . \tag{1.9}$$

Because $\langle d_p f, v\rangle_p = (\nabla_p f \,|\, v)_p$ for $v \in T_pN$, it follows from (1.9) that $\nabla_p f$ is orthogonal to $T_p(\partial B)$.

Finally, let $\lambda\colon (-\varepsilon, \varepsilon) \to N$ be a C^1 path in N with $\lambda(0) = p$ and $\dot\lambda(0) = \nabla_p f$ (see Theorem VII.10.6). Then

$$(f \circ \lambda)^{\cdot}(0) = \langle d_p f, \nabla_p f\rangle = |\nabla_p f|_p^2 > 0 \ .$$

Therefore we derive from the Taylor formula of Corollary IV.3.3 that

$$f(\lambda(t)) = c + t\, |\nabla_p f|_p^2 + o(t) \quad (t \to 0) \ .$$

Therefore $f(\lambda(t)) > c$, that is, $f(\lambda(t)) \notin B$ for sufficiently small positive t. This implies that $\nabla_p f$ is an outward pointing tangent vector of B at p. Now the last claim is also clear. ■

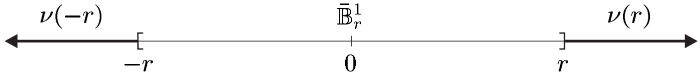
1.16 Remarks **(a)** Because we can locally represent submanifolds as fibers of regular maps (see Remark 1.8(a)), we can also locally represent submanifolds with boundary as inverse images of half open intervals. More precisely, suppose B is an n-dimensional submanifold of N with boundary. Then there is for every point $p \in B$ a neighborhood U in N and a function $f \in C^\infty(U, \mathbb{R})$ such that $B \cap U = f^{-1}((-\infty, 1))$ if $p \in \operatorname{int}(B)$ but $f(p) = 0$ and $B \cap U = f^{-1}((-\infty, 0])$ if $p \in \partial B$, and for which 0 is a regular value.

Proof Suppose (φ, U) is a submanifold chart of N around p for B with $\varphi(p) = 0$. We can assume that $\varphi(U)$ is contained in $\mathbb{B}^n_\infty$. If p is an interior point of B, we set $f(q) := \varphi^n(q)$ for $q \in U$. Then f belongs to $C^\infty(U, \mathbb{R})$, and $f^{-1}((-\infty, 1)) = U$. If p belongs to ∂B, we set $f(q) := -\varphi^n(q)$ for $q \in U$. Then $f(p) = 0$, and $f^{-1}((-\infty, 0]) = U \cap B$. Because $\varphi \in \operatorname{Diff}(U, \varphi(U))$, we know f is a submersion. Therefore 0 is a regular value of f. ■

(b) (regularity) Suppose c is a regular value of $f \in C^k(N, \mathbb{R})$ for some $k \in \mathbb{N}^\times$. Then $f^{-1}((-\infty, c])$ is an n-dimensional submanifold in N with boundary. In this case, one need only assume that N is a C^k manifold. ■

1.17 Examples **(a)** For every $r > 0$, $\overline{\mathbb{B}}^n_r := r\overline{\mathbb{B}}^n = \{\, x \in \mathbb{R}^n \ ;\ |x| \le r \,\}$ is an n-dimensional submanifold of $\mathbb{R}^n$ with boundary. Its boundary coincides with the topological boundary and therefore with the $(n{-}1)$-sphere of radius r, that is, $\partial\overline{\mathbb{B}}^n_r = rS^{n-1}$. The outward normal $\nu(p)$ at $p \in \partial\overline{\mathbb{B}}^n_r$ is given by $(p, p/|p|)$.

In the case $n = 1$, the ball $\bar{\mathbb{B}}_r^1$ is the closed interval $[-r, r]$ in $\mathbb{R}$, and the 0-sphere with radius r is given by $S_r^0 = \{-r\} \cup \{r\}$. The outward normal has $\nu(-r) = (-r, -1)$ and $\nu(r) = (r, 1)$.



Proof This follows from Theorem 1.15 with $N := \mathbb{R}^n$ and $f(x) := |x|^2$ for $x \in \mathbb{R}^n$. ■

(b) Suppose $A \in \mathbb{R}_{\mathrm{sym}}^{(n+1)\times(n+1)}$ and $c \in \mathbb{R}^\times$. Also suppose

$$V_c := \{\, x \in \mathbb{R}^{n+1} \ ; \ (Ax \,|\, x) \le c \,\}$$

is not empty. If A is positive definite and $c > 0$, then V_c is an $(n+1)$-dimensional solid whose boundary is the n-dimensional ellipsoid

$$K_c := \{\, x \in \mathbb{R}^{n+1} \ ; \ (Ax \,|\, x) = c \,\} \ .$$

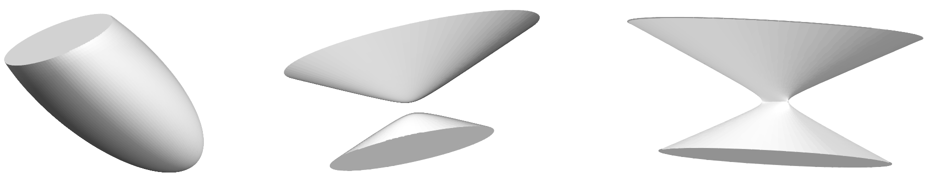
If A is negative definite and $c < 0$, then V_c is the complement of the interior of V_{-c}, and the boundary of V_{-c} is the n-dimensional ellipsoid K_{-c}. If A is indefinite but invertible, then V_c is the "interior" or "exterior" of an appropriate n-dimensional hyperboloid K_c that bounds V_c. In every case, $Ax/|Ax|$ is the outward normal of V_c at K_c. (Compare this with Remark VII.10.18, and interpret the pictures there accordingly.)

(c) Suppose $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is symmetric and $c \in \mathbb{R}^\times$ with $K_c \neq \emptyset$. Also suppose $v \in \mathbb{R}^{n+1}\backslash\{0\}$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Then

$$B := \{\, x \in K_c \ ; \ \alpha \le (v \,|\, x) \le \beta \,\}$$

is the part of K_c that lies between the two parallel hyperplanes

$$H_\gamma := \{\, x \in \mathbb{R}^{n+1} \ ; \ (v \,|\, x) = \gamma \,\} \quad \text{for } \gamma \in \{\alpha, \beta\} \ .$$



If H_α and H_β are not tangent hyperplanes of K_c, then B is an n-dimensional submanifold of K_c with boundary with

$$\partial B = \{\, x \in K_c \ ; \ (v \,|\, x) \in \{\alpha, \beta\} \,\} \ .$$

Proof Because the map $g := (v \,|\, \cdot\,) \,|\, K_c : K_c \to \mathbb{R}$ is smooth by Remark 1.1(j), we know $g^{-1}((\alpha, \beta))$ is open in K_c. Therefore $g^{-1}((\alpha, \beta))$ is an n-dimensional submanifold of K_c.

Hence it suffices to show that every $p \in g^{-1}(\{\alpha,\beta\})$ is a boundary point of B. So let V be an open neighborhood in K_c of $p \in g^{-1}(\beta)$ such that $g^{-1}(\alpha) \cap V = \emptyset$. The assumption at H_β is not a tangent hyperplane implies that β is a regular value of $f := g\,|\,V$ (prove this!). The claim now follows from Theorem 1.15 applied to the manifold V and the function f. A similar argument shows that every $p \in g^{-1}(\alpha)$ is a boundary point of B. ■

(d) (cylinder-like rotational hypersurfaces) Suppose

$$\gamma\colon [0,1] \to (0,\infty) \times \mathbb{R}\ , \quad t \mapsto \big(\rho(t), \sigma(t)\big)$$

is a smooth embedding. Also let $i\colon S^m \hookrightarrow \mathbb{R}^{m+1}$ and

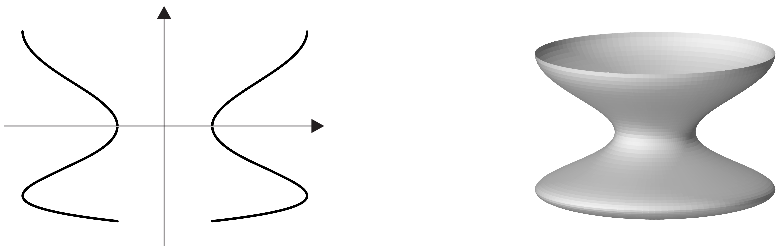
$$f\colon S^m \times [0,1] \to \mathbb{R}^{m+1} \times \mathbb{R}\ , \quad (q,t) \mapsto \big(\rho(t) i(q), \sigma(t)\big)\ .$$

Then f is a smooth embedding, and

$$Z^{m+1} := f\big(S^m \times [0,1]\big)$$

is a hypersurface in $\mathbb{R}^{m+2}$ with boundary which is diffeomorphic to the "spherical cylinder" $S^m \times [0,1]$.

In the case $m = 0$, Z^1 consists of two copies of smooth, non-self-intersecting, compact curves[16] $\gamma([0,1])$ that are symmetric about the y-axis.



For $m = 1$, Z^2 is the surface of rotation in $\mathbb{R}^3$ obtained by rotating the **meridian curve**

$$\Gamma := \{\, (\rho(t), 0, \sigma(t))\ ;\ t \in [0,1]\,\}$$

around the z-axis.

In the general case, we call Z^{m+1} a **cylinder-like surface of rotation with boundary**. Its boundary satisfies

$$\partial Z^{m+1} = f\big(S^m \times \{0\}\big) \cup f\big(S^m \times \{1\}\big)\ ,$$

while its interior has

$$\operatorname{int}(Z^{m+1}) = f\big(S^m \times (0,1)\big)\ .$$

[16] That is, one-dimensional manifolds with boundary.

In particular, $\mathrm{int}(Z^{m+1})$ is a cylinder-type hypersurface of rotation without boundary. In the case $m = 1$, it is generated by rotating the meridian curve

$$\mathrm{int}(\Gamma) := \{ \big(\rho(t), 0, \sigma(t)\big) \; ; \; 0 < t < 1 \}$$

around the z-axis.

Proof It is easy to see[17] that $S^m \times [0,1]$ is a submanifold of $\mathbb{R}^{m+2}$ with boundary and that its boundary is $(S^m \times \{0\}) \cup (S^m \times \{1\})$. An obvious modification of the proof of Example 1.5(b) shows that f is an embedding. Now the claims follow from Remark 1.13(f). ■

One-dimensional manifolds

Obviously every perfect interval J in $\mathbb{R}$ is a one-dimensional submanifold of $\mathbb{R}^n$ with or without boundary, depending on whether J is open or not. Also, we already know that the 1-sphere S^1 is a one-dimensional submanifold of $\mathbb{R}^n$, provided $n \geq 2$. It is easy to see[18] that a nonempty perfect interval is diffeomorphic to $(0,1)$ if it is open, to $[0,1)$ if it is closed on one side, and to $[0,1]$ if it is compact. The following important classification theorem shows that these intervals and S^1 are, up to diffeomorphism, the only one-dimensional connected manifolds.

1.18 Theorem *Suppose C is a connected one-dimensional submanifold N with [or without] boundary. Then C is diffeomorphic to $[0,1]$ or $[0,1)$ [or to $(0,1)$] or S^1.*

Proof For a proof, we refer to Section 3.4 of [BG88], which treats manifolds without boundary. An obvious modifications of the arguments there also covers the case of manifolds with boundary (see the appendix in [Mil65]). ■

1.19 Remarks **(a)** We understand a (smooth) **curve *C* embedded** in N to be the image of a perfect interval of S^1 under a (smooth) embedding. In the last case, we also call C the **1-sphere embedded in *N***. Then Theorem 1.18 says that every connected one-dimensional submanifold of N with or without boundary is an embedded curve, and conversely.

(b) (regularity) Theorem 1.18 remains true for C^1 manifolds. ■

Partitions of unity

We conclude this section by proving a technical result which will be particularly helpful in the transition from local to global (and conversely).

[17] See Exercise 4.
[18] See Exercise 7.

Suppose X is an n-dimensional submanifold of $\mathbb{R}^{\overline{n}}$ with or without boundary for some $\overline{n} \in \mathbb{N}^{\times}$. Also let $\{ U_\alpha \ ; \ \alpha \in \mathsf{A} \}$ be an open cover of X. Then we say that the family $\{ \pi_\alpha \ ; \ \alpha \in \mathsf{A} \}$ is a **smooth partition of unity of unity** subordinate to this cover if it satisfies the properties

(i) $\pi_\alpha \in C^\infty\bigl(X, [0,1]\bigr)$ with $\operatorname{supp}(\pi_\alpha) \subset\subset U_\alpha$ for $\alpha \in \mathsf{A}$;

(ii) the family $\{ \pi_\alpha \ ; \ \alpha \in \mathsf{A} \}$ is **locally finite**, that is, for every $p \in X$ there is an open neighborhood V such that $\operatorname{supp}(\pi_\alpha) \cap V = \emptyset$ for all but finitely many $\alpha \in \mathsf{A}$;

(iii) $\sum_{\alpha \in \mathsf{A}} \pi_\alpha(p) = 1$ for every $p \in X$.

1.20 Proposition *Every open cover X has a smooth partition of unity subordinate to it.*

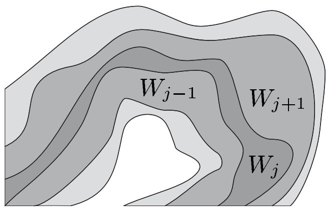
Proof (i) Let (φ, U) be a chart around $p \in X$. Then $\varphi(U)$ is open in $\overline{\mathbb{H}}^n$. Hence there is a compact neighborhood K' of $\varphi(p)$ in $\overline{\mathbb{H}}^n$ such that $K' \subset \varphi(U)$. Because φ is topological, $K := \varphi^{-1}(K')$ is a compact neighborhood of p in X with $K \subset U$, and $(\varphi \,|\, \mathring{K}, \mathring{K})$ is a chart around p. In particular, X is locally compact.

Proposition X.7.14 implies the existence of a $\chi' \in C^\infty\bigl(\varphi(U), [0,1]\bigr)$ with $\chi' \,|\, K' = 1$ and $\operatorname{supp}(\chi') \subset\subset \varphi(U)$. We set $\chi(q) := \varphi^*\chi'(q)$ if $q \in U$ and $\chi(q) := 0$ if q belongs to $X \backslash U$. Then χ lies in $C^\infty\bigl(X, [0,1]\bigr)$ and has compact support, which is contained in U.

(ii) By Corollary IX.1.9(ii) and Remark X.1.16(e), there exists a countable cover $\{ V_j \ ; \ j \in \mathbb{N} \}$ of X consisting of relatively compact open sets. We set $K_0 := \overline{V}_0$. Then there are $i_0, \ldots, i_m \in \mathbb{N}$ such that K_0 is covered by $\{V_{i_0}, \ldots, V_{i_m}\}$. In addition, we set $j_1 := \max\{i_0, \ldots, i_m\} + 1$ and $K_1 := \bigcup_{i=0}^{j_1} \overline{V}_i$. The set K_1 is compact, and $K_0 \subset\subset K_1$. We then inductively obtain a sequence (K_j) of compact sets with $K_j \subset\subset K_{j+1}$, and $\bigcup_{j=0}^\infty K_j = \bigcup_{j=0}^\infty V_j = X$.

(iii) We first assume that $K_j \neq K_{j+1}$ for $j \in \mathbb{N}$, and we set $W_j := K_j \backslash \mathring{K}_{j-1}$ for $j \in \mathbb{N}$ with $K_{-1} := \emptyset$. Then W_j is compact, and $W_j \cap W_k = \emptyset$ for $|j-k| \geq 2$. We also have $\bigcup_{j=0}^\infty W_j = X$.

Let $\mathcal{U} := \{ U_\alpha \ ; \ \alpha \in \mathsf{A} \}$ be an open cover of X. From (i) and the compactness of W_j, it follows that for every $j \in \mathbb{N}$, there is a finite cover $\bigl\{ \widetilde{U}_{j,i} \in \mathcal{U} \ ; \ 0 \leq i \leq m(j) \bigr\}$ of W_j. We set



$$U_{j,i} := \widetilde{U}_{j,i} \cap (\mathring{W}_{j-1} \cup W_j \cup \mathring{W}_{j+1})$$

and choose functions $\chi_{j,i} \in C^\infty\bigl(U_{j,i}, [0,1]\bigr)$ so that

$$\operatorname{supp}(\chi_{j,i}) \subset\subset U_{j,i} \subset \mathring{W}_{j-1} \cup W_j \cup \mathring{W}_{j+1} \quad \text{for } 0 \leq i \leq m(j) \ ,$$

with $W_{-1} := \emptyset$ and

$$\bigcup_{i=0}^{m(j)} [\chi_{j,i} > 0] \supset W_j$$

for $j \in \mathbb{N}$. Then $\{\, \chi_{j,i} \ ;\ 0 \le i \le m(j),\ j \in \mathbb{N} \,\}$ is a locally finite family. Therefore

$$\chi := \sum_{j=0}^{\infty} \sum_{i=0}^{m(j)} \chi_{j,i}$$

is defined, belongs to $C^\infty\big(X, [0,1]\big)$, and satisfies $\chi(p) > 0$ for $p \in X$. Now we set

$$\pi_\alpha := \sum\nolimits_\alpha \chi_{j,i}/\chi \quad \text{for } \alpha \in \mathsf{A} ,$$

where $\sum_\alpha$ means the sum over all index pairs (j,i) for which $U_{j,i}$ is contained in U_α. Then $\{\, \pi_\alpha \ ;\ \alpha \in \mathsf{A}\}$ is a smooth partition of unity subordinate to the cover $\{\, U_\alpha \ ;\ \alpha \in \mathsf{A} \,\}$.

(iv) If there is a $j \in \mathbb{N}$ such that $K_j = K_{j+1}$, then $X = K_j$. Therefore X is compact. In this case, the claim follows by a simple modification of (iii) (as only a single compact set, namely X, must be considered). ■

Remark (a), below, shows that Proposition 1.20 is a wide-reaching generalization of Theorem X.7.16.

1.21 Remarks **(a)** Suppose K is a compact subset of the manifold X, and suppose $\{\, U_j \ ;\ 1 \le j \le m\}$ is an open cover of K. Then there are functions $\pi_j \in C^\infty(X, [0,1])$ such that $\operatorname{supp}(\pi_j) \subset\subset U_j$ for $1 \le j \le m$, and $\sum_{j=1}^m \pi_j(p) = 1$ for $p \in K$.

Proof Let $U_0 := X \backslash K$. Then $\{\, U_j \ ;\ 0 \le j \le m \,\}$ is an open cover of X. Now the claim follows easily from Proposition 1.20. ■

(b) The proof of Proposition 1.20 shows that every submanifold of $\mathbb{R}^{\overline{m}}$ with or without boundary is locally compact, has a countable basis, and is σ-compact.

(c) (regularity) Suppose $k \in \mathbb{N}^\times$. Replacing $\pi_\alpha \in C^\infty(X, [0,1])$ by $\pi_\alpha \in C^k(X, [0,1])$ in part (i) of the definition above, we obtain a C^k partition of unity subordinate to the cover $\{\, U_\alpha \ ;\ \alpha \in \mathsf{A} \,\}$. Then Proposition 1.20 remains true if one replaces "smooth partition" by "C^k partition". In this case, it suffices to assume that X belongs to the class C^k. ■

Convention In the rest of this book, we understand every manifold to be a smooth submanifold with boundary in a suitable "surrounding space" $\mathbb{R}^{\overline{m}}$.

Exercises

1 Suppose $f : M \to N$ is a submersion. Show that f "locally looks like a projection", that is, for every $p \in M$, there are charts (φ, U) of M around p and (ψ, V) of N around

$f(p)$ with $f(U) \subset V$ that satisfy

$$f_{\varphi,\psi} : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n \ , \quad (x,y) \mapsto x \ .$$

2 Suppose $f : M \to N$ is an immersion. Prove that f locally looks like the canonical injection $\mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{n-m}$, $x \mapsto (x,0)$.

3 Show that every diffeomorphism from M to N locally looks like the identity in $\mathbb{R}^m$.

4 Suppose B is a submanifold of N with boundary. Show that $M \times B$ is a submanifold of $M \times N$ with boundary with $\partial(M \times B) = M \times \partial B$.

5 Show that both the cylinder $[0,1] \times M$ with "cross section' M and the "filled" torus $S^1 \times \overline{\mathbb{B}}^2$ are manifolds with boundary. Determine the dimension and boundary of each.

6 Show that the closed r-ball $r\overline{\mathbb{B}}^n$ in $\mathbb{R}^n$ is diffeomorphic to the closed unit ball $\overline{\mathbb{B}}^n$.

7 Show that a perfect interval in $\mathbb{R}$ is diffeomorphic to $(0,1)$, $[0,1)$, or $[0,1]$.

8 Suppose B is a nonempty k-dimensional submanifold of M (with or without boundary). Show that the Hausdorff dimension of B equals k.
(Hints: Exercises 4–6 of IX.3 and Remark 1.21(b).)

9 Suppose B is a submanifold of M with boundary and $f \in C^\infty(B,N)$. Show that graph(f) is a submanifold of $M \times N$ with boundary and determine its boundary.

10 Suppose X is an n-dimensional submanifold of $\mathbb{R}^{\overline{n}}$ with or without boundary, and let $U := \{\, U_\alpha \ ; \ \alpha \in \mathsf{A} \,\}$ and $V := \{\, V_\beta \ ; \ \beta \in \mathsf{B} \,\}$ denote open covers of X. We call $\mathcal{V}$ a **refinement** of U if there is a $j : \mathsf{B} \to \mathsf{A}$ such that $V_\beta \subset U_{j(\beta)}$ for $\beta \in \mathsf{B}$. Show that every smooth partition of unity subordinate to $\mathcal{V}$ induces a smooth partition of unity subordinate to U.

2 Multilinear algebra

To construct and understand the calculus of differential forms of higher degree, we need several results from linear (more precisely, multilinear) algebra, which we provide in this section.

2.1 Remarks Suppose V is a finite-dimensional vector space.

(a) V can be provided with an inner product $(\cdot\,|\,\cdot)_V$, so that $(V, (\cdot\,|\,\cdot)_V)$ is a Hilbert space. All norms on V are equivalent.

Proof By Remark I.12.5, there is a vector space isomorphism $T : \mathbb{K}^m \to V$ such that $m := \dim(V)$. Then

$$(v\,|\,w)_V := (T^{-1}v\,|\,T^{-1}w) \quad \text{for } v, w \in V$$

defines a scalar product on V, where $(\cdot\,|\,\cdot)$ denotes the Euclidean inner product in $\mathbb{K}^m$. Thus $(V, (\cdot\,|\,\cdot)_V)$ is a finite-dimensional inner product space and therefore a Hilbert space, as we know from Remark VII.1.7(b). The second claim follows from Corollary VII.1.5. ■

(b) As usual (in functional analysis), we denote by V^* the space of all (continuous) conjugate linear maps from V to $\mathbb{C}$, while V' is the space dual to V, the space of all (continuous) linear forms on V. Then it follows from (a) and the Riesz representation theorem (Theorem VII.2.14) that the map

$$V \to V^* , \quad v \mapsto (v\,|\,\cdot)_V \tag{2.1}$$

is an isometric isomorphism, whereas

$$V \to V' , \quad v \mapsto (\cdot\,|\,v)_V \tag{2.2}$$

is conjugate linear. If $\mathbb{K} = \mathbb{R}$, then $V^* = V'$, and the maps (2.1) and (2.2) are identical because every real scalar product is symmetric. In the following, we will exclusively treat the real case, and so, for this and some historical reasons, we will write V^* instead of V'. ■

In this section, let

- V and W be finite-dimensional real vector spaces.

Exterior products

For $r \in \mathbb{N}$, we denote by $\mathcal{L}^r(V, \mathbb{R})$ the vector space of all r-linear maps $V^r \to \mathbb{R}$. By Remark 2.1(b) and Theorem VII.4.2(iii), this notation is consistent with that introduced in Section VII.4. In particular, we have

$$\mathcal{L}^0(V, \mathbb{R}) = \mathbb{R} \quad \text{and} \quad \mathcal{L}^1(V, \mathbb{R}) = V^* .$$

An r-linear map $\alpha\colon V^r \to W$ is said to be **alternating** if $r \geq 2$ and

$$\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) = \operatorname{sign}(\sigma)\, \alpha(v_1, \ldots, v_r) \quad \text{for } v_1, \ldots, v_r \in V$$

and for every permutation $\sigma \in \mathsf{S}_r$ (see Exercise I.9.6). We set

$$\textstyle\bigwedge^0 V^* := \mathcal{L}^0(V, \mathbb{R}) = \mathbb{R} \quad \text{and} \quad \bigwedge^1 V^* := \mathcal{L}^1(V, \mathbb{R}) = V^*$$

and

$$\textstyle\bigwedge^r V^* := \big\{\, \alpha \in \mathcal{L}^r(V, \mathbb{R}) \ ; \ \alpha \text{ is alternating} \,\big\} \quad \text{for } r \geq 2 \ .$$

Here $\bigwedge^r V^*$ is called the **r-fold exterior product of V^*** for $r \in \mathbb{N}$, and $\alpha \in \bigwedge^r V^*$ is an **alternating r-form on V** (or, for short, simply an **r-form**).

2.2 Remarks **(a)** $\bigwedge^r V^*$ is a vector subspace of $\mathcal{L}^r(V, \mathbb{R})$, the **vector space of alternating r-forms** on V.

(b) Let $r \geq 2$ and $\alpha \in \mathcal{L}^r(V, \mathbb{R})$. These four statements are equivalent:

(i) $\alpha \in \bigwedge^r V^*$.

(ii) $\alpha(v_1, \ldots, v_r) = 0$ if $v_j = v_k$ for any a pair (j, k) with $j \neq k$.

(iii) $\alpha(\ldots, v_j, \ldots, v_k, \ldots) = -\alpha(\ldots, v_k, \ldots, v_j, \ldots)$ for $j \neq k$, that is, if two entries in $\alpha(v_1, \ldots, v_r)$ are exchanged, its sign reverses.

(iv) If $v_1, \ldots, v_r \in V$ are linearly independent, then $\alpha(v_1, \ldots, v_r) = 0$.

Proof The implication "(i)⇒(iii)⇒(ii)" is obvious.

"(ii)⇒(iv)" Suppose $v_1, \ldots, v_r \in V$ are linearly independent. This means there are $k \in \{1, \ldots, r\}$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ such that $\lambda_k = 0$ and $v_k = \sum_{j=1}^r \lambda_j v_j$. Now it follows from the linearity of α in its k-th variable and from (ii) that

$$\alpha(v_1, \ldots, v_r) = \sum_{j=1}^r \lambda_j \alpha(v_1, \ldots, \underset{(k)}{v_j}, \ldots, v_r) = 0 \ .$$

"(iv)⇒(iii)" From (iv) and the multilinearity, we get

$$\begin{aligned} 0 &= \alpha(\ldots, v_j + v_k, \ldots, v_j + v_k, \ldots) \\ &= \alpha(\ldots, v_j, \ldots, v_j, \ldots) + \alpha(\ldots, v_j, \ldots, v_k, \ldots) \\ &\qquad\qquad + \alpha(\ldots, v_k, \ldots, v_j, \ldots) + \alpha(\ldots, v_k, \ldots, v_k, \ldots) \\ &= \alpha(\ldots, v_j, \ldots, v_k, \ldots) + \alpha(\ldots, v_k, \ldots, v_j, \ldots) \ , \end{aligned}$$

and which proves the claim.

"(iii)⇒(i)" This follows from the fact that every permutation can be written as a product of transpositions (see Exercise I.9.6). ■

(c) $\bigwedge^r V^* = \{0\}$ for $r > \dim(V)$.

Proof This follows from (iv) of (b). ■

For $r \in \mathbb{N}^\times$ and $\varphi^1, \ldots, \varphi^r \in V^*$, the **exterior product**[1]

$$\varphi^1 \wedge \cdots \wedge \varphi^r$$

is defined by

$$\varphi^1 \wedge \cdots \wedge \varphi^r(v_1, \ldots, v_r) := \det\bigl[\langle \varphi^j, v_k \rangle\bigr] = \det \begin{bmatrix} \langle \varphi^1, v_1 \rangle & \cdots & \langle \varphi^1, v_r \rangle \\ \vdots & & \vdots \\ \langle \varphi^r, v_1 \rangle & \cdots & \langle \varphi^r, v_r \rangle \end{bmatrix} \tag{2.3}$$

for $v_1, \ldots, v_r \in V$. It is known from linear algebra that the determinant of an $(r \times r)$-matrix is an alternating r-form in its column vectors. From this and the linearity of $\varphi^1, \ldots, \varphi^r$, it follows immediately that $\varphi^1 \wedge \cdots \wedge \varphi^r$ belongs to $\bigwedge^r V^*$: The exterior product $\varphi^1 \wedge \cdots \wedge \varphi^r$ is an alternating r-form on V.

2.3 Proposition

(i) *Let $m := \dim(V) > 0$. If $(e_1, \ldots, e_m)$ is a basis*[2] *of V and $(\varepsilon^1, \ldots, \varepsilon^m)$ is the associated dual basis of V^*, then*

$$\{\, \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_r} \ ;\ 1 \le j_1 < j_2 < \cdots < j_r \le m \,\}$$

is a basis of $\bigwedge^r V^*$ for $1 \le r \le m$.

(ii) $\dim(\bigwedge^r V^*) = \binom{m}{r}$ *for $r \in \mathbb{N}$.*

Proof For short, we set

$$\mathbb{J}_r := \mathbb{J}_r^m := \{\, (j) := (j_1, \ldots, j_r) \in \mathbb{N}^r \ ;\ 1 \le j_1 < j_2 < \cdots < j_r \le m \,\} .$$

Also, for an ordered multiindex $(j) \in \mathbb{J}_r$, let

$$\varepsilon^{(j)} := \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_r} .$$

(i) Let α be an alternating r-form. Because every vector $v \in V$ has the basis representation $v = \sum_{k=1}^m \langle \varepsilon^k, v \rangle e_k$, it follows from Remark 2.2(b) that

$$\begin{aligned} \alpha(v_1, \ldots, v_r) &= \sum_{k_1=1}^m \cdot \cdots \cdot \sum_{k_r=1}^m \langle \varepsilon^{k_1}, v_1 \rangle \cdot \cdots \cdot \langle \varepsilon^{k_r}, v_r \rangle \alpha(e_{k_1}, \ldots, e_{k_r}) \\ &= \sum_{(j) \in \mathbb{J}_r} a_{(j)} \sum_{\sigma \in \mathsf{S}_r} \operatorname{sign}(\sigma) \langle \varepsilon^{\sigma(j_1)}, v_1 \rangle \cdot \cdots \cdot \langle \varepsilon^{\sigma(j_r)}, v_r \rangle , \end{aligned}$$

where

$$a_{(j)} := \alpha(e_{j_1}, \ldots, e_{j_r}) . \tag{2.4}$$

[1] The exterior product is also called the **wedge** product.

[2] If $\{e_1, \ldots, e_m\}$ is an ordered basis, that is, the order of its elements is fixed, we write $(e_1, \ldots, e_m)$.

By Remark VII.1.19(a) and because the determinant of a square matrix does not change when it is transposed, we can rewrite the inner sum of the last expression as

$$\det\bigl(\bigl[\langle\varepsilon^{j_\mu}, v_\nu\rangle\bigr]_{1\le\mu,\nu\le r}\bigr) = \varepsilon^{(j)}(v_1,\ldots,v_r) \ .$$

Therefore

$$\alpha(v_1,\ldots,v_r) = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\varepsilon^{(j)}(v_1,\ldots,v_r) \quad \text{for } v_1,\ldots,v_r\in V \ ,$$

and hence

$$\alpha = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\varepsilon^{(j)} \ . \tag{2.5}$$

This shows that the set $\{\,\varepsilon^{(j)}\ ;\ (j)\in\mathbb{J}_r\,\}$ spans the vector space $\bigwedge^r V^*$.

Now suppose

$$\alpha = \sum_{(j)\in\mathbb{J}_r} b_{(j)}\varepsilon^{(j)}$$

with $b_{(j)}\in\mathbb{R}$ is another representation of α. Then we have in particular that

$$\alpha(e_{k_1},\ldots,e_{k_r}) = \sum_{(j)\in\mathbb{J}_r} b_{(j)}\varepsilon^{(j)}(e_{k_1},\ldots,e_{k_r}) \quad \text{for } (k)\in\mathbb{J}_r \ .$$

Because

$$\varepsilon^{(j)}(e_{k_1},\ldots,e_{k_r}) = \det\bigl([\delta^{j_\mu}_{k_\nu}]_{1\le\mu,\nu\le r}\bigr) = \begin{cases} 1 & \text{if } (j)=(k) \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

it follows that $b_{(j)} = a_{(j)}$ for $(j)\in\mathbb{J}_r$. Therefore the representation (2.5) is unique.

(ii) This statement is now clear because an m element set contains exactly $\binom{m}{r}$ subsets with r elements (see Exercise I.6.3). ■

In the following, let

$$\alpha^1\wedge\cdots\wedge\widehat{\alpha^j}\wedge\cdots\wedge\alpha^r$$

for $1\le j\le r$ denote the $(r-1)$-form one gets by omitting the linear form α^j from $\alpha^1\wedge\cdots\wedge\alpha^r$. We use like notation, for example

$$\alpha^1\wedge\cdots\wedge\widehat{\alpha^j}\wedge\cdots\wedge\widehat{\alpha^k}\wedge\cdots\wedge\alpha^r \ ,$$

when more linear forms are omitted.

2.4 Examples **(a)** The one-dimensional vector spaces $\bigwedge^0 V^* = \mathbb{R}$ and $\bigwedge^m V^*$ have bases 1 and $\varepsilon^1 \wedge \cdots \wedge \varepsilon^m$, respectively.

(b) $\{\varepsilon^1, \ldots, \varepsilon^m\}$ is a basis of $\bigwedge^1 V^* = V^*$.

(c) $\bigl\{\, \varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^j} \wedge \cdots \wedge \varepsilon^m \ ;\ 1 \le j \le m \,\bigr\}$ is a basis of $\bigwedge^{m-1} V^*$.

(d) For the basis representation

$$\alpha^j = \sum_{k=1}^m a_k^j \varepsilon^k \in V^* \quad \text{for } 1 \le j \le r \ ,$$

and

$$\alpha^1 \wedge \cdots \wedge \alpha^r = \sum_{(j) \in \mathbb{J}_r} a_{(j)} \varepsilon^{(j)} \in \textstyle\bigwedge^r V^* \ ,$$

we have $a_k^i = \langle \alpha^i, e_k \rangle$ for $1 \le i \le r$ and $1 \le k \le m$. Also

$$a_{(j)} = \det\bigl([a_{j_k}^i]_{1 \le i,k \le r}\bigr) \quad \text{for } (j) = (j_1, \ldots, j_r) \in \mathbb{J}_r \ .$$

Proof This follows from (2.3) and (2.4). ■

(e) For $r \ge 1$, we have

$$\textstyle\bigwedge^r V^* = \operatorname{span}\{\, \varphi^1 \wedge \cdots \wedge \varphi^r \ ;\ \varphi^j \in V^*,\ 1 \le j \le r \,\} \ .$$

(f) For $r \ge 2$, $\varphi^1 \wedge \cdots \wedge \varphi^r = 0$ if and only if $\varphi^1, \ldots, \varphi^r$ are linearly independent.

Proof This follows from (2.3). ■

As the next proposition shows, we can define a bilinear map from $\bigwedge^r V^* \times \bigwedge^s V^*$ to $\bigwedge^{r+s} V^*$ using the basis representation.

2.5 Proposition *Let $r, s \in \mathbb{N}^\times$.*

(i) *There is exactly one map*

$$\wedge : \textstyle\bigwedge^r V^* \times \bigwedge^s V^* \to \bigwedge^{r+s} V^* \ , \quad (\alpha, \beta) \mapsto \alpha \wedge \beta \ , \tag{2.6}$$

the **exterior product**, *with the properties that*

(α) *$\wedge$ is bilinear;*

(β) *for $\varphi^1, \ldots, \varphi^r, \psi^1, \ldots, \psi^s \in V^*$,*

$$(\varphi^1 \wedge \cdots \wedge \varphi^r) \wedge (\psi^1 \wedge \cdots \wedge \psi^s) = \varphi^1 \wedge \cdots \wedge \varphi^r \wedge \psi^1 \wedge \cdots \wedge \psi^s \ . \tag{2.7}$$

(ii) *Given the basis representations*

$$\alpha = \sum_{(j) \in \mathbb{J}_r} a_{(j)} \varepsilon^{(j)} \quad \text{and} \quad \beta = \sum_{(k) \in \mathbb{J}_s} b_{(k)} \varepsilon^{(k)} \ , \tag{2.8}$$

we have

$$\alpha \wedge \beta = \sum_{\substack{(j)\in\mathbb{J}_r\\(k)\in\mathbb{J}_s}} a_{(j)} b_{(k)} \varepsilon^{(j)} \wedge \varepsilon^{(k)} \ . \tag{2.9}$$

(iii) *The exterior product is associative and* **graded anticommutative**, *that is,*

$$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha \quad \text{for } \alpha \in \textstyle\bigwedge^r V^* \text{ and } \beta \in \textstyle\bigwedge^s V^* \ .$$

Proof If $\wedge$ is some linear map from $\bigwedge^r V^* \times \bigwedge^s V^*$ to $\bigwedge^{r+s} V^*$ satisfying (2.7), it follows immediately from (2.8) that (2.9) is true. Hence we can use (2.9) and a given basis to uniquely define (that is, by the bilinear continuation of the basis elements to the entire space) the bilinear map (2.6) with the properties (α) and (β). By (2.3), (2.7), and Example 2.4(e), $\wedge$ is independent of chosen basis. (iii) is now an immediate consequence of the properties of the determinant. ■

2.6 Remarks Suppose E_k for $k \in \mathbb{N}$ are vector spaces on the same field K.

(a) The **direct sum**

$$E := \bigoplus_{k=0}^{\infty} E_k =: \bigoplus_{k\geq 0} E_k$$

is defined as follows:

E is the set of all sequences (x_k) in $\bigcup_{k=0}^{\infty} E_k$ with $x_k \in E_k$ for $k \in \mathbb{N}$ that satisfy $x_k = 0$ for almost all $k \in \mathbb{N}$. On E, addition $+$ and multiplication by scalars are defined by

$$(x_k) + \lambda(y_k) := (x_k + \lambda y_k) \quad \text{for } (x_k), (y_k) \in E \text{ and } \lambda \in \mathbb{K} \ .$$

Then E is a K-vector space.[3] In addition, E_k will be identified with a vector subspace by means of the linear map

$$E_k \to E \ , \quad x_k \mapsto (0, \ldots, 0, x_k, 0, \ldots) \ ,$$

where x_k occupies the k-th entry in the sequence at right. Obviously

$$E = \operatorname{span}\{ E_k \ ; \ k \in \mathbb{N} \} \quad \text{and} \quad E_k \cap E_j = \{0\} \quad \text{for } k \neq j \ ,$$

which justifies the name "direct sum" (see Example I.12.3(l)).

(b) Letting $E := \bigoplus_{k\geq 0} E_k$, we define a multiplication

$$E \times E \to E \ , \quad (v, w) \mapsto v \odot w$$

[3] E is the vector space of all maps $f : \mathbb{N} \to \bigcup_{k=0}^{\infty} E_k$ with compact support, with $f(k) \in E_k$ and $k \in \mathbb{N}$, endowed with the pointwise product of Example I.12.3(e).

so that $(E, +, \odot)$ is an algebra (see Section I.12). We call E the **graded algebra** (over K), and we say the multiplication is **graded** if

$$E_k \odot E_\ell \subset E_{k+\ell} \quad \text{for } k, \ell \in \mathbb{N} \ .$$

If the relations

$$v_k \odot v_\ell = (-1)^{k\ell} v_\ell \odot v_k \quad \text{for } k, \ell \in \mathbb{N}$$

are also satisfied, then both the multiplication and the algebra are said to be **graded anticommutative**. ■

We set

$$\bigwedge V^* := \bigoplus_{r \geq 0} \bigwedge\nolimits^r V^*$$

and extend the definition of the exterior product by defining

$$\alpha \wedge \beta := \beta \wedge \alpha := \alpha\beta \quad \text{for } \alpha \in \bigwedge\nolimits^0 V^* = \mathbb{R} \ , \quad \beta \in \bigwedge V^* \ . \tag{2.10}$$

We also let $\mathbb{J}_0 := \{0\}$.

2.7 Theorem

(i) *There is exactly one bilinear, associative, and graded anticommutative map*

$$\bigwedge V^* \times \bigwedge V^* \to \bigwedge V^*$$

that extends the exterior product (2.6) *and/or* (2.10) *to all of* $\bigwedge V^* \times \bigwedge V^*$. *It also will be denoted by* $\wedge$ *and called the* **exterior product** *on* $\bigwedge V^*$.

(ii) $\dim(\bigwedge V^*) = 2^{\dim(V)}$.

Proof (i) This follows immediately from Proposition 2.5 and definition (2.10) by the natural bilinear extension.

(ii) Because $\bigwedge^r V^* = \{0\}$ for $r > \dim(V)$ and because $\bigwedge V^*$ is a direct sum of vector subspaces $\bigwedge^r V^*$, it follows from Proposition 2.3(ii) and the binomial theorem that

$$\dim(\bigwedge V^*) = \sum_{r=0}^{m} \binom{m}{r} = 2^m$$

with $m := \dim(V)$. ■

This theorem shows that $\bigwedge V^*$, when provided with the natural vector space structure and the exterior product, is an associative, graded anticommutative, real algebra of dimension $2^{\dim(V)}$. It is called the **Grassmann algebra** (or the **exterior algebra**) of V^*.

2.8 Remark Because V is finite-dimensional, V can be identified with V^{**} by means of the **canonical isomorphism**

$$\kappa : V \to V^{**} := (V^*)^*$$

defined by

$$\langle \kappa(v), v^* \rangle_{V^*} := \langle v^*, v \rangle \quad \text{for } v \in V \ , \quad v^* \in V^* \ .$$

Therefore the Grassmann algebra

$$\bigwedge V := \bigoplus_{r \geq 0} \bigwedge^r V$$

is also well defined on V.

Proof Clearly $\kappa : V \to V^{**}$ is linear. Suppose $\{e_1, \ldots, e_m\}$ is a basis of V, and suppose $\{\varepsilon_1, \ldots, \varepsilon_m\}$ is the associated dual basis of V^*. For $v \in \ker(\kappa)$, we have

$$\langle \varepsilon_j, v \rangle = \langle \kappa(v), \varepsilon_j \rangle_{V^*} = 0 \quad \text{for } j = 1, \ldots, m \ .$$

Then $v = \sum_{j=1}^m \langle \varepsilon_j, v \rangle e_j$ implies $v = 0$, so κ is injective. Now $\dim(V^{**}) = m$ (see Theorem VII.2.14) implies κ is an isomorphism. ■

Pull backs

For $A \in \mathcal{L}(V, W)$ and $\alpha \in \bigwedge^r W^*$, we define $A^*\alpha$ by

$$A^*\alpha(v_1, \ldots, v_r) := \alpha(Av_1, \ldots, Av_r) \quad \text{for } v_1, \ldots, v_r \in V$$

if $r \geq 1$ and by

$$A^*\alpha := \alpha \quad \text{for } \alpha \in \bigwedge^0 W^* = \mathbb{R}$$

if $r = 0$. Then we call $A^*\alpha$ the **pull back of $\boldsymbol{\alpha}$ by $\boldsymbol{A}$ on $\boldsymbol{V}$**.

2.9 Remarks **(a)** For $\alpha \in \bigwedge^r W^*$, the pull back $A^*\alpha$ belongs to $\bigwedge^r V^*$, and the map A^* is linear:

$$A^* \in \mathcal{L}(\bigwedge W^*, \bigwedge V^*) \qquad \text{with} \quad A(\bigwedge^r W^*) \subset \bigwedge^r V^* \text{ and } r \in \mathbb{N} \ .$$

We call A^* the **pull back transformation** (or usually the **pull back**) by A.

In the case $r = 1$, A^* is the map dual to A (denoted in Section VIII.3 by $A^\top$). Note also that A maps the vector space V to W, while A^* maps $\bigwedge W^*$ to $\bigwedge V^*$ and therefore "in the reverse direction":

$$V \xrightarrow{\ A\ } W$$

$$\bigwedge V^* \xleftarrow{\ A^*\ } \bigwedge W^*$$

(b) If X is another finite-dimensional real vector space and $B \in \mathcal{L}(W, X)$, then

$$(BA)^* = A^* B^* \quad \text{and} \quad (\mathrm{id}_V)^* = \mathrm{id}_{\bigwedge V^*} \ .$$

In other words, the map $A \mapsto A^*$ is **contravariant**.

(c) We have

$$A^*(\alpha \wedge \beta) = A^*\alpha \wedge A^*\beta \quad \text{for } \alpha, \beta \in \textstyle\bigwedge W^* \ .$$

Therefore A^* is an algebra homomorphism from $\bigwedge W^*$ to $\bigwedge V^*$.

Proof These statements follow obviously from the definitions of the pull back and the exterior product. ■

Let $m := \dim(V)$ and $W := V$, and let $\alpha \in \bigwedge^m V^*$. According to Proposition 2.3(ii), $\bigwedge^m V^*$ is one-dimensional, and hence $A^*\alpha$ must be proportional to α; we determine the multiple next.

2.10 Proposition *For $m := \dim(V)$ and $A \in \mathcal{L}(V)$,*

$$A^*\alpha = \det(A)\alpha \quad \text{for } \alpha \in \textstyle\bigwedge^m V^* \ .$$

Proof Let $\{e_1, \ldots, e_m\}$ be a basis of V, and let $[a_k^j] \in \mathbb{R}^{m \times m}$ be the matrix of A in this basis (see Section VII.1). Then

$$Ae_k = \sum_{j=1}^m a_k^j e_j \quad \text{for } 1 \le k \le m \ .$$

From this and the properties of $\alpha \in \bigwedge^m V^*$, it follows that

$$\begin{aligned} A^*\alpha(e_1, \ldots, e_m) &= \alpha(Ae_1, \ldots, Ae_m) \\ &= \sum_{j_1=1}^m \cdot \cdots \cdot \sum_{j_m=1}^m a_1^{j_1} \cdot \cdots \cdot a_m^{j_m} \, \alpha(e_{j_1}, \ldots, e_{j_m}) \\ &= \sum_{\sigma \in \mathsf{S}_m} \mathrm{sign}(\sigma) \, a_1^{\sigma(1)} \cdot \cdots \cdot a_m^{\sigma(m)} \, \alpha(e_1, \ldots, e_m) \\ &= \det(A) \, \alpha(e_1, \ldots, e_m) \ , \end{aligned}$$

where in the last step we have used the signature formula of Remark VII.1.19 and the fact that $\det(A^\top) = \det(A)$. Now the claim follows from the multilinearity of α. ■

The volume element

Suppose $\mathcal{O}r$ now is an orientation of V, that is, $V := (V, \mathcal{O}r)$ is an oriented vector space. For short, we a call positively oriented ordered basis of V (which is therefore an element of $\mathcal{O}r$) a **positive basis** (see Remark VIII.2.4). Also let $m := \dim(V)$.

We call every $\alpha \in \bigwedge^m V^* \backslash \{0\}$ a **volume form** on V. Two volume forms α and β are **equivalent** if there is a $\lambda > 0$ such that $\alpha = \lambda\beta$. We check easily that this definition induces an equivalence relation $\sim$ on the set of all volume forms on V. Because $\dim(\bigwedge^m V^*) = 1$, there are exactly two equivalence classes.

2.11 Remarks **(a)** Suppose $(e_1, \ldots, e_m)$ is a positive basis of V and $(\varepsilon^1, \ldots, \varepsilon^m)$ is the associated dual basis. If $(\widetilde{e}_1, \ldots, \widetilde{e}_m)$ is a basis of V and $\alpha \in \bigwedge^m V^* \backslash \{0\}$ with $\alpha \sim \varepsilon^1 \wedge \cdots \wedge \varepsilon^m$, then $(\widetilde{e}_1, \ldots, \widetilde{e}_m)$ is positive if and only if $\alpha(\widetilde{e}_1, \ldots, \widetilde{e}_m) > 0$. This means that the two equivalence classes of $\bigwedge^m V^* \backslash \{0\}$ can be identified with the two orientations of V. In other words, the volume form α **determines the orientation $\mathcal{O}r$ of V** through the requirement

$$\alpha(e_1, \ldots, e_m) > 0 \Longleftrightarrow (e_1, \ldots, e_m) \in \mathcal{O}r \ .$$

Proof Suppose $B \in \mathcal{L}(V)$ is the change of basis from $(e_1, \ldots, e_m)$ to $(\widetilde{e}_1, \ldots, \widetilde{e}_m)$, that is, $\widetilde{e}_j = Be_j$ for $1 \leq j \leq m$. Then it follows from Proposition 2.10 that

$$\alpha(\widetilde{e}_1, \ldots, \widetilde{e}_m) = \det(B)\, \alpha(e_1, \ldots, e_m) = \det(B)\lambda$$

where $\alpha = \lambda\varepsilon^1 \wedge \cdots \wedge \varepsilon^m$ and $\lambda > 0$. ■

(b) We say an automorphism A of V is **orientation preserving** [or **reversing**] if $\det(A) > 0$ [or $\det(A) < 0$]. We set

$$\mathcal{L}\mathrm{aut}^+(V) := GL^+(V) := \big\{\, A \in \mathcal{L}\mathrm{aut}(V) \ ; \ \det(A) > 0 \,\big\} \ .$$

(i) The following statements are equivalent for $A \in \mathcal{L}\mathrm{aut}(V)$:
  (α) $A \in \mathcal{L}\mathrm{aut}^+(V)$.
  (β) For every basis $(b_1, \ldots, b_m)$, the bases $(b_1, \ldots, b_m)$ and $(Ab_1, \ldots, Ab_m)$ have the same orientation.
  (γ) For every $\alpha \in \bigwedge^m V^* \backslash \{0\}$, the volume forms α and $A^*\alpha$ determine the same orientation of V.

(ii) $\mathcal{L}\mathrm{aut}^+(V)$ is a subgroup of $\mathcal{L}\mathrm{aut}(V) =: GL(V)$.

Proof (i) This follows from $A^*\alpha = \det(A)\alpha$ and the definition of orientation.

(ii) The map

$$\mathcal{L}\mathrm{aut}(V) \to (\mathbb{R}^\times, \cdot\,) \ , \quad A \mapsto \det(A)$$

is a homomorphism. According to Exercise I.7.5, $\mathcal{L}\mathrm{aut}^+(V)$, as the inverse image of the subgroup $((0, \infty), \cdot\,)$ of $(\mathbb{R}^\times, \cdot\,)$, is a subgroup of $\mathcal{L}\mathrm{aut}(V)$. ■

Suppose now $(V, (\cdot\,|\,\cdot), \mathcal{O}r)$ is an oriented inner product space. Also let $(e_1, \ldots, e_m)$ be a positive orthonormal basis (ONB), and let $(\varepsilon^1, \ldots, \varepsilon^m)$ be the associated dual basis of V^*. Then

$$\omega := \omega_V := \varepsilon^1 \wedge \cdots \wedge \varepsilon^m$$

is called the **volume element** of V.

2.12 Remarks **(a)** For every positive ONB $(\widetilde{e}_1, \ldots, \widetilde{e}_m)$ of V, we have

$$\omega(\widetilde{e}_1, \ldots, \widetilde{e}_m) = 1 \ .$$

Proof Let B by the basis change specified by $\widetilde{e}_j = Be_j$ for $1 \le j \le m$. Then B belongs to $\mathcal{L}\mathrm{aut}^+(V) \cap O(m)$, and thus $\det(B) = 1$ (see Exercise VII.9.2). Therefore it follows from Proposition 2.10 that

$$\omega(\widetilde{e}_1, \ldots, \widetilde{e}_m) = B^*\omega(e_1, \ldots, e_m) = \det(B)\varepsilon^1 \wedge \cdots \wedge \varepsilon^m(e_1, \ldots, e_m) = 1 \ ,$$

which proves the claim. ■

(b) The volume element of V is the unique volume form that assigns the value 1 to any, and thus every, positive ONB.

Proof This follows from (a). ■

(c) For $v_1, \ldots, v_m \in \mathbb{R}^m$, let

$$P(v_1, \ldots, v_m) := \left\{ \textstyle\sum_{j=1}^m t^j v_j \ ; \ 0 \le t^j \le 1 \right\} ,$$

that is, $P(v_1, \ldots, v_m)$ is the parallelepiped spanned by $v_1, \ldots, v_m$. Then

$$|\omega_{\mathbb{R}^m}(v_1, \ldots, v_m)| = \mathrm{vol}_m(P(v_1, \ldots, v_m)) := \lambda_m(P(v_1, \ldots, v_m)) \ .$$

In other words, the volume element assigns every m-tuple of vectors the **oriented volume**[4] of the parallelogram they span.

