Multiple Zeta Values and Modular Forms in Quantum Field Theory

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Abstract This article introduces multiple zeta values and alternating Euler sums, exposing some of the rich mathematical structure of these objects and indicating situations where they arise in quantum field theory. Then it considers massive Feynman diagrams whose evaluations yield polylogarithms of the sixth root of unity, products of elliptic integrals, and L-functions of modular forms inside their critical strips.

1 Sums and Nested Sums

We begin by generalizing the single sum of a zeta value to the nested sum that defines a multiple zeta value (MZV) [1–4].

1.1 Zeta Values

For integer $s > 1$, the zeta values

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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divide themselves into two radically different classes. At *even* integers we have

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{\pi^6}{945}$$

$$\zeta(8) = \frac{\pi^8}{9{,}450}$$

$$\zeta(10) = \frac{\pi^{10}}{93{,}555}$$

and hence integer relations such as

$$5\zeta(4) - 2\zeta^2(2) = 0. \tag{1}$$

Yet no such relations have been found for *odd* arguments.

To prove (1), consider the wonderful formula

$$\frac{\cos(z)}{\sin(z)} = \sum_{n=-\infty}^{\infty} \frac{1}{z - n\pi}$$

in which the cotangent function is given by the sum of its pole terms, each with unit residue. Multiplying by z, to remove the singularity at $z = 0$, and then combining the terms with positive and negative n, we obtain

$$\frac{z\cos(z)}{\sin(z)} = 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - z^2}.$$

Expanding about $z = 0$ we obtain

$$\frac{1 - z^2/2! + z^4/4! + O(z^6)}{1 - z^2/3! + z^4/5! + O(z^6)} = 1 - 2\,\zeta(2)\frac{z^2}{\pi^2} - 2\,\zeta(4)\frac{z^4}{\pi^4} + O(z^6)$$

and easily prove that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$.

1.2 Double Sums

For integers $a > 1$ and $b > 0$, let

$$\zeta(a,b) = \sum_{m>n>0} \frac{1}{m^a n^b}$$

which is a multiple zeta value (MZV) with weight $a+b$ and depth 2. Then, when a and b are both greater than 1, the double sum in the product

$$\zeta(a)\zeta(b) = \sum_{m>0} \frac{1}{m^a} \sum_{n>0} \frac{1}{n^b}$$

can be split into three terms, with $m > n > 0$, $m = n > 0$ and $n > m > 0$. Hence

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(a+b) + \zeta(b,a). \tag{2}$$

There are further algebraic relations. Consider the sums

$$T(a,b,c) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(j+k)^a j^b k^c}.$$

Multiplying the numerator by $(j+k) - j - k = 0$ we obtain

$$0 = T(a-1,b,c) - T(a,b-1,c) - T(a,b,c-1)$$

and hence by repeated application of

$$T(a,b,c) = T(a+1,b-1,c) + T(a+1,b,c-1)$$

we may reduce these Tornheim double sums [5] to MZVs. For example

$$T(1,1,1) = 2\zeta(2,1).$$

We also have

$$T(1,1,1) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(j+k)jk} = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{j+k} \right).$$

But now the inner sum has only j terms and hence

$$T(1,1,1) = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{n=1}^{j} \frac{1}{n} = \zeta(2,1) + \zeta(3).$$

Comparing the two results for $T(1,1,1)$, we find that

$$\zeta(2,1) = \zeta(3).$$

More generally, for $a > 1$, Euler found that

$$\zeta(a,1) = \frac{a}{2}\zeta(a+1) - \frac{1}{2}\sum_{b=2}^{a-1}\zeta(a+1-b)\zeta(b). \tag{3}$$

Moreover, Euler found the evaluation of all MZVs with odd weight and depth 2. For odd $a > 1$ and even $b > 0$ we have

$$\begin{aligned}\zeta(a,b) = &-\frac{1+C(a,b,a+b)}{2}\zeta(a+b)\\ &+\sum_{k=1}^{(a+b-3)/2} C(a,b,2k+1)\zeta(a+b-2k-1)\zeta(2k+1)\end{aligned} \tag{4}$$

where

$$C(a,b,c) = \binom{c-1}{a-1} + \binom{c-1}{b-1}.$$

For example, we obtain

$$\begin{aligned}\zeta(3,2) &= -\frac{11}{2}\zeta(5) + \frac{\pi^2}{2}\zeta(3)\\ \zeta(2,3) &= \zeta(2)\zeta(3) - \zeta(5) - \zeta(3,2)\\ &= \frac{9}{2}\zeta(5) - \frac{\pi^2}{3}\zeta(3)\end{aligned}$$

using (4) and (2).

With weight $w = a+b < 8$ there is only one double sum $\zeta(a,b)$ not covered by Euler's explicit formulas, namely

$$\zeta(4,2) = \zeta^2(3) - \frac{4}{3}\zeta(6)$$

with an evaluation whose proof will be considered later.

To obtain such evaluations by empirical methods, you may use the `EZFace` interface[1] which supports the `lindep` function of `Pari-GP`. For example, the input

```
lindep([z(4,2),z(3)^2,z(6)])
```

produces the output

```
3., -3., 4.
```

[1] http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi

corresponding to the integer relation

$$3\zeta(4,2) - 3\zeta^2(3) + 4\zeta(6) = 0.$$

At weight $w = 8$, it appears that $\zeta(5,3)$ cannot be reduced to zeta values and their products, though we have no way of proving that such a reduction cannot exist. We cannot even prove that $\zeta(3)/\pi^3$ is irrational. I shall take $\zeta(5,3)$ as an (empirically) irreducible MZV of weight 8 and depth 2. Then all other double sums of weight 8 may be reduced to $\zeta(5,3)$ and zeta values. For example,

$$20\zeta(6,2) = 40\zeta(5)\zeta(3) - 8\zeta(5,3) - 49\zeta(8).$$

It is proven that the number of irreducible double sums of even weight $w = 2n$ is no greater than $\lceil n/3 \rceil - 1$. Up to weight $w = 12$, we may take the irreducible double sums to be $\zeta(5,3)$, $\zeta(7,3)$ and $\zeta(9,3)$. Later we shall see that the proven reduction

$$\zeta(7,5) = \frac{14}{9}\zeta(9,3) + \frac{28}{3}\zeta(7)\zeta(5) - \frac{24,257\pi^{12}}{2,298,646,350} \qquad (5)$$

sets us a puzzle. There is only one irreducible MZV with weight 12 and depth 2.

1.3 Triple Sums

The first MZV of depth 3 that has not been reduced to MZVs of lesser depth (and their products) occurs at weight 11. It is proven that

$$\zeta(a,b,c) = \sum_{l>m>n>0} \frac{1}{l^a m^b n^c}$$

is reducible when the weight $w = a + b + c$ is even or less than 11. I conjectured that all MZVs of depth 3 are expressible in terms of $\mathbf{Q}$-linear combinations of the set

$$\mathscr{B}_3 = \{\zeta(2a+1, 2b+1, 2c+1) | a \geq b \geq c,\ a > c\}$$

together with double sums, $\zeta(a,b)$, single sums, $\zeta(c)$, and their products. This was borne out by investigations with Borwein and Girgensohn [6] and more recently with Blümlein and Vermaseren in [7], with the associated MZV `DataMine`[2] providing strong evidence for many of the claims made in this article. The conjecture

[2] http://www.nikhef.nl/~form/datamine/

implies that the number of irreducible MZVs of weight $w = 2n + 3$ and depth 3 is $\lceil n^2/12 \rceil - 1$, with the sequence

$$1, 2, 2, 4, 5, 6, 8, 10, 11, 14, 16, 18, 21, 24, 26, 30$$

giving the numbers for odd weights from 11 to 41.

1.4 A Quadruple Sum

The mystery of MZVs really begins here. At weight 12 there first appears a quadruple sum that has not been reduced to MZVs with depths less than 4. In the `DataMine` we take this to be

$$\zeta(6,4,1,1) = \sum_{k>l>m>n>0} \frac{1}{k^6 l^4 mn}$$

and prove, by exhaustion, that the following methods are insufficient to reduce it.

2 Shuffles, Stuffles and Duality for MZVs

Next we consider the sources of relations between MZVs.

2.1 Shuffles of Words

For integers $s_j > 0$ and $s_1 > 1$, the MZV

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1>n_2>\ldots>n_k>0} \frac{1}{n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k}}$$

may be encoded by a word of length $w = \sum_{j=1}^{k} s_j$ in the two letter alphabet (A, B), as follows. We write A, $s_1 - 1$ times, then B, then A, $s_2 - 1$ times, then B, and so on, until we end with B. For example

$$\zeta(5,3) = Z(AAAABAAB)$$
$$\zeta(6,4,1,1) = Z(AAAAABAAABBB)$$

where the function Z takes a word as it argument and evaluates to the corresponding MZV. Note that the word must begin with A and end with B. The weight of the MZV is the length of the word and the depth is the number of B's in the word.

We may evaluate the MZV from an iterated integral defined by its word. For example

$$\zeta(2,1) = Z(ABB) = \int_0^1 \frac{\mathrm{d}x_1}{x_1} \int_0^{x_1} \frac{\mathrm{d}x_2}{1-x_2} \int_0^{x_2} \frac{\mathrm{d}x_3}{1-x_3} \tag{6}$$

where we use the differential form $\mathrm{d}x/x$ whenever we see the letter A and the differential form $\mathrm{d}x/(1-x)$ whenever we see the letter B. Then the equality of the nested sum $\zeta(2,1)$ with the iterated integral $Z(ABB)$ follows from binomial expansion of $1/(1-x_2)$ and $1/(1-x_3)$ in (6).

