Elastodynamic End Effects in Structural Mechanics

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Abstract Current status of research on decay of dynamic end effects in elastic structures aiming at formulation of a dynamic analogue to Saint-Venant's principle (DSVP) are critically reviewed. Article concentrates on isotropic homogeneous linear elastic response over a range of structural geometries including waveguides, with either free or constrained lateral surfaces, half space, wedges and cones. Nearly 200 references are examined in context of DSVP, starting with early ideas by Boley. Special attention is placed on available experimental findings on end effects and decay rate in dynamically excited structures. Current perception of possible versions of DSVP is classified into several categories, one of which, namely that of dynamic equivalence, is compatible with much of known experimental data and has been tacitly applied at various engineering situations. That observation, along with a perspective view on evolution of the traditional SVP, provides inspiring ground for renewed interest in both practical and theoretical aspects of DSVP formulation.

1 Motivation

The principle named after Saint-Venant (SVP) has been commonly accepted as a corner stone assumption, widely employed in structural engineering and theoretical analysis of solid mechanics and related fields. Traditionally, that principle pertains to phenomena localized at the ends of a structure, ends to which a self-equilibrated load is applied. Validity of the principle entails that localization.

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In its original version (Saint-Venant, 1856) the principle argues that "the manner of application of a given resultant force and resultant moment on the two ends of a beam little affected the stress pattern, except very near the ends, and that all the solutions of a given problem, for end conditions having the same resultants, rapidly approached one and the same solution" (Toupin, 1965b, p. 223). The underlying idea, labeled by Toupin as "the principle of vanishing end effects", has been introduced by Saint-Venant to qualify theoretical elasticity solutions for beams for use in actual engineering conditions. In a broad sense, the assumption, when valid, implies that the stress field in the interior of a body is not sensitive to spatial distribution (profile) of prescribed local boundary tractions. Applicability of SVP has been confirmed over the years for several types of structures. Authoritative accounts of available research are given in the review by Horgan and Knowles (1983) and in two subsequent updates by Horgan (1989, 1996). Static stress fields that do not conform with SVP have been exposed for thin shells and statically determinate truss structure (Hoff, 1945), laminate/composite structures (Choi and Horgan, 1977, 1978), monocoque structures (Hoff, 1945; Nerubailo et al., 2005), and prestressed plates near points of bifurcation (Durban and Stronge, 1988a; Karp, 2004).

The applicability of SVP in linear elasticity has inspired research and formulation of similar principles in other branches of mechanics of materials. Among these are non-linear materials (Roseman, 1976), pre-strained plates (Durban and Stronge, 1988b), piezoelectric solids (Ruan et al., 2000), heat transfer phenomena (Oleinik and Iosif'yan, 1976; Chirita and Quintanilla, 1996a; Ignaczak, 2002), and fluid flow (Payne and Song, 1997).

A particularly challenging quest is for possible extension of SVP to include dynamic structural response, aiming at formulation of a dynamic Saint-Venant principle (DSVP). Several progress reviews of the classical Saint-Venant principle contain, inter-alia, short comments related to DSVP which deserve recollection. The review by Horgan and Knowles (1983, p. 261) concludes with: "one would not expect to find unqualified decay estimates of the kind discussed here in problems involving elastic wave propagation". The same conclusion is repeated in the first update of that review (Horgan, 1989). In a second update a few studies, apparently supporting validity of DSVP, are mentioned (Horgan, 1996). A further review by Horgan and Simmonds (1994), on application of SVP to composites, refers to end effects in vibration problems as related to DSVP.

At least five doctoral thesisses have been dedicated to investigation of issues and questions concerning DSVP, including Grandin (1972), Binkowski (1975) (both supervised by S. Little), Karp (1996) (supervised by D. Durban), Foster (2003) (supervised by V. Berdichevski) and Babenkova (2004) (supervised by J. Kaplunov). Needless to say, each of these studies contains a review of relevant research literature as known at the time. A related dissertation, though not directly associated with DSVP, is submitted by Meitzler (1955).

It is the purpose of the present review to agglomerate and classify available research work on elastodynamic versions of Saint-Venant principle (DSVP), both experimental and theoretical. The review, which is an outgrowth of the PhD thesis by Karp (1996), followed by a brief historical account (Karp, 2005), begins with a retrospective of early studies by Bruno Boley along with a short discussion of the notion of self-equilibrated load. Next, in Chapter 3, we review experimental work related to DSVP. Analytical and numerical studies of DSVP in waveguides with free lateral surfaces are examined in Chapter 4. Chapters 5, 6, and 7 are devoted to a few available studies on the validity of DSVP in constrained waveguides, miscellaneous structures, and composites, respectively. Dynamic decay estimates for vibrating structures and in viscous materials are reviewed in Chapter 8. Comparison between the classical SVP and DSVP is suggested in Chapter 9, and finally, concluding remarks are given in Chapter 10.

The present review concentrates on studies concerned with aspects of DSVP and dynamic end effects in linear elastic materials. It is largely based on a recently published review by Karp and Durban (2011) with several extensions and updates. Approximately 200 articles, devoted or related to the DSVP are reviewed though, in fact, only a fraction of these papers was originally intended to investigate directly the DSVP. The papers referenced here were categorized as related to DSVP from the viewpoint of our present understanding of the topic. For that reason, no attempt has been made at an exhaustive review with regard to fields which are beyond linear elastic response. However, within that context, a few neighboring fields, like evanescent waves, are covered here in part.

A few studies attempt to find a connection between range of influence of applied load and DSVP. A theorem of this kind states that a sudden excitation of a body, initially in unperturbed state, will subdivide it into two regions; the region close to the disturbance zone where the perturbation is imposed and the rest of the body which is still intact. The surface separating these two regions is propagating with a characteristic velocity determined by material properties. Self-equilibrated loads and equivalent excitations, the key ingredients of SVP, have no special importance to the essence of that theorem. Rigorous formulation of the theorem can be found in Gurtin (1972) and more recently in Maremonti and Russo (1989). That interpretation of DSVP is not addressed in the present review unless it is accompanied with an estimate of decay. Likewise, left outside the review are studies on end effects of Saint-Venant type related to thermal response and problems governed by parabolic field equations (e.g., Sigillito, 1970), heat conduction in thermo-microstretch elastic solid (Quintanilla, 2002) and in porous solids (Iovane and Passarella, 2004), and other phenomena such as flow in ducts (e.g., Ames et al., 1993).

2 Origins of the Principle

Along the timeline, two aspects have been instrumental in assessing the causes for both stagnation and progress in studies of DSVP: the notion of self-equilibrium and its role in validation of SVP, and the original ideas suggested by Bruno Boley. Accordingly, we proceed with a brief review of both aspects followed by a short summary emphasizing the conceptual difficulty arising in treating DSVP.

2.1 Static and Dynamic Self-Equilibrium

The notion of self-equilibrated load is central to the mathematical formulation and validation of the classical SVP. Self-equilibrium of a quasi-static traction vector $\mathbf{t}$ implies zero total force and moment, generally expressed by

$$\int_{(S)} \mathbf{t} dS = \mathbf{0} \qquad \int_{(S)} (\mathbf{r} \times \mathbf{t}) dS = \mathbf{0} \tag{2.1.1}$$

where S is a small portion of the surface of the body on which the self-equilibrated traction is applied and $\mathbf{r}$ is the position vector. The traction vector $\mathbf{t}$ is the projection of stress tensor , given by $\mathbf{t} = \sigma \cdot \mathbf{n}$ where $\mathbf{n}$ is the outward unit vector normal to dS. Then, SVP is stated as "...the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." (Love, 1944, p. 132). Validity of SVP is considered to be established when the effect of self-equilibrated load can be shown to decay (usually exponentially) with distance resulting in a small depth of non-negligible straining. That is the localization phenomena, the focus of the present volume.

Engineering situations in which a self-equilibrated load is applied are admittedly not common. The association of self-equilibrium with SVP facilitates mathematical analysis in providing proofs or quantitative estimates of its validity. The practical usefulness of the classical SVP lies in the concept of equivalence of loads, as suggested by Saint-Venant himself: "If a certain set of external forces acting on a certain part of a surface of a body is replaced by another system of external forces statically equivalent to the preceeding system and distributed over the same sector, the stresses corresponding to these two loads will be identical at a sufficient distance from the point of application of the forces" (Cherepanov, 1979, p. 40).

Whereas analytical proofs and estimations of validity of SVP rely mainly on the self-equilibrium formulation (Love, 1944), experimental demonstration of SVP are largely based on the equivalence definition (as stated above). A classical demonstration has been provided by Frocht (1948) using the photoelastic method (Fig. 1). The experiment shows that far from the edge on which a concentrated load is applied a uniform stress develops, as in the case of a uniformly distributed load with identical static equivalents.

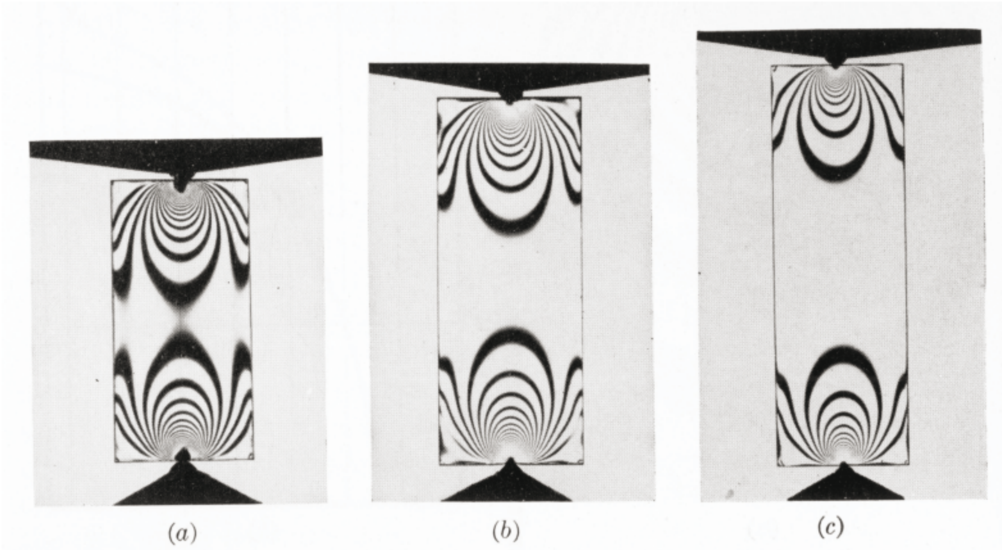


Figure 1. Photoelastic photographs of experiment with rectangular blocks loaded by a concentrated load (from Frocht, 1948, p. 30).

The notion of self-equilibrium is commonly extended to dynamic (time varying) excitation by either replacing the self-equilibrated traction with simple harmonic load fulfilling

$$\int_{(S)} (e^{-i\omega t}\mathbf{t})dS = \mathbf{0} \qquad \int_{(S)} \left(\mathbf{r} \times (e^{-i\omega t}\mathbf{t})\right) dS = \mathbf{0} \tag{2.1.2}$$

for steady state conditions (time denoted by t), or by considering any other time function $f(t)$

$$\int_{(S)} f(t)\mathbf{t}dS = \mathbf{0} \quad \int_{(S)} (\mathbf{r} \times f(t)\mathbf{t})\, dS = \mathbf{0} \tag{2.1.3}$$

for transients. In both cases self-equilibrium is guaranteed to hold at any instant of time.

2.2 Early Ideas by Bruno Boley

Research questions related to DSVP were already addressed in 1948 (see later in Chapter 3), yet Boley was apparently the first to examine explicitly an application of SVP to dynamic problems. His ideas on a possible extension of SVP to dynamic phenomena were expressed in two papers, dating back to 1955 and 1960, along with a wider generalization to problems governed by non-elliptic equation in 1958.

Boley investigated the possibility of extending SVP to dynamic problems by considering two idealized structures; the first consisting of three elastic semi-infinite bars interconnected by shear springs and subjected to longitudinal self-equilibrated loads (Boley, 1955). The second structure consisting of two semi-infinite bars connected by springs on which couples with zero total moment are applied (Boley, 1960a). The original sketches of the problem are recapitulated here in Fig. 2. In both cases the excitation (load or velocity) is self-equilibrated at any instant. The criterion for validity of DSVP is defined as a vanishing ratio between the maximal stresses reaching a distant portion of the beam and the initial stress at the excited end.

The dynamic response of the two structures was determined analytically, using transform method, for applied loads (or velocity) with a ramp variation in time. The time rise of the ramp (t_0) served as a parameter, with the limits of $t_0 = 0$ (representing a step function) and $t_0 \to \infty$ (corresponding to quasi-static conditions) as particular cases. Two main results were obtained, common for all three loading types; longitudinal, shear, and bending (Timoshenko beam equation was employed allowing for shear contribution to the dynamic response). For a suddenly applied load (step), stress with a magnitude of the initial value, or somewhat lower, propagates through the bars indefinitely. That result led to the conclusion: "thus the conventional principle of Saint-Venant certainly does not hold in this case" (Boley, 1955, p. 205).

The second observation made is the convergence of the dynamic solution to the static one as the excitation becomes more and more graduate (shown in Fig. 3). In the limit of the static load, the effect of the self-equilibrated

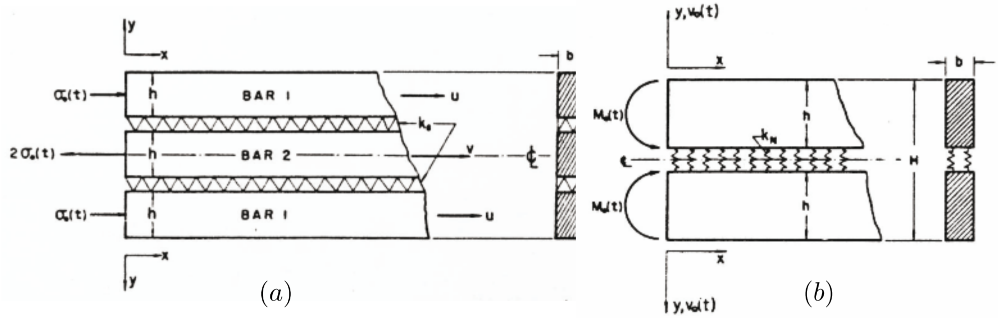


Figure 2. Axial (a) and deflective (b) beam combinations investigated by Boley (1955, 1960a).

