A rational treatment of the relations of balance for mechanical systems with a time-variable mass and other non-classical supplies

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Abstract This contribution intends to present a rational methodology for mechanical systems with a variable mass, represented by a supply of mass. Special emphasis is given to the relations of balance and jump for such systems. In these relations, we also allow for other types of additional, non-classical supplies, e.g., supplies of linear and angular momentum. In doing so, we aim at completing and substantially extending formulations laid down in the famous article by Truesdell and Toupin (1960), who stated local relations of balance of mass and linear momentum in the presence of sources of mass, and, among other formulations with relevance to the present article, gave fundamental formulations for the case that a flow of mass through the surface of the system is present in the global relations of balance.

Our presentation is organized as follows: We remain in the framework of non-relativistic mechanics, referring to a common inertial frame. Throughout the Chapter, we formulate our relations in the Euler or spatial description, in which every entity is understood as a function of the instantaneous place of the material particles under consideration, and of time. In Section 1, the general equation of balance is stated and is applied to the model of a single mass point with a variable mass. This general equation is specified for the fundamental relations of balance of mass, linear momentum, angular momentum and total energy first. The variable mass is associated with a supply of mass. Afterwards, as mathematical consequences of the fundamental statements, we derive the statements of balance of moment of momentum, intrinsic spin, kinetic energy and internal energy for the single mass point. As a rational procedure for formulating the additional, non-classical supplies that are present in the relations of balance, we assume that the single mass point is gaining or losing differential masses by means of continuous impacts, which

H. Irschik, A. K. Belyaev (Eds.), *Dynamics of Mechanical Systems with Variable Mass*,
CISM International Centre for Mechanical Sciences DOI 10.1007/ 978-3-7091-1809-2_1

are again studied in the framework of the general equation of balance. The outcomes of this procedure include a Seeliger-Meschersky type additional supply of linear momentum.

In Section 2, theorems on balance of mass, linear momentum, moment of momentum and kinetic energy for deformable bodies of finite extension with a variable mass are presented. Among these, the first two can be considered as fundamental, while balance of moment of momentum and kinetic energy are derived from balance of mass and linear momentum as mathematical consequences. The supply of mass is associated with distributed sources of mass attached to the material particles, which we call material sources of mass. Both global and local relations of balance are considered, including global and local non-classical supplies of mass and linear momentum. The supplies of moment of momentum and kinetic energy follow as mathematical consequences. A Seeliger-Meschersky type local model for the non-classical supply of linear momentum is presented. Due to limited space, the fundamental relations of balance of angular momentum and total energy for bodies of finite extension are not considered. However, useful global relations concerning the notion of center-of-mass are given, introducing the notions of center-of-mass linear momentum and relative linear momentum, center-of-mass moment of momentum and relative moment of momentum, as well as center-of-mass kinetic energy and relative kinetic energy. Our relations extend some formulations that are well-known for bodies in the absence of a supply of mass. The corresponding relations of balance again follow as mathematical consequences of the fundamental ones, including non-classical supply terms related to the non-classical supplies of mass and linear momentum. In Section 3, global relations of balance for open systems are studied, and are set into analogy to the results of Section 2. An open system is represented by a non-material control volume, the surface of which moves at a velocity different from the velocity of the material particles instantaneously located on that surface, such that a flow of mass takes place. Supplies of mass and linear momentum due to this flow of mass are shown to be analogous to the supplies introduced in Section 2. The theoretically as well as practically important special case of a rigid body that experiences a surface growth is exemplarily treated. Section 4 deals with extended relations of jump for systems with a variable mass. Relations of jump are needed, when certain entities suffer considerable changes across some region of transition. This region of transition is replaced by an equivalent singular surface, for which relations of jump are formulated by including additional non-classical surface supply terms, such as surface supply of mass and linear momentum. Other surface supply terms are derived as mathematical consequences of the latter. As an example for the formulations presented in Sections 3 and 4, the problem of a chain heaped up on a table, the hanging part of the chain being set into motion, is considered in Section 5. It is believed that the corresponding formulation can explain some seemingly controversial results from the literature.

It should be mentioned that our emphasis lies on a rational treatment of the topics under consideration. While our methodology has many important predecessors, but appears to be novel in the systematic manner here presented, we do not intend to give a historical review on the topic due to the limited space available. For the latter, the reader is referred to reviews by Mikhailov (1975), historical presentations to be found, e.g., in the important works of Eke (1998) and Cveticanin (1998), as well as to a review by Irschik and Holl (2004) on balance of mass and momentum for systems with a variable mass. (An extended review by the latter authors concerning balance of moment of momentum and kinetic energy for variable mass systems is being prepared since and hoped to be finished soon). Last but not least, the reader is referred to the other chapters of the present book.

1 An introductory example: The single mass point with a continual time-variation of mass.

1.1 The general and the differential relations of balance

In this Section, we present introductory material on the relations of balance for mechanical systems with a variable mass. The general equation of balance for any system can be written as:

$$Q(t+\Delta t) - Q(t) = \int_{t}^{t+\Delta t} R\,(\tau)\, d\tau \tag{1}$$

In (1), Q denotes some physical meaningful quantity that properly describes the system within the time interval $t \leq \tau \leq t + \Delta t$, and R is the physical cause that is responsible for a time change of Q, also denoted as the source of that time change. When $Q(t + \Delta t) = Q(t)$, i.e., when the integral at the right hand side of (1) does vanish, then $Q(t)$ is said to be conserved with respect to the time instant $t + \Delta t$. When the right hand side of (1) does not vanish, Q is said to be balanced by the integral of the source R over the time interval under consideration.

Eq. (1) represents the most general statement of balance; in order to make physical sense, it is only necessary to require that Q has unique values at the instants t and $t + \Delta t$, and that the integral over the source R does exist, but the source R itself does not need to be continuous in time.

A large part of the history of mechanics (and of physics as a whole) can be understood as an intense struggle for finding physically meaningful quantities Q and corresponding physically meaningful sources R that do satisfy (1). In the Newtonian theory of mechanics, which is used subsequently, time t and quantities Q are defined with respect to an inertial frame, and Q basically stands for the notions of mass, momentum and energy. The notion of energy directly connects the fields of mechanics and thermodynamics, the latter bringing into the play the additional notion of entropy, which, however, will not be addressed in the following due to limited space.

In case R is continuous and bounded, it makes sense to consider an infinitesimal time interval, $\Delta t \to dt$. Then the finite statement in (1) can be replaced by the following differential relation of balance:

$$Q(t + dt) - Q(t) = dQ = R\,dt \tag{2}$$

In the following Subsections 1.2–1.5, we apply the differential statement (2) to the model of a single mass point, also denoted as a point mass, a problem that is elementary for the dynamics of mechanical systems. In the present context, (2) is specified for several relations of balance for the single mass point with a time-varying mass, $m = m(t)$. The source that is responsible for the change in mass will be denoted as a supply of mass. We start with the fundamental relations of balance of mass, and discuss the relations of linear momentum, angular momentum and total energy afterwards. Into these relations, we incorporate further supply terms, additional to classical formulations. For a comprehensive representation of the classical balance statements of mechanics, see, e.g., Ziegler (1998). To add non-classical supply terms to the classical statements can be motivated, e.g., by the theory of multiphase mixtures, in which it is assumed that a particle of a single constituent exchanges mass, momentum and energy with the particles of the other constituents of the mixture, and thus is being supplied with the latter entities, see, e.g., the book by Hutter and Jöhnk (2004).

A continuous impact model then is presented in order to express the non-classical supplies of linear momentum, angular momentum and energy. This model assures that mass, linear momentum and energy is continuously gained from differential masses at an own velocity. As a special case, the Seeliger-Meshchersky formulation for the supply of linear momentum is contained therein.

Having stated the fundamental relations of balance of mass, linear momentum, angular momentum and total energy for a single mass point with a supply of mass and other non-classical supplies, we proceed to consequences of the latter fundamental statements, and derive the statements of balance of moment of momentum, intrinsic spin, kinetic energy and internal energy. All the above mentioned relations of balance will be referred to a common inertial frame without further reference.

1.2 Balance of mass

For balance of mass, (2) reads

$$Q\,(t+dt) - Q\,(t) = dQ = dm, \quad R = s\,[m] \tag{3}$$

The supply of mass from or to the environment is denoted as $s\,[m]$. If mass is added to the point mass, then $s\,[m] > 0$, and if mass is ejected from the point mass, there is $s\,[m] < 0$. Note that the notation $s\,[\lambda]$, say, reads *"the supply of* λ*"*, where the physical dimension of $s\,[\lambda]$ is the dimension of λ per dimension of time. Using (2) and (3), balance of mass becomes:

$$\frac{dm}{dt} = s\,[m] \tag{4}$$

1.3 Balance of linear momentum

The vector of linear momentum is defined as

$$j = m\,v \tag{5}$$

where the absolute velocity vector is $v = dp/dt$, and p is the position vector of the single mass point with respect to the origin of the inertial frame. For linear momentum j, the quantities in (2) are:

$$dQ = dj, \quad R = F + s\,[j] \tag{6}$$

The resultant of imposed and restoring forces that acts upon the single mass point is the vector F, and the vector $s\,[j]$ stands for a non-classical (additional) supply of momentum. Utilizing balance of mass (4), the following relation of balance of linear momentum for a single mass point is obtained from (2) and (6):

$$\frac{dj}{dt} = m\frac{dv}{dt} + s\,[m]\,v = F + s\,[j] \tag{7}$$

1.4 Balance of angular momentum

The vector of angular momentum is defined as

$$\alpha = p \times m\,v + l \tag{8}$$

where the vector product $p \times m\, v$ is called the moment of (linear) momentum, and the vector l denotes an intrinsic spin of the point mass. For angular momentum, the quantities in (2) are:

$$dQ = d\alpha, \quad R = p \times F + M + s\,[\alpha] \tag{9}$$

The resultant of imposed and restoring couples that act upon the mass point is abbreviated by the (free) vector M. The notions of intrinsic spin l and the resultant couple M, while often not taken into account in the mechanics of single mass points, have been introduced in (9) in order to render balance of angular momentum a relation of balance in its own right, bringing additional physical notions into the play that are not present in balance of linear momentum (7). Motivations for introducing l and M can be taken, e.g., from the model of material particles in a continuous polar medium. For a recent comprehensive representation of the theory of micro-polar media, see the book by Eremeyev et al. (2012). Another example is the case that a single mass point shall be used to model the rotational motion of a rigid body of finite extension. From these examples, it makes sense to set

$$l = J \cdot \omega \tag{10}$$

where J is the symmetric second order tensor of inertia, and ω is the angular velocity vector of the mass point. Note that we use the simple single dot product operation that has been introduced in the exposition on tensor fields by Ericksen (1960). Introducing a non-classical (additional) supply of angular momentum $s\,[\alpha]$, using balance of mass (4), and substituting (10), the relation of balance of angular momentum is obtained from (2) and (9) as

$$\frac{d\alpha}{dt} = p \times m \frac{dv}{dt} + J \cdot \frac{d\omega}{dt} + p \times s\,[m]\, v + \frac{dJ}{dt} \cdot \omega = p \times F + M + s\,[\alpha] \tag{11}$$

1.5 Balance of total energy

The total energy E of the mass point is defined as the sum

$$E = E_{kin} + E_{int} \tag{12}$$

where the kinetic energy of the mass point is given by

$$E_{kin} = \frac{1}{2}\left(m\, v \cdot v + \omega \cdot (J \cdot \omega)\right) \tag{13}$$

The internal energy, a notion that stems from thermodynamics, is abbreviated by E_{int}. For total energy, the quantities in (2) are:

$$dQ = dE, \quad R = F \cdot v + M \cdot \omega + r + s\,[E] \tag{14}$$

where r stands for non-mechanical sources of total energy, and $s\,[E]$ is a non-classical (additional) supply of energy. Using balance of mass (4), and substituting (10) and (13), the relation of balance of total energy is obtained from (2) and (12) as

$$\frac{dE}{dt} = m\,v \cdot \frac{dv}{dt} + \omega \cdot \left(J \cdot \frac{d\omega}{dt} \right) + \frac{1}{2} \left(s\,[m]\; v \cdot v + \omega \cdot \left(\frac{dJ}{dt} \cdot \omega \right) \right) + \frac{dE_{int}}{dt}$$
$$= F \cdot v + M \cdot \omega + r + s\,[E] \quad (15)$$

Balance of total energy is also known as the *First Law of Thermodynamics*; the relation in (14) accounts for the case of a variable mass and an additional supply of total energy.

