About Coupling of the Block Elements

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Abstract The block element method gives the representations of the solutions of the partial differential equations more exactly than other numerical methods such as finite and boundary element methods. However it is achieved at expenses of more complicated structure and algorithm of the block element method. This is the reason for limitation of wider use of this method. The block element method provides one with the analytical representation of solution of partial differential equations and opens the way to extract the new regularities of the natural and mechanical process. The main operations for the block element method are the factorization and coupling of the block elements. It is necessary to select small block elements to be coupled to the considered domain. This paper addresses coupling of the block elements having a spherical boundary.

1 Introduction

Let us consider the problem of selection of small block elements which have to be connected to a bounded convex region with smooth boundary to be covered by blocks. For this aim it is necessary to develop the appropriate block elements. First of all we construct the block elements for the ball region.

Following the paper [1, 2], we demonstrate the block element method algorithm. We constructed here the block elements for the boundary-value problem in the ball region Ω_0 with boundary $\partial\Omega_0$ of radius a for the Helmholtz differential equation in the form of

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$$Q(\partial x_1, \partial x_2, \partial x_3)\varphi = \left[\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3 + k^2\right]\psi(x_1, x_2, x_3) = 0 \qquad (1)$$

It is shown in [1, 2] that the pseudo-differential equations for the block element enable us to consider all possible variants of boundary conditions in terms of θ, φ, r for the partial differential equation. For this purpose, we considered both the Dirichlet and Neumann boundary conditions.

In the spherical system of coordinate's θ, φ, r Eq. (1) for the ball has the form

$$(\Delta + k^2)\psi = 0\,,$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \qquad (2)$$

$$r, \theta, \varphi \in \Omega_1$$

The solutions of the boundary-value problems (2) are found in the spaces of slowly increasing generalized functions $\mathbf{H}_s$. For investigating this equation by the differential factorization method, we introduce the Fourier-Bessel transform and write down the spherical functions in the following form

$$\mathbf{B}_2(l, m) = \int_0^{\pi}\int_0^{2\pi} g(\theta, \varphi) Y_l^{m-}(\theta, \varphi)\sin\theta d\theta d\varphi = G(l, m)$$

$$\mathbf{B}_2^{-1}(\theta, \varphi)G = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} G(l, m) Y_l^{m+}(\theta, \varphi) = g(\theta, \varphi)$$

$$\mathbf{B}_3(\lambda, l, m)g = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} g(r, \theta, \varphi) J_{l+\frac{1}{2}}(\lambda r) Y_l^{m-}(\theta, \varphi)\sin\theta d\theta d\varphi r dr = G(\lambda, l, m)$$

$$\mathbf{B}_3^{-1}(r, \theta, \varphi)G = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty} G(\lambda, l, m) J_{l+\frac{1}{2}}(\lambda r) Y_l^{m+}(\theta, \varphi)\lambda d\lambda = g(r, \theta, \varphi)$$

(3)

Here $J_\nu(\lambda r)$ is the Bessel function, and $Y_l^m(\theta, \varphi)$ is the spherical function,

$$Y_l^{m\pm}(\theta, \varphi) = \frac{1}{2}\sqrt{\frac{2l+1}{\pi}\frac{(l-|m|)!}{(l+|m|)!}}\,P_l^{|m|}(\cos\theta)e^{\pm im\varphi}$$

Applying transforms (3) to Eq. (2), we construct the external form [1, 2]

$$\omega = Pb^2\sin\theta d\theta\wedge d\varphi + Qbdr\wedge d\theta + Rb\sin\theta d\varphi\wedge dr$$

where P, Q, R are some functions. We carry out the transition to the functional equation in the form [1,2]

$$K(\lambda)\Psi(l,m,\lambda) = \int\limits_{\partial\Omega_0} \omega \;, \qquad K(\lambda) = \lambda^2 - k^2. \tag{4}$$