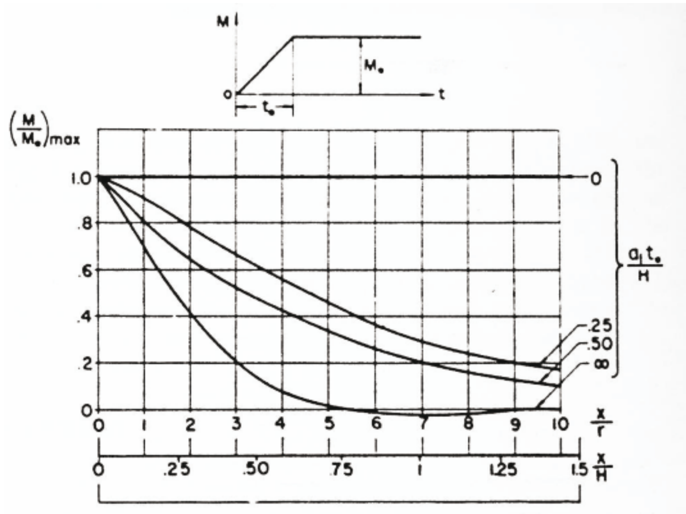


Figure 3. Attenuation patterns of the maximal stress along the axis of the combined strip imposed by moments with different rise time t_0, 0 - for step function and ∞ - for quasi-static case (from Boley, 1960a).

load practically vanishes beyond one width of the composed beam. The conclusion is then: "the conventional usage of the static Saint-Venant principle is not too greatly in error for slowly applied loads" (Boley, 1955, p. 206).

Considering the static case as asymptotic solution for the transient dynamic problem led Boley to suggest that the question of validity of SVP for a dynamic problem is part of the wider issue of how slowly loads must be applied before the static solution ceases to be a good approximation for the dynamic problem (Boley, 1960a). Therefore, it was suggested to redefine the quest for DSVP as a search for the rate of application of the load for which the quasi-static solution will not introduce unacceptable error. That direction of research has been followed by Grandin and Little (discussed later in Chapter 4).

In a short conference paper, which offers newly introduced ideas and terms, Boley (1958) paved the way to much of subsequent research on SVP. A question of general validity of SVP to systems governed by elliptic, parabolic or hyperbolic equations has been raised. For problems governed by elliptic equations a univocal conclusion was drawn: "The existence of the above integral formulas, involving appropriate fundamental solutions, is a general property of elliptic differential equations, which arise in such fields as steady heat conduction, electro-and magneto-statics, non-viscous fluid flow, and so forth, in addition to elasticity; to all these the principle can then be applied" (Boley, 1958, p. 259). This idea of connection between ellipticity of governing equations and SVP, together with the notion of self-equilibrated load, has been employed in many elasticity studies and led to the first genuine proofs of SVP by Knowles (1966) and Toupin (1965a). An illustration for validity of Saint-Venant's principle for a parabolic system, represented by transient heat problem, is given by Boley (1960b). In particular, a generalized notion of "principles of the Saint-Venant type" was suggested along with a recomendation to state them in terms of "upper bound" rather than by order of magnitude. Subsequent studies employing energy inequalities follow these steps.

In a recent correspondence with one of us (BK), Prof. Boley (2006) reflected on the issue from a perspective of 50 years, since he made a start on the topic of DSVP, writing that: "SVP reminds me in spirit of Pirandello's "Six Characters in Search of an Author", it is indeed a principle in search of a theorem. The proofs of SVP, for example, are really proofs of a SVP which may not necessarily be recognized by practicing engineers as the SVP they are actually using. It is probably close enough, it certainly belongs to the same species, and so they may feel confident in using it."

2.3 The Challenge

Boley (1960a, p. 74) concluded his studies on DSVP with an inspiring observation: "A discussion given elsewhere indicated that Saint-Venant's principle is a general property of elliptic boundary value problems, and could not be expected to hold in general in problems of the hyperbolic type, such as, for example, those of the dynamical theory of elasticity. If the loading is applied sufficiently slowly, however, then it is intuitively clear that the static solution will be a good approximation to the dynamic one".

That perception that SVP is a general property of systems, whose behavior is dictated by the nature of their governing equations, has been supported by several independent studies. For example, Horgan and Wheeler (1975) wrote in the abstract: "Third order diffusive type equations, called pseudo-parabolic, are known to govern a wide variety of physical phenomena. A spatial decay estimate is derived for such an equation, similar to the known results in the parabolic case". A similar statement is expressed by Knops et al. (1990, p. 319): "Although the treatment is discussed with special reference to elasticity, it is equally applicable to general systems of elliptic differential equations, and thus reveals a relationship with the classical theorems of Phragmen-Lindelof and Liouville". That view was utilized by Oleinik and Iosif'yan (1978) several years earlier.

The interconnection between decay behavior and the type of the governing equations, together with well known non-decaying phenomena in dynamic problems, are apparently behind the wide spread rejection of SVP validity to dynamic problems in non-dissipative media. With that skepticism in mind, we attempt here to examine the idea of possible formulation of a DSVP in its classical sense, even in a restricted version. Surely, an instructive start of this review is provided by available experimental evidence on decay of dynamic end effects, discussed next.

3 Experimental Evidence

Effects of non-uniformity of dynamic excitation applied at an end of a bar attracted attention of several groups of researchers during the middle of the previous century. Research was driven by growing interest in experimental aspects of the split Hopkinson bar system (Wally and Mason, 2000; Field et al., 2001). Most of these studies were not originally associated with DSVP, yet they are reviewed below as a prelude to later work and, as will be shown, they are of significant value for at least one of the interpretations of DSVP.

Davies (1948) observed in his detailed experiments on split Hopkinson pressure bar system that the pressure distribution over the cross-section of the bar is not uniform at its free end. Two potential sources for that non-uniformity were suspected: end effects and the shape of Pochhammer-Chree modes at high frequencies. No explicit statement on the extent of "close region" near the edge is suggested in that review. In a later study it was found that the edge non-uniformity is smoothed out beyond four to five diameters away from the excited end (Davies, 1956).

Miklowitz (1957) and Miklowitz and Nisewanger (1957) investigated numerically and experimentally the extent of validity of an approximate Mindlin-Herrmann theory for the analysis of propagation of compressive waves in a dispersive elastic rod. Strains in both near (0.75, 1, 1.24, 1.5, and 2 diameters) and far (up to 20 diameters) fields, induced by aerodynamic pressure pulse, were measured. Though limited to uniform excitation over the cross section, the findings suggest that edge effects for that particular loading are limited to several diameters off the end: ”The present experiments give further support for this. They indicate that the initial disturbance, and the phenomena occurring just behind it, are of three-dimensional character and are relatively unimportant several diameters from the source” (Miklowitz and Nisewanger, 1957, p. 244).

Fox and Curtis (1958) devised an experiment aimed to confirm the asymptotic solution of step pulse excitation of a bar obtained by Folk et al. (1958) for strains far from the loaded end. Since the asymptotic solution is valid only beyond a distance of 10-20 diameters from the end, the experimental results do not include data for strains at distances smaller than 20D from the excited end. Due to different specifications of end excitations employed in the asymptotic analysis (mixed condition with no lateral extension) and imposed in the experiments (pure stress condition with no transversal tractions), an additional assumption is required to facilitate the comparison between the two, even at distances beyond 20 diameters from the excited end: ”Failure to satisfy the second end condition is expected to be relatively unimportant for strains at large distances from the end. It is left to experiment to determine the extent to which this expectation is fulfilled” (p. 559). An answer to this question, provided later by several studies, does not refer to that expectation and remained unrelated to it.

Gorham and Ripperger (1959) addressed the same question of non-uniformity by measuring the difference between surface strain and average strain within the cross section of a bar, far from the excited end. The generation of various spatial forms of excitation is achieved by bullets of different size impinged at the end of the bar. They found no substantial difference between the two recordings at a distance of 26 bar diameters. No exact recording of the velocity of the bullets is given, thus preventing any attempt to estimate the frequency spectrum of the excitation (for higher velocities more energy is conveyed by modes of high frequencies). A similar investigation with bars of a square cross section has been detailed by Cunningham and Goldsmith (1959) with the important addition of surface measurement within the near zone. They found that the non-uniformity becomes unimportant at about 2 to 4 bar widths from the impact end.

A carefully conducted study on the extent of non-uniformity, made by Baker and Dove (1962), included embedded strain gauges within the bar at various distances from the impacted end, in addition to surface strain gauges. The core measurement device is shown in Fig. 4.

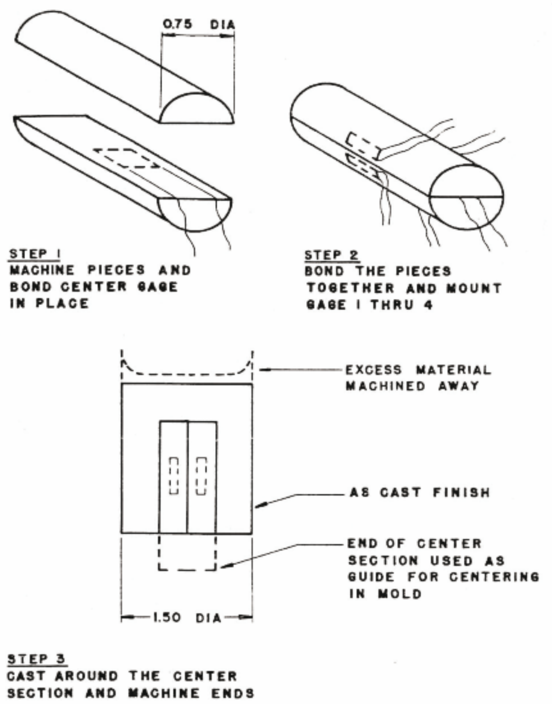


Figure 4. Configuration of the impacted bar with the attached strain gauges (from Baker and Dove, 1962).

The contact end of the impactor had a curvature of radius 3" while the diameter of the impacted bar was 1.5". The findings which resemble those of Cunningham and Goldsmith (1959) have led the authors to conclude that the results obtained earlier by Davies (1956) of 4 diameters as representative distance to which end effect are extended is an overestimation: "It was concluded that, when a pulse in longitudinal bar is initiated by central impact on a small area at one end, the change in the strain profile due to starting conditions ceases in the vicinity of 2 bar diameters from the impact end. This is not in agreement with Davies, who reported that four to five diameters were required" (p. 311). No data on the striker's velocity at the impact is given, though from the experimental set-up it could be estimated to be of the order of one meter per second.

An additional study on the effect of excitation profile on the strain pulse, far from the excited end, is reported by Barton and Volterra (1959). The variation in profile of the excitation was achieved upon employing two strikers in a split Hopkinson bar system. One striker had a flat head and the other a rounded head. The measurement of the surface strain was taken at 24 diameters off the excited end "in order to permit the pulse to travel a sufficient distance to become uniformly distributed over the cross-section of the bar" (p. 321). A typical comparison of recordings for flat and round impacting rods is reproduced here in Fig. 5. It is evident that at such a large distance from the impact end the two strikers had practically an identical effect.

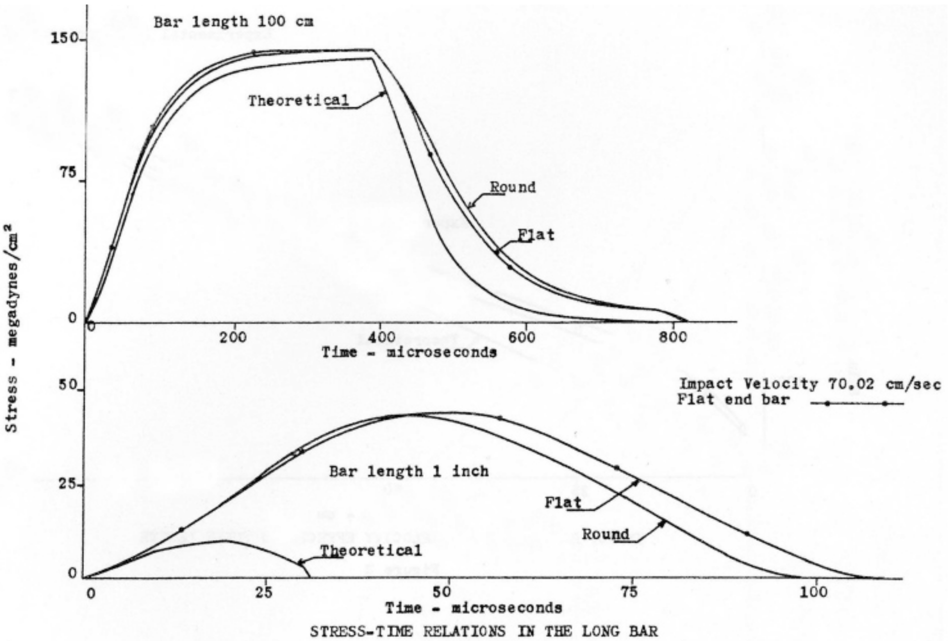


Figure 5. Surface strain for flat and round strikers of length 100 cm (upper) and 2.54 cm (lower), at impact velocity of 0.7 m/s, 24 diameters from the end (from Barton and Volterra, 1959).

Clausing (1959) examined the adequacy of the elementary, one-dimensional theory, to predict results of impact of cylinders of different areas. The response of the impacted rod was recorded at several distances from the impinged end, starting at a distance of 2.4 diameters. The strikers were of various diameters and identical length, all with a radius larger than that of the impacted rod. Since the contact area in all experiments was identical, the results are not directly relevant to the question of the effect of the profile of the contact area.

The influence of different end conditions on the dynamic response of a strip has been examined by Dally et al. (1959) using the photoelastic method. They found that the fringe pattern is almost identical for reflection of waves from a free and a fixed end, except for the region close to the excitation or the end.

The experimental investigation by Flynn and Frocht (1961) appears to be the first experimental work specifically intended to examine possible existence of the DSVP. In this work, dynamic characteristics of near and far fields in a waveguide subjected to uniformly distributed and to concentrated loads were inspected by the photoelastic method. Two basic observations have been made in this study: identical stress distribution is obtained far from the loaded end for both types of transient loadings (uniformly distributed and concentrated); the parameter that determines the stress magnitude in the far field is the impact velocity rather than the force magnitude as in the case of classical SVP. The report was labeled by the authors as a preliminary investigation. A follow up discussion of that paper by Durelli and Dally indeed encouraged further study whereas a discussion by Borg (1961) on such demonstration doubted the possible existence of DSVP.

