Explicit Models for Surface, Interfacial and Edge Waves

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Abstract We derive explicit asymptotic formulations for surface, interfacial and edge waves in elastic solids. The effects of mixed boundary conditions and layered structure are incorporated. A hyperbolic-elliptic duality of surface and interfacial waves is emphasized along with a parabolic-elliptic duality of the edge bending wave on a thin elastic plate. The validity of the model for the Rayleigh wave is illustrated by several moving load problems.

1 Introduction

Surface elastic waves as well as their interfacial and edge analogues seem to be 'hidden' within the general equations of elastodynamics. At the same time the contribution of these waves to the overall dynamic response sometimes is more substantial than that of bulk waves. As an example, we mention a resonant behaviour of elastic solids caused by high speed moving loads.

This chapter is centered on explicit models for surface, interfacial and edge waves, that neglect the effect of bulk waves. We study the classical Rayleigh surface wave (Rayleigh 1885) along with Schölte-Gogoladze (Schölte 1949 and Gogoladze 1948) and Stoneley (1924) interfacial waves, and the edge bending wave on a thin plate discovered by Konenkov (1960), relying on the methodology established in our recent publications (Kaplunov *et al.* 2006, 2010, 2013, Dai *et al.* 2010, Erbaş *et al.* 2012). General formulations for homogenous surface and interfacial waves were also developed last years by Achenbach (1998), Kiselev & Rogerson (2009), Kiselev & Parker (2010), Parker (2012).

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Our approach is based upon a fundamental result by Friedlander (1948) and Chadwick (1976) regarding representation of plane homogenous surface and interfacial waves in terms of harmonic functions. On perturbing the Rayleigh wave eigensolution in slow time we derive in paragraph 2.4 a *hyperbolic-elliptic* model for plane strain near-surface motion. The model consists of a *pseudo-static elliptic* equation governing decay into the interior, subject to the Dirichlet boundary condition in the form of a *hyperbolic* equation describing propagation of the Rayleigh wave under prescribed surface stresses. It reveals a hyperbolic-elliptic duality of the Rayleigh wave and also has obvious advantages for numerical computations. Indeed, we split the original vector hyperbolic problem into a scalar hyperbolic equation and a time independent elliptic problem over the interior.

With the help of the Radon integral transform, we extend the consideration above to 3D case including a pseudo-differential formulation for a coated half-space, see paragraphs 2.4 and 2.6. In addition, the proposed approach appears to be very promising for mixed dynamic problems for cracks and stamps, see paragraph 2.5.

In Section 3 we demonstrate that the *hyperbolic-elliptic* models for interfacial waves are not more difficult than that for the Rayleigh wave. The results of this section may also be easily generalised to 3D problems.

Resonant effect of moving loads studied in Section 4, is virtually the ideal setup for testing derived models. We consider a variety of plane strain problems taking into account mixed boundary conditions along with layered structure. A number of elegant approximate solutions are obtained in a surprisingly straightforward manner.

The dispersive nature of the edge bending wave on a thin plate leads to a *parabolic-elliptic* asymptotic theory. We arrive at a beam-like fourth-order equation modelling propagation of disturbances along the edge, see Section 5.

2 Surface waves

We derive an asymptotic *hyperbolic-elliptic* model for the surface Rayleigh wave. The plane strain motion is studied in great detail including mixed boundary value problems. The obtained results are extended to 3D case.

2.1 Equations of linear elastodynamics

Consider an elastic half-space given by

$$H^{+}_{(3)} = \big\{(x_1; x_2; x_3) \,\big|\, -\infty < x_1 < \infty, \quad -\infty < x_2 < \infty, \quad 0 \le x_3 < \infty\big\}\,.$$

The equations of motion in 3D elasticity are taken in the form (see e.g. Achenbach 1973)

$$\frac{\partial \sigma_{im}}{\partial x_m} = \rho \frac{\partial^2 u_i}{\partial t^2}, \qquad i = 1, 2, 3, \tag{2.1.1}$$

where ρ is volume density, t is time, u_i are displacement vector components, σ_{im} are stress tensor components, and summation over repeated suffices is assumed. In case of a free surface wave homogeneous boundary conditions over the surface $x_3 = 0$ are imposed, yielding

$$\sigma_{3i} = 0. \tag{2.1.2}$$

Below we also consider more general boundary conditions.

The constitutive relations are given by

$$\sigma_{ik} = \delta_{ik} \lambda \, \mathrm{div} \mathbf{u} + 2\mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \tag{2.1.3}$$

where $\mathbf{u} = \{u_1, u_2, u_3\}$, δ_{ik} is the Kronecker delta, and λ and μ are the Lamé elastic moduli. In view of the constitutive relations (2.1.3) the equations of motion take the form

$$(\lambda + \mu) \, \mathrm{grad} \, \mathrm{div} \mathbf{u} + \mu \Delta_3 \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{2.1.4}$$

where $\Delta_3 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$ is the 3D Laplace operator.

2.2 Plane harmonic Rayleigh wave

Let us begin with a 2D problem for a half-plane

$$H^+_{(2)} = \left\{ (x_1; x_3) \,\middle|\, -\infty < x_1 < \infty, \quad 0 \le x_3 < \infty \right\},$$

adapting the plane strain assumptions

$$u_2 = 0, \qquad u_i = u_i(x_1, x_3, t), \quad (i = 1, 3).$$

In this case, the displacement field $\{u_1, u_3\}$ may be expressed through the elastic wave potentials ϕ and ψ as

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3}, \qquad u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1}. \tag{2.2.1}$$

Then, the equations of motion (2.1.4) are rewritten in the form

$$\Delta \phi - \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial t^2} = 0, \qquad \Delta \psi - \frac{1}{c_2^2} \frac{\partial^2 \psi}{\partial t^2} = 0, \tag{2.2.2}$$

where $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ denote the longitudinal and shear wave speeds, respectively, and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}$. The wave potentials may then be found in the form of travelling wave solutions

$$\phi = \phi(x_1 - ct, \alpha x_3) = A \exp\left[ik\left(x_1 - ct\right) - k\alpha x_3\right], \tag{2.2.3}$$

$$\psi = \psi(x_1 - ct, \beta x_3) = B \exp\left[ik\left(x_1 - ct\right) - k\beta x_3\right], \tag{2.2.4}$$

decaying as $x_3 \to \infty$, where c is the sought for wave speed, and as it readily follows from (2.2.2),

$$\alpha = \sqrt{1 - \frac{c^2}{c_1^2}}, \qquad \beta = \sqrt{1 - \frac{c^2}{c_2^2}}. \tag{2.2.5}$$

It is clear that each of the functions ϕ and ψ in (2.2.3) and (2.2.4) are harmonic over the half-plane $H^+_{(2)}$. We also remark that all the speculations in what follows are equally valid for the wave travelling in the opposite direction, i.e. for the functions ϕ and ψ depending on $x_1 + ct$. The boundary conditions (2.1.2) can now be expressed in terms of the wave potentials as

$$\begin{aligned} &2\frac{\partial^2\phi}{\partial x_1 \partial x_3} + \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_3^2} = 0, \\ &\left(\kappa^2 - 2\right)\frac{\partial^2\phi}{\partial x_1^2} + \kappa^2\frac{\partial^2\phi}{\partial x_3^2} + 2\frac{\partial^2\psi}{\partial x_1 \partial x_3} = 0, \end{aligned} \tag{2.2.6}$$

with

$$\kappa = \frac{c_1}{c_2} = \sqrt{\frac{2 - 2\nu}{1 - 2\nu}},$$

where ν is the Poisson ratio. Substitution of the formulae (2.2.3), (2.2.4) into (2.2.6) results in the homogeneous algebraic system in A and B

$$\begin{aligned} 2i\alpha A + (1 + \beta^2)B &= 0 \\ (1 + \beta^2)A - 2i\beta B &= 0 \end{aligned} \tag{2.2.7}$$

which possesses a non-trivial solution provided that the related determinant equals zero, i.e.

$$4\alpha\beta = (1 + \beta^2)^2, \tag{2.2.8}$$

originating from the classical paper by Lord Rayleigh (1885) and having a unique root $c = c_R$, provided that

$$\alpha = \alpha_R = \sqrt{1 - \frac{c_R^2}{c_1^2}}, \qquad \beta = \beta_R = \sqrt{1 - \frac{c_R^2}{c_2^2}}. \tag{2.2.9}$$

2.3 Plane surface wave of arbitrary profile

We follow the approach in Friedlander (1948) and Chadwick (1976) in order to generalize the sinusoidal Rayleigh wave solution derived in the previous subsection to a surface wave of arbitrary profile

$$\phi = \phi(x_1 - ct, \alpha x_3), \qquad \psi = \psi(x_1 - ct, \beta x_3), \tag{2.3.1}$$

with α and β defined above, see (2.2.5), and plane harmonic functions ϕ and ψ satisfying the elliptic equations

$$\frac{\partial^2 \phi}{\partial x_3^2} + \alpha^2 \frac{\partial^2 \phi}{\partial x_1^2} = 0, \qquad \frac{\partial^2 \psi}{\partial x_3^2} + \beta^2 \frac{\partial^2 \psi}{\partial x_1^2} = 0, \tag{2.3.2}$$

arising from the wave equations (2.2.2). On substituting the harmonic functions (2.3.1) into the boundary conditions (2.2.6), we obtain

$$\begin{aligned} &2\frac{\partial^2 \phi(x_1 - ct, 0)}{\partial x_1 \partial x_3} + \left(1 + \beta^2\right) \frac{\partial^2 \psi(x_1 - ct, 0)}{\partial x_1^2} = 0, \\ &- \left(1 + \beta^2\right) \frac{\partial^2 \phi(x_1 - ct, 0)}{\partial x_1^2} + 2\frac{\partial^2 \psi(x_1 - ct, 0)}{\partial x_1 \partial x_3} = 0. \end{aligned} \tag{2.3.3}$$

Throughout this chapter we employ the Cauchy-Riemann identities for a plane harmonic function $f(x, ky)$. They are given by

$$\frac{\partial f}{\partial y} = -k\,\frac{\partial \overline{f}}{\partial x}, \qquad \frac{\partial f}{\partial x} = \frac{1}{k}\,\frac{\partial \overline{f}}{\partial y}, \qquad \overline{\overline{f}} = -f, \tag{2.3.4}$$

where bar indicates a harmonic conjugate.

With the help of these identities the conditions (2.3.3) may be transformed to

$$\begin{aligned} 2\alpha \frac{\partial^2 \phi}{\partial x_1^2} + \left(1 + \beta^2\right) \frac{\partial^2 \overline{\psi}}{\partial x_1^2} = 0, \\ \left(1 + \beta^2\right) \frac{\partial^2 \phi}{\partial x_1^2} + 2\beta \frac{\partial^2 \overline{\psi}}{\partial x_1^2} = 0, \end{aligned} \tag{2.3.5}$$

leading to the Rayleigh equation (2.2.8). In this case the sought for harmonic eigenfunctions

$$\phi = \phi(x_1 - c_R t, \alpha_R x_3), \qquad \psi = \psi(x_1 - c_R t, \beta_R x_3) \tag{2.3.6}$$

are related to each other on the surface $x_3 = 0$ as

$$\frac{\partial \psi}{\partial x_1} = -\frac{2}{1 + \beta_R^2}\frac{\partial \phi}{\partial x_3}, \tag{2.3.7}$$

see $(2.3.3)_1$. Moreover, the last relation specified on the surface may be extended to the whole interior region as

$$\psi(x_1 - c_R t, \beta_R x_3) = \frac{2\alpha_R}{1+\beta_R^2}\,\overline{\phi}(x_1 - c_R t, \beta_R x_3), \tag{2.3.8}$$

or

$$\phi(x_1 - c_R t, \alpha_R x_3) = -\frac{2\beta_R}{1+\beta_R^2}\,\overline{\psi}(x_1 - c_R t, \alpha_R x_3), \tag{2.3.9}$$

for more details see Chadwick (1976). Thus, the wave potentials are related through the Hilbert transform, and consequently the Rayleigh wave field may be expressed through a single harmonic function.