The shuffle algebra of MZVs is the identity

$$Z(U)Z(V) = \sum_{W \in \mathscr{S}(U,V)} Z(W) \tag{7}$$

where $\mathscr{S}(U,V)$ is the set of words obtained by all permutations of the letters of UV that preserve the order of letters in U and the order of letters in V. For example, suppose that $U = ab$ and $V = xy$. Then $\mathscr{S}(U,V)$ consists of the words

$$\mathscr{S}(ab,xy) = \{abxy, axby, xaby, axyb, xayb, xyab\}\,.$$

The only legal two-letter word is AB. Hence setting $a = x = A$ and $b = y = B$ we obtain

$$Z(AB)Z(AB) = 2Z(ABAB) + 4Z(AABB)$$

which shows that

$$\zeta^2(2) = 2\zeta(2,2) + 4\zeta(3,1).$$

2.2 *Stuffles of Nested Sums*

We also have the "stuffle" identity

$$\zeta(2)\zeta(2) = \zeta(2,2) + \zeta(4) + \zeta(2,2)$$

from shuffling the arguments in a product of zetas and adding in the extra "stuff" that originates when summation variables are equal. Hence we conclude that $\zeta(3,1) = \frac{1}{4}\zeta(4)$. The evaluation $\zeta(2,2) = \frac{3}{4}\zeta(4)$ requires the extra piece of information $\zeta^2(2) = \frac{5}{2}\zeta(4)$ obtained from expanding the cotangent function.

Like the shuffle algebra, the stuffle algebra can be used to express any product of MZVs as a sum of MZVs. For example

$$\zeta(3,1)\zeta(2) = \zeta(3,1,2) + \zeta(3,3) + \zeta(3,2,1) + \zeta(5,1) + \zeta(2,3,1).$$

2.3 Duality

By combining shuffles, stuffles and reductions of $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$ to powers of π^2 we may prove that

$$Z(AAABAB) = \zeta(4,2) = \zeta^2(3) - \frac{4}{3}\zeta(6).$$

Moreover, we obtain the same value for the depth-4 MZV

$$Z(ABABBB) = \zeta(2,2,1,1)$$

since $Z(W) = Z(\tilde{W})$, where the dual $\tilde{W}$ of a word W is obtained by writing it backwards and then exchanging A and B. This duality was observed by Zagier. It follows from the transformation $x \to 1-x$ in the iterated integral, which exchanges the differential forms $\mathrm{d}x/x$ and $\mathrm{d}x/(1-x)$ and reverses the ordering of the integrations. Hence

$$\zeta(2,3,1) = Z(ABAABB) = Z(AABBAB) = \zeta(3,1,2).$$

2.4 Conjectured Enumeration of Irreducible MZVs

Thus we arrive at a well-defined question: for a given weight $w > 2$ and a given depth $d > 0$, what is rank-deficiency $D_{w,d}$ of all the algebraic relations that follow from the shuffle and stuffle algebras algebras of MZVs, combined with duality and the reduction of even zeta values to powers of π^2? Note that $D_{w,d}$ is an upper limit for the number of irreducible MZVs at this weight and depth. There may conceivably (but rather improbably) be fewer, since we cannot rule out the possibility of additional integer relations. We cannot even prove that $\zeta(3)/\pi^3$ is irrational.

In 1996, Dirk Kreimer and I conjectured [8] that the answer to this question is given by the generating function

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{D_{w,d}} \stackrel{?}{=} 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)} \tag{8}$$

Table 1 Number of basis elements for MZVs as a function of weight and depth in a minimal depth representation. Underlined are the values we have verified with our programs

w /d	1	2	3	4	5	6	7	8	9	10
3	1									
4										
5	1									
6										
7	1									
8		1								
9	1									
10		1								
11	1		1							
12		1		1						
13	1		2							
14		2		1						
15	1		2		1					
16		2		3						
17	1		4		2					
18		2		5		1				
19	1		5		5					
20		3		7		3				
21	1		6		9		1			
22		3		11		7				
23	1		8		15		4			
24		3		16		14		1		
25	1		10		23		11			
26		4		20		27		5		
27	1		11		36		23		2	
28		4		27		45		16		
29	1		14		50		48		7	
30		4		35		73		37		2

which produces the values of $D_{w,d}$ in Table 1, with underlined values verified by work with Johannes Blümlein and Jos Vermaseren [7].

To explain how I guessed the final term in the generating function (8), we shall need to consider alternating Euler sums.

3 MZVs in QFT

The counterterms in the renormalization of the coupling in ϕ^4 theory, at L loops, may involve MZVs with weights up to $2L - 3$ [9]. Those associated with subdivergence-free diagrams may be obtained from finite massless two-point diagrams with one less loop.

The first irreducible MZV of depth 2, namely $\zeta(5,3)$, occurs in a counterterm coming from the most symmetric six-loop diagram for the ϕ^4 coupling, in which each of the 4 vertices connected to an external line is connected to each of the 3 other vertices, giving 12 internal propagators (or edges, as mathematicians prefer to call them). It hence diverges, at large loop momenta, in the manner of $\int \mathrm{d}^{24}k/k^{24}$. Its contribution to the β-function of ϕ^4-theory is scheme-independent and may be computed to high accuracy by using Gegenbauer polynomial expansions in x-space, which give the counterterm as a four-fold sum that is far from obviously a MZV. Accelerated convergence of truncations of this sum gave an empirical **Q**-linear of combination of $\zeta(5)\zeta(3)$ with

$$\zeta(5,3) - \frac{29}{12}\zeta(8)$$

and the latter combination was found to occur in another six-loop counterterm. I shall attempt to demystify the multiple of $\zeta(8)$ after discussing alternating Euler sums.

At seven loops, Dirk Kreimer and I found the combination

$$\zeta(3,5,3) - \zeta(3)\zeta(5,3)$$

in three different counterterms, where it occurs in combination with rational multiples of $\zeta(11)$ and $\zeta^2(3)\zeta(5)$.

4 Alternating Euler Sums

This second topic is closely related to the first, namely alternating sums of the form

$$\sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{\varepsilon_1^{n_1}\ldots\varepsilon_k^{n_k}}{n_1^{s_1}\ldots n_k^{s_k}}$$

with positive integers s_j and signs $\varepsilon_j = \pm 1$. We may compactly indicate the presence of an alternating sign, when $\varepsilon_j = -1$, by placing a bar over the corresponding integer exponent s_j. Thus we write

$$\zeta(\overline{3},\overline{1}) = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n}$$

$$\zeta(3,\overline{6},3,\overline{6},3) = \sum_{j>k>l>m>n>0} \frac{(-1)^{k+m}}{j^3 k^6 l^3 m^6 n^3}$$

using the same symbol ζ as we did for the MZVs. Such sums may be studied using `EZFace` and the `DataMine`.

4.1 Three-Letter Alphabet

Alternating sums have a stuffle algebra, from their representation as nested sums, and a shuffle algebra, from their representation as iterated integrals. In the integral representation we need a third letter, C, in our alphabet, corresponding to the differential form $\mathrm{d}x/(1+x)$. Consider

$$Z(ABC) = \int_0^1 \frac{\mathrm{d}x}{x} \int_0^x \frac{\mathrm{d}y}{1-y} \int_0^y \frac{\mathrm{d}z}{1+z}.$$

The z-integral gives $\log(1+y) = -\sum_{j>0}(-y)^j/j$ and hence

$$Z(ABC) = -\sum_{j>0} \int_0^1 \frac{\mathrm{d}x}{x} \int_0^x \frac{\mathrm{d}y}{1-y} \frac{(-y)^j}{j}.$$

Expanding $1/(1-y) = \sum_{k>0} y^{k-1}$ and integrating over y we obtain

$$Z(ABC) = -\sum_{k>0} \sum_{j>0} \int_0^1 \frac{\mathrm{d}x}{x} \frac{x^{j+k}}{j+k} \frac{(-1)^j}{j}$$

and the final integration gives

$$Z(ABC) = -\sum_{k>0} \sum_{j>0} \frac{1}{(j+k)^2} \frac{(-1)^j}{j}.$$

Finally, the substitution $k = m - j$ gives

$$Z(ABC) = -\sum_{m>j>0} \frac{(-1)^j}{m^2 j} = -\zeta(2, \overline{1}).$$

It takes a bit of practice to translate between words and sums. Here's another example:

$$Z(ACCAC) = (-1)^3 \sum_{l>0} \sum_{k>0} \sum_{j>0} \frac{(-1)^l}{(j+k+l)^2} \frac{(-1)^k}{j+k} \frac{(-1)^j}{j^2}$$

gives

$$Z(ACCAC) = -\sum_{m>n>j>0} \frac{(-1)^m}{m^2 n j^2} = -\zeta(\overline{2}, 1, 2)$$

after the substitutions $l = m - n$ and $k = n - j$.

Going from sums to words is quite tricky. For example, try to find the word W and the sign $\varepsilon(W)$ such that

$$\zeta(3,\overline{6},3,\overline{6},3) = \varepsilon(W)Z(W).$$

Note that $\varepsilon(W)$ is $+1$ or -1 according as whether there is an odd or even number of letters C in the word W. The word W begins $AABAAAAACAA\ldots$. The next letter is either B or C, but which is it?

4.2 Shuffles and Stuffles for Alternating Sums

The six shuffles in

$$\mathcal{S}(ab,xy) = \{abxy, axby, xaby, axyb, xayb, xyab\}$$

give six different words, with $a = A$, $b = B$, $x = y = C$:

$$\begin{aligned} Z(AB)Z(CC) = {} & Z(ABCC) + Z(ACBC) + Z(CABC) \\ & + Z(ACCB) + Z(CACB) + Z(CCAB) \end{aligned}$$

which translates to

$$\zeta(2)\zeta(\overline{1},1) = \zeta(2,\overline{1},1) + \zeta(\overline{2},\overline{1},\overline{1}) + \zeta(\overline{1},\overline{2},\overline{1}) + \zeta(\overline{2},1,\overline{1}) + \zeta(\overline{1},2,\overline{1}) + \zeta(\overline{1},1,\overline{2}).$$

The stuffles for this product are

$$\zeta(2)\zeta(\overline{1},1) = \zeta(2,\overline{1},1) + \zeta(\overline{3},1) + \zeta(\overline{1},2,1) + \zeta(\overline{1},3) + \zeta(\overline{1},1,2).$$

4.3 Transforming Words

The transformation $x = (1-y)/(1+y)$ gives

$$\begin{aligned} \mathrm{d}\log(x) &= \mathrm{d}\log(1-y) - \mathrm{d}\log(1+y) \\ \mathrm{d}\log(1-x) &= \mathrm{d}\log(y) - \mathrm{d}\log(1+y) \\ \mathrm{d}\log(1+x) &= -\,\mathrm{d}\log(1+y) \end{aligned}$$

and maps $x = 0$ and $x = 1$ to $y = 1$ and $y = 0$. Thus, if we take a word W, write it backwards, and make the transformations

$$A \to (B + C)$$
$$B \to (A - C)$$

we may obtain an expression for $Z(W)$ by expanding the brackets.