1.6 A rational procedure for obtaining expressions for the additional supplies

Additional modeling is necessary in order to describe the supply of mass $s\,[m]$. The same is true for the non-classical supplies $s\,[j]$, $s\,[\alpha]$ and $s\,[E]$. In the present subsection, we introduce a rational model, in which the latter supplies can be expressed using the former. This model may be called an extension of the continuous impact model by Cayley (1856), who assumed that the single mass under consideration *"is continually taking into connexion with itself particles of infinitesimal mass [...], so as not itself to undergo any abrupt change of velocity, but to subject to abrupt changes of velocity the particles so taken into connexion."*

Hence, consider a mass point with an infinitesimal mass of amount $d\bar{m} = dm = s\,[m]\,dt$, with $s\,[m] > 0$. Assume that, at the time instant t, the linear momentum of the infinitesimal mass is $dm\,u$, the intrinsic spin is $dJ \cdot \Omega$, and the internal energy is $d\bar{E}_{int}$, where u is the velocity vector of the infinitesimal mass, and Ω is the angular velocity vector of the infinitesimal mass at time t. Assume further that this infinitesimal mass $d\bar{m}$ is absorbed during the time-interval dt by the mass point with finite mass m, where the linear momentum, angular momentum and energy of the infinitesimal mass are completely transferred to the mass m at time $t + dt$. In accordance with (2), the equations of balance for $d\bar{m}$ become:

Balance of linear momentum

$$dQ = Q\,(t + dt) - Q\,(t) = 0 - dm\,u = -dm\,u, \quad R = s\,[\bar{j}] \quad (16)$$

$$\Rightarrow \quad \frac{dm}{dt}\,u = s\,[m]\;u = -s\,[\bar{j}] \quad (17)$$

Balance of angular momentum

$$dQ = -p \times dm\, u - dJ \cdot \Omega, \quad R = s\left[\bar{\alpha}\right] \tag{18}$$

$$\Rightarrow\; p \times \frac{dm}{dt} \cdot u + \frac{dJ}{dt} \cdot \Omega = p \times s\left[m\right] \cdot u + \frac{dJ}{dt} \cdot \Omega = -s\left[\bar{\alpha}\right] \tag{19}$$

Balance of energy

$$dQ = -\frac{1}{2}\left(dm\, u \cdot u + \Omega \cdot (dJ \cdot \Omega)\right) - d\bar{E}_{int}, \quad R = s\left[\bar{E}\right] \tag{20}$$

$$\Rightarrow\; \frac{1}{2} s\left[m\right] u \cdot u + \frac{1}{2}\Omega \cdot \left(\frac{dJ}{dt} \cdot \Omega\right) + d\bar{E}_{int} = -s\left[\bar{E}\right] \tag{21}$$

Above, no imposed or reactive forces and couples or non-mechanical sources of energy have been considered for the infinitesimal mass for the sake of brevity. Requiring that the supplies for m and $d\bar{m}$ must be mutual, we obtain:

$$s\left[j\right] = -s\left[\bar{j}\right] = s\left[m\right] u \tag{22}$$

$$s\left[\alpha\right] = -s\left[\bar{\alpha}\right] = p \times s\left[m\right] u + \frac{dJ}{dt} \cdot \Omega \tag{23}$$

$$s\left[E\right] = -s\left[\bar{E}\right] = \frac{1}{2} s\left[m\right] u \cdot u + \frac{1}{2}\Omega \cdot \left(\frac{dJ}{dt} \cdot \Omega\right) + d\bar{E}_{int} \tag{24}$$

Substituting into the fundamental equations of balance of linear momentum, angular momentum and total energy stated in Subsections 1.3–1.5, these relations become:

$$\frac{dj}{dt} = m\frac{dv}{dt} + s\left[m\right]\, v = F + s\left[m\right] u \tag{25}$$

$$\begin{aligned}\frac{d\alpha}{dt} &= p \times m\frac{dv}{dt} + J \cdot \frac{d\omega}{dt} + \; p \times s\left[m\right] v + \frac{dJ}{dt} \cdot \omega \\ &= p \times F + M + p \times s\left[m\right] u + \frac{dJ}{dt} \cdot \Omega \end{aligned} \tag{26}$$

$$\begin{aligned}\frac{dE}{dt} &= m\, v \cdot \frac{dv}{dt} + \omega \cdot \left(J \cdot \frac{d\omega}{dt}\right) + \frac{1}{2}\left(s\left[m\right] v \cdot v + \omega \cdot \left(\frac{dJ}{dt} \cdot \omega\right)\right) + \frac{d}{dt}E_{int} \\ &= F \cdot v + M \cdot \omega + r + \frac{1}{2}\left(s\left[m\right] u \cdot u + \Omega \cdot \left(\frac{dJ}{dt} \cdot \Omega\right)\right) + \frac{d}{dt}\bar{E}_{\mathrm{int}} \end{aligned} \tag{27}$$

For the sake of comparison with the literature, the relation of balance of linear momentum is re-written as

$$m\frac{dv}{dt} = F + s\,[m]\,\,(u - v) \tag{28}$$

The term $s\,[m]\,\,(u-v)$ dates back to Seeliger (1890) and Meshchersky (1897); it is called the Meshchersky reactive force, while (28) as a whole is denoted as the Tsiolkovsky-Meshchersky rocket equation. Tsiolkovsky in 1897 independently derived the solution for the case of a constant relative velocity $u-v$, cf. Kosmodemyansky (2000). For historical expositions, see Mikhailov (1975) and Irschik and Holl (2004). The results in (13)–(17) also hold for the case $s\,[m] < 0$. We then assume that the infinitesimal mass $d\bar{m} = -dm > 0$ is ejected during the time-interval dt from the mass point m, where momentum, angular momentum and energy of the infinitesimal mass have been completely released from m at time instant $t + dt$. E.g., balance of linear momentum for the infinitesimal mass reads, cf. (2):

$$dQ = Q\,(t + dt) - Q\,(t) = d\bar{m}\,u - 0 = -dm\,u, \quad R = s\,[\bar{j}] \tag{29}$$

which again yields (17).

It must be emphasized that the above continuous impact model does result in balance relations that are invariant with respect to a change of the common inertial frame. In these relations, we may add a constant position vector to the vector p, a constant velocity vector to v and u, etc., and then subtract the results from the original formulations without obtaining any discrepancies. E.g., adding a constant position vector to p in (26), the difference of the result with respect to the original relation (26) vanishes, since (25) holds. Of course, the supply of mass $s\,[m]$ itself and the internal energies must also be formulated accordingly.

1.7 Consequences of the fundamental relations of balance

The fundamental relations of balance can be mathematically manipulated in order to obtain further relations of balance. This will be done in the following Subsection using the continuous impact model presented in Subsection 1.6 above, i.e., by studying mathematical consequences of the relations stated in (25)–(27). Particularly, we derive balance relations for moment of momentum, intrinsic spin, translational and rotational kinetic energy and internal energy from the fundamental relations. The additional supply terms in the latter derived relations are expressed by the supply terms in the fundamental relations of balance, as it should be. For a systematic treatment of relations between non-classical supply or growth terms in the framework of continuum mechanics, the reader is referred to Irschik (2005, 2007).

Balance of moment of momentum Performing the vector product of the balance of linear momentum (25) with the position vector p yields

$$p \times m\frac{dv}{dt} + p \times s\,[m]\,v = \frac{d}{dt}\,(p \times m\,v) = p \times F + p \times s\,[m]\,u \tag{30}$$

Introducing the moment of momentum as

$$\alpha^* = p \times m\,v = \alpha - l \tag{31}$$

we obtain:

$$\frac{d\alpha^*}{dt} = p \times m\,\frac{dv}{dt} + p \times s\,[m]\;v = p \times F + s\,[\alpha^*] \tag{32}$$

with the non-classical supply of moment of momentum

$$s\,[\alpha^*] = p \times s\,[m]\;u = p \times s\,[j] \tag{33}$$

Balance of intrinsic spin Subtracting (32) from balance of angular momentum (26) gives:

$$\frac{dl}{dt} = J \cdot \frac{d\omega}{dt} + \frac{dJ}{dt} \cdot \omega\; = M + s\,[l] \tag{34}$$

with the non-classical supply of intrinsic spin

$$s\,[l]\; = \frac{dJ}{dt} \cdot \Omega \tag{35}$$

Balance of translational kinetic energy Performing the scalar product of the balance of linear momentum (25) with the velocity v yields

$$v \cdot m\frac{dv}{dt} + v \cdot s\,[m]\,v = \frac{d}{dt}\left(\frac{1}{2}m\,v \cdot v\right) + s\,[m]\,\frac{1}{2}v \cdot v = F \cdot v + s\,[m]\,u \cdot v \tag{36}$$

Introducing the translational kinetic energy as

$$E^{tr}_{kin} = \frac{1}{2}m\,v \cdot v \tag{37}$$

we obtain:

$$\frac{dE^{tr}_{kin}}{dt} = v \cdot m\,\frac{dv}{dt} + \frac{1}{2}s\,[m]\,v \cdot\; v = F \cdot v + s\left[E^{tr}_{kin}\right] \tag{38}$$

with the non-classical supply

$$s\left[E^{tr}_{kin}\right] = s\,[m]\left(u - \frac{1}{2}v\right) \cdot v \tag{39}$$

Balance of rotatory kinetic energy Performing the scalar product of the balance of intrinsic spin (34) with the angular velocity ω yields

$$\begin{aligned}\omega\cdot\frac{dl}{dt}&=\omega\cdot\left(J\cdot\frac{d\omega}{dt}\right)+\omega\cdot\left(\frac{dJ}{dt}\cdot\omega\right)\\&=\frac{d}{dt}\left(\frac{1}{2}\omega\cdot(J\cdot\omega)\right)+\frac{1}{2}\omega\cdot\left(\frac{dJ}{dt}\cdot\omega\right)=M\cdot\omega+\omega\cdot\left(\frac{dJ}{dt}\cdot\Omega\right)\end{aligned}\tag{40}$$

Introducing the rotational kinetic energy as

$$E_{kin}^{rot}=\frac{1}{2}\,\omega\cdot(J\cdot\omega)\tag{41}$$

we obtain:

$$\frac{dE_{kin}^{rot}}{dt}=\omega\cdot\left(J\cdot\frac{d\omega}{dt}\right)+\frac{1}{2}\omega\cdot\left(\frac{dJ}{dt}\cdot\omega\right)=M\cdot\omega+s\left[E_{kin}^{rot}\right]\tag{42}$$

with the non-classical supply of rotatory kinetic energy

$$s\left[E_{kin}^{rot}\right]=\omega\cdot\left(\frac{dJ}{dt}\cdot\left(\Omega-\frac{\omega}{2}\right)\right)\tag{43}$$

Balance of internal energy Subtracting the relations of balance for kinetic energy, (38) and (42), from the relation of balance of total energy in the form of (27), we obtain:

$$\frac{dE_{int}}{dt}=r+s\left[E_{int}\right]\tag{44}$$

with the non-classical supply of internal energy

$$s\left[E_{int}\right]=\frac{1}{2}s\left[m\right](u-v)\cdot(u-v)+\frac{1}{2}\left(\Omega-\omega\right)\cdot\left(\frac{dJ}{dt}\cdot(\Omega-\omega)\right)+\frac{d}{dt}\bar{E}_{int}\tag{45}$$

2 Balance relations for bodies of finite extension with a variable mass

So far, we have dealt with the relations of balance for a single mass point with a variable mass. In the following, we extend these considerations to the case of a material body of finite size. Hence, we subsequently deal with a mechanical system that consists of an infinite set of continuously distributed material particles, for which mass is not conserved. Due to limited space, we remain in the framework of purely mechanical notions in the present section, dealing with balance of mass, linear momentum, moment of momentum and kinetic energy, the latter two as consequences of mathematical manipulations of the former ones. This restriction is also motivated by a remark of Truesdell and Toupin (1960), who stated that the concepts of linear momentum, moment of momentum and kinetic energy of a finite body are *"the stuff of which classical mechanics is made"*, and that they *"deserve the most minute analysis"*. It is the scope of the present section to present a rational methodology for formulating relations of balance for these quantities, considering the case of a body with variable mass and a finite extension.

The relations of balance in their most basic form, see (2), are to be referred to the total of a system under consideration, in the present case by integrating over the finite volume of the body. In order to highlight this fact, the corresponding balance equations are also denoted as global relations of balance. Considering the so-called localization argument, i.e., requiring that the relations of balance must hold for any sub-volume of the body, the global relations can be localized under obvious continuity conditions to so-called local relations of balance.

We start our considerations by dealing with global and local balance of mass. For writing global relations, we consider balance with respect to a material volume at first, in other words, for a finite volume with a surface that moves together with the particles instantaneously located in it. This material volume constitutes the material body under consideration. As the reason for a variable mass of the so defined material body, we assume that the elementary masses carried by the particles of the body do change in time. In order to characterize this situation, we say that material sources of mass are attached to the particles. In our subsequent balance formulations, these sources are associated with global and local supplies of mass, the latter being spatially distributed. Since these supplies are absent in classical formulations of continuum mechanics, we talk about non-classical supplies.