Proof We define $B \in \mathcal{L}(\mathbb{R}^m)$ by $v_j = Be_j$ for $1 \le j \le m$. Then

$$P(v_1, \ldots, v_m) = B([0,1]^m) \ .$$

Then it follows from Proposition 2.10 and (a) that

$$\omega_{\mathbb{R}^m}(v_1, \ldots, v_m) = B^*\omega_{\mathbb{R}^m}(e_1, \ldots, e_m) = \det(B) \ .$$

From Theorem IX.5.25, we know that $\lambda_m(B([0,1]^m)) = |\det(B)|$, as desired. ■

In the following proposition, we represent the volume element ω in terms of an arbitrary positive basis of V.

2.13 Proposition *Suppose $(b_1, \ldots, b_m)$ is a positive basis of V and $(\beta^1, \ldots, \beta^m)$ is its dual basis. Then*

$$\omega = \sqrt{G}\, \beta^1 \wedge \cdots \wedge \beta^m \ ,$$

where $G := \det\big[(b_j \,|\, b_k)\big]$ *is the* **Gram determinant**. *In particular,*

$$\omega(b_1, \ldots, b_m) = \sqrt{G} \ .$$

[4] An oriented volume is positive if and only if $(v_1, \ldots, v_m)$ is positive; it is negative if and only if $(v_1, \ldots, v_m)$ belongs to $-\mathcal{O}r$.

Proof Let $(e_1,\dots,e_m)$ be a positive ONB of V, and define $B\in\mathcal{L}(V)$ by $b_j = Be_j$ for $1\le j\le m$. According to (i) of Remark 2.11(b), we have $\det(B)>0$. From Remark 2.12(a) and (2.3), we get

$$\begin{aligned}\omega(e_1,\dots,e_m) = 1 &= \beta^1\wedge\cdots\wedge\beta^m(b_1,\dots,b_m)\\ &= B^*(\beta^1\wedge\cdots\wedge\beta^m)(e_1,\dots,e_m)\ .\end{aligned}$$

Therefore

$$\omega = \det(B)\beta^1\wedge\cdots\wedge\beta^m\ ,$$

because an m-form is determined by its value on a basis of V, and because of Proposition 2.10. Also

$$(b_j\,|\,b_k) = (Be_j\,|\,Be_k) = (B^*Be_j\,|\,e_k)\quad\text{for } 1\le j,k\le m \tag{2.11}$$

(see Exercise VII.1.5). Because $(e_1,\dots,e_m)$ is an ONB, any $v\in V$ has the representation $v=\sum_{k=1}^m (v\,|\,e_k)e_k$. From this it follows that

$$Te_j = \sum_{k=1}^m (Te_j\,|\,e_k)e_k\quad\text{for } 1\le j\le m \text{ and } T\in\mathcal{L}(V)\ .$$

Hence (2.11) shows that $\big[(b_j\,|\,b_k)\big]\in\mathbb{R}^{m\times m}$ is the matrix of B^*B in the basis $(e_1,\dots,e_m)$. Therefore

$$G = \det\big[(b_j\,|\,b_k)\big] = \det(B^*B) = (\det(B))^2$$

because $\det(B^*) = \det(B)$. The claim follows. ■

The Riesz isomorphism

Suppose $(V,(\cdot\,|\,\cdot))$ is an inner product space and $m:=\dim(V)$. We denote the Riesz isomorphism (2.2) by

$$\Theta := \Theta_V : V\to V^*\ ,\quad v\mapsto(\cdot\,|\,v)\ ,$$

that is,

$$\langle\Theta v, w\rangle = (w\,|\,v)\quad\text{for } v,w\in V\ . \tag{2.12}$$

Then

$$(\alpha\,|\,\beta)_* := (\Theta^{-1}\alpha\,|\,\Theta^{-1}\beta)\quad\text{for }\alpha,\beta\in V^* \tag{2.13}$$

defines an inner product on V^*, the **scalar product dual to** $(\cdot\,|\,\cdot)$. In the following, we always provide V^* with this inner product, so that $V^* := (V^*,(\cdot\,|\,\cdot)_*)$ is an inner product space.

2.14 Remarks Suppose $\{e_1,\ldots,e_m\}$ is a basis in V and $\{\varepsilon^1,\ldots,\varepsilon^m\}$ is its dual basis in V^*.

(a) We set

$$g_{jk} := (e_j \,|\, e_k) \quad \text{for } 1 \le j,k \le m \ , \text{ where } [g^{jk}] := [g_{jk}]^{-1} \in \mathbb{R}^{m\times m} \ .$$

Then

$$\Theta e_j = \sum_{k=1}^m g_{jk}\varepsilon^k \text{ and } \Theta^{-1}\varepsilon^j = \sum_{k=1}^m g^{jk} e_k \quad \text{for } 1 \le j \le m \ .$$

Proof From the basis expansion $\Theta e_j = \sum_{k=1}^m a_{jk}\varepsilon^k$ for $1 \le j \le m$ and from (2.12), we get

$$(e_j \,|\, v) = \langle \Theta e_j, v\rangle = \sum_{k=1}^m a_{jk}\langle \varepsilon^k, v\rangle \quad \text{for } v \in V \text{ and } 1 \le j \le m \ .$$

Replacing v by each of $e_1,\ldots,e_m$, we find $a_{jk} = (e_j \,|\, e_k)$, which proves the first statement. The representation of $\Theta^{-1}\varepsilon^j$ is obvious. ■

(b) For $v = \sum_{j=1}^m \xi^j e_j \in V$ and $w = \sum_{j=1}^m \eta^j e_j \in V$, we have

$$(v \,|\, w) = \sum_{j,k=1}^m g_{jk}\xi^j\eta^k \ .$$

For $\alpha = \sum_{j=1}^m a_j\varepsilon^j \in V^*$ and $\beta = \sum_{j=1}^m b_j\varepsilon^j \in V^*$, we have the relation

$$(\alpha \,|\, \beta)_* = \sum_{j,k=1}^m g^{jk} a_j b_k \ .$$

Proof The first statement is obvious. From (a) and (2.13), we derive

$$(\varepsilon^i \,|\, \varepsilon^\ell)_* = (\Theta^{-1}\varepsilon^i \,|\, \Theta^{-1}\varepsilon^\ell) = \sum_{j,k=1}^m g^{ij}g^{\ell k}(e_j \,|\, e_k) = \sum_{j,k=1}^m g^{ij}g^{\ell k}g_{jk} = g^{i\ell}$$

for $1 \le i,\ell \le m$. Now the second claim follows from the bilinearity of $(\cdot \,|\, \cdot)_*$. ■

(c) If $\{e_1,\ldots,e_m\}$ is an ONB, then $\Theta e_j = \varepsilon^j$ for $1 \le j \le m$, and $\{\varepsilon^1,\ldots,\varepsilon^m\}$ is likewise an ONB.

(d) You may have noticed that we have used upper indices to label the coefficients of a vector in a basis representation, whereas we used lower indices for the expansion coefficients of a 1-form. That is,

$$v = \sum_{j=1}^m \xi^j e_j \in V \text{ and } \alpha = \sum_{j=1}^m a_j\varepsilon^j \in V^* \ .$$

From (a) and the symmetry of $[g_{jk}]$, it follows that

$$\Theta v = \sum_{j=1}^{m} b_j \varepsilon^j \in V^* \text{ and } \Theta^{-1}\alpha = \sum_{j=1}^{m} \eta^j e_j \in V$$

with

$$b_j := \sum_{k=1}^{m} g_{jk}\xi^k \quad \text{and} \quad \eta^j := \sum_{k=1}^{m} g^{jk} a_k \quad \text{for } 1 \le j \le m \ .$$

The application of Θ [or Θ^{-1}] formally effects a **lowering** [or raising] of indices. On these grounds, we may borrow the musical notations

$$g^\flat := \Theta \quad \text{and} \quad g^\sharp := \Theta^{-1} \ ,$$

or simply $v^\flat := \Theta v$ for $v \in V$ and $\alpha^\sharp := \Theta^{-1}\alpha$ for $\alpha \in V^*$. ■

The Hodge star operator[5]

Suppose $(V, (\cdot\,|\,\cdot), \mathcal{O}r)$ is an oriented inner product space, $m := \dim(V)$, and ω is the volume element of V. Also let $\{e_1, \ldots, e_m\}$ be an ONB of V, and let $\{\varepsilon^1, \ldots, \varepsilon^m\}$ be its dual basis.

We now define a scalar product $(\cdot\,|\,\cdot)_r$ on $\bigwedge^r V^*$ as follows:
For $r = 0$, let

$$(\alpha\,|\,\beta)_0 := \alpha\beta \quad \text{for } \alpha, \beta \in \textstyle\bigwedge^0 V^* = \mathbb{R} \ . \tag{2.14}$$

For $1 \le r \le m$, let

$$\alpha = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\varepsilon^{(j)} \quad \text{and} \quad \beta = \sum_{(j)\in\mathbb{J}_r} b_{(j)}\varepsilon^{(j)} \ ,$$

which, according to Proposition 2.3, are valid basis representations of $\alpha, \beta \in \bigwedge^r V^*$. Then we set

$$(\alpha\,|\,\beta)_r := \sum_{(j)\in\mathbb{J}_r} a_{(j)} b_{(j)} \ . \tag{2.15}$$

It is clear that $(\cdot\,|\,\cdot)_r$ is a scalar product on $\bigwedge^r V^*$ for $0 \le r \le m$. By Remarks 2.14(b) and (c), we have $(\cdot\,|\,\cdot)_1 = (\cdot\,|\,\cdot)_*$.

2.15 Remarks **(a)** The basis $\{\, \varepsilon^{(j)} \ ; \ (j) \in \mathbb{J}_r \,\}$ is an ONB of $(\bigwedge^r V^*, (\cdot\,|\,\cdot)_r)$ for $1 \le r \le m$.

(b) For $\alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^r \in V^*$, we have

$$\left(\alpha^1 \wedge \cdots \wedge \alpha^r \,\middle|\, \beta^1 \wedge \cdots \wedge \beta^r\right)_r = \det\left[(\alpha^j\,|\,\beta^k)_*\right] \ .$$

[5] This section and the next may be skipped on first reading.

Proof Suppose

$$\alpha^j = \sum_{i=1}^m a_i^j \varepsilon^i \quad \text{and} \quad \beta^k = \sum_{i=1}^m b_i^k \varepsilon^i \quad \text{for } 1 \le j,k \le r \ .$$

Also let

$$\alpha^1 \wedge \cdots \wedge \alpha^r = \sum_{(j)\in \mathbb{J}_r} a_{(j)} \varepsilon^{(j)} \quad \text{and} \quad \beta^1 \wedge \cdots \wedge \beta^r = \sum_{(k)\in \mathbb{J}_r} b_{(k)} \varepsilon^{(k)}$$

be basis representations. Then according to Example 2.4(d), we have

$$a_{(j)} = \det\bigl([a^i_{j_k}]_{1\le i,k\le r}\bigr) \quad \text{and} \quad b_{(k)} = \det\bigl([b^i_{k_\ell}]_{1\le i,\ell\le r}\bigr)$$

for $(j) = (j_1,\ldots,j_r) \in \mathbb{J}_r$ and $(k) = (k_1,\ldots,k_r) \in \mathbb{J}_r$.

By the bilinearity and symmetry of $(\cdot\,|\,\cdot)_*$ and the fact that the determinant is an alternating r-form in its row vectors, we find (see the proof of Proposition 2.3(i))

$$\begin{aligned}
\det\bigl[(\alpha^j\,|\,\beta^k)_*\bigr] &= \sum_{(j)\in\mathbb{J}_r}\sum_{\sigma\in\mathsf{S}_r} \operatorname{sign}(\sigma)\, a^1_{j_{\sigma(1)}} \cdot \cdots \cdot a^r_{j_{\sigma(r)}} \det\Bigl(\bigl[(\varepsilon^{j_k}\,|\,\beta^\ell)_*\bigr]_{1\le k,\ell\le r}\Bigr) \\
&= \sum_{(j)\in\mathbb{J}_r} \det\bigl([a^i_{j_k}]_{1\le i,k\le r}\bigr) \det\Bigl(\bigl[(\varepsilon^{j_k}\,|\,\beta^\ell)_*\bigr]_{1\le k,\ell\le r}\Bigr) \\
&= \sum_{(j)\in\mathbb{J}_r} a_{(j)} \det\Bigl(\bigl[(\varepsilon^{j_k}\,|\,\beta^\ell)_*\bigr]_{1\le k,\ell\le r}\Bigr) \\
&= \sum_{(j)\in\mathbb{J}_r}\sum_{(k)\in\mathbb{J}_r} a_{(j)} b_{(k)} \det\Bigl(\bigl[(\varepsilon^{j_i}\,|\,\varepsilon^{k_\ell})_*\bigr]_{1\le i,\ell\le r}\Bigr) \ .
\end{aligned}$$

By (a), we have

$$\det\bigl[(\varepsilon^{j_i}\,|\,\varepsilon^{k_\ell})_*\bigr] = \det\bigl([\delta^{j_i,k_\ell}]_{1\le i,\ell\le r}\bigr) = \begin{cases} 1 & \text{if } (j)=(k) \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Thus we get

$$\det\bigl[(\alpha^j\,|\,\beta^k)_*\bigr] = \sum_{(j)\in\mathbb{J}_r} a_{(j)} b_{(j)} \ .$$

By (2.15), this finishes the proof. ■

(c) The scalar product $(\cdot\,|\,\cdot)_r$ on $\bigwedge^r V^*$ does not depend on the special choice of ONB or its orientation, but rather only on the inner product $(\cdot\,|\,\cdot)$ on V.

Proof This follows from (b), Example 2.4(e), and the scalar product's bilinearity. ■

Because

$$\dim\bigl(\textstyle\bigwedge^r V^*\bigr) = \binom{m}{r} = \binom{m}{m-r} = \dim\bigl(\textstyle\bigwedge^{m-r} V^*\bigr) \ , \tag{2.16}$$

$\bigwedge^r V^*$ and $\bigwedge^{m-r} V^*$ are isomorphic vector spaces for $0 \le r \le m$. We now introduce a special (natural) isomorphism from $\bigwedge^r V^*$ to $\bigwedge^{m-r} V^*$, the Hodge star operator.

We first note that for every $\alpha \in \bigwedge^r V^*$, Proposition 2.5 implies

$$(\beta \mapsto \alpha \wedge \beta) \in \mathcal{L}(\textstyle\bigwedge^{m-r} V^*, \bigwedge^m V^*) \ . \tag{2.17}$$

Because $\bigwedge^m V^*$ is one-dimensional, there exists exactly one $f_\alpha(\beta) \in \mathbb{R}$ such that

$$\alpha \wedge \beta = f_\alpha(\beta)\omega_V \quad \text{for } \beta \in \textstyle\bigwedge^{m-r} V^* \ .$$

By (2.17), f_α belongs to $\mathcal{L}(\bigwedge^{m-r} V^*, \mathbb{R})$. Then according to the Riesz representation theorem, there is exactly one $*\alpha \in \bigwedge^{m-r} V^*$ with $f_\alpha(\beta) = (*\alpha \,|\, \beta)_{m-r}$ for $\beta \in \bigwedge^{m-r} V^*$. In other words, every $\alpha \in \bigwedge^r V^*$ has a unique element $*\alpha \in \bigwedge^{m-r} V^*$ such that

$$\alpha \wedge \beta = (*\alpha \,|\, \beta)_{m-r}\omega_V \quad \text{for } \beta \in \textstyle\bigwedge^{m-r} V^* \ . \tag{2.18}$$

Therefore $*\alpha = \Theta^{-1} f_\alpha$, where Θ denotes the Riesz isomorphism Θ of the space $\bigwedge^{m-r} V^*$. Hence

$$(\alpha \mapsto *\alpha) \in \mathcal{L}(\textstyle\bigwedge^r V^*, \bigwedge^{m-r} V^*) \ . \tag{2.19}$$

This map is called the **Hodge star operator** (or simply the Hodge star).

2.16 Remarks **(a)** The Hodge star is an isomorphism.

Proof From $*\alpha = 0$ and (2.18), it follows that $\alpha \wedge \beta = 0$ for every $\beta \in \bigwedge^{m-r} V^*$. For the special choice $\beta := \varepsilon^{r+1} \wedge \cdots \wedge \varepsilon^m$, it follows from

$$\alpha = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\varepsilon^{(j)} \text{ with } (j_0) := (1, \ldots, r)$$

that $0 = \alpha \wedge \beta = a_{(j_0)}\omega_V$, and therefore $a_{(j_0)} = 0$. Analogously we find that $a_{(j)} = 0$ for $(j) \in \mathbb{J}_r$. Therefore (2.19) is injective. Now the claim is implied by (2.16). ■

(b) The Hodge star depends on the scalar product and the orientation of V. ■

2.17 Examples **(a)** For $1 \le j \le m$, we have $*\varepsilon^j = (-1)^{j-1}\varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^j} \wedge \cdots \wedge \varepsilon^m$.

Proof From the alternating property, the associativity of the exterior product, and Example 2.4(f) it follows that

$$\varepsilon^j \wedge (\varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^k} \wedge \cdots \wedge \varepsilon^m) = (-1)^{j-1}\delta^{jk}\varepsilon^1 \wedge \cdots \wedge \varepsilon^m = (-1)^{j-1}\delta^{jk}\omega$$

for $1 \le k \le m$. Now the claim is implied by (2.18) and the fact that, according to Remark 2.15(a),

$$\{\, \varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^k} \wedge \cdots \wedge \varepsilon^m \ ; \ 1 \le k \le m \,\}$$

is an ONB of $\bigwedge^{m-1} V^*$. ■

(b) For $1 \le j \le m$, we have

$$*(\varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^j} \wedge \cdots \wedge \varepsilon^r) = (-1)^{m-j}\varepsilon^j \ .$$

Proof Because

$$(\varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^j} \wedge \cdots \wedge \varepsilon^m) \wedge \varepsilon^k = (-1)^{m-j}\delta^{jk}\omega \ ,$$

the statement follows as in the previous proof. ■

(c) $*1 = \omega$ and $*\omega = 1$.

(d) We now consider the general case covering both (a) and (b). Suppose therefore $1 \le r \le m-1$ and $(j) \in \mathbb{J}_r$. Then there is exactly one $(j^c) \in \mathbb{J}_{m-r}$ such that $(j) \vee (j^c) := (j_1, \ldots, j_r, j_1^c, \ldots, j_{m-r}^c)$ is a permutation of $\{1, \ldots, m\}$. Putting $s(j) := \operatorname{sign}((j) \vee (j^c))$, we then have

$$*\varepsilon^{(j)} = s(j)\varepsilon^{(j^c)} . \tag{2.20}$$

It follows for $\alpha = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\varepsilon^{(j)} \in \bigwedge^r V^*$ that

$$*\alpha = \sum_{(j)\in\mathbb{J}_r} s(j)a_{(j)}\varepsilon^{(j^c)} .$$

Proof For $(k) \in \mathbb{J}_{m-r}$ with $(k) \ne (j^c)$, we have $\varepsilon^{(j)} \wedge \varepsilon^{(k)} = 0$, because at least one ε^{j_i} occurs twice in this product. For $(k) = (j^c)$, we derive from (2.3) that

$$\varepsilon^{(j)} \wedge \varepsilon^{(j^c)} = s(j)\omega . \tag{2.21}$$

Now (2.20) follows from (2.18) and Remark 2.15(a). ■

(e) For $\alpha \in \bigwedge^r V^*$ with $0 \le r \le m$, we have $**\alpha := *(*\alpha) = (-1)^{r(m-r)}\alpha$.

Proof For $(j), (k) \in \mathbb{J}_r$, it follows from (2.18), (d), and Proposition 2.5(iii) that

$$(**\varepsilon^{(j)} \,|\, \varepsilon^{(k)})_r\omega = (*\varepsilon^{(j)}) \wedge \varepsilon^{(k)} = s(j)\varepsilon^{(j^c)} \wedge \varepsilon^{(k)} = (-1)^{r(m-r)}s(j)\varepsilon^{(k)} \wedge \varepsilon^{(j^c)} .$$

Then because $\varepsilon^{(k)} \wedge \varepsilon^{(j^c)} = 0$ for $(k) \ne (j)$ and using (2.21), we find

$$(**\varepsilon^{(j)} \,|\, \beta)_r = (-1)^{r(m-r)}(\varepsilon^{(j)} \,|\, \beta)_r \quad \text{for } \beta \in \textstyle\bigwedge^r V^* .$$

Hence $**\varepsilon^{(j)} = (-1)^{r(m-r)}\varepsilon^{(j)}$ for $(j) \in \mathbb{J}_r$, which, because of Proposition 2.3(i), proves the claim. ■

(f) For $\alpha, \beta \in \bigwedge^r V^*$, we have the relationship

$$\alpha \wedge *\beta = \beta \wedge *\alpha = (\alpha \,|\, \beta)_r\omega . \tag{2.22}$$

Proof Suppose $\alpha := \beta := \varepsilon^{(j)}$. Then by (d) and (2.21), we have

$$\alpha \wedge *\beta = \alpha \wedge *\alpha = \beta \wedge *\alpha = s(j)\varepsilon^{(j)} \wedge \varepsilon^{(j^c)} = \omega = (\varepsilon^{(j)} \,|\, \varepsilon^{(j)})_r\omega = (\alpha \,|\, \beta)_r\omega .$$

Letting $\alpha := \varepsilon^{(j)}$ and $\beta := \varepsilon^{(k)}$ with $(j) \ne (k)$, we obtain from (d) that

$$\alpha \wedge *\beta = s(k)\varepsilon^{(j)} \wedge \varepsilon^{(k^c)} = 0 = s(j)\varepsilon^{(k)} \wedge \varepsilon^{(j^c)} = \beta \wedge *\alpha .$$

In addition, from Remark 2.15(a) we have $(\alpha \,|\, \beta)_r = 0$. Therefore (2.22) also holds in this case. Now this claim also follows from Proposition 2.3(i). ■

Indefinite inner products

For several applications, particularly in physics, one must drop the assumption that the scalar product is positive definite. So we will now shortly go over the modifications needed to handle this case.

A bilinear form $\mathfrak{b}: V \times V \to \mathbb{R}$ is said to be **nondegenerate** if for every $y \in V \backslash \{0\}$ there is an $x \in V$ such that $\mathfrak{b}(x,y) \neq 0$. It is **symmetric** if

$$\mathfrak{b}(x,y) = \mathfrak{b}(y,x) \quad \text{for } x,y \in V \ .$$

Suppose $\mathfrak{b}: V \times V \to \mathbb{R}$ is a nondegenerate symmetric bilinear form on $V := (V, (\cdot \,|\, \cdot))$. In the following remarks, we list some basic properties of $\mathfrak{b}$.

2.18 Remarks **(a)** There is a **$\mathfrak{b}$-orthonormal basis** ($\mathfrak{b}$-ONB) of V, that is, there is a basis $\{b_1, \ldots, b_m\}$ of V such that $\mathfrak{b}(b_j, b_k) = \pm\delta_{jk}$ for $1 \leq j,k \leq m$. If r is the number of plus sign and s is the number of minuses, then $r + s = m$. The number $t := r - s$ is called the **signature** of $\mathfrak{b}$. The signature, as well as r and s, is independent of the choice of $\mathfrak{b}$-ONB. In particular,[6]

$$(-1)^s = \operatorname{sign}(\mathfrak{b}) := \operatorname{sign}\bigl(\det\bigl[\mathfrak{b}(b_j, b_k)\bigr]\bigr) \ .$$

Proof Theorem VII.4.2(iii) clearly implies that $\mathfrak{b}$ is continuous. Then $\mathfrak{b}(x,\cdot): V \to \mathbb{R}$ is a continuous linear form on V. Therefore, the Riesz representation theorem (Theorem VII.2.14) guarantees the existence of a unique $\mathfrak{B}x \in V$ such that

$$\mathfrak{b}(x,y) = \bigl(\mathfrak{B}x \bigm| y\bigr) \quad \text{for } y \in V \ .$$

From the linearity of $\mathfrak{b}(\cdot\,, y)$, it follows that $x \mapsto \mathfrak{B}x$ is linear. Then by Theorem VII.1.6, $\mathfrak{B}$ belongs to $\mathcal{L}(V)$, and

$$\mathfrak{b}(x,y) = (\mathfrak{B}x \,|\, y) \quad \text{for } x,y \in V \ .$$

The map $\mathfrak{B}$ is called the **representation operator** of $\mathfrak{b}$ with respect to $(\cdot \,|\, \cdot)$. Because $\mathfrak{b}$ is nondegenerate, $\mathfrak{B}$ is an automorphism of V (and conversely), and because $\mathfrak{b}$ is symmetric, so is $\mathfrak{B}$.

Remark 2.1(a) allows us to identify V with $\mathbb{R}^m$. Therefore the principal axis transformation theorem[7] guarantees the existence of an ONB $\{v_1, \ldots, v_m\}$ of V and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m$ of $\mathfrak{B}$ such that

$$\mathfrak{B}v_j = \lambda_j v_j \quad \text{for } 1 \leq j \leq m \ . \tag{2.23}$$

Because $\mathfrak{b}$ is nondegenerate, we have $\lambda_j \neq 0$ for $1 \leq j \leq m$. We set $b_j := v_j \big/ \sqrt{|\lambda_j|}$. Then $\{b_1, \ldots, b_m\}$ is a basis of V, and it follows from (2.23) that

$$\mathfrak{b}(b_j, b_k) = (\mathfrak{B}b_j \,|\, b_k) = (\mathfrak{B}v_j \,|\, v_k) \big/ \sqrt{|\lambda_j \lambda_k|} = \lambda_j (v_j \,|\, v_k) \big/ \sqrt{|\lambda_j \lambda_k|} = \operatorname{sign}(\lambda_j)\delta_{jk}$$

for $1 \leq j,k \leq m$.

[6] $\operatorname{sign}(\mathfrak{b})$ should not be confused with the signature t. Obviously $2\operatorname{sign}(\mathfrak{b}) = m - t$.

[7] See Example VII.10.17(b).

To show that $t = r - s$ is independent of the choice of $\mathfrak{b}$-ONB, it suffices, because $r + s = m$, to prove this is true of r. Suppose therefore $\{c_1, \ldots, c_m\}$ is a $\mathfrak{b}$-ONB of V such that $\mathfrak{b}(c_j, c_j) = 1$ for $1 \le j \le \rho$ and $\mathfrak{b}(c_j, c_j) = -1$ for $\rho + 1 \le j \le m$. We want to show that the vectors $b_1, \ldots, b_r, c_{\rho+1}, \ldots, c_m$ are linearly independent, since this would imply $r + (m - \rho) \le m$ and therefore $r \le \rho$; also, by exchanging the two $\mathfrak{b}$-ONB, we would analogously obtain $\rho \le r$, which then determines r.

Suppose therefore

$$\beta_1 b_1 + \cdots + \beta_r b_r = \gamma_{\rho+1} c_{\rho+1} + \cdots + \gamma_m c_m$$

with real numbers $\beta_1, \ldots, \beta_r, \gamma_{\rho+1}, \ldots, \gamma_m$. Every linear dependence relation of the set $\{b_1, \ldots, b_r, c_{\rho+1}, \ldots, c_m\}$ can be so written. Then for $v := \beta_1 b_1 + \cdots + \beta_r b_r$, we have

$$\mathfrak{b}(v, v) = \sum_{j=1}^{r} \beta_r^2 = -\sum_{j=\rho+1}^{m} \gamma_j^2 \ ,$$

which implies $\beta_1 = \cdots = \beta_r = \gamma_{\rho+1} = \cdots = \gamma_m = 0$. The last claim is now clear. ■

(b) (Riesz representation theorem) To every $v^* \in V^*$, there is exactly one $v \in V$ with $\mathfrak{b}(v, w) = \langle v^*, w \rangle$ for $w \in W$. The map

$$\Theta_{\mathfrak{b}} : V \to V^* \ , \quad v \mapsto \mathfrak{b}(v, \cdot\,)$$

is a vector space isomorphism, the **Riesz isomorphism** with respect to $\mathfrak{b}$. The statements of Remark 2.14(a) also hold in this case.

Proof With the representation operator $\mathfrak{B}$ of $\mathfrak{b}$ and the Riesz isomorphism Θ of V, Theorem VII.2.14 implies

$$\mathfrak{b}(v, w) = (\mathfrak{B}v \,|\, w) = \langle \Theta\mathfrak{B}v, w \rangle \quad \text{for } v, w \in V \ .$$

The claim then follows after putting $\Theta_{\mathfrak{b}} := \Theta\mathfrak{B}$. ■

(c) For every basis $\{v_1, \ldots, v_m\}$ of V, the **Gram determinant** with respect to $\mathfrak{b}$, that is,

$$G_{\mathfrak{b}} := \det\bigl(\bigl[\mathfrak{b}(v_j, v_k)\bigr]\bigr) \ ,$$

is nonzero.

Proof The determinant $G_{\mathfrak{b}}$ is zero if and only if the system of linear equations

$$\sum_{k=1}^{m} \mathfrak{b}(v_j, v_k)\xi^k = 0 \quad \text{for } 1 \le j \le m \ , \tag{2.24}$$

has a nontrivial solution. If $v := \sum_{k=1}^{m} \xi^k v_k$, then (2.24) is equivalent to $\mathfrak{b}(v_j, v) = 0$ for $1 \le j \le m$. Because $\{v_1, \ldots, v_m\}$ is a basis of V and $\mathfrak{b}$ is nondegenerate, it follows that $v = 0$, and we are done. ■

(d) Suppose $(b_1, \ldots, b_m)$ is a positive basis of V and $(\beta^1, \ldots, \beta^m)$ is its dual basis. Also assume $\mathfrak{B}$ is unitary. Then

$$\varepsilon^1 \wedge \cdots \wedge \varepsilon^m = \sqrt{|G_{\mathfrak{b}}|}\, \beta^1 \wedge \cdots \wedge \beta^m \ .$$

Proof The first part of the proof of Proposition 2.13 shows that

$$\varepsilon^1 \wedge \cdots \wedge \varepsilon^m = \det(B)\beta^1 \wedge \cdots \wedge \beta^m ,$$

where $B \in \mathcal{L}(V)$ is the change of basis from $(e_1, \ldots, e_m)$ to $(b_1, \ldots, b_m)$. With the representation $\mathfrak{B}$ of $\mathfrak{b}$, we find

$$\mathfrak{b}(b_j, b_k) = (\mathfrak{B}b_j \,|\, b_k) = (\mathfrak{B}Be_j \,|\, Be_k) = (B^*\mathfrak{B}Be_j \,|\, e_k) \quad \text{for } 1 \leq j, k \leq m .$$

As in the proof of Proposition 2.13, this implies

$$G_{\mathfrak{b}} = \det\bigl[\mathfrak{b}(b_j, b_k)\bigr] = \det(B^*\mathfrak{B}B) = \det(\mathfrak{B})(\det(B))^2 .$$

Because $|\det(\mathfrak{B})| = 1$, it follows that $(\det(B))^2 = |G_{\mathfrak{b}}|$, which implies the claim. ■

Suppose now $\mathcal{O}r$ is an orientation of V and $\mathfrak{b}$ is a nondegenerate symmetric bilinear form on V. Also let $(e_1, \ldots, e_m)$ be a positive $\mathfrak{b}$-ONB of V whose dual basis is $(\varepsilon^1, \ldots, \varepsilon^m)$.

On V^*, we define by

$$\mathfrak{b}_*(v^*, w^*) := \mathfrak{b}(\Theta_{\mathfrak{b}}^{-1}v^*, \Theta_{\mathfrak{b}}^{-1}w^*) \quad \text{for } v^*, w^* \in V^* .$$

the nondegenerate symmetric bilinear form $\mathfrak{b}_*$. For $\alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^r \in V^*$, we set

$$\mathfrak{b}_r(\alpha^1 \wedge \cdots \wedge \alpha^r, \beta^1 \wedge \cdots \wedge \beta^r) := \det\bigl[\mathfrak{b}_*(\alpha^j, \beta^k)\bigr] ,$$

and therefore $\mathfrak{b}_1 = \mathfrak{b}_*$; we also define

$$\mathfrak{b}_r : \textstyle\bigwedge^r V^* \times \bigwedge^r V^* \to \mathbb{R} \quad \text{for } r \geq 1$$

by bilinear extension using the basis representation of Proposition 2.3(i). As in (2.16)–(2.19), it follows (with $\mathfrak{b}_0 := (\cdot\,|\,\cdot)_0$) that there is a linear map

$$\textstyle\bigwedge^r V^* \to \bigwedge^{m-r} V^* , \quad \alpha \mapsto *\alpha$$

for $0 \leq r \leq m$, called the **Hodge star operator**, that is characterized by

$$\alpha \wedge \beta = \mathfrak{b}_{m-r}(*\alpha, \beta)\varepsilon^1 \wedge \cdots \wedge \varepsilon^m \quad \text{for } \beta \in \textstyle\bigwedge^{m-r} V^* . \tag{2.25}$$

2.19 Remarks **(a)** The Hodge star is an isomorphism that depends only on the bilinear form $\mathfrak{b}$ and the orientation, not on the $\mathfrak{b}$-ONB.

(b) For $1 \leq r \leq m$, $\{\, \varepsilon^{(j)} \;;\; (j) \in \mathbb{J}_r \,\}$ is a $\mathfrak{b}_r$-ONB, and for $\omega := \varepsilon^1 \wedge \cdots \wedge \varepsilon^m$, we have $\mathfrak{b}_m(\omega, \omega) = \operatorname{sign}(\mathfrak{b})$.

Proof The first statement follows easily from the definition of $\mathfrak{b}_r$. Because

$$\begin{aligned}\mathfrak{b}_m(\omega, \omega) &= \det\bigl(\operatorname{diag}\bigl[\mathfrak{b}_*(\varepsilon^1, \varepsilon^1), \ldots, \mathfrak{b}_*(\varepsilon^m, \varepsilon^m)\bigr]\bigr)\\ &= \det\bigl(\operatorname{diag}\bigl[\mathfrak{b}(e_1, e_1), \ldots, \mathfrak{b}(e_m, e_m)\bigr]\bigr) ,\end{aligned}$$

the second statement is also true. ■

(c) We have $*1 = \operatorname{sign}(\mathfrak{b})\omega$ and $*\omega = 1$. Also

$$*\varepsilon^{(j)} = s(j)\mathfrak{b}_{m-r}(\varepsilon^{(j^c)}, \varepsilon^{(j^c)})\varepsilon^{(j^c)} \quad \text{for } (j) \in \mathbb{J}_r \ ,$$

for $1 \le r \le m-1$.

Proof First $\omega = 1 \wedge \omega = \mathfrak{b}_m(*1, \omega)\omega$ implies $\mathfrak{b}_m(*1, \omega) = 1$. Next $\dim(\bigwedge^m V^*) = 1$ gives $*1 = a\omega$ with $a \in \mathbb{R}$. From this we obtain with (b) that

$$1 = \mathfrak{b}_m(*1, \omega) = a\mathfrak{b}_m(\omega, \omega) = a \operatorname{sign}(\mathfrak{b}) \ ,$$

and therefore $a = \operatorname{sign}(\mathfrak{b})$. This proves the first claim. Analogously, we find $*\omega = 1$.

Suppose $1 \le r \le m-1$ and $(j) \in \mathbb{J}_r$. Then

$$\omega = s(j)\varepsilon^{(j)} \wedge \varepsilon^{(j^c)} = s(j)\mathfrak{b}_{m-r}(*\varepsilon^{(j)}, \varepsilon^{(j^c)})\omega \ ,$$

and therefore $\mathfrak{b}_{m-r}(*\varepsilon^{(j)}, \varepsilon^{(j^c)}) = s(j)$. Note $\{\, \varepsilon^{(k)} \ ; \ (k) \in \mathbb{J}_{m-r} \,\}$ is a $\mathfrak{b}_{m-r}$-ONB of $\bigwedge^{m-r} V^*$. Also $*\varepsilon^{(j)} \in \bigwedge^{m-r} V^*$, and $\mathfrak{b}_{m-r}(*\varepsilon^{(j)}, \varepsilon^{(k)}) = 0$ for $(k) \ne (j^c)$. It follows that $*\varepsilon^{(j)} = a\varepsilon^{(j^c)}$ with $a \in \mathbb{R}$, and therefore

$$a\mathfrak{b}_{m-r}(\varepsilon^{(j^c)}, \varepsilon^{(j^c)}) = \mathfrak{b}_{m-r}(*\varepsilon^{(j)}, \varepsilon^{(j^c)}) = s(j) \ .$$

This implies $a = s(j)\mathfrak{b}_{m-r}(\varepsilon^{(j^c)}, \varepsilon^{(j^c)})$. Now the last claim is clear. ■

(d) For $\alpha \in \bigwedge^r V^*$ with $0 \le r \le m$, we have $**\alpha = \operatorname{sign}(\mathfrak{b})\,(-1)^{r(m-r)}\alpha$.

Proof As in the proof of (c), we obtain from

$$\mathfrak{b}_r(*\varepsilon^{(j^c)}, \varepsilon^{(j)})\omega = \varepsilon^{(j^c)} \wedge \varepsilon^{(j)} = (-1)^{r(m-r)}\varepsilon^{(j)} \wedge \varepsilon^{(j^c)} = s(j)(-1)^{r(m-r)}\omega \ ,$$

that $*\varepsilon^{(j^c)} = s(j)(-1)^{r(m-r)}\mathfrak{b}_r(\varepsilon^{(j)}, \varepsilon^{(j)})\varepsilon^{(j)}$. Therefore we find by (c) that

$$\begin{aligned} *(*\varepsilon^{(j)}) &= *\big(s(j)\mathfrak{b}_{m-r}(\varepsilon^{(j^c)}, \varepsilon^{(j^c)})\varepsilon^{(j^c)}\big) \\ &= s(j)^2(-1)^{r(m-r)}\mathfrak{b}_r(\varepsilon^{(j)}, \varepsilon^{(j)})\mathfrak{b}_{m-r}(\varepsilon^{(j^c)}, \varepsilon^{(j^c)})\varepsilon^{(j)} \ , \end{aligned}$$

from which the claim follows. ■

(e) For $\alpha, \beta \in \bigwedge^r V^*$, we have

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \operatorname{sign}(\mathfrak{b})\, \mathfrak{b}_r(\alpha, \beta)\omega \ .$$

Proof This is true by an obvious modification of the proof of Example 2.17(f). ■

An important use of these ideas is the **Minkowski space** $\mathbb{R}^4_{1,3} := \big(\mathbb{R}^4, (\cdot \,|\, \cdot)_{1,3}\big)$, that is, the “spacetime” of special relativity with the **Minkowski metric**

$$(x \,|\, y)_{1,3} := x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 \ .$$

(In relativity theory, the “0-th coordinate” is the time.) We will elaborate on this later.

An indefinite nondegenerate symmetric bilinear form $\mathfrak{b}$ is also called an **indefinite inner product**; accordingly, $(V, \mathfrak{b})$ is an **indefinite inner product space**.

Tensors

For the sake of completeness, we now briefly introduce the concept of general tensors, which we will encounter in several later sections. Suppose $r, s \in \mathbb{N}$. An $(r+s)$-linear map

$$\gamma : \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \to \mathbb{R}$$

is called a **tensor on V of type (r,s)** or an **(r,s)-tensor**. In particular, γ is **contravariant of order r and covariant of order s** (or **r-contravariant and s-covariant**). We denote by $T^r_s(V)$ the normed[8] vector space of all **(r,s)-tensors** on V.

For $\gamma_1 \in T^{r_1}_{s_1}(V)$ and $\gamma_2 \in T^{r_2}_{s_2}(V)$, the **tensor product** $\gamma_1 \otimes \gamma_2$ is defined by

$$\begin{aligned}\gamma_1 \otimes \gamma_2(\alpha^1, \ldots, \alpha^{r_1}, \beta^1, \ldots, \beta^{r_2}, v_1, \ldots, v_{s_1}, w_1, \ldots, w_{s_2})\\ := \gamma_1(\alpha^1, \ldots, \alpha^{r_1}, v_1, \ldots, v_{s_1})\gamma_2(\beta^1, \ldots, \beta^{r_2}, w_1, \ldots, w_{s_2})\end{aligned}$$

with $\alpha^1, \ldots, \alpha^{r_1}, \beta^1, \ldots, \beta^{r_2} \in V^*$ and $v_1, \ldots, v_{s_1}, w_1, \ldots, w_{s_2} \in V$.

In the following and as usual, we identify V^{**} with V using the canonical isomorphism κ of Remark 2.8.

2.20 Remarks **(a)** $T^1_0(V) = V$, $T^0_1(V) = V^*$, and $T^0_2(V) = \mathcal{L}^2(V, \mathbb{R})$.

(b) For $\gamma \in T^1_1(V)$, there exists exactly one $C \in \mathcal{L}(V)$ with

$$\gamma(v^*, v) = \langle v^*, Cv \rangle \quad \text{for } v \in V \ , \quad v^* \in V^* \ . \tag{2.26}$$

The map

$$T^1_1(V) \to \mathcal{L}(V) \ , \quad \gamma \mapsto C$$

is an isometric isomorphism.

Proof For $v \in V$, the map $\gamma(\,\cdot\,, v)$ belongs to $V^{**} = V$. Because γ is bilinear, we have

$$C := \big(v \mapsto \gamma(\,\cdot\,, v)\big) \in \mathcal{L}(V)$$

with $\langle v^*, Cv \rangle = \gamma(v^*, v)$ for $(v, v^*) \in V \times V^*$. Conversely, every $C \in \mathcal{L}(V)$ defines by virtue of (2.26) a $\gamma \in T^1_1(V)$. The last claim is now clear. ■

(c) The tensor product is bilinear and associative.

(d) Letting $m := \dim(V)$, we have $\dim\big(T^r_s(V)\big) = m^{r+s}$. If $(e_1, \ldots, e_m)$ is a basis of V and $(\varepsilon^1, \ldots, \varepsilon^m)$ is its dual basis, then

$$\big\{\, e_{j_1} \otimes \cdots \otimes e_{j_r} \otimes \varepsilon^{k_1} \otimes \cdots \otimes \varepsilon^{k_s} \ ; \ j_i, k_i \in \{1, \ldots, m\} \,\big\}$$

is a basis of $T^r_s(V)$.

[8] See Theorem VII.4.2.

Proof We leave the simple proof to you. ■

(e) $\bigwedge^r V^*$ is a vector subspace of $T_r^0(V)$.

(f) The dual pairing $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ is a $(1,1)$-tensor on V. ■

Exercises

1 For $T \in \mathcal{L}^r(V, \mathbb{R})$, the **alternator**, $\mathrm{Alt}(T)$, is defined by

$$\mathrm{Alt}(T)(v_1, \ldots, v_r) := \frac{1}{r!} \sum_{\sigma \in \mathsf{S}_r} \mathrm{sign}(\sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(r)})$$

for $v_1, \ldots, v_r \in V$. Show that

(a) $\mathrm{Alt} \in \mathcal{L}(\mathcal{L}^r(V, \mathbb{R}), \bigwedge^r V^*)$;

(b) $\mathrm{Alt}^2 = \mathrm{Alt}$.

2 For $S \in \mathcal{L}^s(V, \mathbb{R})$ and $T \in \mathcal{L}^t(V, \mathbb{R})$, define $S \otimes T \in \mathcal{L}^{s+t}(V, \mathbb{R})$ by

$$S \otimes T(v_1, \ldots, v_s, v_{s+1}, \ldots, v_{s+t}) := S(v_1, \ldots, v_s) T(v_{s+1}, \ldots, v_{s+t}) \ ,$$

where $v_1, \ldots, v_{s+t} \in V$. Show that for $\alpha \in \bigwedge^r V^*$ and $\beta \in \bigwedge^s V^*$,

$$\alpha \wedge \beta = \frac{(r+s)!}{r!\, s!} \, \mathrm{Alt}(\alpha \otimes \beta) \ .$$

In Exercises 3–8, let $(V, (\cdot \,|\, \cdot), \mathcal{O}r)$ be an oriented inner product space, let ω be its volume element, and let Θ be the Riesz isomorphism.

3 Let $\dim(V) = 3$. Then the **vector** or **cross product** $\times$ on V is defined by[9]

$$\times : V \times V \to V \ , \quad (v, w) \mapsto v \times w := \Theta^{-1} \omega(v, w, \cdot) \ .$$

Show the following:

(a) $(v \times w \,|\, u) = \omega(v, w, u)$ for $u, v, w \in V$.

(b) The vector product is bilinear and alternating.

(c) The vector $v \times w$ is different from zero if and only if v and w are linearly independent.

(d) If v and w are linear independent, then $(v, w, v \times w)$ is a positive basis of V.

(e) The vector $v \times w$ is orthogonal to v and w.

(f) For $v, w \in V \backslash \{0\}$, we have

$$|v \times w| = \sqrt{|v|^2 \, |w|^2 - (v \,|\, w)^2} = |v| \, |w| \sin \varphi \ ,$$

where $\varphi \in [0, \pi]$ is the (unoriented) angle between the vectors v and w.

[9] See Remarks VIII.2.14.

(g) Let (e_1, e_2, e_3) be a positive ONB of V. Then for $v = \sum_j \xi^j e_j$ and $w = \sum_j \eta^j e_j$, we have

$$v \times w = (\xi^2\eta^3 - \xi^3\eta^2)e_1 + (\xi^3\eta^1 - \xi^1\eta^3)e_2 + (\xi^1\eta^2 - \xi^2\eta^1)e_3 \ .$$

(h) (Grassmann identity) $v_1 \times (v_2 \times v_3) = (v_1 \,|\, v_3)v_2 - (v_1 \,|\, v_2)v_3$.

(i) The vector product is not associative.

(j) $(v_1 \times v_2) \times (v_3 \times v_4) = \omega(v_1, v_2, v_4)v_3 - \omega(v_1, v_2, v_3)v_4$.

(k) (Jacobi identity) $v_1 \times (v_2 \times v_3) + v_2 \times (v_3 \times v_1) + v_3 \times (v_1 \times v_2) = 0$.

(Hints: (f) Recall Proposition 2.13 and (a). (h) The vector product is determined by its values in the basis (e_1, e_2, e_3).)

4 For $0 \le r \le m$, verify the following formulas:

(a) $(*\alpha \,|\, \beta)_{m-r} = (-1)^{r(m-r)}(\alpha \,|\, *\beta)_r$ for $\alpha \in \bigwedge^r V^*$ and $\beta \in \bigwedge^{m-r} V^*$.

(b) $(*\alpha) \wedge \beta = (*\beta) \wedge \alpha$ for $\alpha, \beta \in \bigwedge^r V^*$.

(c) $*(\Theta v \wedge *\Theta w) = (v \,|\, w)$ for $v, w \in V$.

5 Let $(b_1, \ldots, b_m)$ be a positive basis of V with $(\beta^1, \ldots, \beta^m)$ its dual basis. Prove these:

(a) $\beta^j \wedge *\beta^k = g^{jk}\sqrt{G}\,\beta^1 \wedge \cdots \wedge \beta^m$ for $1 \le j, k \le m$.

(b) $*\beta^j = \sum_{k=1}^m (-1)^{k-1} g^{jk}\sqrt{G}\,\beta^1 \wedge \cdots \wedge \widehat{\beta^k} \wedge \cdots \wedge \beta^m$ for $1 \le j \le m$. If V is three-dimensional, show that

$$*(\beta^j \wedge \beta^k) = \frac{1}{\sqrt{G}} \operatorname{sign}(j, k, \ell) \sum_{i=1}^3 g_{\ell i}\beta^i = \frac{1}{G} \operatorname{sign}(j, k, \ell)\Theta b_\ell$$

for $(j, k, \ell) \in \mathsf{S}_3$.

6 In the case $\dim(V) = 3$, show $v \times w = \Theta^{-1}\big(*(\Theta v \wedge \Theta w)\big)$ for $v, w \in V$.

7 Let (b_1, b_2, b_3) be a positive basis of V with dual basis $(\beta^1, \beta^2, \beta^3)$. Show that

$$b_j \times b_k = \sqrt{G} \operatorname{sign}(j, k, \ell) \sum_{i=1}^3 g^{\ell i} b_i = \sqrt{G} \operatorname{sign}(j, k, \ell)\Theta^{-1}\beta^\ell$$

for $(j, k, \ell) \in \mathsf{S}_3$.

8 Let $\big(W, (\cdot \,|\, \cdot)_W, \mathcal{O}r(W)\big)$ be an oriented inner product, and let $A \in \mathcal{L}(V, W)$ be an orientation-preserving isometry (that is, $A^*\omega_W = \omega_V$). Then show that the diagram

$$\begin{array}{ccc}
\bigwedge^r V^* & \xrightarrow{\ *\ } & \bigwedge^{m-r} V^* \\
{\scriptstyle A^*}\big\uparrow & & \big\uparrow{\scriptstyle A^*} \\
\bigwedge^r W^* & \xrightarrow{\ *\ } & \bigwedge^{m-r} W^*
\end{array}$$

commutes for $0 \le r \le m$.

9 Formulate and prove the claims of Exercises 4 and 5 for indefinite inner product spaces.

10 For $k \in \mathbb{N}$, let $\mathbb{K}_k[X]$ be the vector space of all polynomials of degree $\leq k$ over $\mathbb{K}$. Show that $\mathbb{K}[X] = \bigoplus_{k \geq 0} \mathbb{K}_k[X]$ is a graded commutative algebra with respect to the usual multiplication of polynomials, that is, with respect to the convolution of Section I.8.

11 Let $(\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3)$ be the basis dual to the standard basis of $\mathbb{R}^4$. For $c, E_j, H_j \in \mathbb{R}$, set

$$\begin{aligned}\alpha &:= \ \ (E_1\varepsilon^1 + E_2\varepsilon^2 + E_3\varepsilon^3) \wedge c\varepsilon^0 + (H_1\varepsilon^2 \wedge \varepsilon^3 + H_2\varepsilon^3 \wedge \varepsilon^1 + H_3\varepsilon^1 \wedge \varepsilon^2)\ , \\ \beta &:= -(H_1\varepsilon^1 + H_2\varepsilon^2 + H_3\varepsilon^3) \wedge c\varepsilon^0 + (E_1\varepsilon^2 \wedge \varepsilon^3 + E_2\varepsilon^3 \wedge \varepsilon^1 + E_3\varepsilon^1 \wedge \varepsilon^2)\end{aligned}$$

and calculate $*\alpha$ and $*\beta$ with respect to $(\cdot\,|\,\cdot)_{1,3}$.

3 The local theory of differential forms

In Section VIII.3, we learned much about differential forms of degree 1, the Pfaff forms, and we developed a calculus that forms the foundation for the theory of line integral. Now we extend these ideas to more dimensions. In a first step, to which this section is given, we introduce differential forms of arbitrary degree on open subsets of Euclidean space, and we provide the calculus of differential forms in this "local" situation. In the sections thereafter, we consider the general situation, namely, differential forms on manifolds.

A differential form of degree r on an open subset X of $\mathbb{R}^m$ is nothing other than a set consisting of an alternating r-form on the tangent space T_xX for each $x \in X$. For this reason, the first part of this section is really only a reformulation of the results of linear algebra provided in Section 2. Rather than formulating new theorems, we will explain the definitions with remarks and examples. Analysis will come into play when we introduce an operation on differential forms, the exterior derivative. The exterior derivative makes use of concepts from analysis and goes beyond linear algebra.

In this entire section

- X is open in $\mathbb{R}^m$ and $\mathbb{K} = \mathbb{R}$.

Definitions and basis representations

For $x \in X$, the cotangent space $T_x^*X = \{x\} \times (\mathbb{R}^m)^*$ is the space dual to the tangent space $T_xX = \{x\} \times \mathbb{R}^m$. Therefore the exterior product

$$\textstyle\bigwedge^r T_x^*X = \{x\} \times \bigwedge^r (\mathbb{R}^m)^* \quad \text{for } r \in \mathbb{N} \tag{3.1}$$

and the Grassmann algebra

$$\textstyle\bigwedge T_x^*X = \{x\} \times \bigwedge (\mathbb{R}^m)^*$$

are well defined on T_x^*X. We can generalize the tangent and cotangent bundle by defining the **bundle of alternating *r*-forms** on X by

$$\textstyle\bigwedge^r T^*X := \bigcup_{x \in X} \bigwedge^r T_x^*X = X \times \bigwedge^r (\mathbb{R}^m)^*$$

and by defining the **Grassmann bundle** of X by

$$\textstyle\bigwedge T^*X := \bigcup_{x \in X} \bigwedge T_x^*X = X \times \bigwedge (\mathbb{R}^m)^* \ .$$

A map

$$\boldsymbol{\alpha} \colon X \to \textstyle\bigwedge^r T^*X \qquad \text{with} \quad \boldsymbol{\alpha}(x) \in \bigwedge^r T_x^*X \text{ and } x \in X \ ,$$

that is, a **section**[1] of the Grassmann bundle, is called a **differential form of degree r** (for short, an **r-form**) **on X**. By (3.1), every r-form on X has a unique representation

$$\boldsymbol{\alpha}(x) = (x, \alpha(x)) \quad \text{for } x \in X$$

whose **r-covector part** (for short, covector part) is

$$\alpha \colon X \to \bigwedge^r (\mathbb{R}^m)^* \ .$$

Let $k \in \mathbb{N} \cup \{\infty\}$. The r-form $\boldsymbol{\alpha}$ belongs to the **class C^k** (or is k-times **continuously differentiable**,[2] or **smooth** in case $k = \infty$) if this is true for its covector part, that is, if

$$\alpha \in C^k\bigl(X, \bigwedge^r (\mathbb{R}^m)^*\bigr) \ . \tag{3.2}$$

This definition is meaningful because, according to Remark 2.2(a), $\bigwedge^r (\mathbb{R}^m)^*$ is a (closed) vector subspace of $\mathcal{L}^r(\mathbb{R}^m, \mathbb{R})$.