For example the transformation

$$AB \to (A - C)(B + C) = AB + AC - CB - CC$$

gives

$$Z(AB) = Z(AB) + Z(AC) - Z(CB) - Z(CC).$$

Combining this with the shuffle

$$Z(C)Z(C) = Z(CC) + Z(CC)$$

we obtain

$$0 = Z(AC) - Z(CB) - \frac{1}{2}Z(C)Z(C) = -\zeta(\overline{2}) + \zeta(\overline{1}, \overline{1}) - \frac{1}{2}\zeta(\overline{1})\zeta(\overline{1}).$$

Combining this with the stuffle

$$\zeta(\overline{1})\zeta(\overline{1}) = \zeta(\overline{1}, \overline{1}) + \zeta(2) + \zeta(\overline{1}, \overline{1})$$

we obtain

$$\zeta(\overline{2}) = -\frac{1}{2}\zeta(2)$$

which is also obtainable as follows.

4.4 Doubling Relations

For $a > 1$ we have

$$\zeta(a) + \zeta(\overline{a}) = \sum_{n>0} \frac{1 + (-1)^n}{n^a} = \sum_{k>0} \frac{2}{(2k)^a} = 2^{1-a}\zeta(a)$$

by the substitution $n = 2k$. Hence

$$\zeta(\overline{a}) = (2^{1-a} - 1)\zeta(a).$$

At $a = 2$, we obtain $\zeta(\overline{2}) = -\zeta(2)/2$, as above. Note also that $\zeta(\overline{1}) = -\log(2)$.

We may take any MZV and convert it into a combination of MZVs and alternating sums, by doubling the summation variables. For example, we obtain

$$\begin{aligned} 2^{2-a-b}\zeta(a,b) &= \sum_{m>n>0} \frac{2}{(2m)^a}\frac{2}{(2n)^b} \\ &= \sum_{j>k>0} \frac{1+(-1)^j}{j^a}\frac{1+(-1)^k}{k^a} \\ &= \zeta(a,b) + \zeta(\overline{a},b) + \zeta(a,\overline{b}) + \zeta(\overline{a},\overline{b}) \end{aligned}$$

by the transformations $j = 2m$ and $k = 2n$.

More complicated doubling relations were used in constructing the `DataMine`. With these, it was possible to avoid using the time-consuming transformations $A \to (B + C)$ and $B \to (A - C)$ as algebraic input. It was verified that the output, obtained by shuffling, stuffling and doubling, satisfied the relations that follow from word transformation.

4.5 Conjectured Enumeration of Irreducible Alternating Sums

Before considering the enumeration of irreducible MZVs, in the (A, B) alphabet, I already had a rather simple conjecture for the generator of the number, $E_{w,d}$, of irreducible sums of weight w and depth d in the (A, B, C) alphabet, namely

$$\prod_{w>2}\prod_{d>0}(1 - x^w y^d)^{E_{w,d}} \stackrel{?}{=} 1 - \frac{x^3 y}{(1-xy)(1-x^2)}. \tag{9}$$

If this be true, it is easy to obtain $E_{w,d}$ by Möbius transformation of the binomial coefficients in Pascal's triangle. Let [8]

$$T(a,b) = \frac{1}{a+b}\sum_{c|a,b}\mu(c)\frac{(a/c+b/c)!}{(a/c)!(b/c)!} \tag{10}$$

where the sum is over all positive integers c that divide both a and b and the Möbius function is defined by

$$\mu(c) = \begin{cases} 1 & \text{when } c = 1 \\ 0 & \text{when } c \text{ is divisible by the square of a prime} \\ (-1)^k & \text{when } c \text{ is the product of } k \text{ distinct primes.} \end{cases} \tag{11}$$

When w and d have the same parity, and $w > d$, one obtains from (9)

$$E_{w,d} = T\left(\frac{w-d}{2}, d\right). \tag{12}$$

The DataMine now provides extensive evidence to support this conjecture. It was verified at depth 6 up to weight 12, solving the algebraic input in rational arithmetic, and then up to weight 18, using arithmetic modulo a 31-bit prime. At depth 5, the corresponding weights are 17 and 21. At depth 4, they are 22 and 30.

5 Pushdown from MZVs to Alternating Sums

Now consider the integers $M_{w,d}$ generated by an even simpler process:

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{M_{w,d}} = 1 - \frac{x^3 y}{1-x^2}. \tag{13}$$

But what is the question, to which this is the answer?

I conjectured that $M_{w,d}$ is the number of irreducible sums of weight w and depth d in the (A, B, C) alphabet that suffice for the evaluation of MZVs in the (A, B) alphabet.

5.1 Pushdown at Weight 12

As already hinted, the first place that this conjecture becomes non-trivial is at weight 12, where the enumerations $M_{12,4} = 0$ and $M_{12,2} = 2$ are to be contrasted with the enumerations $D_{12,4} = 1$ and $D_{12,2} = 1$ of irreducible MZVs. The conjecture requires that

$$\zeta(6,4,1,1) = \sum_{k>l>m>n>0} \frac{1}{k^6 l^4 mn}$$

be reducible to sums of lesser depth, if we include an alternating double sum in the basis.

In 1996, I found such a "pushdown" empirically, using the integer-relation search routine PSLQ [10]. It took another decade to prove such an integer relation, by the laborious process of solving all the known algebraic relations in the (A, B, C) alphabet at weight 12 and depths up to 4. Jos Vermaseren derived this proven identity from the DataMine:

$$\begin{aligned}\zeta(6,4,1,1) = &-\frac{64}{27}A(7,5) - \frac{7{,}967}{1{,}944}\zeta(9,3) + \frac{1}{12}\zeta^4(3) + \frac{11{,}431}{1{,}296}\zeta(7)\zeta(5)\\ &-\frac{799}{72}\zeta(9)\zeta(3) + 3\zeta(2)\zeta(7,3) + \frac{7}{2}\zeta(2)\zeta^2(5) + 10\zeta(2)\zeta(7)\zeta(3)\\ &+\frac{3}{5}\zeta^2(2)\zeta(5,3) - \frac{1}{5}\zeta^2(2)\zeta(5)\zeta(3) - \frac{18}{35}\zeta^3(2)\zeta^2(3)\\ &-\frac{5{,}607{,}853}{6{,}081{,}075}\zeta^6(2)\end{aligned}$$

where

$$A(7,5) = Z(AAAAAA(B-C)AAAAB) = \zeta(7,5) + \zeta(\overline{7},\overline{5}).$$

It is now proven that all MZVs of weight up to 12 are reducible to **Q**-linear combinations of $\zeta(5,3)$, $\zeta(7,3)$, $\zeta(3,5,3)$, $\zeta(9,3)$, $\zeta(\overline{7},\overline{5})$, single zeta values, and products of these terms.

5.2 *Enumeration of MZVs Revisited*

I can now explain the rather simple-minded procedure that Dirk Kreimer and I used in 1996 to arrive at the conjecture [8]

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{D_{w,d}} \stackrel{?}{=} 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)}$$

for the number $D_{w,d}$ of irreducible sums in the (A,B) alphabet of pure MZVs. We added the third term to the much simpler conjectured generator for the much complicated question answered by $M_{w,d}$, namely the number of irreducibles in the (A,B,C) alphabet that suffice for reductions of MZVs. The numerator, $x^{12}y^2$ $(1-y^2)$, of this term was determined by the single pushdown observed at weight 12, from an MZV of depth 4 to an alternating sum of depth 2. The denominator, $(1-x^4)(1-x^6)$, was chosen to agree with the empirical number $D_{2n,2} = \lceil n/3 \rceil - 1$ of double non-alternating irreducible sums of weight $2n$. Then we assumed that the enumeration of all other pushdowns would be generated by filtration. It was possible to check this, in a few cases, using PSLQ in 1996.

The list of explicit pushdowns that have now been obtained, in accord with the conjecture, has grown since then.

At weights 15, 16, 17, we have found pushdowns from MZVs to these alternating sums: $\zeta(\overline{6},3,\overline{6})$, $\zeta(\overline{13},\overline{3})$, $\zeta(\overline{6},5,\overline{6})$.
At weight 18, there were pushdowns to $\zeta(\overline{15},\overline{3})$ and $\zeta(6,\overline{5},\overline{4},3)$.
At weight 19, to $\zeta(\overline{8},3,\overline{8})$ and $\zeta(\overline{6},7,\overline{6})$.
At weight 20, to $\zeta(\overline{17},\overline{3})$, $\zeta(8,\overline{5},\overline{4},3)$ and $\zeta(6,\overline{5},\overline{6},3)$.

Our most ambitious efforts were at weight 21, where 3 MZVs of depth 5 are pushed down to the alternating sums $\zeta(\overline{8}, 5, \overline{8})$, $\zeta(\overline{6}, 9, \overline{6})$and $\zeta(\overline{8}, 3, \overline{10})$. Moreover the first pushdown from an MZV of depth 7 to an alternating sum of depth 5 is predicted at weight 21. A demanding `PSLQ` computation gave a relation of the form

$$\zeta(6,2,3,3,5,1,1) = -\frac{326}{81}\zeta(3,\overline{6},3,\overline{6},3) + \ldots \tag{14}$$

where the remaining 150 terms are formed by MZVs with depth no greater than 5, and their products. At such weight and depth, it becomes rather non-trivial to decide on a single alternating sum that might replace a MZV of greater depth. It took several attempts to discover that the alternating sum

$$\zeta(3,\overline{6},3,\overline{6},3) = \sum_{j>k>l>m>n>0} \frac{(-1)^{k+m}}{(jk^2lm^2n)^3}$$

is an "honorary MZV" that performs this task.