In a next step, the notion of the center of mass is introduced in the presence of material sources of mass, which, to a certain extent, connects the formulation for the single mass point in Section 1 above with the problem of a material body of finite spatial extension. Some useful global relations in connection with the notion of center of mass are presented, introducing the notions of center-of-mass linear momentum and relative linear momentum, center-of-mass moment of momentum and relative moment of momentum, as well as center-of-mass kinetic energy and relative kinetic energy. The corresponding relations extend some formulations well-known for bodies in the absence of a supply of mass. The global and local relations of balance of linear momentum are then discussed, where non-classical supplies of linear momentum are additionally introduced. The global relation of balance of linear momentum can be replaced by a center-of-mass oriented form, from which a relation of balance of relative linear momentum follows. The global relations of balance of moment of momentum, center-of-mass moment of momentum and relative moment of momentum are afterwards derived as consequences of mathematical manipulations of the local relation of linear momentum. The same strategy is used for deriving the global relations of balance of kinetic energy, center-of-mass kinetic energy and relative kinetic energy. In all of these derived relations of balance, additional, non-classical supply terms emerge. These, however, are expressed by the non-classical supplies of mass and linear momentum, and which vanish when the latter two are absent. From this, one can conclude that the notion of a non-classical supply as such is necessary in order to ensure consistency of the various relations of balance.

A problem oriented constitutive modeling is needed, in order to properly formulate the non-classical supply terms for mass and linear momentum. In order to provide a rational formulation for the global and local relations of balance of linear momentum, a simple local model for the non-classical supply of linear momentum is presented, which, in a continuum mechanics framework, was suggested by Irschik (2005) for bodies with a growing mass. In this model, mass is locally added to (or removed from) the particles at an own velocity and at a rate equal to the local non-classical supply of mass. For a system consisting of several distinct single mass points, an analogous model was considered by Federhofer (1922). The model assures the invariance of the global and local relations of balance of linear momentum with respect to a Galilean transformation of the inertial frame. Due to the apparent analogy to the relation of balance of linear momentum for the single mass point (28), it can be called a model of the Seeliger-Meshchersky type, cf. Section 1. Using this particular model, the relations of balance of global and center-of-mass moment of momentum and kinetic energy are exemplarily re-formulated.

2.1 Variable mass due to material sources of mass in the interior of a material volume

The instantaneous total mass m of a deformable body of finite extension that occupies the volume V in the current configuration is denoted as

$$m = \int_V \rho \, dV \tag{46}$$

In (46), ρ denotes the current mass density, i.e., the local mass per unit volume in the current configuration, and V is assumed to be a material volume, i.e., a volume that moves together with the particles instantaneously located in it. The case of a flow of mass through the surface of V, i.e., the case of a non-material volume, will be studied in Section 3 below. As the reason for a non-vanishing time-rate of the total mass, for the moment being we assume that distributed sources (sinks) of mass are assigned to the particles in V, where the change of mass is not understood as a change in the number of particles in V, but as a change of the elementary mass of the particles contained in V. We therefore talk about material sources of mass, which result in a non-classical supply of mass, the latter being absent in classical formulations dealing with mechanical systems with a conserved mass. We start our considerations with balance of mass.

2.2 Global and local relations of balance of mass

Assume that the material body under consideration is subjected to a continual, non-classical supply of mass due to material sources of mass. From the general balance relation (2), and in analogy to the relation (4), which has been stated above for a single point mass, we write the global statement of balance of mass as

$$\frac{dm}{dt} = \frac{d}{dt} \int_V \rho \, dV = s\,[m] \tag{47}$$

where $s\,[m]$ is the total supply of mass of the body due to the material sources of mass. We now introduce a local supply of mass $s\,[1]$ by setting

$$s\,[m] = \int_V s\,[1]\, \rho \, dV \tag{48}$$

Interchanging the time derivative and the integral in (47), and requiring that this relation must hold for any sub-volume of the material volume V, the following relation for the time rate of the elementary mass carried by a particle is obtained:

$$\frac{d}{dt} (\rho \, dV) = s\,[1]\, \rho \, dV \tag{49}$$

An example for the material frame indifferent constitutive modeling of $s\,[1]$ can be found in the chapter written by D. Indeitsev in the present book, where the case of a single constituent of a binary mixture is studied.

The left hand side of (49) can be transformed to

$$\frac{d}{dt} (\rho \, dV) = \frac{d\rho}{dt} dV + \rho \frac{d}{dt} (dV) \tag{50}$$

The place of the particles is characterized by their position vectors p with respect to an inertial frame. Particularly, for the mass density we write $\rho = \rho\,(p,t)$. The material time derivative of the mass density in (50) then becomes

$$\frac{d}{dt}\rho\,(p,t) = \frac{\partial\rho}{\partial t} + v \cdot \mathrm{grad}\,\rho \tag{51}$$

where the absolute velocity of the particles with respect to the inertial frame is denoted as $v = dp/dt$. The divergence and gradient operators with respect to the place of the particles in the current configuration are written as "div" and "grad", respectively. Utilizing the Euler expansion formula for the time rate of the elementary volume carried by a particle,

$$\frac{d}{dt}\,(dV) = (\mathrm{div}\,v)\;dV \tag{52}$$

together with the following vector identity,

$$\mathrm{div}\,(\rho\,v) = \rho\,\mathrm{div}\,v + v \cdot \mathrm{grad}\,\rho \tag{53}$$

we can put (50) into the form

$$\frac{d}{dt}\,(\rho\,dV) = \left(\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\,v)\right) dV \tag{54}$$

Substituting (54) into (49) yields the local equation of balance of mass in the presence of material sources of mass:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\,v) = s\,[1]\,\rho \tag{55}$$

Note that the term $s\,[1]\,\rho$ in (55) is denoted as J in D. Indeitsev's Chapter of the present book.

2.3 Some useful relations involving the center of mass

The position vector c of the instantaneous center of mass from the origin of the inertial frame is defined by

$$c\,m = \int_V p\,\rho\,dV \tag{56}$$

where p is the position vector of a particle currently having the elementary mass $\rho\,dV$. Setting

$$p = c + p' \tag{57}$$

with the position vector p' relative to the center of mass, it follows from (56) with (46) that

$$\int_V p' \rho \, dV = 0 \tag{58}$$

Time-wise differentiation of (56) gives, recall that $v = dp/dt$ and $s[m] = dm/dt$, and see (49):

$$\begin{aligned} \frac{d}{dt}(m\,c) = m\frac{dc}{dt} + s\,[m]\,c = \frac{d}{dt}\left(\int_V p\,\rho\,dV\right) \\ = \int_V v\,\rho\,dV + \int_V p\frac{d}{dt}(\rho\,dV) = \int_V v\,\rho\,dV + \int_V p\,s[1]\,\rho\,dV \end{aligned} \tag{59}$$

Note that the center of mass defined in (56) is not to be confused with the place of some material particle, which instantaneously might coincide with it. Indeed, the center of mass might not even be situated within the body, think, e.g., of a hollow sphere. Hence, the velocity dc/dt of the center of mass in general will be different from the velocity of a material particle that instantaneously might be located at the place of the center of mass. (One may ask, why dp/dt in (59) has been identified as the absolute velocity v of a material particle. The reason is that the position of the center of mass of the elementary mass $\rho\,dV$ carried by that material particle is given by its position vector p, which, however, does not move relative to the particle. Hence, the time derivatives are consistently applied in the various expressions presented in (59).)

Now, a further characteristic place $\tilde{c}$ is introduced by defining

$$s\,[m]\,\tilde{c} = \int_V p\frac{d}{dt}(\rho\,dV) = \int_V p\,s\,[1]\,\rho\,dV \tag{60}$$

Substituting into (59) yields

$$\int_V v\,\rho\,dV = m\frac{dc}{dt} + s\,[m]\,(c - \tilde{c}) = m\frac{dc}{dt} - \int_V p' s[1]\,\rho\,dV \tag{61}$$

The left hand side of (61) represents the total linear momentum j of the body, being defined as

$$j = \int_V v\,\rho\,dV = m\frac{dc}{dt} + \int_V v' \rho\,dV = m\frac{dc}{dt} + s\,[m]\,(c - \tilde{c}) \tag{62}$$

In (62), use has been made of the relation

$$v = \frac{dp}{dt} = \frac{dc}{dt} + v' \tag{63}$$

The velocity $v' = dp'/dt$ in (63) represents the difference between the velocity of some material particle and the velocity of the center of mass, dc/dt. With (63), the total linear momentum in (62) can be decomposed into

$$j = j^c + j' \tag{64}$$

with the pseudo- or center-of-mass linear momentum

$$j^c = m\frac{dc}{dt} \tag{65}$$

and the relative linear momentum

$$j' = \int_V v'\rho\, dV = s\,[m]\,(c - \tilde{c}) = -\int_V p' s[1]\,\rho\, dV \tag{66}$$

which follows by comparison of (64) with (61). Hence, the relative linear momentum in a body with a non-vanishing supply of mass $s\,[m]$ in general does not vanish. The relation in (66) thus extends a theorem on the center of mass by Thomson and Tait (1867), who treated bodies without a supply of mass. Note from (48), (57), (58) and (60) that $c = \tilde{c}$, when $s\,[1]$ is uniformly distributed over the body, i.e., $j' = 0$ in (66).

The total moment of momentum of the body with respect to the origin of the inertial frame is defined as

$$\alpha^* = \int_V p \times v\,\rho\, dV \tag{67}$$

Substituting (57), (58), (63) and (66), we get

$$\begin{aligned} \alpha^* &= \int_V p \times \frac{dc}{dt}\,\rho\, dV + \int_V p \times v'\rho\, dV \\ &= c \times m\frac{dc}{dt} + c \times \int_V v'\rho\, dV + \int_V p' \times v'\rho\, dV = \alpha^c + \alpha' + c \times s\,[m]\,(c - \tilde{c}) \end{aligned} \tag{68}$$

The center-of-mass moment of momentum is

$$\alpha^c = c \times j^c = c \times m\frac{dc}{dt} \tag{69}$$

and the relative moment of momentum reads

$$\alpha' = \int_V p' \times v' \rho \, dV \tag{70}$$

Hence, the total moment of momentum of a body with a non-vanishing supply of mass $s\,[m]$ in general is not represented by the sum of the center-of-mass moment of momentum and the relative moment of momentum only. Instead, the last term in (68) needs to be taken into account, in which $c \times s[m]\, c = 0$ holds, of course, but which is kept for formal reasons.

The total kinetic energy of the of the body in the absence of an intrinsic spin of the particles is defined as

$$E_{kin} = \frac{1}{2} \int_V v \cdot v \, \rho \, dV \tag{71}$$

Substituting (57), (63) and (66), we obtain

$$\begin{aligned} E_{kin} &= \frac{1}{2} m \frac{dc}{dt} \cdot \frac{dc}{dt} + \frac{dc}{dt} \cdot \int_V v' \rho \, dV + \frac{1}{2} \int_V v' \cdot v' \rho \, dV \\ &= E^c_{kin} + E'_{kin} + \frac{dc}{dt} \cdot s\,[m]\,(c - \tilde{c}) \end{aligned} \tag{72}$$

with the center-of-mass kinetic energy

$$E^c_{kin} = \frac{1}{2} m \frac{dc}{dt} \cdot \frac{dc}{dt} \tag{73}$$

and the relative kinetic energy

$$E'_{kin} = \frac{1}{2} \int_V v' \cdot v' \rho \, dV \tag{74}$$

Hence, the total kinetic energy of a body with a non-vanishing supply of mass $s\,[m]$ in general is not given by the sum of the center-of-mass kinetic energy and the relative kinetic energy, but the last term in (72) must be taken into account. This represents an extension of a theorem on the center of mass by König (1751), who treated bodies without a supply of mass.