For simplicity and in order to concentrate on the essential aspects of the theory, we consider almost exclusively smooth r-forms and smooth vector fields. We treat the C^k case only briefly in remarks, whose verification we leave to you.

We denote the set of all smooth r-forms on X by $\Omega^r(X)$. For short we set

$$\mathcal{E}(X) := C^\infty(X) \quad \text{and} \quad \mathcal{V}(X) := \mathcal{V}^\infty(X) \ .$$

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are vector fields on X with corresponding vector parts $v_1, \ldots, v_r$, that is, if $\boldsymbol{v}_j(x) = (x, v_j(x))$ for $x \in X$ and $1 \le j \le r$, then we set

$$\boldsymbol{\alpha}(v_1, \ldots, v_r)(x) := \boldsymbol{\alpha}(x)\bigl(\boldsymbol{v}_1(x), \ldots, \boldsymbol{v}_r(x)\bigr) \quad \text{for } x \in X \ . \tag{3.3}$$

Then it follows from (VIII.3.1) that

$$\boldsymbol{\alpha}(x)(\boldsymbol{v}_1(x), \ldots, \boldsymbol{v}_r(x)) = \alpha(x)(v_1(x), \ldots, v_r(x)) \quad \text{for } x \in X \ ,$$

that is,

$$\boldsymbol{\alpha}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \alpha(v_1, \ldots, v_r) \ . \tag{3.4}$$

This shows that, without causing misunderstanding, we can identify an r-form $\boldsymbol{\alpha}$ with its covector part α and a vector field $\boldsymbol{v}$ with its vector part v. For this reason, we will from now on write differential forms and vector fields in a normal font (not boldface). In each instance, you will be able to decide without trouble whether a symbol describes a form or its covector part (or whether it means a vector field or its vector part).

[1] We apply the language of the theory of "vector bundles". We will not elaborate on these here (but see for example [Con93], [Dar94], or [HR72]), although it would lead to a unification of various ideas.

[2] Naturally, we say an r-form of class C^0 is **continuous**.

Addition

$$\Omega^r(X) \times \Omega^r(X) \to \Omega^r(X) , \quad (\alpha, \beta) \mapsto \alpha + \beta$$

and the **exterior product**

$$\wedge : \Omega^r(X) \times \Omega^s(X) \to \Omega^{r+s}(X) , \quad (\alpha, \beta) \mapsto \alpha \wedge \beta$$

are performed pointwise:

$$(\alpha + \beta)(x) := \alpha(x) + \beta(x) \quad \text{and} \quad (\alpha \wedge \beta)(x) := \alpha(x) \wedge \beta(x) \quad \text{for } x \in X .$$

These maps are obviously well defined.

3.1 Remarks **(a)** $\Omega^0(X) = \mathcal{E}(X)$.

(b) $\Omega^1(X) = \Omega_{(\infty)}(X)$, that is, the smooth 1-forms of X are C^∞ Pfaff forms on X.

(c) $\Omega^r(X) = \{0\}$ for $r > m$.

(d) $\Omega^r(X)$ for $0 \leq r \leq m$ is an infinite-dimensional real vector space and a free $\mathcal{E}(X)$-module of dimension $\binom{m}{r}$ (with respect to pointwise multiplication). A module basis for $\Omega^r(X)$ is given by

$$\left\{ dx^{(j)} := dx^{j_1} \wedge \cdots \wedge dx^{j_m} \ ; \ (j) \in \mathbb{J}_r \right\} . \tag{3.5}$$

Proof Because of (a) and the canonical identification of $\mathbb{R}$ with the subring[3] $\mathbb{R}\mathbf{1}$ of $\mathcal{E}(X)$, we have the relation $\alpha \wedge \beta = \alpha\beta$ for $\alpha \in \mathbb{R}$ and $\beta \in \Omega(X)$. The first statement then follows immediately from Remark 2.2(a) and Example I.12.3(e).

According to Remark VIII.3.3, $(dx^1(x), \ldots, dx^m(x))$ is the basis dual to the canonical basis $((e_1)_x, \ldots, (e_m)_x)$ of T_xX. Then the remaining claim follows from Proposition 2.3. ■

(e) An r-form α on X belongs to the class C^k if and only if every r-tuple $v_1, \ldots, v_r$ in $\mathcal{V}^k(X)$ satisfies

$$\alpha(v_1, \ldots, v_r) \in C^k(X) . \tag{3.6}$$

This is the case if and only if the coefficients $a_{(j)}$ of the canonical basis representation[4]

$$\alpha = \sum_{(j) \in \mathbb{J}_r} a_{(j)} dx^{(j)} \tag{3.7}$$

satisfy the relation

$$a_{(j)} \in C^k(X) \quad \text{for } (j) \in \mathbb{J}_r . \tag{3.8}$$

[3] $\mathbf{1}(x) = 1$ for $x \in X$.

[4] It follows from Proposition 2.3, as in the proof of (d), that (3.5) is a basis of the $\mathbb{R}^X$-module of all r-forms on X.

Proof When α belongs to the class C^k, it follows easily from (3.2) and Corollary VII.4.7 that (3.6) is true. Then because (2.4) implies

$$a_{(j)} = \alpha(e_{j_1}, \ldots, e_{j_r}) , \tag{3.9}$$

(3.8) follows from (3.6). If (3.8) is satisfied, we conclude from (3.7) and the constancy of the basis forms $dx^{(j)}$ that α belongs to the class C^k. ■

(f) The exterior product[5] is bilinear, associative, and graded anticommutative. Therefore

$$\Omega(X) := \bigoplus_{r \geq 0} \Omega^r(X)$$

is an (infinite-dimensional) associative, graded anticommutative algebra (with respect to the product $\wedge$). Also $\Omega(X)$ is a free $\mathcal{E}(X)$-module of dimension 2^m (with respect to pointwise multiplication); we call it the **module of differential forms** on X.

Proof These are simple consequences of Theorem 2.7 and (d). ■

(g) Every $\alpha \in \Omega^r(X)$ is an alternating r-form on $\mathcal{V}(X)$.

Proof This follows immediately from the definition (3.3). ■

(h) (regularity) For $k \in \mathbb{N}$, let $\Omega^r_{(k)}(X)$ be the set of r-forms of class C^k on X. Then the previous statements hold analogously for $\Omega^r_{(k)}(X)$ when $\mathcal{E}(X)$ is replaced everywhere by $C^k(X)$. ■

In the following, we will generally not state that the coefficients $a_{(j)}$ of the canonical basis representation (3.7) of $\alpha \in \Omega^r(X)$ belong to $\mathcal{E}(X)$. This will be deemed self-evident.

3.2 Examples **(a)** As we already know, every Pfaff Form $\alpha \in \Omega^1(X)$ has the canonical basis representation

$$\alpha = \sum_{j=1}^{m} a_j \, dx^j \ .$$

(b) For $\alpha \in \Omega^{m-1}(X)$, the basis representation has the form

$$\alpha = \sum_{j=1}^{m} (-1)^{j-1} a_j \, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m \ .$$

(c) In the case $m = 3$, any $\alpha \in \Omega^2(X)$ has the basis representation[6]

$$\alpha = a_1 \, dx^2 \wedge dx^3 + a_2 \, dx^3 \wedge dx^1 + a_3 \, dx^1 \wedge dx^2 \ .$$

Proof Because $dx^3 \wedge dx^2 = -dx^2 \wedge dx^3$, this follows from (b). ■

[5] Sometimes we say **wedge product** instead of exterior product.

[6] Note the cyclic permutation of the indices.

(d) Every $\alpha \in \Omega^m(X)$ has the form $a\,dx^1 \wedge \cdots \wedge dx^m$ with $a \in \mathcal{E}(X)$.

(e) For $m = 3$, the wedge product of

$$\alpha = a_1\,dx^1 + a_2\,dx^2 + a_3\,dx^3 \text{ and } \beta = b_1\,dx^1 + b_2\,dx^2 + b_3\,dx^3$$

is

$$\alpha \wedge \beta = (a_2 b_3 - a_3 b_2)\,dx^2 \wedge dx^3 + (a_3 b_1 - a_1 b_3)\,dx^3 \wedge dx^1 + (a_1 b_2 - a_2 b_1)\,dx^1 \wedge dx^2 \ .$$

Proof This follows from Remark 3.1(f). ■

Pull backs

Let Y be open in $\mathbb{R}^n$ and $\varphi \in C^\infty(X, Y)$. In a generalization of the pull back of Pfaff forms, we introduce the **pull back** of differential forms by φ. It is a map

$$\varphi^* : \Omega(Y) \to \Omega(X) \tag{3.10}$$

defined by

$$(\varphi^*\beta)(x) := (T_x\varphi)^*\beta\bigl(\varphi(x)\bigr) \quad \text{for } x \in X \text{ and } \beta \in \Omega(Y) \ . \tag{3.11}$$

If $\beta \in \Omega^r(Y)$, then, because $T_x\varphi \in \mathcal{L}(T_xX, T_{\varphi(x)}Y)$ and by Remark 2.9(a), both $(T_x\varphi)^*\beta\bigl(\varphi(x)\bigr)$ and $\beta\bigl(\varphi(x)\bigr) \in \bigwedge^r T^*_{\varphi(x)}Y$ lie in $\bigwedge^r T^*_x X$. From $T_x\varphi = \bigl(\varphi(x), \partial\varphi(x)\bigr)$ and

$$\partial\varphi \in C^\infty\bigl(X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)\bigr)$$

and also because (3.4) implies

$$\varphi^*\beta(v_1, \ldots, v_r) = (\beta \circ \varphi)\bigl((\partial\varphi)v_1, \ldots, (\partial\varphi)v_r\bigr) \quad \text{for } v_1, \ldots, v_r \in \mathcal{V}(X) \ ,$$

we see by Remark 3.1(e) that $\varphi^*\beta$ belongs to $\Omega^r(X)$. Therefore (3.10) is well defined through (3.11).

3.3 Remarks **(a)** The map (3.10) is $\mathbb{R}$-linear and satisfies

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* \quad \text{and} \quad (\mathrm{id}_X)^* = \mathrm{id}_{\Omega(X)} \ ,$$

that is, the pull back operates contravariantly. It is also compatible with the exterior product, that is,

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta \quad \text{for } \alpha, \beta \in \Omega(Y) \ .$$

Therefore φ^* is an algebra homomorphism from $\Omega(Y)$ to $\Omega(X)$.

Proof This follows from Remarks 2.9 and the chain rule given in Remark VII.10.2(b). ■

(b) (regularity) The pull back can also be naturally defined for $\varphi \in C^{k+1}(X, Y)$. If $1 \le r \le m$, then an r-form of class C^{k+1} generally becomes an r-form only of class C^k, while an r-form of class C^k remains in the same class. In the case $r = 0$, the pull back preserves the regularity. ■

3.4 Examples Let $(x^1,\ldots,x^m)$ and $(y^1,\ldots,y^n)$ be the Euclidean coordinates of X and Y, respectively.

(a) $\varphi^*\,dy^j = d\varphi^j = \sum_{k=1}^m \partial_k\varphi^j\,dx^k$ and $1\le j\le n$.

Proof See Example VIII.3.14(a). ■

(b) For

$$\beta = \sum_{(j)\in\mathbb{J}_r} b_{(j)}\,dy^{(j)} \in \Omega^r(Y)\ ,$$

we have

$$\varphi^*\beta = \sum_{(j)\in\mathbb{J}_r} (\varphi^* b_{(j)})\,d\varphi^{(j)}\ .$$

Proof This is a consequence of (a) and Remark 3.3(a). ■

(c) In the case $m=n$, we have

$$\varphi^*(dy^1\wedge\cdots\wedge dy^m) = d\varphi^1\wedge\cdots\wedge d\varphi^m = (\det\partial\varphi)\,dx^1\wedge\cdots\wedge dx^m\ .$$

Proof The first equality follows from (b). Because $\det T_x\varphi = \det\partial\varphi(x)$ for $x\in X$, the claim follows from Proposition 2.10 and the constancy of the basis form $dx^1\wedge\cdots\wedge dx^m$ on X. ■

(d) Let $m=2$ and $n=3$, and let (u,v) and (x,y,z) be respective Euclidean coordinates of X and Y. Then[7]

$$\begin{aligned}&\varphi^*(a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy)\\ &\qquad = \Bigl[a\circ\varphi\,\frac{\partial(\varphi^2,\varphi^3)}{\partial(u,v)} + b\circ\varphi\,\frac{\partial(\varphi^3,\varphi^1)}{\partial(u,v)} + c\circ\varphi\,\frac{\partial(\varphi^1,\varphi^2)}{\partial(u,v)}\Bigr]\,du\wedge dv\ .\end{aligned}$$

Proof Because $d\varphi^j = \varphi_u^j\,du + \varphi_v^j\,dv$ for $1\le j\le 3$ and because

$$\frac{\partial(\varphi^2,\varphi^3)}{\partial(u,v)} = \det\begin{bmatrix}\varphi_u^2 & \varphi_v^2\\ \varphi_u^3 & \varphi_v^3\end{bmatrix} = \varphi_u^2\varphi_v^3 - \varphi_u^3\varphi_v^2$$

etc., the claim follows from (b) and Example 3.2(e). ■

(e) (plane polar coordinates) Let

$$f_2:\mathbb{R}^2\to\mathbb{R}^2\ ,\quad (r,\varphi)\mapsto(x,y) := (r\cos\varphi, r\sin\varphi)$$

be the polar coordinate map. Then

$$f_2^*(dx\wedge dy) = r\,dr\wedge d\varphi\ .$$

Proof This follows from (c) and Example X.8.7. ■

[7] See Remark VII.7.9.

(f) (spherical coordinates) For the spherical coordinate map

$$f_3 : \mathbb{R}^3 \to \mathbb{R}^3 \ , \quad (r, \varphi, \vartheta) \mapsto (x, y, z) := (r \cos\varphi \sin\vartheta, r \sin\varphi \sin\vartheta, r \cos\vartheta) \ ,$$

we have

$$f_3^*(dx \wedge dy \wedge dz) = -r^2 \sin\vartheta \, dr \wedge d\varphi \wedge d\vartheta \ .$$

Proof Lemma X.8.8 and (c). ■

(g) (m-dimensional polar coordinates) Let

$$f_m : \mathbb{R}^m \to \mathbb{R}^m \ , \quad (r, \varphi, \vartheta_1, \ldots, \vartheta_{m-2}) \mapsto (x^1, \ldots, x^m)$$

be the m-dimensional polar coordinate map (X.8.17). Then

$$f_m^* \, dx^1 \wedge \cdots \wedge dx^m = (-1)^m r^{m-1} w_m(\vartheta) \, dr \wedge d\varphi \wedge d\vartheta_1 \wedge \cdots \wedge d\vartheta_{m-2} \ ,$$

where $w_m(\vartheta) := \sin\vartheta_1 \sin^2\vartheta_2 \cdots \sin^{m-2}\vartheta_{m-2}$.

Proof This follows from Lemma X.8.8. ■

(h) (cylindrical coordinates) Let

$$f : \mathbb{R}^3 \to \mathbb{R}^3 \ , \quad (r, \varphi, z) \mapsto (x, y, z) := (r \cos\varphi, r \sin\varphi, z)$$

be the cylindrical coordinate map. Then

$$f^*(dx \wedge dy \wedge dz) = r \, dr \wedge d\varphi \wedge dz \ .$$

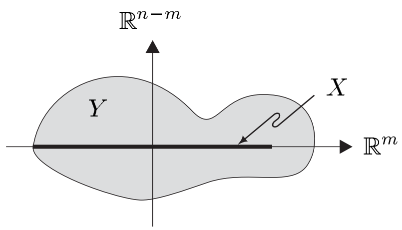
Proof Example VII.9.11(c) and (c). ■

(i) If φ is a constant map, then $\varphi^*\alpha = 0$ for $\alpha \in \Omega^r(Y)$ with $r \geq 1$.

Proof Because $d\varphi^j = 0$ for $1 \leq j \leq n$, the claim is a consequence of (b). ■

(j) Let $m \leq n$, and let $i : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ be the natural embedding that identifies $\mathbb{R}^m$ with $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$. Also let Y be open in $\mathbb{R}^n$ with

$$Y \cap \big(\mathbb{R}^m \times \{0\}\big) \supset i(X) \ .$$



Note that X is an m-dimensional submanifold of Y.

For $\alpha \in \Omega^r(Y)$, define $\alpha \,|\, X$, the **restriction** of α to X, by

$$(\alpha \,|\, X)(x) := \alpha(x, 0) \,|\, (T_x X)^r \quad \text{for } x \in X \ .$$

In other words, when α has the basis representation

$$\alpha = \sum_{(j) \in \mathbb{J}_r^n} a_{(j)} \, dx^{(j)} \ ,$$

it follows that

$$(\alpha \,|\, X)(x) = \sum_{\substack{(j)\in\mathbb{J}_r^n \\ j_r\le m}} a_{(j)}(x,0)\,dx^{(j)} \quad \text{for } x \in X \ .$$

Then $i^*\alpha = \alpha \,|\, X$.

Proof Because of the linearity of i^* and that of the restriction map

$$\Omega^r(Y) \to \Omega^r(X) \ , \quad \alpha \mapsto \alpha \,|\, X \ ,$$

it suffices to consider the case $\alpha = a\,dx^{(j)}$ for $(j) \in \mathbb{J}_r^n$. Then it follows from (b) that

$$i^*\alpha = (i^*a)\,di^{(j)} \ .$$

By $(i^*a)(x) = a(x,0)$ and $i^k = \mathrm{pr}_k\, i = 0$ for $m+1 \le k \le n$ (where $\mathrm{pr}_k : \mathbb{R}^n \to \mathbb{R}$ is the canonical projection), we have $di^{(j)} = 0$ for $j_r > m$. For $j_r \le m$, we find $di^{(j)} = dx^{(j)}$. Now the claim is obvious. ■

(k) Let $(q,p) \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$ be any point of $\mathbb{R}^{2m}$. We define the (standard) **symplectic form** on $\mathbb{R}^{2m}$ by

$$\sigma := \sum_{j=1}^m dp^j \wedge dq^j \ .$$

We denote by $\mathrm{Sp}(2m)$ the set of all $S \in \mathcal{L}(\mathbb{R}^{2m})$ with $S^*\sigma = \sigma$. Then $\mathrm{Sp}(2m)$ is a subgroup of $\mathcal{L}\mathrm{aut}(\mathbb{R}^{2m})$, the symplectic group. Any $S \in \mathrm{Sp}(2m)$ satisfies $\det(S) = 1$.

Proof We define $\alpha \in \Omega^{2m}(\mathbb{R}^{2m})$ by $\alpha := \sigma \wedge \cdots \wedge \sigma$ (with m factors). Then there is an $a \in \mathbb{R}^\times$ such that $\alpha = a\omega$, where ω denotes the volume element of $\mathbb{R}^{2m}$. Suppose now $S \in \mathrm{Sp}(2m)$. Then it follows from $S^*\sigma = \sigma$ and Remark 3.3(a) that

$$S^*\alpha = S^*\sigma \wedge \cdots \wedge S^*\sigma = \sigma \wedge \cdots \wedge \sigma = \alpha \ .$$

Because $S^*\alpha = S^*(a\omega) = aS^*\omega$ and from (c), we find

$$\alpha = S^*\alpha = a\det(S)\omega = \det(S)\alpha \ ,$$

and therefore $\det(S) = 1$. We leave the proof that $\mathrm{Sp}(2m)$ is a subgroup of $\mathcal{L}\mathrm{aut}(\mathbb{R}^{2m})$ to you as an exercise. ■

The exterior derivative

In Section VIII.3, we saw that the differential df of a function $f \in \mathcal{E}(X) = \Omega^0(X)$ is a smooth Pfaff form and therefore an element of $\Omega^1(X)$. Obviously $d \colon \Omega^0(X) \to \Omega^1(X)$ is linear. In addition, we know from Proposition VIII.3.12 that d commutes with pull backs. The following theorem shows that d can be extended to an $\mathbb{R}$-linear map from the module $\Omega(X)$ of differential forms to itself; this map likewise commutes with pull backs.

3.5 Theorem *There is exactly one map*

$$d\colon \Omega(X) \to \Omega(X) \ ,$$

the **exterior derivative**,[8] *with the properties* (i)–(iv):

(i) *d is $\mathbb{R}$-linear and maps $\Omega^r(X)$ to $\Omega^{r+1}(X)$.*

(ii) *d satisfies the* **product rule**

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta \quad \text{for } \alpha \in \Omega^r(X) \text{ and } \beta \in \Omega(X) \ .$$

(iii) $d^2 := d \circ d = 0$.

(iv) *The exterior derivative df for $f \in \mathcal{E}(X)$ equals the differential of f.*

If Y is open in $\mathbb{R}^n$ and $\varphi \in C^\infty(X,Y)$, then

$$d \circ \varphi^* = \varphi^* \circ d \ , \tag{3.12}$$

that is, the exterior derivative commutes with the pull back.

Proof (a) (uniqueness) For

$$\alpha = \sum_{(j) \in \mathbb{J}_r} a_{(j)}\, dx^{(j)} \in \Omega^r(X) \ , \tag{3.13}$$

it follows easily from (i)–(iv) that

$$d\alpha = \sum_{(j) \in \mathbb{J}_r} da_{(j)} \wedge dx^{(j)} \in \Omega^{r+1}(X) \ . \tag{3.14}$$

This implies that at most one map can satisfy the properties (i)–(iv).

(b) (existence) For $\alpha \in \Omega^r(X)$ expanded as in (3.13), we defined $d\alpha$ by (3.14). Then d obviously satisfies the demands (i) and (iv).

To show (ii), realize that (i) means we need only consider the case $\alpha = a\, dx^{(j)}$ and $\beta = b\, dx^{(k)}$ with $(j) \in \mathbb{J}_r$ and $(k) \in \mathbb{J}_s$. Then it follows from (3.14), the properties of the exterior product, and the ordinary product rule of Corollary VII.3.8 that

$$\begin{aligned} d(\alpha \wedge \beta) &= d(ab\, dx^{(j)} \wedge dx^{(k)}) = d(ab) \wedge dx^{(j)} \wedge dx^{(k)} \\ &= da \wedge dx^{(j)} \wedge b\, dx^{(k)} + (-1)^r a\, dx^{(j)} \wedge db \wedge dx^{(k)} \\ &= d(a\, dx^{(j)}) \wedge b\, dx^{(k)} + (-1)^r a\, dx^{(j)} \wedge d(b\, dx^{(k)}) \\ &= d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta \ , \end{aligned}$$

as desired.

[8]Sometimes the exterior derivative is called the **Cartan derivative**.

For the proof of (iii), we can use the linearity of d to again restrict to the case $\alpha = a\,dx^{(j)}$ with $(j) \in \mathbb{J}_r$. Then it follows from (3.14) and (ii) that

$$d(d\alpha) = d(da \wedge dx^{(j)}) = d^2 a \wedge dx^{(j)} - da \wedge d(dx^{(j)}) \ .$$

By successive application of the product rule (ii) to $d(dx^{(j)})$, we see that the claim will follow if we can show $d^2 a = 0$ for $a \in \Omega^0(X) = \mathcal{E}(X)$.

Suppose therefore $a \in \mathcal{E}(X)$. Then we may use (i), (ii), and (iv) to derive the relation

$$\begin{aligned} d(da) &= d\Bigl(\sum_{k=1}^m \partial_k a\, dx^k\Bigr) = \sum_{k=1}^m d(\partial_k a) \wedge dx^k \\ &= \sum_{j,k=1}^m \partial_j \partial_k a\, dx^j \wedge dx^k = \sum_{1 \le j < k \le m} (\partial_j \partial_k a - \partial_k \partial_j a)\, dx^j \wedge dx^k = 0 \ , \end{aligned}$$

where the last equality follows from Schwarz's theorem (Corollary VII.5.5). Therefore (iii) is satisfied.

(c) Suppose $\varphi \in C^\infty(X, Y)$ and $(j) \in \mathbb{J}_r^n$. Let $\beta = b\, dy^{(j)} \in \Omega^r(Y)$. Then according to Example 3.4(b), we have

$$\varphi^* \beta = \varphi^* b\, d\varphi^{(j)} \in \Omega^r(X) \ . \tag{3.15}$$

From (3.14) and the property of the pull back explained in Remark 3.3(a), we get

$$\varphi^*\, d\beta = \varphi^*(db \wedge dy^{(j)}) = \varphi^*\, db \wedge \varphi^*\, dy^{(j)} = \varphi^*\, db \wedge d\varphi^{(j)} \ .$$

Proposition VIII.3.12 implies $\varphi^*\, db = d(\varphi^* b)$. Therefore we find using (i), (iii), and (3.15) that

$$\begin{aligned} \varphi^*\, d\beta = d(\varphi^* b) \wedge d\varphi^{(j)} &= d(\varphi^* b) \wedge d\varphi^{(j)} + (-1)^1 \varphi^* b \wedge d(d\varphi^{(j)}) \\ &= d(\varphi^* b \wedge d\,\varphi^{(j)}) = d(\varphi^* \beta) \ . \end{aligned}$$

Now (3.12) follows from the linearity of φ^* and d and from Remark 3.1(e). ■

3.6 Remarks **(a)** For $\alpha = \sum_{(j) \in \mathbb{J}_r} a_{(j)}\, dx^{(j)} \in \Omega^r(X)$, we have

$$d\alpha = \sum_{(j) \in \mathbb{J}_r} da_{(j)} \wedge dx^{(j)} \ .$$

Proof This is the statement (3.13), (3.14). ■

(b) For $\varphi \in C^\infty(X,Y)$ and $r \in \mathbb{N}$, the diagram

$$\begin{array}{ccc} \Omega^r(Y) & \xrightarrow{\;d\;} & \Omega^{r+1}(Y) \\ {\scriptstyle \varphi^*}\big\downarrow & & \big\downarrow{\scriptstyle \varphi^*} \\ \Omega^r(X) & \xrightarrow{\;d\;} & \Omega^{r+1}(X) \end{array}$$

commutes.

Proof This is (3.12). ■

(c) (regularity) If α is an r-form of class C^{k+1}, then $d\alpha$ is obviously an $(r+1)$-form of class C^k. However, for $\alpha = a\,dx^{(j)}$ with $(j) \in \mathbb{J}_r^m$, we have

$$d\alpha = da \wedge dx^{(j)} = \sum_i \partial_i a\,dx^i \wedge dx^{(j)} ,$$

where we only sum over the indices $i \in \{1,\ldots,m\}$ with $i \neq j_k$ for $1 \leq k \leq r$, because $dx^i \wedge dx^{(j)} = 0$ is true of the remaining indices. Hence there is an r-form α of class C^k for which $d\alpha$ also belongs to the class C^k. ■

3.7 Examples **(a)** For $\alpha = \sum_{j=1}^m a_j\,dx^j \in \Omega^1(X)$, we have

$$d\alpha = \sum_{1\leq j<k\leq m} (\partial_j a_k - \partial_k a_j)\,dx^j \wedge dx^k .$$

(b) For $\alpha = \sum_{j=1}^m (-1)^{j-1} a_j\,dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m \in \Omega^{m-1}(X)$, we get

$$d\alpha = \Bigl(\sum_{j=1}^m \partial_j a_j\Bigr)\,dx^1 \wedge \cdots \wedge dx^m .$$

(c) $d\alpha = 0$ for $\alpha \in \Omega^m(X)$. ■

The Poincaré lemma

A differential form $\alpha \in \Omega(X)$ is said to be **closed** if $d\alpha = 0$. We say it is **exact** if there an **antiderivative** $\beta \in \Omega(X)$ such that[9] $d\beta = \alpha$.

3.8 Remarks and examples **(a)** Example 3.7(a) says $\alpha = \sum_{j=1}^m a_j\,dx^j$ is closed if and only if $\partial_j a_k = \partial_k a_j$ for $1 \leq j,k \leq m$. Therefore this extended notion of closedness reduces to the definition of Section VIII.3 in the case of Pfaff forms.

[9] Saying that a form is exact implies that it has degree at least 1.

(b) Every exact form is closed.

Proof This follows from $d^2 = 0$. ■

(c) Every m-form on X is closed.

Proof Example 3.7(c). ■

(d) (regularity) The definition of closed is clearly meaningful for forms of class C^1; the notion of exact makes sense for continuous differential forms. ■

In Theorem VIII.3.8, we have seen that every closed Pfaff form is exact if X is star shaped. In the following, we will show that this "lemma" of Poincaré is also true in the general case.

Let $I := [0, 1]$, and let t be a generic point in I. For $\ell \in \{0, 1\}$, the injection

$$i_\ell \colon X \to I \times X \ , \quad x \mapsto (\ell, x)$$

is smooth. Obviously i_0 and i_1 identify the X with the "bottom" $\{0\} \times X$ and the "top" $\{1\} \times X$, respectively, of the cylinder $I \times X$ over X. Therefore[10]

$$i_\ell^* \colon \Omega^r(I \times X) \to \Omega^r(X)$$

is defined. For $\alpha \in \Omega(I \times X)$ the form $i_0^*\alpha$ [or $i_1^*\alpha$] is a restriction of α to X. It is obtained by replacing (t, x) by $(0, x)$ [or $(1, x)$] in the coefficients of the canonical basis representation of α, and by removing all terms in which dt occurs (see Example 3.4(j)).

We define a linear map

$$K \colon \Omega^{r+1}(I \times X) \to \Omega^r(X)$$

by

$$K\alpha := \sum_{(j)\in\mathbb{J}_r} \int_0^1 a_{(j)}(t, \cdot\,)\, dt\, dx^{(j)} \tag{3.16}$$

for

$$\alpha = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\, dt \wedge dx^{(j)} + \sum_{(k)\in\mathbb{J}_{r+1}} b_{(k)}\, dx^{(k)} \ . \tag{3.17}$$

3.9 Lemma *K is well defined and satisfies*

$$K \circ d + d \circ K = i_1^* - i_0^* \ . \tag{3.18}$$

Proof The theorem about the differentiability of parameter-dependent integrals (Theorem X.3.18) implies easily that $K\alpha$, defined for the α of (3.17) by (3.16), belongs to $\Omega^r(X)$. Clearly the map K is also linear.

[10] Because the partial derivative ∂_t is defined on I, it is clear how differential forms are defined on $I \times X$. Note that $I \times X$ is a manifold with boundary, and see Section 4.

To show (3.18), it suffices to consider the cases $\alpha = a\,dt \wedge dx^{(j)}$ and $\alpha = b\,dx^{(k)}$ with $(j) \in \mathbb{J}_r$ and $(k) \in \mathbb{J}_{r+1}$.

(i) Let $\alpha = a\,dt \wedge dx^{(j)}$. Then $i_0^*\alpha = i_1^*\alpha = 0$. We also get

$$\begin{aligned} K\,d\alpha = K(da \wedge dt \wedge dx^{(j)}) &= K\Bigl(\sum_{\ell=1}^{m} \partial_{x^\ell} a\,dx^\ell \wedge dt \wedge dx^{(j)}\Bigr) \\ &= -\sum_{\ell=1}^{m} \int_0^1 \partial_{x^\ell} a(t,\cdot\,)\,dt\,dx^\ell \wedge dx^{(j)} \ , \end{aligned}$$

where we have used $dt \wedge dt \wedge dx^{(j)} = 0$. On the other hand, Theorem X.3.18 gives

$$d(K\alpha) = d\Bigl(\int_0^1 a(t,\cdot\,)\,dt\,dx^{(j)}\Bigr) = \sum_{\ell=1}^{m} \int_0^1 \partial_{x^\ell} a(t,\cdot\,)\,dt\,dx^\ell \wedge dx^{(j)} \ .$$

This proves the claim in this case.

(ii) Let $\alpha = b\,dx^{(k)}$ with $(k) \in \mathbb{J}_{r+1}$. Then $K\alpha = 0$, and therefore $dK\alpha = 0$. Also, we find

$$d\alpha = \partial_t b\,dt \wedge dx^{(k)} + \sum_{\ell=1}^{m} \partial_{x^\ell} b\,dx^\ell \wedge dx^{(k)}$$

and

$$K d\alpha = \int_0^1 \partial_t b(\tau,\cdot\,)\,d\tau\,dx^{(k)} = \bigl(b(1,\cdot\,) - b(0,\cdot\,)\bigr)\,dx^{(k)} = i_1^*\alpha - i_0^*\alpha \ ,$$

so the claim holds in this case also. ∎

Let M and N be manifolds. Two maps $f_0, f_1 \in C^\infty(M,N)$ are said to be **homotopic in N** if there is a map[11] $h \in C^\infty(I \times M, N)$, a **homotopy**, such that $h(j,\cdot\,) = f_j$ for $j = 0,1$. A map $f \in C^\infty(M,N)$ is **null-homotopic in N** if it is homotopic in N to a constant map. Finally, we say M is **contractible** if the identity map from M to M is null-homotopic.

3.10 Remarks **(a)** The statement "f_1 is homotopic in N to f_2" defines an equivalence relation in $C^\infty(M,N)$ (or, more generally, in $C^k(M,N)$).

(b) The concept of a (continuous) homotopy obviously generalizes the idea of a loop homotopy (see Section VIII.4).

(c) Every star shaped open set is contractible.

Proof Let X be star shaped with respect to $x_0 \in X$. Then

$$h : I \times X \to X \ , \quad (t,x) \mapsto x_0 + t(x - x_0)$$

is obvious a homotopy with $h(0,\cdot\,) = x_0$ and $h(1,\cdot\,) = \mathrm{id}_X$. ∎

[11] Note that $I \times M$ is a manifold with boundary.

(d) (regularity) For $k \in \mathbb{N}^\times$, the definitions above are meaningful for C^k manifolds M and N if all the functions that appear belong to the class C^k. They are then also meaningful if M and N are topological spaces and all functions considered are continuous. ■

We can now easily prove the generalized Poincaré lemma.

3.11 Theorem (Poincaré lemma) *If X is contractible, then every closed differential form on X is exact.*

Proof Suppose $\alpha \in \Omega^{r+1}(X)$ is closed. Because X is contractible, there exists an $h \in C^\infty(I \times X, X)$ such that $h(1, \cdot\,) = \mathrm{id}_X$ and $h(0, \cdot\,) = p$ for some $p \in X$. Because α is closed, $h^*\alpha \in \Omega^{r+1}(I \times X)$ is also closed because $d \circ h^* = h^* \circ d$. Therefore it follows from Lemma 3.9 that

$$d(Kh^*\alpha) = i_1^* h^*\alpha - i_0^* h^*\alpha = (h \circ i_1)^*\alpha = \alpha\ .$$

This is because $h \circ i_1 = \mathrm{id}_X$ and because $i_0^* h^*\alpha = (h \circ i_0)^*\alpha$ is a null form according to Example 3.4(i). ■

We should point out that the proof of the Poincaré lemma gives an explicit procedure for constructing an antiderivative of a given closed differential form. The situation is particularly simple when X is star shaped, where we can assume without loss of generality (by applying a suitable translation) that X is star shaped with respect to 0.

3.12 Corollary *Suppose X is star shaped with respect to 0. Suppose with $r \in \mathbb{N}^\times$ that*

$$\alpha = \sum_{(j) \in \mathbb{J}_r} a_{(j)}\, dx^{(j)} \in \Omega^r(X)$$

is closed. Also let

$$\beta := \sum_{(j) \in \mathbb{J}_r} \sum_{k=1}^{r} (-1)^{k-1} \int_0^1 t^{r-1} a_{(j)}(tx)\, dt\, x^{j_k}\, dx^{j_1} \wedge \cdots \wedge \widehat{dx^{j_k}} \wedge \cdots \wedge dx^{j_r}\ . \tag{3.19}$$

Then β belongs to $\Omega^{r-1}(X)$, and $d\beta = \alpha$.

Proof In this case, $h(t, x) := tx$ for $(t, x) \in I \times X$ defines a "contraction of X to 0". From $dh^j = x^j\, dt + t\, dx^j$ and Example 3.4(b), it follows that

$$\begin{aligned} h^*\alpha(t, x) = {} & \sum_{(j) \in \mathbb{J}_r} a_{(j)}(tx) t^r\, dx^{(j)} \\ & + \sum_{(j) \in \mathbb{J}_r} \sum_{k=1}^{r} (-1)^{k-1} a_{(j)}(tx) t^{r-1} x^{j_k}\, dt \wedge dx^{j_1} \wedge \cdots \wedge \widehat{dx^{j_k}} \wedge \cdots \wedge dx^{j_r}\ , \end{aligned}$$

because those terms in which dt occurs at least twice vanish. From (3.16) and (3.17), it follows that $\beta = Kh^*\alpha$, and the claim now follows from the proof of the Poincaré lemma. ■

3.13 Remarks **(a)** In the case $r = 1$, that is, when α is a Pfaff form, the formula for β is the case as the one in (VIII.3.4).

(b) Let $m = 3$ and

$$\alpha = a_1\, dx^2 \wedge dx^3 + a_2\, dx^3 \wedge dx^1 + a_3\, dx^1 \wedge dx^2 \in \Omega^2(X) \ .$$

Then the problem of finding a $\beta = \sum_{j=1}^3 b_j\, dx^j$ with $d\beta = \alpha$ is equivalent to the problem of finding three functions $b_1, b_2, b_3 \in \mathcal{E}(X)$ that satisfy the system

$$\begin{aligned} \partial_1 b_2 - \partial_2 b_1 &= a_3 \ , \\ \partial_2 b_3 - \partial_3 b_2 &= a_1 \ , \\ \partial_3 b_1 - \partial_1 b_3 &= a_2 \end{aligned} \tag{3.20}$$

of partial differential equations in X. Then, for given $a_j \in \mathcal{E}(X)$,

$$\partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3 = 0 \tag{3.21}$$

is required for (3.20) to have a solution. If X is contractible (for example $X = \mathbb{R}^3$), then (3.21) is also sufficient.

Proof By Example 3.7(a), (3.20) is equivalent to $d\beta = \alpha$. Example 3.7(b) shows that (3.21) is equivalent to $d\alpha = 0$. Now the claim follows from $d^2 = 0$ and the Poincaré lemma. ■

From Corollary 3.12, it follows in particular that in the case of star shaped domains, (3.20) can be solved by quadrature using the formula (3.19). In the general case, the equation $d\beta = \alpha$ can clearly also be reformulated as an equivalent system of partial differential equations.

Of course, (3.20) does not have a unique solution, because a closed form can be added to β, that is, one can add any solution (b_1, b_2, b_3) of the homogeneous system obtained by zeroing the right side of (3.20).

Tensors

Let $r, s \in \mathbb{N}$. For $x \in X$, we set

$$T^r_s(T_xX) := \{x\} \times T^r_s(\mathbb{R}^m) \tag{3.22}$$

and call $\gamma \in T^r_s(T_xX)$ an r-contravariant and s-covariant **tensor**, or **tensor of type (r,s)** on T_xX. The **bundle of (r,s)-tensors** on X is defined by

$$T^r_s(X) := \bigcup_{x\in X} T^r_s(T_xX) = X \times T^r_s(\mathbb{R}^m) \ .$$

A map

$$\boldsymbol{\gamma}\colon X \to T^r_s(X) \qquad \text{with} \quad \boldsymbol{\gamma}(x) \in T^r_s(T_xX) \ ,$$

that is, a section of the tensor bundle $T^r_s(X)$, is called an **(r,s)-tensor (field)** or tensor of **type (r,s)** on X. By (3.22), every (r,s)-tensor $\boldsymbol{\gamma}$ on X has the unique representation

$$\boldsymbol{\gamma}(x) = \bigl(x, \gamma(x)\bigr) \quad \text{for } x \in X \ ,$$

with the **principal part**[12]

$$\gamma\colon X \to T^r_s(\mathbb{R}^m) \ .$$

Let $k \in \mathbb{N} \cup \{\infty\}$. An (r,s)-tensor $\boldsymbol{\gamma}$ belongs to the **class** C^k (or is k-times **continuously differentiable** or **smooth** if $k=\infty$) if this is true of its principal part, that is, if[13]

$$\gamma \in C^k\bigl(X, \mathcal{L}^{r+s}(\mathbb{R}^m, \mathbb{R})\bigr) \ .$$

We denote the set of all smooth (r,s)-tensors on X by

$$\mathcal{T}^r_s(X) \ .$$

If $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r$ are Pfaff forms and $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_s$ are vector fields on X with corresponding principal parts $\alpha_1,\ldots,\alpha_r$ and $v_1,\ldots,v_s$, then we will set

$$\boldsymbol{\gamma}(\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r,\boldsymbol{v}_1,\ldots,\boldsymbol{v}_s)(x) := \gamma(x)\bigl(\alpha_1(x),\ldots,\alpha_r(x),v_1(x),\ldots,v_s(x)\bigr)$$

for $x \in X$. (This is clearly consistent with (3.4).) For these reasons, we can use the same notational conventions as before with vector fields and differential forms, that is, we identify tensors with their principal parts, and from now on use the ordinary font instead of boldface.

Addition

$$\mathcal{T}^r_s(X) \times \mathcal{T}^r_s(X) \to \mathcal{T}^r_s(X) \ , \quad (\gamma,\delta) \mapsto \gamma + \delta \ ,$$

multiplication by functions

$$\mathcal{E}(X) \times \mathcal{T}^r_s(X) \to \mathcal{T}^r_s(X), \ , \quad (f,\gamma) \mapsto f\gamma$$

and the **tensor product**

$$\mathcal{T}^{r_1}_{s_1}(X) \times \mathcal{T}^{r_2}_{s_2}(X) \to \mathcal{T}^{r_1+r_2}_{s_1+s_2}(X) \ , \quad (\gamma,\delta) \mapsto \gamma \otimes \delta \tag{3.23}$$

[12] In the case $s=0$, we called this the vector part, and for $r=0$, we called it the covector part. A tensor combines vectors and covectors, so such terminology is no longer possible.

[13] As usual, we identify $T_x\mathbb{R}^m$ and $T_x^*\mathbb{R}^m$ with $\mathbb{R}^m$.

will also be defined pointwise:

$$(\gamma+\delta)(x) := \gamma(x)+\delta(x)\ , \quad (f\gamma)(x) := f(x)\gamma(x)\ , \quad (\gamma\otimes\delta)(x) := \gamma(x)\otimes\delta(x)\ .$$

The following remarks are simple consequences of Remarks 2.20 and the chain rule. We leave the detailed proofs to you as exercises.

3.14 Remarks **(a)** $\mathcal{T}_0^1(X)=\mathcal{V}(X)$ and $\mathcal{T}_1^0(X)=\Omega^1(X)$. Also

$$\mathcal{T}_2^0(X)=C^\infty\big(X,\mathcal{L}^2(\mathbb{R}^m)\big)\ ,$$

where we have used the canonical identification of a tensor with its principal part.

(b) The tensor product is $\mathcal{E}(X)$-bilinear and associative.

(c) $\mathcal{T}_s^r(X)$ is an infinite-dimensional $\mathbb{R}$-vector space and an m^{r+s}-dimensional $\mathcal{E}(X)$-module. With the canonical basis $(\partial/\partial x^1,\ldots,\partial/\partial x^m)$ of $\mathbb{R}^m$,

$$\Big\{\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_r}}\otimes dx^{k_1}\otimes\cdots\otimes dx^{k_s}\ ;\ j_i,k_i\in\{1,\ldots,m\}\Big\} \tag{3.24}$$

is a module basis of $\mathcal{T}_s^r(X)$.

(d) An (r,s)-tensor γ on X belongs to $\mathcal{T}_s^r(X)$ if and only if every r-tuple $\alpha_1,\ldots,\alpha_r$ in $\Omega^1(X)$ and every s-tuple $v_1,\ldots,v_s$ in $\mathcal{V}(X)$ satisfy

$$\gamma(\alpha_1,\ldots,\alpha_r,v_1,\ldots,v_s)\in\mathcal{E}(X)\ .$$

This is the case if and only if the coefficients of γ in basis (3.24) belong to $\mathcal{E}(X)$.

(e) (regularity) The definitions and claims above have obvious analogues which remain true for tensors of class C^k. ■

Exercises

1 Let $\alpha,\beta\in\Omega(\mathbb{R}^4)$ be given by

$$\alpha := dx^1+x^2\,dx^2 \quad\text{and}\quad \beta := \sin(x^2)\,dx^1\wedge dx^3+\cos(x^3)\,dx^2\wedge dx^4\ ,$$

and define $h\in C^\infty(\mathbb{R}^4,\mathbb{R}^4)$ by $h(x):=(x^1,x^2,x^3x^4,x^4)$.
Calculate:

(i) $\gamma:=\alpha\wedge\beta$;

(ii) $h^*\gamma$;

(iii) $h^*\gamma(0)(e_1,e_2,e_3+e_4)$, where (e_1,e_2,e_3,e_4) is the standard basis in $\mathbb{R}^4$;

(iv) $d\alpha$, $d\beta$, $d\gamma$, $d(h^*\gamma)$.

2 Let $f_3:\mathbb{R}^3\to\mathbb{R}^3$, $(r,\varphi,\vartheta)\mapsto(x,y,z)$ be the spherical coordinate map.
Calculate

(a) $f_3^*\,dx$, $f_3^*\,dy$, $f_3^*\,dz$;

(b) $f_3^*(dy \wedge dz)$;

(c) $f_3^*\, dx \wedge f_3^*(dy \wedge dz)$.

3 A simple thermodynamic system (for example, an ideal gas) is characterized by its volume V and its temperature T (here $V, T \in \mathbb{R}$). The state of such a system is then described by the pressure $p := p(V,T)$ and the internal energy $E := E(V,T)$. By the second law of thermodynamics, the system has another state function $S := S(V,T)$, the entropy, whose differential is given by

$$dS := \frac{dE + p\,dV}{T} \quad \text{for } T > 0 \ .$$

Show the following facts:

(a) E and p satisfy the relation

$$\frac{\partial E}{\partial V} = T\,\frac{\partial p}{\partial T} - p \ .$$

(b) The internal energy of an ideal gas, which satisfies the equation of state $pV = RT$ with $R \in \mathbb{R}$ the (universal gas) constant, is independent of the volume, that is, $E = E(T)$.

(c) For van der Waals gas, which has the equation of state

$$\Bigl(p + \frac{a}{V^2}\Bigr)(V - b) = cT \quad \text{for } a, b, c \in \mathbb{R}^\times \ , \tag{3.25}$$

the internal energy does depend on volume.

(Hints: (a) $d^2 = 0$. (c) (3.25) $\Rightarrow T\,\partial p/\partial T = p + a/V^2$.)

Remark In the physics literature, $d\alpha$ is often written $\delta\alpha$ when the 1-form α is not exact.

4 An r-form $\alpha \in \Omega^r(X)$ is said to be **decomposable** if there are $\alpha_1, \ldots, \alpha_r \in \Omega^1(X)$ such that

$$\alpha = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_r \ .$$

Let $\alpha, \beta \in \Omega^r(X)$ be decomposable. Calculate $(\alpha + \beta) \wedge (\alpha + \beta)$.

5 Suppose $\alpha = \sum_{j \le k} a_{jk}\, dx^j \wedge dx^k \in \Omega^2(X)$. Show that α is decomposable if and only if

$$a_{ij}a_{k\ell} + a_{jk}a_{i\ell} + a_{ki}a_{j\ell} = 0 \quad \text{for } 1 \le i, j, k, \ell \le n \ ,$$

where $a_{jk} := -a_{kj}$ for $j \ge k$.

6 Let $\alpha = \sum_{i \le j} a_{ij}\, dx^i \wedge dx^j \in \Omega^2(X)$. Show

$$d\alpha = \sum\nolimits_{i<j<k} \Bigl(\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{jk}}{\partial x^i} + \frac{\partial a_{ki}}{\partial x^j}\Bigr)\, dx^i \wedge dx^j \wedge dx^k \ .$$

7 Calculate the exterior derivatives of

(a) $d\alpha \wedge \beta - \alpha \wedge d\beta$ and

(b) $d\alpha \wedge \beta \wedge \gamma + \alpha \wedge d\beta \wedge \gamma + \alpha \wedge \beta \wedge d\gamma$, where in (b) α and β are of even degree.

8 Find $d\alpha$ if $\alpha := \sum_{j=1}^m (-1)^{j-1}\, x^j/|x|^m\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m \in \Omega^{m-1}\bigl(\mathbb{R}^m \setminus \{0\}\bigr)$.

9 Let $\alpha := 2xz\,dy \wedge dz + dz \wedge dx - (z^2 + e^x)\,dx \wedge dy \in \Omega^2(\mathbb{R}^3)$. Show that α is exact and determine an antiderivative.

10 Suppose $\omega \in \Omega^2(X)$ is nondegenerate. Show that

$$\Theta_\omega : \mathcal{V}(X) \to \Omega^1(X) \ , \quad v \mapsto \omega(v, \cdot)$$

is an $\mathcal{E}(X)$-module isomorphism.

11 Prove these three statements:

(a) The symplectic form $\sigma \in \Omega^2(\mathbb{R}^{2m})$ is nondegenerate and closed.

(b) The m-fold product $\sigma^m := \sigma \wedge \cdots \wedge \sigma \in \Omega^{2m}(\mathbb{R}^{2m})$ satisfies $\sigma^m \neq 0$.

(c) According to Exercise 10 and (b) the **symplectic gradient** $\operatorname{sgrad} f := \Theta_\sigma^{-1}\,df \in \mathcal{V}(\mathbb{R}^{2m})$ is defined for every $f \in \mathcal{E}(\mathbb{R}^{2m})$. Calculate $\operatorname{sgrad} f$ in the coordinates $(q,p) \in \mathbb{R}^m \times \mathbb{R}^m$.

12 If σ is the symplectic form on $\mathbb{R}^{2m}$, then

$$\{\cdot,\cdot\} : \mathcal{E}(\mathbb{R}^{2m}) \times \mathcal{E}(\mathbb{R}^{2m}) \to \mathcal{E}(\mathbb{R}^{2m}) \ , \quad (f,g) \mapsto \sigma(\operatorname{sgrad} f, \operatorname{sgrad} g)$$

is called the **Poisson bracket**.

For $f, g, h \in \mathcal{E}(\mathbb{R}^{2m})$ and $c \in \mathbb{R}$ prove

(i) in local coordinates $(q_1, \ldots, q_m, p_1, \ldots, p_m)$, the Poisson bracket reads

$$\{f,g\} = \sum_{j=1}^{m} \Bigl(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \Bigr) \ ;$$

(ii) $\{f, c\,g + h\} = c\{f,g\} + \{f,h\}$;

(iii) $\{f,g\} = -\{g,f\}$;

(iv) $\bigl\{f, \{g,h\}\bigr\} + \bigl\{g, \{h,f\}\bigr\} + \bigl\{h, \{f,g\}\bigr\} = 0$ (Jacobi identity);

(v) $\{f, gh\} = g\{f,h\} + h\{f,g\}$;

(vi) $\operatorname{sgrad}\{f,g\} = (\operatorname{sgrad} f \mid \operatorname{sgrad} g)_{\mathbb{R}^{2m}}$.

13 Show that the Poisson bracket is related to the symplectic form σ on $\mathbb{R}^{2m}$ by the relation

$$df \wedge dg \wedge \sigma^{m-1} = \frac{1}{m}\,\{f,g\}\sigma^m \ .$$

4 Vector fields and differential forms

This section is devoted to the global theory of differential forms, that is, to differential forms on manifolds. The first part, which is essentially a simple transfer of the local theory, requires us to focus on the problem of regularity. With help from a theorem about partitions of unity, we can then extend the important concept of the exterior derivative to the case of manifolds and show that the rules we developed for the local theory still apply.

The global theory brings up an important new idea, the orientability of a manifold. We present various ways to characterize this central concept and consider numerous examples. To prepare for the theory of integration on manifolds, we give explicit representations of the volume elements of many important manifolds.

In this entire section,

- M is an m-dimensional, and N is an n-dimensional manifold;
- $r \in \mathbb{N}$.