Borg's comment was based on comparison of the response of a semi-infinite beam, modeled by Timoshenko theory, with two different loadings of equal moments applied at the close end. In one case the moment is produced by normal stresses distributed linearly over the cross section according to the simple beam theory. In the other case the moment is induced by two identical concentrated forces separated by small distance, acting normal to beam axis. It is argued that in the second case only a shear wave is generated implying wave front propagating with shear velocity, while in the first case both shear and dilatational waves are generated, giving rise to substantial difference in wave fronts of the two cases. On this ground it was concluded that " a dynamic Saint-Venant Principle does not exist (in the form considered herein, which most closely parallels the static formulation) for the Timoshenko representation of the vibrating beam" (Borg, 1961, p. 120). To the Author's best knowledge the work by Flynn and Frocht (1961) has escaped notice in subsequent studies of DSVP, while a recent review by Field et al. (2001) refers to it as confirming the validity of DSVP.

By the end of the sixties sufficient data has accumulated to enable comparison of the spatial extent of dynamic end effects with that of static end effects. Photoelastic fringe pattern of a semi-infinite strip with concentrated static load applied at a center of a strip is given by Theocaris (1959). This pattern is very similar to that reported by Meyer (1964) for a concentrated impact load, taken from Flynn et al. (1962). In a study by Kawata and Hashimoto (1967) static and dynamic concentration factors around irregularities are compared. Fringe patterns exhibit similarity of the affected region for both static and dynamic cases. The same localized dynamic response around a hole in a strip is shown in Flynn et al. (1962) with comparison between uniform and concentrated load excitations of the strip.

Validity of DSVP as a prerequisite for suitability of an experimental setup for acoustic emission study was recognized by Kroll and Tatro (1964). To that end the characteristics of wave propagation in a tensile specimen were examined for later use in a study of correlation between dislocation motion and acoustic emission. The authors investigated uniformity of a wave at the end of the specimen originated by a pointwise source. The results obtained (though limited to 5 diameters from the edge due to electrical interference) confirm earlier results by Bell (p. 130): "Bell has shown that, in three to five bar diameters, the stress waves have reflected many times and their resultant becomes an extensional stress wave, which is uniform across the cross section, travelling at the bar velocity v_b. This establishes a dynamic St. Venant's principle". The cited work by Bell (1960) was not available to us. The authors conclude with (p. 134): "The dynamic St. Venant's principle will insure that the stress wave becomes uniform after several diameters of travel".

Hettche and Au (1967) studied the effect of non-uniformity of the stress field across a semi-infinite plate. Theoretical considerations of that problem were supported experimentally by impacting cylindrical hollow rods. The authors state (p. 308) that " this stress is seen to be maximum at the center line and vanishes at the surface of the plate, and is critical only within the first plate thickness from the impact face".

Bertholf and Karnes (1969) investigated surface and center-line stresses in the immediate vicinity of the impact end, while the impact velocity was designed to generate stresses slightly above the yield stress. The conclusions (p. 541) reported are: "It is clear that a one-dimensional analysis is inadequate for $z < 4R$. ... For the elastic pressure bar numerical solution will determine the length at which the dilatation front becomes negligible and the uniaxial-stress approximation becomes valid. It is anticipated that this length will be between 10 and 20 dia.". That quantitative estimation is not explained, and actually is not in agreement with a comparable statement in an earlier paper by one of the authors (Bertholf, 1967) estimating it to be 4 diameters. Moreover, the experimental results in the latter reference are in good agreement with those of Baker and Dove (1962) and others cited above, while the observation of "10 and 20 dia." is consistent with the practice to be suggested later by Follansbee (1985). No comparison to any previous analogous results is given, nor have the effects of exceeding yield stress and high impact velocity (100 m/s) been examined.

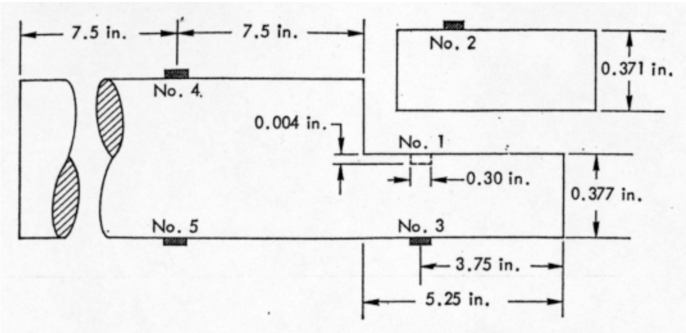


Figure 6. Configuration of the impacted bar with an embedded strain gauge No. 1 and the surface strain gauges Nos. 2-5 (from Habberstad et al., 1972).

Experimental results for the centerline strain within the near-field region are reported by Habberstad et al. (1972). The configuration of the embedded strain gauges is displayed in Fig. 6. The striker velocity was 5 m/s with a flat head in all experiments. Comparison of the center-line strain with surface strain was made for distances of 2/3, 2, and 3 diameters from the impact end. Typical result at the distance of 3 diameters is displayed in Fig. 7, showing clearly that even at that distance considerable difference exists between center-line and surface strains. By using the same bar in inverted position the authors confirmed that the recording at distances of 10 and 22 diameters, from the impact end, are identical. This makes 10 diameters the upper limit for practical uniformity of the signal.

Zemanek (1971, 1972) provided experimental and theoretical insight into the origins and nature of non-uniformity in context of reflection of a wave from a free end of a bar, raised earlier by Davies (1948). These studies are detailed in the next chapter.

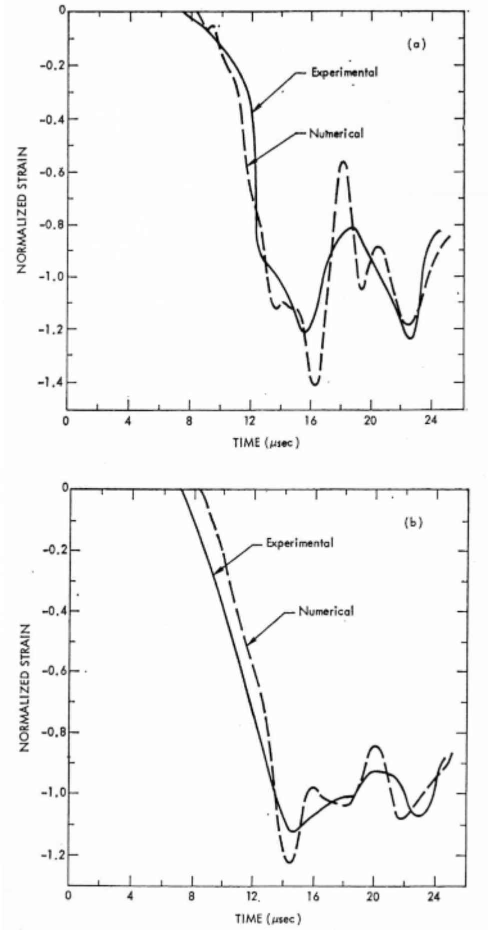


Figure 7. Experimental and numerical results for center (a) and surface (b) strain at a distance of 3 diameters from impact (from Habberstad et al., 1972).

Experimental observations of impacting rods by Bell (1973, p. 351) led the author to explicitly support the validity of DSVP (though no particular reference was given). Bell preserved the velocity of colliding rods with different distributions of the contact area while keeping the total area constant. The experiments revealed that the spatial distribution of the transient load has little effect on the surface strain of the rod at distances larger than half the diameter of the examined rod, a distance much smaller than suggested previously: " impacts of small hollow cylinders of the same area as the solid rod demonstrated that beyond the first half-diameter the experimental results were insensitive to the major changes in the spatial distribution of loading at the impact face" (p. 351).

Though not aiming directly to investigate DSVP, two additional experimental studies using photoelasticity provide further evidence for non sensitivity of the far field to details of end excitations. Miles (1976) examined the effect of surface roughtness on the uniformity of the wave generated upon impact (Fig. 8). Here the impact velocity remains identical when a different profile of the excitation is induced by irregularities at the contact surface.

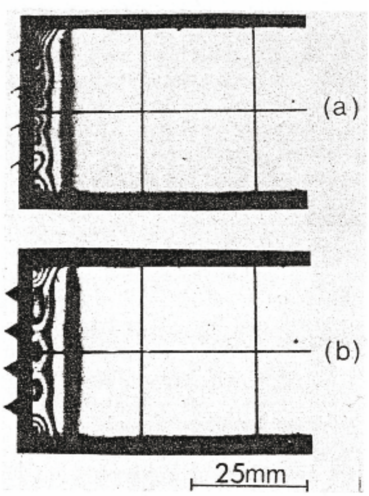


Figure 8. A fringe pattern in a plate impacted by two different materials with different surface irregularities (from Miles, 1976).

A recent study by Kawata et al. (2007) investigated dynamic stress field in a strip, generated by impact on one end, using photoelastic high speed photography. The fringe patterns obtained (Fig. 9) are identical to those generated by static loading, as shown by Frocht (1948) (Fig. 1). Both studies make it clear that the distance at which the non uniformity is preserved is nearly the same as in the static situations.
Following a gap of nearly three decades, experimental research on DSVP has been revived in recent years. Reflection of transient disturbance at a built-in end of a beam, generated by a transversal excitation at the free end, was investigated by Karp et al. (2008). The variation of end conditions was

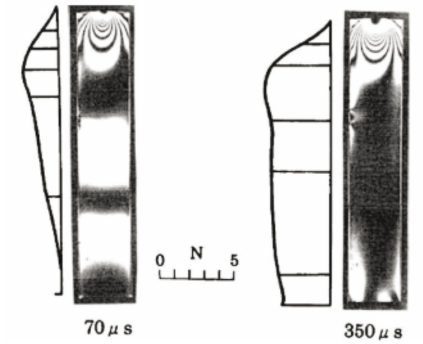


Figure 9. A fringe pattern in a plate impacted at the center of the upper end of a strip at two instants (from Kawata et al., 2007).

achieved by altering the tightness of screws used to fix the built-in end. The measurement of the surface axial strain suggested that extremely small variation in screws tightness can be detected by strain gauges located in the near field (Fig. 10), but not by strain gauges located in the far field. The extent of the near field is estimated to be approximately one width of the beam.

Symmetric excitation of a bar by striker impact in split Hopkinson bar system (SHPB) was studied by Karp et al. (2009) in the spirit of Bell's (1973) comment. The variation of end excitation was realized with various shapes of the contact surface of the striker. Experimental results, limited to surface strain measurements, were accompanied by numerical simulation confirming similarity in behavior between the core and the surface of the bar.

A similar, not yet published, study was undertaken at Nanyang Technological University, Singapore, with SHPB having much larger diameter rods, enabling direct detection of end vibration (Ma et al.). Four different strikers, having the same contact area with different shapes, used in the experiment are shown in Fig. 11.

The typical axial surface strain at a distance of x/D = 0.5 from the impacted end is displayed in Fig. 12. The difference in amplitude for each striker is evident. That difference becomes negligible beyond the distance x/D = 1 (not shown here). An interesting observation are the small oscillations, notable only at that particular distance, after the main signal has died out. That phenomenon is more pronounced for certain strikers and is associated with end vibration consisting of evanescent waves (e.g.,

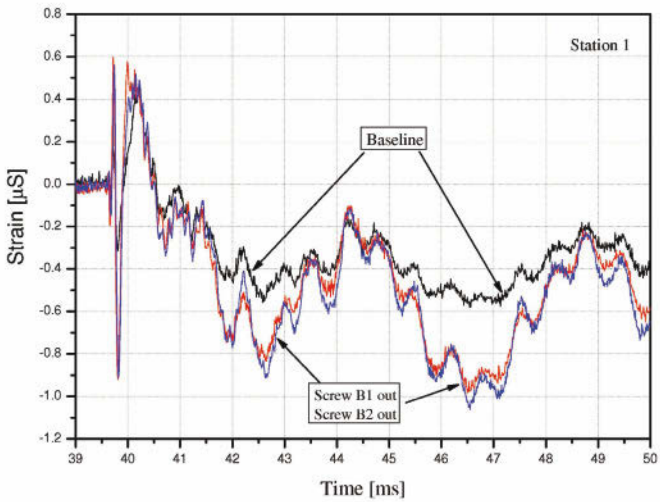


Figure 10. Surface strain recording within the near-field (station 1) for three beam fixation conditions excited by transversal excitation at the free end of a cantilever beam. Baseline is the recording of excitation when all screws are tight in. (from Karp et al., 2008).

Ratassepp et al., 2008).

In these experimental studies on SHPB system the same conclusion was reached, namely that the response of the bar is not sensitive to the form of the excitation beyond 1.5 diameters of the rod (Fig. 13).

The body of experimental research on DSVP can be summarized by the observation that most of experimental investigations aim at understanding the sensitivity of response of waveguides to spatial distribution of the load, or to the type of boundary conditions (either mixed or pure traction). The requirement of self-equilibrium of imposed excitation was not invoked, even in the few studies explicitly addressing the existence of DSVP. The induced excitations are of impact type (with Zemanek (1972) as exception). The results confirm that dynamic response of a beam or a rod excited at its end is not sensitive to the exact stress distribution of the excitation far enough from the excited end. Almost all experimentally different studies suggest that the extent of the non-uniformity of the cross-sectional properties penetrates into the bar less than 2 to 4 bar diameters (or plate thickness). Studies arguing for a larger distance do not report any contradicting results on small distance, but rather refer to an upper limit due to particular experimental limitations. Yet, the necessary conditions for equivalence be-

Figure 11. Surface strain recording (in Volts) within the near field for four strikers with identical contact area and different form (Fig. 11). Small oscillations are notable for certain strikers (from Ma et al.)

tween excitations were not identified (nor questioned). Standing out are Flynn and Frocht (1961), Bell (1973) and Miles (1976) who refer explicitly to the velocity of the impacting rod as a parameter to be kept constant for comparison of effects far from the loaded end. Nevertheless, understanding causes for far field insensitivity remains a major issue for future study. The next Chapter will provide some insight into this aspect of DSVP.

4 Unconstrained Waveguides

Waveguides with free lateral surfaces can be viewed as a dynamic analogue of beam-like and plate-like structures, for which the static version of SVP is most frequently applied. In that sense, the present chapter provides the complementary part of the previous chapter, where studies related or dedicated to the existence and validity of DSVP in such waveguides are reviewed.

The stress free condition is defined by vanishing of traction vector on lateral surfaces

$$\mathbf{t} = \sigma \cdot \mathbf{n} = \mathbf{0} \tag{4.0.1}$$

where $\mathbf{t}$ is the traction vector, σ - the stress tensor, and $\mathbf{n}$ denotes the outward unit normal to the surface.