2.4 Global and local balance of linear momentum

We now specialize the general equation of balance (2) for the linear momentum defined in (62). This yields

$$dQ = dj = d\left(\int_V v \, \rho \, dV \right), \quad R = F + s\,[j] \tag{75}$$

where

$$F = \int_V b\,dV + \int_S \sigma_n\,dS \tag{76}$$

The imposed body forces per unit current volume are denoted by b, and the surface tractions are given by the stress vector σ_n. The latter is connected to the stress tensor by Cauchy's fundamental law, which describes the stress vector at the current material surface S of V as the linear mapping

$$\sigma_n = n \cdot \Sigma \tag{77}$$

The unit outer normal vector at S is denoted by n, and the Cauchy stress tensor is written as Σ. The resultant of the body forces and the surface tractions is the vector F. In (75), the total non-classical supply of linear momentum is defined as

$$s\,[j] = \int_V s\,[v]\,\rho\,dV \tag{78}$$

with the local non-classical supply of linear momentum per unit mass denoted by $s\,[v]$. We thus deduce from (75) that the relation of global balance of linear momentum reads

$$\frac{dj}{dt} = \frac{d}{dt}\left(\int_V v\,\rho\,dV\right) = F + s\,[j] = \int_V b\,dV + \int_V (\operatorname{div}\Sigma)\,dV + \int_V s\,[v]\,\rho\,dV \tag{79}$$

Noting from (49) that

$$\begin{aligned} \frac{d}{dt}\left(\int_V v\,\rho\,dV\right) &= \int_V \frac{d}{dt}\,(v\,\rho\,dV) \\ &= \int_V \frac{dv}{dt}\,\rho\,dV + \int_V v\frac{d}{dt}\,(\rho\,dV) = \int_V \left(\frac{dv}{dt} + s\,[1]\,v\right)\rho\,dV \end{aligned} \tag{80}$$

the global statement (79) localizes to

$$\rho\frac{dv}{dt} + s\,[1]\,\rho\,v = b + \operatorname{div}\Sigma + s\,[v]\,\rho \tag{81}$$

This equation represents the local relation of balance of linear momentum in the presence of a supply of mass and a non-classical supply of linear momentum. Note that in the Chapter by D. Indeitsev, the term $s\,[1]\,\rho\,v$ is denoted as $J\,v$, and $s\,[v]\,\rho$ is written as R.

2.5 Balance of center-of-mass linear momentum

As a center-of-mass oriented formulation of balance of linear momentum, using (64) and (66), in (2) we can also write

$$dQ = dj = d\left(j^c + s\left[m\right]\left(c - \tilde{c}\right)\right) \tag{82}$$

With (65), the relation of balance of linear momentum (79) can be replaced by the following center-of-mass oriented form:

$$\frac{dj^c}{dt} = m\,\frac{d^2c}{dt^2} + s\left[m\right]\frac{dc}{dt} = F + s\left[j^c\right] \tag{83}$$

with the non-classical supply of center-of-mass linear momentum

$$s\left[j^c\right] = -\frac{d}{dt}\left(s\left[m\right]\left(c - \tilde{c}\right)\right) + s\left[j\right] = -s\left[m\right]\left(\frac{dc}{dt} - \frac{d\tilde{c}}{dt}\right) - \frac{d^2m}{dt^2}\left(c - \tilde{c}\right) + s\left[j\right] \tag{84}$$

The relation in (83) provides a formulation of balance of center-of-mass linear momentum. Subtracting (83) from (79) with (64), it is found that

$$\frac{dj'}{dt} = \frac{dj}{dt} - \frac{dj^c}{dt} = s\left[m\right]\left(\frac{dc}{dt} - \frac{d\tilde{c}}{dt}\right) + \frac{d^2m}{dt^2}\left(c - \tilde{c}\right) \tag{85}$$

2.6 Balance of relative linear momentum

Using (64), a relation of balance of relative moment of momentum, see (66), is directly obtained from (85):

$$\frac{dj'}{dt} = s\left[j'\right] \tag{86}$$

with the non-classical supply term

$$s\left[j'\right] = s\left[j\right] - s\left[j^c\right] = s\left[m\right]\left(\frac{dc}{dt} - \frac{d\tilde{c}}{dt}\right) + \frac{d^2m}{dt^2}\left(c - \tilde{c}\right) \tag{87}$$

Note that the time rate of the relative linear momentum in general does not vanish if a supply of mass is present. Only if m is conserved, or in the exceptional case of $c = \tilde{c}$, $dj'/dt = 0$ holds.

2.7 Balance of moment of momentum

Performing the vector product of the local form of balance of linear momentum (81) with the position vector p and integrating over the volume

V gives

$$\int_V p \times \frac{d}{dt}(v\,\rho\,dV) = \frac{d}{dt}\int_V p \times \rho\,v\,dV = \int_V p \times (b + \operatorname{div}\Sigma + s\,[v]\,\rho)\,dV \tag{88}$$

Substituting (67) and (80), and using an extended divergence theorem, this can be re-written as:

$$\begin{aligned}\frac{d\alpha^*}{dt} &= \int_V p \times \left(\rho\frac{dv}{dt} + s\,[1]\,\rho\,v\right) dV \\ &= \int_V p \times (b - \Sigma_\times)\,dV + \int_S p \times \sigma_n dS + s\,[\alpha^*]\end{aligned} \tag{89}$$

The so-called Gibbsian cross-vector of the Cauchy stress tensor is denoted as $\Sigma_\times$. It is twice the axial vector of the skew-symmetric part of the Cauchy stress tensor Σ, and thus vanishes if Σ is symmetric. With (57), the non-classical supply of moment of momentum in (89) becomes

$$s\,[\alpha^*] = \int_V p \times s\,[v]\,\rho\,dV = c \times s\,[j] + \int_V p' \times s\,[v]\,\rho\,dV \tag{90}$$

Note that (89) and (90) represent pure consequences of a mathematical manipulation of the local relation of balance of linear momentum (81). Now, in the absence of an intrinsic spin and of applied body and surface couples, the fundamental relation of balance of angular momentum in principle does coincide with balance of moment of momentum (89), compare the analogous relation (32) for the angular momentum of a single mass point. The only exceptions are that the term with $\Sigma_\times$ is not present in the relation of balance of angular momentum, and that the non-classical supply of moment of momentum $s\,[\alpha^*]$ is to be replaced by a possibly different non-classical supply of angular momentum. Hence, if the non-classical supplies of moment of momentum and of angular momentum can be assumed to be equal, it follows that the volume integral over $\Sigma_\times$ in (89) vanishes. The localization argument then leads to the conclusion that the stress tensor must be symmetric, such that $\Sigma_\times = 0$ in (89). This can be considered as an extension of an axiom for non-polar bodies without a supply of mass, see Ziegler (1998)

2.8 Balance of center-of-mass moment of momentum

Performing the vector product of the center-of-mass form of balance of linear momentum (83) with c gives

$$c \times \frac{dj^c}{dt} = \frac{d}{dt}\left(c \times m\frac{dc}{dt}\right) = c \times F + c \times s\left[j^c\right] \tag{91}$$

Substituting (69), this can be re-written into the following relation of balance for the center-of-mass moment of momentum:

$$\frac{d\alpha^c}{dt} = c \times m\frac{d^2c}{dt^2} + c \times s\left[m\right]\frac{dc}{dt} = c \times F + s\left[\alpha^c\right] \tag{92}$$

with the non-classical supply, see (84),

$$s\left[\alpha^c\right] = c \times s\left[j^c\right] \tag{93}$$

Subtracting (92) from (89), cf. (84) and (85), gives

$$\frac{d\alpha^*}{dt} - \frac{d\alpha^c}{dt} = \int\limits_V p' \times b\, dV + \int\limits_S p' \times \sigma_n dS - \int\limits_V p \times \Sigma_\times dV + c \times \frac{d}{dt}\left(s\left[m\right](c - \tilde{c})\right) + \int\limits_V p' \times s\left[v\right]\rho\, dV \tag{94}$$

2.9 Balance of relative moment of momentum

Utilizing (68), a relation of balance of relative moment of momentum, see (70), is obtained from (94):

$$\frac{d\alpha'}{dt} = \int\limits_V p' \times b\, dV + \int\limits_S p' \times \sigma_n dS - \int\limits_V p \times \Sigma_\times dV + s\left[\alpha'\right] \tag{95}$$

with the non-classical supply term

$$s\left[\alpha'\right] = -\frac{dc}{dt} \times s\left[m\right](c - \tilde{c}) + \int\limits_V p' \times s\left[v\right]\rho\, dV \tag{96}$$

2.10 Balance of kinetic energy

Performing the scalar product of the local relation of balance of linear momentum (81) with the absolute velocity vector v, and integrating over the material volume V yields

$$\int_V v \cdot \frac{dv}{dt}\,\rho\,dV + \int_V s\,[1]\,v \cdot v\,\rho\,dV = \frac{d}{dt}\left(\int_V \frac{1}{2} v \cdot v\,\rho\,dV\right) + \int_V \frac{1}{2} s\,[1]\,v \cdot v\,\rho\,dV = \int_V (b \cdot v + v \cdot \mathrm{div}\,\Sigma + \rho\,s\,[v] \cdot v) dV \quad (97)$$

Using (71) and an extended divergence theorem, we obtain the relation of balance of kinetic energy as

$$\frac{dE_{kin}}{dt} = \int_V v \cdot b\; dV + \int_S v \cdot \sigma_n dS - \int_V \mathrm{tr}\,(\Sigma \cdot \mathrm{grad}\,v)\,dV + s\,[E_{kin}] \quad (98)$$

The trace of a second order tensor is denoted by "tr". Using (63), the relation in (98) can be re-written to

$$\frac{dE_{kin}}{dt} = F \cdot \frac{dc}{dt} + \int_V v' \cdot b\,dV + \int_S v' \cdot \sigma_n dS - \int_V \mathrm{tr}\,(\Sigma \cdot \mathrm{grad}\,v')\,dV + s\,[E_{kin}] \quad (99)$$

The non-classical supply of kinetic energy in (98) and (99) turns out to be

$$s\,[E_{kin}] = -\frac{1}{2}\int_V v \cdot v\,s\,[1]\,\rho\,dV + \int_V v \cdot s\,[v]\,\rho\,dV \quad (100)$$

Again with (63), this becomes

$$s\,[E_{kin}] = -\frac{1}{2}\,s\,[m]\,\frac{dc}{dt} \cdot \frac{dc}{dt} - \frac{dc}{dt} \cdot \int_V v' s\,[1]\,\rho\,dV - \int_V \frac{1}{2} v' \cdot v' s\,[1]\,\rho\,dV + \frac{dc}{dt} \cdot s\,[j] + \int_V v' \cdot s\,[v]\,\rho\,dV \quad (101)$$

2.11 Balance of center-of-mass kinetic energy

The scalar product of the center-of-mass form of balance of linear momentum (83) with the absolute velocity of the center of mass, dc/dt, gives

$$\frac{dc}{dt}\cdot m\frac{d^2c}{dt^2}+\,s\,[m]\,\frac{dc}{dt}\cdot\frac{dc}{dt}=\frac{d}{dt}\left(\frac{1}{2}m\frac{dc}{dt}\cdot\frac{dc}{dt}\right)+\frac{1}{2}\,s\,[m]\,\frac{dc}{dt}\cdot\frac{dc}{dt}$$
$$=F\cdot\frac{dc}{dt}+s\,[j^c]\cdot\frac{dc}{dt}\qquad(102)$$

Using (73), we obtain the following relation of balance for the center-of-mass kinetic energy:

$$\frac{dE^c_{kin}}{dt}=F\cdot\frac{dc}{dt}+s\,[E^c_{kin}]\qquad(103)$$

with the non-classical supply of the center-of-mass kinetic energy

$$s\,[E^c_{kin}]=-\frac{1}{2}\,s\,[m]\,\frac{dc}{dt}\cdot\frac{dc}{dt}+s\,[j^c]\cdot\frac{dc}{dt}\qquad(104)$$

Substituting (84), this can be re-written to

$$s\,[E^c_{kin}]=-s\,[m]\left(\frac{3}{2}\frac{dc}{dt}-\frac{d\tilde{c}}{dt}\right)\cdot\frac{dc}{dt}-\frac{d^2m}{dt^2}\,(c-\tilde{c})\cdot\frac{dc}{dt}+s\,[j]\cdot\frac{dc}{dt}\qquad(105)$$

Subtracting (103) from (99), cf. (101) and (105), it is found that

$$\frac{dE_{kin}}{dt}-\frac{dE^c_{kin}}{dt}=\int\limits_V v'\cdot b\,dV+\int\limits_S v'\cdot\sigma_n dS$$
$$-\int\limits_V \mathrm{tr}\,(\Sigma\cdot\mathrm{grad}\,v')\,dV+s\,[m]\left(\frac{dc}{dt}-\frac{d\tilde{c}}{dt}\right)\cdot\frac{dc}{dt}$$
$$+\frac{d^2m}{dt^2}\,(c-\tilde{c})\cdot\frac{dc}{dt}-\frac{dc}{dt}\cdot\int\limits_V v'\,s\,[1]\,\rho\,dV$$
$$-\int\limits_V\frac{1}{2}v'\cdot v'\,s\,[1]\,\rho\,dV+\int\limits_V v'\cdot s\,[v]\,\rho\,dV\qquad(106)$$