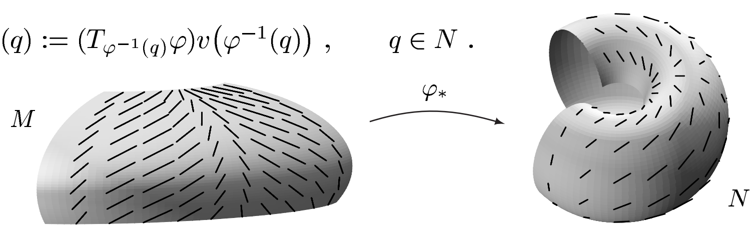
Vector fields

By a **vector field** v on M, we mean a map

$$v : M \to TM \qquad \text{with} \quad v(p) \in T_pM \text{ for } p \in M \ ,$$

that is, a section of the tangent bundle. If v is a vector field on M, then we can "transplant" it using a diffeomorphism from M to N. So we define for $\varphi \in \mathrm{Diff}^1(M, N)$ the **push forward** $\varphi_* v$ of v by φ by letting

$$\varphi_* v(q) := (T_{\varphi^{-1}(q)}\varphi) v\big(\varphi^{-1}(q)\big) \ , \qquad q \in N \ .$$



Therefore $\varphi_* v$ is a vector field on N. For functions on M, the **push forward** by a bijection $\psi : M \to N$ is the assignment

$$\psi_* : \mathbb{R}^M \to \mathbb{R}^N \ , \quad a \mapsto \psi_* a := a \circ \psi^{-1} \ .$$

4.1 Remarks **(a)** For functions, the push forward ψ_* is obviously the same as the pull back ψ^{-1}: $\psi_* = (\psi^{-1})^*$. Note however that, in contrast to the pull back, the push forward is only defined for bijections. In particular, it must be true that $\dim(M) = \dim(N)$.[1]

(b) Let $\varphi \in \mathrm{Diff}^1(M,N)$. Then

$$\varphi_*(a+b) = \varphi_* a + \varphi_* b \ , \quad \varphi_*(v+w) = \varphi_* v + \varphi_* w \ ,$$

and

$$\varphi_*(av) = \varphi_* a \, \varphi_* v$$

for $a, b \in \mathbb{R}^M$ and vector fields v and w on M.

(c) Let $\varphi \in \mathrm{Diff}(M,N)$ and $\psi \in \mathrm{Diff}(N,L)$, where L is another manifold. Then

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_* \quad \text{and} \quad (\mathrm{id}_M)_* = \mathrm{id}_{\mathcal{F}(M)} \tag{4.1}$$

for $\mathcal{F}(M) := \mathcal{E}(M)$ or $\mathcal{F}(M) := \mathcal{V}(M)$. The rule (4.1) means that the push forward operates **covariantly**.

Proof The statement is obvious for push forwards of functions. For vector fields, (4.1) follows from the chain rule of Remark VII.10.9(b) and from Remark 1.14(c). ■

Let $k \in \mathbb{N} \cup \{\infty\}$. The vector field v on M belongs to the class C^k (that is, it is k-times **continuously differentiable**, or **smooth** in case $k = \infty$) if every point p of M has a chart (φ, U) around p such that[2] $\varphi_* v \in \mathcal{V}^k\bigl(\varphi(U)\bigr)$. We denote the set of all vector fields on M of class C^k by $\mathcal{V}^k(M)$. For simplicity of notation, we set

$$\mathcal{V}(M) := \mathcal{V}^\infty(M) \quad \text{and} \quad \mathcal{E}(M) := C^\infty(M) \ .$$

4.2 Remarks **(a)** The definition of C^k vector fields is coordinate-independent. If v is a C^k vector field and (ψ, V) is an arbitrary chart of M, then $\psi_* v$ belongs to the class C^k.

Proof Suppose therefore (ψ, V) is a chart of M. Then we need to show that $\psi_* v$ belongs to the class C^k. Every $q \in V$ has a chart (φ, U) of M around it such that $\varphi_* v \in \mathcal{V}^k\bigl(\varphi(U)\bigr)$. Then $\psi_* v = (\psi \circ \varphi^{-1})_* \varphi_* v$ follows from (4.1). Because

$$\psi \circ \varphi^{-1} \in \mathrm{Diff}\bigl(\varphi(U \cap V), \psi(U \cap V)\bigr)$$

and $\varphi_* v \in \mathcal{V}^k\bigl(\varphi(U)\bigr)$, we find $\psi_* v \in \mathcal{V}^k\bigl(\psi(U \cap V)\bigr)$. Because this holds for every $q \in V$ and because differentiability is a local property, we get $\psi_* v \in \mathcal{V}^k\bigl(\psi(V)\bigr)$. ■

(b) The pointwise-defined operations

$$\mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M) \ , \quad (v, w) \mapsto v + w$$

[1] See Exercise VII.10.9.

[2] See Section VIII.3. What was said there holds without change for open subsets of $\overline{\mathbb{H}}^m$ as well.

and

$$\mathcal{E}(M) \times \mathcal{V}(M) \to \mathcal{V}(M) , \quad (a,v) \mapsto av$$

make $\mathcal{V}(M)$ into an $\mathcal{E}(M)$-module. In particular, $\mathcal{E}(M)$ and $\mathcal{V}(M)$ are (infinite-dimensional) $\mathbb{R}$-vector spaces.

If $\varphi \in \mathrm{Diff}(M,N)$, then φ_* is a module isomorphism from $\mathcal{E}(M)$ to $\mathcal{E}(N)$ and from $\mathcal{V}(M)$ to $\mathcal{V}(N)$.

Proof It follows from (4.1) that

$$\mathrm{id}_M = (\varphi^{-1} \circ \varphi)_* = (\varphi^{-1})_* \varphi_* \quad \text{and} \quad \mathrm{id}_N = (\varphi \circ \varphi^{-1})_* = \varphi_* (\varphi^{-1})_* \ .$$

Therefore φ_* is bijective, and $\varphi_*^{-1} = (\varphi^{-1})_*$. The remaining claims are simple consequences of Remark 4.1(b) and the properties of vector fields on open subsets of $\overline{\mathbb{H}}^m$ (see Section VIII.3). ■

(c) Let X_0 and X_1 be open in $\mathbb{R}^m$, and suppose $\varphi \in \mathrm{Diff}(X_0, X_1)$. Also denote by $\Theta_j : \mathcal{V}(X_j) \to \Omega^1(X_j)$ for $j = 0, 1$ the canonical module isomorphism that was defined in Remark VIII.3.3(g). Then

$$(\varphi^{-1})^* \circ \Theta_0 = \Theta_1 \circ \varphi_* \ ,$$

that is, the diagram

$$\begin{array}{ccc} \mathcal{V}(X_0) & \xrightarrow{\ \varphi_*\ } & \mathcal{V}(X_1) \\ {\scriptstyle \Theta_0}\big\downarrow & & \big\downarrow{\scriptstyle \Theta_1} \\ \Omega^1(X_0) & \xrightarrow{\ (\varphi^{-1})^*\ } & \Omega^1(X_1) \end{array}$$

commutes.

(d) (regularity) Let $k \in \mathbb{N}$. For $\varphi \in \mathrm{Diff}^{k+1}(M,N)$ and $0 \le \ell \le k$, the push forward φ_* maps $C^\ell(M)$ to $C^\ell(N)$ and $\mathcal{V}^\ell(M)$ to $\mathcal{V}^\ell(N)$, but this statement (without the inequality) does not hold for $\ell = k+1$.

If M is a C^{k+1} manifold, then the $C^\ell(M)$-modules $C^\ell(M)$ and $\mathcal{V}^\ell(M)$ are defined for $0 \le \ell \le k$; however[3] the modules $C^{k+1}(M)$ and $\mathcal{V}^{k+1}(M)$ are not.

Proof This is because the tangential “loses” one derivative. ■

Local basis representation

Let (φ, U) be a chart of M around p. Then we denote by

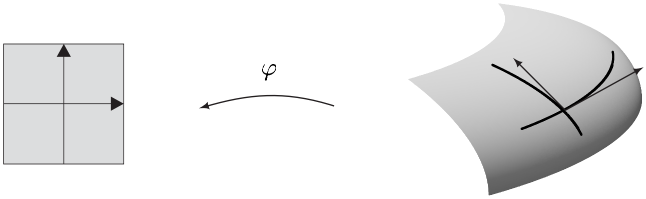
$$\partial_j|_p = \frac{\partial}{\partial x^j}\Big|_p \in T_pM \quad \text{for } 1 \le j \le m$$

[3]except for trivial cases

the basis vectors of T_pM corresponding to the local coordinates $\varphi = (x^1, \ldots, x^m)$. In other words, $\partial_j|_p$ is the tangent vector on the coordinate path $t \mapsto \varphi^{-1}\bigl(\varphi(p) + te_j\bigr)$ at the point[4] p, that is,

$$\partial_j|_p := (T_p\varphi)^{-1}\bigl(\varphi(p), e_j\bigr) \quad \text{for } 1 \le j \le m\ , \tag{4.2}$$

where $(e_1, \ldots, e_m)$ is the canonical basis of $\mathbb{R}^m$.



4.3 Remarks **(a)** Let $i_M : M \hookrightarrow \mathbb{R}^{\overline{m}}$, and let $g_\varphi := i_M \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^{\overline{m}}$ be the parametrization belonging to φ. Then

$$(T_p i_M)\partial_j|_p = \bigl(p, \partial_j g_\varphi\bigl(\varphi(p)\bigr)\bigr) \in T_p\mathbb{R}^{\overline{m}} \quad \text{for } 1 \le j \le m\ .$$

This means that, if we identify $\partial_j|_p \in T_pM$ with its image in $T_p\mathbb{R}^{\overline{m}}$ under the canonical injection

$$T_p i_M : T_pM \to T_p\mathbb{R}^{\overline{m}}\ ,$$

then we find $\partial_j|_p = \bigl(p, \partial_j g_\varphi\bigl(\varphi(p)\bigr)\bigr)$.

Proof From Example VII.10.9(b) and Remark 1.14(c), we get

$$T_{\varphi(p)} g_\varphi = T_{\varphi(p)}(i_M \circ \varphi^{-1}) = T_p i_M \circ T_{\varphi(p)}(\varphi^{-1}) = T_p i_M \circ (T_p\varphi)^{-1}\ .$$

Then it follows from (4.2) that

$$(T_p i_M)\partial_j|_p = (T_{\varphi(p)} g_\varphi)\bigl(\varphi(p), e_j\bigr) = \bigl(p, \partial g_\varphi\bigl(\varphi(p)\bigr)e_j\bigr) = \bigl(p, \partial_j g_\varphi\bigl(\varphi(p)\bigr)\bigr)\ . \ \blacksquare$$

(b) The maps

$$\partial_j = \frac{\partial}{\partial x^j} : U \to TU\ , \quad p \mapsto \partial_j|_p \quad \text{for } 1 \le j \le m$$

are smooth vector fields on U.

Proof This is clear because

$$(\varphi_*\partial_j)\bigl(\varphi(p)\bigr) = (T_p\varphi)(T_p\varphi)^{-1}\bigl(\varphi(p), e_j\bigr) = \bigl(\varphi(p), e_j\bigr) \quad \text{for } 1 \le j \le m$$

for $p \in U$. $\blacksquare$

[4] If p is in the interior of M. If p is a boundary point, we must make φ a submanifold chart of $\mathbb{R}^{\overline{m}}$ around p for M.

(c) For $p \in U$, we have a basis $(\partial_1|_p, \ldots, \partial_m|_p)$ of T_pM and a module basis $(\partial_1, \ldots, \partial_m)$ of $\mathcal{V}(U)$. A vector field v on U belongs to $\mathcal{V}(U)$ if and only if the coefficients v_j of the basis representation

$$v = \sum_{j=1}^{m} v^j \partial_j$$

all belong to $\mathcal{E}(U)$.

Proof The first statement follows from Remark VII.10.5 and the definition of the tangent space at a boundary point. The second claim is a consequence of

$$\varphi_* v = \varphi_* \Bigl(\sum_{j=1}^{m} v^j \partial_j\Bigr) = \sum_{j=1}^{m} (\varphi_* v^j) \varphi_* \partial_j \ ,$$

of (b), and of Remark VIII.3.3(c). ■

(d) (regularity) Let $k \in \mathbb{N}$, and let M be a C^{k+1} manifold. In this case, $(\partial_1, \ldots, \partial_m)$ is a $C^k(U)$-module basis of $\mathcal{V}^k(U)$. A vector field v on U belongs to $\mathcal{V}^k(U)$ if and only if its coefficients with respect to this basis representation lie in $C^k(U)$. ■

Differential forms

To generalize the cotangent space T_p^*X and the cotangent bundle T^*X of an open subset X of $\mathbb{R}^m$, we now define the **cotangent space** of M at the point p by

$$T_p^*M := (T_pM)^* = \mathcal{L}(T_pM, \mathbb{R}) \ .$$

We define the **cotangent bundle** of M by

$$T^*M := \bigcup_{p \in M} T_p^*M \ .$$

We denote by

$$\langle \cdot , \cdot \rangle_p : T_p^*M \times T_pM \to \mathbb{R} \quad \text{for } p \in M$$

the dual pairing[5] and call

$$\langle \cdot , \cdot \rangle : T^*M \times TM \to \mathcal{E}(M) \ , \quad (\alpha, v) \mapsto \bigl[p \mapsto \bigl\langle \alpha(p), v(p) \bigr\rangle_p\bigr]$$

the **dual pairing** as well.

Because T_pM is an m-dimensional vector space, so is T_p^*M. Hence for $r \in \mathbb{N}$ and $p \in M$, the r-fold exterior product $\bigwedge^r T_p^*M$ of T_p^*M and the Grassmann

[5] See Section VIII.3.

algebra

$$\bigwedge T_p^* M = \bigoplus_{r \geq 0} \bigwedge^r T_p^* M$$

of $T_p^* M$ are defined. To extend the concepts introduced in the previous section, we define the **bundle of alternating *r*-forms** on M by

$$\bigwedge^r T^* M := \bigcup_{p \in M} \bigwedge^r T_p^* M \ .$$

We define the **Grassmann bundle** of M by

$$\bigwedge T^* M := \bigcup_{p \in M} \bigwedge T_p^* M \ .$$

A **differential form** on M is then a map

$$\alpha \colon M \to \bigwedge T^* M \qquad \text{with} \quad \alpha(p) \in \bigwedge T_p^* M \text{ for } p \in M \ ,$$

that is, a section of the Grassmann bundle. It has **degree *r*** (or is called an ***r*-form**) if $\alpha(M) \subset \bigwedge^r T^* M$. Sometimes we call a 1-form a **Pfaff form**.

If α and β are differential forms on M, then the **sum** $\alpha + \beta$ and the **exterior product**[6] $\alpha \wedge \beta$ are defined pointwise:

$$(\alpha + \beta)(p) := \alpha(p) + \beta(p) \text{ and } \alpha \wedge \beta(p) := \alpha(p) \wedge \beta(p) \quad \text{for } p \in M \ .$$

If α is an r-form on M, then its effect on vector fields is also defined pointwise:

$$\alpha(v_1, \ldots, v_r)(p) := \alpha(p)\big(v_1(p), \ldots, v_r(p)\big) \quad \text{for } p \in M \text{ and } v_1, \ldots, v_r \in \mathcal{V}(M) \ .$$

Finally let $\varphi \in C^1(M, N)$, and let β be a differential form on N. Then the **pull back** of β by φ is again defined pointwise:

$$\varphi^* \beta(p) := (T_p \varphi)^* \beta\big(\varphi(p)\big) \quad \text{for } p \in M \ .$$

Obviously $\varphi^* \beta$ is a differential form on M, the pull back of β by φ. If φ is a C^1 diffeomorphism from M to N, then

$$\varphi_* \alpha := (\varphi^{-1})^* \alpha$$

is the **push forward** of the differential form α on M.

Let $k \in \mathbb{N} \cup \{\infty\}$. The differential form α on M belongs to the **class** C^k (or is k-times **continuously differentiable**,[7] or **smooth** in the case $k = \infty$) if there is a

[6] $\wedge$ is also called the **wedge product**.

[7] Of course, we say a differential form of class C^0 is continuous.

chart (φ, U) around every point of M such that $\varphi_* \alpha$ is a differential form of class C^k on $\varphi(U)$. We denote the set of all r-forms of class C^k on M by

$$\Omega^r_{(k)}(M) \ ,$$

and

$$\Omega^r(M) := \Omega^r_{(\infty)}(M)$$

is the set of all smooth r-forms on M. Finally

$$\Omega(M) := \Omega_{(\infty)}(M)$$

is the set of all smooth differential forms on M.

Following our treatment of vector fields, we will generally restrict our attention to the study of smooth differential forms. We leave it to you to prove that all the statements we prove about smooth forms also hold analogously for forms of class C^k provided k has been restricted as the case may require.

4.4 Remarks **(a)** The above notion of differentiability of differential forms is coordinate-independent.

If α is an r-form of class C^k on M and (ψ, V) is a chart on M, then $\psi_* \alpha$ is an r-form of class C^k on $\psi(V)$.

Proof Every $p \in M$ has a chart (φ, U) around it with $\varphi_* \alpha \in \Omega^r_{(k)}\big(\varphi(U)\big)$. From Remark 3.3(a) and the pointwise definition of the push forward, it follows that

$$\psi_* \alpha = (\psi \circ \varphi^{-1})_* \varphi_* \alpha \ .$$

After this, the claim follows in analogy to the proof of Remark 4.2(a). ■

(b) $\Omega(M)$ and $\Omega^r(M)$ are $\mathcal{E}(M)$-modules and therefore in particular $\mathbb{R}$-vector spaces. Also

$$\Omega(M) = \bigoplus_{r \geq 0} \Omega^r(M) \ .$$

The exterior product is $\mathbb{R}$-bilinear, associative, and graded anticommutative, that is, it satisfies these rules:

(i) The map

$$\Omega^r(M) \times \Omega^s(M) \to \Omega^{r+s}(M) \ , \quad (\alpha, \beta) \mapsto \alpha \wedge \beta$$

is well defined and $\mathbb{R}$-bilinear.

(ii) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ for $\alpha, \beta, \gamma \in \Omega(M)$.

(iii) $\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha$ for $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^s(M)$.

Proof This follows from the definition of smoothness, from the pointwise definition of $\wedge$, and from Theorem 2.7. ■

(c) Every $\alpha \in \Omega^r(M)$ is an alternating r-form on $\mathcal{V}(M)$.

(d) $\Omega^0(M) = \mathcal{E}(M)$, and $\Omega^r(M) = \{0\}$ for $r > m$.

(e) For $h \in C^\infty(M,N)$, the pull back $h^* : \Omega(N) \to \Omega(M)$ is an algebra homomorphism, that is,

$$h^*(\alpha + \beta) = h^*\alpha + h^*\beta \ , \quad h^*(\alpha \wedge \beta) = h^*\alpha \wedge h^*\beta$$

for $\alpha, \beta \in \Omega(N)$. If $\alpha \in \Omega^r(N)$, then $h^*\alpha$ belongs to $\Omega^r(M)$. Also

$$(k \circ h)^* = h^* \circ k^* \text{ and } (\mathrm{id}_M)^* = \mathrm{id}_{\Omega(M)} \ .$$

If h is a diffeomorphism, then h^* is bijective, and $(h^*)^{-1} = (h^{-1})^* = h_*$.

Proof We leave the simple checks to you. ■

(f) Suppose M is a submanifold of N and $i : M \hookrightarrow N$ is the natural embedding.[8] Then for $\alpha \in \Omega^r(N)$,

$$\alpha \,|\, M := i^*\alpha \in \Omega^r(M)$$

is the **restriction**[9] of α to M. Let $p \in M$. Because the tangent space T_pM can be regarded as a vector subspace of T_pN, we have $(\alpha \,|\, M)(p) = \alpha(p) \,|\, (T_pM)^r$. ■

Local representations

Suppose $f \in C^1(M) := C^1(M,\mathbb{R})$. As in Section VII.10, we define the **differential** df of f by

$$df(p) := \mathrm{pr} \circ T_pf \quad \text{for } p \in M \ ,$$

where

$$\mathrm{pr} := \mathrm{pr}_2 : T_{f(p)}\mathbb{R} = \bigl\{f(p)\bigr\} \times \mathbb{R} \to \mathbb{R}$$

is the canonical projection.

Let (φ, U) be a chart around $p \in M$. Then it follows from the definitions of $df(p)$ and $\partial_j|_p$ as well as the chain rule of Remarks VII.10.9(b) and 1.14(c) that

$$\begin{aligned}
\bigl\langle df(p), \partial_j|_p \bigr\rangle_p &= \bigl\langle df(p), (T_{\varphi(p)}\varphi^{-1})\bigl(\varphi(p), e_j\bigr)\bigr\rangle_p \\
&= \mathrm{pr} \circ T_pf \circ T_{\varphi(p)}\varphi^{-1}\bigl(\varphi(p), e_j\bigr) \\
&= \mathrm{pr} \circ T_{\varphi(p)}(f \circ \varphi^{-1})\bigl(\varphi(p), e_j\bigr) \\
&= \partial(f \circ \varphi^{-1})\bigl(\varphi(p)\bigr)e_j \\
&= \partial_j(f \circ \varphi^{-1})\bigl(\varphi(p)\bigr) \\
&= \partial_j(\varphi_*f)\bigl(\varphi(p)\bigr)
\end{aligned}$$

for $1 \le j \le m$. With the abbreviation

$$\partial_j f(p) := \frac{\partial f}{\partial x^j}(p) := \partial_j(f \circ \varphi^{-1})\bigl(\varphi(p)\bigr) = \partial_j(\varphi_*f)\bigl(\varphi(p)\bigr) \tag{4.3}$$

[8] In this situation, we always assume that N is without boundary.

[9] See Example 3.4(j).

for $1 \le j \le m$ and $p \in U$, we thus have

$$\left\langle df(p), \partial_j|_p \right\rangle_p = \partial_j f(p) \quad \text{for } 1 \le j \le m\ , \quad p \in U\ . \tag{4.4}$$

Therefore

$$\langle df, \partial_j \rangle = \partial_j f \quad \text{for } 1 \le j \le m\ . \tag{4.5}$$

Note that the usual partial derivative $\partial_j f$ on M (in the sense of Remark VII.2.7(a)) is not defined when M is not “flat”, that is, not an open subset of $\mathbb{R}^m$. Because derivatives of functions on manifolds can only be defined in terms of local representations, $\partial_j f$ in (4.5) is meaningless unless it is interpreted as the partial derivative of the function “pushed down” by φ to the parameter domain $\varphi(U)$, that is, the partial derivative of the $\varphi_* f$ appearing in (4.3). This rules out any misinterpretation in practice. The notation $\partial f/\partial x^j$ has the advantage that it gives the “name of the coordinates” $(x^1, \ldots, x^m) = \varphi$ in which f is locally written.

In Section VII.2, for the case of open subsets of $\mathbb{R}^m$, we defined the partial derivative $\partial_j f(p)$ as the image of the j-th coordinate unit vector e_j under the (total) derivative $\partial f(p)$ (that is, the linearization of f at p). Since $df(p)$ is just the tangent part of the tangential $T_p f$ and therefore the “linearization of f at the point p”, and since $\partial_j|_p$ is the j-th coordinate basis vector of T_pM, (4.4) shows that $\partial_j f(p)$ is the tangent part of the image of these coordinate vectors under the tangential of f. Therefore (4.3) is indeed the correct generalization of the concept of partial derivative to functions defined on manifolds.

Finally, it is clear that (4.3) agrees with the classical partial derivative when M is open in $\mathbb{R}^m$ and φ denotes the trivial chart id_M.

4.5 Remarks Let (φ, U) be a chart of M.

(a) For $f \in \mathcal{E}(M) = \Omega^0(M)$, the differential df belongs to $\Omega^1(M)$. The map

$$d\colon \Omega^0(M) \to \Omega^1(M)\ , \quad f \mapsto df$$

is $\mathbb{R}$-linear.

(b) Let $(x^1, \ldots, x^m) = \varphi$ be the local coordinates on U induced by φ, so that

$$x^j := \mathrm{pr}_j \circ \varphi \in \mathcal{E}(U) \quad \text{for } 1 \le j \le m\ ,$$

where $\mathrm{pr}_j : \mathbb{R}^m \to \mathbb{R}$ are the canonical projections. Then $\Omega^1(U)$ is a free $\mathcal{E}(U)$-module of dimension m, and $(dx^1, \ldots, dx^m)$ is a module basis with

$$\left\langle dx^j, \frac{\partial}{\partial x^k} \right\rangle = \delta^j_k \quad \text{for } 1 \le j, k \le m \tag{4.6}$$

and is the **dual basis** to the basis $(\partial/\partial x^1, \ldots, \partial/\partial x^m)$ of $\mathcal{V}(U)$. The basis representations

$$v = \sum_{j=1}^m v^j \frac{\partial}{\partial x^j} \in \mathcal{V}(U) \quad \text{and} \quad \alpha = \sum_{j=1}^m a_j\, dx^j \in \Omega^1(U) \tag{4.7}$$

require the relations

$$v^j = \langle dx^j, v\rangle \in \mathcal{E}(U) \quad \text{and} \quad a_j = \Big\langle \alpha, \frac{\partial}{\partial x^j}\Big\rangle \in \mathcal{E}(U) \tag{4.8}$$

for $1 \le j \le m$. In particular, for $f \in \mathcal{E}(U)$, we have

$$df = \sum_{j=1}^m \frac{\partial f}{\partial x^j}\, dx^j \in \Omega^1(U)\ .$$

Proof (4.3) and (4.5) imply

$$\langle dx^j, \partial_k\rangle = \partial_k x^j = \varphi^*\partial_k(\varphi_* x^j) = \varphi^*\partial_k\big[(\mathrm{pr}_j \circ\varphi)\circ\varphi^{-1}\big] = \varphi^*\partial_k\,\mathrm{pr}_j = \delta^j_k$$

and hence (4.6). For v with the representation given in (4.7), we obtain

$$\big\langle dx^j(p), v(p)\big\rangle_p = \sum_{k=1}^m v^k(p)\Big\langle dx^j(p), \frac{\partial}{\partial x^k}\Big|_p\Big\rangle_p = \sum_{k=1}^m v^k(p)\delta^j_k = v^j(p) \tag{4.9}$$

for $p \in U$ and $1 \le j \le m$, because $dx^j(p)$ is a linear form on $T_pM = T_pU$. Therefore the first part of (4.7) and Remark 4.3(c) imply the first claim of (4.8).

For the push forward dx^j by φ, we find by applying Remarks VII.10.9(b) and 1.14(c) as well as (4.2) and (4.6) that

$$\begin{aligned}\big\langle(\varphi_* dx^j)\big(\varphi(p)\big), \big(\varphi(p), e_k\big)\big\rangle_{\varphi(p)} &= \big\langle dx^j(p), (T_{\varphi(p)}\varphi^{-1})\big(\varphi(p), e_k\big)\big\rangle_p \\ &= \big\langle dx^j(p), (T_p\varphi)^{-1}\big(\varphi(p), e_k\big)\big\rangle_p \\ &= \Big\langle dx^j(p), \frac{\partial}{\partial x^k}\Big|_p\Big\rangle_p = \delta^j_k\ .\end{aligned} \tag{4.10}$$

This shows that $(\varphi_*\, dx^1, \ldots, \varphi_*\, dx^m)$ is, at every point $\varphi(p) \in \varphi(U)$, the basis dual to the canonical basis of $T_{\varphi(p)}\varphi(U)$. In particular, the covector part of $\varphi_*\, dx^j$ is constant on $\varphi(U)$.

Remark 4.4(e) guarantees that φ_* is a vector space isomorphism from $\Omega^1(U)$ to $\Omega^1(\varphi(U))$. From this, (4.10), and Proposition 2.3, we conclude that every $\alpha \in \Omega^1(U)$ has a representation of the form given in (4.7) by real-valued functions a_j on U. Because of

$$\varphi_*\alpha = \sum_{j=1}^m (\varphi_* a_j)\,\varphi_* dx^j \tag{4.11}$$

and due to the constancy of the covector part of 1-forms $\varphi_* dx^j$ on $\varphi(U)$, we learn from Remark 3.1(e) that α belongs to $\Omega^1(U)$ if and only if $a_j \in \mathcal{E}(U)$ for $1 \le j \le m$. Finally $a_j = \langle \alpha, \partial_j\rangle$ follows by a calculation analogous to (4.9). ■

(c) For $r \in \mathbb{N}$, $\Omega^r(U)$ is a free $\mathcal{E}(U)$-module of dimension $\binom{m}{r}$, and

$$\big\{\, dx^{(j)} = dx^{j_1} \wedge \cdots \wedge dx^{j_r}\ ;\ (j) = (j_1, \ldots, j_r) \in \mathbb{J}_r \,\big\} \tag{4.12}$$

is a basis. An r-form α on U has a uniquely determined **basis representation in local coordinates**

$$\alpha = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\, dx^{(j)} \tag{4.13}$$

whose coefficients are

$$a_{(j)} = \alpha\Big(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_r}}\Big) \quad \text{for } (j) \in \mathbb{J}_r\ . \tag{4.14}$$

If $k \in \mathbb{N} \cup \{\infty\}$, then α belongs to the class C^k on U if and only if $a_{(j)} \in C^k(U)$ for $(j) \in \mathbb{J}_r$.

Proof From (4.10) and the properties of the pull back $(\varphi^{-1})^* = \varphi_*$ given in Remark 4.4(e), it follows that

$$\varphi_*\, dx^{(j)} = \varepsilon^{(j)} \quad \text{for } (j) \in \mathbb{J}_r\ , \tag{4.15}$$

where $(\varepsilon^1, \ldots, \varepsilon^m)$ denotes the basis dual to the canonical basis of $T_{\varphi(p)}\varphi(U)$ for $p \in U$. Because φ_* is a vector space isomorphism from $\Omega^r(U)$ to $\Omega^r\big(\varphi(U)\big)$, we derive from Proposition 2.3 and (4.2) that every r-form α on U has a unique representation of the form (4.13), whose coefficients are given by (4.14). Because

$$\varphi_*\alpha = \sum_{(j)\in\mathbb{J}_r} (\varphi_* a_{(j)})\varphi_*\, dx^{(j)}$$

and by (4.15), the definition of the differentiability of an r-form of class C^k implies that α belongs to the class C^k if and only if the $a_{(j)}$ lie in $C^k(U)$. ■

(d) Note that we have only shown that $\mathcal{V}(U)$ and $\Omega(U)$ are free modules, while we have made no such statements about $\mathcal{V}(M)$ and $\Omega(M)$. Indeed, corresponding statements are not generally true in the global case, that is, for manifolds that cannot be described by a single chart. For example, it is known[10] that the n-sphere does not support n (nontrivial) linearly independent vector fields (that is, $\mathcal{V}(S^n)$ is not a free module of dimension n) unless $n = 0, 1, 3$, or 7.

(e) (regularity) If $k \in \mathbb{N}$, then the statements of (c) remain true if M is a C^{k+1} manifold. ■

The local coordinates $x^1, \ldots, x^m$ on U belonging to a chart φ are smooth functions on U; namely, they are the maps $\mathrm{pr}_j \circ \varphi \in \mathcal{E}(U)$ for $1 \le j \le m$. On the other hand, we also use $(x^1, \ldots, x^m)$ as the notation for a general point of $\varphi(U)$, that is, the coordinates of $\mathbb{R}^m$ are also called $x^1, \ldots, x^m$. This use of the same notation for two different things is deliberate. It simplifies calculations with (local) coordinates considerably, if it is clear from context which interpretation is correct. For example, the expression

$$\alpha = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\, dx^{(j)} \tag{4.16}$$

[10] By work of Bott, Kervaire, and Milnor.

has two meanings if no other specification is made (which is usual in practice). First, we can regard (4.16) as the basis representation of an r-form on the open subset $X = \varphi(U)$ of $\overline{\mathbb{H}}^m$, as we have done in the previous sections. Or, we can interpret (4.16) as the basis representation of an r-form on U with respect to the local coordinates in the corresponding chart. This is the standpoint we have taken here. In the first case, the $a_{(j)}$ are functions on X, and the $dx^{(j)}$ are the constant basis forms of $\mathbb{R}^m$. In the second, the $a_{(j)}$ are functions on $U \subset M$, and the $dx^{(j)}$ are the position-dependent r-forms that "live" on U. *Because of* (4.15), *we must, in order to pass from second interpretation to the first, "pass down" the coefficient functions* $a_{(j)} = a_{(j)}(p)$ *to the parameter domain using* φ. *That is,* $a_{(j)}$ *must be interpreted as* $\varphi_* a_{(j)} = a_{(j)} \circ \varphi^{-1}$, *and we must think* $a_{(j)} = a_{(j)}(x)$ *for* $x \in X$.

4.6 Examples **(a)** Denote the upper and lower hemispheres of the m-sphere S^m in $\mathbb{R}^{m+1}$ by S^m_+ and S_-, respectively. That is, let

$$S^m_\pm := \left\{ x \in \mathbb{R}^{m+1} \ ; \ |x| = 1,\ \pm x^{m+1} > 0 \right\} .$$

Also let

$$\varphi_\pm : S^m_\pm \to \mathbb{B}^m \ , \quad x \mapsto x' := (x^1, \ldots, x^m)$$

be the projection of $\mathbb{B}^m = \mathbb{B}^m \times \{0\}$ onto the hyperplane orthogonal to the x^{m+1}-axis. Then (φ_+, S^m_+) and (φ_-, S^m_-) are charts of S^m. For

$$\alpha := \sum_{j=1}^{m+1} (-1)^{j-1} x^j \, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{m+1} \in \Omega^m(\mathbb{R}^{m+1}) \ ,$$

the restriction to S^m reads, in the local coordinates induced by $\varphi_\pm$, as

$$\alpha \,|\, S^m_\pm = \pm \frac{(-1)^m}{\sqrt{1 - |x'|^2}} \, dx^1 \wedge \cdots \wedge dx^m \ .$$

Proof Let $g_\pm(x') := \bigl(x', \pm\sqrt{1-|x'|^2}\bigr)$ for $x' \in \mathbb{B}^m$. Then $g_\pm$ is smooth and is the parametrization belonging to $\varphi_\pm$ of the hemisphere $S^m_\pm$ as a graph over $\mathbb{B}^m$. Also $g_\pm = i \circ \varphi_\pm^{-1}$ with $i \colon S^m \hookrightarrow \mathbb{R}^{m+1}$. Therefore $(\varphi_\pm, S^m_\pm)$ are charts of S^m. For these we find

$$\begin{aligned}
(\varphi_\pm)_*(\alpha \,|\, S^m_\pm) &= (\varphi_\pm^{-1})^* \circ i^* \alpha = g_\pm^* \alpha \\
&= \sum_{j=1}^{m+1} (-1)^{j-1} g_\pm^j \, dg_\pm^1 \wedge \cdots \wedge \widehat{dg_\pm^j} \wedge \cdots \wedge dg_\pm^{m+1} \\
&= \sum_{j=1}^{m} (-1)^{j-1} x^j \, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m \wedge \sum_{k=1}^{m} \frac{\mp x^k \, dx^k}{\sqrt{1-|x'|^2}} \\
&\quad \pm (-1)^m \sqrt{1-|x'|^2} \, dx^1 \wedge \cdots \wedge dx^m \\
&= \pm \frac{(-1)^m}{\sqrt{1-|x'|^2}} \Bigl[-\sum_{j=1}^{m} (-1)^{m+j-1+m-j} (x^j)^2 + 1 - |x'|^2 \Bigr] \, dx^1 \wedge \cdots \wedge dx^m \ .
\end{aligned}$$

The claim follows because the expression in the square brackets reduces to 1. ■

(b) Let $\omega_{S^1} := (x\,dy - y\,dx)\,|\,S^1$, and make

$$g_1 : (0,2\pi) \to S^1 \setminus \{(1,0)\}\ , \quad t \mapsto (\cos t, \sin t)$$

a parametrization of $S^1 \setminus \{(1,0)\}$. Then with respect to the local coordinates induced by the chart (φ, U) with $\varphi := g_1^{-1}$ and $U := S^1 \setminus \{(1,0)\}$, we have

$$\omega_{S^1}\,|\,U = dt\ .$$

Proof This follows from $\varphi_* \omega_{S^1} = (g_1^1 \dot g_1^2 - g_1^2 \dot g_1^1)\,dt$. ■

(c) Let $U := S^2 \setminus H_3$ be the 2-sphere S^2 minus the half circle where it intersects the half plane $H_3 := \mathbb{R}^+ \times \{0\} \times \mathbb{R}$.[11] Also let

$$(0,2\pi) \times (0,\pi) \to U\ , \quad (\varphi, \vartheta) \mapsto (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta)$$

be the parametrization of U by spherical coordinates. Finally, let

$$\alpha := x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy \in \Omega^2(\mathbb{R}^3)\ .$$

Then the form $\omega_{S^2} := \alpha\,|\,S^2 \in \Omega^2(S^2)$ has the representation

$$\omega_{S^2}\,|\,U = -\sin\vartheta\,d\varphi \wedge d\vartheta$$

with respect to the local coordinates (φ, ϑ).

Proof After a simple calculation,[12] we obtain this from Example 3.4(d). ■

Coordinate transformations

To carry out concrete calculations efficiently, it is important to choose the coordinates best suited to the problem. So, for example, we use polar coordinates when we want to describe rotationally symmetric problems, as we have already done in our treatment of integration theory in Section X.8.

Because a given problem is usually already described in a coordinate system, we must be able to change to another coordinate system without undue trouble. This background frames the following transformation theorem for vector fields and Pfaff forms.

Let (φ, U) and (ψ, V) be charts of M with $U \cap V \neq \emptyset$. Let $\varphi = (x^1, \ldots, x^m)$ and $\psi = (y^1, \ldots, y^m)$. On $U \cap V$, we can regard the y^j as functions of local coordinates $x = (x^1, \ldots, x^m)$; we could also regard the x^j as functions of $y = (y^1, \ldots, y^m)$. Here is it usual and expedient not to introduce new symbols but rather to write simply $y = y(x)$ and $x = x(y)$. Clearly the map $y(\cdot)$ is a

[11] See Example VII.9.11(b).

[12] Note that (for example) $\partial(x,y)/\partial(\varphi,\vartheta)$ is the determinant of the matrix obtained from (VII.9.3) by removing the last row.

diffeomorphism from $U \cap V$ to itself, a **coordinate transformation**, which we also denote by $x \mapsto y$. The inverse map is $x(\cdot)$, that is, the coordinate transformation $y \mapsto x$. However, we can also regard x [or y] as a generic point in $X := \varphi(U \cap V)$ [or $Y := \psi(U \cap V)$] in $\overline{\mathbb{H}}^m$. Then the coordinate transformation $x \mapsto y$ is nothing but the transition function $\psi \circ \varphi^{-1} \in \mathrm{Diff}(X, Y)$. It will always be clear from context which of these two interpretations is to be chosen.

In the following formulas, we leave it to you to determine from context whether x^j means an independent variable or the function $x^j(\cdot)$. The double meaning, which is scarcely a problem in practice, is used on purpose since it helps to cast formulas into a form that is more intuitively understandable and easier to remember.

4.7 Proposition *For the coordinate transformation $x \mapsto y$, we have*

$$\frac{\partial}{\partial y^j} = \sum_{k=1}^m \frac{\partial x^k}{\partial y^j}\,\frac{\partial}{\partial x^k} \quad \textit{and} \quad dy^j = \sum_{k=1}^m \frac{\partial y^j}{\partial x^k}\,dx^k$$

for $1 \le j \le m$.

Proof From Remark 4.5(c), it follows that

$$\frac{\partial}{\partial y^j} = \sum_{k=1}^m v_j^k\,\frac{\partial}{\partial x^k} \text{ and } v_j^k = \Big\langle dx^k, \frac{\partial}{\partial y^j}\Big\rangle \quad \text{for } 1 \le j,k \le m\ .$$

With $x = f(y)$ and (4.5), we find

$$\Big\langle dx^k, \frac{\partial}{\partial y^j}\Big\rangle = \frac{\partial x^k}{\partial y^j} \quad \text{for } 1 \le j,k \le m\ , \tag{4.17}$$

which proves the first claim.

Analogously, we have

$$dy^j = \sum_{k=1}^m a_k\,dx^k \quad \text{and} \quad a_k = \Big\langle dy^j, \frac{\partial}{\partial x^k}\Big\rangle = \frac{\partial y^j}{\partial x^k}$$

for $1 \le j,k \le m$, which proves the second. ■

4.8 Corollary (a) *The Jacobi matrix of the coordinate transformation $x \mapsto y$ satisfies*

$$\Big[\frac{\partial y^j}{\partial x^k}\Big] = \Big[\frac{\partial x^j}{\partial y^k}\Big]^{-1}\ .$$

(b) $$dy^1 \wedge \cdots \wedge dy^m = \frac{\partial(y^1,\ldots,y^m)}{\partial(x^1,\ldots,x^m)}\,dx^1 \wedge \cdots \wedge dx^m .$$

Proof (a) Because

$$y(\cdot) = \psi \circ \varphi^{-1} \in \mathrm{Diff}(X, Y) \quad \text{and} \quad y(\cdot)^{-1} = x(\cdot) = \varphi \circ \psi^{-1} \in \mathrm{Diff}(Y, X) ,$$

the claim is immediate.

(b) This is a consequence of Example 3.4(c), the considerations after Remark 4.5(e), and the fact that

$$\frac{\partial(y^1, \dots, y^m)}{\partial(x^1, \dots, x^m)}$$

is the Jacobian of the coordinate transformation $x \mapsto y$ (see Remark VII.7.9(a)). ■

4.9 Examples **(a)** (plane polar coordinates) Using the polar coordinate transformation

$$V_2 \to \mathbb{R}^2 , \quad (r, \varphi) \mapsto (x, y) := (r \cos \varphi, r \sin \varphi)$$

with $V_2 := (0, \infty) \times (0, 2\pi)$, we have

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} = -r \sin \varphi \frac{\partial}{\partial x} + r \cos \varphi \frac{\partial}{\partial y} .$$

(b) (spherical coordinates) Let $V_3 := (0, \infty) \times (0, 2\pi) \times (0, \pi)$. Using the spherical coordinate transformation

$$V_3 \to \mathbb{R}^3 , \quad (r, \varphi, \vartheta) \mapsto (x, y, z) = (r \cos \varphi \sin \vartheta, r \sin \varphi \sin \vartheta, r \cos \vartheta) ,$$

we find

$$\begin{aligned}
\frac{\partial}{\partial r} &= \cos \varphi \sin \vartheta \frac{\partial}{\partial x} + \sin \varphi \sin \vartheta \frac{\partial}{\partial y} + \cos \vartheta \frac{\partial}{\partial z} \\
\frac{\partial}{\partial \varphi} &= -r \sin \varphi \sin \vartheta \frac{\partial}{\partial x} + r \cos \varphi \sin \vartheta \frac{\partial}{\partial y} \\
\frac{\partial}{\partial \vartheta} &= r \cos \varphi \cos \vartheta \frac{\partial}{\partial x} + r \sin \varphi \cos \vartheta \frac{\partial}{\partial y} - r \sin \vartheta \frac{\partial}{\partial z} .
\end{aligned}$$

(c) (cylindrical coordinates) Let $X := (0, \infty) \times (0, 2\pi) \times \mathbb{R}$. For the cylindrical coordinate transformation

$$X \to \mathbb{R}^3 , \quad (r, \varphi, \zeta) \mapsto (x, y, z) := (r \cos \varphi, r \sin \varphi, \zeta) ,$$

we find

$$\frac{\partial}{\partial r} = \cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y} , \quad \frac{\partial}{\partial \varphi} = -r \sin \varphi \frac{\partial}{\partial x} + r \cos \varphi \frac{\partial}{\partial y} , \quad \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial z} . \ ■$$

The exterior derivative

The next theorem shows that the exterior derivative can be generalized so that it is defined globally on manifolds.

4.10 Theorem *There is exactly one map*

$$d\colon \Omega(M) \to \Omega(M) \ ,$$

the **exterior** (*or* **Cartan**) **derivative**, *with these four properties*:

(i) *d is $\mathbb{R}$-linear and maps $\Omega^r(M)$ to $\Omega^{r+1}(M)$.*

(ii) *d satisfies the* **product rule**

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta \quad \text{for } \alpha \in \Omega^r(M) \text{ and } \beta \in \Omega(M) \ .$$

(iii) $d^2 = d \circ d = 0$.

(iv) *The differential df of $f \in \mathcal{E}(M) = \Omega^0(M)$ is the same as the differential of f.*

Also

$$d \circ h^* = h^* \circ d \tag{4.18}$$

for $h \in C^\infty(M, N)$.

Proof (a) (existence) Let (φ, U) be a chart of M. According to Theorem 3.5, there is exactly one map $d\colon \Omega\bigl(\varphi(U)\bigr) \to \Omega\bigl(\varphi(U)\bigr)$ with the properties (i)–(iv). We define $d_U : \Omega(U) \to \Omega(U)$ by requiring the commutativity of the diagram

$$\begin{array}{ccc} \Omega(U) & \xrightarrow{\ d_U\ } & \Omega(U) \\ {\scriptstyle \varphi_*}\big\downarrow & & \big\uparrow{\scriptstyle \varphi^*} \\ \Omega\bigl(\varphi(U)\bigr) & \xrightarrow{\ d\ } & \Omega\bigl(\varphi(U)\bigr) \ . \end{array} \tag{4.19}$$

Equivalently, we set $d_U := \varphi^* \circ d \circ \varphi_*$. We learn from Remark 4.4(e) that φ_* is an algebra isomorphism with $(\varphi_*)^{-1} = (\varphi^{-1})_*$. With this and (4.19), we verify easily that d_U has the properties (i)–(iv) and is uniquely defined.

Let (ψ, V) be another chart of M such that $U \cap V \neq \emptyset$. Then it follows from $\varphi = (\varphi \circ \psi^{-1}) \circ \psi$, the properties of pull backs, and (3.12) that

$$\begin{aligned} d_U = \varphi^* \circ d \circ \varphi_* &= \psi^* \circ (\varphi \circ \psi^{-1})^* \circ d \circ (\varphi \circ \psi^{-1})_* \circ \psi_* \\ &= \psi^* \circ d \circ \psi_* = d_V \end{aligned} \tag{4.20}$$

(of course, on $U \cap V$). Therefore d_U is independent of the special coordinates chosen.

Let $\{(\varphi_\kappa, U_\kappa) \ ; \ \kappa \in \mathsf{K}\}$ be an atlas for M, and let $i_\kappa : U_\kappa \hookrightarrow M$ be the natural embedding. Then we define $d \colon \Omega(M) \to \Omega(M)$ by

$$d\alpha(p) := d_{U_\kappa}\bigl[(i_\kappa)^*\alpha\bigr](p) \quad \text{for } \alpha \in \Omega(M) \ ,$$

where $\kappa \in \mathsf{K}$ is chosen so that p lies in U_κ. By (4.20) this definition is meaningful, and it is clear that d has the properties (i)–(iv).

(b) (uniqueness) Let $\alpha \in \Omega^r(M)$ and $p \in M$. Also let (φ, U) be a chart around p. According to Remark 4.5(c), $\alpha \,|\, U$ can be written in local coordinates as

$$\alpha \,|\, U = \sum_{(j)\in\mathbb{J}_r} a_{(j)}\, dx^{(j)}$$

with $a_{(j)} \in \mathcal{E}(U)$. Now it follows from (4.19) and Remarks 3.6(a) and 4.4(e) that

$$\begin{aligned} d_U(\alpha \,|\, U) &= \varphi^* \, d\varphi_*(\alpha \,|\, U) = \varphi^* \sum_{(j)\in\mathbb{J}_r} d(\varphi_* a_{(j)}) \wedge \varphi_* \, dx^{(j)} \\ &= \sum_{(j)\in\mathbb{J}_r} d_U a_{(j)} \wedge dx^{(j)} = \sum_{(j)\in\mathbb{J}_r} da_{(j)} \wedge dx^{(j)} \ , \end{aligned} \tag{4.21}$$

because $d_U a_{(j)}$ is the differential of $a_{(j)} \in \mathcal{E}(U)$.

Let V be an open neighborhood of p with $V \subset\subset U$. Then $\varphi(V) \subset\subset \varphi(U)$. Hence Remark 1.21(a) implies the existence of $\widetilde{\chi} \in \mathcal{D}\bigl(\varphi(U)\bigr)$ such that $\widetilde{\chi} \,|\, \varphi(V) = 1$. For

$$\chi := \begin{cases} \varphi^*\widetilde{\chi} & \text{on } U \ , \\ 0 & \text{on } M\backslash U \ , \end{cases}$$

we have $\chi \in \mathcal{E}(M)$ and $\chi \,|\, V = 1$. This implies that both

$$b_{(j)} := \chi a_{(j)} \text{ for } (j) \in \mathbb{J}_r \quad \text{and} \quad \xi^j := \chi x^j \text{ for } 1 \le j \le m$$

belong to $\mathcal{E}(M)$. Therefore the differentials $d\xi^j \in \Omega^1(M)$ are defined, which implies that

$$\beta := \sum_{(j)\in\mathbb{J}_r} b_{(j)}\, d\xi^{(j)}$$

is also defined and belongs to $\Omega(M)$.

Now suppose $\widetilde{d}$ is a map from $\Omega(M)$ to itself satisfying (i)–(iv). Then we find easily that

$$\widetilde{d}\beta = \sum_{(j)\in\mathbb{J}_r} db_{(j)} \wedge d\xi^{(j)} \ .$$

For $a \in \mathcal{E}(U)$, the product rule gives

$$d(\chi a) = a\, d\chi + \chi\, da$$

(see Corollary VII.3.8 and the definition of the tangential). Because $\chi \,|\, V = 1$, we may use the natural embedding $i\colon V \hookrightarrow M$ to conclude

$$\bigl\langle i^* \, d(\chi a)(q), v(q)\bigr\rangle_q = \bigl\langle d(\chi a)(q), v(q)\bigr\rangle_q = \bigl\langle da(q), v(q)\bigr\rangle_q \quad \text{for } q \in V \ ,$$

for $v \in \mathcal{V}(M)$. That is, $d(\chi a) \,|\, V = da \,|\, V$. This and (4.21) imply $\beta \,|\, V = \alpha \,|\, V$ and

$$\widetilde{d}\beta \,|\, V = d_V(\alpha \,|\, V) \ . \tag{4.22}$$

Because d_V is unique and every $p \in M$ has an open coordinate neighborhood V for which (4.22) holds, we see that $\widetilde{d} = d$.

(c) To prove (4.18), we can use our previous work to restrict to the local situation, that is, we can assume that $M = U$. Then the claim follows from (4.19), (3.12), and Theorem 3.5. ■

4.11 Remarks **(a)** Let

$$\alpha \,|\, U = \sum_{(j)\in\mathbb{J}_r} a_{(j)} \, dx^{(j)}$$

be the representation of $\alpha \in \Omega^r(M)$ in the local coordinates of the chart (φ, U). Then

$$d(\alpha \,|\, U) = \sum_{(j)\in\mathbb{J}_r} da_{(j)} \wedge dx^{(j)} \ .$$

Proof This follows from (4.21). ■

(b) (regularity) For $k \in \mathbb{N}$, the map

$$d\colon \Omega^r_{(k+1)}(M) \to \Omega^{r+1}_{(k)}(M) \quad \text{for } r \in \mathbb{N}$$

is defined and $\mathbb{R}$-linear. This remains true when M is a C^{k+2} manifold. ■

Closed and exact forms

As in the local theory, we say $\alpha \in \Omega(M)$ is **closed** if $d\alpha = 0$. We say it is **exact** if there is a $\beta \in \Omega(M)$, an **antiderivative**, such that $d\beta = \alpha$.

4.12 Remarks and examples **(a)** Because $d^2 = 0$, every exact form is closed.

(b) Every m-form on M is closed.

Proof This is because $\Omega^{m+1}(M) = \{0\}$. ■

(c) (Poincaré lemma) Let $r \in \mathbb{N}^\times$, and let $\alpha \in \Omega^r(M)$ be closed. Then α is **locally exact**, that is, every $p \in M$ has an open neighborhood U and a $\beta \in \Omega^{r-1}(U)$ such that $d\beta = \alpha \,|\, U$.

Proof Let (φ, U) be a chart around p, in which $\varphi(U)$ is star shaped. Because $d\alpha = 0$ and $d\varphi_*\alpha = \varphi_* \, d\alpha$, the form $\varphi_*\alpha \in \Omega^r\bigl(\varphi(U)\bigr)$. Since $\varphi(U)$ is contractible, it follows from the Poincaré lemma (Theorem 3.11) that there exists a $\beta_0 \in \Omega^{r-1}\bigl(\varphi(U)\bigr)$ such that $d\beta_0 = \varphi_*\alpha$. For $\beta := \varphi^*\beta_0 \in \Omega^{r-1}(U)$, we then have $d\beta = \varphi^* \, d\beta_0 = \varphi^*\varphi_*\alpha = \alpha \,|\, U$. ■

Contractions

Let $\alpha \in \Omega^{r+1}(M)$ and $v \in \mathcal{V}(M)$. Then the **contraction $v \lrcorner \alpha$ of α by v** is defined by

$$v \lrcorner \alpha(v_1, \ldots, v_r) := \alpha(v, v_1, \ldots, v_r) \quad \text{for } v_j \in \mathcal{V}(M) \text{ and } 1 \le j \le r \ .$$

We sometimes write $v \lrcorner \cdot$ as i_v and call $i_v \alpha$ the **interior product of v by α**. We verify easily that $v \lrcorner \alpha$ belongs to $\Omega^r(M)$. For completeness and to avoid a bothersome special case, we simply set

$$v \lrcorner \alpha := 0 \quad \text{for } \alpha \in \Omega^0(M) \ .$$

4.13 Remarks and examples **(a)** If $\varphi \colon M \to N$ is a diffeomorphism, then

$$v \lrcorner (\varphi^* \alpha) = \varphi^* (\varphi_* v \lrcorner \alpha)$$

for $\alpha \in \Omega(N)$ and $v \in \mathcal{V}(M)$. In particular, for every r the diagram

$$\begin{array}{ccc} \Omega^{r+1}(M) & \xleftarrow{\ \varphi^*\ } & \Omega^{r+1}(N) \\ {\scriptstyle v \lrcorner}\downarrow & & \downarrow{\scriptstyle \varphi_* v \lrcorner} \\ \Omega^r(M) & \xleftarrow{\ \varphi^*\ } & \Omega^r(N) \end{array}$$

commutes.