2.4 Hyperbolic-elliptic model

Plane strain problem Consider now non-homogeneous boundary conditions

$$\sigma_{31} = Q(x_1, t), \qquad \sigma_{33} = P(x_1, t), \tag{2.4.1}$$

imposed along the surface $x_3 = 0$ of the half-plane $H^+_{(2)}$. These may be reformulated in terms of the wave potentials as

$$\begin{aligned} 2\frac{\partial^2\phi}{\partial x_1 \partial x_3} + \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_3^2} &= \frac{Q}{\mu}, \\ \left(\kappa^2 - 2\right)\frac{\partial^2\phi}{\partial x_1^2} + \kappa^2\frac{\partial^2\phi}{\partial x_3^2} + 2\frac{\partial^2\psi}{\partial x_1 \partial x_3} &= \frac{P}{\mu}. \end{aligned} \tag{2.4.2}$$

Let us we perturb the surface wave eigensolutions (2.3.6) in slow time

$$\tau = \varepsilon t, \qquad (\varepsilon \ll 1). \tag{2.4.3}$$

Throughout this paragraph we assume that the deviation of the analysed perturbed motion $\{\phi(x_1 - c_R t, x_3, \tau),\ \psi(x_1 - c_R t, x_3, \tau)\}$ from the homogeneous Rayleigh wave field (2.3.6) is small. On inserting slow time τ into the original equations of motion (2.2.2) at $\alpha = \alpha_R$ and $\beta = \beta_R$, and taking into account the operator identity $\dfrac{\partial}{\partial t} = -c_R\dfrac{\partial}{\partial x_1} + \varepsilon\dfrac{\partial}{\partial \tau}$, we have

$$\begin{aligned} \frac{\partial^2\phi}{\partial x_3^2} + \alpha_R^2\frac{\partial^2\phi}{\partial x_1^2} + 2\frac{\varepsilon}{c_R}\left(1-\alpha_R^2\right)\frac{\partial^2\phi}{\partial x_1 \partial \tau} - \frac{\varepsilon^2}{c_R^2}\left(1-\alpha_R^2\right)\frac{\partial^2\phi}{\partial \tau^2} &= 0, \\ \frac{\partial^2\psi}{\partial x_3^2} + \beta_R^2\frac{\partial^2\psi}{\partial x_1^2} + 2\frac{\varepsilon}{c_R}\left(1-\beta_R^2\right)\frac{\partial^2\psi}{\partial x_1 \partial \tau} - \frac{\varepsilon^2}{c_R^2}\left(1-\beta_R^2\right)\frac{\partial^2\psi}{\partial \tau^2} &= 0. \end{aligned} \tag{2.4.4}$$

Next, we expand the potentials in asymptotic series as

$$\begin{aligned}\phi(x_1-c_Rt,x_3,\tau)&=\phi_0(x_1-c_Rt,\alpha_Rx_3,\tau)+\varepsilon\phi_1(x_1-c_Rt,x_3,\tau)+\ldots,\\ \psi(x_1-c_Rt,x_3,\tau)&=\psi_0(x_1-c_Rt,\beta_Rx_3,\tau)+\varepsilon\psi_1(x_1-c_Rt,x_3,\tau)+\ldots,\end{aligned} \tag{2.4.5}$$

where the leading order terms ϕ_0 and ψ_0 coincide with the surface wave eigensolutions (2.3.6) to within a parametric dependence of slow time.

On substituting the expansion (2.4.5) into the perturbed equations of motion (2.4.4) we get expressions for $O(\varepsilon)$ terms. They are written as

$$\begin{aligned}\phi_1 &= \phi_{10} - x_3\frac{1-\alpha_R^2}{\alpha_R c_R}\frac{\partial\overline{\phi}_0}{\partial\tau},\\ \psi_1 &= \psi_{10} - x_3\frac{1-\beta_R^2}{\beta_R c_R}\frac{\partial\overline{\psi}_0}{\partial\tau},\end{aligned} \tag{2.4.6}$$

where $\phi_{10} = \phi_{10}(x_1 - c_Rt, \alpha_Rx_3, \tau)$ and $\psi_{10} = \psi_{10}(x_1 - c_Rt, \beta_Rx_3, \tau)$ are arbitrary functions, harmonic in the first two variables, for more details see Kaplunov *et al.* (2006).

It is convenient to treat the two sub-problems for boundary conditions, namely, the cases of vertical ($Q = 0, P \neq 0$) and horizontal ($P = 0, Q \neq 0$) loading. Let us consider first the effect of a vertical force normalizing it by $P = \varepsilon P_\varepsilon$. On introducing the formulae (2.4.5) and (2.4.6) into the boundary conditions (2.4.2) we get at $x_3 = 0$

$$\begin{aligned}&2\frac{\partial^2\phi_{10}}{\partial x_1\partial x_3} + \left(1+\beta_R^2\right)\frac{\partial^2\psi_{10}}{\partial x_1^2} - \frac{2(1-\alpha_R^2)}{c_R\alpha_R}\frac{\partial^2\overline{\phi}_0}{\partial x_1\partial\tau}\\ &\qquad\qquad\qquad\qquad + \frac{2(1-\beta_R^2)}{c_R\beta_R}\frac{\partial^2\overline{\psi}_0}{\partial x_3\partial\tau} = 0,\\ &-\left(1+\beta_R^2\right)\frac{\partial^2\phi_{10}}{\partial x_1^2} + 2\frac{\partial^2\psi_{10}}{\partial x_1\partial x_3} - \frac{2(1-\alpha_R^2)\kappa^2}{c_R\alpha_R}\frac{\partial^2\overline{\phi}_0}{\partial x_3\partial\tau}\\ &\qquad\qquad\qquad\qquad - \frac{2(1-\beta_R^2)}{c_R\beta_R}\frac{\partial^2\overline{\psi}_0}{\partial x_1\partial\tau} = \frac{P_\varepsilon}{\mu}.\end{aligned} \tag{2.4.7}$$

Then, using the Cauchy-Riemann identities (2.3.4) along with the relations

$$\overline{\psi}_0(x_1-c_Rt,0) = -\frac{2\alpha_R}{1+\beta_R^2}\,\phi_0(x_1-c_Rt,0) = -\frac{1+\beta_R^2}{2\beta_R}\,\phi_0(x_1-c_Rt,0), \tag{2.4.8}$$

following from (2.3.8) and (2.3.9), we rewrite the boundary conditions (2.4.7) as

$$
\begin{aligned}
&2\alpha_R \frac{\partial^2 \phi_{10}}{\partial x_1^2} + \left(1+\beta_R^2\right) \frac{\partial^2 \overline{\psi}_{10}}{\partial x_1^2} = \frac{2}{c_R}\left[\frac{1-\beta_R^4}{2\beta_R} - \frac{1-\alpha_R^2}{\alpha_R}\right] \frac{\partial^2 \phi_0}{\partial x_1 \partial \tau}, \\
&-\left(1+\beta_R^2\right) \frac{\partial^2 \phi_{10}}{\partial x_1^2} - 2\beta_R \frac{\partial^2 \overline{\psi}_{10}}{\partial x_1^2} = \frac{2}{c_R}\left[1-\beta_R^2 - \frac{1-\beta_R^4}{2\beta_R^2}\right] \frac{\partial^2 \phi_0}{\partial x_1 \partial \tau} + \frac{P_\varepsilon}{\mu}.
\end{aligned}
\tag{2.4.9}
$$

It is clear that the determinant of the left hand side of (2.4.9) equals zero. The solvability condition is

$$
\frac{2}{c_R} \frac{\partial^2 \phi_0}{\partial x_1 \partial \tau} = \frac{1+\beta_R^2}{2\mu B} P_\varepsilon, \tag{2.4.10}
$$

where

$$
B = \frac{\beta_R}{\alpha_R}(1-\alpha_R^2) + \frac{\alpha_R}{\beta_R}(1-\beta_R^2) - (1-\beta_R^4). \tag{2.4.11}
$$

Let the load on the right hand side of (2.4.10) evolve in slow time as

$$
P_\varepsilon(x_1, t) = \frac{\partial^2 p_\varepsilon}{\partial \tau \partial x_1}, \tag{2.4.12}
$$

with $p_\varepsilon = p_\varepsilon(x_1 - c_R t, \tau)$. Then we readily infer from (2.4.10) that

$$
\phi_0 = \frac{\left(1+\beta_R^2\right) c_R}{4\mu B} p_\varepsilon, \tag{2.4.13}
$$

i. e. $\phi_0 = \phi_0(x_1 - c_R t, \tau)$ as was initially assumed.

It is evident, however, that for an arbitrary vertical load P the solution of the equation (2.4.10) may demonstrate a more general time dependence. Nevertheless, this equation always enables a correct evaluation of the Rayleigh wave contribution to the overall dynamic response. Moreover, the developed perturbation procedure is a counterpart of a routine relying on computation of the residues corresponding to the Rayleigh wave poles, see the Appendix in Kaplunov *et al.* (2006). It is also very crucial that the solution of (2.4.10) will often dominate in the near-surface zone, in particular for impulse and near-resonant moving loads. For the latter the slow time may be defined as

$$
\tau = \left|1 - \frac{c}{c_R}\right| t, \quad c \approx c_R, \tag{2.4.14}
$$

where c is the speed of the load.

Finally, applying the operator asymptotic relationship

$$\frac{2\varepsilon}{c_R}\frac{\partial^2}{\partial x_1 \partial \tau} = \frac{\partial^2}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2}{\partial t^2} + O\left(\varepsilon^2\right),$$

we present the equation (2.4.10) for $\phi = \phi_0$ in terms of the original time t as

$$\frac{\partial^2 \phi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{1+\beta_R^2}{2\mu B} P. \tag{2.4.15}$$

Thus, the asymptotic formulation for the Rayleigh wave has been reduced to a scalar problem for the pseudo-static elliptic equation $(2.3.2)_1$ derived in the previous subsection subject to the Dirichlet boundary condition at $x_3 = 0$ in the form of the wave equation (2.4.15). The shear potential ψ may then be restored from the relation (2.3.8).

In case of tangential loading a similar asymptotic model consists of a scalar problem for the elliptic equation $(2.3.2)_2$ subject to a boundary condition at $x_3 = 0$, given by the following hyperbolic equation

$$\frac{\partial^2 \psi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{1+\beta_R^2}{2\mu B} Q, \tag{2.4.16}$$

with the potential ϕ determined through the Hilbert transform from (2.3.9).

We remark that the established approximate formulation is oriented to the Rayleigh wave only and does not incorporate the effect of bulk waves. The range of validity of the model (see (2.3.2), (2.3.8), (2.3.9), (2.4.15), and (2.4.16)) covers the problems of near-surface dynamics with the dominant contribution of the Rayleigh wave.

The consideration above reveals a dual *hyperbolic-elliptic* nature of the Rayleigh wave. It is worth noting however that not all the displacement components demonstrate a wave behaviour along the surface. In particular, in case of vertical loading only the horizontal displacement u_1 is governed by a hyperbolic equation. The latter follows from (2.4.15) (see also (2.3.3)) and can be written as

$$\frac{\partial^2 u_1}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 u_1}{\partial t^2} = \frac{1-\beta_R^4}{4\mu B}\frac{\partial P}{\partial x_1}. \tag{2.4.17}$$

3D problem Let us generalise the plane strain formulation obtained in the previous subsection, to the 3D case. We start from the equations of motion (2.1.4), in case of vertical surface loading modelled by the boundary conditions at $x_3 = 0$

$$\sigma_{31} = \sigma_{32} = 0, \qquad \sigma_{33} = P(x_1, x_2, t). \tag{2.4.18}$$

The Radon integral transform

$$f^{(\alpha)}(\chi, \alpha, x_3, t) = \int_{-\infty}^{\infty} f\left(\chi \cos\alpha - \eta \sin\alpha, \chi \sin\alpha + \eta \cos\alpha, x_3, t\right) d\zeta, \quad (2.4.19)$$

where

$$\chi = x_1 \cos\alpha + x_2 \sin\alpha, \qquad \eta = -x_1 \sin\alpha + x_2 \cos\alpha,$$

with the angle α varying over the interval $0 \leq \alpha < 2\pi$, reduces the original 3D elastodynamics problem to a 2D problem for associated transforms, for more details see Georgiadis & Lykotrafitis (2001) and references therein. In (2.4.19) and below the Radon transforms are denoted by suffice (α). We also define transformed displacements in the Cartesian frame (χ, η) as

$$u_\chi^{(\alpha)} = u_1^{(\alpha)} \cos\alpha + u_2^{(\alpha)} \sin\alpha, \qquad u_\eta^{(\alpha)} = -u_1^{(\alpha)} \sin\alpha + u_2^{(\alpha)} \cos\alpha, \quad (2.4.20)$$

and set $u_\eta^{(\alpha)} = 0$ assuming that the the anti-plane motion does not induce surface waves.