5.3 Suppression of π in Massless Diagrams

Now I can demystify, somewhat, the combination

$$\zeta(5,3) - \frac{29}{12}\zeta(8)$$

that occurs in scheme-independent counterterms of ϕ^4 theory at six loops. Dirk Kreimer and I discovered that the combinations [8]

$$N(a,b) = \zeta(\overline{a}, b) - \zeta(\overline{b}, a),$$

with distinct odd integers a and b, simplify the results for counterterms. In particular, the use of

$$N(3,5) = \frac{27}{80}\left(\zeta(5,3) - \frac{29}{12}\zeta(8)\right) + \frac{45}{64}\zeta(3)\zeta(5)$$

removes all powers of π from both subdivergence-free diagrams that contribute to the six-loop β-function. In each case, the contribution is a $\mathbf{Z}$-linear combination of $N(3,5)$ and $\zeta(3)\zeta(5)$.

At higher loop numbers, Oliver Schnetz has found that $N(3,7)$ suppresses the appearance π^{10}. However, at 8 loops he found that $N(3,9)$ and $N(5,7)$ are not sufficient to remove π^{12}. Like the maths, the physics becomes different at weight 12.

6 Magnetic Moment of the Electron

The magnetic moment of an electron, with charge $-e$ and mass m, is slightly greater than the Bohr magneton

$$\frac{e\hbar}{2m} = 9.274 \times 10^{-24}\ \mathrm{J\,T^{-1}}$$

which was the value predicted by Dirac. Here I included $\hbar = h/(2\pi)$, which we usually set to unity in QFT.

Using perturbation theory, we may expand in powers of the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.035999\ldots}.$$

In QFT, we usually set $\varepsilon_0 = 1$ and $c = 1$ and expand in powers of $\alpha/\pi = e^2/(4\pi^2)$, obtaining a perturbation expansion

$$\frac{\text{magnetic moment}}{\text{Bohr magneton}} = 1 + A_1\frac{\alpha}{\pi} + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + \ldots$$

which is known up to three loops.

In 1947, Schwinger [11] found the first correction term $A_1 = \frac{1}{2}$. In 1950, Karplus and Kroll [12] claimed the value

$$28\zeta(3) - 54\zeta(2)\log(2) + \frac{125}{6}\zeta(2) - \frac{2{,}687}{288} \ = \ -2.972604271\ldots$$

for the coefficient of the next correction. It turned out that they had made a mistake in this rather difficult calculation. The correct result

$$A_2 \ = \ \frac{3}{4}\zeta(3) - 3\zeta(2)\log(2) + \frac{1}{2}\zeta(2) + \frac{197}{144} \ = \ -0.3284789655\ldots$$

was not obtained until 1957 [13, 14]. Not until 1996 was the next coefficient

$$\begin{aligned} A_3 = &-\frac{215}{24}\zeta(5) + \frac{83}{12}\zeta(3)\zeta(2) - \frac{13}{8}\zeta(4) - \frac{50}{3}\zeta(\overline{3},\overline{1}) \\ &+ \frac{139}{18}\zeta(3) - \frac{596}{3}\zeta(2)\log(2) + \frac{17{,}101}{135}\zeta(2) + \frac{28{,}259}{5{,}184} \qquad (15) \\ = {}& 1.181241456\ldots \end{aligned}$$

found, by Stefano Laporta and Ettore Remiddi [15]. The irrational numbers appearing on the second line are those already seen in A_2. On the first line we see zeta values and a new number, namely the alternating double sum

$$\zeta(\overline{3},\overline{1}) = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} \approx -0.117875999650509326841013950834 13761\ldots$$

I visited Stefano and Ettore in Bologna when they were working on this formidable calculation and recommended to them a method of integration by parts, in D dimensions, that I had found useful for related calculations in the quantum field theory of electrons and photons [16]. Here $D = 4 - 2\varepsilon$ is eventually set to 4, the number of dimensions of space-time. But it turns out to be easier if we keep it as a variable until the final stage of the calculation. Then if we find parts of the result that are singular at $\varepsilon = 0$ we need not worry: all that matters is that the complete result is finite. Based on D-dimensional experience, I expected their final result to look simplest when written in terms of $\zeta(\overline{3},\overline{1})$.

The D-dimensional calculation that informed this intuition involved three-loop massive diagrams contributing to charge renormalization in QED [16]. These yielded Saalschützian F_{32} hypergeometric series, with parameters differing from $\frac{1}{2}$ by multiples of ε, namely

$$W(a_1, a_2; a_3, a_4) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - a_1\varepsilon)_n (\frac{1}{2} - a_2\varepsilon)_n}{(\frac{1}{2} + a_3\varepsilon)_{n+1} (\frac{1}{2} + a_4\varepsilon)_{n+1}}$$

with $(\alpha)_n \equiv \Gamma(\alpha + n)/\Gamma(\alpha)$. In particular, I needed the expansions of $W(1,1;1,0)$ and $W(1,0;1,1)$ in ε. The result for the most difficult three-loop diagram had the value $\pi^2 \log(2) - \frac{3}{2}\zeta(3)$ at $\varepsilon = 0$. Noting that this also occurs in the two-loop contribution to the magnetic moment, I expanded the charge-renormalization result to $O(\varepsilon)$, where I found only $\zeta(\overline{3},\overline{1})$ and $\zeta(4)$. I thus hazarded the guess that these two sums would exhaust the weight-4 contributions to the magnetic moment at three loops, which happily is the case.

One may also write (15) in terms of a polylog that is not evaluated on the unit circle, such as

$$\mathrm{Li}_4(1/2) = \sum_{n=1}^{\infty} \frac{1}{n^4} \left(\frac{1}{2}\right)^n = -\frac{1}{24}\log^4(2) + \frac{1}{4}\zeta(2)\log^2(2) + \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(\overline{3},\overline{1}),$$

but then the result for A_3 will acquire extra terms, involving powers of $\log^2(2)$.

7 Three-Loop Massive Bubble Diagrams

Here we consider three-loop diagrams with a massive particle in at least one of the internal lines. If this mass is much larger than the scales set by external momenta, we may set the latter to zero, and obtain vacuum bubbles.

7.1 Tetrahedral Bubbles from Two-Loop Propagators

There are ten distinct colourings of a tetrahedron by mass, shown in Fig. 1.

The massive lines in V_{2A} and V_{2N} are adjacent and non-adjacent, respectively; in the dual cases, V_{4A} and V_{4N}, it is the massless lines that are adjacent and non-adjacent; in cases V_{3T}, V_{3S} and V_{3L}, the massive lines form a triangle, star and line, and hence the massless lines form a star, triangle and line.

Defining the finite two-point function (with space-like p^2)

$$I(r_1 \ldots r_5; p^2/m^2) := \frac{p^2}{\pi^4} \int d^4k \int d^4l \quad P_1(k)P_2(p+k)P_3(k-l)P_4(l)P_5(p+l) \tag{16}$$

with $P_j(k) := 1/(k^2 + m^2 r_j)$, in 4 dimensions, we obtain

$$V(r_1 \ldots r_5, 0) - V(r_1 \ldots r_5, 1) = \int_0^\infty dx\, I(r_1 \ldots r_5; x) \left\{ \frac{1}{x} - \frac{1}{x+1} \right\} + O(\varepsilon) \tag{17}$$

for the difference of vacuum diagrams with a massless and massive sixth propagator. This difference is finite in 4 dimensions.

Suppressing the parameters $r_1 \ldots r_5$, temporarily, we exploit the dispersion relation

$$I(x) = \int_{s_0}^\infty ds\, \sigma(s) \left\{ \frac{1}{s+x} - \frac{1}{s} \right\} \tag{18}$$

where $-2\pi i \sigma(s) = I(-s + i0) - I(-s - i0)$ is the discontinuity across the cut $[-\infty, -s_0]$ on the negative axis. Integration by parts then gives

$$I(x) = \int_{s_0}^\infty ds\, \sigma'(s) \left\{ -\log\left(1 + \frac{x}{s}\right) + \log\left(1 + \frac{x}{s_0}\right) \right\} \tag{19}$$

where the constant term in the logarithmic weight function may be dropped if $\sigma(s_0) = 0$, as occurs when $s_0 = 0$. As $x \to \infty$, we obtain the universal asymptotic value

$$6\zeta(3) = I(\infty) = \int_{s_0}^\infty ds\, \sigma'(s) \left\{ \log(s) - \log(s_0) \right\} \tag{20}$$

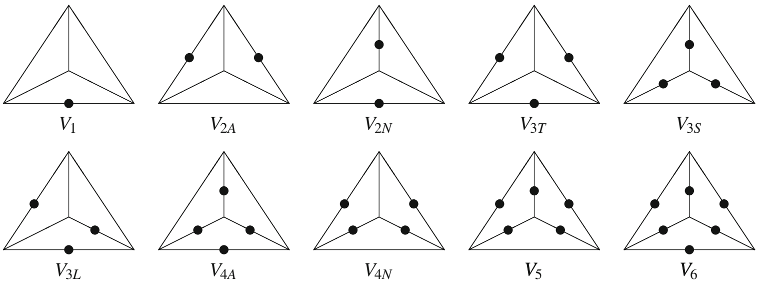

Fig. 1 Colourings of a tetrahedron by mass, denoted by a blob

with the $\log(s_0)$ term dropped when $s_0 = 0$. The finite difference in (17) is obtained from (19) as

$$\int_0^\infty dx\, I(x) \left\{ \frac{1}{x} - \frac{1}{x+1} \right\} = \int_{s_0}^\infty ds\, \sigma'(s) \left\{ L_2(s) - L_2(s_0) \right\} \tag{21}$$

with a dilogarithmic weight function

$$L_2(s) := \int_0^\infty \frac{dx}{x(x+1)} \log\left(\frac{1+x}{1+x/s}\right) = \mathrm{Li}_2(1-1/s) = -\frac{1}{2}\log^2(s) - \mathrm{Li}_2(1-s) \tag{22}$$

that is chosen to satisfy $L_2(1) = 1$, thus enabling one to drop $L_2(s_0)$ for $s_0 = 0$ and $s_0 = 1$, which covers all the cases with $N \le 3$ massive particles in the two-point function, and hence $N + 1 \le 4$ massive particles in vacuum diagrams.