Proof If α is a null form, then the claim is trivially true. Therefore we can assume $\alpha \in \Omega^{r+1}(N)$. Then we find for $p \in M$ and $v_1, \ldots, v_r \in T_pM$ that

$$\begin{aligned} v \lrcorner (\varphi^*\alpha)(p)(v_1, \ldots, v_r) &= (\varphi^*\alpha)(p)\bigl(v(p), v_1, \ldots, v_r\bigr) \\ &= \alpha\bigl(\varphi(p)\bigr)\bigl((T_p\varphi)v(p), (T_p\varphi)v_1, \ldots, (T_p\varphi)v_r\bigr) \\ &= \alpha\bigl(\varphi(p)\bigr)\bigl(\varphi_* v\bigl(\varphi(p)\bigr), (T_p\varphi)v_1, \ldots, (T_p\varphi)v_r\bigr) \\ &= (\varphi_* v \lrcorner \alpha)\bigl(\varphi(p)\bigr)\bigl((T_p\varphi)v_1, \ldots, (T_p\varphi)v_r\bigr) \\ &= \varphi^*(\varphi_* v \lrcorner \alpha)(p)(v_1, \ldots, v_r) \ , \end{aligned}$$

which proves the claim. ■

(b) Suppose X is open in $\overline{\mathbb{H}}^m$ and $\omega := dx^1 \wedge \cdots \wedge dx^m$. For $v = \sum_{j=1}^m v^j \partial_j$, we have

$$v \lrcorner \omega = \sum_{j=1}^m (-1)^{j-1} v^j \, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m \ .$$

Proof We set $v_1 := v$. Then for $v_2, \ldots, v_m \in \mathcal{V}(X)$, we have

$$(v_1 \lrcorner \omega)(v_2, \ldots, v_m) = \omega(v_1, \ldots, v_m) = \det\bigl[\langle dx^j, v_k \rangle\bigr] \ .$$

By expanding this determinant in the first column, we find it has the value

$$\sum_{j=1}^{m}(-1)^{j+1}\langle dx^j, v_1\rangle \det(A_j) \ ,$$

where A_j is the matrix obtained by striking the first column and the j-th row from $[\langle dx^j, v_k\rangle]$. From this it follows that

$$\det(A_j) = dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m(v_2, \ldots, v_m) \ .$$

The claim now follows because

$$\langle dx^j, v_1\rangle = \sum_{k=1}^{m} v^k \langle dx^j, \partial_k\rangle = v^j \ . \ \blacksquare$$

(c) Let

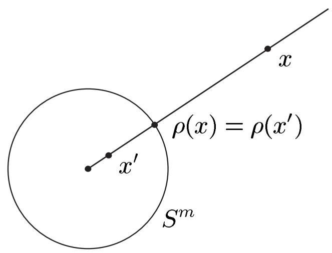
$$\rho\colon \mathbb{R}^{m+1}\backslash\{0\} \to S^m \ , \quad x \mapsto x/|x|$$

be the **radial retraction**[13] on the m-sphere in $\mathbb{R}^{m+1}$. Also let

$$\alpha := \sum_{j=1}^{m+1}(-1)^{j-1}x^j\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{m+1}$$

and

$$\omega_{S^m} := \alpha \,|\, S^m \ .$$



Then letting $r(x) := |x|$ for $x \in \mathbb{R}^{m+1}$, we have

$$\rho^*\omega_{S^m} = \frac{1}{r^{m+1}}\alpha = \sum_{j=1}^{m+1}(-1)^{j-1}\frac{x^j}{|x|^{m+1}}\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{m+1} \ ,$$

and $\rho^*\omega_{S^m}$ is closed.

Proof Because $\rho \in C^\infty(\mathbb{R}^{m+1}\backslash\{0\}, \mathbb{R}^{m+1})$ with $\mathrm{im}(\rho) = S^m$, we know ρ is a smooth map from $\mathbb{R}^{m+1}\backslash\{0\}$ to S^m. Therefore $\rho^*\omega_{S^m} \in \Omega^m(\mathbb{R}^{m+1}\backslash\{0\})$ is defined. It is closed by Remark 4.12(b) and because $d(\rho^*\omega_{S^m}) = \rho^*\, d\omega_{S^m} = 0$.

To show that $\rho^*\omega_{S^m} = r^{-(m+1)}\alpha$, we must verify that for every $p \in \mathbb{R}^{m+1}\backslash\{0\}$, both sides agree on every m-tuple from a system of basis vectors of $T_p\mathbb{R}^{m+1}$. Suppose therefore $p \in \mathbb{R}^{m+1}\backslash\{0\}$. A basis of $T_p\mathbb{R}^{m+1}$ is given by the vectors $\{(p)_p, (v_1)_p, \ldots, (v_m)_p\}$, where

[13]If X is topological space and A is a subset of X, then a continuous map $\rho\colon X \to A$ is called a **retraction** of X on A if $\rho(a) = a$ for $a \in A$. If there is a retraction of X on A, then A is a **retract** of X.

$\{(v_1)_p, \ldots, (v_m)_p\}$ is a basis of $T_p\big(r(p)S^m\big)$. If the m-tuple $(w_1)_p, \ldots, (w_m)_p$ contains the vector $(p)_p$, then by letting $\omega := dx^1 \wedge \cdots \wedge dx^{m+1}$, we can use (b) to find

$$\begin{aligned}\alpha(p)\big((w_1)_p, \ldots, (w_m)_p\big) &= \big((p)_p \mathbin{\lrcorner} \omega\big)\big((w_1)_p, \ldots, (w_m)_p\big) \\ &= \omega\big((p)_p, (w_1)_p, \ldots, (w_m)_p\big) = 0 \ ,\end{aligned}$$

because two entries are equal. Therefore $r^{-(m+1)}(p)\alpha(p)$ also vanishes on this m-tuple. We also find

$$\rho^*\omega_{S^m}(p)\big((w_1)_p, \ldots, (w_m)_p\big) = \omega_{S^m}\big(\rho(p)\big)\big((T_p\rho)(w_1)_p, \ldots, (T_p\rho)(w_m)_p\big)$$

with

$$(T_p\rho)(w_j)_p = \big(\rho(p), \partial\rho(p)w_j\big) \ ,$$

where, according to Proposition VII.2.5, we have

$$\partial\rho(p)w_j = \partial_t\rho(p + tw_j)\big|_{t=0} \quad \text{for } 1 \le j \le m \ .$$

Because $\rho(p+tp) = \rho(p)$ for $t \in (-1, 1)$, it follows in particular that $(T_p\rho)(p)_p = 0$. Therefore $\rho^*\omega_{S^m}(p)\big((w_1)_p, \ldots, (w_m)_p\big)$ also vanishes if the m-tuple $(w_1)_p, \ldots, (w_m)_p$ contains the vector $(p)_p$.

It remains to show

$$\rho^*\omega_{S^m}(p)\big((v_1)_p, \ldots, (v_m)_p\big) = \frac{1}{r(p)^{m+1}}\,\alpha(p)\big((v_1)_p, \ldots, (v_m)_p\big) \ . \tag{4.23}$$

For $(v)_p \in T_p\big(r(p)S^m\big)$, Theorem VII.10.6 gives an $\varepsilon > 0$ and a $\gamma \in C^1\big((-\varepsilon, \varepsilon), r(p)S^m\big)$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Now we use $\rho \circ \gamma(t) = \gamma(t)/r(p)$ to get

$$\partial\rho(p)v = (\rho \circ \gamma)^{\cdot}(0) = v/r(p) \ .$$

From this we derive

$$\rho^*\omega_{S^m}(p)\big((v_1)_p, \ldots, (v_m)_p\big) = r(p)^{-m}\alpha\big(\rho(p)\big)\big((v_1)_p, \ldots, (v_m)_p\big) \ ,$$

which implies (4.23), thus finishing the proof. ■

(d) (regularity) Let $k \in \mathbb{N}$, and suppose M is a C^{k+1} manifold. For $\alpha \in \Omega^{r+1}_{(k)}(M)$ and $v \in \mathcal{V}^k(M)$, the contraction $v \mathbin{\lrcorner} \alpha$ belongs to $\Omega^r_{(k)}(M)$. ■

Orientability

As we learned in Section 2, T_pM can be oriented by choosing a volume form $\alpha(p) \in \bigwedge^m T_p^*M$. Thereby one gets an m-form α on M with $\alpha(p) \ne 0$ for $p \in M$. Conversely, every map $p \mapsto \alpha(p) \in \Lambda^m T_p^*M$ such that $\alpha(p) \ne 0$ for $p \in M$ induces an orientation on every T_pM. However, such α will generally not be continuous. Intuitively, this means that the orientation of the tangent spaces is not "coherent", that is, the tangent spaces can "flip over" in moving from one point to the next. To avoid this, we also require that α be smooth (more precisely, as regular as permitted by the regularity of the manifold).

A manifold M is said to be **orientable** if there is an $\alpha \in \Omega^m(M)$ such that $\alpha(p) \neq 0$ for every $p \in M$; such an m-form α is called a **volume form on M**.

4.14 Remarks **(a)** If M is orientable, then $\Omega^m(M)$ is a one-dimensional $\mathcal{E}(M)$-module.

Proof Let α be a volume form on M, and let $\beta \in \Omega^m(M)$. Because $\dim \bigwedge^m T_p^* M = 1$ for $p \in M$, there is an $f : M \to \mathbb{R}$ such that $\beta = f\alpha$. We must show that f is smooth. In local coordinates, we have

$$\alpha \,|\, U = a\, dx^1 \wedge \cdots \wedge dx^m \quad \text{and} \quad \beta \,|\, U = b\, dx^1 \wedge \cdots \wedge dx^m$$

with $a, b \in \mathcal{E}(U)$ and $a(p) \neq 0$ for $p \in U$. From this we deduce that $\beta \,|\, U = f\alpha \,|\, U$, where $f := b/a$ belongs to $\mathcal{E}(U)$. ■

(b) (regularity) Suppose $k \in \mathbb{N}$ and M is a C^{k+1} manifold. Then M is orientable if and only if there is an $\alpha \in \Omega^m_{(k)}(M)$ such that $\alpha(p) \neq 0$ for $p \in M$. This is the case if and only if the $C^k(M)$-module $\Omega^m_{(k)}(M)$ is one-dimensional. ■

The next proposition shows that one can also characterize the orientability of a manifold by its charts.

If X and Y are open in $\overline{\mathbb{H}}^m$, then we say $\varphi \in \mathrm{Diff}(X, Y)$ is **orientation-preserving** [or **orientation-reversing**] if $\det \partial\varphi(x) > 0$ [or $\det \partial\varphi(x) < 0$] for every $x \in X$, that is, if $\partial\varphi(x) \in \mathcal{L}(\mathbb{R}^m)$ is an orientation-preserving [or orientation-reversing] automorphism for every $x \in X$. An atlas of M is said to be **oriented** if all of its transition functions are orientation-preserving.

4.15 Proposition *A manifold of dimension ≥ 2 is orientable if and only if it has an oriented atlas.*

Proof (a) Suppose M is orientable and $\alpha \in \Omega^m(M)$ is a volume form. In addition, let $\{ (\varphi_\kappa, U_\kappa) \,;\, \kappa \in \mathsf{K} \}$ be an atlas of M. Then $(\varphi_\kappa)_* \alpha = a_\kappa\, dx^1 \wedge \cdots \wedge dx^m$ on $X_\kappa := \varphi_\kappa(U_\kappa) \subset \overline{\mathbb{H}}^m$, with $a_\kappa(x) \neq 0$ for $x \in X_\kappa$. Because we can change coordinates (if necessary) as $x \mapsto (-x^1, x^2, \ldots, x^m)$, we can assume that $a_\kappa(x_\kappa)$ is strictly positive for some $x_\kappa \in X_\kappa$. Because we can assume that U_κ and therefore also X_κ are connected, it follows from the intermediate value theorem (Theorem III.4.7) that $a_\kappa(x) > 0$ for all $x \in X_\kappa$ and every $\kappa \in \mathsf{K}$.

Suppose now $(\varphi_\kappa, U_\kappa)$ and $(\varphi_\lambda, U_\lambda)$ are local charts with $U_\kappa \cap U_\lambda \neq \emptyset$. Also let $\varphi_\kappa = (x^1, \ldots, x^m)$ and $\varphi_\lambda = (y^1, \ldots, y^m)$. Then we find

$$\begin{aligned}
&(\varphi_\lambda \circ \varphi_\kappa^{-1})^* \big(a_\lambda\, dy^1 \wedge \cdots \wedge dy^m \,\big|\, \varphi_\lambda(U_\kappa \cap U_\lambda)\big) \\
&\quad = (\varphi_\kappa)_* \varphi_\lambda^* \big(a_\lambda\, dy^1 \wedge \cdots \wedge dy^m \,\big|\, \varphi_\lambda(U_\kappa \cap U_\lambda)\big) \\
&\quad = (\varphi_\kappa)_* \alpha \,\big|\, (U_\kappa \cap U_\lambda) = a_\kappa\, dx^1 \wedge \cdots \wedge dx^m \ .
\end{aligned} \tag{4.24}$$

By Example 3.4(c), we have

$$(\varphi_\lambda \circ \varphi_\kappa^{-1})^*\, dy^1 \wedge \cdots \wedge dy^m = \det \partial(\varphi_\lambda \circ \varphi_\kappa^{-1})\, dx^1 \wedge \cdots \wedge dx^m \ .$$

By comparing with (4.24), we see

$$(\varphi_\lambda \circ \varphi_\kappa^{-1})^* a_\lambda(x) \det \partial(\varphi_\lambda \circ \varphi_\kappa^{-1})(x) = a_\kappa(x) > 0 \quad \text{for } x \in \varphi_\kappa(U_\kappa \cap U_\lambda) \ .$$

Because a_λ is positive, it follows that M has an oriented atlas.

(b) Let $\{\, (\varphi_\kappa, U_\kappa) \ ; \ \kappa \in \mathsf{K} \,\}$ be an oriented atlas. Proposition 1.20 guarantees the existence of a smooth partition of unity $\{\, \pi_\kappa \ ; \ \kappa \in \mathsf{K} \,\}$ that is subordinate to the cover $\{\, U_\kappa \ ; \ \kappa \in \mathsf{K} \,\}$ of M. For $\kappa \in \mathsf{K}$, define $\alpha_\kappa \in \Omega^m(U_\kappa)$ by

$$\alpha_\kappa := \begin{cases} \pi_\kappa \varphi_\kappa^* \, dx^1 \wedge \cdots \wedge dx^m & \text{in } U_\kappa \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

We can verify easily that the definition

$$\alpha := \sum_{\kappa \in \mathsf{K}} \alpha_\kappa \in \Omega^m(M)$$

is meaningful. We must show that $\alpha(p) \neq 0$ for $p \in M$.

Let $p \in M$, and choose $\kappa \in \mathsf{K}$ so that $\pi_\kappa(p) > 0$. For $\lambda \in \mathsf{K}$ with $\lambda \neq \kappa$ and $U_\kappa \cap U_\lambda \neq \emptyset$, it follows, as in (a), that

$$\begin{aligned} \alpha_\lambda = \pi_\lambda \varphi_\lambda^* \, dy^1 \wedge \cdots \wedge dy^m &= \pi_\lambda \varphi_\kappa^* (\varphi_\lambda \circ \varphi_\kappa^{-1})^* \, dy^1 \wedge \cdots \wedge dy^m \\ &= \pi_\lambda \big(\varphi_\kappa^* \det\big(\partial(\varphi_\lambda \circ \varphi_\kappa^{-1})\big)\big) \, \varphi_\kappa^* \, dx^1 \wedge \cdots \wedge dx^m \ . \end{aligned}$$

From this we obtain

$$\alpha(p) = \Big(\pi_\kappa(p) + \sum_{\substack{\lambda \in \mathsf{K} \\ \lambda \neq \kappa}} \pi_\lambda(p) \det\big(\partial(\varphi_\lambda \circ \varphi_\kappa^{-1})\big)\big(\varphi_\kappa(p)\big) \Big) \varphi_\kappa^* \, dx^1 \wedge \cdots \wedge dx^m(p) \ ,$$

where only finitely many summands differ from zero. Because $\pi_\lambda(p) \geq 0$ and because the transition functions are orientation-preserving, we see that $\alpha(p) \neq 0$. Therefore α is a volume form, and M is orientable. ■

Suppose M is orientable. Then we say two volume forms $\alpha, \beta \in \Omega^m(M)$ are **equivalent** if there is an $f \in \mathcal{E}(M)$ such that $f(p) > 0$ for $p \in M$, and $\alpha = f\beta$. This is obviously an equivalence relation on the set of all volume forms on M. Every equivalence class with respect to this relation is called an **orientation** on M. Given $\mathcal{O}r := \mathcal{O}r(M)$ an orientation of M, then we call $(M, \mathcal{O}r)$ an **oriented manifold**. If the orientation of M is clear from context, we may write M for $(M, \mathcal{O}r)$.

If $\alpha \in \mathcal{O}r$, then $-\alpha$ is a volume form that does not belong to $\mathcal{O}r$. We denote the associated equivalence class by $-\mathcal{O}r$ and call it the **orientation opposite to** $\mathcal{O}r$. It is clear that $-\mathcal{O}r$ is independent of its particular representative.

4.16 Remarks **(a)** An orientable manifold is connected if and only if it has exactly two orientations.

Proof Suppose M is connected, and α and β are two volume forms. By Remark 4.14(a), there is an $f \in \mathcal{E}(M)$ such that $\alpha = f\beta$. Because α vanishes nowhere, we have $f(p) \neq 0$ for $p \in M$. Because M is connected, the intermediate value theorem (see Theorem III.4.7) implies that either $f(p) > 0$ or $f(p) < 0$ for every $p \in M$. Hence α is equivalent either to β or to $-\beta$. Therefore M has precisely two orientations.

Now suppose M is connected. Proposition III.4.2 guarantees the existence of a nonempty, open, and closed proper subset X of M. For α a volume form on M, we set

$$\beta(p) := \begin{cases} \alpha(p) & \text{if } p \in X \ , \\ -\alpha(p) & \text{if } p \in M \backslash X \ . \end{cases}$$

Then β is obviously a volume form with $\beta \notin \mathcal{O}r \cup (-\mathcal{O}r)$, where $\mathcal{O}r$ is the equivalence class of α. Therefore M has more than two orientations. ■

(b) Let $M = (M, \mathcal{O}r)$ be an oriented manifold. A chart (φ, U) of M is said to be **positive(ly oriented)** if $\varphi_*(\alpha \,|\, U)$ for $\alpha \in \mathcal{O}r$ is equivalent to the m-form $dx^1 \wedge \cdots \wedge dx^m \,|\, \varphi(U)$. Otherwise it is **negative(ly oriented)**. M has an atlas consisting only of positive charts, an **oriented atlas**.

Proof For $\beta \in \mathcal{O}r$, we have $\alpha = f\beta$ with $f \in \mathcal{E}(M)$ and $f(p) > 0$ for $p \in M$. With $i : U \hookrightarrow M$, it follows from this that

$$\varphi_*\alpha \,|\, U = \varphi_* i^*\alpha = \varphi_* i^*(f\beta) = (\varphi_* i^* f)(\varphi_* i^*\beta) = g\varphi_*(\beta \,|\, U) \ ,$$

where $g := f \circ \varphi^{-1} \in \mathcal{E}\big(\varphi(U)\big)$ and $g(x) > 0$ for $x \in \varphi(U)$. This shows that the definition does not depend on the chosen representative. That there is indeed an atlas with positive charts was shown in part (a) of the proof of Proposition 4.15. ■

(c) Let M be oriented. Then (φ, U) is a positive chart if and only if $(\partial_1|_p, \ldots, \partial_m|_p)$ is a positive basis of T_pM for $p \in U$.

Proof For $\alpha \in \Omega^m(M)$, Remark 4.5(c) says that the basis representation in local coordinates is

$$\alpha \,|\, U = a \, dx^1 \wedge \cdots \wedge dx^m$$

with $a(p) = \alpha(p)(\partial_1|_p, \ldots, \partial_m|_p)$ for $p \in U$. The claim is now clear. ■

4.17 Examples **(a)** Every open subset U of an orientable manifold M is itself orientable.[14]

Proof For $\alpha \in \mathcal{O}r(M)$, the restriction $\alpha \,|\, U$ is a volume form on U. ■

(b) If M and N are orientable and one of these manifolds is without boundary, then the product manifold[15] $M \times N$ is orientable.

[14] We stipulate that the empty set is orientable.

[15] See Exercise VII.9.4 and Exercise 3. Why do we assume that one of these two manifolds is without boundary?

Proof If $\{ (\varphi_\kappa, U_\kappa) ; \kappa \in \mathsf{K} \}$ and $\{ (\psi_\lambda, V_\lambda) ; \lambda \in \mathsf{L} \}$ are oriented atlases of M and N, respectively, then it is easy to see that $\{ \varphi_\kappa \times \psi_\lambda ; (\kappa, \lambda) \in \mathsf{K} \times \mathsf{L} \}$ with

$$\varphi_\kappa \times \psi_\lambda(p,q) := \big(\varphi_\kappa(p), \psi_\lambda(q)\big) \in \mathbb{R}^m \times \mathbb{R}^n \quad \text{for } (p,q) \in U_\kappa \times V_\lambda ,$$

is an oriented atlas of $M \times N$. Because we can assume without loss of generality that M and N are at least one-dimensional, the claim follows from Proposition 4.15. ■

(c) Any manifold that can be described by a single chart (that is, one that has an atlas with only one chart) is orientable.

Proof This is trivial (see the first part of the proof of Proposition 4.15). ■

(d) (graphs) Suppose X is open in $\mathbb{R}^m$ and $f \in C^\infty(X, \mathbb{R}^n)$. Then $\operatorname{graph}(f)$ is an m-dimensional orientable submanifold of $\mathbb{R}^{m+n}$.

Proof For Proposition VII.9.2, we know that $\operatorname{graph}(f)$ is an m-dimensional submanifold of $\mathbb{R}^{m+n}$. The proof of that result shows that

$$\varphi \colon \operatorname{graph}(f) \to X , \quad \big(x, f(x)\big) \mapsto x$$

is a chart that describes $\operatorname{graph}(f)$. Therefore the claim follows from (c). ■

(e) (fibers of regular maps) Suppose X is open in $\mathbb{R}^m$ and $\ell \in \{0, \dots, m-1\}$. Also let q be regular value of $f \in C^\infty(X, \mathbb{R}^{m-\ell})$. Then the ℓ-dimensional submanifold $f^{-1}(q)$ of X is orientable.

Proof Let $\omega := dx^1 \wedge \cdots \wedge dx^m \,\big|\, X$ and

$$\nabla f^k := \sum_{j=1}^m \partial_j f^k \frac{\partial}{\partial x^j} \in \mathcal{V}(X) \quad \text{for } 1 \le k \le m - \ell .$$

With the notations of Remark VII.10.11(a), we have $\nabla f^k(p) = \nabla_p f^k$ for $p \in X$. We can assume that $L := f^{-1}(q)$ is not empty. Then

$$\alpha := \nabla f^1 \lrcorner \Big(\nabla f^2 \lrcorner \big(\cdots \lrcorner (\nabla f^{m-\ell} \lrcorner\, \omega) \cdots\big)\Big) \,\Big|\, L \in \Omega^\ell(L) .$$

Proposition VII.10.13 guarantees that $\nabla f^1(p), \dots, \nabla f^{m-\ell}(p)$ are linearly independent. Therefore

$$\alpha(p) = \omega\big(\nabla f^{m-\ell}(p), \dots, \nabla f^1(p), \dots\big) \ne 0 \quad \text{for } p \in L ,$$

that is, α is a volume form on L. ■

(f) If M and N are diffeomorphic, then M is orientable if and only if N is orientable.

Proof Let $f \in \operatorname{Diff}(M, N)$, and let (φ, U) be a chart of M. Then $\psi := \varphi \circ f^{-1}$ is a chart of N with $V := f(U) = \operatorname{dom}(\psi)$. Because M and N are diffeomorphic, $m = n$. Suppose now $\beta \in \Omega^m(N)$ is a volume form on N. The local coordinates $(y^1, \dots, y^m) = \psi$ give β the representation $\beta \,|\, V = b\, dy^1 \wedge \cdots \wedge dy^m$ with $b(q) \ne 0$ for $q \in V$. From this it follows that

$$f^*(\beta \,|\, V) = (f^*b) f^*(dy^1 \wedge \cdots \wedge dy^m) = b \circ f \, dx^1 \wedge \cdots \wedge dx^m ,$$

because $f^*y^j = \operatorname{pr}_j \circ \psi \circ f = \operatorname{pr}_j \circ \varphi = x^j$ with $(x^1, \dots, x^m) = \varphi$. Because $b \circ f(p) \ne 0$ for $p \in U$, we see that $f^*\beta$ is a volume form on M. Now the claim is immediate. ■

(g) Every one-dimensional manifold is orientable.

Proof We can assume that the manifold M is connected since it suffices to show that every connected component in orientable. Then by Theorem 1.18, M is diffeomorphic to an interval J or to S^1. Because J and S^1 are orientable (where the orientability of S^1 follows from (e), for example), the claim is implied by (f). ■

(h) (hypersurfaces) A hypersurface M in $\mathbb{R}^{m+1}$ is orientable if and only if there is a smooth **unit normal field** on M, that is, a $\nu \in C^\infty(M, \mathbb{R}^{m+1})$ such that $|\nu(p)| = 1$ and $\boldsymbol{\nu}(p) = \big(p, \nu(p)\big) \in T_p^\perp M$ for $p \in M$.

Proof If ν is a unit normal field on M, then $(\boldsymbol{\nu} \mathbin{\lrcorner} dx^1 \wedge \cdots \wedge dx^{m+1}) \,\big|\, M$ is a volume form on M. Therefore M is orientable.

Let M be orientable. If (φ, U) is a positive chart with $\varphi = (x^1, \ldots, x^m)$, then, because $\dim(T_p^\perp M) = 1$, there is for every $p \in U$ exactly one $\boldsymbol{\nu}(p) = \big(p, \nu(p)\big) \in T_p^\perp M$ with $|\nu(p)| = 1$ such that

$$\Big(\boldsymbol{\nu}(p), \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p\Big)$$

is a positive basis of $T_p\mathbb{R}^{m+1}$. By shrinking U, we can assume that there are open sets $\widetilde{U}$ and $\widetilde{V}$ of $\mathbb{R}^{m+1}$, with $U = \widetilde{U} \cap M$, and a $\Phi \in \mathrm{Diff}(\widetilde{U}, \widetilde{V})$ such that $U = f^{-1}(0)$ for $f := \Phi^{m+1} \in \mathcal{E}(\widetilde{U})$. It follows because $\nabla f(p) \neq 0$ for $p \in \widetilde{U}$ that f is regular. Hence it follows from Proposition VII.10.13, that

$$\nu(p) = \varepsilon \nabla f(p) / |\nabla f(p)| \quad \text{for } p \in U$$

with $\varepsilon \in \{\pm 1\}$. This shows that ν is smooth.

Now let (ψ, V) be a second positive chart with $U \cap V \neq \emptyset$ and $\psi = (y^1, \ldots, y^m)$, and suppose $\boldsymbol{\mu}(q) = \big(q, \mu(q)\big) \in T_q^\perp M$ satisfies $\mu \in C^\infty(V, \mathbb{R}^{m+1})$ and $|\mu(q)| = 1$ for $q \in V$. Also suppose $\big(\boldsymbol{\mu}(q), \frac{\partial}{\partial y^1}\big|_q, \ldots, \frac{\partial}{\partial y^m}\big|_q\big)$ is a positive basis of $T_q\mathbb{R}^{m+1}$ for $q \in V$. Because the two bases

$$\Big(\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p\Big) \quad \text{and} \quad \Big(\frac{\partial}{\partial y^1}\Big|_p, \ldots, \frac{\partial}{\partial y^m}\Big|_p\Big)$$

have the same orientation for $p \in U \cap V$, it follows that $\mu(p) = \nu(p)$ for $p \in U \cap V$. Now the existence of a unit normal field follows from the existence of an oriented atlas of M. ■

(i) (Möbius strip) Suppose $R > 0$, and define

$$f : [-\pi, \pi) \times (-1, 1) \to \mathbb{R}^3$$

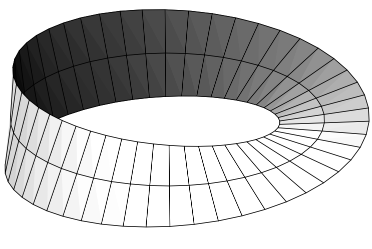
by

$$f(\theta, t) := \Big(\Big(R + t\cos\frac{\theta}{2}\Big)\cos\theta, \Big(R + t\cos\frac{\theta}{2}\Big)\sin\theta, t\sin\frac{\theta}{2}\Big) .$$

Then the image M of f is a nonorientable surface, the **Möbius strip**. Visually, the map f works as follows: Because

$$f(\pm\pi, t) = (-R, 0, \pm t) ,$$

it twists the end $\{\pi\} \times (-1, 1)$ of the rectangle $[-\pi, \pi] \times (-1, 1)$ by 180 degrees relative to the start $\{-\pi\} \times (-1, 1)$. These two ends are then glued together.



Representing f in the form

$$f(\theta,t)=R(\cos\theta,\sin\theta,0)+tg(\theta)$$

with $g(\theta):=\bigl(\cos(\theta/2)\cos\theta,\sin(\theta/2)\sin\theta,\sin(\theta/2)\bigr)$, we obtain an interpretation of the parametrization of f: A point with angular velocity 1 traces a circle in the (x,y)-plane with center 0 and radius R; this describes the first summand. The midpoint of a rod of length 2 is affixed to this point (along its length) and is allowed to simultaneously rotate about its own midpoint with an angular velocity of 1/2, so that its direction is reversed after one rotation; this is the second summand.

Proof The proof that M is a smooth surface is left to you.

For $-\pi\le\theta\le\pi$, we have

$$\begin{aligned}v_1(\theta)&:=\partial_1 f(\theta,0)=R(-\sin\theta,\cos\theta,0)\ ,\\ v_2(\theta)&:=\partial_2 f(\theta,0)=\bigl(\cos(\theta/2)\cos\theta,\cos(\theta/2)\sin\theta,\sin(\theta/2)\bigr)\ .\end{aligned}$$

It follows that for every $\theta\in[-\pi,\pi)$ the vectors $v_1(\theta)$, $v_2(\theta)$ attached to $p(\theta):=f(\theta,0)$ form a basis of $T_{p(\theta)}M$. Therefore the vector

$$n(\theta):=\bigl(-v_1(\theta)\times v_2(\theta)\bigr)/R=\bigl(-\cos\theta\sin(\theta/2),-\sin\theta\sin(\theta/2),\cos(\theta/2)\bigr)$$

attached to $p(\theta)$ is a unit normal vector for $-\pi\le\theta<\pi$. In particular, $n(0)=e_3$.

We assume that $\nu: M\to\mathbb{R}^3$ is a unit normal field with $\nu\bigl(p(0)\bigr)=e_3$. Then because $T^{\perp}_{p(\theta)}M$ is continuous and one-dimensional, it follows that the vectors $\nu\bigl(p(\theta)\bigr)$ and $n(\theta)$ coincide in $-\pi\le\theta<\pi$. From this and the relation $p(-\pi)=p(\pi)$, we find as $\theta\to\pi$ that

$$-e_1=n(-\pi)=\nu\bigl(p(-\pi)\bigr)=\nu\bigl(p(\pi)\bigr)=n(\pi)=e_1\ ,$$

which is not possible. Thus there is no smooth (or even continuous) unit normal field on M; this, by (h), shows that M is not orientable. ■

(j) (regularity) With obvious modifications, the statements above remain true for C^1 manifolds. ■

Tensor fields

Let $r,s\in\mathbb{N}$. Then, according to Section 2, the vector space $T^r_s(T_pM)$, which consists of r-contravariant and s-covariant tensors, is well defined on T_pM. Therefore the **bundle of (r,s)-tensors** on M,

$$T^r_s(M):=\bigcup_{p\in M}T^r_s(T_pM)\ ,$$

is also well defined. An **(r,s)-tensor** (more precisely, an r-contravariant and s-covariant tensor) on M is a section of this bundle, that is, it is a map

$$\gamma: M\to T^r_s(M)\qquad\text{with } \gamma(p)\in T^r_s(T_pM)\text{ for } p\in M\ .$$

If γ and δ are (r,s)-tensors on M and $f \in \mathbb{R}^M$, then the **sum**, $\gamma + \delta$, the **product with functions**, $f\gamma$, and the **tensor product**, $\gamma \otimes \delta$, are again defined pointwise as

$$(\gamma+\delta)(p) := \gamma(p) + \delta(p) \ , \quad (f\gamma)(p) := f(p)\gamma(p) \ , \quad \gamma \otimes \delta(p) := \gamma(p) \otimes \delta(p)$$

for $p \in M$. Likewise, the effect of $\gamma \in T_s^r(M)$ on an r-tuple $\alpha_1, \ldots, \alpha_r$ of Pfaff forms and an s-tuple $v_1, \ldots, v_s$ of vector fields is defined pointwise by

$$\gamma(\alpha_1, \ldots, \alpha_r, v_1, \ldots, v_s)(p) := \gamma(p)\big(\alpha_1(p), \ldots, \alpha_r(p), v_1(p), \ldots, v_s(p)\big) \text{ for } p \in M \ .$$

Finally let $\varphi \in \mathrm{Diff}^1(M,N)$. Then we define the **push forward** by φ of $\gamma \in T_s^r(M)$ through

$$(\varphi_*\gamma)(\alpha_1, \ldots, \alpha_r, v_1, \ldots, v_s) := (\gamma \circ \varphi^{-1})(\varphi^*\alpha_1, \ldots, \varphi^*\alpha_r, \varphi^* v_1, \ldots, \varphi^* v_s) \ ,$$

where $\alpha_1, \ldots, \alpha_r$ are Pfaff forms and $v_1, \ldots, v_s$ are vector fields on N, and we have set

$$\varphi^* v := (\varphi^{-1})_* v \tag{4.25}$$

with v a vector field on N. Naturally, $\varphi^*\gamma := (\varphi^{-1})_*\gamma$ is then the **pull back** of $\gamma \in T_s^r(N)$.

Let $k \in \mathbb{N} \cup \{\infty\}$. Then an (r,s)-tensor γ belongs to the class C^k (or, is k-times **continuously differentiable** or **smooth** in the case $k = \infty$) if every point of M has a chart (φ, U) such that $\varphi_*\gamma$ is a (r,s)-tensor on $\varphi(U)$ of class C^k. We denote the set of all smooth (r,s)-tensors on M by

$$\mathcal{T}_s^r(M) \ .$$

The proofs of the following remarks are straightforwardly transferred from the corresponding proofs for differential forms in Sections 2 and 3.[16] Therefore we leave these proofs to you.

4.18 Remarks **(a)** The definition of differentiability is coordinate independent.

(b) $\mathcal{T}_s^r(M)$ is an $\mathcal{E}(M)$-module. The tensor product map

$$\otimes \colon \mathcal{T}_{s_1}^{r_1}(M) \times \mathcal{T}_{s_2}^{r_2}(M) \to \mathcal{T}_{s_1+s_2}^{r_1+r_2}(M) \ , \quad (\gamma, \delta) \mapsto \gamma \otimes \delta$$

is $\mathcal{E}(M)$-bilinear and associative.

(c) An (r,s)-tensor γ on M is smooth if and only if

$$\gamma(\alpha_1, \ldots, \alpha_r, v_1, \ldots, v_s) \in \mathcal{E}(M)$$

for all $v_1, \ldots, v_s \in \mathcal{V}(M)$ and $\alpha_1, \ldots, \alpha_r \in \Omega^1(M)$.

[16]This repetition can be avoided if one first develops the (elementary) theory of vector bundles.

(d) Let (φ, U) be a chart of M. Then

$$\left\{ \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \otimes dx^{k_1} \otimes \cdots \otimes dx^{k_s} \ ; \ j_i, k_i \in \{1, \ldots, m\} \right\} \tag{4.26}$$

is a module basis of $\mathcal{T}_s^r(M)$. Then $\gamma \in \mathcal{T}_s^r(U)$ if and only if the coefficients of γ in the basis representation (4.26) are smooth.

(e) Let $\varphi \in \mathrm{Diff}(M, N)$. Then φ_* maps the module $\mathcal{T}_s^r(M)$ to $\mathcal{T}_s^r(N)$ and operates covariantly as

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_* \text{ and } (\mathrm{id}_M)_* = \mathrm{id}_{\mathcal{T}_s^r(M)} \ .$$

Analogously, $\varphi^*\big(\mathcal{T}_s^r(N)\big) = \mathcal{T}_s^r(M)$, and φ^* operates contravariantly. Finally φ_* and therefore also φ^* is compatible with the tensor product map, that is,

$$\varphi_*(\gamma \otimes \delta) = \varphi_*\gamma \otimes \varphi_*\delta \ .$$

(f) For $f \in C^\infty(M, N)$ and $\gamma \in \mathcal{T}_s^0(N)$, the **pull back** $f^*\gamma$ of γ by f is determined by

$$f^*\gamma(v_1, \ldots, v_s) := (\gamma \circ f)\big((Tf)v_1, \ldots, (Tf)v_s\big) \quad \text{for } v_1, \ldots, v_s \in \mathcal{V}(M)$$

with $((Tf)v)(p) := (T_p f)v(p)$ for $p \in M$. Then the map

$$f^* : \mathcal{T}_s^0(N) \to \mathcal{T}_s^0(M) \ , \quad \gamma \mapsto f^*\gamma$$

is well defined and $\mathbb{R}$-linear, operates contravariantly, and is compatible with the tensor product map. For $f \in \mathrm{Diff}(M, N)$, it is the same as the previously defined pull back.

(g) $\mathcal{T}_0^1(M) = \mathcal{V}(M)$, $\mathcal{T}_1^0(M) = \Omega^1(M)$, and dual pairing $\langle \cdot, \cdot \rangle$ is a $(1,1)$-tensor on M.

(h) (regularity) Let $k \in \mathbb{N}$. Then the statements above hold analogously if M is a C^{k+1}manifold and C^∞ is replaced by C^k. ■

Exercises

1 Let N be a submanifold of a manifold without boundary. Show that (with the canonical identification) $\mathcal{V}^k(N) \subset \mathcal{V}^k(M)$ for $k \in \mathbb{N} \cup \{\infty\}$.

2 Suppose $\alpha \in \Omega^r(M)$ and $\beta \in \Omega(M)$, and let $v \in \mathcal{V}(M)$. Show that

$$v \,\lrcorner\, (\alpha \wedge \beta) = (v \,\lrcorner\, \alpha) \wedge \beta + (-1)^r \alpha \wedge (v \,\lrcorner\, \beta) \ .$$

3 Verify the statements made in the proof of Example 4.17(b).

4 For $\alpha \in \Omega^1(M)$ and $v \in \mathcal{V}(M)$, calculate $d\langle \alpha, v \rangle$ in local coordinates.

5 Riemannian metrics

We already know from Section VII.10 that the Euclidean inner product $(\cdot\,|\,\cdot)$ on $\mathbb{R}^{\overline{m}}$ can be used to define another inner product by restricting it to the tangent space T_pM of a submanifold M. This gives a way to measure lengths and angles on T_pM. So, for example, we can determine if two curves Γ_1 and Γ_2 on M intersect orthogonally at a point p by verifying that the tangent spaces $T_p\Gamma_1$ and $T_p\Gamma_2$ are themselves orthogonal in T_pM.

That the Euclidean structure of $\mathbb{R}^{\overline{m}}$ induces one on M, or precisely on the tangent bundle of M, is the foundation for the theory of integration on manifolds, which we will treat in the next chapter. In this section, we explore a few consequences of the existence of a Euclidean structure on M, and we study several examples. We also introduce the Hodge star operator and the codifferential, which are of significance for a deeper incursion into the theory of differential forms — in particular, these concepts are important in (theoretical) physics.

To ease the introduction to the material, we consider first the case of the Euclidean structure on M induced by $(\cdot\,|\,\cdot)$. It will be apparent, however, that all abstract theorems remain true in an essentially more general framework, namely, that of Riemannian geometry. Because these facts are of great theoretical and practical importance, we will introduce the concept of a (pseudo) Riemannian metric, which forms the general framework for our subsequent considerations.

For the entire section, suppose the following:

- M is an m-dimensional submanifold of $\mathbb{R}^{\overline{m}}$; N is an n-dimensional submanifold of $\mathbb{R}^{\overline{n}}$.
- The indices i, j, k, l always range from 1 to m unless otherwise stated, and $\sum_j$ means that j is summed from 1 to m.

The volume element

Suppose M is oriented. Then $\mathcal{O}r$ induces an orientation on every tangent space T_pM. Also T_pM is an inner product space with inner product $(\cdot\,|\,\cdot)_p$ induced by the Euclidean scalar product of the surrounding space $\mathbb{R}^{\overline{m}}$. Thus, by Remark 2.12(b), there is a unique volume element ω_p on T_pM. Therefore

$$\omega_M(p) := \omega_p \quad \text{for } p \in M$$

defines an m-form on M, the **volume element** on M.

5.1 Proposition *Suppose M is oriented. Then ω_M belongs to $\mathcal{O}r(M)$. If (φ, U) is a positive chart with $\varphi = (x^1, \ldots, x^m)$, then*

$$\omega_M \,|\, U = \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^m \ , \tag{5.1}$$

where $G := \det[g_{jk}] \in \mathcal{E}(U)$ *is the* **Gram determinant** *and*

$$g_{jk}(p) := \bigl(\partial_j|_p \bigm| \partial_k|_p\bigr)_p \quad \text{for } 1 \le j,k \le m \text{ and } p \in U \ .$$

Letting $g_\varphi := i \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^{\overline{m}})$, *with* $i\colon M \hookrightarrow \mathbb{R}^{\overline{m}}$, *be the parametrization belonging to* φ, *we have*

$$\varphi_* g_{jk}(x) = \bigl(\partial_j g_\varphi(x) \bigm| \partial_k g_\varphi(x)\bigr) \quad \text{for } 1 \le j,k \le m \text{ and } x \in \varphi(U) \ . \qquad (5.2)$$

Proof Because $(\partial_1|_p, \ldots, \partial_m|_p)$ is a positive basis of T_pM, it follows from Proposition 2.13 that

$$\omega_p = \sqrt{G(p)}\, dx^1 \wedge \cdots \wedge dx^m(p) \quad \text{for } p \in U \ .$$

Therefore (5.1) holds. Because

$$\varphi_*(\omega_M \,|\, U) = \varphi_* \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^m \bigm| \varphi(U)$$

with $\varphi_* \sqrt{G} = \sqrt{G \circ \varphi^{-1}}$, it follows from Remark 4.3(a) that (5.2) is satisfied. Because the scalar product and the determinant function are smooth (see Proposition VII.4.6 and Exercise VII.4.2) and because $G(p) > 0$ for $p \in U$, the chain rule gives $\varphi_* \sqrt{G} \in \mathcal{E}\bigl(\varphi(U)\bigr)$. Therefore $\omega_M \,|\, U$ is smooth, which proves $\omega_M \in \mathcal{O}r$. ■

5.2 Remark (regularity) By modifying the statement of this proposition in the obvious way, we find it remains true when M is a C^1 manifold. ■

5.3 Examples **(a)** (open sets in $\overline{\mathbb{H}}^m$) Let X be a nonempty open subset of $\overline{\mathbb{H}}^m$. Then X is endowed with a **natural orientation** with respect to which every tangent space $T_pX = T_p\mathbb{R}^m$ with $p \in X$ is naturally oriented, that is, this orientation makes the canonical basis $\bigl((e_1)_p, \ldots, (e_m)_p\bigr)$ positive. Then the volume element of X is given by

$$\omega_X = dx^1 \wedge \cdots \wedge dx^m \bigm| X \ .$$

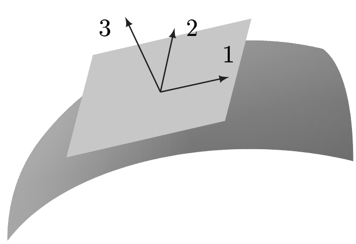
The trivial chart (id_X, X) is positive.

(b) (fibers of regular maps) Suppose X is open in $\mathbb{R}^m$ and q is a regular value of $f \in \mathcal{E}(X)$ with $M := f^{-1}(q) \ne \emptyset$. We provide the hypersurface M with the **orientation** $\mathcal{O}r(M, \nabla f)$ **induced by** ∇f, as follows: For every $p \in M$, the basis $(v_1, \ldots, v_{m-1})$ of T_pM is positive if and only if the basis

$$\bigl(\nabla f(p), v_1, \ldots, v_{m-1}\bigr)$$

of $T_pX = T_p\mathbb{R}^m$ is positive with $\nabla f = \sum_k \partial_k f\, \partial/\partial x^k$. With

$$\nu := \nabla f / |\nabla f|$$



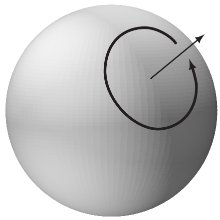
the unit normal field of M, the volume element of $\bigl(M, \mathcal{O}r(M, \nabla f)\bigr)$ is given by $\omega_M := (\nu \mathbin{\lrcorner} \omega_X) \,|\, M$.

If $m = 3$, a basis $\big(v_1(p), v_2(p)\big)$ of T_pM is positive if and only if the three vectors $\big(v_1, v_2, \nu(p)\big)$ form a "right handed basis" of $T_p\mathbb{R}^3$. Here (w_1, w_2, w_3) is a **right handed basis** if one can stretch out the thumb, first, and second fingers of one's right hand (with the middle finger bent palmward) so that these three fingers point in the direction (and the same order) of these three vectors. This is called the **right hand rule**.

Proof Because q is a regular point, $\nabla f(q) \neq 0$ for $q \in M$. By the regular value theorem, M is a smooth hypersurface in X. The proof of Example 4.17(e) shows that ω_M is a smooth volume form. Now all is clear. ■

(c) (spheres) The m-sphere S^m in $\mathbb{R}^{m+1}$ for $m \in \mathbb{N}$ is **canonically** oriented by the outward unit normal field

$$\nu(x) := (x, x) \in T_x\mathbb{R}^{m+1} .$$



If $m = 0$, S^0 consists of the two points $\{\pm 1\} \subset \mathbb{R}$, and the outward unit normal field at 1 [or -1] is given by $(1, 1) \in T_1\mathbb{R}$ [or $(-1, -1) \in T_{-1}\mathbb{R}$].[1]

When $m = 1$, the canonical orientation of S^1 is the same as the one given in Remark VIII.5.8. Therefore "one traverses S^1 is the positive direction" exactly when the traversal is counterclockwise. In this case, ν coincides with the negative unit normal vector $-\mathfrak{n}$ in the sense of the Frenet two-frame.

The volume element of the canonically oriented m-sphere is the m-form[2]

$$\begin{aligned}\omega_{S^m} &= (\nu \lrcorner\, \omega_{\mathbb{R}^{m+1}}) \,\big|\, S^m \\ &= \sum_{j=1}^{m+1} (-1)^{j-1} x^j \, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{m+1} \,\Big|\, S^m .\end{aligned}$$

The chart $(\varphi_\pm, S^m_\pm)$ describes the upper [or lower] hemisphere $S^m_\pm$ that is projected along the x^{m+1}-axis onto $\mathbb{B}^m \times \{0\}$; this chart is positively oriented when m is even [or odd] and is negatively oriented for odd [or even] m.

The spherical coordinate chart of S^1 is positive, whereas that of S^2 is negative.

Proof The formula for ω_{S^m} is a special case of (b). The statements about the various charts of S^m follow from Examples 4.6(a)–(b). ■

(d) (graphs) Let X be open in $\overline{\mathbb{H}}^m$ and $f \in C^\infty(X, \mathbb{R}^n)$. Then the **natural orientation of the graph** $M := \mathrm{graph}(f)$ is the one for which the natural chart (φ, M) with

$$\varphi \colon M \to \mathbb{R}^m , \quad \big(x, f(x)\big) \mapsto x$$

[1]See Example 1.17(a).

[2]This justifies the notations used in Examples 4.6 and 4.13(c).

is positive. In the case $n = 1$, the volume element ω_M has the local representation

$$\omega_M \,|\, M = \sqrt{1 + |\nabla f|^2}\, dx^1 \wedge \cdots \wedge dx^m$$

with $\nabla f = \sum_{j=1}^m \partial_j f \,\partial_j$.

Proof Because $g_\varphi(x) = \bigl(x, f(x)\bigr)$ for $x \in X = \varphi(M)$, it follows from Remark 4.3(a) that

$$\partial_j|_p = \bigl(p, (e_j, \partial_j f(x)\bigr) \in T_p\mathbb{R}^{m+1} \quad \text{for } p = \bigl(x, f(x)\bigr) \in M \text{ and } 1 \le j \le m\ ,$$

where T_pM is identified canonically with the vector subspace $(T_p i_M)(T_pM)$ of $T_p\mathbb{R}^{m+1}$. Putting $d_j := \partial_j f$, we then get $g_{jk} = \delta_{jk} + d_j d_k$.

Let $D_m := [\varphi_* g_{jk}]$. Then it follows that

$$G = \det D_m = \det\begin{bmatrix} 1+d_1^2 & d_1 d_2 & \cdots & d_1 d_m \\ d_2 d_1 & 1+d_2^2 & \cdots & d_2 d_m \\ \vdots & \vdots & \ddots & \vdots \\ d_m d_1 & d_m d_2 & \cdots & 1+d_m^2 \end{bmatrix}$$

$$= \det\left[\begin{array}{ccc:c} & D_{m-1} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hdashline d_m d_1 & \cdots & d_m d_{m-1} & 1 \end{array}\right] + d_m^2 \det\left[\begin{array}{ccc:c} & D_{m-1} & & \begin{matrix} d_1 \\ \vdots \\ d_{m-1} \end{matrix} \\ \hdashline d_1 & \cdots & d_{m-1} & 1 \end{array}\right] .$$

To compute the last determinant, we subtract d_j times the last column from the j-th column for $1 \le j \le m-1$, and so find that its value is 1. This then gives the recursion formula

$$\det D_m = \det D_{m-1} + d_m^2\ .$$

Because $\det D_1 = 1 + d_1^2$, the recursion yields

$$G = \det D_m = 1 + d_1^2 + \cdots + d_m^2 = 1 + |\nabla f|^2$$

and hence the claim. ■

(e) (curves) Suppose J is a perfect interval in $\mathbb{R}$, and $\gamma\colon J \to \mathbb{R}^m$ is a smooth embedding. Then $M := \gamma(J)$ is an embedded curve in $\mathbb{R}^m$. Also let M be **oriented by** γ, that is, let $\bigl(\gamma(t), \dot\gamma(t)\bigr)$ be a positive basis of $T_{\gamma(t)}M$ for $t \in J$. Finally, suppose $\varphi\colon M \to \mathbb{R}$, with $\gamma = i_M \circ \varphi^{-1}$ the chart of M belonging to γ. Then $\omega_M = |\dot\gamma|\, dt$.

Proof This is an immediate consequence of Proposition 5.1. ■

(f) (parametrized surfaces) Let X be open in $\overline{\mathbb{H}}^2$, and $h\colon X \to \mathbb{R}^n$ let be a smooth embedding. Then $M := h(X)$ is a two-dimensional submanifold in $\mathbb{R}^n$, a **surface** in $\mathbb{R}^n$, which is described by a single chart. Therefore M is orientable. By the **orientation induced by the parametrization h**, we mean that orientation for which $\bigl(\partial_1 h(x), \partial_2 h(x)\bigr)$ is a positive basis of $T_{h(x)}M$ for every $x \in X$.