2.12 Balance of relative kinetic energy

Using (72), a relation of balance of relative kinetic energy, see (74), is obtained from (106):

$$\frac{dE'_{kin}}{dt}=\int\limits_V v'\cdot b\,dV+\int\limits_S v'\cdot\sigma_n dS-\int\limits_V \mathrm{tr}\,(\Sigma\cdot\mathrm{grad}\,v')\;dV+s\,[E'_{kin}]\qquad(107)$$

with the non-classical supply of relative kinetic energy

$$s\left[E'_{kin}\right] = -s\left[m\right]\frac{d^2c}{dt^2}\cdot(c-\tilde{c}) - \frac{dc}{dt}\cdot\int_V v'\,s\left[1\right]\rho\,dV - \int_V \frac{1}{2}v'\cdot v'\,s\left[1\right]\rho\,dV + \int_V v'\cdot s\left[v\right]\rho\,dV \quad (108)$$

2.13 Material volume with distributed material sources of mass of the Seeliger-Meshchersky type

Motivated by the procedure for obtaining a rational expression for the additional, non-classical supply of linear momentum presented in Subsection 1.3 above, see (17), we set

$$s\left[v\right] = s\left[1\right]u \quad (109)$$

where the velocity u, at which mass is locally gained or lost by the particles, in general will be different from the particle velocity v. The local relation of balance (81) then can be written as

$$\rho\frac{dv}{dt} = b + \operatorname{div}\Sigma + s\left[1\right]\rho\left(u-v\right) \quad (110)$$

The relation of balance of global linear momentum (79) follows to

$$\int_V \frac{dv}{dt}\,\rho dV = \int_V b\,dV + \int_S \sigma_n\,dS + \int_V s\left[1\right](u-v)\,\rho\,dV \quad (111)$$

The continuum mechanics based model for the additional supply of linear momentum in (109) was suggested and embedded into the literature by Irschik (2005) in the framework of growing materials. Due to the apparent analogy to the relation of balance of linear momentum for the single mass point, (27), we denote it as a model of the Seeliger-Meshchersky type. Note that this model meets the requirements of the so-called Galileian invariance of the relations of balance of linear momentum in (110) and (111). In other words, seen from an observer, who moves with a constant velocity relative to the global inertial frame, the relations given in (110) and (111) do remain invariant. The same remains true, if a translational rigid-body motion with constant velocity is superimposed upon the actual motion. Moreover, would we add to the model in (109) some vector, which does not depend on the velocity but acts like a body force, this term could be treated as an additional body force and would not interfere with the Galileian invariance.

Introducing (109) into (90), the relation of balance of moment of momentum (89) becomes

$$\int\limits_V p \times \rho \frac{dv}{dt}\, dV = \int\limits_V p \times (b - \Sigma_\times)\, dV + \int\limits_S p \times \sigma_n dS + \int\limits_V p \times s\,[1]\,(u - v) dV \quad (112)$$

In order to write down further global statements, we now introduce another characteristic velocity w as

$$s\,[m]\,w = s\,[j] = \int\limits_V s\,[v]\,\rho\, dV = \int\limits_V s\,[1]\,u\,\rho\, dV \quad (113)$$

With the definition stated in (113), the various relations of balance, which have been presented above, reduce to the following useful forms:

Balance of total linear momentum, see (79):

$$\frac{dj}{dt} = F + s\,[m]\,w \quad (114)$$

This coincides with the Seeliger-Meshchersky relation (25) for the single point mass.

Balance of center-of-mass linear momentum, see (83) and (84):

$$\frac{dj^c}{dt} = F + s\,[m] \left(w - \frac{dc}{dt} + \frac{d\tilde{c}}{dt} \right) + \frac{d^2 m}{dt^2}\,(\tilde{c} - c) \quad (115)$$

With (65), this becomes

$$m \frac{d^2 c}{dt^2} = F + s\,[m] \left(w - 2\frac{dc}{dt} + \frac{d\tilde{c}}{dt} \right) - \frac{d^2 m}{dt^2}\,(\tilde{c} - c) \quad (116)$$

Note that this relation is Galilei-invariant. For $c = \tilde{c}$, (116) reduces to (27) for the single mass point. For a mechanical system consisting of a set of discrete mass points, (116) is due to Federhofer (1922).

Balance of moment of momentum, see (89) and (90):

$$\frac{d\alpha^*}{dt} = c \times F + \int\limits_V p' \times (b - \Sigma_\times)\, dV + \int\limits_S p' \times \sigma_n dS + c \times s\,[m]\, w + \int\limits_V p' \times s\,[1]\, u\, \rho\, dV \quad (117)$$

If the material volume V shrinks to a single mass point, such that the relative position vectors p' shrink to zero, this Galilei-invariant form reduces to (32).

Balance of center-of-mass moment of momentum, see (92), (93) and (84):

$$\frac{d\alpha^c}{dt} = c \times F + c \times s\,[m] \left(w - \frac{dc}{dt} + \frac{d\tilde{c}}{dt} \right) + c \times \frac{d^2 m}{dt^2} (\tilde{c} - c) \quad (118)$$

or, substituting (69),

$$c \times m \frac{d^2 c}{dt^2} = c \times F + c \times s\,[m] \left(w - 2\frac{dc}{dt} + \frac{d\tilde{c}}{dt} \right) + c \times \frac{d^2 m}{dt^2} (\tilde{c} - c) \quad (119)$$

When $c = \tilde{c}$, this reduces to (32).

Balance of kinetic energy, see (99)–(101):

$$\frac{dE_{kin}}{dt} = F \cdot \frac{dc}{dt} + \int\limits_V v' \cdot b\, dV + \int\limits_S v' \cdot \sigma_n dS - \int\limits_V \mathrm{tr}\,(\Sigma \cdot \mathrm{grad}\, v')\, dV + s\,[m] \left(w - \frac{1}{2}\frac{dc}{dt} \right) \cdot \frac{dc}{dt} - \frac{dc}{dt} \cdot \int\limits_V v' s\,[1]\, \rho\, dV + \int\limits_V v' \cdot s\,[1] \left(u - \frac{1}{2} v' \right) \rho\, dV \quad (120)$$

When V shrinks towards a single mass point, this reduces to the balance of translational kinetic energy (38).

Balance of center-of-mass kinetic energy, see (103)–(105):

$$\frac{dE^c_{kin}}{dt} = F \cdot \frac{dc}{dt} + s\,[m] \left(w - \frac{3}{2}\frac{dc}{dt} + \frac{d\tilde{c}}{dt} \right) \cdot \frac{dc}{dt} - \frac{d^2 m}{dt^2} (c - \tilde{c}) \cdot \frac{dc}{dt} \quad (121)$$

When $c = \tilde{c}$, this reduces to (38).

3 Global relations of balance written for a non-material control volume

3.1 The Reynolds transport theorem

We now release the assumption of a material volume V that moves together with the material particles contained in it. Instead, we consider a so-called open system with a non-material control volume V, the surface S of which moves at a velocity u that is different from the velocity v of the material particles instantaneously located on that non-material control surface S. In this situation, a flow of mass through S will be present implying that the total mass contained in V in general will not be conserved. As is shown below, the above methodology for taking into account material sources of mass inside a material volume can be adopted in the present case by analogy. Moreover, the presence of material sources of mass as well as of non-classical supply terms for other entities can be additionally taken into account, when there is a flow of mass through the non-material control surface S. In demonstrating this, we use the transport theorem by Reynolds (1903), cf. Truesdell and Toupin (1960). As has been discussed in some detail by Irschik and Holl (2004), the transport theorem can be put into a form that involves both, the rate of the total of some entity contained in a non-material control volume, as well as the rate of this entity contained in the material volume that instantaneously coincides with the non-material control volume.

In a generalized form, this version of the Reynolds transport theorem can be stated as follows. Consider a scalar or vector quantity Ψ, which is the total of a local entity $\psi\,\rho$ carried by the particles in the volume V under consideration:

$$\Psi = \int_V \psi\,\rho\,dV \tag{122}$$

Then the transport theorem can be written as

$$\frac{d_u}{dt}\Psi = \frac{d\Psi}{dt} + s_u\,[\Psi] \tag{123}$$

with the non-classical supply

$$s_u\,[\Psi] = \int_S n \cdot (u - v)\psi\,\rho\,dS \tag{124}$$

The operator d_u/dt in (123) indicates that the time-rate refers to the motion of the non-material control volume V, while d/dt means the time rate considering the motion of the material volume that instantaneously coincides with the latter. The surface integral in (124) vanishes, when V is a material volume, since then $u = v$. Various useful versions of (123) and (124) are listed below, where the rates with respect to the motion of the material volume, $d\psi/dt$ in (123), can be substituted directly from the equations of balance presented above in Subsections 2.2–2.12. Note that additional considerations are needed in the presence of a singular surface, on which $\psi\,\rho$ takes on different values at the two sides of this surface. This case will be discussed in Section 4 below.

3.2 Balance of mass

For balance of mass, there is $\psi = 1$, $\Psi = m$, see (46). With (47), we obtain from (123) and (124) that

$$\frac{d_u}{dt} m = s\,[m] + s_u\,[m] \tag{125}$$

with the additional, non-classical supply of mass

$$s_u\,[m] = \int_S n \cdot (u - v)\rho\, dS \tag{126}$$

3.3 Balance of linear momentum

With the definition of total linear momentum j stated in (62), we set $\psi = v$, $\Psi = j$ in (123) and (124). Substituting (79), this yields

$$\frac{d_u}{dt} j = \frac{d}{dt} j + s_u\,[j] = F + s\,[j] + s_u\,[j] \tag{127}$$

with the additional non-classical supply of linear momentum

$$s_u\,[j] = \int_S n \cdot (u - v)\rho\, v\, dS \tag{128}$$

The apparent analogy between the global equations of balance of mass and linear momentum for a material volume with material sources of mass and for a non-material control volume with a flow of mass trough its surface along with various applications, particularly in fluid mechanics and rocketry, has been discussed in the review by Irschik and Holl (2004). Note that the cases of material sources of mass in the interior and of a flow of mass through the surface have been treated separately in Irschik and Holl (2004). In the present case of taking into account both, material sources in the interior and a flow of mass through the surface, in the equations of balance presented above in Subsections 2.2–2.12, it is only necessary to replace the time rate $d\Psi/dt$ by $d_u\Psi/dt$, in order to indicate that there is a flow of mass through the surface S, and to replace $s[\Psi]$ by the sum $s[\Psi] + s_u[\Psi]$. This is subsequently shown for various relations of balance. A further remark seems to be in order: It is evident from (127) that a relation of balance for a non-material control volume with a flow of mass through the surface can be brought into the form valid for the material volume that instantaneously coincides with the former non-material volume. This follows from the canceling of the non-classical supply $s_u[\Psi]$ in (127). However, in many practical applications it is necessary to compute the time rate $d_u\Psi/dt$, following the motion of the volume with a flow of mass through the surface, instead of $d\Psi/dt$.

3.4 Balance of moment of momentum

We set $\psi = p \times v$ and $\Psi = \alpha^*$ in (123) and (124). Using (89), this gives

$$\frac{d_u}{dt}\alpha^* = \frac{d}{dt}\alpha^* + s_u[\alpha^*] = \int_V p \times (b - \Sigma_\times)\, dV + \int_S p \times \sigma_n dS + s[\alpha^*] + s_u[\alpha^*] \quad (129)$$

with the additional non-classical supply of moment of momentum

$$s_u[\alpha^*] = \int_S n \cdot (u - v)\, p \times \rho\, v\, dS \quad (130)$$

3.5 Balance of kinetic energy

Setting $\psi = v \cdot v/2$ and $\Psi = E_{kin}$ in (123) and (124) and substituting (98) gives:

$$\frac{d_u}{dt}E_{kin} = \frac{d}{dt}E_{kin} + s_u[E_{kin}] = \int_V v \cdot b\, dV + \int_S v \cdot \sigma_n dS - \int_V \mathrm{tr}(\Sigma \cdot \mathrm{grad}\, v)\, dV + s[E_{kin}] + s_u[E_{kin}], \quad (131)$$

with the non-classical additional supply of kinetic energy

$$s_u[E_{kin}] = \int_S n \cdot (u - v)\, \frac{1}{2}\rho\, v \cdot v\, dS \quad (132)$$

Similar extensions can be written down for the center-of-mass oriented relations of balance stated above. Particularly, balance of center-of-mass linear momentum has been treated in some detail in Irschik and Holl (2004), but separately from the case of sources of mass in the interior. In the following, we give extended and additional relations.

3.6 Some useful relations involving the center of mass

Subsequently, we repeatedly utilize balance of mass in the form (123)–(124). The definition of the position vector c of the instantaneous center of mass from the origin of the inertial frame stated above in (56), the split of the position vector in (57) and its consequence (58) remain unchanged.