We now prove that the two terms in the weight function (22) can be separated to yield the finite parts of the vacuum diagrams combined in (17), as follows:

$$F(r_1 \ldots r_5, 0) = \frac{1}{2} \int_{s_0}^\infty ds\, \sigma'(r_1 \ldots r_5; s) \left\{ \log^2(s) - \log^2(s_0) \right\} \tag{23}$$

$$F(r_1 \ldots r_5, 1) = -\int_{s_0}^\infty ds\, \sigma'(r_1 \ldots r_5; s)\ \left\{ \mathrm{Li}_2(1-s) - \mathrm{Li}_2(1-s_0) \right\} \tag{24}$$

with constant terms in the weight functions that are inert when $s_0 = 0$ and when $s_0 = 1$. The proof uses the representation

$$I(x) = 6\zeta(3) + \int_{s_0}^\infty ds\, \sigma'(s) \left\{ -\log(x+s) + \log(x+s_0) \right\} \tag{25}$$

in which the asymptotic value (20) is subtracted. Then one obtains

$$\int_0^\infty dx\, \frac{I(\infty) - I(x)}{x+1} = -\int_{s_0}^\infty ds\, \sigma'(s) \left\{ \mathrm{Li}_2(1-s) - \mathrm{Li}_2(1-s_0) \right\}. \tag{26}$$

Specializing the analysis to cases with $r_j = 0$ or 1, we obtain from [17]

$$\sigma'(r_1 \ldots r_5; s) = \left\{\sigma'_a(r_1 \ldots r_5; s)\, \Theta\left(s - (r_1 + r_2)^2\right) + (1 \leftrightarrow 4, 2 \leftrightarrow 5)\right\} + \left\{\sigma'_b(r_1 \ldots r_5; s)\, \Theta\left(s - (r_2 + r_3 + r_4)^2\right) + (1 \leftrightarrow 2, 4 \leftrightarrow 5)\right\}$$

$$\sigma'_a(r_1 \ldots r_5; s) := 2\,\Re \int_{(r_4+r_5)^2}^{\infty} dx\, \frac{T(x, r_1, r_2, r_3, r_4, r_5)}{\Delta(s, r_1, r_2)} \frac{\partial}{\partial x}\left(\frac{\Delta(x, r_1, r_2)}{x - s + i0}\right) \tag{27}$$

$$\sigma'_b(r_1 \ldots r_5; s) := 2\,\Re \int_{(r_3+r_4)^2}^{(\sqrt{s}-r_2)^2} dx\, \frac{\partial}{\partial s}\left(\frac{T(x, s, r_2, r_5, r_4, r_3)}{x - r_1 + i0}\right) \tag{28}$$

$$T(s, a, b, c, d, e) := \operatorname{arctanh}\left(\frac{\Delta(s, a, b)\Delta(s, d, e)}{x^2 - x(a + b - 2c + d + e) + (a - b)(d - e)}\right) \tag{29}$$

$$\Delta(a, b, c) := \sqrt{a^2 + b^2 + c^2 - 2ab - 2bc - 2ca} \tag{30}$$

with integration by parts in (27) giving a logarithmic result, in all cases, and differentiation in (28) giving a logarithmic result when $r_1 r_3 r_5 = r_2 r_3 r_4 = 0$, i.e. when there is no intermediate state with three massive particles.

7.2 *The Totally Massive Case*

We were able to hand nine cases by methods that avoided intermediate states with three massive particles. Now there is no option, since

$$F_6 = -\int_4^{\infty} ds\, \overline{\sigma}'(s)\, \mathrm{Li}_2(1 - s) \tag{31}$$

involves intermediate states with two and three massive particles in

$$\overline{\sigma}'(s) = \overline{\sigma}'_a(s)\, \Theta(s - 4) + \overline{\sigma}'_b(s)\, \Theta(s - 9)\,. \tag{32}$$

We may, however, simplify matters by separating these contributions in

$$F_6 - F_5 = \int_4^{\infty} ds\, \overline{\sigma}'(s)\, \mathrm{Li}_2(1 - 1/s) = F_a + F_b \tag{33}$$

$$F_a := \int_4^{\infty} ds\, \overline{\sigma}'_a(s)\, \{\mathrm{Li}_2(1 - 1/s) - \zeta(2)\} \tag{34}$$

$$F_b := \int_9^{\infty} ds\, \overline{\sigma}'_b(s)\, \{\mathrm{Li}_2(1 - 1/s) - \zeta(2)\} \tag{35}$$

where F_5 may be evaluated without encountering elliptic integrals.

The two-particle cut gives a logarithm in

$$\overline{\sigma}'_a(s) = \frac{2}{s-3}\left\{\operatorname{arccosh}(s/2-1) - \frac{2\pi}{\sqrt{3s(s-4)}}\right\} \tag{36}$$

while the three-particle cut gives the elliptic[3] integral

$$\overline{\sigma}'_b(s) = -2\int_4^{(\sqrt{s}-1)^2} \frac{dx}{x-1}\,\frac{\Delta(x,1,1)}{\Delta(x,s,1)}\,\frac{x+s-1}{\Delta^2(x,s,1)+xs}\,. \tag{37}$$

At large s, contributions (36) and (37) are each $O(\log(s)/s)$, while their sum is $O(\log(s)/s^2)$. The integrals (34) and (35) converge separately, thanks to the $\zeta(2)$ in their weight functions, to which the combination (33) is blind.

It appears that we need to integrate the product of a dilog and an elliptic integral. To avoid this, we may we reverse the order of integration. Setting $x = 1/u^2 \in [4,\infty]$ in (37), which now becomes the outer integration, and $s = (1/u+v)(1/u+1/v) \in [(1/u+1)^2,\infty]$ in the inner, we then integrate by parts on $v \in [0,1]$ to convert the dilog to a product of logs, with the result [18]

$$F_b = 2\int_0^{\frac{1}{2}} du\left(\frac{dA(u)}{du}\right)\int_0^1 dv\left(\frac{\partial B(u,v)}{\partial v}\right) C(u,v)\,D(u,v) \tag{38}$$

$$A(u) := \log\left(\frac{u^2}{1-u^2}\right) \tag{39}$$

$$B(u,v) := \log\left(\frac{(1+uv)(u+v)}{u+v+uv^2}\right) \tag{40}$$

$$C(u,v) := \log\left(\frac{(1+uv)(u+v)}{u^2v}\right) \tag{41}$$

$$D(u,v) := \log\left(\frac{1+2uv+v^2+(1-v^2)\sqrt{1-4u^2}}{1+2uv+v^2-(1-v^2)\sqrt{1-4u^2}}\right) \tag{42}$$

which establishes that F_b is the integral of a trilogarithm.

The NAG routine D01FCF is notably efficient at evaluating rectangular double integrals in double-precision FORTRAN, which was ample to to discover the remarkable relation

$$F_6 = F_{3S} + F_{4N} - F_{2N} = 4\left(\mathrm{Cl}_2^2(\pi/3) + 4\zeta(4) + 2\zeta(\overline{3},\overline{1})\right) \tag{43}$$

[3] I am told that Källén was disappointed to find that the two-loop electron propagator involves an elliptic integral, unlike the simpler photon propagator.

where $\mathrm{Cl}_2(\pi/3) = \sum_{n>0} \sin(n\pi/3)/n^2$. This corresponds to a direct relation between diagrams

$$V_6 + V_{2N} = V_{3S} + V_{4N} + O(\varepsilon) \tag{44}$$

verified to 15 digits. It stands as testament to the oft remarked fact that results in quantum field theory have a simplicity that tends to increase with the labour expended.

8 Massive Banana Diagrams

To progress beyond diagrams that yield polylogs and elliptic integrals we now turn attention to vacuum diagrams with merely two vertices. I shall call these "banana" diagrams. The L-loop banana diagram has $L+1$ edges, each representing a massive propagator with unit mass. To avoid ultra-violate divergences, let us consider these in two space-time dimensions.

8.1 Schwinger's Bananas

Let A be the diagonal $N \times N$ matrix with entries $A_{i,j} = \delta_{i,j}\alpha_i$. Let U be the column vector of length N with unit entries, $U_i = 1$. Then $B = U\tilde{U}$ is the $N \times N$ matrix with unit entries, $B_{i,j} = 1$. The banana diagram with $N+1$ edges of unit mass, in two space-time dimensions, may be evaluated by Schwinger's trick as a multiple of the N-dimensional integral

$$\overline{V}_{N+1} = \int_{\alpha_i>0} \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N}{\mathrm{Det}(A+B)(\mathrm{Tr}(A)+1)} \tag{45}$$

where

$$\mathrm{Det}(A+B) = \sum_{i=0}^{N} \frac{1}{\alpha_i} \prod_{j=0}^{N} \alpha_j$$

is the first Symanzik polynomial, with $\alpha_0 = 1$ fixed by momentum conservation, and the second Symanzik polynomial

$$\mathrm{Tr}(A) + 1 = \sum_{i=0}^{N} \alpha_i$$

results from the fact that the $N+1$ edges are propagators with unit mass.

8.2 Bessels's Bananas

We may also evaluate banana diagrams in x-space, since the two-dimensional Fourier transform of the p-space Euclidean propagator $1/(p^2 + m^2)$, with $p^2 = p_0^2 + p_1^2$, yields the Bessel function $K_0(mx)$, with $x^2 = x_0^2 + x_1^2$. The normalization in (45) corresponds to

$$\overline{V}_{N+1} = 2^N \int_0^\infty [K_0(t)]^{N+1} t \, \mathrm{d}t \tag{46}$$

which differs by a power of 2 from the Bessel moments that I studied with Bailey, Borwein and Glasser [19].