Let $\varphi\colon M \to \mathbb{R}^2$ with $\varphi = (u, v)$ charge belonging to h, that is, $h = i_M \circ \varphi^{-1}$. With the classical notations

$$\mathsf{E} := |\partial_1 h|^2 \ , \quad \mathsf{F} := (\partial_1 h \,|\, \partial_2 h) \ , \quad \mathsf{G} := |\partial_2 h|^2 \ ,$$

we have

$$\omega_M = \sqrt{\mathsf{EG} - \mathsf{F}^2}\, du \wedge dv \ .$$

Proof This follows from $\varphi_* G = \mathsf{EG} - \mathsf{F}^2$. ■

(g) (boundaries) Suppose M is an oriented manifold with boundary and $\nu(p)$ is the outward (unit) normal vector ∂M at $p \in \partial M$. Then we say a basis $(v_1, \ldots, v_{m-1})$ of $T_p\partial M$ is positive if $\bigl(\nu(p), v_1, \ldots, v_{m-1}\bigr)$ is a positive basis of T_pM. This is turn determines an orientation on ∂M, the **orientation induced by the outward normal**. The volume element $\omega_{\partial M}$ of ∂M satisfies

$$\omega_{\partial M} = (\nu \lrcorner\, \omega_M) \,|\, \partial M = i^*_{\partial M}(\nu \lrcorner\, \omega_M) \ ,$$

where $i_{\partial M} : \partial M \hookrightarrow M$ is the natural embedding.

Obviously (c) is a special case of this situation. Note also that the orientation induced by the outward normal must not agree with that induced by ∇f if ∂M can be represented as in (b) as the fiber of a regular map.

Proof From Theorem 1.15 and Remark 1.16(a), we know that ν can be locally described in the form $\nu(p) = \nabla f(p)/|\nabla f(p)|$, where f is a smooth function satisfying $\nabla f(p) \neq 0$. This shows the unit normal vector field is smooth. From this it follows easily that $(\nu \lrcorner\, \omega_M) \,|\, \partial M$ belongs to $\Omega^{m-1}(\partial M)$. If $(v_1, \ldots, v_{m-1})$ is an ONB of $T_p\partial M$, then $\bigl(\nu(p), v_1, \ldots, v_{m-1}\bigr)$ is an ONB of T_pM. When $\bigl(\nu(p), v_1, \ldots, v_{m-1}\bigr)$ is a positive ONB of T_pM,

$$1 = \omega_M\bigl(\nu(p), v_1, \ldots, v_{m-1}\bigr) = (\nu \lrcorner\, \omega_M)(p)(v_1, \ldots, v_{m-1}) \ . \ ■$$

(h) Let $m \geq 2$. Then the orientation of $\partial\mathbb{H}^m = \mathbb{R}^{m-1}$ induced by the outward normal $\nu = -e_m$ coincides with the natural orientation of $\mathbb{R}^{m-1}$ if and only if m is even.

Proof This we may read off from

$$\det[\nu, e_1, \ldots, e_{m-1}] = (-1)^{m-1} \det[e_1, \ldots, e_{m-1}, -e_m] = (-1)^m \ . \ ■$$

Riemannian manifolds

The proof of Proposition 5.1 depends on the fact that every tangent space T_pM is endowed naturally with an inner product that varies differentiably with $p \in M$. Such situations appear quite frequently, although the scalar product on TM is often generated in another way. Therefore it is useful to explore these issues somewhat more precisely.

A **Riemannian metric** on M is a tensor $g \in \mathcal{T}_2^0(M)$ such that $g(p)$ is an inner product on T_pM for every $p \in M$. Then (M, g) is a **Riemannian manifold**.

Let (M, g) be a Riemannian manifold. We will often write $\bigl((x^1, \ldots, x^m), U\bigr)$ for the chart (φ, U) of M with $\varphi = (x^1, \ldots, x^m)$. Then we set

$$g_{jk} := g\Bigl(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Bigr) \in \mathcal{E}(U) \tag{5.3}$$

and put

$$[g^{jk}] := [g_{jk}]^{-1} \in C^\infty(U, \mathbb{R}^{m\times m}_{\mathrm{sym}}) \quad \text{and} \quad G := \det[g_{jk}] \in \mathcal{E}(U)\ . \tag{5.4}$$

We also call g the (**first**) **fundamental tensor**. Here $[g_{jk}]$ is the representation matrix (or simply, the matrix) of g in the local coordinates $(x^1, \ldots, x^m)$; it is also called the (**first**) **fundamental matrix**.[3] As before, G is called the **Gram determinant**.

5.4 Remarks **(a)** If g is a Riemannian metric on M, the map

$$\mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{E}(M)\ , \quad (v, w) \mapsto g(v, w) \tag{5.5}$$

is well defined, bilinear, symmetric, and **positive** in the sense that

$$g(v, v) \geq 0 \quad \text{and} \quad g(v, v) = 0 \Longleftrightarrow v = 0\ . \tag{5.6}$$

Proof That the map (5.5) is well defined follows immediately from Remark 4.18(c). The remaining claims are direct consequences of the properties of scalar products. ■

(b) Let $\bigl((x^1, \ldots, x^m), U\bigr)$ be a chart of a Riemannian manifold (M, g). Then

$$g\,|\,U = \sum\nolimits_{j,k} g_{jk}\, dx^j \otimes dx^k\ .$$

In this context, we usually write $dx^j dx^k$ for $dx^j \otimes dx^k$.

Proof According to (4.8), any $v \in \mathcal{V}(U)$ has a basis representation

$$v = \sum\nolimits_j \langle dx^j, v\rangle \frac{\partial}{dx^j}\ .$$

The claim follows from this, the bilinearity of the map (5.5), and the definitions of $dx^j \otimes dx^k$ and g_{jk}. ■

(c) Let $\bigl((x^1, \ldots, x^m), U\bigr)$ be a positive chart of an oriented Riemannian manifold (M, g). Then the volume element ω_M of M satisfies

$$\omega_M\,|\,U = \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^m\ .$$

Proof This follows from Proposition 2.13. ■

[3] See Remark VII.10.3(b).

(d) Let g be a Riemannian metric on M, and let $(x^1,\ldots,x^m)$ and $(y^1,\ldots,y^m)$ be local coordinates on an open set U of M. Then

$$g\,|\,U=\sum\nolimits_{j,k} g_{jk}\,dx^j\otimes dx^k=\sum\nolimits_{r,s}\overline{g}_{rs}\,dy^r\otimes dy^s$$

with

$$\overline{g}_{rs}=\sum\nolimits_{j,k}\frac{\partial x^j}{\partial y^r}\,\frac{\partial x^k}{\partial y^s}\,g_{jk}\quad \text{for } 1\le r,s\le m\ .$$

Proof Because

$$dx^j=\sum\nolimits_r \frac{\partial x^j}{\partial y^r}\,dy^r\quad \text{for } 1\le r\le m\ ,$$

this is a consequence of (b). ■

(e) If we only require of $g\in\mathcal{T}_2^0(M)$ that the bilinear form $g(p)$ on T_pM is symmetric and nondegenerate for every $p\in M$, then we call g an **indefinite Riemannian metric**, and (M,g) is a **pseudo-Riemannian manifold**. In this case, we again use the notations (5.3) and (5.4). Then (a), (b) and (d), with the exception of (5.6), remain true. Every Riemannian manifold is also pseudo-Riemannian.

(f) Let (M,g) be a (pseudo-)Riemannian manifold, and suppose W is open in M. If $v_1,\ldots,v_m\in\mathcal{V}(W)$ satisfy

$$g(v_j,v_k)=\pm\delta_{jk}\quad \text{for } 1\le j,k\le m\ ,$$

we say $(v_1,\ldots,v_m)$ is an **orthonormal frame** on W. Of course, Riemannian manifolds have $g(v_j,v_j)=1$ for $1\le j\le m$. An orthonormal frame $(v_1,\ldots,v_m)$ on W is therefore an m-tuple of (smooth) vector fields W that form an ONB (with respect to the (indefinite) inner product $g(p)$ of T_pM) at every point $p\in W$. Such an orthonormal frame does not exist in general, because, according to Remark 4.5(d), one cannot generally find m vector fields that are everywhere linearly independent.

If (φ,U) is a chart of M, then there is an orthonormal frame on U.

Proof The basis vector fields $\partial_1,\ldots,\partial_m\in\mathcal{V}(U)$ are linearly independent at every point. Because g is nondegenerate, the Gram–Schmidt orthonormalization procedure (see for example [Art93, §§ 7.1 and 7.2]) then generates an orthonormal frame. The details are left to you. ■

(g) Let (M,g) be an oriented pseudo-Riemannian manifold. If (φ,U) is a positive chart of M with $\varphi=(x^1,\ldots,x^m)$, we set

$$\omega_M\,|\,U:=\sqrt{|G|}\,dx^1\wedge\cdots\wedge dx^m\ .$$

This then defines a volume form $\omega_M\in\Omega^m(M)$ on M, which we call the **volume element** of M. Every positive orthonormal frame $(v_1,\ldots,v_m)$ on U satisfies

$$\omega_M(v_1,\ldots,v_m)=1\ .$$

Proof We show first that $\omega_M \in \Omega^m(M)$ is well defined. So let $(e_1,\ldots,e_m)$ be any orthonormal frame on U, with $(\varepsilon^1,\ldots,\varepsilon^m)$ its dual frame; that is, $\varepsilon^j \in \Omega^1(U)$ and $\langle \varepsilon^j, e_k\rangle = \delta^j_k$ for $1 \le j,k \le m$. Then it follows from Remark 2.18(d) that

$$\varepsilon^1 \wedge \cdots \wedge \varepsilon^m = \sqrt{|G|}\, dx^1 \wedge \cdots \wedge dx^m \ .$$

Because this is true for every positive coordinate system $(x^1,\ldots,x^m)$ on U, it follows that $\omega_M \,|\, U \in \Omega^m(U)$ is well defined and independent of the special choice of local coordinates. Suppose now $\{ (\varphi_\alpha, U_\alpha) \ ;\ \alpha \in \mathsf{A} \}$ is a positive atlas of M and $(v_{\alpha,1},\ldots,v_{\alpha,m})$ is a positive orthonormal frame on U_α with dual frame $(\varepsilon^1_\alpha,\ldots,\varepsilon^m_\alpha)$. Then we define ω_M on M by $\omega_M \,|\, U_\alpha := \varepsilon^1_\alpha \wedge \cdots \wedge \varepsilon^m_\alpha$. From the previous considerations, it follows that ω_M is well defined and belongs to $\Omega^m(M)$. The last claim is now obvious. ■

(h) (regularity) Let $k \in \mathbb{N}$, and let M be a C^{k+1} manifold. Then the definitions and statements above remain true if $\mathcal{V}(M)$ and $\mathcal{E}(M)$ are replaced everywhere by $\mathcal{V}^k(M)$ and $C^k(M)$, respectively. ■

Suppose $(N, \bar g)$ is a Riemannian manifold and $f\colon M \to N$ is an immersion. Then $f^*\bar g$ (the pull back of $\bar g$ by f) is a Riemannian metric on M. If M is a submanifold of N and $i\colon M \hookrightarrow N$ is the natural embedding, then $i^*\bar g$ is the **Riemannian metric induced by N** (more precisely, by $(N,\bar g)$).

Let (M,g) and $(N,\bar g)$ be Riemannian manifolds. An immersion $f\colon M \to N$ is said to be an **isometry** if $g = f^*\bar g$. If f is an isometric diffeomorphism, that is, both an isometry and a diffeomorphism, then M and N are **isometrically isomorphic**.

5.5 Examples **(a)** Suppose (M,g) is a Riemannian manifold and (φ, U) is a chart with $\varphi = (x^1,\ldots,x^m)$. Then (U,g) and $\bigl(\varphi(U), \varphi_* g\bigr)$ are isometrically isomorphic, and

$$\varphi_* g = \sum\nolimits_{j,k} g_{jk}\, dx^j dx^k \ .$$

Proof This follows immediately from the definition of the fundamental matrix. ■

(b) $\mathbb{R}^{\overline{m}}$ is a Riemannian manifold with the Euclidean metric $g_{\overline{m}} := (\cdot\,|\,\cdot)$, the standard metric. Therefore $\mathbb{R}^{\overline{m}}$ induces a Riemannian metric g on M, which we also call the **standard metric**. It is obviously independent of $\mathbb{R}^{\overline{m}}$ in the sense that $\mathbb{R}^n$ induces the same metric on M when M lies in $\mathbb{R}^n$. In particular, $g(p) = (\cdot\,|\,\cdot)_p$ for $p \in M$ (with the notation we have been using) for the scalar product induced by $(\cdot\,|\,\cdot)$ in T_pM.

If (φ, U) is a chart of M with $\varphi = (x^1,\ldots,x^m)$ and $h := i \circ \varphi^{-1}$ with $i\colon M \hookrightarrow \mathbb{R}^{\overline{m}}$ is the associated parametrization, then

$$\varphi_* g = \sum\nolimits_{j,k} (\partial_j h \,|\, \partial_k h)\, dx^j dx^k \ .$$

In other words, the first fundamental matrix $[g_{jk}]$ is given in local coordinates $(x^1,\ldots,x^m)$ by

$$\bigl[(\partial_j h \,|\, \partial_k h)\bigr] \in C^\infty\bigl(\varphi(U), \mathbb{R}^{m\times m}\bigr) \ .$$

This is consistent with Remark 5.4(b) and also shows that Proposition 5.1 is a special case of Remark 5.4(c).

Proof From $g = i^* g_{\overline{m}}$, it follows that $\varphi_* g = (\varphi^{-1})^* i^* g_{\overline{m}} = h^* g_{\overline{m}}$. Let $(y^1, \ldots, y^{\overline{m}})$ be Euclidean coordinates of $\mathbb{R}^{\overline{m}}$. Then

$$g_{\overline{m}} = \sum_{j=1}^{\overline{m}} (dy^j)^2 \quad \text{and} \quad h^* g_{\overline{m}} = \sum_{j=1}^{\overline{m}} h^*(dy^j \otimes dy^j) = \sum_{j=1}^{\overline{m}} dh^j \otimes dh^j \ ,$$

which follow easily from the definition of the pull back of $(0,2)$-tensors $dy^j \otimes dy^j$ and from Example 3.4(a). Now the claim follows easily from the bilinearity of $(\alpha, \beta) \mapsto \alpha \otimes \beta$ for $\alpha, \beta \in \Omega^1\big(\varphi(U)\big)$ and from $dh^j = \sum_k \partial_k h^j \, dx^k$. ■

(c) (graphs) Suppose X is open in $\overline{\mathbb{H}}^m$ and $f \in C^\infty(X, \mathbb{R}^n)$. Let M be the graph of f, and let

$$\varphi \colon M \to \mathbb{R}^m \ , \quad \big(x, f(x)\big) \mapsto x$$

be the natural chart (φ, M). Then the standard metric g of M satisfies

$$g = \sum\nolimits_j (dx^j)^2 + \sum\nolimits_{j,k} (\partial_j f \,|\, \partial_k f) \, dx^j \, dx^k \ .$$

In particular, in the case of a surface $(m = 2)$,

$$g = (1 + |\partial_1 f|^2)(dx)^2 + 2(\partial_1 f \,|\, \partial_2 f) \, dxdy + (1 + |\partial_2 f|^2)(dy)^2 \ .$$

Proof Because $g_{jk} = \delta_{jk} + (\partial_j f \,|\, \partial_k f)$ for $1 \le j, k \le m$, this follows from (b). ■

(d) (parametrized surfaces) Suppose X is open in $\overline{\mathbb{H}}^2$ and $h \colon X \to \mathbb{R}^n$ is an embedding. Then the standard metric of the surface $M := h(X)$ is given by

$$g = \mathsf{E}(du)^2 + 2\mathsf{F} \, dudv + \mathsf{G}(dv)^2 \ ,$$

where we have used the notations of Example 5.3(f).

(e) (plane polar coordinates) Let $V_2 := (0, \infty) \times (0, 2\pi)$. Then the polar coordinate map

$$f_2 \colon V_2 \to \mathbb{R}^2 \ , \quad (r, \varphi) \mapsto (x, y) := (r \cos \varphi, r \sin \varphi)$$

is an embedding with $M := f_2(V_2) = \mathbb{R}^2 \backslash \big(\mathbb{R}^+ \times \{0\}\big)$, and

$$g_2 \,|\, M = (dx)^2 + (dy)^2 = (dr)^2 + r^2 (d\varphi)^2 \ .$$

Proof This follows easily from (d). ■

(f) (circular coordinates) With respect to the parametrization

$$h \colon (0, 2\pi) \to \mathbb{R}^2 \ , \quad t \mapsto (\cos t, \sin t)$$

of $S^1 \backslash \big\{(1,0)\big\}$, the standard metric on the circle satisfies $g_{S^1} = (dt)^2$.

Proof Because $|\partial h| = 1$, this follows from (b). ■

(g) (m-dimensional polar coordinates) With $m \geq 3$, let[4]

$$f_m : V_m \to \mathbb{R}^m \ , \quad (r, \varphi, \vartheta_1, \ldots, \vartheta_{m-2}) \mapsto (x^1, x^2, x^3, \ldots, x^m)$$

be the (restriction to V_m of the) polar coordinate map (X.8.17). Then f_m is a parametrization of $\mathbb{R}^m \backslash H_{m-1}$, and

$$\sum_{j=1}^m (dx^j)^2 = (dr)^2 + r^2 \Big[a_{m,0}(d\varphi)^2 + \sum_{k=1}^{m-2} a_{m,k}(d\vartheta_k)^2 \Big]$$

with

$$a_{m,k} := \prod_{i=k+1}^{m-2} \sin^2 \vartheta_i \quad \text{for } 0 \leq k \leq m-3 \quad \text{and} \quad a_{m,m-2} := 1 \ .$$

In particular, spherical coordinates satisfy ($m = 3$)

$$(dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2 \big[\sin^2 \vartheta (d\varphi)^2 + (d\vartheta)^2 \big] \ .$$

Proof With $y = (r, z) \in \mathbb{R} \times \mathbb{R}^{m-1}$, we read off from (X.8.14) that

$$\partial_1 f_m(y) = h_{m-1}(z) \quad \text{and} \quad \partial_j f_m(y) = r \partial_{j-1} h_{m-1}(z) \quad \text{for } 2 \leq j \leq m \ . \tag{5.7}$$

Therefore (X.8.13) implies

$$|\partial_1 f_m|^2 = 1 \ . \tag{5.8}$$

Differentiation of $|h_{m-1}|^2 = 1$ gives $(h_{m-1} \,|\, \partial_k h_{m-1}) = 0$ for $1 \leq k \leq m-1$. Then it follows from (5.7) that

$$\big(\partial_1 f_m(y) \,\big|\, \partial_k f_m(y) \big) = r \big(h_{m-1}(z) \,\big|\, \partial_{k-1} h_{m-1}(z) \big) = 0 \quad \text{for } 2 \leq k \leq m \ . \tag{5.9}$$

From (5.7) we also get

$$\big(\partial_j f_m(y) \,\big|\, \partial_k f_m(y) \big) = r^2 \big(\partial_{j-1} h_{m-1}(z) \,\big|\, \partial_{k-1} h_{m-1}(z) \big) \quad \text{for } 2 \leq j, k \leq m \ . \tag{5.10}$$

The recursion formula (X.8.12) with $z = (z', z_{m-1}) \in \mathbb{R}^{m-2} \times \mathbb{R}$ leads to

$$\partial_j h_{m-1}(z) = \big(\partial_j h_{m-2}(z') \sin z_{m-1}, 0 \big) \quad \text{for } 1 \leq j \leq m-2 \ , \tag{5.11}$$

and

$$\partial_{m-1} h_{m-1}(z) = \big(h_{m-2}(z') \cos z_{m-1}, -\sin z_{m-1} \big) \ .$$

From this and (X.8.13), it follows that

$$|\partial_{m-1} h_{m-1}(z)|^2 = |h_{m-2}(z')|^2 \cos^2 z_{m-1} + \sin^2 z_{m-1} = 1 \ ,$$

and, in analogy to the above, we have

$$\big(\partial_j h_{m-1}(z) \,\big|\, \partial_{m-1} h_{m-1}(z) \big) = \sin z_{m-1} \cos z_{m-1} \big(h_{m-2}(z') \,\big|\, \partial_j h_{m-2}(z') \big) = 0$$

[4] We use the notations of (X.8.11)–(X.8.24).

for $1 \le j \le m-2$. With (5.7), this proves

$$|\partial_m f_m(z)|^2 = r^2 \quad \text{and} \quad (\partial_j f_m \,|\, \partial_m f_m) = 0 \quad \text{for } 2 \le j \le m-1 \ . \tag{5.12}$$

Finally (5.11) implies

$$\big(\partial_j h_{m-1}(z) \,\big|\, \partial_k h_{m-1}(z)\big) = \sin^2 z_{m-1}\big(\partial_j h_{m-2}(z') \,\big|\, \partial_k h_{m-2}(z')\big) \quad \text{for } 1 \le j \le m-2 \ .$$

Thus (5.7) and (5.10) give the recursion formula

$$\big(\partial_j f_m(y) \,\big|\, \partial_k f_m(y)\big) = \sin^2 z_{m-1}\big(\partial_j f_{m-1}(y') \,\big|\, \partial_k f_{m-1}(y')\big) \tag{5.13}$$

for $2 \le j,k \le m-1$, with $y = (y', y_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$. Because (5.8) and (5.12) are true for all $m \ge 3$, induction on (5.13) gives

$$|\partial_j f_m|^2 = r^2 a_{m,j-2} \quad \text{for } 2 \le j \le m-1 \ , \tag{5.14}$$

and

$$(\partial_j f_m \,|\, \partial_k f_m) = 0 \quad \text{for } 2 \le j,k \le m-1 \text{ and } j \ne k \ . \tag{5.15}$$

Now the claim follows from (5.8), (5.9), (5.12), (5.14), (5.15), and (b). ■

(h) (m-dimensional spherical coordinates) For $m \ge 2$, let

$$h_m : W_m \to \mathbb{R}^{m+1} \ , \quad (\varphi, \vartheta_1, \ldots, \vartheta_{m-1}) \mapsto (y^1, y^2, \ldots, y^{m+1}) \ ,$$

where $W_m := (0, 2\pi) \times (0, \pi)^{m-1}$ and

$$\begin{aligned}
y^1 &= \cos\varphi \sin\vartheta_1 \sin\vartheta_2 \cdots \sin\vartheta_{m-1} \ , \\
y^2 &= \sin\varphi \sin\vartheta_1 \sin\vartheta_2 \cdots \sin\vartheta_{m-1} \ , \\
y^3 &= \cos\vartheta_1 \sin\vartheta_2 \cdots \sin\vartheta_{m-1} \ , \\
&\ \vdots \\
y^m &= \cos\vartheta_{m-2} \sin\vartheta_{m-1} \ , \\
y^{m+1} &= \cos\vartheta_{m-1}
\end{aligned}$$

are (***m*-dimensional**) **spherical coordinates**.[5] Then h_m is a parametrization of the open subset $U_m := S^m \backslash H_m$ of the m-sphere. The standard metric g_{S^m} of S^m satisfies

$$g_{S^m} = a_{m+1,0}(d\varphi)^2 + \sum_{k=1}^{m-1} a_{m+1,k}(d\vartheta_k)^2 \ .$$

In the case of the 2-sphere (with $\vartheta := \vartheta_1$), this becomes

$$g_{S^2} = \sin^2\vartheta (d\varphi)^2 + (d\vartheta)^2 \ .$$

Proof Because $h_m = f_{m+1}(1, \cdot\,)$, the claim is a simple consequence of (g). ■

[5] See Example VII.9.11(b).

(i) (Minkowski metric) We denote the Euclidean coordinates of $\mathbb{R}^4$ by (t,x,y,z) or (x^0,x^1,x^2,x^3) and set $\mathbb{R}^4_{1,3} := \bigl(\mathbb{R}^4, (\cdot\,|\,\cdot)_{1,3}\bigr)$ with the Minkowski metric

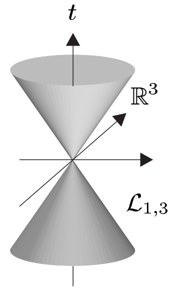
$$(\cdot\,|\,\cdot)_{1,3} = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 = (dx^0)^2 - \sum_{j=1}^{3}(dx^j)^2 \ .$$

Then $\mathbb{R}^4_{1,3}$ is a pseudo-Riemannian manifold, the **spacetime** or **Minkowski space** of (special) relativity theory.

For $v=(v^0,\ldots,v^3)\subset\mathbb{R}^4_{1,3}$, we call

$$|v|^2_{1,3} := (v\,|\,v)_{1,3} = (v^0)^2 - \sum\nolimits_{j=1}^{3}(v^j)^2$$

the **Minkowski norm** of the vector v. Vectors with positive Minkowski norm are said to be timelike; those with negative norm are **spacelike**. Those whose Minkowski norm is zero are lightlike; in $\mathbb{R}^4_{1,3}$, the lightlike vectors form a (double) cone, the **light cone** $\mathcal{L}_{1,3}$.



(j) (pseudospherical coordinates) Let $V_{1,3} := \mathbb{R}\times V_3$ and

$$f_{1,3}: V_{1,3}\to\mathbb{R}^4\ ,\quad (\rho,\chi,\varphi,\vartheta)\mapsto(x^0,x^1,x^2,x^3)$$

with

$$\begin{aligned} x^0 &= \rho\cosh\chi\ ,\\ x^1 &= \rho\sinh\chi\cos\varphi\sin\vartheta\ ,\\ x^2 &= \rho\sinh\chi\sin\varphi\sin\vartheta\ ,\\ x^3 &= \rho\sinh\chi\cos\vartheta\ ; \end{aligned}$$

this is the **pseudospherical coordinate map**. Then $f_{1,3}$ is a smooth diffeomorphism from $V_{1,3}\backslash\{0\}$ to the **interior**

$$\mathring{\mathcal{L}}_{1,3} := \bigl\{\, x\in\mathbb{R}^4\ ;\ |x|^2_{1,3}>0 \,\bigr\}$$

of the **light cone**, and

$$(\cdot\,|\,\cdot)_{1,3} = (d\rho)^2 - \rho^2\bigl[(d\chi)^2 + \sinh^2\chi\sin^2\vartheta(d\varphi)^2 + \sinh^2\chi(d\vartheta)^2\bigr]\ .$$

Proof This follows easily from the properties of sinh and cosh (see Exercises III.6.5 and IV.2.5), and from Remark 5.4(d). ■

(k) (hyperbolic spaces) To generalize the Minkowski space, we set

$$(\cdot\,|\,\cdot)_{1,m} := (dx^0)^2 - \sum_{j=1}^{m}(dx^j)^2$$

for $n\in\mathbb{N}^\times$. Then $\mathbb{R}^{m+1}_{1,m} := \bigl(\mathbb{R}^{m+1},(\cdot\,|\,\cdot)_{1,m}\bigr)$ is an m-dimensional pseudo-Riemannian manifold.

Let

$$M^m := \left\{ (x^0, x) \in \mathbb{R} \times \mathbb{R}^m \ ; \ (x^0)^2 - |x|^2 = 1,\ x^0 > 0 \right\} ,$$

that is, M^m is the upper connected component of the m-dimensional two-shelled hyperboloid

$$K_1 := \left\{ x \in \mathbb{R}^{m+1} \ ; \ (Ax \,|\, x) = 1 \right\} , \quad \text{where} \quad A := \operatorname{diag}(1, -1, \ldots, -1)$$

(see Example 1.17(b)). Also let $i\colon M^m \hookrightarrow \mathbb{R}^{m+1}$ be the canonical embedding, and let

$$g_{H^m} := -i^*(\,\cdot\,|\,\cdot\,)_{1,m} \ .$$

Then

$$H^m := (M^m, g_{H^m})$$

is an m-dimensional Riemannian manifold, the m-dimensional **hyperbolic space**. If $N := (N, g)$ is isometrically isomorphic to H^m, we say N is a **model** of H^m. In particular, if we provide $\mathbb{R}^m$ with the metric

$$\frac{(dr)^2}{1 + r^2} + r^2 g_{S^{m-1}} \ ,$$

written in the “polar coordinates” $(r, \sigma) \in \mathbb{R}^+ \times S^{m-1}$, then $\mathbb{R}^m$ is a model of H^m.

Proof For $u\colon \mathbb{R}^m \to \mathbb{R}^{m+1}$, $x \mapsto \sqrt{1 + |x|^2}$, we have $M^m = \operatorname{graph}(u)$. Therefore

$$\varphi\colon M^m \to \mathbb{R}^m \ , \quad \big(h(x), x\big) \mapsto x$$

is a diffeomorphism from the hypersurface M^m in $\mathbb{R}^{m+1}$ to $\mathbb{R}^m$. Hence we have only to show that the bilinear form $g_{H^m}(0)$ induced on M by $-(\,\cdot\,|\,\cdot\,)_{1,m}$ is positive definite and that $\varphi_* g_{H^m}$ has the form indicated, because one could read off from this that $g_{H^m}(p)$ is positive definite for every $p \in M \backslash \{\varphi^{-1}(0)\}$.

With $h(x) := \big(u(x), x\big)$ for $x \in \mathbb{R}^m$, we have $h = i \circ \varphi^{-1}$ and

$$\partial_j h = (\partial_j u, e_j) \quad \text{for } 1 \le j \le m \ ,$$

where e_j is the j-th standard basis vector of $\mathbb{R}^m$. Because $\partial_j u(x) = x^j / u(x)$, it follows that

$$(\varphi_* g_{H^m})_{jk}(x) = (\partial_j h \,|\, \partial_k h)_{1,m}(x) = (\delta_{jk} - x^j x^k)/u^2(x) \quad \text{for } x \in \mathbb{R}^m \ ;$$

in particular $(\varphi_* g_{H^m})(0) = \sum_j (dx^j)^2$.

As in (g), let

$$f_m\colon (0, \infty) \times W_{m-1} \to \mathbb{R}^m \ , \quad (r, \vartheta) \mapsto r h_{m-1}(\vartheta)$$

be the m-dimensional polar coordinate map. Then $\psi := f_m^{-1} \circ \varphi$ is a local chart of M, and $a := i \circ \psi^{-1} = h \circ f_m = f_m^* h$ is the associated parametrization. This has

$$a(r, \vartheta) = \big(\sqrt{1 + r^2}, r h_{m-1}(\vartheta)\big) \ ,$$

and therefore

$$\partial_r a(r,\vartheta) = \Bigl(\frac{r}{\sqrt{1+r^2}}, h_{m-1}(\vartheta)\Bigr) , \quad \partial_{\vartheta^j} a(r,\vartheta) = \bigl(0, r\partial_j h_{m-1}(r,\vartheta)\bigr)$$

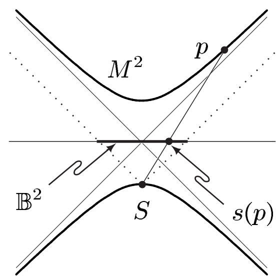
for $(r,\vartheta) \in (0,\infty) \times W_{m-1}$. Because $|h_{m-1}| = 1$, we derive

$$\begin{aligned}\psi_* g_{H^m} &= -a^*(\,\cdot\,|\,\cdot\,)_{1,m} \\ &= r^2 \sum\nolimits_{j,k} (\partial_j h_{m-1} \,|\, \partial_k h_{m-1})\, dx^j\, dx^k + \Bigl(1 - \frac{r^2}{\sqrt{1+r^2}}\Bigr)\,(dr)^2 .\end{aligned}$$

The claim now follows from this because of (h) and because the part still missing from $M^m \backslash \{\varphi^{-1}(0)\}$ can be analogously parametrized by rotating M^m around the x^0-axis. ■

(l) (the Poincaré model) In analogy to the stereographic projection of the sphere onto the plane, consider the stereographic projection of the **pseudosphere**

$$S^2_{1,3} := \{\, (t,x,y) \in \mathbb{R}^3 \ ; \ t^2 - x^2 - y^2 = 1 \,\} .$$



We set $N := (1,0,0)$, the **north pole** of $S^2_{1,3}$, and define the **south pole** as $S := (-1,0,0)$. Then the value $s(p)$ of the point $p \in M^2$ of the **stereographic projection** $s\colon M^2 \to \mathbb{R}^2$ is defined as the point where the line from S to p intersects the plane $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^3$. If the (Euclidean) coordinates of $p \in M^2$ are (t,x,y), and those of $s(p)$ are (u,v), we learn from the figure above that

$$\frac{x}{u} = \frac{t+1}{1} \quad \text{and} \quad \frac{y}{v} = \frac{t+1}{1} .$$

Because $t^2 - x^2 - y^2 = 1$, it follows that $t^2 - (u^2+v^2)(t+1)^2 = 1$. From this we calculate

$$t = \frac{1+u^2+v^2}{1-u^2-v^2} , \quad x = \frac{2u}{1-u^2-v^2} , \quad y = \frac{2v}{1-u^2-v^2} .$$

This shows that

$$\pi : \mathbb{B}^2 \to M^2 , \quad (u,v) \mapsto \Bigl(\frac{1+u^2+v^2}{1-u^2-v^2}, \frac{2u}{1-u^2-v^2}, \frac{2v}{1-u^2-v^2}\Bigr)$$

is a parametrization of M^2 over $\mathbb{B}^2$. It satisfies

$$\pi^* g_{H^2} = 4\, \frac{(du)^2 + (dv)^2}{(1-u^2-v^2)^2} .$$

Therefore

$$\Bigl(\mathbb{B}^2, 4\, \frac{(dx)^2 + (dy)^2}{(1-x^2-y^2)^2}\Bigr)$$

is a model of the hyperbolic plane, the **Poincaré model**.

Proof The proof that $\pi^* g_{H^2}$ has the form given is left to you as an exercise. ■

(m) (the Lobachevsky model) Following what we did in (h) and (j), we can parametrize M^2 by the **pseudospherical coordinates**

$$h_{1,2} : \mathbb{R}^+ \times [0, 2\pi) \to \mathbb{R}^3 \ , \quad (\chi, \varphi) \mapsto (t, x, y)$$

with

$$t = \cosh\chi \ , \quad x = \sinh\chi \cos\varphi \ , \quad y = \sinh\chi \sin\varphi \ .$$

These satisfy $h_{1,2}^* g_{H^3} = (d\chi)^2 + \sinh^2\chi\,(d\varphi)^2$. Therefore

$$\bigl(\mathbb{R}^+ \times [0, 2\pi), (d\chi)^2 + \sinh^2\chi\,(d\varphi)^2\bigr)$$

is a model of the hyperbolic plane H^2, the **Lobachevsky model**.

Proof The verification of the given formulas is again left to you as an exercise. ■

(n) (general pseudo-Riemannian metrics) Let X be open in $\overline{\mathbb{H}}^m$, and suppose $g_{jk} = g_{kj} \in \mathcal{E}(X)$ for $1 \le j,k \le m$ and $\det\bigl[g_{jk}(x)\bigr] \ne 0$ for $x \in X$. Then

$$g := \sum\nolimits_{j,k} g_{jk}\, dx^j\, dx^k$$

defines a pseudo-Riemannian metric on X. If the matrix $\bigl[g_{jk}(x)\bigr]$ is positive definite for every $x \in X$, then g is a Riemannian metric on X.

Now suppose $\bigl\{\, (\varphi_\alpha, U_\alpha) \ ; \ \alpha \in \mathsf{A} \,\bigr\}$ is an atlas for M,

$$g_{\alpha,jk} = g_{\alpha,kj} \in \mathcal{E}\bigl(\varphi_\alpha(U_\alpha)\bigr) \quad \text{for } 1 \le j,k \le m \ ,$$

and $\det\bigl[g_{\alpha,jk}(x)\bigr] \ne 0$ for $x \in \varphi_\alpha(U_\alpha)$ and $\alpha \in \mathsf{A}$. Then there is exactly one pseudo-Riemannian metric g on M such that

$$g \,|\, U_\alpha = g_\alpha := \sum\nolimits_{j,k} g_{\alpha,jk}\, dx^j\, dx^k$$

if the transition function $h := \varphi_\beta^{-1} \circ \varphi_\alpha$ satisfies

$$g_{\beta,rs} = \sum\nolimits_{j,k} \frac{\partial h^j}{\partial x^r}\, \frac{\partial h^k}{\partial x^s}\, g_{\alpha,jk}$$

for $\alpha, \beta \in \mathsf{A}$ with $U_\alpha \cap U_\beta \ne \emptyset$.

Proof This is a consequence of Remark 5.4(d). ■

A Riemannian manifold has a Euclidean structure on every tangent space, which allows lengths and angles to be measured. This allows many concepts from Euclidean geometry to be extended. For example, we presented in Section VIII.1

a formula for the length of a curve. It can now be naturally generalized: a curve $\gamma : I \to M$ on a Riemannian manifold M has length

$$\int_I \sqrt{g\big(\dot\gamma(t), \dot\gamma(t)\big)}\, dt \ ,$$

where $\dot\gamma(t) \in T_{\gamma(t)}M$ is the "velocity vector"

$$\dot\gamma(t) = (T_t\gamma)(t,1) \quad \text{for } t \in I$$

at the point $\gamma(t)$. We will not expand here on this subject, as the questions raised are best treated in the framework of *Riemannian geometry* (see however Exercise 5).

The Hodge star[6]

Suppose (M, g) is an Riemannian manifold and ω_M is the volume element of M.

For $0 \le r \le m$, we define bilinear maps

$$(\cdot\,|\,\cdot)_{g,r} : \Omega^r(M) \times \Omega^r(M) \to \mathcal{E}(M) \tag{5.16}$$

by

$$(\alpha\,|\,\beta)_{g,r}(p) := \big(\alpha(p)\;\big|\;\beta(p)\big)_{g(p),r} \quad \text{for } p \in M \text{ and } \alpha, \beta \in \Omega^r(M)\ , \tag{5.17}$$

where $(\cdot\,|\,\cdot)_{g(p),r}$ denotes the scalar product on $\bigwedge^r T_p^*M$ introduced in (2.14) and (2.15). The **Hodge star operator** (or simply Hodge star)

$$* : \Omega^r(M) \to \Omega^{m-r}(M)\ , \quad \alpha \mapsto *\alpha \tag{5.18}$$

is also defined pointwise:

$$(*\alpha)(p) := *\alpha(p) \quad \text{for } p \in M \text{ and } \alpha \in \Omega(M)\ .$$

5.6 Remarks (a) The map (5.16) is well defined, bilinear, symmetric, and positive.

Proof We need only show that $(\alpha\,|\,\beta)_{g,r}$ belongs to $\mathcal{E}(M)$ for $\alpha, \beta \in \Omega^r(M)$, because the other statements follow from the properties of $(\cdot\,|\,\cdot)_{g(p),r}$. Suppose therefore (φ, U) is a positive chart of M. According to Remark 5.4(f), we can choose an oriented orthonormal frame $(v_1, \ldots, v_m)$ on U. Let $(\eta^1, \ldots, \eta^m)$ be its dual frame. Then Remark 3.1(e) implies

$$\alpha\,|\,U = \sum_{(j)\in\mathbb{J}_r} \alpha_{(j)}\eta^{j_1} \wedge \cdots \wedge \eta^{j_r} \tag{5.19}$$

with

$$\alpha_{(j)} = \alpha(v_{j_1}, \ldots, v_{j_r}) \in \mathcal{E}(U) \quad \text{for } (j) \in \mathbb{J}_r\ . \tag{5.20}$$

Now the claim follows from (2.14), (2.15), and Remark 2.15(c). ■

[6] The rest of this chapter can be skipped on first reading.

(b) The star operator is a well-defined $\mathcal{E}(M)$-module isomorphism with

$$**\alpha = (-1)^{r(m-r)}\alpha \quad \text{for } \alpha \in \Omega^r(M) \ . \tag{5.21}$$

Proof Because (5.21) follows from Example 2.17(e) and the pointwise definition (5.18), and because (5.21) also shows that the star operator is bijective, it only remains to show that $*\alpha$ is smooth. So let (φ, U) be a positive chart of M. As in the proof of (a), let $(v_1, \ldots, v_m)$ be an orthonormal frame on U, and let $(\eta^1, \ldots, \eta^m)$ be its dual frame. Then $*\alpha \,|\, U \in \mathcal{E}(U)$ follows from (5.19), (5.20), and the explicit representation of $*\alpha$ in Example 2.17(d). ■

(c) For $\alpha, \beta \in \Omega^r(M)$, we have

$$\alpha \wedge *\beta = \beta \wedge *\alpha = (\alpha \,|\, \beta)_{g,r}\omega_M \ . \tag{5.22}$$

Proof This follows immediately from Example 2.17(f) and the pointwise definition of all operations involved. ■

(d) $*1 = \omega_M$ and $*\omega_M = 1$.

(e) (regularity) It is clear that the statements above are still true when M is a C^{k+1} manifold and $\Omega(M)$ is replaced by $\Omega_{(k)}(M)$. ■

Using the pointwise definition of the star operator, we can transfer the other formulas of Example 2.17 to the present case. The following examples gather several rules so obtained.

Let $\big((x^1, \ldots, x^m), U\big)$ be a chart of M. When $(\partial_1, \ldots, \partial_m)$ is an orthonormal frame on U, we say $(x^1, \ldots, x^m)$ are **orthonormal coordinates** on U. If the ∂_j are not necessarily normalized, that is, we only know $g(\partial_j, \partial_k) = 0$ for $j \neq k$, then the coordinates are **orthogonal**.

5.7 Examples In these examples, $(x^1, \ldots, x^m)$ are orthonormal coordinates on $U \subset M$, and $\alpha \in \Omega(U)$.

(a) Euclidean coordinates are orthonormal coordinates. Polar, spherical, and pseudospherical coordinates are orthogonal.

Proof This follows from Examples 5.5. ■

(b) $*\sum_j a_j\, dx^j = \sum_{j=1}^m (-1)^{j-1} a_j\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m$.

(c) $*\sum_j (-1)^{j-1} a_j\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m = (-1)^{m-1} \sum_j a_j\, dx^j$.

(d) For $m = 3$, we have

$$*d\Big(\sum\nolimits_j a_j\, dx^j\Big) = (\partial_2 a_3 - \partial_3 a_2)\, dx^1 + (\partial_3 a_1 - \partial_1 a_3)\, dx^2 + (\partial_1 a_2 - \partial_2 a_1)\, dx^3 \ .$$

Proof This follows from Example 3.7(a) (and the remarks following (4.16)), because (2.20) implies the relations

$$*(dx^2 \wedge dx^3) = dx^1 \ , \quad *(dx^3 \wedge dx^1) = dx^2 \ , \quad *(dx^1 \wedge dx^2) = dx^3 \ . \ \blacksquare$$

Of course, we can also explicitly calculate $* \sum_{(j)\in\mathbb{J}_r} a_{(j)}\, dx^{(j)}$ even if we are not using orthonormal coordinates. For simplicity, we only consider the case of 1-forms.

5.8 Proposition *Let $\big((x^1,\ldots,x^m),U\big)$ be a positive chart of M. Then*

$$*dx^j = \sum\nolimits_k (-1)^{k-1} g^{jk} \sqrt{G}\, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^m \ .$$

Proof Because $*dx^j \in \Omega^{m-1}(U)$, Example 3.2(b) guarantees that there are $a_{j\ell}$ in $\mathcal{E}(U)$ such that

$$*dx^j = \sum\nolimits_\ell (-1)^{\ell-1} a_{j\ell}\, dx^1 \wedge \cdots \wedge \widehat{dx^\ell} \wedge \cdots \wedge dx^m \ .$$

This gives

$$\begin{aligned} dx^k \wedge *dx^j &= \sum\nolimits_\ell (-1)^{\ell-1} a_{j\ell}\, dx^k \wedge dx^1 \wedge \cdots \wedge \widehat{dx^\ell} \wedge \cdots \wedge dx^m \\ &= a_{jk}\, dx^1 \wedge \cdots \wedge dx^m \ . \end{aligned} \tag{5.23}$$

From Remark 2.14(b) we get $(dx^j \mid dx^k)_{g,1} = g^{jk}$. Thus Remark 5.6(c) gives

$$dx^k \wedge *dx^j = g^{kj} \omega_M = g^{jk} \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^m \ , \tag{5.24}$$

where the last equality follows from Remark 5.4(c). Now the claim follows from (5.23) and (5.24). ■

The codifferential

Let (M,g) be an oriented Riemannian manifold. To avoid an exceptional case, we set $\Omega^{-1}(M) := \{0\}$ so that, because $\Omega^{m+1}(M) = \{0\}$, we can also define the star operation $* : \Omega^{m+1}(M) \to \Omega^{-1}(M)$. With help of the (so extended) star operator and the exterior derivative, we define for $0 \le r \le m$ the **codifferential**

$$\delta : \Omega^r(M) \to \Omega^{r-1}(M)$$

by[7]

$$\delta\alpha := (-1)^{m(r+1)} {*d*}\alpha \quad \text{for } \alpha \in \Omega^r(M) \ .$$

[7]The normalization factor $(-1)^{m(r+1)+1}$ is often used instead of $(-1)^{m(r+1)}$, particularly in differential geometry. The reason for our choice will be made clear in Remark 6.23(c).

In other words, we require that the diagram

$$\begin{array}{ccc} \Omega^r(M) & \xrightarrow{\ \ *\ \ } & \Omega^{m-r}(M) \\ {\scriptstyle (-1)^{m(r+1)}\delta}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ \Omega^{r-1}(M) & \xleftarrow{\ \ *\ \ } & \Omega^{m-r+1}(M) \end{array}$$

commutes.

The following remarks list several properties of the codifferential.

5.9 Remarks **(a)** $\delta^2 = 0$.

Proof Because $**\alpha = (-1)^{r(m-r)}\alpha$, we have $\delta\delta\alpha = \pm *d**d*\alpha = \pm *d^2*\alpha = 0$ because $d^2 = 0$. ■

(b) $*\delta d = d\delta*$ and $*d\delta = \delta d*$.

Proof If $\alpha \in \Omega^r(M)$, then $d\alpha$ belongs to $\Omega^{r+1}(M)$. Therefore

$$*\delta d\alpha = (-1)^{m(r+2)}**d*d\alpha = (-1)^{mr}**d*d\alpha \ .$$

Because $d*d\alpha \in \Omega^{m-r}(M)$, we thus find $*\delta d\alpha = (-1)^{-r^2}d*d\alpha$. Analogously,

$$d\delta*\alpha = (-1)^{m(m-r+1)}d*d**\alpha = (-1)^{m(m+1)-r^2}d*d\alpha \ .$$

Because $m(m+1)$ is even, this proves the first claim. The second follows analogously. ■

(c) $d*\delta = \delta*d = 0$.

Proof We leave the simple proof to you. ■

(d) $*\delta\alpha = (-1)^{r+1}d*\alpha$ and $\delta(*\alpha) = (-1)^r*d\alpha$ for $\alpha \in \Omega^r(M)$.

Proof The first statement follows from

$$*\delta\alpha = (-1)^{m(r+1)}**d*\alpha = (-1)^{mr+m}(-1)^{(m-r+1)(r-1)}d*\alpha = (-1)^{r+1}d*\alpha \ .$$

The second follows from an analogous calculation. ■

(e) (regularity) From the definition of δ and Remarks 4.11(b) and 5.6(e), it follows immediately that δ is an $\mathbb{R}$-linear map from $\Omega^r_{(k)}$ to $\Omega^{r-1}_{(k-1)}$ for $1 \le r \le m$ and $k \in \mathbb{N}^\times$. This remains true for C^{k+1} manifolds. ■

5.10 Examples Let $\big((x^1,\dots,x^m),U\big)$ be a positive chart on M.

(a) For $\alpha = \sum_j a_j\,dx^j \in \Omega^1(U)$, we have

$$\delta\alpha = \frac{1}{\sqrt{G}}\sum\nolimits_{j,k}\frac{\partial}{\partial x^j}\big(g^{jk}a_k\sqrt{G}\big) \in \mathcal{E}(U) \ .$$

Proof It follows from Proposition 5.8 that

$$*\alpha = \sum\nolimits_j a_j \sum\nolimits_k (-1)^{k-1}g^{jk}\sqrt{G}\,dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^m \ .$$

From this we derive (because $r=1$) that

$$\begin{aligned}\delta\alpha = {*d*}\alpha &= *\sum_j\sum_k\sum_\ell(-1)^{k-1}\frac{\partial}{\partial x^\ell}\big(a_j g^{jk}\sqrt{G}\big)\,dx^\ell\wedge dx^1\wedge\cdots\wedge\widehat{dx^k}\wedge\cdots\wedge dx^m\\ &= *\sum_{j,k}\frac{\partial}{\partial x^j}\big(g^{jk}a_k\sqrt{G}\big)\,dx^1\wedge\cdots\wedge dx^m\ .\end{aligned}$$

The claim now follows from

$$dx^1\wedge\cdots\wedge dx^m = \frac{1}{\sqrt{G}}\,\omega_M \tag{5.25}$$

and from Remark 5.6(d). ■

(b) For orthonormal coordinates $(x^1,\ldots,x^m)$, it follows from (a) that

$$\delta\Big(\sum_j a_j\,dx^j\Big) = \sum_j \partial_j a_j\ .$$

(c) $\delta a = 0$ for $a\in\mathcal{E}(M)$.

(d) $\delta(a\,dx^1\wedge\cdots\wedge dx^m)$

$$=\sum_{j,k}(-1)^{k-1}\frac{\partial}{\partial x^j}\Big(\frac{a}{\sqrt{G}}\Big)g^{jk}\sqrt{G}\,dx^1\wedge\cdots\wedge\widehat{dx^k}\wedge\cdots\wedge dx^m\ .$$

Proof Using (5.25) and Remark 5.6(d), we get

$$*(a\,dx^1\wedge\cdots\wedge dx^m) = a/\sqrt{G}\ .$$

Therefore

$${*d*}(a\,dx^1\wedge\cdots\wedge dx^m) = *\sum_j\frac{\partial}{\partial x^j}\Big(\frac{a}{\sqrt{G}}\Big)\,dx^j\ .$$

Now the claim follows from Proposition 5.8. ■

(e) With orthonormal coordinates, we have

$$\delta\sum_{(j)\in\mathbb{J}_r} a_{(j)}\,dx^{(j)} = \sum_{(j)\in\mathbb{J}_r}\sum_{k=1}^r(-1)^{k-1}\partial_{j_k}a_{(j)}\,dx^1\wedge\cdots\wedge\widehat{dx^{j_k}}\wedge\cdots\wedge dx^{j_r}\ .$$

Proof Because of the linearity, it suffices to consider $\alpha = a\,dx^{(j)}$ with $(j)\in\mathbb{J}_r$. From (2.20) and Theorem 4.10(ii), we obtain

$$d{*}\alpha = s(j)\,da\wedge dx^{(j^c)} = s(j)\sum_{k=1}^r\partial_{j_k}a\,dx^{j_k}\wedge dx^{(j^c)}\ .$$

Therefore Example 2.17(d) implies

$${*d*}\alpha = s(j)\sum_{k=1}^r s\big(j_k,(j^c)\big)\partial_{j_k}a\,dx^{j_1}\wedge\cdots\wedge\widehat{dx^{j_k}}\wedge\cdots\wedge dx^{j_r}$$

with $s\big(j_k,(j^c)\big) := \operatorname{sign}\big(j_k,(j^c),j_1,\ldots,\widehat{j_k},\ldots,j_r\big)$. Because (j^c) consists of $m-r$ elements, it follows that

$$\begin{aligned} s\big(j_k,(j^c)\big) &= (-1)^{(m-r)(r-1)} \operatorname{sign}\big(j_k,j_1,\ldots,\widehat{j_k},\ldots,j_r,(j^c)\big) \\ &= (-1)^{(m-r)(r-1)+k-1} s(j) \ . \end{aligned}$$

Due to the (mod 2) congruences

$$(m-r)(r-1)+k-1+m(r+1) \equiv k-r(r+1)-1 \equiv k-1 \ ,$$

the claim then follows from the definition of δ. ■

5.11 Remarks **(a)** By making appropriate modifications, the above properties of the star operator and the codifferential can be extended to the case of pseudo-Riemannian manifolds.

More precisely, suppose (M,g) is an oriented pseudo-Riemannian manifold. We can provide T_pM with the inner product induced by that of $\mathbb{R}^{\overline{m}}$. By Remark 2.18(a), it follows that the representation matrix g of $g(p)$ at every $p\in M$ is diagonal in an appropriately chosen basis, and its diagonal entries are ± 1. Now $(-1)^s = \operatorname{sign} g(p)$ is uniquely determined by $g(p)$, where s denotes the number of negative elements. *We now assume that* $\operatorname{sign}(g) = \operatorname{sign} g(p)$ *is constant on* M, that is, it is independent of p. From (the proof of) Remark 5.4(f), it follows that this assumption is satisfied if M can be described by a single chart.

Under this assumption the star operator, as defined through (2.25), can also be defined pointwise. Then (5.21) and (5.22) must be replaced by

$$**\alpha = \operatorname{sign}(g)(-1)^{r(m-r)}\alpha \quad \text{for } \alpha\in\Omega^r(M) \ ,$$

and

$$\alpha\wedge *\beta = \beta\wedge *\alpha = \operatorname{sign}(g)(\alpha\,|\,\beta)_{g,r}\omega_M \quad \text{for } \alpha,\beta\in\Omega^r(M) \ ,$$

as we learn from Remarks 2.19(d) and (e), respectively. Here ω_M is the volume element of M defined in Remark 5.4(g). Remark 2.19(c) also implies

$$*1 = \operatorname{sign}(g)\omega_M \text{ for } *\omega_M = 1 \ .$$

The codifferential is defined in this case by

$$\delta\alpha := \operatorname{sign}(g)(-1)^{m(r+1)} *d*\alpha \quad \text{for } \alpha\in\Omega^r(M) \ . \tag{5.26}$$

We verify easily that with these modifications, the statements of Remark 5.6 hold as written.