It is clear that the aforementioned 2D problem for Radon transforms is formally identical to that in the theory of plane strain. Then, we introduce an analogue of wave potentials

$$u_\chi^{(\alpha)} = \frac{\partial \phi^{(\alpha)}}{\partial \chi} - \frac{\partial \psi^{(\alpha)}}{\partial x_3}, \qquad u_3^{(\alpha)} = \frac{\partial \phi^{(\alpha)}}{\partial x_3} + \frac{\partial \psi^{(\alpha)}}{\partial \chi} \quad (2.4.21)$$

and follow the perturbation procedure developed in the previous subsection. The asymptotic formulation for the Rayleigh wave (expressed through Radon transforms) contains the elliptic equations

$$\frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} + \alpha_R^2 \frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} = 0, \qquad \frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2} + \beta_R^2 \frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} = 0, \quad (2.4.22)$$

over the interior, along with the hyperbolic equation

$$\frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} - \frac{1}{c_R^2} \frac{\partial^2 \phi^{(\alpha)}}{\partial t^2} = \frac{1 + \beta_R^2}{2\mu B} P^{(\alpha)}, \quad (2.4.23)$$

specified on the surface $x_3 = 0$. The relation between the potentials $\phi^{(\alpha)}$ and $\psi^{(\alpha)}$ on the surface now becomes

$$\frac{\partial \psi^{(\alpha)}}{\partial \chi} = -\frac{2}{1 + \beta_R^2} \frac{\partial \phi^{(\alpha)}}{\partial x_3}. \quad (2.4.24)$$

Next, we introduce a pair of the potentials $\psi_1^{(\alpha)} = \psi^{(\alpha)} \cos\alpha$ and $\psi_2^{(\alpha)} = \psi^{(\alpha)} \sin\alpha$ in order to invert the formulae (2.4.22)-(2.4.24). As a result, we get

$$\frac{\partial^2 \phi}{\partial x_3^2} + k_1^2 \Delta\phi = 0, \tag{2.4.25}$$

$$\frac{\partial^2 \psi_i}{\partial x_3^2} + k_2^2 \Delta\psi_i = 0, \tag{2.4.26}$$

where now $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, and $i = 1, 2$, subject to the boundary conditions $(x_3 = 0)$

$$\Delta\phi - \frac{1}{c_R^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{1+\beta_R^2}{2\mu B} P, \tag{2.4.27}$$

and

$$\frac{\partial \psi_1}{\partial x_1} = \frac{\partial \psi_2}{\partial x_2} = -\frac{2}{1+\beta_R^2}\frac{\partial \phi}{\partial x_3}. \tag{2.4.28}$$

In the formulae above the potentials ϕ, ψ_1 and ψ_2 satisfy the vector relation (Dai *et al.* 2010)

$$\mathbf{u} = \mathrm{grad}\phi + \mathrm{curl}\Psi, \tag{2.4.29}$$

where $\Psi = (-\psi_2, \psi_1, 0)$.

2.5 Plane mixed problems

The methodology in 2.4.1 may also be adapted for mixed boundary value problems arising in dynamics of cracks and stamps. Consider first a *vertical stamp* applied to the surface of the elastic half-plane $H_{(2)}^+$. The boundary conditions at $x_3 = 0$ include zero tangential stresses

$$\sigma_{31} = 0, \tag{2.5.1}$$

along with normal stresses P and vertical displacements U_3 prescribed on the disjoint parts of the surface S_1 and S_2, respectively ($S_1 \cup S_2 = R$). Thus

$$\begin{aligned} \sigma_{33} &= P(x_1, t), \quad \text{at} \quad x_1 \in S_1, \\ u_3 &= U_3(x_1, t), \quad \text{at} \quad x_1 \in S_2, \end{aligned} \tag{2.5.2}$$

see Fig. 1.

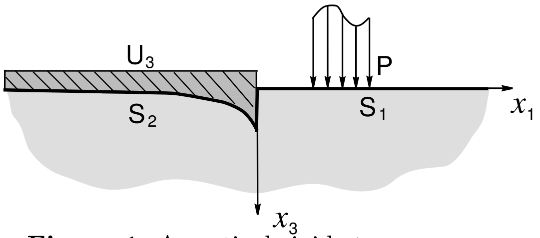

Figure 1. A vertical rigid stamp

On utilizing the formulae (2.3.2), (2.3.8) and (2.4.15), we arrive at a scalar mixed problem for the elliptic equation (see Erbaş *et al.* 2012)

$$\frac{\partial^2 \phi}{\partial x_3^2} + \alpha_R^2 \frac{\partial^2 \phi}{\partial x_1^2} = 0 \tag{2.5.3}$$

subject to the boundary conditions ($x_3 = 0$)

$$\frac{\partial^2 \phi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{1+\beta_R^2}{2\mu B} P, \quad \text{at} \quad x_1 \in S_1, \tag{2.5.4}$$

and

$$\frac{\partial \phi}{\partial x_3} = \frac{1+\beta_R^2}{1-\beta_R^2} U_3, \quad \text{at} \quad x_1 \in S_2. \tag{2.5.5}$$

As before, the shear potential ψ is expressed by the relation (2.3.8).

A similar formulation may be deduced for an elastic half-plane, part of which is coated by a flexible inextensible membrane not resisting to vertical motion. In this case the boundary conditions on the surface $x_3 = 0$ may be written as

$$\begin{aligned} &\sigma_{33} = 0, \quad \text{at} \quad x_3 = 0, \\ &\sigma_{31} = Q(x_1, t), \quad \text{at} \quad x_1 \in S_1, \\ &u_1 = U_1(x_1, t) \quad \text{at} \quad x_1 \in S_2. \end{aligned} \tag{2.5.6}$$

where Q and U_1 denote the given horizontal stresses and displacements, respectively, see Fig. 2.

Now a scalar setup for the shear potential ψ is given by the equation

$$\frac{\partial^2 \psi}{\partial x_3^2} + \beta_R^2 \frac{\partial^2 \psi}{\partial x_1^2} = 0 \tag{2.5.7}$$

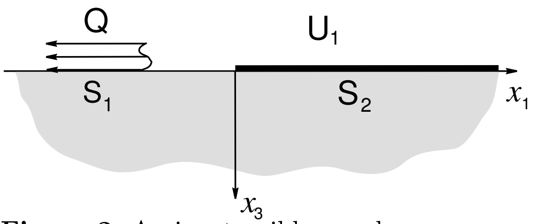

Figure 2. An inextensible membrane

along with the boundary conditions ($x_3 = 0$)

$$\frac{\partial^2\psi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2\psi}{\partial t^2} = \frac{1+\beta_R^2}{2\mu B}Q, \quad \text{at} \quad x_1 \in S_1, \tag{2.5.8}$$

and

$$\frac{\partial\psi}{\partial x_3} = \frac{1+\beta_R^2}{1-\beta_R^2}U_1, \quad \text{at} \quad x_1 \in S_2. \tag{2.5.9}$$

2.6 Long wave asymptotic model for a surface wave on a coated half-space

The asymptotic formulation for the Rayleigh wave may also be extended to a coated half-space in the framework of long-wave approximation. Consider the elastic half-space $H^+_{(3)}$ coated by an elastic layer occupying the region $-h \leq x_3 \leq 0$, see Fig. 3.

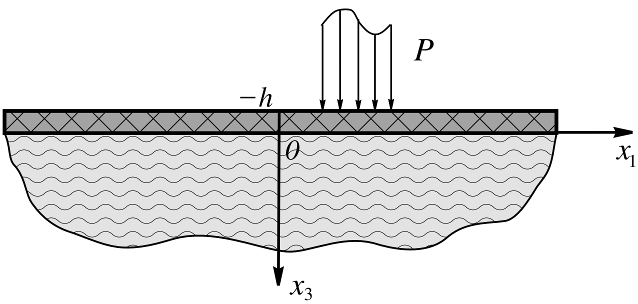

Figure 3. A half-space coated by an elastic layer

As in subsection 2.4.2, we impose the boundary conditions (2.4.18) on the upper face of the coating $x_3 = -h$. We also assume continuity of all displacements and stresses at the interface $x_3 = 0$.

A standard asymptotic long-wave technique applied to the coating (here and below in this subsection for more details see Dai *et al.* (2010) and references therein) results in effective boundary conditions on the interface $x_3 = 0$, namely

$$\begin{aligned}\sigma_{i3} &= \rho_0 h \left\{ \frac{\partial^2 u_i}{\partial t^2} - c_{20}^2 \left[\frac{\partial^2 u_i}{\partial x_j^2} + 4\left(1-\kappa_0^{-2}\right) \frac{\partial^2 u_i}{\partial x_i^2} \right.\right. \\ &\qquad\qquad\left.\left. + \left(3 - 4\kappa_0^{-2}\right) \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right] \right\}, \\ \sigma_{33} &= \rho_0 h \frac{\partial^2 u_3}{\partial t^2} + P, \qquad\qquad 1 \le i \ne j \le 2,\end{aligned} \tag{2.6.1}$$

where ρ_0 is density of the coating, c_{10} and c_{20} are associated bulk wave speeds, and $\kappa_0 = c_{10}/c_{20}$. The boundary conditions (2.6.1) coincide with those earlier proposed by Tiersten (1969).

Thus, the initial problem is reduced to analysis of the uncoated half-space H_3^+ subject to the boundary conditions (2.6.1) imposed on its surface $x_3 = 0$. In this case the transformed equations

$$\begin{aligned}\frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} + \frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} - \frac{1}{c_1^2} \frac{\partial^2 \phi^{(\alpha)}}{\partial t^2} = 0, \\ \frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} + \frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2} - \frac{1}{c_2^2} \frac{\partial^2 \psi^{(\alpha)}}{\partial t^2} = 0,\end{aligned} \tag{2.6.2}$$

are accompanied by the boundary conditions ($x_3 = 0$)

$$\begin{aligned}\mu \left[2\frac{\partial^2 \phi^{(\alpha)}}{\partial \chi \partial x_3} + \frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} - \frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2} \right] &= \mu_0 h \left[c_{20}^{-2} \left(\frac{\partial^3 \phi^{(\alpha)}}{\partial \chi \partial t^2} - \frac{\partial^3 \psi^{(\alpha)}}{\partial x_3 \partial t^2} \right) \right. \\ &\qquad \left. -4\left(1-\kappa_0^{-2}\right) \left(\frac{\partial^3 \phi^{(\alpha)}}{\partial \chi^3} - \frac{\partial^3 \psi^{(\alpha)}}{\partial x_3 \partial \chi^2} \right) \right], \\ \mu \left[\left(\kappa^2 - 2\right) \frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} + \kappa^2 \frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} + 2\frac{\partial^2 \psi^{(\alpha)}}{\partial \chi \partial x_3} \right] & \\ = \mu_0 h c_{20}^{-2} \left(\frac{\partial^3 \phi^{(\alpha)}}{\partial x_3 \partial t^2} + \frac{\partial^3 \psi^{(\alpha)}}{\partial \chi \partial t^2} \right) - P^{(\alpha)}.&\end{aligned} \tag{2.6.3}$$

A perturbation procedure similar to that in subsection 2.4.1, leads to a singularly perturbed hyperbolic equation on the surface. It is given by

$$\frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} - \frac{1}{c_R^2} \frac{\partial^2 \phi^{(\alpha)}}{\partial t^2} + \frac{bh}{\alpha_R} \frac{\partial^3 \phi^{(\alpha)}}{\partial \chi^2 \partial x_3} = \frac{1 + \beta_R^2}{2\mu B} P^{(\alpha)}, \tag{2.6.4}$$

with

$$b = \frac{\mu_0}{2\mu B}(1 - \beta_R^2) \left[(1 - \beta_{R0}^2)(\alpha_R + \beta_R) - 4\beta_R(1 - \kappa_0^{-2}) \right]. \tag{2.6.5}$$

In the original variables, we get from (2.6.4)

$$\Delta\phi - \frac{1}{c_R^2}\frac{\partial^2\phi}{\partial t^2} + \frac{bh}{\alpha_R}\frac{\partial}{\partial x_3}(\Delta\phi) = \frac{1+\beta_R^2}{2\mu B}P, \tag{2.6.6}$$

which is a boundary condition for the elliptic equation (2.4.25), where now $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

The perturbed hyperbolic equation (2.6.6) can also be presented in a pseudo-differential form, i.e.

$$\Delta\phi - \frac{1}{c_R^2}\frac{\partial^2\phi}{\partial t^2} - bh\sqrt{-\Delta}\,(\Delta\phi) = \frac{1+\beta_R^2}{2\mu B}P. \tag{2.6.7}$$

In the plane strain case the last equation becomes

$$\frac{\partial^2\phi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2\phi}{\partial t^2} - bh\sqrt{-\frac{\partial^2}{\partial x_1^2}}\frac{\partial^2\phi}{\partial x_1^2} = \frac{1+\beta_R^2}{2\mu B}P. \tag{2.6.8}$$

This equation may also be written through the Hilbert transform. Therefore, the presence of a coating inevitably leads to an integro-differential formulation.

In addition, the equation (2.6.8) enables a simple approximation of the exact dispersion relation, see e.g. Shuvalov & Every (2008) and references therein. Indeed, we easily deduce from (2.6.8) that

$$v = c_R\left(1 - \frac{b}{2}|kh| + \dots\right), \tag{2.6.9}$$

demonstrating that the Rayleigh wave speed c_R is a local extremum over the long wave domain $kh \ll 1$, where k denotes wave number.

3 Interfacial waves

The results obtained for the Rayleigh wave are now generalized to interfacial waves. In view of the existing representation in terms of a single harmonic function (Kiselev & Parker 2010), we may expect similar hyperbolic-elliptic formulations for both Schölte-Gogoladze and Stoneley waves, see also Prikazchikov (2011). In this section we restrict ourselves to plane strain assumptions, however, 3D formulations may be easily derived using the Radon transform similarly to what has been done for the Rayleigh wave. We show that the analysis of interfacial wave fields may be also reduced to scalar problems for the elliptic equations. As a result, a tedious algebra, traditionally associated with investigation of interfacial waves, seems to be mainly overcome.