Hence I but a bar over V and use the subscript $N + 1$ to indicate the number of Bessel functions.

8.3 Known Bananas

It is proven that [19]

$$\overline{V}_1 = 1 \tag{47}$$

$$\overline{V}_2 = 1 \tag{48}$$

$$\overline{V}_3 = 3L_{-3}(2) \tag{49}$$

$$\overline{V}_4 = 7\zeta(3) \tag{50}$$

where

$$L_{-3}(s) = \sum_{n \geq 0} \left(\frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s} \right)$$

is the Dirichlet L function with conductor -3.

The zero-loop evaluation (47) merely checks our normalization.

The one-loop evaluation

$$\overline{V}_2 = \int_0^\infty \frac{\mathrm{d}\alpha_1}{(\alpha_1 + 1)^2} = 1$$

follows neatly from (45), since with $N = 1$ we have $\mathrm{Det}(A + B) = \mathrm{Tr}(A) + 1 = \alpha_1 + 1$.

I shall now use $\{a, b, c, \ldots\}$ for the Schwinger parameters $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$.

8.4 Three-Edge Banana and Sixth Root of Unity

At two loops, the Schwinger method gives the banana diagram with 3 edges as

$$\overline{V}_3 = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\, \mathrm{d}b}{(ab + a + b)(a + b + 1)}.$$

To proceed we may take partial fractions with respect to b. Then

$$\frac{a^2 + a + 1}{(ab + a + b)(a + b + 1)} = \frac{a + 1}{ab + a + b} - \frac{1}{a + b + 1} = \frac{\partial}{\partial b} \log\left(\frac{ab + a + b}{a + b + 1}\right)$$

enables integration over b. Hence we obtain

$$\overline{V}_3 = \int_0^\infty \frac{G(a)\, \mathrm{d}a}{a^2 + a + 1} \tag{51}$$

with contributions to

$$G(a) = \log(1 + a) + \log(1 + 1/a) \tag{52}$$

at $b = \infty$ and $b = 0$. It is apparent from (51) that the sixth root of unity $\lambda = (1 + \mathrm{i}\sqrt{3})/2$ is implicated, since $a^2 + a + 1 = (a + \lambda)(a + \overline{\lambda})$, where $\overline{\lambda} = (1 - \mathrm{i}\sqrt{3})/2 = 1 - \lambda$ is the conjugate root. Working out the corresponding dilogarithms we obtain

$$\overline{V}_3 = \frac{4}{\sqrt{3}} \Im \mathrm{Li}_2(\lambda) = 3L_{-3}(2)$$

in agreement with the well known result (49).

8.5 Four-Edge Banana and $\zeta(3)$

To evaluate

$$\overline{V}_4 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c}{(abc + ab + bc + ca)(a + b + c + 1)}$$

we take partial fractions with respect to c and then integrate over c, to obtain

$$\overline{V}_4 = \int_0^\infty \int_0^\infty \frac{L(a, b)\, \mathrm{d}a\, \mathrm{d}b}{(a + 1)(b + 1)(a + b)}$$

with

$$L(a,b) = \log\left(\frac{(ab+a+b)(a+b+1)}{ab}\right).$$

Hence with

$$F(a) = \int_0^\infty \frac{(a-1)L(a,b)\,\mathrm{d}b}{(b+1)(a+b)}$$

we have

$$\overline{V}_4 = \int_0^\infty \frac{F(a)\,\mathrm{d}a}{a^2-1} = \int_0^1 \frac{(F(a)-F(1/a))\,\mathrm{d}a}{a^2-1}. \tag{53}$$

I shall need only the derivative of $F(a)$. Let

$$K(a,b) = \frac{b\,L(a,b)}{a+b} + \log(ab+a+b) - 2\log(a+b+1).$$

Then, by construction,

$$\frac{\partial}{\partial b}K(a,b) = a\frac{\partial}{\partial a}\left(\frac{(a-1)L(a,b)}{(b+1)(a+b)}\right)$$

and hence

$$a\frac{\mathrm{d}}{\mathrm{d}a}F(a) = K(a,\infty) - K(a,0) = 2G(a)$$

where $G(a)$ was given in (52). We now integrate (53) by parts, to obtain

$$\overline{V}_4 = \int_0^1 \frac{\mathrm{d}a}{a}\log\left(\frac{1+a}{1-a}\right)(G(a)+G(1/a))$$

and use Nielsen's evaluations

$$-\int_0^1 \frac{\mathrm{d}a}{a}\log(1-a)\log(1+a) = \frac{5}{8}\zeta(3)$$

$$-\int_0^1 \frac{\mathrm{d}a}{a}\log(a)\log(1+a) = \frac{3}{4}\zeta(3)$$

$$\int_0^1 \frac{\mathrm{d}a}{a}\log^2(1+a) = \frac{1}{4}\zeta(3)$$

$$\int_0^1 \frac{\mathrm{d}a}{a}\log(a)\log(1-a) = \zeta(3)$$

to obtain

$$\overline{V}_4 = \left(4 \times \frac{5}{8} + 2 \times \frac{3}{4} + 4 \times \frac{1}{4} + 2\right) \zeta(3) = 7\zeta(3)$$

in agreement with the previously known result (50).

8.6 Unknown Banana

The next diagram has 5 edges and hence 4 loops. After an easy first integration, we obtain

$$\overline{V}_5 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{M(a,b,c)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(ab+a+b)c^2 + (ab+a+b)(a+b)c + (a+b)ab}$$

with

$$M(a,b,c) = \log(a+b+c+1) + \log\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

But then integration over c will produce complicated dilogarithms with arguments involving the square root of the discriminant

$$D(a,b) = (ab+a+b)(a+b)(ab(a+b) + (a-b)^2)$$

of the quadratic in c. The result will have the form

$$\overline{V}_5 = \int_0^\infty \int_0^\infty \frac{L_2(a,b)\mathrm{d}a\,\mathrm{d}b}{\sqrt{D(a,b)}}$$

with undisclosed dilogs in $L_2(a,b)$. Integration by parts, to reduce the dilogs to logs, would require us to introduce an elliptic function, since $D(a,b)$ is a quartic in b.

We know nothing about the number theory of $\overline{V}_5$. Its value is known to 1,000 decimal places.

9 Cut Bananas: On-Shell Sunrise Diagrams

For $N > 2$ we may cut an edge in $\overline{V}_N$ and set the two external half edges on the unit mass shell, which is at $p^2 = -1$. I call the result $\overline{S}_N$. It has $N-1$ internal edges and hence $N-2$ loops. Thus $\overline{V}_3$ and $\overline{S}_4$ correspond to the two-loop diagrams of Fig. 2, with the "sunrise" diagram $\overline{S}_4$ obtained by cutting an edge of $\overline{V}_4$.

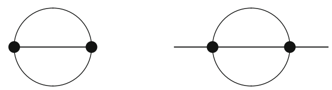

Fig. 2 Two-loop banana and sunrise diagrams

9.1 Schwinger's Cut Bananas

At N loops, the integral over Schwinger parameters is

$$\overline{S}_{N+2} = \int_{\alpha_i > 0} \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N}{\mathrm{Det}(A+B)\mathrm{Tr}(A) + \tilde{U}CU}. \tag{54}$$

where C is the adjoint of $A+B$, with

$$(A+B)C = \mathrm{Det}(A+B)I$$

where I is the unit matrix with $I_{i,j} = \delta_{i,j}$. The denominator in (54) is the second Symanzik polynomial.

9.2 Bessels's Cut Bananas

In x-space, cutting an edge and putting it on the mass shell corresponds to replacing one instance of the Bessel function $K_0(t)$ by $I_0(t)$, to obtain

$$\overline{S}_{N+2} = 2^N \int_0^\infty I_0(t)[K_0(t)]^{N+1} t \, \mathrm{d}t \tag{55}$$

at N loops. Note that $\overline{S}_2$ is divergent, since

$$I_0(t) = \sum_{k \geq 0} \left(\frac{t^k}{2^k k!} \right)^2$$

grows exponentially, with

$$I_0(t) = \frac{\exp(t)}{\sqrt{2\pi t}} \left(1 + \frac{1}{8t} + O(1/t^2) \right)$$

as $t \to \infty$, while

$$K_0(t) = \sqrt{\frac{\pi}{2t}} \exp(-t) \left(1 - \frac{1}{8t} + O(1/t^2)\right)$$

is exponentially damped.

9.3 Known Cut Bananas

It is proven that [19]

$$\overline{S}_3 = 2L_{-3}(1) = \frac{2\pi}{3\sqrt{3}} \tag{56}$$

$$\overline{S}_4 = \mathrm{Li}_2(1) - \mathrm{Li}_2(-1) = \frac{\pi^2}{4} \tag{57}$$

and it is conjectured that [19]

$$\overline{S}_5 \stackrel{?}{=} \frac{1}{30\sqrt{5}} \Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right) \tag{58}$$

which holds to at least 1,000 decimal places.

9.4 Cut Banana with Sixth Root of Unity

The Schwinger formula (54) at one loop gives

$$\overline{S}_3 = \int_0^\infty \frac{\mathrm{d}a}{a^2 + a + 1} = \frac{\log(\lambda) - \log(\overline{\lambda})}{\lambda - \overline{\lambda}} = \frac{2\arctan(\sqrt{3})}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$$

as claimed in (56).

9.5 Cut Banana with π^2

At two loops, we have

$$\overline{S}_4 = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{(a+b)(a+1)(b+1)}$$

with a convenient factorization of the second Symanzik polynomial. Hence

$$\overline{S}_4 = \int_0^\infty \frac{\log(a)\,\mathrm{d}a}{a^2-1} = 2\int_0^1 \frac{\log(a)\,\mathrm{d}a}{a^2-1}$$

yields dilogs at square roots of unity, namely

$$\overline{S}_4 = \mathrm{Li}_2(1) - \mathrm{Li}_2(-1) = \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{\pi^2}{4}$$

as claimed in (57).