Proof The claims follows from Remarks 2.19. ■

(b) Let $\big((x^1,\ldots,x^m),U\big)$ be a positive chart of M. Then

$$\delta \sum\nolimits_j a_j\,dx^j = \frac{1}{\sqrt{|G|}} \sum\nolimits_{j,k} \frac{\partial}{\partial x^j}\big(g^{jk}a_k\sqrt{|G|}\,\big) \in \mathcal{E}(U) \ .$$

Proof This follows, in analogy to the proof of Example 5.10(a), from Remark 5.4(g). ■

5.12 Examples We consider now the Minkowski space $\mathbb{R}^4_{1,3}$ with the metric $(\cdot\,|\,\cdot)_{1,3}$ and therefore with $g := (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$.

(a) If (i,j,k) is a cyclic permutation of $(1,2,3)$, then with $(x^1,x^2,x^3) := (x,y,z)$, we have

$$*(dx^i \wedge dt) = dx^j \wedge dx^k \quad \text{and} \quad *(dx^i \wedge dx^j) = -dx^k \wedge dt \ .$$

Proof Let (e_0,e_1,e_2,e_3) be the canonical basis of $\mathbb{R}^4_{1,3}$. Then

$$g(e_0,e_0) = 1 \quad \text{and} \quad g(e_j,e_j) = -1 \quad \text{for } 1 \le j \le 3 \ .$$

This implies

$$(dt\,|\,dt)_{g,1} = 1 \quad \text{and} \quad (dx^j\,|\,dx^j)_{g,1} = -1 \ .$$

Therefore

$$(dt \wedge dx^j\,|\,dt \wedge dx^j)_{g,2} = -1 \quad \text{and} \quad (dx^j \wedge dx^k\,|\,dx^j \wedge dx^k)_{g,2} = 1 \quad \text{for } 1 \le j < k \le 3 \ .$$

Now the claim follows from Remark 2.19(c). ■

(b) Let $E_j, H_j \in \mathcal{E}(\mathbb{R}^4_{1,3})$ and

$$\begin{aligned}\alpha := {}&(E_1 dx^1 + E_2 dx^2 + E_3 dx^3) \wedge dt \\ &+ H_1 dx^2 \wedge dx^3 + H_2 dx^3 \wedge dx^1 + H_3 dx^1 \wedge dx^2 \ .\end{aligned}$$

Then

$$\begin{aligned}*\alpha = {}&-(H_1 dx^1 + H_2 dx^2 + H_3 dx^3) \wedge dt \\ &+ E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2 \ .\end{aligned}$$

Proof This is an immediate consequence of (a) and the $\mathcal{E}(\mathbb{R}^4_{1,3})$-linearity of the star operator. ■

(c) The α from (b) satisfies

$$\delta\alpha = \sum_{j=1}^{3} \frac{\partial E_j}{\partial t}\,dx^j - \sum_{k=1}^{3} \frac{\partial E_k}{\partial x^k}\,dt + \sum_{(i,j,k)} \Bigl(\frac{\partial H_i}{\partial x^j} - \frac{\partial H_j}{\partial x^i}\Bigr)\,dx^k \ ,$$

where the last term is summed over all cyclic permutations of $(1,2,3)$.

Proof From (b), we know that

$$*\alpha = -\sum_{i=1}^{3} H_i\,dx^i \wedge dt + \sum_{(i,j,k)} E_i\,dx^j \wedge dx^k \ .$$

This implies

$$\begin{aligned}d{*}\alpha = {}&-\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\ne i}}^{3} \frac{\partial H_i}{\partial x^j}\,dx^j \wedge dx^i \wedge dt \\ &+ \sum_{(i,j,k)} \Bigl(\frac{\partial E_i}{\partial t}\,dt \wedge dx^j \wedge dx^k + \frac{\partial E_i}{\partial x^i}\,dx^i \wedge dx^j \wedge dx^k\Bigr) \ .\end{aligned}$$

From Remark 2.19(c), we derive

$$*(dt \wedge dx^i \wedge dx^j) = -dx^k \quad \text{and} \quad *(dx^i \wedge dx^j \wedge dx^k) = dt \ .$$

With this we get

$$*d*\alpha = -\sum_{(i,j,k)} \Bigl(\frac{\partial H_i}{\partial x^j} - \frac{\partial H_j}{\partial x^i}\Bigr)\, dx^k - \sum_{i=1}^3 \frac{\partial E_i}{\partial t}\, dx^i + \sum_{k=1}^3 \frac{\partial E_k}{\partial x^k}\, dt \ .$$

Now the claim follows because $m = 4$ and $\operatorname{sign}(g) = -1$. ■

Exercises

1 Let (M_j, g_j) for $j = 1, 2$ be pseudo-Riemannian manifolds with $\partial M_1 = \emptyset$, and denote by $\pi_j : M_1 \times M_2 \to M_j$ the canonical projection onto M_j. Prove these statements:

(i) $(M_1 \times M_2, \pi_1^* g_1 + \pi_2^* g_2)$ is a Riemannian manifold, the **product** of M_1 and M_2.

(ii) Two points (p_1, p_2) yield submanifolds $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ of $M_1 \times M_2$.

(iii) $T_{(p_1,p_2)}(M_1 \times M_2) = T_{(p_1,p_2)}\bigl(M_1 \times \{p_2\}\bigr) \oplus T_{(p_1,p_2)}\bigl(\{p_1\} \times M_2\bigr)$.

(iv) $\omega_{M_1 \times M_2} = \pi_1^* \omega_{M_1} \wedge \pi_2^* \omega_{M_2}$.

2 Let M be an oriented hypersurface in $\mathbb{R}^{m+1}$. We call $\nu : M \to T\mathbb{R}^{m+1}$ a **positive unit normal field** when ν is a unit normal of M such that, for every $p \in M$ and every positive basis $(v_1, \ldots, v_m)$ of T_pM, the $(m+1)$-tuple $\bigl(\nu(p), v_1, \ldots, v_m\bigr)$ is a positive basis of $T_p\mathbb{R}^{m+1}$.

(a) Show that ν is well defined and unique.

(b) Determine the unit normal on these surfaces in $\mathbb{R}^3$:

(i) graph f, with X open in $\mathbb{R}^2$ and $f \in \mathcal{E}(X)$, (ii) $\mathbb{R} \times S^1$, (iii) S^2 , (iv) $\mathsf{T}^2_{a,r}$.

(Hint: (iv) Exercise VII.10.10 and Example VII.9.11(f).)

3 Let M be an oriented hypersurface in $\mathbb{R}^{m+1}$, provided with the standard metric. Denote by ν the positive unit normal of M. Show these facts:

(i) ν defines a smooth map from M to S^m, the **Gauss map** (which is also denoted by ν).

(ii) For $p \in M$ and $v \in T_pM$, we have $\bigl((T_p\nu)v \,|\, \nu(p)\bigr)_{\mathbb{R}^{m+1}} = 0$. Therefore $(T_p\nu)v$ belongs to T_pM.

(iii) The map

$$L : M \to \bigcup_{p \in M} \mathcal{L}(T_pM) \in \mathcal{L}(T_pM) \ , \quad p \mapsto T_p\nu$$

is well defined. This is called the **Weingarten map** of M.

(iv) For $p \in M$ and $v, w \in T_pM$, we have

$$g(p)\bigl(L(p)v, w\bigr) = g(p)\bigl(v, L(p)w\bigr) \ ,$$

that is, $L(p)$ is symmetric on the inner product space $\bigl(T_pM, g(p)\bigr)$. The tensor $h \in \mathcal{T}_2^0(M)$ defined by

$$h(p)(v, w) := g(p)\bigl(L(p)v, w\bigr) \quad \text{for } p \in M \text{ and } v, w \in T_pM$$

is called the **second fundamental tensor** of M.

(v) In local coordinates (U,φ), with the natural embedding $i\colon M \hookrightarrow \mathbb{R}^{m+1}$, and with $f := i \circ \varphi^{-1}$, we have

$$h_{jk} = (\partial_j \nu \,|\, \partial_k f) = -(\nu \,|\, \partial_j \partial_k f) \ ,$$

where $h_{jk} := h\big(\partial/\partial x^j, \partial/\partial x^k\big)$.

4 Calculate the second fundamental forms of $\mathbb{R}^2$, S^2, $\mathbb{R}\times S^1$, and $\mathsf{T}^2_{a,r}$ as submanifolds of $\mathbb{R}^3$.

5 Suppose I is a compact interval in $\mathbb{R}$ and M is a Riemannian manifold. Also let $\gamma \in C^1(I,M)$. Let $i\colon M \hookrightarrow \mathbb{R}^{\overline{m}}$ be the natural embedding, and put $\widetilde{\gamma} := i \circ \gamma$. Then the length $L(\widetilde{\gamma})$ of $\widetilde{\gamma}$ is defined as in Section VIII.1. Show that if $\dot\gamma(t) := (T_t\gamma)(t,1)$ for $t \in I$, then

$$L(\widetilde{\gamma}) = \int_I \sqrt{g\big(\dot\gamma(t), \dot\gamma(t)\big)}\, dt \ .$$

When $L(\widetilde{\gamma}) = L(I)$, we say γ is **parametrized by arc length**.

6 Suppose M is an oriented surface in $\mathbb{R}^3$ and $\gamma \in C^2(I,M)$ is parametrized by arc length. Also denote by ν the positive unit normal bundle of M. Then we call

$$\kappa_g(\gamma) := \det[\dot\gamma, \ddot\gamma, \nu]$$

the **curvature of γ in M** or the **geodesic curvature** of γ.

(a) Verify in the Euclidean case $M = \mathbb{R}^2$ that the geodesic curvature is the same as the (usual) curvature from Section VIII.2.

(b) Suppose $M = S^2$ and (x,y,z) are the Euclidean coordinates in $\mathbb{R}^3$. Also let γ_z for $z \in (-1,1)$ be a parametrization by arc length of $L_z := \sqrt{1-|z|^2}\, S^1 \times \{z\}$ (see Example 1.5(a)). Show then that

$$\kappa_g(\gamma_z) = \frac{z}{\sqrt{1-|z|^2}} \ .$$

Therefore the geodesic curvature is constant on the circle L_z and vanishes at the equator.

7 Prove the equality $\rho^*\omega_{S^m} = r^{-(m+1)}\alpha$ from Example 4.13(c) by direct calculation for $m = 2$ and 3.

8 Prove the statements made in the proof of Example 5.5(b).

9 Show that

$$\{\, z \in \mathbb{C} \ ;\ \operatorname{Im} z > 0 \,\} \to \mathbb{D} \ , \quad z \mapsto (1 + i\,z)/(1 - i\,z)$$

gives a diffeomorphism from the "upper half of complex half plane" to the unit disc. Then use this map to show that

$$\Big(H^2, \frac{(dx)^2 + (dy)^2}{y^2}\Big)$$

is a model of the hyperbolic plane, the **Klein model**.

10 Show that the Lobachevsky plane from Example 5.5(m) is a model of H^2.

11 For $\alpha \in \Omega^{r-1}(M)$ and $\beta \in \Omega^r(M)$, show

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + \alpha \wedge *\delta\beta \ .$$

12 Show that the codifferential does not depend on the orientation of the underlying Riemannian manifold.

13 Suppose M is oriented, $(N, \bar{g})$ is another oriented m-dimensional Riemannian manifold, and $f : M \to N$ is an isometric diffeomorphism. Show that $f^*\omega_N = \pm\omega_M$. Also show that $f^*\omega_N = \omega_M$ if and only if f is orientation-preserving.

14 Suppose M and N are as in Exercise 13 and $f : M \to N$ is an orientation-preserving isometric diffeomorphism. Show that the diagram

$$\begin{array}{ccc} \Omega^r(M) & \xrightarrow{\ *\ } & \Omega^{m-r}(M) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \Omega^r(N) & \xrightarrow{\ *\ } & \Omega^{m-r}(N) \end{array}$$

is commutative for $0 \le r \le m$.

15 Suppose M and N are as in Exercise 13, and $f : M \to N$ is an isometric diffeomorphism.[8] Show that the diagram

$$\begin{array}{ccc} \Omega^r(M) & \xrightarrow{\ \delta\ } & \Omega^{r-1}(M) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \Omega^r(N) & \xrightarrow{\ \delta\ } & \Omega^{r-1}(N) \end{array}$$

commutes for $0 \le r \le m$.

---

[8]Note Exercise 12.

6 Vector analysis

Vector fields and Pfaff forms can be interchanged using the Riesz isomorphism. While vector fields have an immediate geometrical interpretation, the calculus of differential forms is of great value in calculations. The exterior product and derivative obey relatively simple rules, which themselves stand for a more complicated set of prescriptions for how to change from one system of local coordinates to another. In this section, we will use the Riesz isomorphism to translate some of the concepts and theorems of differential forms into the language of classical vector analysis. In so doing, we will learn about the divergence and curl of vector fields, which are of fundamental significance in physics and the theory of partial differential equations.

For the entire section suppose the following:

- M is an m-dimensional manifold; N is an n-dimensional manifold.
- The indices i, j, k, ℓ always range from 1 to m unless stated otherwise, and $\sum_j$ means that j is summed from 1 to m.

The Riesz isomorphism

Let g be a pseudo-Riemannian metric on M. Then we define the **Riesz isomorphism**, Θ_g, by

$$\Theta_g : \mathcal{V}(M) \to \Omega^1(M) \ , \quad v \mapsto \Theta_g v \tag{6.1}$$

and

$$(\Theta_g v)(p) := \Theta_{g(p)} v(p) \quad \text{for } p \in M \ ,$$

where $\Theta_{g(p)} : T_pM \to T_p^*M$ is the Riesz isomorphism of (2.12) (or Remark 2.18(b)) and is defined by

$$\langle \Theta_{g(p)} u, w \rangle = g(p)(u, w) \quad \text{for } u, w \in T_pM \ .$$

When no confusion is expected, we may write Θ instead of Θ_g.

6.1 Remarks **(a)** The map (6.1) is well defined.

Proof We must show that Θv belongs to $\Omega^1(M)$ for $v \in \mathcal{V}(M)$. In local coordinates, we have

$$v \,|\, U = \sum_j v^j \frac{\partial}{\partial x^j}$$

with $v^j \in \mathcal{E}(U)$. From this and from Remarks 2.14(a) and 2.18(b), it follows that

$$\begin{aligned}\Theta v(p) &= \Theta_{g(p)} \sum_j v^j(p) \frac{\partial}{\partial x^j}\Big|_p = \sum_j v^j(p) \Theta_{g(p)} \frac{\partial}{\partial x^j}\Big|_p \\ &= \sum_j v^j(p) \sum_k g_{jk}(p)\, dx^k(p) = \sum_j a_j(p)\, dx^j(p) \ ,\end{aligned}$$

where

$$a_k := \sum\nolimits_j g_{kj} v^j \in \mathcal{E}(U) \ .$$

Now the claim follows from Remarks 4.5(c) and 5.4(e). ■

(b) In local coordinates,

$$\Theta\Bigl(\sum\nolimits_j v^j \frac{\partial}{\partial x^j}\Bigr) = \sum\nolimits_j a_j \, dx^j \qquad \text{with} \quad a_j := \sum\nolimits_k g_{jk} v^k \ . \tag{6.2}$$

Instead of Θv, we often write $v^\flat$ or $g^\flat v$, because, as seen in (6.2), Θ effects a "lowering of indices" (see Remark 2.14(d)).

Proof This was shown in the proof of (a). ■

(c) The map $\Theta\colon \mathcal{V}(M) \to \Omega^1(M)$ is an $\mathcal{E}(M)$-module isomorphism.

Proof Let $\alpha \in \Omega^1(M)$. Then $\alpha(p) \in T_p^*M$ for $p \in M$. From Section 2, we know that $\Theta_{g(p)}$ is a vector space isomorphism. Therefore $\Theta_{g(p)}^{-1}\alpha(p) \in T_pM$ is well defined. We set

$$(\overline{\Theta}_g \alpha)(p) := \Theta_{g(p)}^{-1}\alpha(p) \quad \text{for } p \in M \text{ and } \alpha \in \Omega^1(M) \ .$$

In local coordinates, we know from Remarks 2.14(a) and 2.18(b) that

$$\begin{aligned}\overline{\Theta}_g \alpha(p) &= \Theta_{g(p)}^{-1} \sum\nolimits_j a_j(p)\, dx^j(p) = \sum\nolimits_j a_j(p) \Theta_{g(p)}^{-1}\, dx^j(p) \\ &= \sum\nolimits_j a_j(p) \sum\nolimits_k g^{jk}(p) \frac{\partial}{\partial x^k}\Big|_p = \sum\nolimits_j v^j(p) \frac{\partial}{\partial x^j}\Big|_p \ ,\end{aligned}$$

where

$$v^j := \sum\nolimits_k g^{jk} a_k \in \mathcal{E}(U) \ .$$

Thus it follows from Remark 4.3(c) that $\overline{\Theta}_g \alpha$ belongs to $\mathcal{V}(M)$. From the definitions of Θ_g and $\overline{\Theta}_g$, it follows immediately that $\Theta_g \overline{\Theta}_g = \mathrm{id}_{\Omega^1(M)}$ and $\overline{\Theta}_g \Theta_g = \mathrm{id}_{\mathcal{V}(M)}$. Therefore Θ_g is bijective, and $\Theta_g^{-1} = \overline{\Theta}_g$.

Finally we see that for $a \in \mathcal{E}(M)$ and $v \in \mathcal{V}(M)$, we have

$$\Theta_g(av)(p) = \Theta_{g(p)} a(p) v(p) = a(p) \Theta_{g(p)} v(p) = (a \Theta_g v)(p) \quad \text{for } p \in M \ .$$

Therefore Θ is an $\mathcal{E}(M)$-Module isomorphism. ■

(d) In local coordinates,

$$\Theta^{-1}\Bigl(\sum\nolimits_j a_j \, dx^j\Bigr) = \sum\nolimits_j v^j \frac{\partial}{\partial x^j} \qquad \text{with} \quad v^j := \sum\nolimits_k g^{jk} a_k \ . \tag{6.3}$$

Instead of $\Theta^{-1}\alpha$, we often write $\alpha^\sharp$ or $g^\sharp \alpha$, because Θ^{-1} "raises indices".

Proof This was shown in the proof of (c). ■

(e) (orthogonal coordinates) If $(x^1, \ldots, x^m)$ are orthogonal coordinates, that is, if

$$g\Bigl(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Bigr) = 0 \quad \text{for } j \neq k \ ,$$

then (6.2) and (6.3) simplify respectively to

$$\Theta v = \sum\nolimits_j g_{jj} v^j \, dx^j \quad \text{and} \quad \Theta^{-1}\alpha = \sum\nolimits_j g^{jj} a_j \frac{\partial}{\partial x^j}$$

for $v = \sum_j v^j \, \partial/\partial x^j$ and $\alpha = \sum_j a_j \, dx^j$.

(f) Let $(N, \bar{g})$ be a pseudo-Riemannian manifold, and let $\varphi \in \mathrm{Diff}(M, N)$ with $\varphi_* g = \lambda \bar{g}$ for some $\lambda \neq 0$. Then the diagram

$$\begin{array}{ccc} \mathcal{V}(M) & \xleftarrow[\cong]{\varphi^*} & \mathcal{V}(N) \\ {\scriptstyle \Theta_M} \Big\downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \Big\downarrow {\scriptstyle \lambda\Theta_N} \\ \Omega^1(M) & \xleftarrow[\cong]{\varphi^*} & \Omega^1(N) \end{array}$$

commutes. Therefore $\Theta_M \varphi^* = \lambda \varphi^* \Theta_N$.

Proof Using the definition and properties of the push forward and the pull back of vector fields and forms (see in particular (4.25)), we find for $v, w \in \mathcal{V}(N)$ that

$$\begin{aligned} \lambda \bar{g}(v, w) &= \varphi_* g(v, w) = g(\varphi^* v, \varphi^* w) = \langle \Theta_M \varphi^* v, \varphi^* w \rangle_M = \langle \varphi_* \Theta_M \varphi^* v, w \rangle_N \\ &= \bar{g}(\Theta_N^{-1} \varphi_* \Theta_M \varphi^* v, w) \ . \end{aligned}$$

Because $\bar{g}$ is nondegenerate and $\mathbb{R}$-linear, it follows that

$$\lambda v = \Theta_N^{-1} \varphi_* \Theta_M \varphi^* v \quad \text{for } v \in \mathcal{V}(M) \ ,$$

which proves the claim. ■

(g) (regularity) Suppose $k \in \mathbb{N}$ and M is a C^{k+1} manifold. Then the definitions and statements above remain true when smooth vector fields, differential forms, and functions are replaced by C^k vector fields, C^k differential forms, and C^k functions. ■

6.2 Examples **(a)** (Euclidean coordinates) Let M be open in $\mathbb{R}^m$. We denote Euclidean coordinates by $(x^1, \ldots, x^m)$, that is, $(\cdot\,|\,\cdot) = \sum_j (dx^j)^2$. Then

$$\Theta\Bigl(\sum\nolimits_j v^j \frac{\partial}{\partial x^j}\Bigr) = \sum\nolimits_j a_j \, dx^j \quad \text{for } a_j := v^j \ .$$

The last assignment means that in the Euclidean case we do not need to introduce new notation; instead we normally write $\sum_j v^j \, dx^j$ for the image of $\sum_j v^j \, \partial/\partial x^j$ under Θ. That is, Θ allows us to regard the components v^j of the vector field $\sum_j v^j \, \partial/\partial x^j$ as those of the Pfaff form $\sum_j v^j \, dx^j$.

Proof Because $g_{jk} = \delta_{jk}$, this follows from Remark 6.1(b). ■

(b) (spherical coordinates) Let $V_3 := (0, \infty) \times (0, 2\pi) \times (0, \pi)$, and let

$$f : V_3 \to \mathbb{R}^3 \ , \quad (r, \varphi, \vartheta) \mapsto (x, y, z)$$

be the spherical coordinate transformation of Example VII.9.11(a). Then with respect to the standard metric, we have

$$\Theta\Bigl(v^1 \frac{\partial}{\partial r} + v^2 \frac{\partial}{\partial \varphi} + v^3 \frac{\partial}{\partial \vartheta}\Bigr) = v^1\, dr + r^2 \sin^2(\vartheta) v^2\, d\varphi + r^2 v^3\, d\vartheta\ .$$

Proof This follows immediately from Remark 6.1(b). ■

(c) (Minkowski metric) On $\mathbb{R}^4_{1,3}$, we have

$$\Theta_g\Bigl(\sum\nolimits_{\mu=0}^{3} v^\mu \frac{\partial}{\partial x^\mu}\Bigr) = v^0\, dx^0 - \sum\nolimits_{j=1}^{3} v^j\, dx^j$$

for $g := (\cdot\,|\,\cdot)_{1,3}$. ■

The gradient

If $f \in \mathcal{E}(M)$, then df belongs to $\Omega^1(M)$. Therefore

$$\operatorname{grad}_g f := \Theta_g^{-1}\, df \in \mathcal{V}(M)$$

is a well-defined vector field on M, the **gradient** of f on the (pseudo-)Riemannian manifold (M, g) (or with respect to g). We may also write it as $\operatorname{grad}_M f$ or $\operatorname{grad} f$ if no misunderstanding is expected. Therefore $\operatorname{grad} f$ is defined by the commutativity of the diagram

$$\begin{array}{ccc} & \mathcal{E}(M) = \Omega^0(M) & \\ {}^{\operatorname{grad}}\swarrow & & \searrow{}^{d} \\ \mathcal{V}(M) & \xrightarrow[\cong]{\ \Theta\ } & \Omega^1(M)\ . \end{array} \tag{6.4}$$

6.3 Remarks **(a)** The map $\operatorname{grad}\colon \mathcal{E}(M) \to \mathcal{V}(M)$, $f \mapsto \operatorname{grad} f$ is $\mathbb{R}$-linear.

(b) For $f \in \mathcal{E}(M)$, the vector field $\operatorname{grad} f$ is characterized by the relation

$$g(\operatorname{grad} f, w) = \langle df, w\rangle \quad \text{for } w \in \mathcal{V}(M)\ .$$

(c) In local coordinates, we have

$$\operatorname{grad} f = \sum\nolimits_j \Bigl(\sum\nolimits_k g^{jk} \frac{\partial f}{\partial x^k}\Bigr) \frac{\partial}{\partial x^j}\ . \tag{6.5}$$

Proof Because we know from (4.5) and (4.8) that $df = \sum_j \partial f/\partial x^j\, dx^j$, the claim follows from Remark 6.1(d). ■

(d) (orthogonal coordinates) In orthogonal coordinates, (6.5) simplifies to

$$\operatorname{grad} f = \sum\nolimits_j g^{jj} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^j} .$$

Because in this case g has the form

$$g = \sum\nolimits_j g_{jj} \, (dx^j)^2 , \tag{6.6}$$

that is, because the fundamental matrix is diagonal, we have $g^{jj} = 1/g_{jj}$. Thus the coefficients g^{jj} can be read directly from the representation (6.6).

(e) Suppose $(N, \bar{g})$ is a pseudo-Riemannian manifold and $\varphi \in \mathrm{Diff}(M, N)$ with $\varphi_* g = \lambda \bar{g}$ for some $\lambda \neq 0$. Then the diagram

$$\begin{array}{ccc} \mathcal{E}(M) & \xrightarrow{\lambda \operatorname{grad}_M} & \mathcal{V}(M) \\ \varphi^* \uparrow & & \uparrow \varphi^* \\ \mathcal{E}(N) & \xrightarrow{\operatorname{grad}_N} & \mathcal{V}(N) \end{array}$$

commutes. Therefore $\operatorname{grad}_M \circ \varphi^* = \lambda^{-1} \varphi^* \circ \operatorname{grad}_N$.

Proof Because the relation $\lambda \Theta_M^{-1} \varphi^* = \varphi^* \Theta_N^{-1}$ follows from Remark 6.1(f), we find for $f \in \mathcal{E}(N)$ that

$$\lambda \operatorname{grad}_M (\varphi^* f) = \lambda \Theta_M^{-1} d(\varphi^* f) = \lambda \Theta_M^{-1} \varphi^* df = \varphi^* \Theta_N^{-1} df = \varphi^* \operatorname{grad}_N f ,$$

where we have used (4.19). ■

(f) (regularity) Let $k \in \mathbb{N}$. For $f \in C^{k+1}(M)$, we have $\operatorname{grad} f \in \mathcal{V}^k(M)$. Here it suffices to assume that M is a C^{k+1} manifold. ■

6.4 Examples **(a)** (Euclidean coordinates) Let M be open in $\mathbb{R}^m$. Denoting Euclidean coordinates by $(x^1, \ldots, x^m)$, we have $g_{jk} = \delta_{jk}$ and therefore

$$\operatorname{grad} f = \sum\nolimits_j \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^j} .$$

This representation obviously coincides with that of Proposition VII.2.16. For an arbitrary locally Riemannian metric, we have already confirmed (6.4) in Remark VII.2.17(c).

(b) (spherical coordinates) Let $V_3 \to \mathbb{R}^3$, $(r, \varphi, \vartheta) \mapsto (x, y, z)$ be the spherical coordinate map. In these coordinates, the gradient with respect to the standard metric reads

$$\operatorname{grad} f = \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial f}{\partial \varphi} \frac{\partial}{\partial \varphi} + \frac{1}{r^2} \frac{\partial f}{\partial \vartheta} \frac{\partial}{\partial \vartheta} .$$

Proof Because spherical coordinates are orthogonal, this follows from Example 5.5(g). ■

(c) (spherical coordinates) Suppose $h_2 : W_2 \to \mathbb{R}^3$, $(\varphi, \vartheta) \mapsto (x, y, z)$ is the parametrization of the open subset $U_2 := S^2 \backslash H_2$ of the 2-sphere. Then for $f \in C^1(U_2, \mathbb{R})$, we have

$$\operatorname{grad}_{S^2} f = \frac{1}{\sin^2 \vartheta} \frac{\partial f}{\partial \varphi} \frac{\partial}{\partial \varphi} + \frac{\partial f}{\partial \vartheta} \frac{\partial}{\partial \vartheta} .$$

Proof This can be read from the representation of g_{S^2} in Example 5.5(h). ■

(d) (Minkowski metric) Suppose X is open in $\mathbb{R}^4_{1,3}$ and $f \in C^1(X, \mathbb{R})$. Then, with respect to the Minkowski metric, we have

$$\operatorname{grad} f = \frac{\partial f}{\partial t} \frac{\partial}{\partial t} - \frac{\partial f}{\partial x} \frac{\partial}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial}{\partial y} - \frac{\partial f}{\partial z} \frac{\partial}{\partial z} ,$$

as we see immediately from the definition of $(\cdot \,|\, \cdot)_{1,3}$. ■

The divergence

Now suppose M is oriented and that ω_M denotes the volume element of (M, g). Then the maps

$$\bullet\, \omega_M : \mathcal{E}(M) \to \Omega^m(M) , \qquad a \mapsto a\omega_M \tag{6.7}$$

and

$$\lrcorner\, \omega_M : \mathcal{V}(M) \to \Omega^{m-1}(M) , \quad v \mapsto v \lrcorner\, \omega_M \tag{6.8}$$

are defined pointwise.

6.5 Lemma *The maps* (6.7) *and* (6.8) *are well-defined $\mathcal{E}(M)$-module isomorphisms. If $\bigl((x^1, \ldots, x^m), U\bigr)$ is a chart of M, then*

$$a\omega_M \,|\, U = \pm a \sqrt{|G|}\, dx^1 \wedge \cdots \wedge dx^m \tag{6.9}$$

and

$$\begin{aligned} &\Bigl(\sum\nolimits_j v^j \frac{\partial}{\partial x^j}\Bigr) \lrcorner\, \omega_M \\ &\qquad = \sum\nolimits_j (-1)^{j-1} v^j \sqrt{|G|}\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m , \end{aligned} \tag{6.10}$$

where the positive sign is used in (6.9) *when the chart is positively oriented, and the negative is used otherwise.*

Proof (i) From the pointwise definition of $\bullet\, \omega_M$ and from Remarks 5.4(c) and (g), the truth of (6.9) follows immediately. From this and Remark 4.5(c), we conclude that $a\omega_M$ belongs to $\Omega^m(M)$ for $a \in \mathcal{E}(M)$. Therefore the map (6.7) is well defined.

It is clearly $\mathcal{E}(M)$-linear. By Remark 4.14(a), every $\alpha \in \Omega^m(M)$ has exactly one $a \in \mathcal{E}(M)$ such that $\alpha = a\omega_M$. Thus (6.7) is also bijective.

(ii) The validity of (6.10) follows from Remark 4.13(b) if the chart is positive. Otherwise we replace[1] x^1 by $-x^1$. Then v^1 is substituted by $-v^1$. This shows that (6.10) is independent of the chart's orientation.

Because $\sqrt{|G|} \in \mathcal{E}(U)$, (6.10) and Remark 4.5(c) show that $v \lrcorner\, \omega_M$ belongs to $\Omega^{m-1}(M)$ for $v \in \mathcal{V}(M)$. Thus the map (6.8) is well defined and clearly $\mathcal{E}(M)$-linear.

Let $\alpha \in \Omega^{m-1}(M)$. Then it follows from Example 3.2(b) and Remark 4.5(c) that there is a unique $a_j \in \mathcal{E}(U)$ such that

$$\alpha \,|\, U = \sum\nolimits_j (-1)^{j-1} a_j \, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^m \ .$$

Then $v^j := a_j \big/ \sqrt{|G|}$ belongs to $\mathcal{E}(U)$. Therefore

$$v := \sum\nolimits_j v^j \, \frac{\partial}{\partial x^j} \in \mathcal{V}(U) \ ,$$

and (6.10) shows $(v \lrcorner\, \omega_M) \,|\, U = \alpha \,|\, U$. This implies that the map $\lrcorner\, \omega_M$ is surjective. Because its injectivity is clear, we see that it is an isomorphism from $\mathcal{V}(M)$ to $\Omega^{m-1}(M)$. ■

6.6 Remarks **(a)** Let $(N, \bar{g})$ be an oriented pseudo-Riemannian manifold, and suppose $\varphi \in C^\infty(M,N)$ with $\varphi^*\omega_N = \mu\omega_M$ for some $\mu \neq 0$. Then the diagram

$$\begin{array}{ccc} \mathcal{E}(M) & \xrightarrow{\ \mu(\bullet\,\omega_M)\ } & \Omega^m(M) \\ \varphi^* \big\uparrow & & \big\uparrow \varphi^* \\ \mathcal{E}(N) & \xrightarrow{\ \bullet\,\omega_N\ } & \Omega^n(N) \end{array}$$

commutes, that is,

$$\mu(\varphi^* a) \bullet \omega_M = \varphi^*(a \bullet \omega_N) \quad \text{for } a \in \mathcal{E}(N) \ .$$

Proof This follows immediately from the behavior of (exterior) products under pull backs. ■

[1]Consider how this proof should be modified for the case of a one-dimensional manifold with boundary.

(b) Let $(N, \bar{g})$ be an oriented pseudo-Riemannian manifold, and suppose φ belongs to $\mathrm{Diff}(M, N)$ and satisfies $\varphi^*\omega_N = \mu\omega_M$ for some $\mu \neq 0$. Then

$$\begin{array}{ccc} \mathcal{V}(M) & \xrightarrow[\cong]{\mu(\lrcorner\,\omega_M)} & \Omega^{m-1}(M) \\ \varphi^* \Big\uparrow & & \Big\uparrow \varphi^* \\ \mathcal{V}(N) & \xrightarrow[\cong]{\lrcorner\,\omega_N} & \Omega^{m-1}(N) \end{array}$$

is a commutative diagram, that is, $\mu\big((\varphi^*v) \lrcorner\, \omega_M\big) = \varphi^*(v \lrcorner\, \omega_N)$ for $v \in \mathcal{V}(N)$.

Proof We derive from Remark 4.13(a) that

$$\mu\big((\varphi^*v) \lrcorner\, \omega_M\big) = \varphi^*v \lrcorner\, (\mu\omega_M) = \varphi^*v \lrcorner\, \varphi^*\omega_N = \varphi^*(\varphi_*\varphi^*v \lrcorner\, \omega_N) = \varphi^*(v \lrcorner\, \omega_N)$$

for $v \in \mathcal{V}(N)$. ■

(c) (regularity) Let $k \in \mathbb{N}$. Clearly then

$$\bullet\,\omega_M : C^k(M) \to \Omega^m_{(k)}(M)$$

and

$$\lrcorner\,\omega_M : \mathcal{V}^k(M) \to \Omega^{m-1}_{(k)}(M) \ ,$$

and these maps are $C^k(M)$-module isomorphisms. Thus it suffices to assume that M is a C^{k+1} manifold. ■

With help of the isomorphisms (6.7) and (6.8), we define a map

$$\mathrm{div}_g : \mathcal{V}(M) \to \mathcal{E}(M) \ , \quad v \mapsto \mathrm{div}_g\, v \tag{6.11}$$

by demanding that the diagram

$$\begin{array}{ccc} \mathcal{V}(M) & \xrightarrow{\mathrm{div}_g} & \mathcal{E}(M) \\ \lrcorner\,\omega_M \Big\downarrow \cong & & \cong \Big\downarrow \bullet\,\omega_M \\ \Omega^{m-1}(M) & \xrightarrow{d} & \Omega^m(M) \end{array} \tag{6.12}$$

commutes. In other words, for $v \in \mathcal{V}(M)$, the **divergence** $\mathrm{div}_g\, v$ of a vector field v on an oriented pseudo-Riemannian manifold (M, g) (or, with respect to g) is defined by the relation

$$(\mathrm{div}_g\, v)\omega_M = d(v \lrcorner\, \omega_M) \ . \tag{6.13}$$

Instead of div_g, we may also write div_M or, if no confusion is anticipated, simply div.

6.7 Remarks **(a)** The map (6.11) is $\mathbb{R}$-linear.

(b) Let $\big((x^1,\ldots,x^m),U\big)$ be a chart of M. For $v := \sum_j v^j\,\partial/\partial x^j \in \mathcal{V}(U)$, we have

$$\operatorname{div} v = \frac{1}{\sqrt{|G|}}\sum\nolimits_j \frac{\partial}{\partial x^j}\big(\sqrt{|G|}\,v^j\big)\ . \tag{6.14}$$

In orthogonal coordinates, we also have $\sqrt{|G|} = \sqrt{|g_{11}\cdot g_{22}\cdot\cdots\cdot g_{mm}|}$.

Proof Let $\varepsilon := 1$ if the chart is positive; use $\varepsilon := -1$ if it is negative. From (6.9), (6.10), and (6.13), we obtain (on U) that

$$\begin{aligned}\operatorname{div}(v)\omega_M = d(v \,\lrcorner\, \omega_M) &= \varepsilon d\Big(\sum\nolimits_j (-1)^{j-1}v^j\sqrt{|G|}\,dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^m\Big)\\ &= \varepsilon\sum\nolimits_{j,k}(-1)^{j-1}\,\frac{\partial\big(v^j\sqrt{|G|}\big)}{\partial x^k}\,dx^k\wedge dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^m\\ &= \varepsilon\Big(\sum\nolimits_j\frac{\partial\big(v^j\sqrt{|G|}\big)}{\partial x^j}\Big)\,dx^1\wedge\cdots\wedge dx^m\\ &= \Big(\frac{1}{\sqrt{|G|}}\sum\nolimits_j\frac{\partial\big(v^j\sqrt{|G|}\big)}{\partial x^j}\Big)\omega_M\end{aligned}$$

for $v\in\mathcal{V}(M)$. ■

(c) Suppose $(N,\bar g)$ is an oriented pseudo-Riemannian manifold and a map $\varphi\in\operatorname{Diff}(M,N)$ satisfies $\varphi^*\omega_N = \mu\omega_M$ for some $\mu\neq 0$. Then

$$\begin{array}{ccc}\mathcal{V}(M) & \xrightarrow{\ \operatorname{div}_M\ } & \mathcal{E}(M)\\ {\scriptstyle\varphi^*}\uparrow & & \uparrow{\scriptstyle\varphi^*}\\ \mathcal{V}(N) & \xrightarrow{\ \operatorname{div}_N\ } & \mathcal{E}(N)\end{array}$$

is a commutative diagram, that is, $\operatorname{div}_M\circ\varphi^* = \varphi^*\circ\operatorname{div}_N$.

Proof From Remark 6.6(b) and from (6.13) we obtain, by using $d\circ\varphi^* = \varphi^*\circ d$, that

$$\begin{aligned}\mu\operatorname{div}_M(\varphi^*v)\omega_M &= \mu\,d(\varphi^*v\,\lrcorner\,\omega_M) = d\varphi^*(v\,\lrcorner\,\omega_N) = \varphi^*d(v\,\lrcorner\,\omega_N)\\ &= \varphi^*\big[(\operatorname{div}_N v)\omega_N\big] = \varphi^*(\operatorname{div}_N v)\varphi^*\omega_N = \mu\varphi^*(\operatorname{div}_N v)\omega_M\end{aligned}$$

for $v\in\mathcal{V}(N)$. Now the claim follows from Lemma 6.5. ■

(d) (regularity) Let $k\in\mathbb{N}$. Then $\operatorname{div} v$ belongs to $C^k(M)$ for $v\in\mathcal{V}^{k+1}(M)$, and the map

$$\operatorname{div}:\mathcal{V}^{k+1}(M)\to C^k(M)\ ,\quad v\mapsto\operatorname{div} v$$

is $\mathbb{R}$-linear. So it suffices to assume that M is a C^{k+2} manifold.

Proof This is a consequence of Remarks 4.11(b) and 6.6(c). ■

As we shall see in the next chapter, the divergence of a vector field has interesting geometric and physical interpretations.

6.8 Examples **(a)** (Euclidean coordinates) Suppose U is open in $\mathbb{R}^m$. Denoting Euclidean coordinates by $(x^1,\ldots,x^m)$, we have

$$\operatorname{div} v = \sum\nolimits_j \frac{\partial v^j}{\partial x^j}$$

for $v = \sum_j v^j\, \partial/\partial x^j$. This formula also holds when $\big((x^1,\ldots,x^m),U\big)$ are any other orthonormal coordinates on (M,g).

(b) (plane polar coordinates) Let $V_2 := (0,\infty)\times(0,2\pi)$, and let

$$f_2 : V_2 \to \mathbb{R}^2 \ , \quad (r,\varphi) \mapsto (x,y) := (r\cos\varphi, r\sin\varphi)$$

be the plane polar coordinate map. Then with respect to the standard metric, we have

$$\operatorname{div}\Big(v^1\,\frac{\partial}{\partial r} + v^2\,\frac{\partial}{\partial\varphi}\Big) = \frac{1}{r}\,\frac{\partial(rv^1)}{\partial r} + \frac{\partial v^2}{\partial\varphi} = \frac{v^1}{r} + \frac{\partial v^1}{\partial r} + \frac{\partial v^2}{\partial\varphi}\ .$$

Proof This follows from $\sqrt{G} = r$, as can be read off the representation of g_2 given in Example 5.5(e). ∎

(c) (spherical coordinates) Let $V_3 := (0,\infty)\times(0,2\pi)\times(0,\pi)$, and let

$$f_3 : V_3 \to \mathbb{R}^3 \ , \quad (r,\varphi,\vartheta) \mapsto (x,y,z)$$

be the spherical coordinate map of Example 5.5(g). With respect to the standard metric $g_3 := (dx)^2 + (dy)^2 + (dz)^2$, we have

$$\begin{aligned}\operatorname{div}\Big(v^1\,\frac{\partial}{\partial r} + v^2\,\frac{\partial}{\partial\varphi} + v^3\,\frac{\partial}{\partial\vartheta}\Big) &= \frac{1}{r^2}\,\frac{\partial(r^2v^1)}{\partial r} + \frac{\partial v^2}{\partial\varphi} + \frac{1}{\sin\vartheta}\,\frac{\partial(v^3\sin\vartheta)}{\partial\vartheta}\\ &= \frac{2}{r}\,v^1 + \frac{\partial v^1}{\partial r} + \frac{\partial v^2}{\partial\varphi} + \cot(\vartheta)v^3 + \frac{\partial v^3}{\partial\vartheta}\ .\end{aligned}$$

Proof Example 5.5(g) gives $\sqrt{|G|} = r^2\sin\vartheta$, as the claim requires. ∎

(d) (Minkowski metric) Let $M := \mathbb{R}^4_{1,3}$ and $g := (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$. Then

$$\operatorname{div}\Big(v^0\,\frac{\partial}{\partial t} + v^1\,\frac{\partial}{\partial x} + v^2\,\frac{\partial}{\partial y} + v^3\,\frac{\partial}{\partial z}\Big) = \frac{\partial v^0}{\partial t} - \frac{\partial v^1}{\partial x} - \frac{\partial v^2}{\partial y} - \frac{\partial v^3}{\partial z}$$

for $v^j \in \mathcal{E}(\mathbb{R}^4_{1,3})$ with $0 \le j \le 3$. ∎

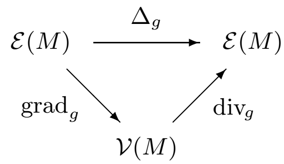
The Laplace–Beltrami operator

By combining the two first order differential operators grad and div, we obtain the most important second order differential operator, the **Laplace–Beltrami operator** Δ_g.

Let (M,g) be an oriented pseudo-Riemannian manifold. Then we define Δ_g by

$$\Delta_g := \mathrm{div}_g\,\mathrm{grad}_g$$

or, equivalently, by requiring that the diagram



commutes. Instead of Δ_g, we may also write Δ_M or simply Δ if g is clear from context.

6.9 Remarks **(a)** The map $\Delta_M : \mathcal{E}(M) \to \mathcal{E}(M)$ is $\mathbb{R}$-linear.

(b) If $\big((x^1,\ldots,x^m),U\big)$ is a chart of M, then

$$\Delta_M f = \frac{1}{\sqrt{|G|}} \sum\nolimits_{j,k} \frac{\partial}{\partial x^j}\Big(\sqrt{|G|}\,g^{jk}\,\frac{\partial f}{\partial x^k}\Big) \quad \text{for } f \in \mathcal{E}(U) \ . \tag{6.15}$$

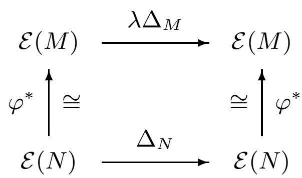
In orthogonal coordinates, (6.15) simplifies to

$$\Delta_M f = \frac{1}{\sqrt{|G|}} \sum\nolimits_{j} \frac{\partial}{\partial x^j}\Big(\sqrt{|G|}\,g^{jj}\,\frac{\partial f}{\partial x^j}\Big) \quad \text{for } f \in \mathcal{E}(U) \ , \tag{6.16}$$

where $\sqrt{|G|} = \sqrt{|g_{11}\cdot g_{22}\cdot\cdots\cdot g_{mm}|}$.

Proof This follows from Remarks 6.3(c) and (d) and Remark 6.7(b). ■

(c) Suppose $(N,\bar g)$ is an oriented pseudo-Riemannian manifold. Also let $\varphi \in \mathrm{Diff}(M,N)$, and suppose there are $\lambda \neq 0$ and $\mu \neq 0$ such that $\varphi_* g = \lambda \bar g$ and $\varphi^*\omega_N = \mu\omega_M$. Then the diagram



commutes: $\lambda\Delta_M \circ \varphi^* = \varphi^* \circ \Delta_N$.

Proof This is a consequence of Remarks 6.3(e) and 6.7(c). ■

(d) (regularity) Let $k \in \mathbb{N}$. Then obviously

$$\Delta_M : C^{k+2}(M) \to C^k(M) \ ,$$

and this map is $\mathbb{R}$-linear. Here it suffices to assume that M is a C^{k+2} manifold. ■

6.10 Examples **(a)** (Euclidean coordinates) Suppose M is open in $\mathbb{R}^m$, with Euclidean coordinates $\bigl((x^1,\ldots,x^m),M\bigr)$. Then Δ_M is the same as the (usual) ***m*-dimensional Laplace operator**

$$\Delta_m := \sum\nolimits_j \partial_j^2 \ .$$

See Exercise VII.5.3.

(b) (circular coordinates) With respect to the parametrization

$$h\colon (0,2\pi)\to\mathbb{R}^2\ ,\quad \varphi\mapsto(\cos\varphi,\sin\varphi)$$

of $S^1\backslash\bigl\{(1,0)\bigr\}$ (and the standard metric), we have $\Delta_{S^1}=\partial_\varphi^2$.

Proof Remark 6.9(b) and Example 5.5(f). ■

(c) (plane polar coordinates) In plane polar coordinates

$$(0,\infty)\times(0,2\pi)\to\mathbb{R}^2\ ,\quad (r,\varphi)\mapsto(r\cos\varphi,r\sin\varphi)\ ,$$

the Laplace–Beltrami operator (with respect to the standard metric $\mathbb{R}^2$) is

$$\Delta_2=\frac{1}{r}\,\partial_r(r\partial_r\,\cdot\,)+\frac{1}{r^2}\,\partial_\varphi^2=\partial_r^2+\frac{1}{r}\,\partial_r+\frac{1}{r^2}\,\partial_\varphi^2=\frac{1}{r^2}\,\bigl[(r\partial_r)^2+\Delta_{S^1}\bigr]\ .$$

Proof This follows from Remark 6.9(b), Example 5.5(e), and (b). ■

(d) (m-dimensional spherical coordinates) For $m\geq 2$, the Laplace–Beltrami operator of S^m (with respect to the standard metric) in the spherical coordinates of Example 5.5(h) assumes the form

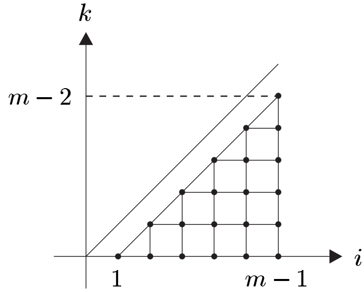
$$\begin{aligned}\Delta_{S^m}=&\ \frac{1}{\sin^2\vartheta_1\cdot\,\cdots\,\cdot\sin^2\vartheta_{m-1}}\,\frac{\partial^2}{\partial\varphi^2}\\ &+\sum_{k=1}^{m-1}\frac{1}{\sin^k\vartheta_k\sin^2\vartheta_{k+1}\cdot\,\cdots\,\cdot\sin^2\vartheta_{m-1}}\,\frac{\partial}{\partial\vartheta_k}\Bigl(\sin^k\vartheta_k\,\frac{\partial}{\partial\vartheta_k}\Bigr)\ .\end{aligned}$$

In particular,

$$\Delta_{S^2}=\frac{1}{\sin^2\vartheta}\,\partial_\varphi^2+\frac{1}{\sin\vartheta}\,\partial_\vartheta(\sin\vartheta\,\partial_\vartheta\,\cdot\,)=\frac{1}{\sin^2\vartheta}\,\partial_\varphi^2+\partial_\vartheta^2+\cot\vartheta\,\partial_\vartheta\ .$$

Proof From Examples 5.5(g) and (h), it follows

$$G=\prod_{k=0}^{m-1}a_{m+1,k}=\prod_{k=0}^{m-2}\prod_{i=k+1}^{m-1}\sin^2\vartheta_i\ .$$



Exchanging the order of the two products gives

$$G=\prod_{i=1}^{m-1}\sin^{2i}\vartheta_i=\bigl[w_{m+1}(\vartheta)\bigr]^2\ , \tag{6.17}$$

where we use the abbreviated notation introduced in Proposition X.8.9.

From the orthogonality of the spherical coordinates, it also follows from the given examples that

$$g^{jj} = \frac{1}{a_{m+1,j-1}} = \frac{1}{\prod_{i=j}^{m-1} \sin^2 \vartheta_i} \quad \text{for } 1 \le j \le m\ .$$

From this we read off

$$\sqrt{G}\, g^{jj} = \Big(\prod_{\substack{i=1 \\ i \ne j-1}}^{m-1} \sin^i \vartheta_i \prod_{k=j}^{m-1} \frac{1}{\sin^2 \vartheta_k} \Big) \sin^{j-1} \vartheta_{j-1}$$

for $2 \le j \le m$. Thus we find

$$\begin{aligned} &\frac{1}{\sqrt{G}} \frac{\partial}{\partial \vartheta_{j-1}} \Big(\sqrt{G}\, g^{jj} \frac{\partial}{\partial \vartheta_{j-1}} \Big) \\ &\qquad = \frac{1}{\sin^{j-1} \vartheta_{j-1} \prod_{i=j}^{m-1} \sin^2 \vartheta_i} \frac{\partial}{\partial \vartheta_{j-1}} \Big(\sin^{j-1} \vartheta_{j-1} \frac{\partial}{\partial \vartheta_{j-1}} \Big) \end{aligned}$$

for $2 \le j \le m$. Now the claim is clear. ■

(e) (m-dimensional polar coordinates) In m-dimensional polar coordinates with $m \ge 2$, the m-dimensional Laplace operator reads

$$\begin{aligned} \Delta_m &= \frac{1}{r^{m-1}} \partial_r (r^{m-1} \partial_r \cdot) + \frac{1}{r^2} \Delta_{S^{m-1}} = \partial_r^2 + \frac{m-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S^{m-1}} \\ &= \frac{1}{r^2} \big[(r\partial_r)^2 + (m-2) r \partial_r + \Delta_{S^{m-1}} \big]\ . \end{aligned}$$

Proof From Examples 5.5(g) and (h), we read off $g_m = (dr)^2 + r^2 g_{S^{m-1}}$. This then implies $G = r^{2(m-1)} G_{S^{m-1}}$. It also implies $g^{11} = 1$ and

$$g^{jj} = \frac{1}{r^2} g_{S^{m-1}}^{(j-1)(j-1)} \quad \text{for } 2 \le j \le m\ .$$

Now the claim follows from (6.16) because of the orthogonality of the coordinates. ■

(f) (Minkowski metric) In orthonormal coordinates, the Laplace–Beltrami operator of the Minkowski space $\mathbb{R}^4_{1,3}$ has the form $\partial_t^2 - \Delta_3$, where Δ_3 is the three-dimensional (Euclidian) Laplace operator. That is, the Laplace–Beltrami operator in the Minkowski space is just the wave operator.[2]

Proof This is an immediate consequence of (6.16). ■

In the next proposition, we list some basic properties of differential operators used in vector analysis. Here and in the following, we denote the pseudo-Riemannian metric of M by $(\cdot\,|\,\cdot)_M$.

[2]See Exercise VII.5.10.