3.1 Schölte-Gogoladze wave

Consider an elastic half-plane $H^+_{(2)}$, joint with a fluid half-plane

$$H^-_{(2)} = \left\{(x_1; x_3)\,\middle|\, -\infty < x_1 < \infty, \quad x_3 < 0\right\},$$

and concentrate on the interfacial Schölte-Gogoladze wave propagating along the line $x_3 = 0$ and decaying away from it. This wave has been discovered independently by Schölte (1949) and Gogoladze (1948).

The equations of motion for an elastic medium are given by (2.2.2), whereas fluid motion is governed by the Helmholz equation

$$\Delta\theta - \frac{1}{c_f^2}\frac{\partial^2\theta}{\partial t^2} = 0, \tag{3.1.1}$$

where θ is the displacement potential, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_3^2}$, and c_f is the fluid wave speed. Below we assume zero tangential stresses and continuity of normal displacements along the interface $x_3 = 0$, leading to the boundary conditions

$$\sigma_{31} = 0, \qquad u_3 = v, \qquad \sigma_{33} - p = P(x_1, t), \tag{3.1.2}$$

where v and p are the vertical displacement and pressure in fluid, respectively, given by

$$v = \frac{\partial\theta}{\partial x_3}, \qquad p = \rho_f\,\frac{\partial^2\theta}{\partial t^2},$$

with ρ_f denoting the volume density of the fluid, and P standing for prescribed vertical stresses along the interface. The boundary conditions (3.1.2) expressed in terms of the potentials ϕ, ψ and θ become

$$2\frac{\partial^2\phi}{\partial x_1\partial x_3} - \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi_0}{\partial x_3^2} = 0,$$

$$\frac{\partial\phi_0}{\partial x_3} + \frac{\partial\psi}{\partial x_1} - \frac{\partial\theta}{\partial x_3} = 0, \tag{3.1.3}$$

$$\mu\left[\left(\kappa^2 - 2\right)\frac{\partial^2\phi_0}{\partial x_1^2} + \kappa^2\frac{\partial^2\phi}{\partial x_3^2} - 2\frac{\partial^2\psi}{\partial x_1\partial x_3}\right] - \rho_f\frac{\partial^2\theta}{\partial t^2} = P.$$

The equation for the interfacial Schölte-Gogoladze wave speed follows from (2.2.2), (3.1.1), and (3.1.3) at $P = 0$. It takes the form

$$4\alpha_{SG}\beta_{SG} - \left(1 + \beta_{SG}^2\right)^2 = \frac{\rho_f}{\rho}\,\frac{\alpha_{SG}}{\gamma_{SG}}\left(1 - \beta_{SG}^2\right)^2, \tag{3.1.4}$$

where

$$\alpha_{SG} = \sqrt{1 - \frac{c_{SG}^2}{c_1^2}}, \qquad \beta_{SG} = \sqrt{1 - \frac{c_{SG}^2}{c_2^2}}, \qquad \gamma_{SG} = \sqrt{1 - \frac{c_{SG}^2}{c_f^2}}, \quad (3.1.5)$$

and c_{SG} is the sought for speed of the Schölte-Gogoladze wave. Similarly to the consideration in 2.4.1, we obtain an approximate hyperbolic-elliptic formulation for the contribution of the Schölte-Gogoladze wave to the general dynamic response. The decay into the interior is again governed by the *elliptic* equation

$$\frac{\partial^2 \phi}{\partial x_3^2} + \alpha_{SG}^2 \frac{\partial^2 \phi}{\partial x_1^2} = 0, \qquad (3.1.6)$$

while the interfacial dynamics is described by the *hyperbolic* equation ($x_3 = 0$)

$$\frac{\partial^2 \phi}{\partial x_1^2} - \frac{1}{c_{SG}^2}\frac{\partial^2 \phi}{\partial t^2} = AP, \qquad (3.1.7)$$

where

$$A = \frac{1 + \beta_{SG}^2}{\mu \left[2B_{SG} - \dfrac{\rho_f}{\rho} \dfrac{(1 - \beta_{SG}^2)^2 \left(\gamma_{SG}^2 - \alpha_{SG}^2 - 4\alpha_{SG}^2 \gamma_{SG}^2\right)}{2\alpha_{SG}\gamma_{SG}^3} \right]}, \qquad (3.1.8)$$

and B_{SG} takes the form (2.4.11) to within the substitutions $\alpha_R = \alpha_{SG}$ and $\beta_R = \beta_{SG}$. It is readily observed that at $\rho_f = 0$ the equation (3.1.7) is identical to that for the Rayleigh wave, see (2.4.15).

The potentials ψ and θ are related to the potential ϕ as

$$\psi(x_1 - c_{SG}t, \beta_{SG}x_3) = \frac{2\alpha_{SG}}{1 + \beta_{SG}^2} \overline{\phi}(x_1 - c_{SG}t, \beta_{SG}x_3), \qquad (3.1.9)$$

and

$$\theta(x_1 - c_{SG}t, \gamma_{SG}x_3) = -\frac{1 - \beta_{SG}^2}{1 + \beta_{SG}^2} \phi(x_1 - c_{SG}t, \gamma_{SG}x_3). \qquad (3.1.10)$$

3.2 Stoneley wave

Next, we study two joint elastic half-planes $H_{(2)}^+$ and $H_{(2)}^-$ in order to develop an asymptotic model for the Stoneley interfacial wave, see Stoneley (1924). The equations of motion are now expressed in terms of two sets of elastic potentials $\phi^{(k)}$ and $\psi^{(k)}$ ($k = 1, 2$) as

$$\Delta\phi^{(k)} - \frac{1}{c_{1k}^2}\frac{\partial^2 \phi^{(k)}}{\partial t^2} = 0, \qquad \Delta\psi^{(k)} - \frac{1}{c_{2k}^2}\frac{\partial^2 \psi^{(k)}}{\partial t^2} = 0, \qquad (3.2.1)$$

where $c_{1k} = \sqrt{(\lambda_k + 2\mu_k)/\rho_k}$ and $c_{2k} = \sqrt{\mu_k/\rho_k}$ are the associated bulk wave speeds for the medium k; in doing so, all the elastic parameters have to satisfy pretty sophisticated existence conditions for the Stoneley wave examined by Schölte (1947).

As before, we only consider a jump of normal stresses at the interface. Thus, we have at $x_3 = 0$

$$\begin{aligned}
&\frac{\partial\phi^{(1)}}{\partial x_1} - \frac{\partial\phi^{(2)}}{\partial x_1} + \frac{\partial\psi^{(1)}}{\partial x_3} - \frac{\partial\psi^{(2)}}{\partial x_3} = 0,\\
&\frac{\partial\phi^{(1)}}{\partial x_3} - \frac{\partial\phi^{(2)}}{\partial x_3} - \frac{\partial\psi^{(1)}}{\partial x_1} + \frac{\partial\psi^{(2)}}{\partial x_1} = 0,\\
&2\mu_1\frac{\partial^2\phi^{(1)}}{\partial x_1\partial x_3} - 2\mu_2\frac{\partial^2\phi^{(2)}}{\partial x_1\partial x_3} + \mu_1\left[\frac{\partial^2\psi^{(1)}}{\partial x_3^2} - \frac{\partial^2\psi^{(1)}}{\partial x_1^2}\right]\\
&\qquad\qquad - \mu_2\left[\frac{\partial^2\psi^{(2)}}{\partial x_3^2} - \frac{\partial^2\psi^{(2)}}{\partial x_1^2}\right] = 0,\\
&\lambda_1\frac{\partial^2\phi^{(1)}}{\partial x_1^2} + (\lambda_1 + 2\mu_1)\frac{\partial^2\phi^{(1)}}{\partial x_3^2} - \lambda_2\frac{\partial^2\phi^{(2)}}{\partial x_1^2}\\
&- (\lambda_2 + 2\mu_2)\frac{\partial^2\phi^{(2)}}{\partial x_3^2} - 2\mu_1\frac{\partial^2\psi^{(1)}}{\partial x_1\partial x_3} + 2\mu_2\frac{\partial^2\psi^{(2)}}{\partial x_1\partial x_3} = P,
\end{aligned} \tag{3.2.2}$$

where $P = P(x_1, t)$ is a given vertical force.

The transcedental equation for the Stoneley wave speed $c = c_S$ (Stoneley 1924) is

$$c_S^4\left((\rho_1 - \rho_2)^2 - a_1a_2\right) + 2c_S^2 m_{12}(\rho_2 b_1 - \rho_1 b_2) + m_{12}^2 b_1 b_2 = 0, \tag{3.2.3}$$

in which

$$\begin{aligned}
a_1 &= (\rho_1\alpha_{2S} + \rho_2\alpha_{1S}), \qquad a_2 = (\rho_1\beta_{2S} + \rho_2\beta_{1S}),\\
b_k &= 1 - \alpha_{kS}\beta_{kS}, \qquad m_{12} = 2(\mu_1 - \mu_2),
\end{aligned} \tag{3.2.4}$$

and

$$\alpha_{kS} = \sqrt{1 - \frac{c_S^2}{c_{1k}^2}}, \qquad \beta_{kS} = \sqrt{1 - \frac{c_S^2}{c_{2k}^2}}, \qquad (k = 1, 2). \tag{3.2.5}$$

The asymptotic model for the Stoneley wave arising from the boundary value problem (3.2.1) and (3.2.2), contains the *elliptic* equation

$$\frac{\partial^2\phi^{(1)}}{\partial x_3^2} + \alpha_{1S}^2\frac{\partial^2\phi^{(1)}}{\partial x_1^2} = 0 \tag{3.2.6}$$

governing the decay into the interior. The rest of wave potentials is determined by the following relations between the potentials at the interface $x_3 = 0$

$$\begin{aligned}\psi^{(2)}(x_1 - c_S t, \beta_{2S} x_3) &= \frac{g_4}{g_1 \beta_{2S}} \overline{\phi}^{(1)}(x_1 - c_S t, \beta_{2S} x_3),\\ \phi^{(2)}(x_1 - c_S t, \alpha_{2S} x_3) &= \frac{g_2}{g_1} \phi^{(1)}(x_1 - c_S t, \alpha_{2S} x_3),\\ \psi^{(1)}(x_1 - c_S t, \beta_{1S} x_3) &= \frac{g_3}{g_4} \psi^{(2)}(x_1 - c_S t, \beta_{1S} x_3),\end{aligned} \tag{3.2.7}$$

where

$$\begin{aligned} g_1 &= (m_{12} - \rho_1 c_S^2) b_2 + \rho_2 c_S^2 (1 + \alpha_{2S} \beta_{1S}),\\ g_2 &= (\rho_2 c_S^2 + m_{12}) b_1 - \rho_1 c_S^2 (1 + \alpha_{1S} \beta_{2S}),\\ g_3 &= \rho_2 c_S^2 (\alpha_{1S} + \alpha_{2S}) - m_{12} \alpha_{1S} b_2,\\ g_4 &= \rho_1 c_S^2 (\alpha_{1S} + \alpha_{2S}) - m_{12} \alpha_{2S} b_1.\end{aligned} \tag{3.2.8}$$

Finally, the *hyperbolic* equation for $\phi^{(1)}$ on the interface $x_3 = 0$ is written as

$$\frac{\partial^2 \phi^{(1)}}{\partial x^2} - \frac{1}{c_S^2} \frac{\partial^2 \phi^{(1)}}{\partial t^2} = \frac{g_1 P_S}{c_S^2 B_S}, \tag{3.2.9}$$

where the constant B_S is given by

$$\begin{aligned} B_S = &-2c_S^2 \left[(\rho_1 - \rho_2)^2 - a_1 a_2 \right] - m_{12} c_S^2 \left(\rho_2 f_2 - \rho_1 f_1 \right)\\ &- \frac{m_{12}^2}{2} \left(b_2 f_1 + b_1 f_2 \right) - \frac{c_S^4}{2} \left(d_1 a_2 + d_2 a_1 \right) + 2 m_{12} \left(\rho_2 b_1 - \rho_1 b_2 \right),\end{aligned} \tag{3.2.10}$$

with

$$d_k = \frac{\rho_2}{\alpha_{kS} c_{1k}^2} + \frac{\rho_1}{\beta_{kS} c_{2k}^2}, \qquad f_k = \frac{\alpha_{kS}}{\beta_{kS} c_{k2}^2} + \frac{\beta_{kS}}{\alpha_{kS} c_{k1}^2}, \qquad (k = 1, 2).$$

It is remarkable that the models for the interfacial Stoneley and Schölte-Gogoladze waves are not more difficult than that for the Rayleigh wave due to the relations for wave potentials, see (3.1.9), (3.1.10), and (3.2.7).

4 Moving load problems

We illustrate the efficiency of the derived hyperbolic-elliptic formulations for the Rayleigh wave by modelling near-resonant regimes of moving loads. As might be expected, the dynamic response caused by a load travelling at a speed close to the Rayleigh wave speed is not strongly affected by bulk waves.

4.1 Steady-state motion of a point force

We begin with the the classical plane strain problem for a steadily moving vertical point force, see Fig. 4 (see e.g. Cole & Huth 1958).