9.6 Cut Banana at the 15th Singular Value

At three loops, we have

$$\overline{S}_5 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{P(a,b,c)}$$

where

$$P(a,b,c) = (abc + ab + bc + ca)(a + b + c) + (ab + bc + ca)$$

with the final term, $(ab + bc + ca)$, resulting from the adjoint matrix. Grouping powers of c, we see that

$$P(a,b,c) = (ab + a + b)c^2 + (ab + a + b)(a + b + 1)c + (a + b + 1)ab$$

yields a discriminant

$$\Delta(a,b) = (ab + a + b)(a + b + 1)((ab + a + b)(a + b + 1) - 4ab)$$

and the integral over c gives

$$\overline{S}_5 = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{\Delta(a,b)}} \log\left(\frac{1 + X(a,b)}{1 - X(a,b)}\right)$$

with

$$X(a,b) = \sqrt{1 - \frac{4ab}{(ab + a + b)(a + b + 1)}}\,.$$

Conjecture (58) was stimulated by a proven result for

$$\overline{T}_5 \equiv 4\int_0^\infty [I_0(t)]^2[K_0(t)]^3 t\,\mathrm{d}t = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{\Delta(a,b)}}$$

namely

$$\overline{T}_5 = \frac{\sqrt{3}}{120\pi}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) \tag{59}$$

obtained at the 15th singular value, by diamond mining [19].

Numerical evaluation suggested that

$$\frac{\overline{S}_5}{\overline{T}_5} \stackrel{?}{=} \frac{4\pi}{\sqrt{15}}$$

and this has been confirmed at 1,000-digit precision. Yet it remains to be proved that

$$\int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{\Delta(a,b)}}\left(\log\left(\frac{1+X(a,b)}{1-X(a,b)}\right) - \frac{4\pi}{\sqrt{15}}\right) \tag{60}$$

vanishes. It has been shown that its magnitude is smaller than $10^{-1{,}000}$.

10 Diagrams Evaluating as L-Series of Modular Forms

Finally, I indicate how sunrise diagrams lead to evaluations in terms of the Dirichlet L-functions of modular forms, evaluated at integers inside their critical strips.

10.1 L-Series of a K3 Surface

For $s > 2$ let

$$L(s) = \prod_p \frac{1}{1 - \frac{A_p}{p^s} + \left(\frac{p}{15}\right)\frac{p^2}{p^{2s}}}$$

where $\left(\frac{\cdot}{15}\right)$ is a Kronecker symbol and the product is over all primes p, with integers

$$A_3 = -3,$$
$$A_5 = 5,$$
$$A_p = 0, \text{ for } \left(\frac{p}{15}\right) = -1,$$

$$A_p = 2x^2 + 2xy - 7y^2, \text{ for } x^2 + xy + 4y^2 = p \equiv 1, 4 \pmod{15}, \qquad (61)$$

$$A_p = x^2 + 8xy + y^2, \text{ for } 2x^2 + xy + 2y^2 = p \equiv 2, 8 \pmod{15}, \qquad (62)$$

with pairs of integers (x, y) defined, for $x > 0$, by the quadratic forms in (61) and (62).

As shown by Peters, Top and van der Vlugt [20], the L-series

$$L(s) = \sum_{n>0} \frac{A_n}{n^s}$$

is generated by the weight-3 modular form

$$f_3(q) = \eta(q)\eta(q^3)\eta(q^5)\eta(q^{15})R(q) = \sum_{n>0} A_n q^n \qquad (63)$$

where

$$\frac{\eta(q)}{q^{1/24}} = \prod_{j>0}(1-q^j) = \sum_{n\in\mathbf{Z}}(-1)^n q^{n(3n+1)/2}, \qquad (64)$$

$$R(q) = \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+4n^2}. \qquad (65)$$

Note that $A_1 = 1$, since $1 + 3 + 5 + 15 = 24$. If $q = p^r$ is a prime power, then

$$A_{pq} = A_p A_q - \left(\frac{p}{15}\right) p^2 A_{q/p}.$$

If $n = \prod_j q_j$, with prime powers $q_j = p_j^{r_j}$, then $A_n = \prod_j A_{q_j}$. Thus (61) and (62) suffice to compute A_n and are easily programmed using the `qfbsolve` command of `Pari-GP`.

I now describe how I was able to evaluate 20,000 good digits of the conditionally convergent series $L(2) = \sum_{n>0} A_n/n^2$. Let

$$\Lambda(s) = \frac{\Gamma(s)}{c^s} L(s), \text{ with } c = \frac{2\pi}{\sqrt{15}}.$$

Then the functional equation $\Lambda(s) = \Lambda(3-s)$ may be used to extend the Mellin transform

$$\Lambda(s) = \sum_{n>0} A_n \int_0^\infty \frac{\mathrm{d}x}{x} x^s \exp(-cnx) \qquad (66)$$

throughout the complex s-plane, as follows

$$\Lambda(s) = \sum_{n>0} A_n \left(\frac{\Gamma(s, cn\lambda)}{(cn)^s} + \frac{\Gamma(3-s, cn/\lambda)}{(cn)^{3-s}} \right) \tag{67}$$

where

$$\Gamma(s, y) = \int_y^\infty \frac{\mathrm{d}x}{x} x^s \exp(-x)$$

is the incomplete Γ function and $\lambda \geq 0$ is an arbitrary real parameter. To establish (67), I remark that it agrees with (66), at $\lambda = 0$, and that its derivative with respect to λ vanishes by virtue of the inversion symmetry

$$M(\lambda) \equiv \lambda^{3/2} \sum_{n>0} A_n \exp(-cn\lambda) = M(1/\lambda).$$

Optimal convergence is achieved at $\lambda = 1$, where

$$\Lambda(s) = \sum_{n>0} A_n \int_1^\infty \frac{\mathrm{d}x}{x} \left(x^s + x^{3-s}\right) \exp\left(-\frac{2\pi n x}{\sqrt{15}}\right) \tag{68}$$

makes the relation $\Lambda(s) = \Lambda(3-s)$ explicit. Zeros on the critical line $\Re s = 3/2$ occur when

$$\Lambda(3/2 + \mathrm{i}s_0) = 2 \sum_{n>0} A_n \int_1^\infty \mathrm{d}x\, x^{1/2} \cos(s_0 \log(x)) \exp\left(-\frac{2\pi n x}{\sqrt{15}}\right)$$

vanishes. I have computed 100 good digits of the first zero, obtaining

$$\begin{aligned} s_0 = {} & 4.84192581422996258804553371124717544839994584063476 \\ & 6693950953608563348168047411353721585251883775250 05\ldots \end{aligned}$$

At $s = 2$, the integral in (68) is elementary and we have dramatically improved convergence for

$$L(2) \equiv \sum_{n>0} \frac{A_n}{n^2} = \sum_{n>0} \frac{A_n}{n^2} \left(1 + \frac{4\pi n}{\sqrt{15}}\right) \exp\left(-\frac{2\pi n}{\sqrt{15}}\right) \tag{69}$$

from which I obtained more than 20,000 good digits in less than a minute, by computing the first 30,000 terms, with the aid of (61) and (62). The result is consistent with the conjecture

$$
\begin{aligned}
3L(2) &\stackrel{?}{=} \overline{T}_5 & (70)\\
&\equiv 4\int_0^\infty [I_0(t)]^2[K_0(t)]^3 t\,\mathrm{d}t & (71)\\
&= \int_0^\infty\int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{(ab+a+b)(a+b+1)((ab+a+b)(a+b+1)-4ab)}} & (72)\\
&= \frac{\pi^2}{8}\left(\sqrt{15}-\sqrt{3}\right)\left(1+2\sum_{n>0}\exp\left(-\sqrt{15}\pi n^2\right)\right)^4 & (73)\\
&= \frac{\sqrt{3}}{120\pi}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) & (74)\\
&\stackrel{?}{=} \frac{\sqrt{15}}{4\pi}\overline{S}_5 & (75)
\end{aligned}
$$

where $\overline{T}_5$ is defined in (71) as a Bessel moment, with a proven integral representation over Schwinger parameters in (72), a proven evaluation at the 15th singular value in (73), a proven reduction to Γ values in (74) and a conjectural relation to $\overline{S}_5$ in (75).

Unfortunately, I did not succeed in relating $\overline{V}_5$ to $L(3)$ and/or $L(4)$.

10.2 L-Series for 6 Bessel Functions

We are interested in relating Bessel moments of the form

$$\overline{V}_N = 2^{N-1}\int_0^\infty [K_0(t)]^N t\,\mathrm{d}t,\ \text{for } N>0, \tag{76}$$

$$\overline{S}_N = 2^{N-2}\int_0^\infty I_0(t)[K_0(t)]^{N-1} t\,\mathrm{d}t,\ \text{for } N>2, \tag{77}$$

$$\overline{T}_N = 2^{N-3}\int_0^\infty I_0^2(t)[K_0(t)]^{N-2} t\,\mathrm{d}t,\ \text{for } N>4, \tag{78}$$

$$\overline{U}_N = 2^{N-4}\int_0^\infty I_0^3(t)[K_0(t)]^{N-3} t\,\mathrm{d}t,\ \text{for } N\ge 6, \tag{79}$$

$$\overline{W}_N = 2^{N-5}\int_0^\infty I_0^4(t)[K_0(t)]^{N-4} t\,\mathrm{d}t,\ \text{for } N\ge 8, \tag{80}$$

to L-series derived from modular forms. In [19] it was conjectured that

$$\overline{S}_5 \stackrel{?}{=} \frac{4\pi}{\sqrt{15}}\,\overline{T}_5 \tag{81}$$

$$\overline{S}_6 \stackrel{?}{=} \frac{4\pi^2}{3}\,\overline{U}_6 \tag{82}$$

$$\overline{T}_8 \stackrel{?}{=} \frac{18\pi^2}{7}\,\overline{W}_8 \tag{83}$$

with a notable appearance of 7 in the denominator on the right hand side of (83).