The common features of wave propagation in cylindrical and strip waveguides (e.g., Miklowitz, 1978, p. 222) enable joint treatment of both geometries. Similarity in behavior of cylindrical and rectangular cross-sections of waveguides is also noted (e.g., Hertelendy, 1968). On these ground, no

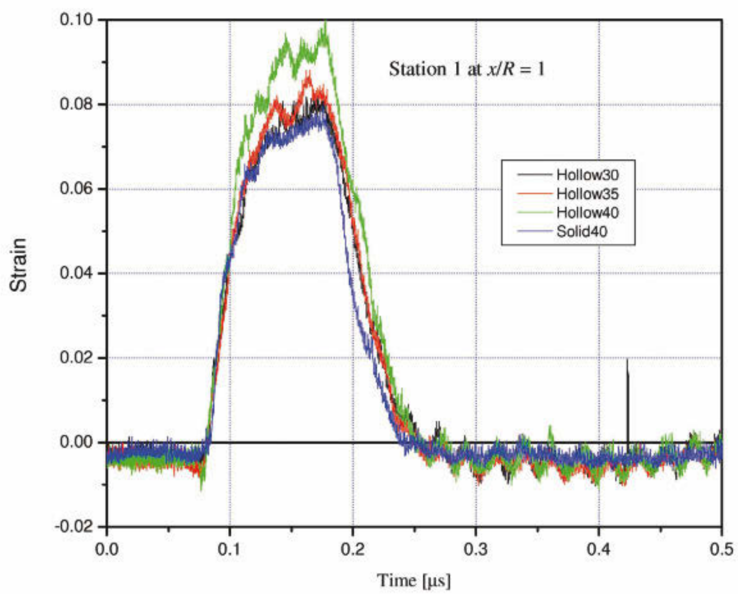


Figure 12. Surface strain recording (in Volts) within the near field for four strikers with identical contact area and different form (Fig. 11). Small oscillations are notable for certain strikers (from Ma et al.)

distinction between waveguides with different cross-sections will be made in the following review.

DeVault and Curtis (1962) examined the relevance of asymptotic solutions obtained for mixed type end data for prediction of actual results in experiments performed with pure end conditions (e.g., Miklowitz and Nisewanger, 1957). An example of the mixed data used therein for asymptotic evaluation is shown in Fig. 14. They reported (pp. 431-432): " all the main features of the observed pulse were correctly predicted despite the difference between the experimental and the assumed end conditions. if there is a real difference between predictions and experiment, it is at least small. This statement refers, of course, to a particular type of load and only to the behavior either at distances greater then a few diameters from the end of the bar or at a considerable time after the pressure is applied".

McCoy (1964) solves analytically the problem of a semi-infinite elastic rod subjected to a shear stress, with arbitrary radial variation, applied to an otherwise free end. The solution is obtained by series expansion which consists of propagating and evanescent waves. That analysis leaded to the conclusion that (p. 463): "This fact allows an evaluation of Saint Venant's

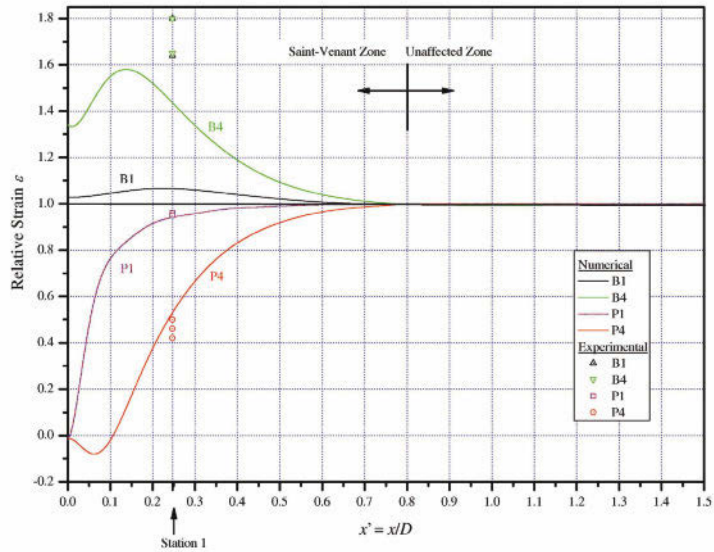


Figure 13. Surface strain recording (normalized by far field strain) in experiment and finite element calculation of a rod, versus distance from the excited end, with different strikers (P1 and P4 pin type, B1 and B4 bore type) (from Karp et al., 2009).

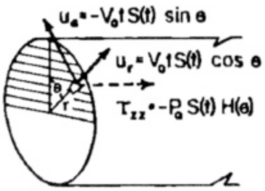


Figure 14. Mixed boundary conditions used by DeVault and Curtis (1962).

principle as applied to dynamic problems. The portion of the energy in a signal that excites a frequency above the cutoff frequency will propagate into the rod, whereas the energy which excites frequencies below the cutoff frequency will set up a vibration confined to the end of the rod. The lower the frequency the more closely confined to the end is the vibration".

Novozhilov and Slepian (1965) were apparently the first after Boley to dedicate a paper for examination of DSVP. Their interest in DSVP was motivated by practical aspects of use of Timoshenko's flexural beam equations with non-ideal end data. For that purpose they studied decay of end effects generated in a beam by dynamic (time varying) self-equilibrated load. It was shown that a steady state (harmonic) load generates a non-zero in-flow of energy associated with propagating waves. Therefore, no dynamical counterpart of the static SVP appears to exist in dynamic steady state fields. For transients, on the other hand, by comparing self-equilibrated and non-self-equilibrated step loads it was confirmed that the static version of SVP is applicable even to rapidly changing transients due to localization of the stress near the wave front. Consequently, a restricted interpretation (Novozhilov and Slepian, 1965, p. 313) of the principle is suggested: "The Saint-Venant principle is applicable for the study of transient process in beam dynamics since deformation corresponding to suddenly applied self-equilibrating load localize themselves in the neighborhood of the wave fronts and in the neighborhood of the cross section over which the load is applied. This assertion does not extend to self-equilibrating disturbances with the continuous in-flow of energy into the beam (for example, to periodic disturbances)".

Torvik (1967) used a variational method to find the actual amplitude of reflecting modes, from a stress-free end of a plate, generated by a single-mode single-frequency incoming wave. Investigation of both propagating and evanescent waves was concluded with an interpretation of DSVP (Torvik, 1967, p. 352) stating that "Below the frequency where more than one propagating mode is possible, an extension of St. Venant's principle is possible but extremely restrictive. The energy put into the system will have to be carried away by the first mode; therefore any two dynamic loadings (at a given frequency) will give rise to the same amplitude in the first (and only) propagating mode if they do work on the same displacements of the first propagating mode at the same rate, even if the stress distribution of the loaded region differs.". The author further suggests estimating the distance beyond which such loads are equivalent by considering the decay distance of the first evanescent mode. Diligent et. al. (2003) realized experimentally the configuration calculated by Torvik. They measured directly the excitation of evanescent waves generated at a free end of a plate upon reflection of the first symmetrical mode. Research conclusion was that evanescent modes can be neglected beyond distance of five times the plate's width.

Jones and Norwood (1967) compared the stress at the wave-front generated by an end excitation of a cylindrical bar under two loadings of mixed type; step pressure with zero radial displacement and step velocity with zero shear. By using asymptotic solutions of the exact elasticity equations (valid beyond 20 diameters from the end) they found, for equivalent pressure and velocity applied at the end, that the stress distribution at distances greater than 20 times the diameter is the same despite the different end data. This result was regarded as confirming the validity of DSVP in such problems (p. 723): "Because the lateral end conditions are markedly different, this constitutes at least an upper bound on a dynamic Saint-Venant's principle for these problems in the range of low frequencies". The equivalence of the two types of excitation was judged by the equality $P_0 = EV_0/C_0$ where P_0 is the amplitude of the step pressure, V_0 is the amplitude of the step velocity, E and C_0 are Young modulus and velocity of longitudinal waves in a bar, respectively.

Bertholf (1967) used numerical integration to evaluate the near field in a bar subjected to steady state uniform displacement with no shear at its end. The solution of Pochhammer-Chree frequency equation was considered as valid only at remote distance from the ends and was used to confirm (p. 734) correctness and accuracy of the numerical solution: "The results of applying a plane, harmonic displacement to the end of a semi-infinite bar compare satisfactory with the Pochhammer-Chree solution at points not near either the end or the wave front". The estimate for the distance at which a reasonable agreement (p. 728) is obtained was 4 bar diameters: " the Pochhammer-Chree solution is correct for distances of more than 4 dia from the end of a semi-infinite bar".

Kennedy and Jones (1969) investigated the effect of spatial distribution of a suddenly applied load on the far field in a circular bar. While the study is in the spirit of the original Saint-Venant's formulation, connection to DSVP was not discussed. The resultant of the applied loads remained constant while their distribution varied, implying self-equilibrium in a static sense obtained from the difference between these loads. The imposed excitations are given by

$$\sigma_x(0,r,t) = -P(r)H(t) \qquad u_r(0,r,t) = 0 \quad at \quad x = 0 \tag{4.0.2}$$

with

$$P(r) = P_0(p+1)\left[1 - \left(\frac{r}{a}\right)^2\right]^p \qquad p \geq 0 \tag{4.0.3}$$

where P_0 is kept constant, parameter p describes the degree of the non-uniformity of the load, a is the radius of the bar and x is the axial coordinate. For $p = 0$ the load is uniformly distributed. Numerical integration results were compared to asymptotic solutions at the far field to gain confidence in the numerical prediction for the near field. The numerical evaluation was limited to distances of 5, 10, and 20 diameters from the end. In view of the findings summarized in chapter 3, the smallest distance of 5D is beyond the extent of near field. Hence, the article can be viewed as an investigation of the effect of the profile of the excitation on the far field. It has been shown that the double transform solution for the first mode, valid in the far field, is independent of p. The conclusion is that the excitation profile has no effect on the average stress in cross sections at distances beyond 5D. The cross section peak stress is insensitive to excitation profile only beyond 20D.

Karal and Alterman (1971) examined the extent of the strain difference in the far field of a bar due to application of pure and mixed shocks at the end. They concluded that already beyond 2D from the excited end it is immaterial whether the data is pure or mixed (p. 10): "the output response for distances equal or greater than two diameters from the end of the rod exhibits the same general features whether the boundary conditions are pure or mixed".

Two numerical and experimental studies by Zemanek (1971, 1972) are instrumental in providing a possible interpretation of DSVP as related to evanescent waves. In the first study a clear distinction is made between near and far fields, along with explanation of the nature of the near field. In the second paper, results of an experimental investigation of end effects are modeled by wave reflection from a free end. A clear insight into the correlation between dynamic end effects, evanescent waves, and complex wave numbers is suggested.

Yeung Wey Kong et al. (1974) solved numerically the exact elasticity equations for impact of a rod as part of a study on the effect of mismatch of contact area between specimen and bars in a split Hopkinson bar system. Their concern was the validity of interpretation of experimental results based on one dimensional theory. Four ratios between the diameter of the contact area r0 (representing the specimen) and the bar diameter D were examined ($r_0/D = 0.18$, 0.36, 0.72, 1.0). The strains at the center and surface of the bar were extracted for stations located at $4D$ and $8D$. Comparison of the calculated surface strain at $8D$ for the four specimens discloses considerable difference, both in first order and second order response. It is not stated explicitly how the imposed boundary conditions were adjusted for the four specimens, whether it is the normal stress or the total force that was preserved for all four simulations.

Grandin and Little (1974) adopted the mathematical interpretation (p. 145) according to which the principle does not exist if a self-equilibrated oscillating load produces non-decaying waves: "The definition of what is meant by a Saint-Venant boundary region might lead to different interpretations as to whether a dynamic Saint-Venant principle exists or not. The approach taken here is to apply an edge stress distribution with null integrated force at any instant of time and determine if non-decaying waves are produced. This would indicate that beyond a certain distance the wave fronts are independent of edge stress distribution". That interpretation led the authors to conclude that DSVP does not exist in steady state problems, as already suggested by Boley (1955, 1960a) and by Novozhilov and Slepian (1965). The authors conclude (p. 146) with: "Examination of the results at lower frequencies indicated that the stress magnitude of the non-decaying modes was greatly reduced and the results tend toward the static solution. It must, however, be noted that the frequencies must be very low before the non-decaying mode may be neglected contributing to the argument against the existence of a dynamic Saint-Venant region".

Binkowski (1975) examined the dynamic response of a circular waveguide subjected to three different end excitations, two of which were self-equilibrated and harmonic in time. Comparison of the stress field of the propagating waves generated by these two self-equilibrated loads revealed a "radically" different response. Based on that finding the author concludes (p. 60) that "A dynamic Saint-Venant region does not exist for a solid circular semi-infinite cylinder". It should be noted that the comparison is made at a frequency above the first cut-off, where two propagating modes are available.

Apparently the first formulation of Saint-Venant type energy inequality for dynamic response of a cylinder with free lateral surfaces was given by Ignaczak (1974). For the proof of spatial decay of the total cross-section energy two assumptions (p. 313) were employed: "we assume that B is a semi-infinite nonhomogeneous and isotropic elastic cylinder loaded smoothly on the end face of the cylinder, and that the stress field is to vanish in a fast way at infinity". However, in view of the results obtained by Boley (1955, 1960a) and by Novozhilov and Slepian (1965) it is not clear how the second assumption can be fulfilled for a general response of a waveguide with free surfaces.

Sinclair and Miklowitz (1975) considered a plate in plane strain with free faces excited by suddenly applied normal symmetric loads at the end. They used double transform for long time solution of two different loads, uniform and concentrated, on the centerline. It was estimated that the actual response of a plate to any other form of excitation will lie within the limits of the solution to these two forms. The authors report that if the total force is identical, the far field is practically identical as well.

Orazov (1983) investigated the validity of DSVP in an elastic semi-infinite waveguide with free or clamped surfaces subjected to displacement or stress on the near end and zero displacement at the remote end (at infinity). Under some restricting conditions on the end excitation, the author proved complete decay of the response at some distance from the end, even for a waveguide with free surfaces, thus suggesting a proof for DSVP in a particular case. It is noted that in general, such conditions give rise also to non-evanescent waves radiating energy to infinity, resembling Sommerfeld's condition.