Time-wise differentiation of (56) in the presence of a flow of mass through the surface gives

$$\begin{aligned}\frac{d_u}{dt}(m\,c) &= \frac{d_u}{dt}\int_V c\,\rho\,dV = m\,\frac{dc}{dt} + \left(s\,[m] + s_u\,[m]\right)c \\ &= \frac{d_u}{dt}\left(\int_V p\,\rho\,dV\right) = \frac{d}{dt}\left(\int_V p\,\rho\,dV\right) + \int_S n\cdot(u-v)\,\rho\,p\,dS \\ &= \int_V v\,\rho\,dV + \int_V p\,s\,[1]\,\rho\,dV + \int_S n\cdot(u-v)\,\rho\,p\,dS \qquad (133)\end{aligned}$$

The transport theorem (125) and (126) has been used for expressing the time-rate of the volume integral in (133). Analogous to (60), a further characteristic place $\tilde{c}$ is introduced by defining

$$\begin{aligned}\left(s\,[m] + s_u\,[m]\right)\tilde{c} &= \int_V p\,s\,[1]\,\rho\,dV + \int_S n\cdot(u-v)\,\rho\,p\,dS \\ &= \left(s\,[m] + s_u\,[m]\right)c + \int_V p' s\,[1]\,\rho\,dV + \int_S n\cdot(u-v)\,\rho\,p' dS \qquad (134)\end{aligned}$$

see (49). Substituting into (133) yields

$$\begin{aligned}\int_V v\,\rho\,dV &= m\frac{dc}{dt} + \left(s\,[m] + s_u\,[m]\right)(c-\tilde{c}) \\ &= m\frac{dc}{dt} - \int_V p' s\,[1]\,\rho\,dV - \int_S n\cdot(u-v)\,\rho\,p' dS \qquad (135)\end{aligned}$$

Recalling that the split of the absolute velocity in (63), the resulting representation of the linear momentum in (62), as well as the definition of the center-of-mass linear momentum in (65) and the split of the linear momentum in (64) remain unchanged, we now have, in extension of (62):

$$j = \int_V v\,\rho\,dV = m\frac{dc}{dt} + \int_V v'\rho\,dV \\ = m\frac{dc}{dt} + \left(s\,[m] + s_u\,[m]\right)(c - \tilde{c}) = j^c + j' \quad (136)$$

The relative linear momentum becomes

$$j' = \int_V v'\rho\,dV = \left(s\,[m] + s_u\,[m]\right)(c - \tilde{c}) \\ = -\int_V p'\,s\,[1]\,\rho\,dV - \int_S n\cdot(u - v)\,\rho\,p'dS \quad (137)$$

Analogously, we get for the total moment of momentum of the body with respect to the origin of the inertial frame, see (67):

$$\alpha^* = \alpha^c + \alpha' + c \times \left(s\,[m] + s_u\,[m]\right)(c - \tilde{c}) \quad (138)$$

The definitions of the center-of-mass and the relative moment of momentum in (69) and (70) remain unchanged.

The total kinetic energy of the body in the absence of an intrinsic spin of the particles, see (71), becomes

$$E_{kin} = E^c_{kin} + E'_{kin} + \frac{dc}{dt}\cdot\left(s\,[m] + s_u\,[m]\right)(c - \tilde{c}) \quad (139)$$

with the center-of-mass and the relative kinetic energy as given in (73) and (74).

3.7 Balance of center-of-mass linear momentum

Using the definition of the center-of-mass linear momentum in (65) and substituting (136), the relation of balance of linear momentum (127) can be reformulated to the following form:

$$\frac{d_u}{dt}j = \frac{d_u}{dt}\left(j^c + \left(s\,[m] + s_u\,[m]\right)(c - \tilde{c})\right) = F + s\,[j] + s_u\,[j] \quad (140)$$

Performing the operation d_u/dt, we obtain the following center-of-mass oriented formulation of balance of linear momentum:

$$\frac{d_u}{dt} j^c = m \frac{d^2 c}{dt^2} + (s[m] + s_u[m]) \frac{dc}{dt} = F + s[j^c] + s_u[j^c] \tag{141}$$

with the non-classical supply of center-of-mass linear momentum

$$s_u[j^c] = -s_u[m] \left(\frac{dc}{dt} - \frac{d\tilde{c}}{dt} \right) - \left(\frac{d}{dt} s_u[m] \right) (c - \tilde{c}) + s_u[j] \tag{142}$$

For the case without a flow of mass through the surface, see (84). Having derived (141)–(142) in some detail, we subsequently list additional results by direct analogy to the equations of balance, which have been presented above in Subsections 2.2–2.12 for the case without a flow of mass through the surface.

3.8 Balance of relative moment of momentum

$$\frac{d_u}{dt} j' = s[j'] + s_u[j'] \tag{143}$$

$$s_u[j'] = s_u[j] - s_u[j^c] = s_u[m] \left(\frac{dc}{dt} - \frac{d\tilde{c}}{dt} \right) + \left(\frac{d}{dt} s_u[m] \right) (c - \tilde{c}) \tag{144}$$

with $s_u[j^c]$ of (142).

3.9 Balance of center-of-mass moment of momentum

$$\frac{d_u}{dt} \alpha^c = c \times m \frac{d^2 c}{dt^2} + c \times (s[m] + s_u[m]) \frac{dc}{dt} = c \times F + s[\alpha^c] + s_u[\alpha^c] \tag{145}$$

$$s_u[\alpha^c] = c \times s_u[j^c] \tag{146}$$

with $s_u[j^c]$ of (142).

3.10 Balance of relative moment of momentum

$$\frac{d_u}{dt} \alpha' = \int_V p' \times b \, dV + \int_S p' \times \sigma_n dS - \int_V p \times \Sigma_\times dV + s[\alpha'] + s_u[\alpha'] \tag{147}$$

$$s_u[\alpha'] = -\frac{dc}{dt} \times (s_u[m] (c - \tilde{c})) + \int_S n \cdot (u - v) \, p' \times \rho v \, dS \tag{148}$$

3.11 Balance of center-of-mass kinetic energy

$$\frac{d_u}{dt}E^c_{kin} = F \cdot \frac{dc}{dt} + s\left[E^c_{kin}\right] + s_u\left[E^c_{kin}\right] \tag{149}$$

$$s_u\left[E^c_{kin}\right] = -s_u\left[m\right]\left(\frac{3}{2}\frac{dc}{dt} - \frac{d\tilde{c}}{dt}\right) \cdot \frac{dc}{dt} - \left(\frac{d}{dt}s_u\left[m\right]\right)\left(c - \tilde{c}\right) \cdot \frac{dc}{dt} + s_u\left[j\right] \cdot \frac{dc}{dt} \tag{150}$$

3.12 Balance of relative kinetic energy

$$\begin{aligned}\frac{d_u}{dt}E'_{kin} = \int_V v' \cdot b\, dV + \int_S v' \cdot \sigma_n dS \\ - \int_V \mathrm{tr}\left(\Sigma \cdot \mathrm{grad}\, v'\right)\, dV + s\left[E'_{kin}\right] + s_u\left[E'_{kin}\right]\end{aligned} \tag{151}$$

$$\begin{aligned}s_u\left[E'_{kin}\right] = -s_u\left[m\right]\frac{d^2c}{dt^2} \cdot \left(c - \tilde{c}\right) \\ - \int_S n \cdot \left(u - v\right)\rho\, v' dS - \int_S n \cdot \left(u - v\right)\rho\left(\frac{1}{2}v' \cdot v'\right) dS\end{aligned} \tag{152}$$

3.13 Application to rigid bodies

The above formulations do hold irrespective of the specific deformation behavior of the bodies under consideration. As a theoretically as well as practically important special case, which also allows an exemplary comparison of our formulations with results from the literature, we consider a rigid body in what follows. An interesting case is represented, e.g., by the problem of a rigid body that experiences a surface growth, see Ong and O'Reilly (2004). In this case, the surface of the body moves at a velocity u that in general is different from the velocity v of the particles instantaneously located on it. Thus, our above methodology is applicable. For a rigid body, the velocity of a material particle is given by the Euler velocity formula, see Ziegler (1998):

$$v = v_c + \omega \times p' \tag{153}$$

Here, v_c denotes the velocity vector of the material particle, which *"instantaneously coincides with the position vector c of the center of mass"*. Of course, it is a delicate matter to talk about such a particle, since the center of mass may not even be situated inside the body, see the remarks given above. However, for a rigid body it is quite straightforward to consider a fictitious rigid extension of the body towards the instantaneous location of the center of mass. In general, the velocity v_c is different from the velocity of the center of mass, dc/dt. The angular velocity of the rigid body in (153) is denoted as ω.

We first present some useful formulas involving the velocity of the material particle that instantaneously coincides with the center of mass c. Substituting (153) into (62), we obtain for the linear momentum of the rigid body

$$j = \int_V v\,\rho\,dV = m\,v_c + \omega \times \int_V p'\rho\,dV = m\,v_c \tag{154}$$

which coincides with (4.2) of Ong and O'Reilly (2004).

Moreover, substituting (153) into (67), the moment of momentum for the rigid body becomes

$$\alpha^* = \int_V p \times v\,\rho\,dV = c \times m\,v_c + \int_V p' \times (\omega \times p')\,\rho\,dV = c \times j + J' \cdot \omega \tag{155}$$

with

$$J' = \int_V ((p' \cdot p')\,I - p' \otimes p')\,\rho\,dV \tag{156}$$

The tensor of inertia relative to the center of mass is denoted as J', see Ziegler (1998), compare (10) for the single mass point. Note that (155) coincides with (4.4) and (4.6) of Ong and O'Reilly (2004).

For kinetic energy, substituting (153) into (71) yields

$$\begin{aligned} E_{kin} = \frac{1}{2}\int_V v \cdot v\,\rho\,dV =& \frac{1}{2} m\,v_c \cdot v_c + v_c \cdot \omega \times \int_V p'\rho\,dV \\ &+ \frac{1}{2}\int_V (\omega \times p') \cdot (\omega \times p')\,\rho\,dV = \frac{1}{2} m\,v_c \cdot v_c + \frac{1}{2}\,(J' \cdot \omega) \cdot \omega \end{aligned} \tag{157}$$

see (4.8) of Ong and O'Reilly (2004).

It is to be noted that the relations (154)–(157) are well-known for rigid bodies without sources of mass in the interior and without a flow of mass trough the surface, cf. Ziegler (1998). In order to establish a connection to the more general relations stated above, we equate (153) and (63), from which it is found that

$$v' = v_c + \omega \times p' - \frac{dc}{dt} \tag{158}$$

Substituting into the relation for the relative linear momentum (137) yields

$$j' = \int_V v' \rho \, dV = m \left(v_c - \frac{dc}{dt} \right) = \left(s\,[m] + s_u\,[m] \right) (c - \tilde{c}) \tag{159}$$

from which it follows that

$$m\, v_c = m \frac{dc}{dt} + \left(s\,[m] + s_u\,[m] \right) (c - \tilde{c}) \tag{160}$$

This is also obtained from a direct comparison of (136) and (154).

Having stated the useful expressions (154)–(157) for linear momentum, moment of momentum and kinetic energy, which are based on the entities v_c, J' and ω of the rigid body, we proceed with formulating the relations of balance taking into account the assumption of rigidity.