Francis Brown suggested that the weight-4 modular form

$$f_4(q) = \left[\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)\right]^2 = \sum_{n>0} A_{4,n} q^n \tag{84}$$

of Hulek, Spandaw, van Geemen, and van Straten [21] might yield an L-series

$$L_4(s) = \sum_{n>0} \frac{A_{4,n}}{n^s} = \frac{1}{1+2^{1-s}}\,\frac{1}{1+3^{1-s}} \prod_{p>3} \frac{1}{1 - \frac{A_{4,p}}{p^s} + \frac{p^3}{p^{2s}}}$$

with values related to the problem with 6 Bessel functions. Note that $A_{4,1} = 1$, since $2(1+2+3+6) = 24$.

The Mellin transform

$$\Lambda_4(s) = \frac{\Gamma(s)}{(2\pi/\sqrt{6})^s}\, L_4(s) = \sum_{n>0} A_{4,n} \int_0^\infty \frac{\mathrm{d}x}{x} x^s \exp\left(-\frac{2\pi n x}{\sqrt{6}}\right)$$

may be analytically continued to give

$$\Lambda_4(s) = \sum_{n>0} A_{4,n} \int_1^\infty \frac{\mathrm{d}x}{x} \left(x^s + x^{4-s}\right) \exp\left(-\frac{2\pi n x}{\sqrt{6}}\right)$$

by virtue of the inversion symmetry

$$M_4(\lambda) \equiv \lambda^2 \sum_{n>0} A_{4,n} \exp\left(-\frac{2\pi n \lambda}{\sqrt{6}}\right) = M_4(1/\lambda)$$

that gives the reflection symmetry $\Lambda_4(s) = \Lambda_4(4-s)$.

Then, at $s = 2$ and $s = 3$, we obtain the very convenient formulas

$$L_4(2) = \sum_{n>0} \frac{A_{4,n}}{n^2} \left(2 + \frac{4\pi n}{\sqrt{6}}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right) \tag{85}$$

$$L_4(3) = \sum_{n>0} \frac{A_{4,n}}{n^3} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right) \tag{86}$$

without resort to incomplete Γ functions that entail exponential integrals. By this means, I was able to compute 20,000 good digits of (85) and (86) in less than 100 s. Then the conjectural evaluations

$$\overline{S}_6 \stackrel{?}{=} 48\zeta(2)L_4(2) \tag{87}$$

$$\overline{T}_6 \stackrel{?}{=} 12L_4(3) \tag{88}$$

$$\overline{U}_6 \stackrel{?}{=} 6L_4(2) \tag{89}$$

were discovered and checked at 1,000-digit precision.

I remark that Francis Brown had expected a result of form (88), for $\overline{T}_6$, with an unknown rational coefficient, which I here evaluate as 12. The existence of a relation of the form (89), for $\overline{U}_6$, had not been predicted, since I had been unable to provide an expression for this Bessel moment as an integral over Schwinger parameters of an algebraic or polylogarithmic function. However, it was quite natural to guess that a reduction of $\overline{T}_6$ to $L_4(3)$ would be accompanied by a reduction of $\overline{U}_6$ to $L_4(2)$. Then the reduction of $\overline{S}_6$ to $\zeta(2)L_4(2)$ follows from conjecture (82), which I had already checked at 1,000-digit precision in [19].

10.3 L-Series for 8 Bessel Functions

Next, Francis Brown provided the first 100 Fourier coefficients of a weight-6 modular form $f_6(q) = \sum_{n>0} A_{6,n} q^n$, whose L-series

$$L_6(s) = \sum_{n>0} \frac{A_{6,n}}{n^s} = \frac{1}{1-2^{2-s}} \frac{1}{1+3^{2-s}} \prod_{p>3} \frac{1}{1-\frac{A_{6,p}}{p^s}+\frac{p^5}{p^{2s}}}$$

was expected to yield values related to the problem with 8 Bessel functions. His data may be condensed down to the values

```
-66, 176, -60, -658, -414, 956, 600, 5574, -3592, -8458,
19194, 13316, -19680, -31266, 26340, -31090, -16804, 6120,
-25558, 74408, -6468, -32742, 166082
```

of $A_{6,p}$ for the primes $p = 5, 7, \ldots, 97$.

From this I inferred that the explicit modular form is given by

$$f_6(q) = g(q)g(q^2) \tag{90}$$

$$g(q) = \left[\eta(q)\eta(q^3)\right]^2 \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2} \tag{91}$$

with $f_6(q)/f_4(q)$ given by the θ function of the strongly 6-modular lattice [22] indexed by `QQF.4.g`[4] with expansion coefficients in entry `A125510` of Neil Sloane's On-Line Encyclopedia of Integer Sequences.[5]

Proceeding along the lines of the previous section, I accelerated the convergence of

$$\Lambda_6(s) = \frac{\Gamma(s)}{(2\pi/\sqrt{6})^s} L_6(s) = \sum_{n>0} A_{6,n} \int_0^\infty \frac{\mathrm{d}x}{x} x^s \exp\left(-\frac{2\pi n x}{\sqrt{6}}\right)$$

by using the functional relation $\Lambda_6(s) = \Lambda_6(6-s)$ to obtain

$$\Lambda_6(s) = \sum_{n>0} A_{6,n} \int_1^\infty \frac{\mathrm{d}x}{x} \left(x^s + x^{6-s}\right) \exp\left(-\frac{2\pi n x}{\sqrt{6}}\right)$$

and hence the convenient formulas

$$L_6(3) = \sum_{n>0} \frac{A_{6,n}}{n^3} \left(2 + \frac{4\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right), \tag{92}$$

$$L_6(4) = \sum_{n>0} \frac{A_{6,n}}{n^4} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{4\pi^2 n^2}{9} + \frac{4\pi^3 n^3}{9\sqrt{6}}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right), \tag{93}$$

$$L_6(5) = \sum_{n>0} \frac{A_{6,n}}{n^5} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{\pi^2 n^2}{3} + \frac{2\pi^3 n^3}{9\sqrt{6}} + \frac{\pi^4 n^4}{27}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right). \tag{94}$$

The resulting fits

$$\overline{T}_8 \stackrel{?}{=} 216 L_6(5) \tag{95}$$

$$\overline{U}_8 \stackrel{?}{=} 36 L_6(4) \tag{96}$$

$$\overline{W}_8 \stackrel{?}{=} 8 L_6(3) \tag{97}$$

are rather satisfying. They leave the conjectural relation

$$L_6(5) \stackrel{?}{=} \frac{4}{7} \zeta(2) L_6(3) \tag{98}$$

as a restatement of the notable conjecture (83) given in [19].

[4] http://www2.research.att.com/~njas/lattices/QQF.4.g.html

[5] http://oeis.org/A125510

Thanks to the explicit formula (90) for the weight-6 modular form, conjecture (98) has now been checked to 20,000-digit precision.

11 Open Questions

This article has provided examples of single-scale Feynman diagrams that evaluate to five types of number: multiple zeta values, alternating sums, polylogarithms of the sixth root of unity, products of elliptic integrals, and L-functions of modular forms. In each case, I indicate an open question concerning the physics and an open puzzle concerning the mathematics.

Q1: At which loop-number do the counterterms of QFT cease to evaluate to MZVs?
There remains a single subdivergence-free counterterm in ϕ^4 theory at seven loops that has not been reduced to MZVs, but might be expected to evaluate to polylogs. At eight loops there is a diagram for which there is good reason [23] to suppose that no reduction to polylogs will be possible, yet there is no concrete guess of the type of new number that might emerge.
On the mathematical side, the conjectural enumeration [8] of irreducible MZVs by weight and depth, in (8), is still unproven. Might it be that the conjecture fails at weights higher than those in the table of Sect. 2.4, notwithstanding the notable agreement so far achieved?
Q2: At which loop-number do the diagrams for the magnetic moment of the electron case to evaluate in terms of alternating sums?
One might guess that this will happen at four loops, since there one has diagrams with five electrons in an intermediate state and the corresponding on-shell sunrise diagram in two dimensions, with 6 Bessel functions, evaluates to the L function of a modular form, as seen in Eq. (87).
On the mathematical side, one would like to understand why a depth-5 alternating sum like $\zeta(3,\overline{6},3,\overline{6},3)$ in (14) is an honorary MZV of depth 7.
Q3: Does a polylogarithm of the sixth root of unity appear in the seven-loop beta-function of ϕ^4 theory?
It has been argued [24] that this may happen, for one special diagram. However, comparable arguments suggested the appearance of alternating sums from a pair of simpler seven-loop diagrams and these were found to evaluate to MZVs.
On the mathematical side, one would like to have an economical basis for polylogs of the sixth root of unity up to weight 11, so as to tackle the seven-loop problem in QFT. However, that seems to be a daunting task.
Q4: What type of number results from the four-loop banana diagram $\overline{V}_5$, with 5 Bessel functions?
We have seen that the three-loop on-shell sunrise diagram $\overline{S}_5$ evaluates, empirically, to the square of an elliptic integral at the 15th singular value. Yet the

simplest result so far achieved for $\overline{V}_5$ is the integral of the product of a dilogarithm and a complete elliptic integral [19].

On the mathematical side, one would like to be able to prove the vanishing of the remarkable integral (60).

Q5: Is there a modular form whose L-function gives an evaluation of the on-shell five-loop sunrise diagram $\overline{S}_7$?

It is frustrating to have identified modular forms for problems with 5, 6 and 8 Bessel functions, yet to have failed to do so for any 7-Bessel problem.

On the mathematical side, one would like to understand the relation between integrals of powers of Bessel functions and Kloosterman sums [25, 26] that evaluate to rational numbers.

In conclusion, these open questions arose from fertile meetings of number theory, algebraic geometry and quantum field theory, reported in part by this article. While much remains to be understood, we may still rejoice that mathematicians and physicists continue to learn how to share their understanding and their puzzles at the work-face of perturbative quantum field theory.

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