Kim and Steele (1989) demonstrated the advantage of stiffness matrix method over collocation method for estimation of end effects associated with time-harmonic excitation of a bar. Various forms of excitations were solved to illustrate the method, yet no explicit evaluation of the extent of the end effects is given.

Gomilko et al. (1995) compared the amplitude of waves induced by a self-equilibrated excitation to those induced by non-self-equilibrated excitations. The ratio obtained turned out to be extremely small for excitations with a low frequency, confirming a version of DSVP. Gomilko et al. (1995, p. 1153) formulated DSVP as follows: "When a self-balanced system of forces acts on the end of semi-infinite strip, stresses arise as a result of this system only near the end. At a significant distance from the point of application of the forces the effect of such a load is practically zero".

Karunasena et al. (1995) have presented an explicit verification of the amplitude and depth of penetration of evanescent waves induced by reflection of the first propagating mode at a fixed edge. The authors showed that the evanescent waves generated upon reflection from a fixed end of a composite plate are negligible at distances larger than twice the plate's width. This analysis and its results are similar to those suggested by Torvik (1967), though no connection to DSVP was noted.

Cherukuri and Shawki (1996) confirm, by using finite difference solutions, the results obtained by Fox and Curtis (1958), according to which beyond two diameters off the impacted end the type of BC (either mixed of pure) has no effect. The same conclusion, based on energy partition among propagating modes, was derived by Karp (2008) with the aid of bi-orthogonality relations for an elastic strip.

Chirita and Quintanilla (1996b) treated both transient and steady state excitations using energy inequalities. To establish decay for a transient load it was assumed that the excitations are self-equilibrated at any instant (this work appears to be among the few studies using differential inequalities where that condition is required) and allowed for lateral surfaces to remain free of traction. Under these conditions it has been proved that the cross-section energy, within the domain of influence, decays linearly with distance.

Similar energy inequalities were established by Iesan and Scalia (1997) for microstretch material, and by Borrelli and Patria (2000) for a piezoelectric beam with clamped or free surfaces. Borrelli and Patria remark (p. 74) that: "the decay result concerning the energy does not require assumptions on the boundary data on the base". Knops and Payne (2005) derived equivalent inequalities for a nonhomogeneous, anisotropic material with constrained excitation at the base.

Karp and Durban (1997) enhanced several existing approaches, mainly similar to Torvik (1967), following Karp (1996). The central idea was to abandon the quasi-static notion of the "self-equilibrated load", and to replace it with a "system of equivalent loads". That idea is further developed in Karp (2009), where, in particular, it was shown that DSVP formulation based on "dynamic equivalents" coincides with the "static equivalents" in the limit of zero frequency. Moreover, it was demonstrated that the extent of non-uniformity associated with the end effect is the same for both static and dynamic situations, as demonstrated by the photoelastic fringes shown here in Figs. 1 and 10.

Tyas and Watson (2000) examined numerically the transient response of a bar to concentrated and arbitrary distributed loads in the context of reconstruction of the applied load out of measured strain history. Employing a finite element code they showed that for low frequency load, its magnitude can be deduced from measurements taken far enough from the edge (five times the radius). It is stated (p. 1549) that the study is not intended to postulate a dynamic version of SVP: "Unlike previous work of this type, these findings have not been used to postulate a dynamic Saint-Venant's principle for the pressure bar".

Meng and Li (2001) suggested improvements of the interpretation of data from split Hopkinson pressure bar tests by invoking DSVP. Their view of DSVP resembles a direct extension of static SVP stated as insensitivity to the spatial distribution of the applied surface load. Using finite element code the authors found that the surface response of the output bar (the second bar in SHPB system) beyond 1.5 rod diameters is insensitive to spatial distribution of the end load. For the sake of comparison, the average pressure was held constant. Application of that conclusion to the improvement of the split Hopkinson bar system is detailed in a subsequent paper (Meng and Li, 2003).

Berdichevsky and Foster (2003) considered (p. 3293) the lack of orthogonality of the eigenfunctions as a major reason for difficulties in establishing DSVP: "In dynamics, Saint-Venant's principle of exponential decay of stress resulting from a self-equilibrating load is not valid. It is not clear how to formulate the conditions that eliminate the penetrating modes". Such conditions have been derived later by Babenkova and Kaplunov (2005) and by Karp (2009). The conclusion (p. 3296) is that "An unpleasant consequence is that, in general, one cannot trust the predictions of dynamical one-dimension beam theory that takes into account only the total force and moment at the beam end".

In order to bypass that difficulty Foster and Berdichevsky (2000) and Berdichevsky and Foster (2003) suggested a novel approach to measure quantitatively, by statistical average, the degree of violation of SVP in structural dynamics. Using statistical distribution of a self-equilibrated load, they evaluated quantitatively the frequency range of a harmonic, self-equilibrated load, for which the error involved in assuming SVP is acceptable. It was shown with high probability that for a frequency region below some value, the error of classical theory is very low. In a subsequent study Foster and Berdichevsky (2004) enhanced that work to estimate (p. 2551) the effect of end effects in end vibration of a semi-infinite beam: "In the case of a dynamic load, Lamb (1916) showed that a traveling wave is also excited, so that a self-equilibrated end load will cause some level of stress to penetrate into the beam: Saint-Venant's principle is violated". Furthermore (p. 2552), "Our major conclusion is that over a wide range of frequencies, the maximum propagating stress is small compared with the maximum applied stress. Saint-Venant's principle may be said to apply in this problem, until the frequency approaches a critical high level. Below this frequency of vibration, the error involved is considerably smaller for flexural vibrations than it is for longitudinal vibrations".

This topic was further investigated by Babenkova and Kaplunov (2004) who examine conditions on a low frequency excitation for not generating propagating waves. The condition that the symmetric non-self-equilibrated excitation $\sigma_0(y)$, with y as a normalized coordinate in the transverse direction, will not induce propagating waves is given by

$$\int_0^1 \left(1 - \frac{1}{4}\nu\lambda^2 y^2\right) \sigma_0(y) dy = 0 \tag{4.0.4}$$

where ν is Poisson's ratio and λ the non-dimensional frequency. This formula involves a low-frequency corrector to the applied self-equilibrated load required to ensure validity of Saint-Venant principle and can be considered as a deviation from the self-equilibrium conditions for static decay, as the authors write (p. 2168): "The derived low-frequency decay conditions represent a starting point for an asymptotic refinement of 2D boundary conditions in dynamics of thin plates and shells. It is important that these conditions allow us to take into account deviations from the classical formulation of the Saint-Venant principle". An earlier work by these authors (Babenkova and Kaplunov, 2003) examines the application of DSVP involving similar correction to the quasi-static self equilibrated load with low frequency while applied to a finite strip. Extension for high frequency oscillating load was formulated by Babenkova and Kaplunov (2005).

A follow up study by Babenkova et al. (2005) evaluates the ratio of the power generated by self-equilibrated loads to the power generated by non-self-equilibrated loads for low frequencies; the interpretation given to DSVP resembles that of Gomilko et al. (1995). An interesting (p. 405) analogy is proposed: "In the problem of the propagation of harmonic waves in a half-strip, homogeneous (non-decaying) modes, which are determined by the real roots of the well-known dispersion equations , can serve as an analogue of the Saint-Venant solution". With that analogy the authors extend the principle (p. 1165) for high frequency excitations: "However, in contrast to statics, high-frequency behaviour is often characterized by short-wave propagating sinusoidal modes that do not decay along with polynomial terms. These propagating modes have to satisfy the Sommerfeld condition at infinity. Thus, we do not require a total decay. We require only the absence of polynomial modes that do not satisfy the radiation condition at infinity".

It is probably a telling sign, indicating that research on DSVP is still in the formative period, that none of the studies discussed in Chapter 3 were referenced in papers reviewed up to this point in Chapter 4. In particular, there is no reference to the investigations by Davies (1956), Baker and Dove (1962), and Cunningham and Goldsmith (1959), which bear direct implication to DSVP. This comment applies also to review articles by Horgan and Knowles (1983), Horgan (1989, 1996), Field et al. (2001), Karp (2005). Understanding of DSVP has evolved along more than one avenue, not in a linear pattern, with several diversities over the time line.

Karp (2008) investigated the sensitivity of far field response to the form of end excitation of an elastic, semi-infinite strip. Since, as stated by Torvik (1967), below the first cut-off frequency only one mode can be generated regardless of the form of the excitation, the study examined higher frequencies. It was found that moderately non-uniform excitations exhibit similar energy partition among the propagating modes, suggesting a degree of insensitivity to form even at high frequencies. This can explain similar far-field response detected in experiments by Barton and Voltera (1959) with rounded head strikers, albeit the high frequency spectrum of the excitation.

Adherence to the equivalence of loads required by classical SVP led Karp (2009) to formulate a DSVP based on dynamic equivalence of loads. Dynamically equivalent loads are defined as loads generating identical far-field response within the waveguide. Such formulation of DSVP, based on dynamically equivalent loads, is consistent with experimental results on insensitivity of the far field to details of end excitation and can be related directly to evanescent waves. It was also shown that the requirement on excitation for no-radiation (no far field response) is mathematically reduced to the static requirement of self-equilibrium of load, as necessary for decay in the static case. The suggested (p. 3072) formulation of the principle reads: "If a certain set of external excitations acting on a certain part of a surface of a body is replaced by another system of external excitations dynamically equivalent to the preceeding system and distributed over the same sector, the stresses corresponding to these two excitations will be practically identical at a sufficient distance from the point of application of the excitations". Further unification of classical SVP and DSVP is proposed (p. 3075): " unification of static and dynamic formulations of SVP can be achieved by noting that in both cases the validity of the principle stems from far-field response being not sensitive to the form of the excitation".

That formulation of DSVP enables one to mitigate the objection, raised by Borg, to counter-example the validity of DSVP. Borg compared two anti-symmetric responses of a Timoshenko beam model. His argument is based on the observation that a pure bending disturbance of a beam propagates with higher velocity than a shear disturbance, and therefore two equal-magnitude moments will generate different far-field response. While this observation is correct it does not contradict the DSVP formulation suggested since the shear mode is a second anti-symmetric mode, available only above the first anti-symmetric cut-off frequency (e.g., Abramson et. al., 1958, p. 157), whereas in Karp (2009) it is suggested to restrict the validity of DSVP to frequencies below the first cut-off. In other words, these two equal-magnitude moments do not comply with the requirements for equivalent excitations (as defined in Karp, 2009).

The studies reviewed in this chapter cover analytical research of solutions relevant to experimental findings reviewed in Chapter 3. The papers reviewed in the following sections are of less direct connection to experiments, though various versions of DSVP are discussed therein.

5 Constrained Waveguides

The search for DSVP in constrained waveguides has concentrated on two types of constrains: clamped faces and energy leaking surfaces. Though the dynamic response of waveguides with constrained surfaces has been treated by several methods, those dedicated to DSVP are limited to use of energy inequality methods (with Orazov, 1983; Karp and Durban, 2005; and Karp, 2011 as exceptions). It is worth noting that elastodynamic solutions for a strip with clamped surfaces (e.g., Mindlin, 1960, Karp and Durban, 2005) suggest that any end disturbance, with frequency below the first cut-off frequency, will generate response decaying in the axial direction leaving no response at all far from the loaded end.

5.1 Clamped Lateral Surfaces

Clamped surfaces are defined by zero displacement $\mathbf{u}$ over the generators of the waveguide

$$\mathbf{u} = \mathbf{0}. \tag{5.1.1}$$

Orazov (1983) used the general formulation of elasticity equations along with eigenfunction analysis to derive decay estimates in a waveguide with general cross-section. The decay rate was associated with the wave number having the lowest imaginary part. As mentioned already, the author emphasizes that the same result is valid also for a waveguide with free lateral surfaces.

Knops (1989) determined spatial decay estimate of cross-sectional energy for a quasi-linear, semi-infinite cylinder, with anisotropy induced by finite prestress. In that study clamped faces were chosen for simplicity of the analysis (p. 193). The end excitation was taken as harmonic in time. As an intermediate result, facilitating an energy inequality, it was proven (corollary 3.1) that the cross-sectional work function Φ vanish far from the loaded end. Energy inequalities derived from that corollary lead to a somewhat complicated mathematical result, from which it follows that (p. 202): "We have demonstrated that within the disturbed region $0 < z < \beta t$ the energy is bounded above by the sum of a constant term and a term that decays exponentially with distance from the base of the beam".

A similar result has been derived by Flavin et al. (1990) for non-linear materials, by Borrelli and Patria (1995) for a mixture of two linear elastic solids, by Borrelli and Patria (1996) for a magnetoelastic cylinder, Chirita and Quintanilla (1996b) for elastic materials, Iesan and Scalia (1997) for microstretch elastic bodies and by Aron and Chirita (1997) for micropolar elastic cylinders.

Quintanilla (1999) established energy decay estimates for the spatial behavior in thermoelasticity without energy dissipation. The derivation of the inequality was made for clamped lateral surfaces with the concluding (p. 221) remark that: "The analysis presented in this section also works if we substitute the boundary conditions imposed at the beginning by $\mathbf{t} = \mathbf{0}$" (where $\mathbf{t}$ is the traction vector on the lateral surfaces). However, that statement has not been supported by other evidence and indeed contradicts the known phenomenon of propagation of non-attenuating waves under such conditions.

Borrelli and Patria (2000) derive a Saint-Venant type decay relation for piezoelectric material, similar to those obtained by Chirita and Quintanilla (1996b) for clamped surfaces excited by harmonic excitation and for a body with free surfaces excited by a transient force. Yilmaz (2007) derived similar energy decay estimates for a system of coupled parabolic-hyperbolic equations with clamed surfaces under non-linear conditions.

Chirita and Ciarleta (2008) gave spatial decay estimate for an anisotropic homogeneous and compressible cylinder. The lateral surface and far end of the cylinder are constrained by zero displacement condition. An exponential decay result for excitation frequency below some critical frequency has been derived. Algebraic decay has been obtained for frequencies above that critical frequency though it is not explained how such decay is possible at high frequencies without exclusion of propagating waves.

Tibullo and Vaccaro (2008) derive a theorem of influence domain and decay estimation for strongly elliptic, anisotropic materials. They conclude (p. 993) that: " inside of the influence domain, a spatial estimate of Saint-Venant type has been established, which describes the exponential decay of solutions with respect to the distance from the loaded end".