Balance of linear momentum With (154), the time rate of linear momentum in the presence of sources of mass in the interior and a flow of mass through the surface can be formulated as

$$\begin{aligned} \frac{d_u}{dt} j = \frac{d_u}{dt} \left(m\, v_c \right) &= \frac{d_u}{dt} \int_V v_c \, \rho \, dV \\ = \frac{d}{dt} \int_V v_c \, \rho \, dV + \int_S n \cdot (u - v) \; \rho \, v_c \, dV &= m \frac{dv_c}{dt} + \; \left(s\,[m] + s_u\,[m] \right) v_c \end{aligned} \tag{161}$$

where the transport theorem (123), (124) has been utilized. Equating (161) and (127), we obtain

$$m \frac{dv_c}{dt} + \left(s\,[m] + s_u\,[m] \right) v_c = F + s\,[j] + s_u\,[j] \tag{162}$$

Substituting the Euler velocity formula (153), the non-classical supplies of momentum $s_u\,[j]$, see (128), for the rigid body becomes:

$$\begin{aligned} s_u\,[j] &= v_c \int_S n \cdot (u - v) \, \rho \, dS + \omega \times \int_S n \cdot (u - v) \, \rho \, p' dS \\ &= v_c \, s_u\,[m] + \omega \times \int_S n \cdot (u - v) \, \rho \, p' dS \end{aligned} \tag{163}$$

Relation (162) thus reduces to

$$m\frac{dv_c}{dt} + s\,[m]\,v_c = F + s\,[j] + \omega \times \int\limits_S n \cdot (u - v)\,\rho\,p' dS \qquad (164)$$

This coincides with (7.3) and (7.4) of Ong and O'Reilly (2004) for the case of $s\,[m] = 0$ and $s\,[j] = 0$. Now, since v_c is the velocity of the material particle that instantaneously coincides with the center of mass, and since the latter in general is moving with respect to the rigid body, the time derivative dv_c/dt in general can not be expected to coincide with the acceleration a_c of the material point that instantaneously is located at the center of mass. A further clarification of the mechanical meaning of the derivative dv_c/dt therefore is deemed desirable. For that sake, we use the Euler formula for the acceleration a of a material particle of a rigid body, see Ziegler (1998):

$$a = \frac{dv}{dt} = a_c + \beta \times p' + \omega \times (\omega \times p') \qquad (165)$$

The angular acceleration of the body is denoted as β, and a_c is the absolute acceleration of the material particle that instantaneously is situated at the place of the center of mass. Utilizing (80), the relation of balance of linear momentum (127) becomes:

$$\frac{d_u}{dt} j = \frac{d}{dt} j + s_u\,[j] = \int\limits_V \frac{dv}{dt}\,\rho\,dV + \int\limits_V v\,s\,[1]\;\rho\,dV + s_u\,[j] = F + s\,[j] + s_u\,[j] \qquad (166)$$

Cancelling out the non-classical supply $s_u\,[j]$, substituting (166) and (153) and noting (58) yields

$$m\,a_c + s\,[m]\,v_c = F + s\,[j] - \omega \times \int\limits_V p' s\,[1]\,\rho\,dV \qquad (167)$$

Comparing (164) and (167), the following illustrative explanation of the difference between a_c and dv_c/dt is obtained:

$$\begin{aligned} m\,a_c - m\frac{dv_c}{dt} &= -\omega \times \left(\int\limits_V p' s\,[1]\,\rho\,dV + \int\limits_S n \cdot (u - v)\,\rho\,p' dS \right) \\ &= \omega \times j' = \omega \times (s\,[m] + s_u\,[m])\,(c - \tilde{c}) \qquad (168) \end{aligned}$$

see (137) for the relative linear momentum j'. Only when $\omega \times j'$ vanishes, this difference disappears.

A further remark seems to be in order. As already discussed above, a relation of balance can be brought into a form, in which the non-classical supply due to the flow of mass through the surface formally cancels out. This is demonstrated for balance of relative moment of momentum next.

Balance of relative moment of momentum For comparison's sake, we study balance of moment of momentum relative to the canter of mass. From (155), it is immediately seen that the relative moment of momentum (70) for the rigid body is

$$\alpha' = \int_V p' \times (\omega \times p')\ \rho\, dV = J' \cdot \omega \tag{169}$$

This follows by substituting (156) into (70), and noting (58). Using $\Psi = p' \times (\omega \times p')$ in (123)–(124), the transport theorem yields the relation of balance of relative moment of momentum as

$$\begin{aligned} \frac{d_u}{dt}\alpha' = \int_V p' \times b\, dV + \int_S p' \times \sigma_n dS - \int_V p \times \Sigma_\times dV \\ + s\,[\alpha'] + \int_S n \cdot (u - v)\rho\,(p' \times (\omega \times p'))\, dS \end{aligned} \tag{170}$$

where the non-classical supply due to a flow of mass through the surface in case of a rigid body is given by

$$s_u[\alpha'] = \int_S n \cdot (u - v)\rho\,(p' \times (\omega \times p'))\, dS \tag{171}$$

For this relation, compare (6.10) of Ong and O'Reilly (2004), and the literature cited there. For the rigid body, we moreover obtain

$$\frac{d}{dt}\alpha' = \frac{d}{dt}\,(J' \cdot \omega) = \left(\frac{d}{dt}J'\right) \cdot \omega + J' \cdot \dot{\omega}, \tag{172}$$

see (156) and (169). Taking into account (49), the time derivative of the tensor of relative inertia J' can be written as

$$\begin{aligned} \frac{d}{dt}J' = \frac{d}{d} \int_V ((p' \cdot p')\, I - p' \otimes p')\, \rho\, dV \\ = \omega \times J' + \omega \cdot \int_V ((p' \cdot p')\, I - p' \otimes p')\, s[1] \rho\, dV \end{aligned} \tag{173}$$

We thus can re-write (170) in the following form, which no longer contains $s_u[\alpha']$:

$$\left(\frac{d}{dt}J'\right)\cdot\omega+'\cdot\dot{\omega}=\int_V p'\times b\,dV+\int_S p'\times\sigma_n dS-\int_V p\times\Sigma_\times dV+s[\alpha'] \quad (174)$$

In case of $\Sigma_\times = 0$, $s[1] = 0$ and $s\,[\alpha'] = 0$, this coincides with (7.7) of Ong and O'Reilly (2004). Analogous derivations can be performed for the other relations of balance, e.g., for balance of kinetic energy.

Balance of kinetic energy For the rigid body, using (153) and (58) and some vector algebra, the relation of balance of kinetic energy in (131) reduces to

$$\begin{aligned}\frac{d_u}{dt}E_{kin} &= \frac{d}{dt}E_{kin}+s_u[E_{kin}]\\ &= v_c\cdot F+\omega\cdot\left(\int_V p'\times b\,dV+\int_S p'\times\sigma_n dS\right)+s[E_{kin}]+s_u[E_{kin}] \quad (175)\end{aligned}$$

Setting $s[E_{kin}] = 0$, and substituting $s_u[E_{kin}]$ of (130), this coincides with (8.3) and (8.5) of Ong and O'Reilly (2004).

4 Presence of a singular surface in a material volume

So far, we have tacitly assumed that the entities under consideration are distributed continuously throughout the volume V. Now assume that a (smooth) singular surface $\bar{S}$ is present within V, such that $\bar{S}$ subdivides V into two non-material sub-volumes. At a singular surface $\bar{S}$, certain local entities $\rho\,\psi$, e.g., mass and linear momentum, and their supplies may take on different values at the two sides of the singular surface $\bar{S}$. Moreover, the points of $\bar{S}$ may move at a velocity w, which in general is different from the velocities of the particles that instantaneously are located at the two sides of $\bar{S}$. In order to connect the local forms of the balance relations at the two sides of the singular surface, jump relations are needed. A classical strategy for deriving jump relations was presented by Truesdell and Toupin (1960): The Reynolds transport theorem is applied to the two non-material sub-volumes, taking into account the motion of $\bar{S}$ relative to the material particles and adding the results. The volume V then is shrunken down to the singular surface $\bar{S}$, assuming that the integrands of the respective volume integrals remain bounded, and letting the surface $\bar{S}$ be finite in the limit. The result of this procedure is the jump relation of Kotchine (1926). The Kotchine jump relation takes into account the jump of the entity $\rho\,\psi$ across $\bar{S}$ in a straightforward manner, however, this procedure is at the cost of not considering the particles that instantaneously are passing the singular surface $\bar{S}$, as well as some non-classical concentrated supplies that may travel with $\bar{S}$. This conceptual drawback can be removed by applying the following more general strategy, which was described in some detail by Irschik (2003). This strategy is discussed and adopted for variable mass systems in the following.

4.1 On a generalized form of the general relation of jump at a singular surface

The methodology proposed by Irschik (2003) consists of several steps. First, instead of starting with the assumption of a singular surface, one studies a non-material shell-type layer of transition, within which the entity $\rho\,\psi$ and its sources are subjected to considerable changes in their spatial distribution. One then replaces the shell-type layer of transition by an equivalent singular surface $\bar{S}$, where several terms in the transport theorem for the non-material layer are represented by surface supply terms that equivalently describe the behavior of the respective quantity contained in the non-material shell-type layer. Motivated by fundamental studies of Slattery (1990) on interfacial transport theorems, a rational mechanical and thermodynamic formulation for this general strategy was presented in Irschik (2003), see also Irschik (2004) and Irschik (2007). This formulation contains the classical Kotchine jump relation as a special case. The additional equivalent surface supply terms have been denoted as surface growth terms by Irschik (2003). In the following, we shortly review the latter rational formulation with special emphasis on the presence of both, a surface supply of mass and linear momentum at the equivalent singular surface $\bar{S}$, and we demonstrate that and show how a surface supply of kinetic energy must be introduced for the sake of consistency, even when the surface supplies of mass and linear momentum do vanish. In the subsequent Section 5, we will exemplarily apply these extended jump relations to the case of a chain heaped up on the edge of a table, the hanging part of the chain being set into motion. In this study, the transition from the heaped part to the moving part of the chain is described by the extended jump relations, while the equation of motion for the moving part is obtained by the relations of balance written for a non-material control volume, which have been stated in Section 3 above.

4.2 Jump of mass

The extended relation of jump of mass at an equivalent singular surface $\bar{S}$ as derived by Irschik (2003) can be written as

$$n_{\bar{S}} \cdot (i_{\bar{S}}\,[\rho] + [\![(v - w)\,\rho\,]\!]) = 0 \tag{176}$$

The outer unit normal vector at the equivalent singular surface is $n_{\bar{S}}$, and the vector $i_{\bar{S}}\,[\rho]$ represents the non-classical equivalent surface supply of mass at $\bar{S}$. Note that the dimension "dim" of any non-classical surface supply $i_{\bar{S}}\,[k]$, say, is given by:

$$\dim\,(i_{\bar{S}}\,[k]) = \frac{\dim\,(k)}{\dim\,(time)\ \dim\,(area)} \tag{177}$$

The jump operator at $\bar{S}$ is defined as the difference of the entities k at the two sides of:

$$[\![k\,]\!] = k^+ - k^- \tag{178}$$

In the following, the unit outer normal vectors at the two sides of $\bar{S}$ are taken such that $n^+ = n_{\bar{S}}$ and $n^- = -n_{\bar{S}}$.

4.3 Jump of linear momentum

The extended relation of jump of linear momentum reads

$$n_{\bar{S}} \cdot (i_{\bar{S}}\,[\rho\,v] + [\![(v - w) \otimes \rho\,v - \Sigma]\!]) = 0 \tag{179}$$

where the second-order tensor $i_{\bar{S}}\,[v\,\rho]$ represents the non-classical equivalent surface supply of linear momentum at $\bar{S}$.

4.4 Jump of kinetic energy

The extended relation of jump of kinetic energy reads

$$n_{\bar{S}} \cdot \left(i_{\bar{S}} \left[\rho\,\frac{v \cdot v}{2} \right] + \left[\!\!\left[(v - w)\,\rho\,\frac{v \cdot v}{2} - v \cdot \Sigma \right]\!\!\right] \right) = 0 \tag{180}$$

The non-classical surface supply of kinetic energy at $\bar{S}$ is a vector denoted as $i_{\bar{S}}\,[\rho\,v \cdot v/2]$. It has been shown by Irschik (2003) that the non-classical surface supplies of mass, linear momentum and kinetic energy are not independent. Rather, $i_{\bar{S}}\,[\rho\,v \cdot v/2]$ must obey the following relation:

$$\begin{aligned} n_{\bar{S}} \cdot i_{\bar{S}} \left[\rho\,\frac{v \cdot v}{2} \right] = {} & \left(\frac{\langle v \cdot v \rangle}{2} - \langle v \rangle \cdot \langle v \rangle \right) n_{\bar{S}} \cdot i_{\bar{S}}\,[\rho\,] \\ & + \langle v \rangle \cdot (n_{\bar{S}} \cdot i_{\bar{S}}\,[\rho\,v]) + [\![v]\!] \cdot (n_{\bar{S}} \cdot \langle \Sigma \rangle) \end{aligned} \tag{181}$$

The mean value operator of some entity k across $\bar{S}$ is given by

$$\langle k \rangle = \frac{1}{2}\left(k^{+} + k^{-}\right) \tag{182}$$

It is seen from (181) that, even if the non-classical equivalent surface supplies of mass and linear momentum are absent, it is generally necessary to consider a non-classical equivalent surface supply of kinetic energy. Subsequently, we give an example, in which a non-classical equivalent surface supply of mass must be introduced for modeling reasons.

5 Example for the formulations in Sections 3 and 4: Caley's chain set into motion

The example of a chain heaped up on a table with the hanging part being set into motion has been chosen by Cayley (1856) in a fundamental contribution on the dynamics of what he called continuous impact problems, see Fig. 1 for a sketch.