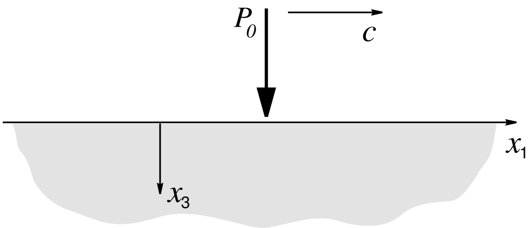

Figure 4. A point force travelling along the surface of a half-plane.

The equations of motion are given by (2.2.2), with boundary conditions on the surface $x_3 = 0$ written as

$$\sigma_{31} = 0, \qquad \sigma_{33} = P_0 \delta(x_1 - ct), \tag{4.1.1}$$

where c is a constant speed of the load.

The asymptotic model for the Rayleigh wave developed in subsection 2.4, now consists of the scalar boundary value problem

$$\frac{\partial^2 \phi}{\partial x_3^2} + \alpha_R^2 \frac{\partial^2 \phi}{\partial s^2} = 0, \tag{4.1.2}$$

subject to the boundary condition ($x_3 = 0$)

$$\left(1 - \frac{c^2}{c_R^2}\right) \frac{\partial^2 \phi}{\partial s^2} = \frac{1 + \beta_R^2}{2\mu B} P_0 \, \delta(s), \tag{4.1.3}$$

where $s = x_1 - ct$ is a moving coordinate. Remarkably, a resonant effect may be immediately observed from (4.1.3) due to degeneration at $c = c_R$. This scalar problem may be reformulated as a Dirichlet problem for the derivative $\phi_s = \dfrac{\partial \phi}{\partial s}$ as

$$\frac{\partial^2 \phi_s}{\partial x_3^2} + \alpha_R^2 \frac{\partial^2 \phi_s}{\partial s^2} = 0, \tag{4.1.4}$$

subject to

$$\phi_s(s, 0) = \frac{\left(1 + \beta_R^2\right) c_R^2 P_0}{2\mu B \left(c_R^2 - c^2\right)} \left(H(s) - \frac{1}{2}\right), \tag{4.1.5}$$

where a constant of integration is chosen because of symmetry. In fact, the 2D steady-state solution is defined to within the rigid body motion of a half-plane, which may be determined from the associated transient problem only, see Kaplunov (1986).

The problem (4.1.4) is easily solved by exploiting the Poisson formula (e.g. Courant & Hilbert 1989), giving

$$\phi_s(s,x_3) = \frac{\left(1+\beta_R^2\right) P_0 c_R^2}{2\pi\mu B\left(c_R^2-c^2\right)} \arctan \frac{s}{\alpha_R\, x_3}. \tag{4.1.6}$$

Therefore (see (2.3.8)),

$$\psi_s(s,x_3) = \frac{\partial\psi}{\partial s} = -\frac{\alpha_R P_0 c_R^2}{2\pi\mu B\left(c_R^2-c^2\right)} \ln\left(s^2+\beta_R^2\, x_3^2\right). \tag{4.1.7}$$

As a result, the steady-state displacement field is given by

$$\begin{aligned} u_1^{st}(\xi) &= \frac{(1+\beta_R^2)P_0 v_R^2}{2\mu\pi B(v_R^2-v^2)}\left[\arctan\frac{\xi}{\alpha_R} - \frac{1+\beta_R^2}{2}\arctan\frac{\xi}{\beta_R}\right],\\ u_2^{st}(\xi) &= -\frac{(1+\beta_R^2)P_0 v_R^2\alpha_R}{4\mu\pi B(v_R^2-v^2)}\left[\ln\left(\xi^2+\alpha_R^2\right) - \frac{2}{1+\beta_R^2}\ln\left(\xi^2+\beta_R^2\right)\right], \end{aligned} \tag{4.1.8}$$

with the following dimensionless parameters

$$\xi = \frac{s}{x_3}, \qquad v = \frac{c}{c_2}, \qquad v_R = \frac{c_R}{c_2}.$$

It may be verified that the displacement components in (4.1.8) are the leading order terms in the Taylor expansion of the exact solution in Cole & Huth (1958) around the resonant Rayleigh wave speed $c = c_R$.

4.2 Transient motion of a point force

Let us now consider the associated transient problem. In this case the same equation (4.1.2) is subject to the following hyperbolic boundary condition on the surface $x_3 = 0$

$$\frac{\partial^2\phi}{\partial s^2} - \frac{1}{c_R^2}\frac{\partial^2\phi}{\partial t^2} = \frac{1+\beta_R^2}{2\mu B} P_0\delta(s). \tag{4.2.1}$$

The solution of the latter can be written as (here and below in this subsection see Kaplunov *et al.* (2010) for more detail)

$$\phi(s,0,t) = B_1 \int_0^t \left[H\left(s+(c-c_R)r\right) - H\left(s+(c+c_R)r\right)\right] dr. \tag{4.2.2}$$

with

$$B_1 = \frac{\left(1 + \beta_R^2\right) c_R P_0}{4\mu B}, \tag{4.2.3}$$

and the *resonant* ($c = c_R$) case arising immediately from the analysis of the integrand.

For the *sub-Rayleigh* regime ($c < c_R$) we get from (4.2.2)

$$\phi(s,0,t) = \begin{cases} B_1 \dfrac{s - s_1}{c_R - c}, & 0 \le s < s_1; \\ B_1 \dfrac{s - s_2}{c_R + c}, & s_2 < s < 0; \\ 0, & \text{otherwise}, \end{cases} \tag{4.2.4}$$

with the values s_1 and s_2 given by

$$s_1 = t(c_R - c), \qquad s_2 = -t(c_R + c). \tag{4.2.5}$$

For the *super-Rayleigh regime* ($c > c_R$) we have

$$\phi(s,0,t) = \begin{cases} 2B_1 \dfrac{c_R s}{c^2 - c_R^2}, & s_1 \le s \le 0; \\ -B_1 \dfrac{s - s_2}{c_R + c}, & s_2 < s < s_1; \\ 0, & \text{otherwise}. \end{cases} \tag{4.2.6}$$

Finally, if $c = c_R$, we obtain

$$\phi(s,0,t) = \begin{cases} -B_1 \dfrac{s - s_2}{2c_R}, & s_2 \le s \le 0; \\ 0, & \text{otherwise}, \end{cases} \tag{4.2.7}$$

with s_2 given now by $s_2 = -2c_R t$.

The solutions on the surface (4.2.4), (4.2.6) and (4.2.7) provide an immediate insight into the physics of the original problem. In particular, Fig. 5 shows a clear distinction of the resonant regime from the two others. In this figure the function $\phi(s,0,t)$ at a fixed time t is plotted for all three aforementioned cases. If $c \neq c_R$, the solution in question is continuous in s, see Figs 5(a) and 5(b). At the same time, the limiting resonant solution in Fig. 5(c), demonstrates a discontinuity under a line moving force ($s = 0$), which is linearly increasing in time. As a result we should not expect a steady-state regime at $c = c_R$. Thus, a rather straightforward analysis of an infinite string under a moving load immediately reveals the resonant phenomena associated with the Rayleigh wave.

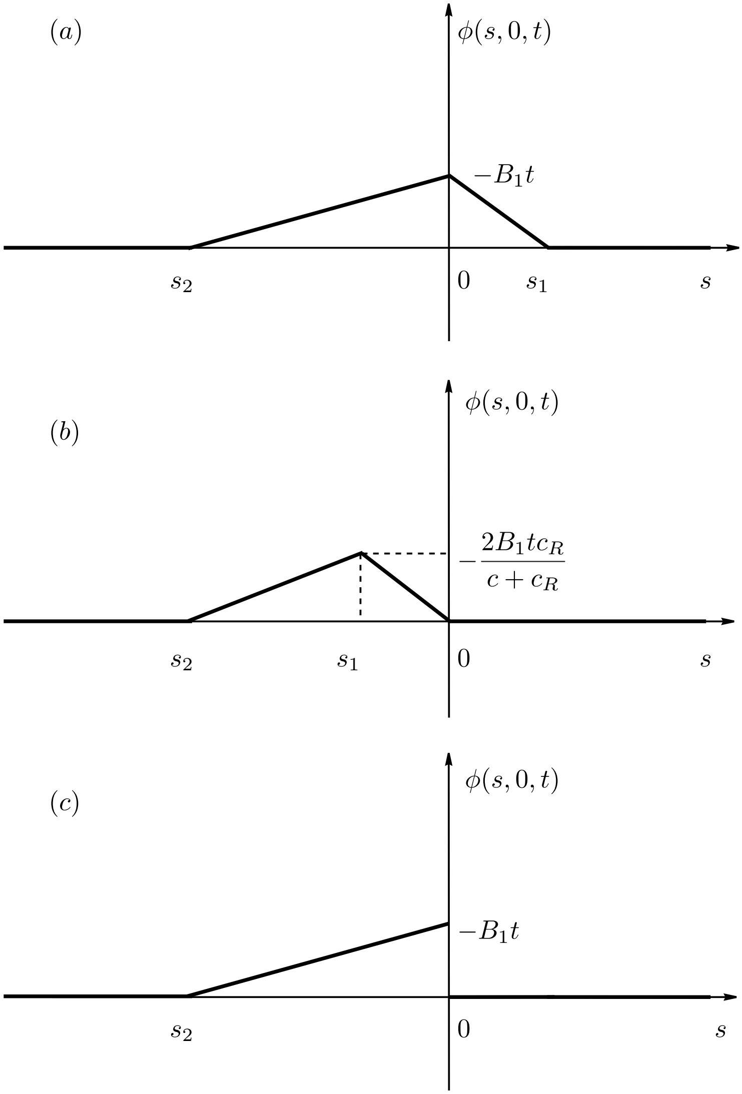

Figure 5. The wave potential ϕ vs. the moving co-ordinate s on the surface $x_3 = 0$: (a) the sub-Rayleigh regime ($c < c_R$); (b) the super-Rayleigh regime ($c > c_R$); (c) resonant regime ($c = c_R$).

Once the potential is determined at the surface $x_3 = 0$, the solution is then restored over the interior through the Poisson formulae. In the *sub-*

Rayleigh and *super-Rayleigh* regimes the displacement components become

$$
\begin{aligned}
u_1(\xi,\tau) = {} & \frac{2B_1\, v_R}{\pi c_2(v_R^2 - v^2)}\left[\arctan\frac{\xi}{\alpha_R} - \frac{1+\beta_R^2}{2}\arctan\frac{\xi}{\beta_R}\right] \\
& - \frac{B_1}{\pi c_2(v_R + v)}\left[\arctan\frac{\xi-\xi_2}{\alpha_R} - \frac{1+\beta_R^2}{2}\arctan\frac{\xi-\xi_2}{\beta_R}\right] \\
& - \frac{B_1}{\pi c_2(v_R - v)}\left[\arctan\frac{\xi-\xi_1}{\alpha_R} - \frac{1+\beta_R^2}{2}\arctan\frac{\xi-\xi_1}{\beta_R}\right],
\end{aligned}
\tag{4.2.8}
$$

$$
\begin{aligned}
u_3(\xi,\tau) = {} & \frac{B_1\,\alpha_R}{2\pi c_2(v_R+v)}\left[\ln\frac{(\xi-\xi_2)^2+\alpha_R^2}{\xi^2+\alpha_R^2} - \frac{2}{1+\beta_R^2}\ln\frac{(\xi-\xi_2)^2+\beta_R^2}{\xi^2+\beta_R^2}\right] \\
& + \frac{B_1\,\alpha_R}{2\pi c_2(v_R-v)}\left[\ln\frac{(\xi-\xi_1)^2+\alpha_R^2}{\xi^2+\alpha_R^2} - \frac{2}{1+\beta_R^2}\ln\frac{(\xi-\xi_1)^2+\beta_R^2}{\xi^2+\beta_R^2}\right],
\end{aligned}
\tag{4.2.9}
$$

with

$$
\tau = \frac{c_2 t}{x_3}, \qquad \xi_1 = \frac{s_1}{x_3} = (v_R - v)\tau, \qquad \xi_2 = \frac{s_2}{x_3} = -(v + v_R)\tau, \tag{4.2.10}
$$

and s_1, s_2 defined by (4.2.5). In the *resonant* regime the corresponding displacement components may be found as

$$
\begin{aligned}
u_1(\xi,\tau) = {} & \frac{B_1\alpha_R\tau}{\pi c_2}\left[\frac{1}{\xi^2+\alpha_R^2} - \frac{2\beta_R^2}{(1+\beta_R^2)(\xi^2+\beta_R^2)}\right] \\
& + \frac{\beta}{2\pi c_2 v_R}\left[\arctan\frac{\xi}{\alpha_R} - \arctan\frac{\xi-\xi_2}{\alpha_R}\right] \\
& - \frac{B_1(1+\beta_R^2)}{4\pi c_2 v_R}\left[\arctan\frac{\xi}{\beta_R} - \arctan\frac{\xi-\xi_2}{\beta_R}\right],
\end{aligned}
\tag{4.2.11}
$$

$$
\begin{aligned}
u_3(\xi,\tau) = {} & \frac{B_1\alpha_R\xi\tau}{\pi c_2}\left[\frac{2}{(1+\beta_R^2)(\xi^2+\beta_R^2)} - \frac{1}{\xi^2+\alpha_R^2}\right] \\
& + \frac{B_1\alpha_R}{4\pi c_2 v_R}\left[\ln\frac{(\xi-\xi_2)^2+\alpha_R^2}{\xi^2+\alpha_R^2} - \frac{2}{1+\beta_R^2}\ln\frac{(\xi-\xi_2)^2+\beta_R^2}{\xi^2+\beta_R^2}\right],
\end{aligned}
\tag{4.2.12}
$$

with $\xi_2 = -2v_R\tau$.