A study on evanescent waves characteristics in a strip with various boundary data on the faces is undertaken by Karp and Durban (2005) in the context of incremental finite elasticity. In particular, the authors point out that the response of a strip with clamped surfaces consist of decaying fields regardless of the self-equilibrium of the excitation, provided the frequency is below the first cut-off frequency. Frequency map (Fig. 15) shows that for the symmetric fields, the first non-dimensional frequency is $\Omega = 1$. That result is in agreement with previously reviewed results for decaying fields in waveguides with clamped surfaces below a specific frequency.

5.2 Energy Leaking Surfaces

The boundary conditions for waveguides with energy leaking surfaces are expressed by the inequality

$$\mathbf{t} \cdot \mathbf{u} \neq 0 \tag{5.2.1}$$

on lateral surfaces, where $\mathbf{t}$ is surface traction and $\mathbf{u}$ is the displacement vector. Nappa (1998) establish energy decay estimates for both bounded and unbounded bodies with boundary condition of this type. The suggested interpretation of the DSVP is again a combination of domain of influence theorem and spatial energy decay relation within that domain. Extension of this interpretation of the DSVP and energy estimates was made for various domains, among them by Chirita and Nappa (1999) for incremental response of a non-linear material, by Chirita and Ciarletta (1999) for thermodynamic processes, and by Gales (2002) for swelling porous elastic solids.

In several studies it was assumed that the displacement or the load at

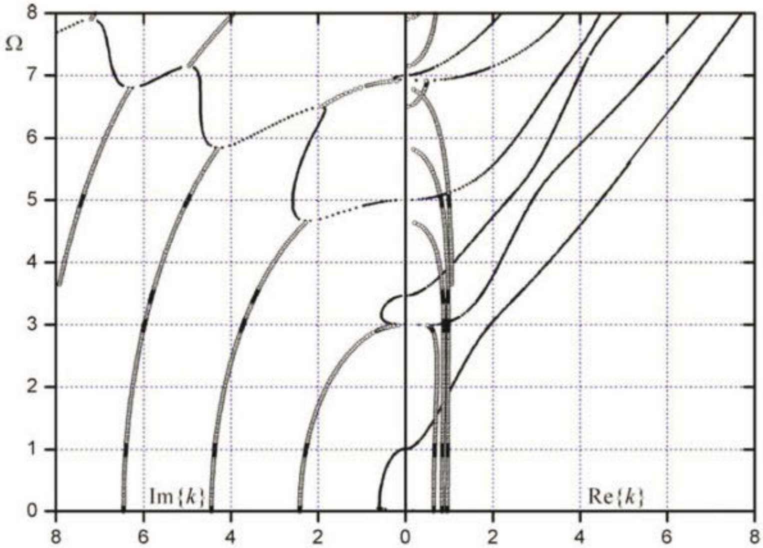


Figure 15. Frequency map (wave number k versus frequency Ω) for symmetric fields in a strip with clamped surfaces made of elastic material with Poisson's ratio $\nu = 0.25$ (Blatz-Ko material without prestretch). Thin lines (composed of black dots) indicate real and purely imaginary branches. Thick lines (composed of hollow circles) indicate complex branches (two curves for each eigenvalue). Purely real wave numbers are associated with propagating waves. (from Karp and Durban, 2005).

the surfaces are imposed, e.g.

$$\mathbf{u} = \widetilde{\mathbf{u}} \quad \text{or} \quad \mathbf{t} = \widetilde{\mathbf{t}}. \tag{5.2.2}$$

If this prescribed data is constant over time, an imposed displacement is actually sort of clamping while imposed traction represents leaking energy. If the data has time dependence, both stand for leaking energy. Such boundary condition was examined by Scalia (2001) to establish energy decay estimates for anisotropic, inhomogeneous linear material with voids. Ciarletta et al. (2002) extended that analysis for porous elastic mixtures and, in Ciarletta (2002), for a thin plate with transverse shear deformation in steady state excitation under clamped lateral conditions, and also for a transient excitation with a dictated displacement at the lateral surfaces. Additional spatial estimates in linear thermoelasto-dynamics for imposed displacement at the lateral surface were derived by Chirita and Ciarletta

(2003), and energy decay estimates for various boundary conditions are reported by Knops and Payne (2005).

Sliding and inextensional surfaces for a waveguide under plane strain (in x, y coordinates) conditions are defined by

$$\tau_{xy} = u_y = 0 \qquad \text{sliding lateral surface} \tag{5.2.3}$$

$$\sigma_y = u_x = 0 \qquad \text{inextensional lateral surface} \tag{5.2.4}$$

Dynamic response of such waveguides has a simple solution which appears in text books on elastic waves (e.g., Achenbach, 1973, Graff, 1975, Miklowitz, 1978).

The study of evanescent waves in such waveguides, within the context of incremental elasticity, is reported by Karp and Durban (2005). An interesting result which has emerged is that non-self-equilibrating excitations induce decaying fields within the strip, as has already been noted in the equivalent static case by Karp and Durban (2002). The decay rates are of the same order as for waveguides with free surfaces. Similar work has been done by Wijeyewickrema et al. (2008), with emphasis on propagating waves.

Recently, Karp (2011) combined the mathematical simplicity of waveguide analysis with sliding boundaries conditions analysis to demonstrate validity and practical aspects of DSVP. In that study the sensitivity of surface strain within the near field (Saint Venant region) to the fine details of end excitation has been confirmed. Results were interpreted with a new measure, the Saint-Venant's ratio (SVR), defined as the ratio of surface axial strain to strain amplitude in the far-field associated with propagating wave. That measure represents deviation of the near field from the far field. For example, SVR = 1 when there is no end effect at all (the profile of the excitation is identical to the profile of the propagating wave at a given frequency). The variation of SVR with non-dimensional distance x/h for various excitations is reproduced here in Fig. 16. The resemblance of these curves to those obtained for a rod subjected to a transient excitation in Fig. 13, is notable.

6 Special Geometries

Extension of SVP to half-space, wedge and a cone is not a straightforward task since these geometries lack any natural length scale, as opposed to a beam to which Saint Venant originally referred to (de Saint-Venant, 1886). Therefore, for formulation of SVP for such cases the length scale is taken from the load spatial parameters, as demonstrated in statics by von-Mises (1945) and Sternberg (1954) for a half space and by Horvay (1957),

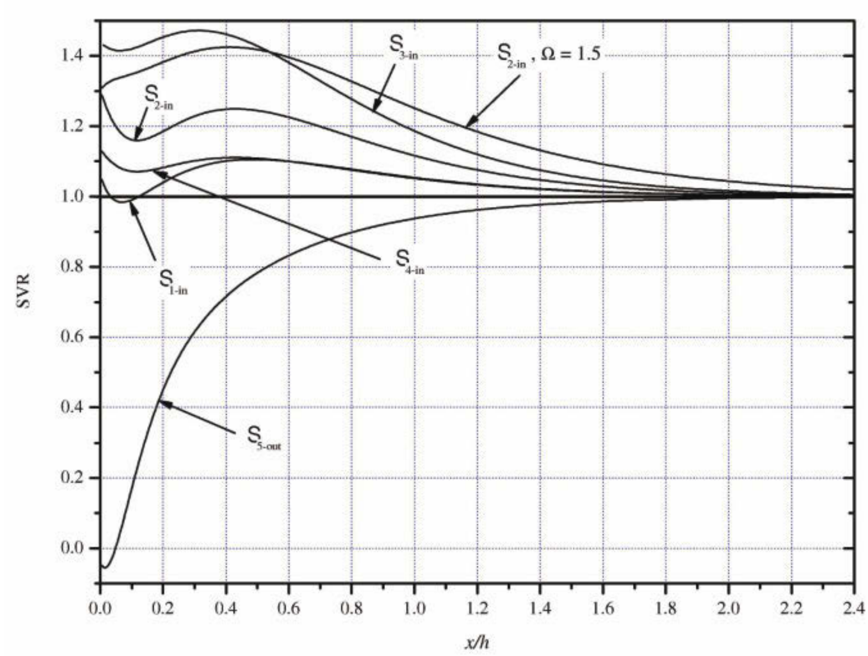


Figure 16. Variation of SVR for five excitation forms S (resembling damaged joint) with frequency $\Omega = 0.5$ (below first cut-off) and S_{2-in} excitation also with frequency $\Omega = 1.5$ (close to first cut-off). Four S_{i-in} excitations represent damaged joint at the center line of the strip while S_{5-out} excitation represents damage at the outer edges. (from Karp, 2011).

Markenscoff (1994) and Stephen (2008) for a wedge. Few studies examine the possibilities to extend findings in these geometries to the dynamic case.

6.1 Half-Space

Miyao et al. (1975) studied the application of SVP to dynamic response of a semi-infinite body subjected to an impulsive torque on the surface of a hemispherical pit. The temporal variation of the excitation applied to the body was a step load with several additional cases of gradually profiled loads. It was found that the stresses just behind the wave front are strongly influenced by the spatial distribution of shear forces on the pit. Stresses behind the wave-front, far enough from the pit, are not sensitive to such variations of load distribution. The distance at which that insensitivity manifests is smaller for excitations that gradually change with time. It is interesting to bring the note (p. 963) stating: "Compared with the results previously obtained for the rod, the difference among stress systems produced by several systems of applied load distributions is more remarkable for the semi-infinite body".

Kim and Soedel (1988) solved the problem of dynamic response of a half-space on which a step load is applied. Emphasis of the article is on a novel method enabling a simple solution for arbitrary spatial distribution of the step load. Upon preserving the equivalent static load, the result is that far enough from the loaded area, the strains behind the wave-front are insensitive to the spatial distribution of the load. The same conclusion was obtained later by Awrejcewicz and Pyryev (2003). No explicit reference to DSVP was made.

Wang and Kim (1997), on the other hand, analyzed the effect of an impulse load with varying contact area acting on a half-space (modeling impact against a stop). Comparison of the response included the full time history of the stress at a distant point while preserving the total load. Conclusion was that at distances greater than five times the diameter of the loaded area, the size of the loaded area has a fairly small effect on the stress generated. This conclusion is directly related to DSVP, and has been used in the study to confirm the method suggested for analysis of such problems.

Awrejcewicz and Pyryev (2003) compared the response of a half-space to a step load with different spatial distributions preserving the integral intensity of the load. They conclude that Saint-Venant's principle cannot be applied to the wave front, but rather to its trail (behind the wave front) - after the lapse of time ensuring the passage of a Rayleigh wave at a point of consideration.

A more definite conclusion regarding the non-validity of SVP to dynamic excitation of a half-space has been put forward by Ziv (2002, 2003): " half-space response is hypersensitive to the type of loading, to the way it is distributed on the source rim, and to the geometry of the source rim under the load" (Ziv, 2002, p. 402). Therefore (Ziv, 2003, p. 254-255): "Saint-Venant's principle for wave propagation problems cannot be formulated. The source geometry and its load must be tackled directly as they are prescribed; i.e. two different configurations sharing the same resultant are not interchangeable".

Ignaczak (2002) considered the issue of SVP in microperiodic, layered, thermoelastic semi-space, thus formulating a time dependent energy decay estimate.

6.2 Wedge and Cone

Budaev et al. (1996) summarize two previously published studies, Budayev et al. (1994) and Morozov and Narbut (1995) on Saint-Venant's principle in a wedge and cone, both for static and dynamic excitations. Three types of end excitations, applied on the surface generators of the wedge or cone, were considered: torsion, anti-plane, and normal traction. The criterion for validity of both SVP and DSVP is whether differently distributed loads with identical moment generate the same asymptotic result far from the wedge apex. For the static normal loads they find (p. 32) that SVP is not valid since: "... it is possible to find forces $f_1(r)$ and $f_2(r)$, having the same couple M_1, for which the solutions will be quite different". For anti-plane shear excitation it is demonstrated (p. 33) that SVP does not hold, neither for the static case, nor for the dynamic case: "... so the error of substitution of one system by another is not small...". For torsion, both static and dynamic, the principle does hold (p. 36): "...then the validity of the Saint-Venant principle is deduced from previous analysis. In fact, the stresses in the cone under torsion at some large distance from the apex are mainly characterized by the moment of boundary forces".

7 Composites and Laminates

From studies on the validity of the classical SVP it is known that the decay rate in laminates and in composite materials can be much lower than in isotropic elastic materials (e.g., Horgan and Simmonds, 1994). This makes the depth of penetration of end effects to be significantly larger as is evident from studies on dynamic response of composite waveguides available in the literature. Apart from that extended depth of penetration of end effects, there is no substantial difference in analysis between elastic composite and isotropic homogeneous waveguides. The papers cited below were chosen due to specific comments related to DSVP, and for explicit association made between evanescent waves, edge vibrations and DSVP.

End effects in anisotropic cylindrical shells were discussed by Bhattacharyya and Vendhan (1991). Detailed mathematical and physical interpretation of evanescent waves was given, followed by the observation of low spatial attenuation at frequencies near cut-off frequencies. The evanescent waves (p. 71) were associated with Saint-Venant's zone: "...The effect of the attenuating modes on the dynamic stress field is localized near the end zone of the shell, the extent of the zone being dependent on the roots for a specific "end" problem. This is simply the Saint-Venant zone, so well known in static problems, and hence the end effect is essentially a dynamic Saint-Venant effect". The term "roots" refers here to wave numbers which are the roots of the Pochhammer-Chree equation.

Dong and Huang (1985) applied the finite element method to investigate edge vibrations in laminated composite plates by considering explicitly the evanescent waves. They regard that treatment as related (p. 437) to DSVP: "The analysis procedure may be considered as the dynamic counterpart of the quantitative analysis of Saint-Venant's principle". That view is an extension of a similar static analysis made earlier by Dong and Goetschel (1982). The outcome of this work is that dynamic edge vibrations are analogous to static end effects regardless of self-equilibrium of the excitation, as suggested already by Torvik (1967). The authors derive a characteristic equation for use in finite element analysis to find (p. 435) that: "... equation (16) represents the dynamic counterpart of Saint-Venant's principle for the determination of the decay rates into the plate's interior of self-equilibrated edge vibrations". A similar study and connection with DSVP for an anisotropic composite cylinder can be found in the paper by Huang and Dong (1984). It is not clarified in what sense the "edge vibrations" are "self-equilibrated" except for these evanescent waves being a natural extension of the static eigenfunctions which are indeed self-equilibrated in a static problem.

The papers by Scalia (2001) and Chirita and Ciarleta (2008) mentioned above, in a different context, analyzed anisotropic materials and therefore belong as well to this group of studies.

8 Related Studies

In this chapter two neighboring areas in which DSVP is questioned or invoked are reviewed. Neither is a natural part of the categories detailed here, yet both could be integrated in further studies on DSVP.