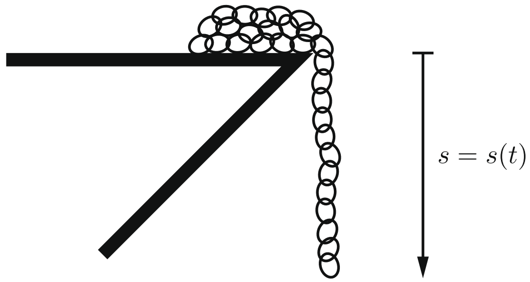

Figure 1. Chain hanging over the edge of a table.

Cayley wrote: *"A problem of the sort arises when a portion of a heavy chain hangs over the edge of the table, the remainder of the chain being coiled or heaped up close to the edge of the table, the part hanging over constitutes the moving system, and in each element of time the system takes into connexion with itself, and sets into motion with a finite velocity an infinitesimal length of the chain."* Cayley used this problem in order to demonstrate the application of a novel variational formulation, as well as of a corresponding extended form of Lagrange's equations. For a contemporary discussion on Cayley (1856), see Irschik (2012).

Subsequently, we use Cayley's example of a chain set into motion in order to apply the formulations stated in Sections 3 and 4. For extended forms of Lagrange's equations for mechanical systems with a variable mass, see Irschik and Holl (2002), Pesce (2003) and the Chapter by Pesce in the present book.

5.1 The hanging part of the chain as a system with a time-varying mass

The chain is assumed to be inextensible and to be coiled up loosely at the table, the hanging part having the instantaneous length $s = s(t)$. In a first step, the hanging part of the chain is described as an open system. For that sake, it is enclosed by a non-material control surface S, see the dashed surface in Fig. 2. For this surface, the formulas of Section 3 above do apply. The upper horizontal part of S is located immediately under the edge of the table and is fixed in space, such that there is $u = 0$; the particles of the inextensible chain enter there the control volume at the velocity $v = (ds/dt)e_x = \dot{s}\, e_x$, see Fig. 2. The outer unit normal vector n at this location is opposite to the global x-direction, $n = -e_x$; a superimposed dot denotes the time derivative. Hence, there is $n \cdot (u - v) = \dot{s}$ in (124). Note that the chain in general is stressed at this upper horizontal location of the control surface S. Denoting the mass of the chain per unit length as μ, and the tensile force in the chain at the upper horizontal part of the control surface as N, Cauchy's fundamental theorem on stresses yields $n \cdot \Sigma = -N(\rho/\mu)e_x$. The lower horizontal part of the control surface moves together with the tip of the chain, $u = v = \dot{s}\, e_x$, such that there is $u - v = 0$ in (124); also, the chain is free of stress there, $n \cdot \Sigma = 0$, see Fig. 2. The vertical portions of S do not contribute to the relations of balance, since no material is present there.

The instantaneous mass of the hanging part of the chain is $m = \mu\, s$, such that

$$\frac{d_u}{dt} m = \mu\, \dot{s} \tag{183}$$

Recall that the operator d_u/dt means that we consider the motion of the non-material control volume; in other words, $s = s(t)$ in $m = \mu\, s$ is not kept fixed, which leads to (183). The additional, non-classical supply of mass due to the flow of mass through the control surface becomes, see (126),

$$s_u[m] = \int_S n \cdot (u - v)\rho\, dS = \mu\, \dot{s} \tag{184}$$

Further recall that only the upper part of the control surface in Fig. 2 contributes to the surface integral. Substituting (183) and (184), it is seen

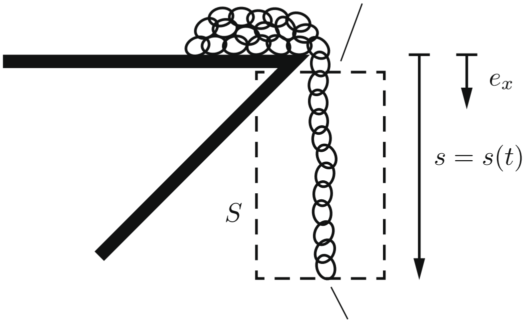

Figure 2. Non-material control volume S enclosing the hanging part of the inextensible chain.

that balance of mass (125) indeed is satisfied. No sources of mass in the interior are present, $s[m] = 0$.

We now proceed to balance of linear momentum (127) which we apply for the open system depicted in Fig. 2. The instantaneous linear momentum of the hanging part of the chain in x-direction is

$$j = \mu\, s\, \dot{s}\, e_x \tag{185}$$

such that

$$e_x \cdot \frac{d_u}{dt} j = \mu\, s\, \ddot{s} + \mu\, \dot{s}^2 \tag{186}$$

For the additional, non-classical supply of linear momentum due to the flow of mass through the control surface, see (128), one obtains analogous to (184) that

$$e_x \cdot s_u[j] = e_x \cdot \int_S n \cdot (u - v) \rho\, v\, dS = \mu\, \dot{s}^2 \tag{187}$$

The resultant force acting upon the hanging part of the chain is

$$e_x \cdot F = -N + \mu\, g\, s \tag{188}$$

where the last term represents the instantaneous weight of the hanging part and g denotes the gravitational acceleration. With $s[j] = 0$, the relation of balance of linear momentum (127) thus yields the following equation of motion:

$$\mu\, s\,(\ddot{s} - g) + N = 0 \tag{189}$$

Under the assumptions of an inextensible homogeneous chain in a vertical motion due to its own weight, this relation is to be considered as exact; however the normal force N in the chain at the upper horizontal part of the control surface in Fig. 2 is yet unspecified.

5.2 Modeling of the region of transition between the moving and resting parts of the chain by an equivalent singular surface

In order to set up an additional relation for N, the region of transition between the part of the chain in motion and the heaped up part at rest now is modeled by means of an equivalent singular surface $\bar{S}$, see the dashed line in Fig. 3, where the region of transition with the two outer surfaces S^+ and S^- is also sketched.

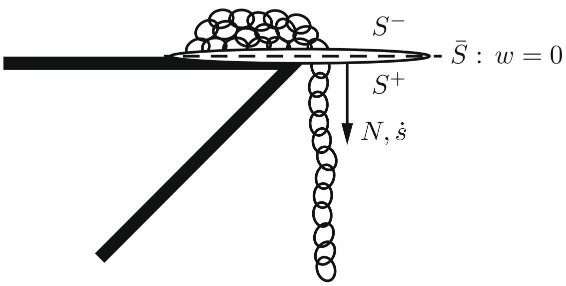

Figure 3. Equivalent singular surface modeling the region of transition between the moving and the resting parts of the chain.

Since the equivalent singular surface $\bar{S}$ is at rest, there is $w = 0$ in the formulas of Section 4 above. Outer unit normal vectors, velocities and stresses of the particles at the two sides of $\bar{S}$ are as follows, see also Fig. 4:

$$\begin{aligned}
&\bar{S}: \quad && w = 0, && n_{\bar{S}} = n^+ = e_x, && \\
&S^-: && v^- = 0, && n^- = -e_x, && n^- \cdot \Sigma^- = 0, \\
&S^+: && v^+ = \dot{s}\, e_x, && n^+ \cdot \Sigma^+ = N \frac{\rho}{\mu} e_x &&
\end{aligned} \tag{190}$$

The heaped part of the chain, which is at rest, can be taken as unstressed for the present purpose, $N = 0$ at S^-, while the force in the chain at the side

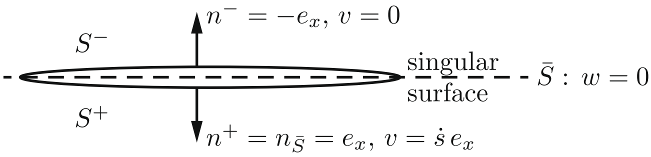

Figure 4. Details of equivalent singular surface.

S^+ is N. For the situation sketched in Fig. 4, (190) yields for the relation of jump of mass (176) that

$$n_{\bar{S}} \cdot i_{\bar{S}}[\rho] = -\rho\, \dot{s} \tag{191}$$

Note that $n_{\bar{S}} \cdot i_{\bar{S}}[\rho]$ only vanishes, if the hanging part is at rest. This example clearly demonstrates the necessity of extending the classical Kotchine-type relations of jump by non-classical equivalent surface supply terms, as proposed by Irschik (2003).

The next step is the relation of jump of linear momentum. (179). Substituting (190), one obtains for the non-classical equivalent surface supply of linear momentum

$$e_x \cdot i_{\bar{S}}[\rho\, v] = \left(-\rho\, \dot{s}^2 + N\, \frac{\rho}{\mu} \right) e_x \tag{192}$$

Analogously, the relation of jump of kinetic energy (180) gives for the non-classical equivalent surface supply of kinetic energy that

$$e_x \cdot i_{\bar{S}} \left[\rho\, \frac{v \cdot v}{2} \right] = -\frac{\rho\, \dot{s}^3}{2} + \frac{N\, \rho}{\mu}\, \dot{s} \tag{193}$$

It is easily checked that (181), which states the relation between the surface supplies of mass, linear momentum and kinetic energy derived by Irschik (2003), indeed is satisfied, since, at the equivalent singular surface $\bar{S}$, there is

$$\bar{S}: \quad [\![v]\!] = \dot{s}\, e_x, \quad \langle v \rangle = \frac{\dot{s}}{2} e_x, \quad \langle v \cdot v \rangle = \frac{\dot{s}^2}{2}, \quad [\![v]\!] \cdot (n_{\bar{S}} \cdot \langle \Sigma \rangle) = \frac{N\, \rho}{\mu} \frac{\dot{s}}{2} \tag{194}$$

We now may distinguish two special cases: The first is obtained by assuming that there is no non-classical surface supply of linear momentum,

$$e_x \cdot i_{\bar{S}}\, [\rho\, v] = 0 \tag{195}$$

in (192), from which the chain force N becomes

$$N = N_{nc} = \mu\, \dot{s}^2 \tag{196}$$

Substituting into the equation of motion (189) gives

$$\mu\, s_{nc}\, (\ddot{s}_{nc} - g) + \mu\, \dot{s}_{nc}^2 = 0 \tag{197}$$

The solution with homogeneous intial conditions is

$$s_{nc} = \frac{g\, t^2}{6} \tag{198}$$

The solution stated in (197) and (198) has been obtained by Cayley (1856) in the framework of another methodology. In the literature, this type of solution has been called a non-conservative solution, see, e.g., Wong and Yasui (2006). From this naming, the index "*nc*" has been introduced in (196)–(198). Indeed, substituting (196), from (193) we obtain the following equivalent surface supply of kinetic energy:

$$e_x \cdot i_{\bar{S}_{nc}} \left[\rho\, \frac{v \cdot v}{2}\right] = \rho\, \dot{s}^3 \tag{199}$$

On the other hand, if we assume that the equivalent surface supply of kinetic energy vanishes,

$$e_x \cdot i_{\bar{S}} \left[\rho\, \frac{v \cdot v}{2}\right] = 0 \tag{200}$$

then (193) yields

$$N = N_c = \mu\, \frac{\dot{s}^2}{2} \tag{201}$$

Substituting into the equation of motion (197), we now obtain

$$\mu\, s_c\, (\ddot{s}_c - g) + \frac{1}{2}\, \mu\, \dot{s}_c^2 = 0 \tag{202}$$

The solution for homogeneous initial conditions this time becomes

$$s_c = \frac{g\, t^2}{4} \tag{203}$$

This solution, which predicts a fall of the chain faster than Cayley's non-conservative solution (198), has been called a conservative solution in the literature, see Wong and Yasui (2006). Note, however, that this conservative solution is associated with a non-classical supply of linear momentum, which follows by substituting (201) into (192):

$$e_x \cdot i_{\bar{S}_c}[\rho\, v] = \frac{\rho\, \dot{s}^2}{2}\, e_x \tag{204}$$

The discrepancy between conservative and non-conservative solutions in Cayley's problem, as well as in related problems of falling chains, ropes, cables and strings, has given raise to a long-term controversy in the literature. Here, we mention the more recent theoretical and/or experimental works by Tomaszewski et al. (2006), Wong and Yasui (2006), Wong et al. (2007), Grewal et al. (2011) and Irschik (2012).

It is hoped that the rational methodology given in the present section, which is based on relations of balance of mass, linear momentum and kinetic energy for open systems, and on generalized corresponding relations of jump, will contribute to a further clarification of the different results in the literature. It should have become clear from our reasoning that a more detailed modeling of the region of transition between the heaped and the moving parts of the chain in general will lead to both, a non-vanishing equivalent surface supply of linear momentum, as well as a non-vanishing equivalent surface supply of kinetic energy, and thus will give raise to solutions different from the above discussed two cases, cf. O'Reilly and Varadi (1999) for a study on shocks.

Acknowledgments

This work has been supported by the Linz Center of Mechatronics (LCM) in the framework of the Austrian COMET-K2 program.

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