The obtained displacements (4.2.8)-(4.2.12) are expressed in elementary functions in contrast to the integral exact solution of the problem, see Appendix of Kaplunov *et al.* (2010). Nevertheless, the approximate solution demonstrates key features of the problem, in particular, an important large time limit as $\tau \to \infty$ immediately follows from the formulae above. In the sub-Rayleigh regime we have

$$
u_i(\xi,\tau) \sim u_i^\infty(\xi,\tau), \qquad u_i^\infty(\xi,\tau) = u_i^{st}(\xi) + u_i^r(\tau) \qquad (i = 1, 2), \tag{4.2.13}
$$

where u_i^{st} $(i = 1, 2)$ are displacements in the related steady-state problem (4.1.8), and

$$u_1^r(\tau) = u_1^{r0}, \qquad u_2^r(\tau) = u_2^{r0} + u_2^{r\tau}(\tau), \tag{4.2.14}$$

with

$$\begin{aligned} u_1^{r0} &= \frac{B_1 v \left(1 - \beta_R^2\right)}{2c_2(v_R^2 - v^2)}, \\ u_2^{r0} &= -\frac{B_1 \alpha_R (1 - \beta_R^2)}{\pi c_2 (1 + \beta_R^2)} \left[\frac{\ln(v_R + v)}{v_R + v} + \frac{\ln |v_R - v|}{v_R - v} \right], \\ u_2^{r\tau}(\tau) &= -\frac{2 B_1 v_R \alpha_R (1 - \beta_R^2)}{\pi c_2 (v_R^2 - v^2)(1 + \beta_R^2)} \ln \tau. \end{aligned} \tag{4.2.15}$$

Here u_i^r $(i = 1, 2)$ are components of the rigid body motion of the half-plane. It is remarkable that the rigid body motion along the vertical axe demonstrates a logarithmic growth in time, see (4.2.14) and (4.2.15), observed earlier in Kaplunov (1986). This means that a steady-state regime in subsection 4.1 cannot be achieved at a large time limit.

The formulae (4.2.13)-(4.2.15) are also valid for the super-Rayleigh case, except the expression for the rigid body motion component along the horizontal axe, which now becomes

$$u_1^{r0} = -\frac{B_1 v_R \left(1 - \beta_R^2\right)}{2c_2(v_R^2 - v^2)}. \tag{4.2.16}$$

In the resonant case the limiting behaviour as $\tau \to \infty$ is

$$u_i(\xi, \tau) \sim u_i^\infty(\xi, \tau) \qquad (i = 1, 2), \tag{4.2.17}$$

with

$$u_1^\infty(\xi, \tau) = \frac{B_1 \alpha_R \tau}{\pi c_2} \left[\frac{1}{\xi^2 + \alpha_R^2} - \frac{2\beta_R^2}{(1 + \beta_R^2)(\xi^2 + \beta_R^2)} \right], \tag{4.2.18}$$

$$\begin{aligned} u_2^\infty(\xi, \tau) = {} & \frac{B_1 \alpha_R \xi \tau}{\pi c_2} \left[\frac{2}{(1 + \beta_R^2)(\xi^2 + \beta_R^2)} - \frac{1}{\xi^2 + \alpha_R^2} \right] \\ & + \frac{B_1 \alpha_R (\beta_R^2 - 1)}{4\pi c_2 v_R (\beta_R^2 + 1)} \ln \tau. \end{aligned} \tag{4.2.19}$$

Thus, the displacements demonstrate linear growth in time apart from the vertical displacement at $\xi = 0$, which increases as $\ln \tau$.

Another interesting observation is related to the resonant regime of a moving semi-infinite strip, in which $P = P_0 H(x - c_R t)$. In this case the

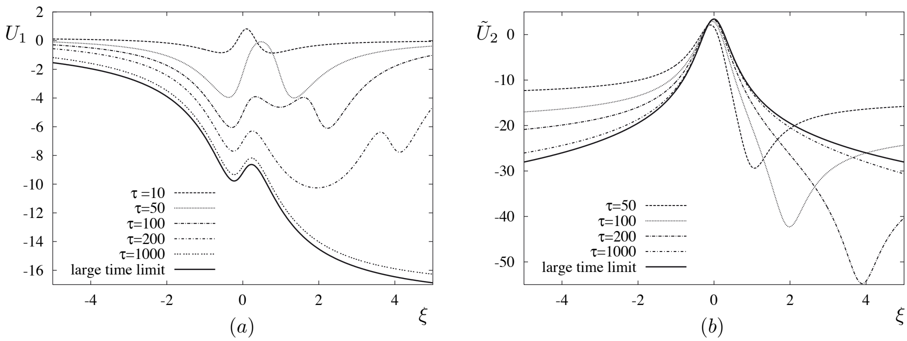

Figure 6. The sub-Rayleigh transient (a) horizontal and (b) vertical displacements (4.2.8) and (4.2.9) and the large time limits (4.2.13) for $v = 0.9$.

asymptotic model recovers the classical result of Goldstein (1965) with less effort.

Numerical illustrations are presented in Fig. 6-8 for the Poisson ratio $\nu = 0.25$ corresponding to $v_R \approx 0.9194$.

We plot the dimensionless displacements

$$U_k = \frac{\pi \mu u_k}{P_0}, \quad \tilde{U}_2 = \frac{\pi \mu}{P_0} \left(u_2(\xi, \tau) - u_2^{r\tau}(\tau) \right).$$

Here we subtract from the vertical displacement $u_2(\xi, \tau)$ the function $u_2^{r\tau}(\tau)$ having a logarithmic growth in time, see (4.2.14) and (4.2.15). In this case we depict only a bounded in time component in order to show convergence at a large time limit.

The sub-Rayleigh displacements of the half-space (4.2.8) and (4.2.9) are plotted in Fig. 6 for $v = 0.9$ and several values of time τ. Similar results for the super-Rayleigh regime ($v = 0.95$) are presented in Fig. 7. The solid line corresponds to the limits (4.2.13) with (4.2.14)–(4.2.15) and (4.2.16). As might be expected, transient displacements tend to their large time values as time increases. The resonant displacements (4.2.11) and (4.2.12) are displayed in Fig. 8 for $\tau = 10$, 30, 50 and 100. They demonstrate a linear growth in time according to the formulae (4.2.11) and (4.2.12).

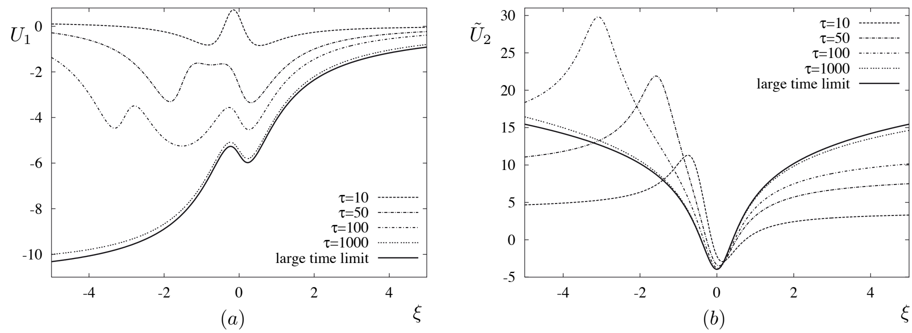

Figure 7. The super-Rayleigh transient (a) horizontal and (b) vertical displacements (4.2.8) and (4.2.9) and the large time limits (4.2.13) for $v = 0.95$.

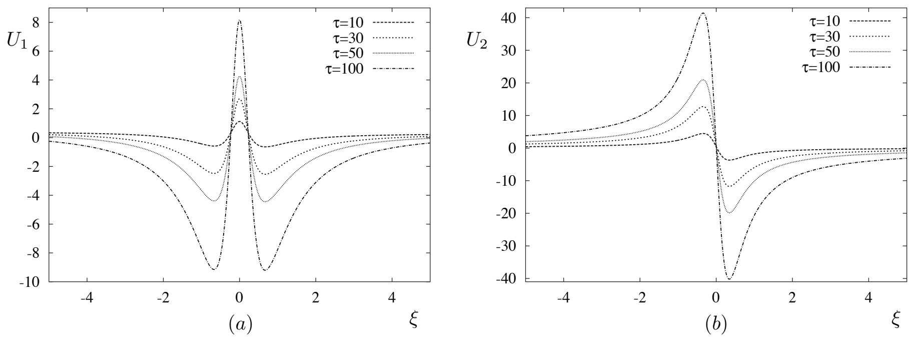

Figure 8. The resonant transient (a) horizontal and (b) vertical displacements (4.2.11) and (4.2.12) for $v = v_R$.

4.3 Steady-state motion of a stamp

Consider the steady-state motion of a rigid stamp, see Fig. 9, assuming

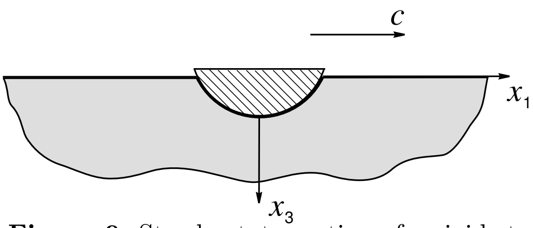

Figure 9. Steady-state motion of a rigid stamp

that its effect results in a prescribed surface displacements $U_3(x_1, t) = f(x_1 - ct)$. We also set $P = 0$ in the equation (2.5.4) governing the surface motion outside the stamp. The formulation of the mixed boundary value problem, obtained in subsection 2.5, may then be specified for the scaled normal derivative

$$\chi(s,p) = \frac{\beta_R^2 - 1}{\beta_R^2 + 1}\frac{\partial \phi}{\partial s}, \tag{4.3.1}$$

where $s = x_1 - ct$, $p = \alpha_R x_3$, for more details see Erbaş *et al.* (2012). Thus, we arrive at a canonical problem for the Laplace equation

$$\frac{\partial^2 \chi}{\partial p^2} + \frac{\partial^2 \chi}{\partial s^2} = 0, \tag{4.3.2}$$

with the mixed boundary conditions ($p = 0$)

$$\chi = f(s), \quad \text{at } \ s \in S_2 \tag{4.3.3}$$

and

$$\frac{\partial \chi}{\partial p} = 0, \quad \text{at } \ s \in S_1, \tag{4.3.4}$$

where and S_1 and S_2 are the traction free and constrained parts of the surface $p = 0$, respectively.

As an example, we consider an exponential stamp $f(s) = be^{-as}$, where a and b are positive constants. In this case (e.g. see Sveshnikov & Tikhonov 1978)

$$\chi(s,p) = b\mathrm{Re}\left\{e^{-aq}\left[1 - \mathrm{erf}\left(\sqrt{-aq}\right)\right]\right\}, \tag{4.3.5}$$

where $q = s + ip$ and $\mathrm{erf}(q)$ is the error function.

A resonant nature of the Rayleigh wave is clearly seen from the formula for the normal stress under the stamp

$$\sigma_{33}(s,0) = \frac{2\mu B\alpha_R^3 \left(c^2 - c_R^2\right)}{(\beta_R^2 - 1)c_R^2}\,\chi(s,0)\,, \quad \text{at } \ s \in S_2. \tag{4.3.6}$$

Thus, the resonant limit as $c \to c_R$, corresponds to an asymptotically vanishing stress induced by a displacement of finite magnitude.

4.4 Moving load on a coated half-plane

Let a coated half-plane be loaded by a distributed force of the form (see Fig. 10)

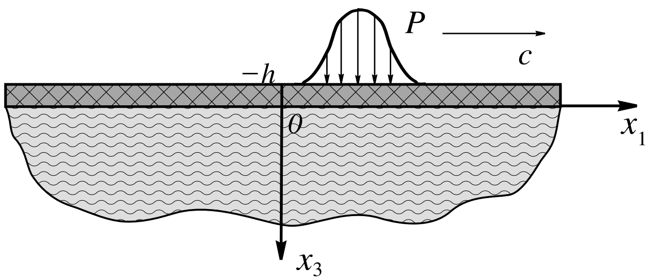

Figure 10. Distributed moving load on a coated half-plane.

$$P(x_1,t) = \frac{P_0 l}{\pi\left[l^2 + s^2\right]},$$

where l is a typical length, and $s = x_1 - ct$. We restrict ourselves to analysis of the surface motion ($x_3 = 0$), governed by the perturbed hyperbolic equation (2.6.8). We introduce the dimensionless moving coordinate $s_l = s/l$ along with the parameters

$$g = 1 - \frac{c^2}{c_R^2}, \qquad h_l = \frac{bh}{l} \tag{4.4.1}$$

and rewrite (2.6.8) as

$$g\theta - h_l\sqrt{-\frac{\partial^2}{\partial s_l^2}}\theta = \frac{1}{1+s_l^2}, \tag{4.4.2}$$

where

$$\theta = \frac{AP_0}{\pi l}\frac{\partial^2 \phi}{\partial s_l^2}$$

denotes scaled normal surface stresses, here and below in this subsection see Dai *et al.* (2010) for further detail.