8.1 Structural Vibrations

Vibration, by its nature, is not associated with spatial propagation of energy. Two distinct phenomena of structural vibration can be identified. One is the global vibration of beam, plate or shell-like structures, typical to a finite structure. The second is edge vibration associated with evanescent waves and can exist also in a semi-infinite structure (e.g., Kaul and McCoy, 1964). Since end vibration consists of evanescent waves, references to that phenomenon have been reviewed in chapter 4, including for example the work of Foster and Berdichevski (2004), albeit the vibration oriented title. Yet, to make a clear distinction between studies of finite and semi-infinite structures, this chapter reviews both types of vibration: structure vibration and end vibration, if treated in the context of a finite structure (e.g., Gales, 2003).

Duva and Simmonds (1991, 1992) studied possible ways for obtaining accurate natural frequencies of beams, especially those weak in shear. The first order approximate frequency values were deduced from elementary beam theory. Two methods for arriving at more accurate values were examined: refined beam theories and implementing end effects. Based on analysis of two-dimensional end effects in vibration of a cantilevered beam the authors demonstrated that the contribution of end effects to correction of natural frequency of a beam (either weak in shear or not) is more meaningful in comparison with the correction obtained with higher order beam theory. Accordingly, a correction factor for the natural frequency is suggested.

A technique to bypass the need to consider end effects in analysis of vibration was suggested by Chen et al. (2003). The treatment of dynamic response of a laminated beam by conventional state space method combined with differential quadrature method shows (p. 75) that: "It also can cope with arbitrary boundary conditions without applying Saint-Venant's principle".

End effects in a rectangular plate of thickness h and dimensions axb, are considered by Kathnelson (1997). To clarify the extent and the magnitude of end effects an asymptotic analysis of the exact shear edge effect solution near a free side of a rectangular isotropic linear elastic plate is carried out. It is argued that in the dynamic case the end effects are identical.

Differential inequalities leading to energy estimates were derived by Flavin and Knops (1987) for a finite cylinder, either with clamped faces or made of viscoelastic material. That work was followed by a series of subsequent studies employing differential inequalities (cited above in the context of waveguides with free or constrained surfaces). Combining the treatment of waveguides with clamped faces with viscoelastic material response in one paper appears to reflect the common feature of spatial decay of energy (as opposed to a previous paper by Ignaczak, 1974). The authors succeeded in proving that the effect of dynamic excitation remains localized at the vicinity of the excited end for both cases.

Two remarks made by Flavin and Knops (1987) extend the validity of that result to waveguides with free lateral surfaces or without viscous dissipation. The first (Remark 2, p. 255) reads: "The results remain valid if $\eta = 0$ (no viscosity) provided that special initial conditions, appropriate to the forced oscillation are adopted.". The second (Remark 4, p. 261) states: "Theorem 2 continues to be valid in the absence of damping provided that the complementary oscillation (undumped in the ideal elastic case) which co-exists with the forced oscillation is subtracted out". It is suggested in

Karp (2009) that the "complementary oscillation" can be considered as an equivalent excitation having the same average power and frequency as the main load (and opposite phase), and that the "special initial condition" is an excitation that does not generate propagating modes.

Chirita (1995) considered spatial decay in problems governed by parabolic and hyperbolic equations. The principle derived consists of two parts: domain of influence and energy decay estimate within that domain.

Iesan and Scalia (1997) established a spatial decay estimate for a finite cylinder, made of micro stretch elastic solid, and excited at one of its ends. Both free and clamped lateral surfaces were considered, leading to exponential decay below some critical frequency. Since both formulations refer to steady state response of a finite rod, it is clear that the end excitation should have zero average power. Otherwise, energy inflow will not allow for the response to settle into a steady state. Then, the decay measures obtained reflect upon the evanescent waves.

Essentially an identical problem was studied by Gales (2003). Here, the amplitude of the steady-state vibration of a finite cylinder made of a mixture consisting of three components (an elastic solid, a viscous fluid and a gas) was investigated. An exponential decay estimate of Saint-Venant type in terms of the distance from one end of the cylinder was obtained with the decay constant depending on excitation frequency, constitutive coefficients and the first positive eigenvalue of the fixed membrane problem for the given cross-sectional geometry (as derived by Toupin, 1965a). The author concludes (p. 152) with: "To the amplitude of the steady-state vibration we associate a cross-sectional measure and, provided the exciting frequency is lower than a certain critical frequency, we derive a first-order differential inequality, which by integration leads to a spatial decay estimate of Saint-Venant type. The result proves that the above cross-sectional measure decays faster than a certain exponential function of the distance from the loaded end".

The problem of thermoelastic vibrating plate was addressed by D'Apice (2005). Saint-Venant type decay is derived for frequency of vibration below a specific value, with an exponential decay of energy contents in the cross section.

Experimental investigation of end effects on the frequency of vibration of a cantilever elastic beam was conducted by Karp et al. (2008). An aluminum beam was clamped by six screws at one end and excited by a lateral impact at its free end. The level of tightness of the screws was the controlled variable and considered as a variation of end conditions without changing the global characteristics of the structure as a cantilever beam. It was found that complete release of any of the six screws had significant effect on near field response (Fig. 17) but not on the far field. FFT (fast Fourier transform) analysis of the far field response revealed no sensitivity to absence of one screw. Removal of two or more screws did have an effect on the vibration frequencies of the beam (not reported in the article).

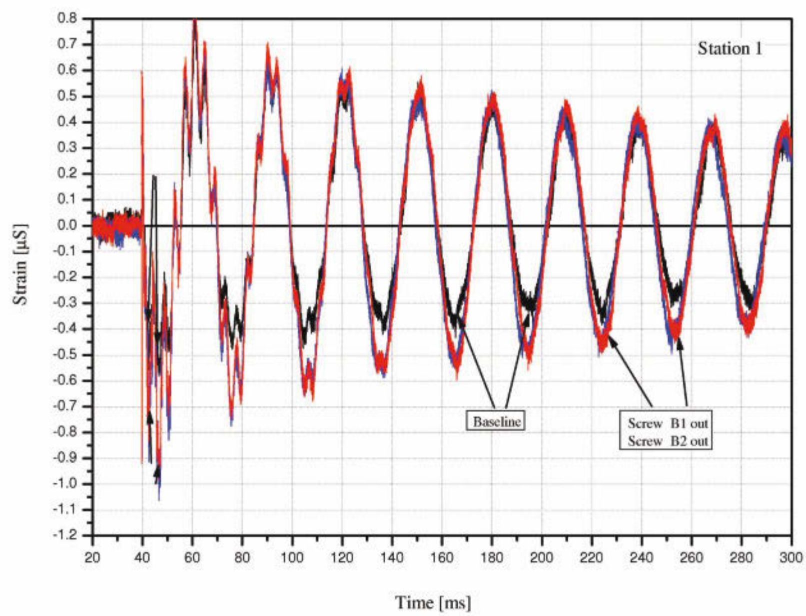


Figure 17. Axial surface strain recording in the close vicinity to clamped end of a beam, with three different "clamping" conditions subjected to transversal excitation at the far end of the beam (from Karp et al., 2008).

Evans and Porter (2008) used Green's function to demonstrate existence of edge waves for a semi-infinite plate within the context of plate theory. Specifically, the authors have shown that plane waves incident on a pinned point on the straight edge of an elastic plate can generate edge waves which radiate energy to infinity along the edge. Various aspects of edge waves are discussed in a recent volume of Mathematics and Mechanics of Solids accumulate several reports on edge vibration and resonance, including Kaplunov and Lawrie (2012), Zacharov (2012), Pichugin and Rogerson (2012), Pagneux (2006, 2012), Kaplunov and Fu (2012).

8.2 Viscoelastic Materials

Since spatial decay of energy can not be granted in elastic materials due to existence of propagating modes, even when a self-equilibrated load is applied, viscosity was introduced to provide a dissipating mechanism ensuring the required decay. That type of material is mainly studied by authors who considered the spatial decay of end effects as a criterion for a valid version of Saint-Venant's principle.

Murray (1970) looked at the question of spatial and temporal decay of discontinuities induced at a surface in a mildly nonlinear Maxwell rod with finite nonlinear viscous damping governed by a first order partial differential equation. Various degrees of spatial decay behavior were derived for different characteristics of the problem.

Rauch (1976) investigated the qualitative behavior of dissipative wave equations of a bounded domain with a general cross-section. Munoz Rivera et al. (1996) employed integral theorems to establish decay rates for viscoelastic plates with memory. The decay considered is a function of time, which formally excludes it from being of Saint-Venant's type.

In the work by Chirita (1997) energy decay estimates were obtained for transient response of a finite length bar, made of anisotropic viscoelastic material, with lateral surfaces free of traction, for both self-equilibrated and general dynamic load. An analogous asymptotic result was obtained for a semi-infinite cylinder. Existence of exponential spatial decay for both self-equilibrated and generally non-self-equilibrated loads, derived in that work, emphasize the question of relevance of self-equilibrium for a dynamic version of SVP. It also exposes the substantial difference between dissipating and non-dissipating media in the context of DSVP.

Ciarletta and Chirita (1999) establish decay estimate for a viscoelastic material with voids. De Cicco and Nappa (1999) derived an exponential decay estimate for a micropolar viscoelastic finite cylinder in a form similar to Toupin's (1965a) decay estimate for a quasi-static case.

More than ten studies on decay of dynamic disturbances in viscoelastic materials appeared in the literature (reviewed here in previous chapters). Because the damping coefficient enters implicitly into decay estimations, it is difficult to figure out mathematically the role of damping and possibly relate these works to decay estimates derived originally for elastic materials without any dissipating mechanisms. No equivalence criterion is invoked in these studies.

9 Comparing SVP with DSVP

It is a standard practice in physics to require a general formulation and solution of a dynamic problem to degenerate to its quasi-static equivalent by taking the limit of vanishing frequency $\omega \to 0$. Following that practice, it is expected that any valid version of DSVP will degenerate to the classical SVP at that limit. Successful degeneration of one the DSVP version can provide some credibility to that version.

Classification of the articles reviewed in Chapter 4 discloses five approaches to what DSVP should be (summarized in Section 10.1 below). Among these five only the dynamic equivalence approach appears to be legitimate for degeneration to SVP. Excluded are approaches denying existence of DSVP and those introducing viscosity. That comparison between SVP and DSVP and degeneration of DSVP to SVP is suggested below. The comparison is not complete due to the currently early stages of the research on that version of DSVP. Yet, it might be valuable in pointing to potentially constructive research direction on the topic.

9.1 Mathematical Formulation and Foundation

Mathematical foundation of the classical SVP is based mainly of two methods: energy inequality (e.g., Knowles, 1966, Toupin, 1965a) and eigenfunction expansion (e.g., Horvay, 1953). Let us begin the comparison with the eigenfunction expansion method.

Eigenfunction expansion method commonly regarded as an accurate quantitative estimation of validity of SVP (e.g., Goetshel and Hu, 1985; Horgan, 1989). It is commonly related to a semi-infinite strip, resembling a beam like or plate like structure, with a typical width of $2h$. The response of the strip to a self-equilibrated load is captured by eigenfunctions, decaying exponentially from the loaded end, providing the effect of localization. The complete displacement response of the strip in the (x, y) plane with x as an axial direction is given by

$$\mathbf{u}(x, y) = \sum_n A_n \mathbf{U}_n(y) e^{-\xi_n x} \tag{9.1.1}$$

where the sum is taken over the infinite set of eigenfunctions $\mathbf{U}_n(y)$ each associated with eigenvalue ξ_n. Positive real part of the eigenvalue $Re\{\xi_n\}$ dictates the rate of spatial decay of the amplitude of each mode n. The dynamic response of the same semi-infinite strip is described in a similar way by (e.g., Achenbach, 1973)

$$\mathbf{u}(x, y, t) = \sum_n A_n \mathbf{U}_n(y) e^{i(\xi_n x - \omega t)} \tag{9.1.2}$$

Except for the additional time variable and frequency parameter, there are only semantic differences between the quasi-static solution and the dynamic one. Here $\mathbf{U}_n(y)$ are wave modes and ξ_n are wave numbers. The decaying wave modes are termedevanescent, spatial decay of which is dictated now by $Im\{\xi_n\}$ due to the i factor in the exponent. By taking the limit of zero frequency, one derives the quasi-static solution from the dynamic one. The spatial decay rate of both static and dynamic fields is governed by the same variable ξ_n.

The values of the decay rates ξ_n of the static fields are obtained from the Fadle-Papkovich equation (Timoshenko and Goodier, 1972). The values of the decay rates ξ_n of the evanescent waves are obtained from the Rayleigh-Lamb equation (Mindlin, 1960; Graff, 1975). That equation is degenerated to Fadle-Papkovich equation by taking the limit of zero frequency (Miklowitz, 1978).

Let us contrast now SVP and DSVP as they are formulated using energy methods. In the static case the decay of energy contained in the body due to application of self-equilibrated load is exponential (Toupin, 1965a; Knowles, 1966). The decay rate obtained is considered to be an approximate to the exact one obtained from eigenfunction expansion ξ_n.

An equivalent examination of energy decay due to dynamic self-equilibrated load revealed non-decaying (propagating) modes that deliver energy to infinity without attenuation. Yet, since self-equilibrated loads are not a result of difference between dynamically equivalent loads, that result does not necessarily disprove DSVP. Indeed, it has been shown by Karp (2009) that under some limiting conditions a difference between any two dynamically equivalent loads result in excitation having zero average power. From waveguide analysis it is obtained that for such excitations the energy content in a waveguide will indeed decay with rate dictated by ξ_n. Such decay was obtained using energy inequalities only for waveguide with clamped lateral surfaces (as reviewed in Sec. 5.1) and a more general proof is remained to be wanted.

9.2 Practical Application

The application of SVP consists of replacing the actual system of loads by other system having identical static equivalents, namely, same total force and couple. According to the dynamic equivalence version the usage of DSVP is by replacing the original excitation by other excitation having identical dynamic equivalents, namely, total average power and frequency (below first cut-off frequency). Apparently, static and dynamic equivalences are different. Difference between two statically equivalent loads results in self-equilibrated load whereas difference between two dynamically equivalent excitations lead to excitation with zero input power. Yet, it was shown for a simple waveguide (Karp, 2009) that any excitation having zero net power applied to a wave guide is self equilibrated in the static sense in the limit of zero frequency.