6.11 Proposition *Suppose $\big(M, (\cdot\,|\,\cdot)_M\big)$ is an oriented pseudo-Riemannian manifold, $f, g \in \mathcal{E}(M)$, and $v, w \in \mathcal{V}(M)$. Then*

(i) $\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f$;

(ii) $\operatorname{div}(fv) = f \operatorname{div} v + (\operatorname{grad} f \,|\, v)_M$;

(iii) $\Delta(fg) = f\Delta g + 2(\operatorname{grad} f \,|\, \operatorname{grad} g)_M + g\Delta f$;

(iv) $f\Delta g - g\Delta f = \operatorname{div}(f \operatorname{grad} g) - \operatorname{div}(g \operatorname{grad} f)$.

Proof (i) Because Θ is a module isomorphism, it follows from (6.4) that (i) is equivalent to

$$d(fg) = f\,dg + g\,df \ . \tag{6.18}$$

Because (6.18) is a local statement, it suffices to prove this formula in local coordinates. In this case, it is an immediate consequence of the product rule.

(ii) From $(fv) \,\lrcorner\, \omega_M = f(v \,\lrcorner\, \omega_M) = f \wedge (v \,\lrcorner\, \omega_M)$ and the product rule of Theorem 4.10, it follows that

$$d\big((fv) \,\lrcorner\, \omega_M\big) = d\big(f \wedge (v \,\lrcorner\, \omega_M)\big) = df \wedge (v \,\lrcorner\, \omega_M) + f\,d(v \,\lrcorner\, \omega_M) \ . \tag{6.19}$$

Because this is also a local statement, we can use local representations. Then, with $v = \sum_j v^j\, \partial/\partial x^j$ and a positive chart, we obtain from (6.9) and (6.10) that

$$\begin{aligned} &df \wedge (v \,\lrcorner\, \omega_M) \\ &\quad= \Big(\sum\nolimits_j \frac{\partial f}{\partial x^j}\, dx^j\Big) \wedge \sum\nolimits_k (-1)^{k-1} v^k \sqrt{|G|}\, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^m \\ &\quad= \Big(\sum\nolimits_j \frac{\partial f}{\partial x^j}\, v^j\Big) \sqrt{|G|}\, dx^1 \wedge \cdots \wedge dx^m = \sum\nolimits_j \frac{\partial f}{\partial x^j}\, v^j \omega_M \ . \end{aligned} \tag{6.20}$$

We then read from Remark 6.3(b) and (4.4) that

$$(\operatorname{grad} f \,|\, v)_M = \langle df, v\rangle = \sum\nolimits_j \Big\langle df, \frac{\partial}{\partial x^j}\Big\rangle v^j = \sum\nolimits_j \frac{\partial f}{\partial x^j}\, v^j \ . \tag{6.21}$$

Therefore it follows from (6.19)–(6.21) and the definition (6.13) that

$$\operatorname{div}(fv)\omega_M = d\big((fv) \,\lrcorner\, \omega_M\big) = (\operatorname{grad} f \,|\, v)_M \omega_M + f \operatorname{div} v\, \omega_M \ ,$$

which implies the claim.

(iii) This we get immediately from $\Delta = \operatorname{div} \operatorname{grad}$ and (i), (ii).

(iv) From (ii), it follows that

$$\operatorname{div}(f \operatorname{grad} g) = f\Delta g + (\operatorname{grad} f \,|\, \operatorname{grad} g)_M \ . \tag{6.22}$$

Exchanging f and g and subtracting the result from (6.22) then yields (iv). ■

The curl

Suppose now (M,g) is a 3-dimensional oriented pseudo-Riemannian manifold. Then we define the **curl**[3] $\operatorname{curl} v$ of the vector field $v \in \mathcal{V}(M)$ by requiring that the diagram

$$\begin{array}{ccc} \mathcal{V}(M) & \xrightarrow[\cong]{\ \Theta\ } & \Omega^1(M) \\ {\scriptstyle \operatorname{curl}}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ \mathcal{V}(M) & \xrightarrow[\cong]{\ \lrcorner\, \omega_M\ } & \Omega^2(M) \end{array} \tag{6.23}$$

commutes, that is, by requiring

$$(\operatorname{curl} v) \lrcorner\, \omega_M = d(\Theta v) \quad \text{for } v \in \mathcal{V}(M)\ . \tag{6.24}$$

The definition is clearly only possible in the case $m = 3$.

6.12 Remarks **(a)** The map $\operatorname{curl}\colon \mathcal{V}(M) \to \mathcal{V}(M),\ \ v \mapsto \operatorname{curl} v$ is $\mathbb{R}$-linear.

(b) Let $\big((x^1,x^2,x^3),U\big)$ be a chart of M. Then

$$\operatorname{curl} v = \frac{1}{\sqrt{|G|}} \sum_{i=1}^{3} \sum_{(j,k,\ell)\in \mathsf{S}_3} \operatorname{sign}(j,k,\ell)\, \frac{\partial}{\partial x^j}\,(g_{ki}v^i)\, \frac{\partial}{\partial x^\ell}$$

for $v = \sum_{j=1}^{3} v^j\, \partial/\partial x^j$. If the coordinates are orthogonal, this expression simplifies to

$$\begin{aligned} \operatorname{curl} v &= \frac{1}{\sqrt{|G|}} \sum\nolimits_{(j,k,\ell)\in \mathsf{S}_3} \operatorname{sign}(j,k,\ell)\, \frac{\partial}{\partial x^j}\,(g_{kk}v^k)\, \frac{\partial}{\partial x^\ell} \\ &= \frac{1}{\sqrt{|G|}} \Big[\big(\partial_2(g_{33}v^3) - \partial_3(g_{22}v^2)\big)\frac{\partial}{\partial x^1} + \big(\partial_3(g_{11}v^1) - \partial_1(g_{33}v^3)\big)\frac{\partial}{\partial x^2} \\ &\qquad\qquad + \big(\partial_1(g_{22}v^2) - \partial_2(g_{11}v^1)\big)\frac{\partial}{\partial x^3}\Big] \end{aligned}$$

with $\sqrt{|G|} = \sqrt{|g_{11}g_{22}g_{33}|}$. If the coordinates are orthonormal, this becomes

$$\operatorname{curl} v = (\partial_2 v^3 - \partial_3 v^2)\frac{\partial}{\partial x^1} + (\partial_3 v^1 - \partial_1 v^3)\frac{\partial}{\partial x^2} + (\partial_1 v^2 - \partial_2 v^1)\frac{\partial}{\partial x^3}\ .$$

Proof Remark 6.1(b) and the properties of the exterior derivative give

$$d(\Theta v) = d\sum\nolimits_k \Big(\sum\nolimits_i g_{ki}v^i\Big)\, dx^k = \sum\nolimits_k \sum\nolimits_{j\neq k} \frac{\partial}{\partial x^j}\Big(\sum\nolimits_i g_{ki}v^i\Big)\, dx^j \wedge dx^k\ .$$

[3] Sometimes written rot, short for "rotation".

From (6.10), we read off

$$\operatorname{curl} v \lrcorner\, \omega_M = \sqrt{|G|}\,\bigl((\operatorname{curl} v)^1\, dx^2\wedge dx^3 + (\operatorname{curl} v)^2\, dx^3\wedge dx^1 + (\operatorname{curl} v)^3\, dx^1\wedge dx^2\bigr)\ . \tag{6.25}$$

Therefore the claim follows from (6.24). ■

(c) (regularity) Let $k \in \mathbb{N}$. Then $\operatorname{curl} v \in \mathcal{V}^k(M)$ for $v \in \mathcal{V}^{k+1}(M)$. So it suffices here to assume that M is a C^{k+2} manifold. ■

In the case $m = 3$, there are important relations between the operators grad, div, and curl. These are summarized diagrammatically in the following theorem.

6.13 Theorem *Let (M, g) be a three-dimensional oriented (pseudo-)Riemannian manifold.*

(i) *The diagram*

$$\begin{array}{ccccccc}
\mathcal{E}(M) & \xrightarrow{\ \operatorname{grad}\ } & \mathcal{V}(M) & \xrightarrow{\ \operatorname{curl}\ } & \mathcal{V}(M) & \xrightarrow{\ \operatorname{div}\ } & \mathcal{E}(M) \\
\Big\| & & \cong \Big\downarrow \Theta_M & & \cong \Big\downarrow \lrcorner\, \omega_M & & \cong \Big\downarrow \bullet\, \omega_M \\
\Omega^0(M) & \xrightarrow{\ d\ } & \Omega^1(M) & \xrightarrow{\ d\ } & \Omega^2(M) & \xrightarrow{\ d\ } & \Omega^3(M)
\end{array} \tag{6.26}$$

commutes.

(ii) $\operatorname{curl} \circ \operatorname{grad} = 0$.

(iii) $\operatorname{div} \circ \operatorname{curl} = 0$.

Proof (i) follows immediately from the commutativity of the diagrams (6.4), (6.12), and (6.23).

(ii) and (iii) are now direct consequence of $d^2 = 0$. ■

6.14 Corollary *Let X be open and contractible in $\mathbb{R}^3$. Also let v be a smooth vector field on X.*

(i) *If $\operatorname{curl} v = 0$, then there is an $f \in \mathcal{E}(X)$ such that $v = \operatorname{grad} f$, a* **potential** *for v.*

(ii) *If $\operatorname{div} v = 0$, then there is a $w \in \mathcal{V}(X)$ with $v = \operatorname{curl} w$, a* **vector potential** *for v.*

Proof (i) From (6.26) we learn that $\operatorname{curl} v = 0$ is equivalent to $d(\Theta_X v) = 0$. Therefore the 1-form $\Theta_M v$ is closed, and the Poincaré lemma (Theorem 3.11) guarantees the existence of an $f \in \Omega^0(X) = \mathcal{E}(X)$ such that $\Theta_X v = df$. From this it follows that $v = \Theta_X^{-1}\, df = \operatorname{grad} f$.

(ii) Analogously to (i), it follows from $\operatorname{div} v = 0$ that the 2-form $v \lrcorner\, \omega_X$ is closed and therefore exact, again by the Poincaré lemma. Thus there is an $\alpha \in \Omega^1(X)$ with $d\alpha = v \lrcorner\, \omega_X$. Then $w := \Theta_X^{-1}\alpha \in \mathcal{V}(X)$, by the commutativity of the middle "loop" of (6.26), satisfies $\operatorname{curl} w = v$. ■

6.15 Remarks Suppose X is open in $\mathbb{R}^3$.

(a) In Euclidean coordinates, the equality $\operatorname{curl} v = 0$ is equivalent to the integrability conditions

$$\partial_j v^k = \partial_k v^j \quad \text{for } 1 \le j, k \le 3 \ ,$$

which can be seen from Remark 6.12(b). Therefore Corollary 6.14(i) is a special case of Remark VIII.4.10(a).

(b) (classical notation) In Euclidean coordinates, we know from Example 6.4(a) that $\operatorname{grad} f$ agrees with the ∇f from Proposition VII.2.16. The physics and engineering literatures, and many mathematical texts, use the formal **nabla vector**

$$\nabla := \Bigl(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Bigr) \ .$$

With the notation $x \cdot y$ for the Euclidean scalar product in $\mathbb{R}^3$ and $x \times y$ for the vector product, the nabla vector notation leads to the (formal) relations

$$\operatorname{div} v = \nabla \cdot v \ , \quad \operatorname{curl} v = \nabla \times v \ , \quad \Delta v = (\nabla \cdot \nabla) v =: \nabla^2 v \ .$$

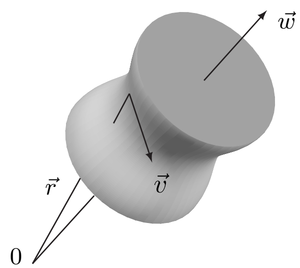
These follow easily from the corresponding local representations of these operators and from Remark VIII.2.14(d). In particular, the components of the vector $\operatorname{curl} v$ can be found by expanding the (formal) determinant

$$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v^1 & v^2 & v^3 \end{vmatrix}$$

in its first row. Here $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are the standard basis vectors of $\mathbb{R}^3$, and $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ are *not* interpreted as tangent vectors, but as differential operators.

Because the symbol ∇ has another meaning in the context of Riemannian geometry, we will rarely use the nabla vector in the rest of this book.

(c) (the physical meaning of the curl[4]) We consider a rigid body rotating at constant (angular) velocity about a fixed axis. We then choose an orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and the coordinate origin so that $\vec{e}_3$ points along the rotation axis. Also let ω be the **angular velocity**, that is, ω is the speed of any point P fixed in the rotating body at unit distance from the axis of rotation. If $\vec{r}$ is the radius vector of the point P, that



[4]A deeper interpretation of the curl of a vector field is given in Section XII.3.

is, the position vector of the point P in the coordinate system $(O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$ (see the statements after Remarks I.12.6) and if θ is the angle between $\vec{e}_3$ and $\vec{r}$ (in the plane spanned by $\vec{e}_3$ and $\vec{r}$), then the distance a from P to the rotation axis satisfies $a=|\vec{r}|\sin\theta$. Therefore the modulus of the velocity vector $\vec{v}$ of the point P is given by

$$|\vec{v}|=\omega a=\omega\,|\vec{r}|\,\sin\theta\ .$$

Denote by $\vec{w}:=\omega\vec{e}_3$ the "angular velocity vector" and orient it so that the body rotates clockwise about it. Then it follows from the properties of the vector product that

$$\vec{v}=\vec{w}\times\vec{r}\ , \tag{6.27}$$

since the point P moves with constant speed ω in a circle centered at and in a plane orthogonal to the $\vec{e}_3$-axis.[5]

Let (x,y,z) be the coordinates of P with respect to $(O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$. Then

$$\vec{r}=x\,\frac{\partial}{\partial x}+y\,\frac{\partial}{\partial y}+z\,\frac{\partial}{\partial z}\quad\text{and}\quad\vec{w}=\omega\,\frac{\partial}{\partial z}\ .$$

Therefore

$$\vec{v}=\vec{w}\times\vec{r}=-\omega y\,\frac{\partial}{\partial x}+\omega x\,\frac{\partial}{\partial y}\ .$$

For the curl of the vector field $\vec{v}$, we find $\operatorname{curl}\vec{v}=2\omega\,\partial/\partial z=2\vec{w}$. In words, for a rigid body rotating about a fixed axis, the curl of the velocity vector is a vector field whose elements are parallel to the rotation axis and have absolute value twice the angular velocity.

(d) (regularity) The statements of Theorem 6.13 and Corollary 6.14 can be proved with weaker differentiability assumptions that are easily derived from earlier remarks about regularity. ■

The Lie derivative

Now suppose M is again an arbitrary manifold. For $f\in\mathcal{E}(M)$ and $v\in\mathcal{V}(M)$, we set

$$L_vf:=\langle df,v\rangle\in\mathcal{E}(M)$$

and call L_vf the **Lie derivative** of f with respect to v.

6.16 Proposition

(i) *The map $L_v:\mathcal{E}(M)\to\mathcal{E}(M)$, the* **Lie derivative** *with respect to v, has the properties that*

(α) *L_v is $\mathbb{R}$-linear;*

(β) *$L_v(fg)=L_v(f)g+fL_vg$ for $f,g\in\mathcal{E}(M)$.*

[5] We leave the formal proof of (6.27) to you.

(ii) *In local coordinates,*

$$L_v f = \sum\nolimits_j v^j \frac{\partial f}{\partial x^j} \quad \text{and} \quad v = \sum\nolimits_j v^j \frac{\partial}{\partial x^j} \ .$$

Proof (i) follows immediately from the properties of d (see (6.18)).

(ii) is a consequence of (4.4). ■

6.17 Remarks **(a)** Proposition 6.16(ii) makes it clear that the Lie derivative generalizes the directional derivative of Section VII.2.

(b) Let A be an $\mathbb{R}$-algebra. A map $D\colon A \to A$ is said to be a **derivation** (of A) if D is $\mathbb{R}$-linear and satisfies the product rule

$$D(ab) = (Da)b + a(Db) \quad \text{for } a, b \in A \ .$$

Therefore the Lie derivative with respect to $v \in \mathcal{V}(M)$ is a derivation of the algebra $\mathcal{E}(M)$.

(c) If A is an algebra with unity e and D is a derivation of A, then $De = 0$.

Proof The product rule gives

$$De = D(ee) = (De)e + e(De) = De + De = 2De$$

and hence the claim. ■

The next theorem shows that every derivation of $\mathcal{E}(M)$ is given by a Lie derivative.

6.18 Theorem *Let D be a derivation of $\mathcal{E}(M)$. Then there is exactly one $v \in \mathcal{V}(M)$ such that $D = L_v$.*

Proof (i) We show first that D is a "local operator". Let U be an open and K a compact neighborhood of $p \in M$ with $K \subset\subset U$. Remark 1.21(a) guarantees the existence of a $\chi \in \mathcal{E}(M)$ with $\chi \,|\, K = 1$ and $\operatorname{supp}(\chi) \subset\subset U$.

Let $f \in \mathcal{E}(M)$ with $f \,|\, U = 0$. Then $f = f\chi + f(1-\chi) = f(1-\chi)$, and thus

$$Df(p) = Df(p)\bigl(1 - \chi(p)\bigr) + f(p)D(1-\chi)(p) = 0 \ .$$

Because this is true for every $p \in U$, it follows that $D(f) \,|\, U = 0$. If $\chi_1 \in \mathcal{E}(M)$ is another function with $\operatorname{supp}(\chi_1) \subset\subset U$ and which is identically equal to 1 in a neighborhood of p, then $f\chi - f\chi_1 \in \mathcal{E}(M)$ for $f \in \mathcal{E}(U)$ vanishes in a neighborhood of p. Then it follows from the above that $D(f\chi) = D(f\chi_1)$ for $f \in \mathcal{E}(U)$. Hence the "restriction of D to U" is well defined by

$$D_U f := D(f\chi) \quad \text{for } f \in \mathcal{E}(U)$$

and is independent of the special choice of χ.

(ii) Suppose now (φ, U) is a chart with $\varphi = (x^1, \ldots, x^m)$. We can assume that $X := \varphi(U)$ is convex. For every fixed $p \in U$, it follows from the mean value theorem in integral form (Theorem VII.3.10) with $a := \varphi(p)$ that

$$(\varphi_* f)(x) = (\varphi_* f)(a) + \sum\nolimits_j (x^j - a^j)\widetilde{f}_j(x) \quad \text{for } x \in X \ ,$$

where we have set

$$\widetilde{f}_j(x) := \int_0^1 \partial_j f\bigl(a + t(x-a)\bigr)\, dt \quad \text{for } x \in X \ .$$

Therefore

$$f_j := \varphi^* \widetilde{f}_j \in \mathcal{E}(U) \ , \qquad f_j(p) = \frac{\partial f}{\partial x^j}\,(p) \ ,$$

and

$$f(q) = f(p) + \sum\nolimits_j \bigl(\varphi^j(q) - \varphi^j(p)\bigr) f_j(q) \quad \text{for } q \in U \ .$$

From this, the properties of D, and Remark 6.17(c), it follows that

$$Df(p) = \sum\nolimits_j D\varphi^j(p)\, \frac{\partial f}{\partial x^j}\,(p) \quad \text{for } p \in U \ , \tag{6.28}$$

where we have written D instead of D_U.

(iii) Let (ψ, V) be a second chart around p with $\psi = (y^1, \ldots, y^m)$. Now define the transition function $k := \psi \circ \varphi^{-1}$. Then, in analogy to (ii) and because we can assume $U = V$, we have

$$k^j(x) = k^j(a) + \sum\nolimits_\ell (x^\ell - a^\ell) k_\ell^j(x) \quad \text{for } x \in X \ , \tag{6.29}$$

with $k_\ell^j \in \mathcal{E}(X)$ and $k_\ell^j(a) = \partial_\ell k^j(a)$. Because $\varphi^* k = \psi$, applying φ^* to (6.29) gives

$$\psi^j(q) = \psi^j(p) + \sum\nolimits_\ell \bigl(\varphi^\ell(q) - \varphi^\ell(p)\bigr) h_\ell^j(q) \quad \text{for } q \in U \ , \tag{6.30}$$

with $h_\ell^j := \varphi^* k_\ell^j \in \mathcal{E}(U)$ and

$$h_\ell^j(p) = (\varphi^* \partial_\ell k^j)(p) = \frac{\partial y^j}{\partial x^\ell}\,(p) \ .$$

This and (6.30) imply

$$D\psi^j(p) = \sum\nolimits_k D\varphi^k(p)\, \frac{\partial y^j}{\partial x^k}\,(p) \quad \text{for } p \in U \text{ and } 1 \le j,k \le m \ . \tag{6.31}$$

Now we set

$$v_\varphi := \sum\nolimits_j D\varphi^j\, \frac{\partial}{\partial x^j} \quad \text{and} \quad v_\psi := \sum\nolimits_j D\psi^j\, \frac{\partial}{\partial y^j} \ . \tag{6.32}$$

Then it follows from (6.31) and Proposition 4.7 that

$$v_\psi = \sum\nolimits_j \sum\nolimits_k D\varphi^k \frac{\partial y^j}{\partial x^k}\frac{\partial}{\partial y^j} = \sum\nolimits_k D\varphi^k \frac{\partial}{\partial x^k} = v_\varphi \ .$$

This shows that (6.32) defines a vector field $v_U \in \mathcal{V}(U)$ on U that is independent of the coordinates chosen. From (6.28), (6.31), and Proposition 6.16(ii), we read off $D_U f = L_{v_U} f$ for $f \in \mathcal{E}(U)$.

(iv) Suppose now $\{ (\varphi_\alpha, U_\alpha) \ ; \ \alpha \in \mathsf{A} \}$ is an atlas for M. Then it follows from (iii) that for every $\alpha \in \mathsf{A}$ there is an $v_\alpha \in \mathcal{V}(U_\alpha)$ such that $D_{U_\alpha} f = L_{v_\alpha} f$ for $f \in \mathcal{E}(U_\alpha)$. Moreover, the observations in (iii) show that there is exactly one $v \in \mathcal{V}(M)$ such that $v \,|\, U_\alpha = v_\alpha$ for $\alpha \in \mathsf{A}$. Now (i) and Proposition 6.16(ii) give $D = L_v$.

(v) Suppose $v, w \in \mathcal{V}(M)$ with $D = L_v$ and $D = L_w$. Then $L_v f = L_w f$ for every $f \in \mathcal{E}(M)$. In an arbitrary local chart $\bigl((x^1, \ldots, x^m), U\bigr)$, we then have

$$\sum\nolimits_j (v^j - w^j) \frac{\partial f}{\partial x^j} = 0 \quad \text{for } f \in \mathcal{E}(U) \ .$$

Choosing $f := x^k$, we find $\partial f / \partial x^j = \delta_j^k$ and therefore $v^k - w^k = 0$. That this is true for $1 \le k \le m$ implies $v \,|\, U = w \,|\, U$ and hence $v = w$. Thus we are done. ■

6.19 Lemma *For $L_v L_w - L_w L_v$ is a derivation of $\mathcal{E}(M)$ for $v, w \in \mathcal{V}(M)$.*

Proof Clearly $L_v L_w - L_w L_v$ is an $\mathbb{R}$-linear map of $\mathcal{E}(M)$ to itself. For $f, g \in \mathcal{E}(M)$, we know because $\mathcal{E}(M)$ is commutative that

$$\begin{aligned} L_v L_w (fg) &= L_v \bigl(L_w(f) g + f L_w g \bigr) \\ &= g L_v L_w f + L_v f L_w g + L_v g L_w f + f L_v L_w g \ . \end{aligned}$$

The claim is now obvious. ■

Let $v, w \in \mathcal{V}(M)$. Then it follows from Theorem 6.18 and Lemma 6.19 that there is exactly one smooth vector field $[v, w]$ on M such that

$$L_{[v,w]} = L_v L_w - L_w L_v \ . \tag{6.33}$$

We call $[v, w]$ the **Lie bracket** or the **commutator** of v and w.

6.20 Proposition

(i) *The map $\mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M)$, $(v, w) \mapsto [v, w]$ has the properties that*

(α) (bilinearity) $[\cdot, \cdot]$ *is $\mathbb{R}$-bilinear.*

(β) (skew-symmetry) $[v, w] = -[w, v]$ *for $v, w \in \mathcal{V}(M)$.*

(γ) (Jacobi identity) *$u, v, w \in \mathcal{V}(M)$ satisfy the relation*

$$\big[u,[v,w]\big] + \big[v,[w,u]\big] + \big[w,[u,v]\big] = 0 \ .$$

(ii) *In local coordinates,*

$$[v,w] = \sum\nolimits_{j,k} \Big(v^k \frac{\partial w^j}{\partial x^k} - w^k \frac{\partial v^j}{\partial x^k} \Big) \frac{\partial}{\partial x^j} \qquad (6.34)$$

for $v = \sum_j v^j \, \partial/\partial x^j$ and $w = \sum_j w^j \, \partial/\partial x^j$.

Proof The simple proofs are left to you. ■

6.21 Remarks (a) Suppose M is open in $\mathbb{R}^m$ and $(x^1, \ldots, x^m)$ are Euclidean coordinates on M. Using the nabla vector ∇, (6.34) can be written symbolically in the intuitive form

$$[v,w] = (v \cdot \nabla) w - (w \cdot \nabla) v \ .$$

(b) Suppose V is a vector space and $[\cdot\,,\cdot] : V \times V \to V$ is a map with the properties (α)–(γ) of Proposition 6.20(i). Then $\big(V, [\cdot\,,\cdot]\big)$ is called a **Lie algebra**. Because of (β), the "multiplication" $[\cdot\,,\cdot]$ is generally not commutative. It follows from (β) and (γ) that

$$\big[a,[b,c]\big] - \big[[a,b],c\big] = \big[[c,a],b\big] \quad \text{for } a,b,c \in V \ .$$

So the multiplication is generally not associative either. Thus a Lie algebra is generally a noncommutative, nonassociative algebra.[6] Therefore $\big(\mathcal{V}(M), [\cdot\,,\cdot]\big)$ is a Lie algebra.

(c) (regularity) Let $k \in \mathbb{N}$ and $v, w \in \mathcal{V}^k(M)$. Let M be a manifold of class C^{k+1}. Then L_v is not a derivation on $\mathcal{E}^{k+1}(M)$, because $L_v f$ for $f \in \mathcal{E}^{k+1}(M)$ generally only belongs to $\mathcal{E}^k(M)$. Therefore the Lie bracket cannot be defined through (6.33) either. We are left to choose local coordinates and to define $[v,w]$ for $v, w \in \mathcal{V}^k(M)$ through (6.34).[7] Then $[v,w] \in \mathcal{V}^{k-1}(M)$. ■

The Hodge–Laplace operator

In the rest of this section, we use the codifferential and the star operator to derive some more important relations from vector analysis.

Let $\big(M, (\cdot\,|\,\cdot)_M\big)$ be an oriented pseudo-Riemannian manifold. First we write the divergence in terms of the codifferential.

[6] In the trivial commutative case in which $[a,b] = 0$ for $a, b \in V$, it is of course commutative and associative.

[7] You may want to consider why $[v,w]$ so defined is well defined on all of M.

6.22 Proposition *The diagram*

$$\begin{array}{ccc} \mathcal{V}(M) & \xrightarrow{\ \Theta\ } & \Omega^1(M) \\ & {\scriptstyle \mathrm{div}} \searrow \quad \swarrow {\scriptstyle \delta} & \\ & \mathcal{E}(M) = \Omega^0(M) & \end{array}$$

is commutative, that is, $\operatorname{div} = \delta \circ \Theta$.

Proof If suffices to prove this equation locally. So let $\bigl((x^1,\ldots,x^m),U\bigr)$ be local coordinates. Then for $v = \sum_j v^j\,\partial/\partial x^j \in \mathcal{V}(U)$, Remarks 6.1(b) and 5.11(b) imply

$$\delta\Theta v = \delta \sum\nolimits_j \Bigl(\sum\nolimits_k g_{jk} v^k\Bigr)\,dx^j = \frac{1}{\sqrt{|G|}} \sum\nolimits_j \frac{\partial}{\partial x^j}\bigl(\sqrt{|G|}\,v^j\bigr)\ .$$

Therefore the claim follows from (6.14). ∎

Using the exterior derivative and the codifferential, we define for $0 \le r \le m$ an $\mathbb{R}$-linear map on $\Omega^r(M)$ by

$$\Delta_M := d\delta + \delta d\colon \Omega^r(M) \to \Omega^r(M)\ . \tag{6.35}$$

This is the **Hodge–Laplace operator**. For $a \in \mathcal{E}(M)$, it follows from (6.4) and Proposition 6.22 that

$$(d\delta + \delta d)a = \delta d a = \delta\Theta(\Theta^{-1} da) = \operatorname{div}\operatorname{grad} a\ .$$

Therefore the Hodge–Laplace operator on $\Omega^0(M) = \mathcal{E}(M)$ is the same as the Laplace–Beltrami operator, which justifies the notation. When M is clear from context, we write Δ for Δ_M. ∎

6.23 Remarks **(a)** $*\Delta = \Delta *$.

Proof Remarks 5.9(b) and 5.11(a) give

$$*\Delta = *d\delta + *\delta d = \delta d * + d\delta * = \Delta *$$

and therefore the claim. ∎

(b) $d\Delta = \Delta d = d\delta d$ and $\delta\Delta = \Delta\delta = \delta d\delta$.

Proof From $d^2 = 0$ we get

$$d\Delta = dd\delta + d\delta d = d\delta d = d\delta d + \delta dd = \Delta d\ .$$

The second claim follows analogously. ∎

(c) Suppose M is open in $\mathbb{R}^m$ and $(x^1, \ldots, x^m)$ are Euclidean coordinates. Then

$$\Delta\Bigl(\sum\nolimits_{(j)\in\mathbb{J}_r} a_{(j)}\, dx^{(j)}\Bigr) = \sum\nolimits_{(j)\in\mathbb{J}_r} \Delta a_{(j)}\, dx^{(j)}$$

for $1 \le r \le m$.

Proof Because of the linearity, it suffices to show the statement for $\alpha := a\, dx^{(j)}$ with $(j) \in \mathbb{J}_r$. Using Example 5.10(e), we find

$$\begin{aligned} d\delta\alpha &= d\Bigl(\sum_{k=1}^{r}(-1)^{k-1}\partial_{j_k} a\, dx^{j_1}\wedge\cdots\wedge\widehat{dx^{j_k}}\wedge\cdots\wedge dx^{j_r}\Bigr)\\ &= \sum_{k=1}^{r}(-1)^{k-1}\sum_{\ell=1}^{m}\partial_\ell\partial_{j_k} a\, dx^\ell\wedge dx^{j_1}\wedge\cdots\wedge\widehat{dx^{j_k}}\wedge\cdots\wedge dx^{j_r}\\ &= \sum_{k=1}^{r}\partial_{j_k}^2 a\, dx^{(j)} + \sum_{k=1}^{r}(-1)^{k-1}\sum_{\substack{\ell=1\\ \ell\notin\{j_1,\ldots,j_r\}}}^{m}\partial_\ell\partial_{j_k} a\, dx^\ell\wedge dx^{j_1}\wedge\cdots\wedge\widehat{dx^{j_k}}\wedge\cdots\wedge dx^{j_r}\ . \end{aligned}$$

Analogously, we get

$$\begin{aligned} \delta d\alpha = \delta\sum_{\substack{\ell=1\\ \ell\notin\{j_1,\ldots,j_r\}}}^{m}\partial_\ell a\, dx^\ell\wedge dx^{(j)} &= \sum_{\substack{\ell=1\\ \ell\notin\{j_1,\ldots,j_r\}}}^{m}\partial_\ell^2 a\, dx^{(j)}\\ &\quad - \sum_{\substack{\ell=1\\ \ell\notin\{j_1,\ldots,j_r\}}}^{m}\sum_{k=1}^{r}(-1)^{k-1}\partial_{j_k}\partial_\ell a\, dx^\ell\wedge dx^{j_1}\wedge\cdots\wedge\widehat{dx^{j_k}}\wedge\cdots\wedge dx^{j_r}\ . \end{aligned}$$

This implies[8]

$$\Delta_M\alpha = (d\delta+\delta d)\alpha = \Bigl(\sum\nolimits_k \partial_k^2 a\Bigr)\, dx^{(j)} = (\Delta a)\, dx^{(j)}$$

and therefore the claim. ■

(d) (regularity) Clearly Δ_M is an $\mathbb{R}$-linear map from $\Omega^r_{(k)}(M)$ to $\Omega^r_{(k-2)}(M)$ when $0 \le r \le m$ and $k \in \mathbb{N}$ with $k \ge 2$. In this case, it suffices to assume that M is a C^{k+2} manifold. ■

Finally, we define the **Laplace operator for vector fields**, namely, $\vec{\Delta}$, by

$$\vec{\Delta} := \vec{\Delta}_M := \Theta_M^{-1}\circ\Delta_M\circ\Theta_M : \mathcal{V}(M)\to\mathcal{V}(M)$$

and therefore by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{V}(M) & \xrightarrow{\ \vec{\Delta}\ } & \mathcal{V}(M)\\ \Theta\downarrow & & \downarrow\Theta\\ \Omega^1(M) & \xrightarrow{\ \Delta\ } & \Omega^1(M)\ . \end{array}$$

[8]We chose the sign in the definition of δ to be $(-1)^{m(r+1)}$ so that this formula would take this form. For the sign convention typically used in geometry, the formula is $(d\delta+\delta d)\alpha = -(\Delta a)\, dx^{(j)}$.

6.24 Remarks **(a)** Proposition 6.22 and (6.4) give $\vec{\Delta} = \operatorname{grad}\operatorname{div} + \Theta^{-1}\delta d\Theta$.

(b) Suppose M is open in $\mathbb{R}^m$, and $(x^1,\ldots,x^m)$ are Euclidean coordinates on M. Then

$$\vec{\Delta}\Bigl(\sum\nolimits_j v^j \frac{\partial}{\partial x^j}\Bigr) = \sum\nolimits_j \Delta v^j \frac{\partial}{\partial x^j} .$$

If we identify as usual the vector field $v = \sum_j v^j\, \partial/\partial x^j$ with $(v^1,\ldots,v^m)$, then $\vec{\Delta}v$ means that the Laplace operator can be applied componentwise

$$\vec{\Delta}v = (\Delta v^1,\ldots,\Delta v^m) .$$

In this case, we usually write Δ, not $\vec{\Delta}$.

Proof This follows from Example 6.2(a) and Remark 6.23(c). ■

(c) (regularity) Let $k \in \mathbb{N}$. Then $\vec{\Delta}$ maps the $\mathbb{R}$-vector space $\mathcal{V}^{k+2}(M)$ linearly into $\mathcal{V}^k(M)$. So here it suffices to assume that M is a C^{k+2} manifold. ■

The vector product and the curl

In this last section, we derive the most important properties of the curl operator.

Let $\bigl(M, (\cdot\,|\,\cdot)_M\bigr)$ be a three-dimensional oriented Riemannian[9] manifold with volume element ω_M.

On $\mathcal{V}(M)$, we define the **vector product** or **cross product**,

$$\times : \mathcal{V}(M)\times\mathcal{V}(M)\to\mathcal{V}(M) , \quad (v,w)\mapsto v\times w , \tag{6.36}$$

by

$$v\times w := \Theta_M^{-1}\omega_M(v,w,\cdot\,) . \tag{6.37}$$

Clearly this map is well defined.

6.25 Remarks **(a)** Suppose $\bigl(M,(\cdot\,|\,\cdot)_M\bigr) = \bigl(\mathbb{R}^3,(\cdot\,|\,\cdot)\bigr)$. Then in the case of a constant vector field, (6.37) agrees with the definition of Section VIII.2.

(b) The vector product is bilinear, alternating (skew symmetric), and satisfies

$$(u\,|\,v\times w)_M = \omega_M(u,v,w) \quad \text{for } u,v,w\in\mathcal{V}(M) . \tag{6.38}$$

For $p\in M$, the vector product $(v\times w)(p)$ is orthogonal to $v(p)$ and $w(p)$ with respect to the inner product $(\cdot\,|\,\cdot)_M(p)$ of T_pM. Letting $|v|_M := \sqrt{(v\,|\,v)_M}$, we have

$$|v\times w|_M = \sqrt{|v|_M^2\,|w|_M^2 - (v\,|\,w)_M^2} = |v|_M\,|w|_M\,\sin\varphi ,$$

where $\varphi(p)\in[0,\pi]$ is the angle between the vectors $v(p)$ and $w(p)$ for $p\in M$ and $v,w\in\mathcal{V}(M)$.

[9] For simplicity, we will restrict here to Riemannian metrics, as they are the most important in applications.

The vector product satisfies the **Grassmann identity**

$$v_1 \times (v_2 \times v_3) = (v_1 \,|\, v_3)_M v_2 - (v_1 \,|\, v_2)_M v_3 \ ,$$

the **Jacobi identity**

$$v_1 \times (v_2 \times v_3) + v_2 \times (v_3 \times v_1) + v_3 \times (v_1 \times v_2) = 0$$

and the relation

$$(v_1 \times v_2) \times (v_3 \times v_4) = \omega_M(v_1, v_2, v_4) v_3 - \omega_M(v_1, v_2, v_3) v_4$$

for $v_1, v_2, v_3, v_4 \in \mathcal{V}(M)$. In particular, $\big(\mathcal{V}(M), \times\big)$ is a Lie algebra.

Proof All of these reduce easily to pointwise statements already proved in Exercise 2.3. ■

(c) Suppose $\big((x^1, x^2, x^3), U\big)$ are positive orthonormal coordinates[10] on M. Then the cross product of vector fields $v = \sum_j v^j\, \partial/\partial x^j$ and $w = \sum_j w^j\, \partial/\partial x^j$ takes the form

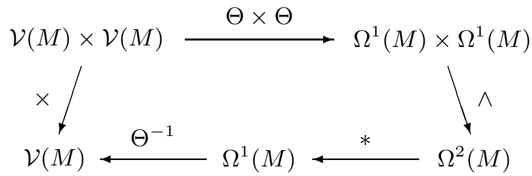
$$v \times w = (v^2 w^3 - v^3 w^2)\, \frac{\partial}{\partial x^1} + (v^3 w^1 - v^1 w^3)\, \frac{\partial}{\partial x^2} + (v^1 w^2 - v^2 w^1)\, \frac{\partial}{\partial x^3} \ .$$

Proof Exercise 2.3. ■

(d) (regularity) For $k \in \mathbb{N}$, the statements above remain true for C^k vector fields, and it suffices to assume that M is a C^{k+1} manifold. ■

The next theorem shows how the vector product is related to the exterior product of 1-forms.

6.26 Proposition *For $v, w \in \mathcal{V}(M)$, we have $v \times w = \Theta^{-1} {*} (\Theta v \wedge \Theta w)$, that is, the diagram*



commutes.

Proof It suffices to prove the equality locally, where we can choose positive orthonormal coordinates $\big((x^1, x^2, x^3), U\big)$. If (v^1, v^2, v^3) and (w^1, w^2, w^3) are the components of vector fields $v, w \in \mathcal{V}(M)$, then it follows from Remark 6.1(e) that

$$\begin{aligned} \Theta v \wedge \Theta w &= \sum\nolimits_j v^j\, dx^j \wedge \sum\nolimits_k w^k\, dx^k \\ &= (v^2 w^3 - v^3 w^2)\, dx^2 \wedge dx^3 + (v^3 w^1 - v^1 w^3)\, dx^3 \wedge dx^1 \\ &\qquad + (v^1 w^2 - v^2 w^1)\, dx^1 \wedge dx^2 \ . \end{aligned}$$

[10]That is, $(\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3)$ is a positive orthonormal frame.

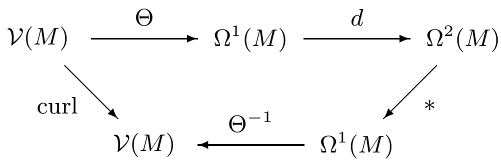
From the proof of Example 5.7(d), we know that

$$*(dx^2 \wedge dx^3) = dx^1 \ , \quad *(dx^3 \wedge dx^1) = dx^2 \ , \quad *(dx^1 \wedge dx^2) = dx^3 \ . \qquad (6.39)$$

Now the claim follows from Remarks 6.1(e) and 6.25(c). ■

We next derive a representation of the curl operator.

6.27 Proposition *The diagram*



commutes, that is, $\text{curl} = \Theta^{-1} {*} d \Theta$.

Proof It again suffices to prove the equality locally in positive orthonormal coordinates $\bigl((x^1, x^2, x^3), U\bigr)$. Then for $v = \sum_{j=1}^3 v^j \ \partial/\partial x^j$, we find using Remark 6.1(e) that

$$\begin{aligned} d(\Theta v) = d\Bigl(\sum\nolimits_j v^j \, dx^j\Bigr) &= \sum\nolimits_{j,k} \frac{\partial v^j}{\partial x^k} \, dx^k \wedge dx^j \\ &= \Bigl(\frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3}\Bigr) dx^2 \wedge dx^3 + \Bigl(\frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1}\Bigr) dx^3 \wedge dx^1 \\ &\qquad + \Bigl(\frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2}\Bigr) dx^1 \wedge dx^2 \ . \end{aligned}$$

Then the claim follows from (6.39) and Remarks 6.1(e) and 6.12(b). ■

We are now ready to deduce several important differential identities involving three-dimensional vector fields.

6.28 Proposition *For* $f \in \mathcal{E}(M)$ *and* $v, w \in \mathcal{V}(M)$*,*

(i) $\text{div}(v \times w) = (\text{curl}\, v \,|\, w)_M - (v \,|\, \text{curl}\, w)_M$;

(ii) $\text{curl}(fv) = f \,\text{curl}\, v + \text{grad}\, f \times v$;

(iii) $\text{curl}(v \times w) = (\text{div}\, w)v - (\text{div}\, v)w - [v, w]$;

(iv) $\text{curl}(\text{curl}\, v) = \text{grad}\,\text{div}\, v - \vec{\Delta} v$.

Proof (i) Putting $m = 3$ in Remark 5.9(d), we obtain

$$*\delta\alpha = (-1)^{m(r+1)} {*}{*} d {*} \alpha = d {*} \alpha \quad \text{for } \alpha \in \Omega^2(M) \ .$$

Now we use Propositions 6.22 and 6.26 to deduce

$$\begin{aligned}\operatorname{div}(v \times w) &= \delta\Theta\bigl(\Theta^{-1}{*}(\Theta v \wedge \Theta w)\bigr) = \delta{*}(\Theta v \wedge \Theta w)\\ &= {*}d(\Theta v \wedge \Theta w) = {*}(d\Theta v \wedge \Theta w - \Theta v \wedge d\Theta w)\ .\end{aligned}$$

From Proposition 6.27, it follows that $\Theta \operatorname{curl} = {*}d\Theta$. Now Remark 2.19(d) implies that $d\Theta = {*}\Theta \operatorname{curl}$, because $m = 3$ and $r = 2$. Hence we get

$$\begin{aligned}\operatorname{div}(v \times w) &= {*}\bigl(({*}\Theta \operatorname{curl} v) \wedge \Theta w - \Theta v \wedge {*}\Theta \operatorname{curl} w\bigr)\\ &= {*}(\Theta w \wedge {*}\Theta \operatorname{curl} v - \Theta v \wedge {*}\Theta \operatorname{curl} w)\ ,\end{aligned}$$

where we have used ${*}\Theta \operatorname{curl} v \in \Omega^2(M)$. Now (2.22), with $r = 1$, and (2.13) give

$$\operatorname{div}(v \times w) = {*}\bigl[(w \,|\, \operatorname{curl} v)_M - (v \,|\, \operatorname{curl} w)\bigr]\omega_M\ .$$

The claim now follows from ${*}\omega_M = \mathbf{1}$.

(ii) Proposition 6.27 gives

$$\begin{aligned}\operatorname{curl}(fv) &= \Theta^{-1}{*}d\Theta(fv) = \Theta^{-1}{*}d(f\Theta v)\\ &= \Theta^{-1}{*}(df \wedge \Theta v + f d\Theta v)\\ &= \Theta^{-1}{*}(\Theta \operatorname{grad} f \wedge \Theta v) + f\Theta^{-1}{*}d\Theta v\\ &= \operatorname{grad} f \times v + f \operatorname{curl} v\ .\end{aligned}$$

Here we have also made use of Proposition 6.26 and properties of d.

(iii) It suffices to prove the statement locally. We can use positive orthonormal coordinates. Then the claim follows from the local representations of Remarks 6.12(b) and 6.25(c) and from Proposition 6.20 after a simple calculation, which we leave to you.

(iv) It follows from Proposition 6.27 and the definition of δ that

$$\begin{aligned}\operatorname{curl}\operatorname{curl} v &= \Theta^{-1}{*}d\Theta\Theta^{-1}{*}d\Theta v = \Theta^{-1}{*}d{*}d\Theta v\\ &= (-1)^{3(2+1)}\Theta^{-1}\delta d\Theta v = -\Theta^{-1}\delta d\Theta v\ .\end{aligned}$$

Now the claim follows from Remark 6.24(a). ■

To demonstrate the power of the new calculus, we proved part (i) using the properties of the codifferential and the star operator. Of course, we could also have worked in the orthonormal coordinates of a positive chart. In other words, we can assume that M is open in $\mathbb{R}^3$ and $(\cdot\,|\,\cdot)_M$ is the standard metric $(\cdot\,|\,\cdot)$. Using the (formal) nabla operator in (6.38), we obtain

$$\begin{aligned}\nabla \cdot (v \times w) &= \det[\nabla, v, w]\\ &= \partial_1(v^2w^3 - v^3w^2) + \partial_2(v^3w^1 - v^1w^3) + \partial_3(v^1w^2 - v^2w^1)\end{aligned}$$

by expanding the (formal) determinant in its first row. By using the product rule, we see easily that the last row agrees with the expression $w \cdot \operatorname{curl} v - v \cdot \operatorname{curl} w$, as claimed in (i).

However, the formal calculus with the nabla operator must be used with caution. For example, if we formally calculate $\operatorname{curl}(v \times w) = \nabla \times (v \times w)$ using the Grassmann identity, we find the *false* statement

$$\nabla \times (v \times w) = (\nabla \cdot w)v - (\nabla \cdot v)w \ .$$

Where was the mistake?

Exercises

1 Find the representation of the Laplace–Beltrami operator with respect to

(i) the cylindrical coordinates $(0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$, $(\varphi, z) \mapsto (\cos\varphi, \sin\varphi, z)$;

(ii) the parametrization

$$(0, 2\pi)^2 \to \mathbb{R}^3 \ , \quad (\alpha, \beta) \mapsto \big((2 + \cos\alpha)\cos\beta, (2 + \cos\alpha)\sin\beta, \sin\alpha\big)$$

of the 2-torus $\mathsf{T}^2_{2,1}$ of Example VII.9.11(f);

(iii) the parametrization $X \to \mathbb{R}^3$, $x \mapsto \big(x, f(x)\big)$ of the graph of $f \in \mathcal{E}(X)$, when X is open in $\mathbb{R}^2$.

2 Let (M_j, g_j) for $j = 1, 2$ be Riemannian manifolds with $\partial M_1 = \emptyset$, and let π_j denote the canonical projection $M_1 \times M_2 \to M_j$. Show that

$$\Delta_{M_1 \times M_2} = \pi_1^* \Delta_{M_1} + \pi_2^* \Delta_{M_2} \ .$$

3 Suppose M and N are Riemannian manifolds and $f : M \to N$ is an isometric diffeomorphism. Then for $0 \le r \le m$, show that the diagram

$$\begin{array}{ccc} \Omega^r(M) & \xrightarrow{\ \Delta_M\ } & \Omega^r(M) \\ {\scriptstyle f^*}\big\uparrow & & \big\uparrow{\scriptstyle f^*} \\ \Omega^r(N) & \xrightarrow{\ \Delta_N\ } & \Omega^r(N) \end{array}$$

commutes.

4 Let (M, g) be a pseudo-Riemannian manifold. Show the commutativity of the diagram

$$\begin{array}{ccccc} \Omega^1(M) & & \xrightarrow{\ \ *\ \ } & & \Omega^{m-1}(M) \\ & {\scriptstyle \Theta}\nwarrow & & \nearrow{\scriptstyle \lrcorner} & \\ & & \mathcal{V}(M) & & \end{array}$$

and derive the relations

(i) $\operatorname{div} = {*d*}\Theta$;

(ii) $\operatorname{curl} = \Theta^{-1}{*d}\Theta \quad (m = 3)$;

(iii) $\Delta_M = *d*d$,

where Δ_M is the Laplace–Beltrami operator of M.

5 Let Ω be open in $\mathbb{R}^3$. For $E, B, j \in C^\infty(\mathbb{R}\times\Omega, \mathbb{R}^3)$, $\rho \in C^\infty(\mathbb{R}\times\Omega, \mathbb{R})$, and $c > 0$, set

$$F := \Theta_e E \wedge (c\,dt) + *\bigl(\Theta_e B \wedge (c\,dt)\bigr)\ , \quad J := \Theta_e j - \rho\,dt \in \Omega(\mathbb{R}^4_{1,3})\ ,$$

where E, B, and j are seen as time-dependent vector fields, ρ is seen as a time-dependent function on Ω, and $\Theta_e : \mathcal{V}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ denotes the (Euclidean) Riesz isomorphism. Also let dt be the first standard basis vector in $\Omega^1(\mathbb{R}^4_{1,3})$. Now show these facts:

(a) The statements

(i) $dF = 0$;

(ii) $\partial B/\partial t + c \operatorname{curl} E = 0$ and $\operatorname{div} B = 0$

are equivalent. (These are the homogeneous Maxwell's equations.) That is, the 2-form F is closed if and only if the vector fields E and B satisfy these two of Maxwell's equations.

(b) The statements

(i) $dF = 4\pi J$;

(ii) $\partial E/\partial t - c \operatorname{curl} B = 4\pi j$ and $\operatorname{div} E = 4\pi\rho$

are equivalent. (These are the Maxwell's equations with sources.)

(c) The statments

(i) $\Delta_{\mathbb{R}^4_{1,3}} F = 0$;

(ii) $\partial B/\partial t + c \operatorname{curl} E = 0$, $\partial E/\partial t - c \operatorname{curl} B = 0$, $\operatorname{div} E = 0$, $\operatorname{div} B = 0$

are equivalent. Therefore the 2-form F is harmonic if and only if the vector fields E and B satisfy the homogeneous Maxwell's equations.

(d) If $dF = 0$, then j and ρ satisfy the **continuity equation**

$$\partial\rho/\partial t + \operatorname{div} j = 0\ ,$$

which can also be written $\operatorname{grad}_{\mathbb{R}^4_{1,3}} J = 0$.

(e) These two statements are equivalent:

(i) F is exact;

(ii) There are an $A \in C^\infty(\mathbb{R}\times\Omega, \mathbb{R})$, a **vector potential**, and $\Phi \in C^\infty(\mathbb{R}\times\Omega, \mathbb{R})$, a **scalar potential**, with

$$\operatorname{curl} A = B \quad \text{and} \quad -\partial A/\partial t - \operatorname{grad}\Phi = E\ .$$

6 Suppose X is open in $\mathbb{R}^3$ and contractible. Also suppose $f, g \in \mathcal{E}(X)$. Show:

(i) There is a $v \in \mathcal{V}(X)$ such that $\operatorname{grad} f \times \operatorname{grad} g = \operatorname{curl} v$.

(ii) If $f(x) \neq 0$ for $x \in X$, then there is an $h \in \mathcal{E}(X)$ with $(\operatorname{grad} f)/f = \operatorname{grad} h$.

7 Verify that

$$\vec{\Delta}_M(f \operatorname{grad} f) = \operatorname{grad}\operatorname{div}(f \operatorname{grad} f) = \Delta_M f \operatorname{grad} f + \operatorname{grad} |\operatorname{grad} f|^2_M + f \operatorname{grad} \Delta_M f$$

for $f \in \mathcal{E}(M)$, where $|v|^2_M := (v\,|\,v)_M$ for $v \in \mathcal{V}(M)$.

8 Show that $\alpha \in \Omega^1(M)$ and $v, w \in \mathcal{V}(M)$ satisfy

$$d\alpha(v,w) = \mathcal{L}_v \langle \alpha, w \rangle - \mathcal{L}_w \langle \alpha, v \rangle - \big\langle \alpha, [v,w] \big\rangle \ .$$

9 Let M and N be m-dimensional manifolds, and let $\varphi \in \mathrm{Diff}(M,N)$. Show that

$$\varphi_*[v,w] = [\varphi_* v, \varphi_* w] \quad \text{for } v, w \in \mathcal{V}(M) \ .$$

10 Let $T^2 := S^1 \times S^2 \subset \mathbb{R}^4$, and let $\alpha, \beta \in \Omega^1(T^2)$ with

$$\alpha := -x^2\, dx^1 + x^1\, dx^2 \ , \quad \beta := -x^4\, dx^3 + x^3\, dx^4 \ .$$

Show that $\Delta\alpha = \Delta\beta = 0$.

11 Show that for $H \in \mathcal{E}(\mathbb{R}^{2m})$ the vector field $\mathrm{sgrad}\, H \in \mathcal{V}(\mathbb{R}^{2m})$ is divergence free.