We remark that g and h_l are key problem parameters characterising the thickness of the coating and the proximity of the speed of the load to the Rayleigh wave speed, respectively. Then, using the Fourier transform, we present the solution of (4.4.2) as

$$\theta = -\frac{1}{2h_l}\sum_{n=1}^{2} e^{q_n}\,\mathrm{Ei}(1, q_n) \tag{4.4.3}$$

where

$$q_n = -\frac{g}{h_l}\left[1 + (-1)^n i s_l\right], \qquad n = 1, 2,$$

and Ei is the integral exponent.

Two limiting cases may then be investigated. The first limit corresponds to the solution for an uncoated half-plane as $h_l/g \to 0$, whereas the second one $g/h_l \to 0$ reveals that the presence of a coating does not remove the resonance at $c = c_R$. The reason is that, despite of the dispersion due to the influence of the coating, the maximum or minimum of the phase speed is still given by the Rayleigh wave speed.

5 Edge bending wave

In this section we apply the proposed philosophy to the bending wave propagating along the edge of a semi-infinite thin elastic plate. We show that the dispersive edge bending wave has a *parabolic-elliptic* duality in contrast to a *hyperbolic-elliptic* duality of the non-dispersive surface and interfacial waves considered above.

5.1 Dispersion relation

Let the geometry of the plate of thickness $2h$ be given by $-\infty < x_1 < \infty$, $0 \le x_2 < \infty$, $-h \le x_3 \le h$, see Fig. 11. We start from the approximate 2D equation in the classical Kirchhoff theory for plate bending, given by

$$D\Delta^2 W + 2\rho h \frac{\partial^2 W}{\partial t^2} = 0, \tag{5.1.1}$$

where $W(x_1, x_2, t)$ is the deflection of the plate, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, and

$$D = \frac{2Eh^3}{3\left(1 - \nu^2\right)} \tag{5.1.2}$$

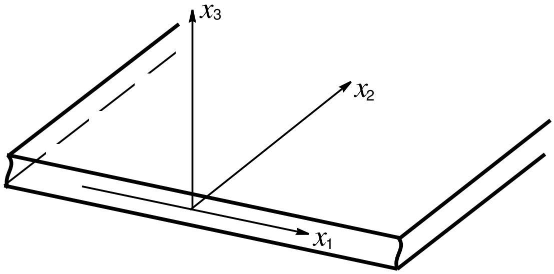

Figure 11. Thin elastic plate.

is bending stiffness, with E and ν denoting the Young modulus and the Poisson ratio, respectively.

The boundary conditions at the edge $x_2 = 0$ can be written as

$$\begin{aligned} &\frac{\partial^2 W}{\partial x_2^2} + \nu \frac{\partial^2 W}{\partial x_1^2} = -\frac{M}{D}, \\ &\frac{\partial^3 W}{\partial x_2^3} + (2-\nu)\frac{\partial^3 W}{\partial x_1^2 \partial x_2} = -\frac{N}{D}, \end{aligned} \tag{5.1.3}$$

where $M = M(x_1, t)$ and $N = N(x_1, t)$ are prescribed bending moment and shear force, respectively.

The travelling wave solution of (5.1.1) may be found in the form

$$W(x_1, x_2, t) = \sum_{j=1}^{2} C_j e^{i(kx_1 - \omega t) - k\lambda_j x_2}, \tag{5.1.4}$$

where

$$\lambda_j = \sqrt{1 + (-1)^j \sqrt{\frac{2\rho h}{D}} \frac{\omega}{k^2}}, \qquad j = 1, 2. \tag{5.1.5}$$

Substitution of (5.1.4) into the homogeneous edge boundary conditions ($M = N = 0$ in (5.1.3)) leads to the dispersion relation

$$Dk^4 \gamma_e^4 = 2\rho h \omega^2, \tag{5.1.6}$$

originating from Konenkov (1960) and subsequent contributions, see also Lawrie & Kaplunov (2012) and Norris *et al.* (2000) and references therein. Here the coefficient

$$\gamma_e = \left[(1-\nu)\left(3\nu - 1 + 2\sqrt{2\nu^2 - 2\nu + 1}\right)\right]^{1/4} \tag{5.1.7}$$

depends on the Poisson ratio only. In view of the dispersion relation (5.1.6), we have

$$\lambda_j = \lambda_{j0} = \sqrt{1 + (-1)^j \gamma_e^2}, \qquad j = 1, 2. \tag{5.1.8}$$

5.2 Edge bending wave of arbitrary profile

Here we generalise the travelling wave solution to that expressed through arbitrary plane harmonic function. The equation (5.1.1), written in terms of the dimensionless variables

$$\zeta_i = \frac{x_i}{h}, \qquad t_h = \frac{t}{h}\sqrt{\frac{E}{3\rho(1-\nu^2)}}, \qquad i = 1, 2 \tag{5.2.1}$$

becomes

$$\Delta^2 W + \frac{\partial^2 W}{\partial t_h^2} = 0, \tag{5.2.2}$$

where $\Delta = \dfrac{\partial^2}{\partial \zeta_1^2} + \dfrac{\partial^2}{\partial \zeta_2^2}$.

Let us assume that

$$\gamma^4 \frac{\partial^4 W}{\partial \zeta_1^4} + \frac{\partial^2 W}{\partial t_h^2} = 0, \tag{5.2.3}$$

where γ is a dimensionless parameter. This is a key assumption, leading below to transformation of the parabolic equation (5.2.2) to an elliptic equation and finally resulting in the sought for representation in terms of a plane harmonic function. The philosophy underlying (5.2.3) essentially mirrors that of subsection 2.3 (see Chadwick 1976), where the surface wave solution was obtained in the form of a travelling wave of arbitrary profile. Indeed, while an elastic string seems to be a right 1D object for understanding surface wave propagation, see the classical wave equation (2.4.15), its counterpart for the edge bending wave is a beam.

The equation (5.2.2) then becomes

$$\left(1 - \gamma^4\right) \frac{\partial^4 W}{\partial \zeta_1^4} + 2\frac{\partial^4 W}{\partial \zeta_1^2 \partial \zeta_2^2} + \frac{\partial^4 W}{\partial \zeta_2^4} = 0. \tag{5.2.4}$$

It also may be expressed in an operator form as

$$\Delta_1 \Delta_2 W = 0, \tag{5.2.5}$$

where

$$\Delta_j = \frac{\partial^2}{\partial \zeta_2^2} + \lambda_j^2 \frac{\partial^2}{\partial \zeta_1^2}, \qquad j = 1, 2, \tag{5.2.6}$$

and

$$\lambda_j^2 = 1 + (-1)^j \gamma^2. \tag{5.2.7}$$

The solution of (5.2.5) is expressed through two decaying as $\zeta_2 \to \infty$ plane harmonic functions W_j as

$$W = \sum_{j=1}^{2} W_j\left(\zeta_1, \lambda_j \zeta_2, t_h\right). \tag{5.2.8}$$

Let us substitute the latter into the homogeneous boundary conditions (5.1.3) rewritten in terms of dimensionless variables, using the Cauchy-Riemann identities (2.3.4). The result is

$$\begin{aligned}
&\left(\nu - \lambda_1^2\right) \frac{\partial^2 W_1}{\partial \zeta_1^2} + \left(\nu - \lambda_2^2\right) \frac{\partial^2 W_2}{\partial \zeta_1^2} = 0, \\
&\lambda_1 \left(\lambda_1^2 - 2 + \nu\right) \frac{\partial^3 W_1}{\partial \zeta_1^3} + \lambda_2 \left(\lambda_2^2 - 2 + \nu\right) \frac{\partial^3 W_2}{\partial \zeta_1^3} = 0,
\end{aligned} \tag{5.2.9}$$

leading to

$$\lambda_2(\nu - \lambda_1^2)^2 - \lambda_1(\nu - \lambda_2^2)^2 = 0. \tag{5.2.10}$$

Due to (5.2.7), the last relation may be re-cast in the form

$$1 - \gamma^4 - (2\nu - 2)\sqrt{1 - \gamma^4} - \nu^2 = 0. \tag{5.2.11}$$

Then,

$$\gamma^4 = (1 - \nu)\left(3\nu - 1 + 2\sqrt{2\nu^2 - 2\nu + 1}\right) = \gamma_e^4, \tag{5.2.12}$$

which coincides with the root of the dispersion relation (5.1.6) implying $\lambda_j = \lambda_{j0}$.

Similarly to subsection 2.3.1, the harmonic functions W_1 and W_2 may be related to each other. Consequently, a representation in terms of a single harmonic function may be established from the boundary conditions (5.2.9), namely

$$W\left(x_1, x_2, t\right) = W_j\left(x_1, \lambda_{j0} x_2, t\right) - \frac{\nu - \lambda_{j0}^2}{\nu - \lambda_{m0}^2} W_j\left(x_1, \lambda_{m0} x_2, t\right). \tag{5.2.13}$$

where $1 \leq j \neq m \leq 2$.

It is remarkable that (5.2.13) is even simpler than its Rayleigh wave counterpart (Chadwick 1976), since it does not involve harmonic conjugate functions.

5.3 Parabolic-elliptic model

Now we proceed with the development of an explicit model for the edge bending wave. In parallel with subsection 2.4, we perturb the equation (5.1.1) around the edge wave eigensolution constructed in the subsection 5.2. However, this procedure is now less trivial because of a multi-scale behaviour in time. Accordingly, we introduce fast ($\tau_f = t_h$) and slow ($\tau_s = \varepsilon t_h$) time variables, where as before $\varepsilon \ll 1$ is a small parameter. The equation (5.2.2) may now be written in terms of specified two time-scales as

$$\Delta^2 W + \left(\frac{\partial^2 W}{\partial \tau_f^2} + 2\varepsilon \frac{\partial^2 W}{\partial \tau_f \partial \tau_s} + \varepsilon^2 \frac{\partial^2 W}{\partial \tau_s^2} \right) = 0. \tag{5.3.1}$$

The deflection W may be then expanded in an asymptotic series, i. e.

$$W = \frac{h^2}{D} \left(W^{(0)} + \varepsilon W^{(1)} + ... \right). \tag{5.3.2}$$

Next, we substitute the expansion (5.3.2) into the governing equation (5.3.1), having at leading order

$$\Delta^2 W^{(0)} + \frac{\partial^2 W^{(0)}}{\partial \tau_f^2} = 0, \tag{5.3.3}$$

which may be readily transformed to

$$\left(1 - \gamma_e^4\right) \frac{\partial^4 W^{(0)}}{\partial \zeta_1^4} + 2 \frac{\partial^4 W^{(0)}}{\partial \zeta_1^2 \partial \zeta_2^2} + \frac{\partial^4 W^{(0)}}{\partial \zeta_2^4} = 0, \tag{5.3.4}$$

by making use of the assumption (5.2.3) at $\gamma = \gamma_e$. The solution of (5.3.4) is then given by a combination of harmonic functions, yielding

$$W^{(0)} = \sum_{j=1}^{2} W_j^{(0)} \left(\zeta_1, \lambda_{j0} \zeta_2, \tau_f, \tau_s \right), \tag{5.3.5}$$

where the scaling factors λ_{j0} ($j = 1, 2$) are defined by (5.1.8).