The mathematical derivation of the classical SVP out of DSVP by taking the limit of zero frequency enable one to generalize the statements of the principle, emphasizing their oneness. One common reference to SVP can now be generalized to be written as: "Principle of elastic equivalence of a *statically/dynamically* equivalent system of *loads/excitations*": Engineers customary refer to a different statement, reading now: "The far field strain produced in a body, by application of *statically/dynamically* equivalent *loads/excitations*, is the same". These two statements, combining SVP and DSVP, are in a complete agreement with the spirit of Saint-Venant's ideas as expressed by Ericksen (1979, p. 7): "St.-Venant's principle gave a rule of thumb for dividing all solutions into equivalence classes ...".

Finally, in both static and dynamic problems the meaning of the principle is that the far field response is not sensitive to the details of the applied excitation, rather to its integral properties. In the static case these integral properties are the static equivalents. In the dynamic case it is the time average input power. The unified statement of the principle is thus: "A property of a structure according to which the strain far from the loaded end has low sensitivity to the spatial distribution of *static/dynamic loads/excitations*".

10 Concluding Comments

10.1 Theoretical Formulation

Most of existing studies on DSVP look either for conditions under which spatial decay can be granted, or search for the distance beyond which the fine details of the excitation has only minor importance, if at all. Each of the two approaches has brunched into several views as to what DSVP should be. These views can be grouped roughly into five categories: DSVP is not valid even if self-equilibrated excitation is applied (Boley, 1955, 1960a; Slepiyan and Novhozilov, 1965; Grandin and Little, 1974; Ziv, 2003; Foster and Berdichevsky, 2004); DSVP is valid regardless of the self-equilibrium of the excitation provided some attenuating conditions are added, such as clamping of the lateral surfaces, energy leaking surfaces, or viscosity (Flavin and Knops, 1987; Nappa, 1998; Ciarletta and Chirita, 1999); DSVP is valid only approximately when the frequency is low enough (Boley 1955, 1960a; Grandin and Little, 1974) or when the excitation deviates slightly from self-equilibrium (Gomilko et al., 1995; Babenkova and Kaplunov, 2004); DSVP is valid only statistically (Foster and Berdichevsky, 2004); and finally, DSVP is valid for dynamically equivalent loads, equivalence that can be defined rigorously at any frequency below the first cut-off frequency (Torvik, 1967; Karp, 2009).

At present, there is apparently no guiding idea as how to select one approach over the other. Nevertheless, it should be possible to follow the original spirit of classical SVP in considering primarily practical aspects of DSVP. As vividly described by Toupin (1965b) and by Benvenuto (1997), Saint Venant used heuristics to propose his assumption aiming at liberating engineers from dealing with beam problems associated with either unknown boundary details or intractable analytically. In Chapter 3 we have tried to provide clear experimental evidence for that quest suggesting that similar practical approach should be valid in the case of dynamic excitation as well. The comparison between the classical SVP and the dynamic equivalence interpretation of DSVP brought in Chapter 9 provide a further demonstration of the possible formulation of DSVP and probably for its potential benefit.

10.2 Application of DSVP

A review by Walley and Mason (2000) on the history of the split Hopkinson bar system for material characterization exemplifies the need for DSVP and the interpretation associated with that (p. 2): "This issue was resolved by theoretical work and experimental checks on whether the Saint Venant Principle could be extended to dynamic 'non-equilibrium' loading problems ... And from about 1953 onwards it became standard to use strain gauges bonded to the outside of Hopkinson bars to measure strain pulses propagating down them". The same view was expressed by Field et al. (2001) in summary of main developments in such systems. According to that summary (p. 112), the 50's are characterized by: "Experimental checks of the St Venant hypothesis and hence legitimation of the use of surface strain gauges to measure stress pulse propagation". Typical example for application of DSVP is found in Pope and Field (1984, p. 817): "Miniature semiconductor strain gauges are sited 10 bar diameters from the input end and by the dynamic equivalent of St Venant's principle the bar can accurately record the total force on the end face, independent of the pressure distribution".

These assertions, along with additional considerations of separation of signals, are the basis for guidance in locating strain gauges in split Hopkinson system by Follansbee (1985, p. 199) "However, these end effects quickly dampen after the wave has propagated about ten bar diameters", and in the newer version by Gray III (2000, p. 463): "The length of the pressure bars must first ensure one-dimensional wave propagation for a given pulse length; for experimental measurements on most engineering materials, this propagation requires approximately 10 bar diameters". From experiments detailed in chapter 3 and from recent studies (e.g., Meng and Li, 2001; Karp et al., 2008, 2009) it can be safely stated that the distance at which a one-dimensional wave is attained is much smaller (2-3 diameters). From Pochhammer-Chree solution and Davies' (1948) studies it is evident that an additional restriction of low frequency should be imposed.

Another practical aspect of DSVP is detailed by Duva and Simmonds (1991). The authors examined influence of end effects on the lower natural vibration frequencies of a laminated beam and the correction to the classical prediction that should follow. A direct association of end effects with DSVP (p. 178) was made: "...It is senseless to proceed without discussing end effects. For the relatively low frequencies of vibration we are considering, these effects should be confined to end zones of width $O(H)$ as suggested by the useful discussions by Boley (1955, 1960) and Grandin and Little (1974) of a dynamic St-Venant's Principle for a semi-infinite elastic strip".

In a few additional engineering situations researchers have relied upon DSVP, either implicitly or explicitly, deliberately or tacitly. The dynamic equivalence version of DSVP is applied in the field of active noise and vibration control (without reference to DSVP). For example, the cancellation of an unwanted sound field is achieved by an array of sources activated to generate a secondary acoustic field having the same frequency and average power with an inverse polarity to the main source, resulting in a destructive interference (Rosenhouse, 2002). Kuznetsov and Stepanov (2007) used the idea of equivalence for dealing with source replacement (p. 326), stating that: "The equivalence of a model is understood in the sense that the pressure levels and interference structures of the amplitude and phase of a volume low-frequency source and of a point multipole should be sufficiently close to provide the required accuracy of measurements.". Another practical aspect of source equivalence is found in ambient noise modeling of urban landscape (e.g., Hornikx and Forssen, 2009). Kundu et al. (1991) used equivalent source replacement based on validity of SVP (p. 153): "In the proposed method unknown sources are placed not at the near field boundary but at the location of the structure. Then the Saint-Venant's principle is utilized to justify that at a distant point the effect of structure's vibration can be effectively modeled by an equivalent vibrating point force and vibrating moment at the structure's position". It is worthy to note that such application of SVP to dynamic response of a half-space was later regarded by several researches as not valid (see section 6.1).

The possible application of the concept of DSVP to the emerging field of structural health monitoring was demonstrated by Karp et al. (2008)

and Karp (2011). It was shown that by estimating the extent of end effects, along with the identification of dynamic equivalence, it is possible to expose incipient damage at joints of beam-like and plate-like structures.

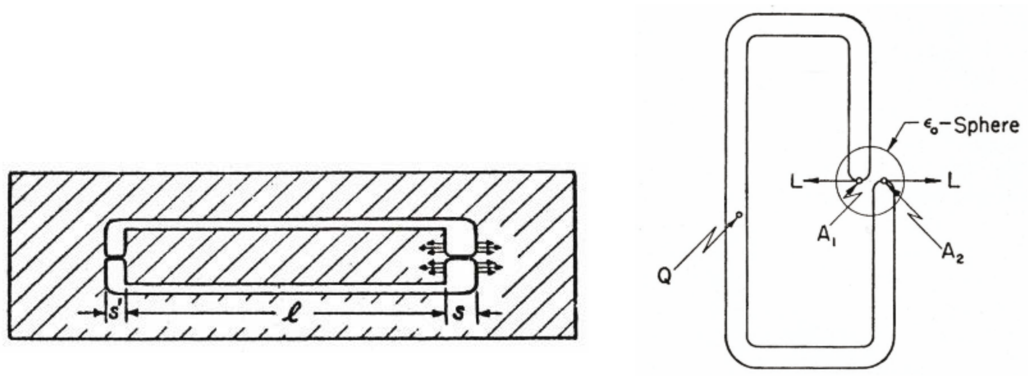
It can be extrapolated from those representative examples that several engineering fields should benefit from applying the dynamic equivalence version of DSVP. It is conceivable that DSVP can be used in other engineering situations as well. This might include: source recognition in acoustics and acoustic emission (e.g., Kroll and Tatro, 1964), earthquakes analysis (e.g., Kundu et al., 1991), force reconstruction in measurement systems relying on wave guiding (e.g., Tyas and Watson, 2000), energy trapping at discontinuities associated with evanescent waves (e.g., Evans, 1992; Kaplunov and Sorokin, 1995; Aslanyan et al., 2000; McIver et al., 2002; Chamberlain, 2004), and dynamic material characterization (e.g., Waldman and Lee, 2002; Sasso et al., 2008; Gilat et al. 2009) where uniformity of the field within the cross-section is required.

10.3 Vision of DSVP

There are definite contradictions and lack of clarity related to the possible validity of DSVP in elastic problems. In particular, this is due to convincing demonstrations for non validity of DSVP (non-decaying field due to self-equilibrated excitation), and general the inherent non-decaying property of fields governed by hyperbolic partial differential equations. That unease can somewhat be mitigated by appreciating that even the study of the classical SVP is not yet complete. A brief review of the history of the ideas related to the classical SVP expose parallel counter examples and mixed attitudes to the essence of SVP.

Examples for structures for which the classical SVP is not valid were discussed years before, and also after, proofs for validity of SVP were derived. Four such structures are shown here in Figures 18 and 19. Additional counter examples for SVP are given by Horvay (1957), Toupin (1965b) and more recently by Huang (1989) and by Markenscoff (1994). It appears that these counter examples for validity of the classical SVP did not lead the community to doubt its existence, nor its usefulness.

It is instructive to quote several prominent scientists referring to the meaning of the classical SVP. von-Mises (1945) wrote a century after Saint-Venant introduced his assumption (p. 562): "What Saint Venant originally had in mind was doubtlessly the case of a long cylinder with infinite ratio of length to diameter. The purpose of the present paper was to show that an extension of the principle to bodies of finite dimensions is not legitimate". In a similar spirit, Sternberg (1954, p. 401) wrote: "For Saint-Venant's prin-



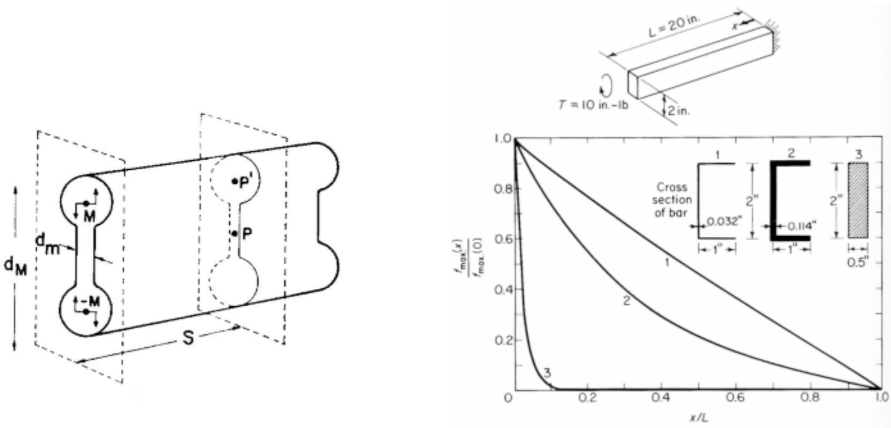
(*a*) (*b*)

Figure 18. Two structures for which the classical SVP is not valid: (a) from Donnell (1962), (b) from Sternberg (1954).

ciple is a statement about relative orders of magnitude and does not tell us anything about the extent of the region within which a self-equilibrated system of tractions, applied to a portion of the surface of an elastic body, "materially" influences the stress distribution in the body". The non univocal meaning of SVP is also evidenced from discussion brought by Naghdi (1960). Even three decades later Levine and Quintanilla (1989, p., 71) noted that: "we believe that the study of the principle and problem is not finished even in the simple case of cylinders".

These citations, along with several counter examples, related to the classical SVP may suggest that existence of clear examples for non-validity of DSVP and some disagreement concerning its very essence, does not necessarily mean the search for DSVP in linearly elastic material is hopeless. Moreover, the wide range of experimental situations in which one of the versions of DSVP appears to be valid, might motivate one to search for ways to relax the apparent contradictions, made explicit in Section 2.3. The use of energy inequalities is one of avenues to be followed in providing a rigorous proof for decaying fields generated by excitations having zero net power. Such a research is expected to provide clearer definitions of the conditions under which DSVP is expected to be valid, and when it does not.

Unification of the classical SVP with the equivalence version of DSVP raises an additional question on whether that version can also be generalized to other Saint-Venant type decay estimates (not reviewed here) such as in quasi-linear or non-linear elasticity, non-mechanical waveguides (op-



(*a*) (*b*)

Figure 19. Two structures for which the classical SVP is not valid: (a) from Toupin, (1965a), (b) from Hoff (1945).

tical), and to heat problems. For example, in the proof of the static SVP, correspondence between the energy inequalities (Toupin, 1965a; Knowles 1966) and direct elasticity solutions with eigenfunctions (Timoshenko and Goodier, 1972) has been established. Can such correspondence be shown to hold for dynamic or heat problems? In that context, it is an open question how to settle the apparent contradiction between established energy decay, obtained by several authors, and the non-decaying propagating modes in a non-dissipating structure with free surfaces.

Since DSVP is related to localizied phenomena, it is natural to expect existence of connections with the various topics covered in this volume. Such possible connections are ought to be pursued, both in continuum mechanics and electromagnetic fields.

Apparent inconsistencies, between several views of DSVP and engineering practice, define clear objectives for additional future research. Since dynamic phenomena encompass a richer spectrum in comparison with static cases, it is expected that applications will exceed the well established limits of the classical SVP. Applying the principle should be supported by firm experimental evidence, both in validity and in quantitative estimation. The work reviewed in chapter 3 provides a promising start for fruitful research to come. Engineers in particular are expected to recognize the validity of the DSVP (at least in one of its versions) and to assimilate its use.

Both theory and engineering practice call for such an undertaking, with research program including related issues like stability of solutions and well posedness. The theoretical basis of the DSVP should be widened and it is hoped that the present review will contribute towards formulating a unified theory, compatible with the body of knowledge already available, on a dynamic analogue of Saint-Venant's principle. Note: This article is an extended and updated version of a review published few years ago in Applied Mechanics Reviews (Karp and Durban, 2011).

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