At next order we obtain from (5.3.1)

$$\Delta^2 W^{(1)} + \frac{\partial^2 W^{(1)}}{\partial \tau_f^2} + 2 \frac{\partial^2 W^{(0)}}{\partial \tau_f \partial \tau_s} = 0, \tag{5.3.6}$$

which, in view of (5.2.3), may be re-written as

$$\Delta_1 \Delta_2 W^{(1)} = -2 \frac{\partial^2 W^{(0)}}{\partial \tau_f \partial \tau_s}. \tag{5.3.7}$$

For the sake of definiteness, we specify (5.2.3) as

$$i\gamma_e^2 \frac{\partial^2 W_j^{(0)}}{\partial \zeta_1^2} + \frac{\partial W_j^{(0)}}{\partial \tau_f} = 0, \qquad j = 1, 2. \tag{5.3.8}$$

Further analysis of (5.3.7) also requires separate consideration for both plane harmonic functions $W_j^{(1)}$, $j = 1, 2$. Let us first concentrate on $W_1^{(1)}$. Using properties of harmonic functions we deduce that

$$\Delta_2 W_1^{(0)} = (\lambda_{20}^2 - \lambda_{10}^2) \frac{\partial^2 W_1^{(0)}}{\partial \zeta_1^2} = 2\gamma_e^2 \frac{\partial^2 W_1^{(0)}}{\partial \zeta_1^2} = 2i \frac{\partial W_1^{(0)}}{\partial \tau_f}. \tag{5.3.9}$$

Therefore, the equation (5.3.7) may be presented as

$$\Delta_1 W_1^{(1)} = i\, \frac{\partial W^{(0)}}{\partial \tau_s}. \tag{5.3.10}$$

It is convenient now to define the function $\Phi_1^{(1)} = \dfrac{\partial W_1^{(1)}}{\partial \zeta_2}$. Then the equation (5.3.10) is rewritten as

$$\Delta_1 \Phi_1^{(1)} = i\, \frac{\partial^2 W_1^{(0)}}{\partial \zeta_2 \partial \tau_s}. \tag{5.3.11}$$

Similarly to (2.4.6), the solution of (5.3.11) may be found as

$$\Phi_1 = \frac{\partial W_1}{\partial \zeta_2} = \frac{\partial W_1^{(0)}}{\partial \zeta_2} + \varepsilon \left(\Phi_1^{(1,0)} + \frac{1}{2}\, i\zeta_2\, \frac{\partial W_1^{(0)}}{\partial \tau_s} \right) + \ldots \quad . \tag{5.3.12}$$

We also have for W_2

$$\Delta_2 W_2^{(1)} = -i\, \frac{\partial W_2^{(0)}}{\partial \tau_s}, \tag{5.3.13}$$

resulting in

$$\Phi_2 = \frac{\partial W_2}{\partial \zeta_2} = \frac{\partial W_2^{(0)}}{\partial \zeta_2} + \varepsilon \left(\Phi_2^{(1,0)} - \frac{1}{2}\, i\zeta_2\, \frac{\partial W_2^{(0)}}{\partial \tau_s} \right) + \ldots \quad . \tag{5.3.14}$$

Finally, we obtain for the normal derivative

$$\begin{aligned} \frac{\partial W}{\partial \zeta_2} = \frac{h^2}{D} \Bigg[& \frac{\partial \left(W_1^{(0)} + W_2^{(0)} \right)}{\partial \zeta_2} \\ & + \varepsilon \left(\Phi_1^{(1,0)} + \Phi_2^{(1,0)} + i \frac{\zeta_2}{2}\, \frac{\partial \left(W_1^{(0)} - W_2^{(0)} \right)}{\partial \tau_s} \right) + \ldots \Bigg]. \end{aligned} \tag{5.3.15}$$

Now we are in position to treat the non-homogeneous boundary conditions (5.1.3). As above in 2.4.1, the problem may be decomposed into two separate sub-problems involving a prescribed edge bending moment or shear force only. First, we study the effect of an edge bending moment normalized as $M = \varepsilon M_\varepsilon$. The boundary conditions rewritten in terms of dimensionless coordinates, are

$$\begin{aligned} &\frac{\partial^2 W}{\partial \zeta_2^2} + \nu \frac{\partial^2 W}{\partial \zeta_1^2} = -\frac{\varepsilon h^2}{D} M_\varepsilon, \\ &\frac{\partial^3 W}{\partial \zeta_2^3} + (2-\nu)\,\frac{\partial^3 W}{\partial \zeta_1^2 \partial \zeta_2} = 0. \end{aligned} \tag{5.3.16}$$

On substituting the asymptotic expansion (5.3.15) into the latter we obtain at leading order

$$\begin{aligned} &\left(\nu - \lambda_{10}^2\right) \frac{\partial^2 W_1^{(0)}}{\partial \zeta_1^2} + \left(\nu - \lambda_{20}^2\right) \frac{\partial^2 W_2^{(0)}}{\partial \zeta_1^2} = 0, \\ &\lambda_{10}\left(\lambda_{10}^2 - 2 + \nu\right) \frac{\partial^3 W_1^{(0)}}{\partial \zeta_1^3} + \lambda_{20}\left(\lambda_{20}^2 - 2 + \nu\right) \frac{\partial^3 W_2^{(0)}}{\partial \zeta_1^3} = 0, \end{aligned} \tag{5.3.17}$$

which is an analogue of (5.2.9). It results in the dispersion relation (5.2.10), implying $\lambda_j = \lambda_{j0}$, $j = 1, 2$, see also (5.1.8).

At next order, the boundary conditions (5.3.16) are given by

$$\begin{aligned} &\frac{\partial^2 W^{(1)}}{\partial \zeta_2^2} + \nu \frac{\partial^2 W^{(1)}}{\partial \zeta_1^2} = -M_\varepsilon, \\ &\frac{\partial^3 W^{(1)}}{\partial \zeta_2^3} + (2-\nu)\,\frac{\partial^3 W^{(1)}}{\partial \zeta_1^2 \partial \zeta_2} = 0. \end{aligned} \tag{5.3.18}$$

The relations (5.3.10) and (5.3.13) may be used to deduce that

$$\left.\frac{\partial^2 W^{(1)}}{\partial \zeta_1^2}\right|_{\zeta_2=0} = \frac{1}{\lambda_{10}^2}\left(\frac{i}{2}\frac{\partial W_1^{(0)}}{\partial \tau_s} - \frac{\partial \Phi_1^{(1,0)}}{\partial \zeta_2}\right) - \frac{1}{\lambda_{20}^2}\left(\frac{i}{2}\frac{\partial W_2^{(0)}}{\partial \tau_s} + \frac{\partial \Phi_2^{(1,0)}}{\partial \zeta_2}\right). \tag{5.3.19}$$

The boundary conditions (5.3.18) taking into account (5.3.15) and (5.3.19), become

$$\left(1-\frac{\nu}{\lambda_{10}^2}\right)\frac{\partial\Phi_1^{(1,0)}}{\partial\zeta_2}+\left(1-\frac{\nu}{\lambda_{20}^2}\right)\frac{\partial\Phi_2^{(1,0)}}{\partial\zeta_2}$$
$$+\frac{i}{2}\left(1+\frac{\nu}{\lambda_{10}^2}\right)\frac{\partial W_1^{(0)}}{\partial\tau_s}-\frac{i}{2}\left(1+\frac{\nu}{\lambda_{20}^2}\right)\frac{\partial W_2^{(0)}}{\partial\tau_s}=-M_\varepsilon,$$
$$\left(2-\nu-\lambda_{10}^2\right)\frac{\partial^2\Phi_1^{(1,0)}}{\partial\zeta_1^2}+\left(2-\nu-\lambda_{20}^2\right)\frac{\partial^2\Phi_2^{(1,0)}}{\partial\zeta_1^2}$$
$$+i\frac{\partial^2 W_1^{(0)}}{\partial\zeta_2\partial\tau_s}-i\frac{\partial^2 W_2^{(0)}}{\partial\zeta_2\partial\tau_s}=0. \qquad (5.3.20)$$

Finally, we obtain using also the formula (5.2.13),

$$\frac{2i\gamma_e^2}{Q}\frac{\partial^2 W^{(0)}}{\partial\zeta_1\partial\tau_s}=-\frac{\partial M_\varepsilon}{\partial\xi}, \qquad (5.3.21)$$

where

$$Q=\frac{\eta\left(\nu+\eta\right)}{1-\nu+\eta}, \qquad (5.3.22)$$

with

$$\eta=\lambda_{10}\lambda_{20}=\sqrt{1-\gamma_e^4}. \qquad (5.3.23)$$

Here the coefficient Q depends on the Poisson ratio only, see Fig. 12.

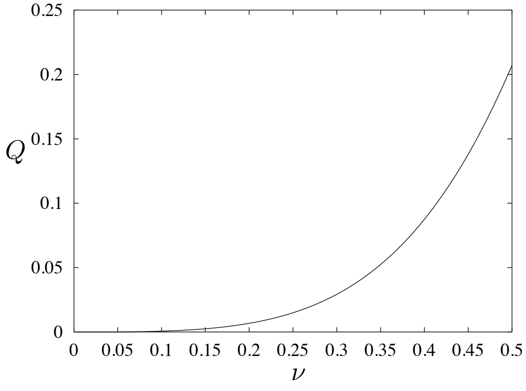

Figure 12. Coefficient Q vs. the Poisson ratio ν

The equation (5.3.21) enables calculation of the edge wave contribution to the overall dynamic response. This observation also follows from a similarity of the developed perturbation procedure and the routine in Kaplunov et al. (2013a) relying on computation of the residues corresponding to edge wave poles.

The operator relationship

$$\frac{\partial}{\partial \tau_s} = \varepsilon^{-1}\left(i\gamma_e^2 \frac{\partial^2}{\partial \zeta_1^2} + \frac{\partial}{\partial t_h}\right), \tag{5.3.24}$$

along with the condition (5.3.8), allow transformation of (5.3.21) to a parabolic equation at the edge $\zeta_2 = 0$. It is given by

$$\gamma_e^4 \frac{\partial^4 W^{(0)}}{\partial \zeta_1^4} + \frac{\partial^2 W^{(0)}}{\partial t_h^2} = Q \frac{\partial^2 M}{\partial \zeta_1^2}. \tag{5.3.25}$$

The equation (5.3.25) (see (5.3.2)) may be re-cast in terms of original variables as ($x_2 = 0$)

$$D\gamma_e^4 \frac{\partial^4 W}{\partial x_1^4} + 2\rho h \frac{\partial^2 W}{\partial t^2} = Q \frac{\partial^2 M}{\partial x_1^2}. \tag{5.3.26}$$

The established approximate formulation also contains the elliptic equation

$$\Delta_1 \Delta_2 W = 0, \tag{5.3.27}$$

where

$$\Delta_j = \frac{\partial^2}{\partial x_2^2} + \lambda_{j0}^2 \frac{\partial^2}{\partial x_1^2}, \qquad j = 1, 2, \tag{5.3.28}$$

which should be solved together with the parabolic equation (5.3.26).

In fact, the representation in terms of a single harmonic function (5.2.13) simplifies things even further since

$$W(x, 0, t) = \frac{\lambda_{i0}^2 - \lambda_{j0}^2}{\nu - \lambda_{j0}^2} W_i(x, 0, t), \quad 1 \le i \ne j \le 2. \tag{5.3.29}$$

The explicit model for the edge bending wave is then given by a Dirichlet problem for any of the following two pseudo-static elliptic equations

$$\frac{\partial^2 W_j}{\partial y^2} + \lambda_{j0}^2 \frac{\partial^2 W_j}{\partial x^2} = 0, \qquad (j = 1, 2) \tag{5.3.30}$$

with the boundary data originating from the parabolic equation (5.3.26). Then, we exploit the relations (5.2.13) and (5.3.29) to restore the overall 2D bending field. Thus, we reveal a dual *parabolic-elliptic nature* of the studied wave.

The second type of boundary conditions is given by

$$\begin{aligned} &\frac{\partial^2 W}{\partial \zeta_2^2} + \nu \frac{\partial^2 W}{\partial \zeta_1^2} = 0, \\ &\frac{\partial^3 W}{\partial \zeta_2^3} + (2-\nu)\,\frac{\partial^3 W}{\partial \zeta_1^2 \partial \zeta_2} = -\frac{h^3}{D} N, \end{aligned} \tag{5.3.31}$$

leading to a parabolic beam-like equation. The analysis is rather similar to that presented in the previous case. Remarkably, now the boundary conditions lead to a parabolic equation for the rotation angle $\theta = \dfrac{\partial W}{\partial x_2}$ evaluated at the edge $x_2 = 0$, namely

$$D\gamma_e^4 \frac{\partial^4 \theta}{\partial x_1^4} + 2\rho h \frac{\partial^2 \theta}{\partial t^2} = -Q \frac{\partial^2 N}{\partial x_1^2}, \tag{5.3.32}$$

with the coefficient Q defined by (5.3.22).

The explicit model for a prescribed shear force contains the elliptic equation (5.3.30) which is to be solved in conjunction with the parabolic equation (5.3.32), and also the relations (5.2.13) and (5.3.29) as above.

6 Concluding remarks

The context of this chapter is restricted to the framework of linear isotropic elasticity. We expect various extensions of the developed asymptotic methodology to elastic solids demonstrating a more sophisticated constitutive behaviour, arising from numerous insights into the properties of surface, interfacial and edge waves, taking into consideration pre-stress (Dowaikh & Ogden 1990, 1991, Rogerson & Sandiford 1999, Pichugin & Rogerson 2012) and anisotropy (Fu 2003, 2005, Destrade 2004, 2007, Zakharov 2004, Norris 1994), see also Prikazchikov 2013. The illustrative examples presented in Section 4 are limited to plane moving load problems associated with the Rayleigh wave. There is a clear potential for 3D generalisations (Kaplunov *et al.* 2013b) and also for analysis of near-interfacial dynamics (Kennedy & Herrmann 1973a,b). In addition, we mention important industrially motivated problems involving viscoelastic coatings.

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