In the case of a ball, we have

$$(\lambda^2 - k^2)\Psi(l,m,\lambda) = L_{lm}(\lambda),$$
$$L_{lm}(\lambda) = a^2\psi'_{lm}(a)T_{lm}(\lambda,a) - a^2\psi_{lm}(a)T'_{lm}(\lambda,a),$$
$$\psi_{lm}(r) = \mathbf{B}_2(l,m)\psi(r,\theta,\varphi) \;, \quad T'_{lm}(\lambda,r) = \frac{1}{\sqrt{r}} J_{l+\frac{1}{2}}(\lambda r).$$

In order to ensure the automorphism and obtain the pseudo-differential equation, we write down the solution of the boundary-value problem as follows

$$\psi(r,\theta,\varphi) = \mathbf{B}_3^{-1}(r,\theta,\varphi)\frac{L_{lm}(\lambda)}{(\lambda^2 - k^2)}. \tag{5}$$

The automorphism requirement implies satisfaction of the equality [1,2]

$$\psi(r,\theta,\varphi) = 0, \quad r > a.$$

As a result of transformations for the simple problem under consideration we obtain a pseudo-differential equation that degenerates into algebraic one in the form of

$$L_{lm}(k) = 0. \tag{6}$$

In complex spatial problems, this equation turns out to be pseudo-differential.

On this example we can observe the difference of the generalized factorization from the simple one: although the characteristic equation $K(\lambda)$ has two roots, Eq. (6) should be satisfied only for one root. A similar problem considered by simple factorization in a layer would require satisfying Eq. (6) for both roots.

Using pseudo-differential equation (6), we consider the formulation of the boundary-value problems for Eq. (2). For example, in the case of Dirichlet conditions for the boundary $\partial\Omega$ in the form of

$$\psi(a,\theta,\varphi) = \psi_0(a,\theta,\varphi) \tag{7}$$

the solution of pseudo-differential equation (6) is obtained in the form of

$$\psi'_{lm}(a) = \frac{\psi_{lm0}(a)T'_{lm}(k,a)}{T_{lm}(k,a)} \;, \quad \text{where} \quad \psi_{lm0}(a) = \mathbf{B}_2(l,m)\psi_0(a,\theta,\varphi).$$

In the complex spatial problems, we obtain either integral or integro-differential equation instead of algebraic one. For example, in [3] we applied the integral factorization method.

Introducing this relation into Eq. (5) and carrying out the necessary calculations, we obtain for $r \to a$ the following result $\psi(a,\theta,\varphi) \to \psi_0(a,\theta,\varphi)$. By using the same algorithm, we have solved the problem with the Neumann boundary condition. In this case, instead of boundary condition (7), the derivative is prescribed on the boundary; i.e.

$$\psi'(a,\theta,\varphi) = \psi_1(a,\theta,\varphi). \tag{8}$$

The solution of the pseudo-differential equation has the following form

$$\psi_{lm}(b) = \frac{\psi'_{lm1}(b)T_{lm}(k,b)}{T'_{lm}(k,b)}, \quad \text{where} \quad \psi_{lm1}(a) = \mathbf{B}_2(l,m)\psi_1(a,\theta,\varphi).$$

Inserting this relation into Eq. (5) and carrying out the transformations, we again obtain that the boundary conditions are satisfied in the Dirichlet problem; i.e., $\psi'(r,\theta,\varphi) \to \psi_1(a,\theta,\varphi)$, $r \to a$ however for the classical component of the solution.

2 The Nonplanar Boundary Block Elements

In order to construct the block elements for the convex domain with smooth boundary it is necessary to use the block elements of the complicated forms calculated in [4]. Let us consider two spheres of the radii a and b, $a < b$ in the Cartesian system of coordinates with the centers on Oz axis, the spheres occupying the regions Ω_1 and Ω_2 of the space Ω, respectively. Let the distance between the centers be h. We designate the region obtained by the intersection of the spheres as Ω_3; i.e.

$$\Omega_3 = \Omega_1 \cap \Omega_2\,, \quad \Omega_4 = \Omega_2 - \Omega_3\,, \quad h < a < b < h + a. \tag{9}$$

Let us construct the block element in this region for the inner boundary-value problem for the Helmholtz differential equation in the form (2).

Following the algorithm of the differential factorization method [4] we introduce two spherical systems of coordinates with the origins in the centers of spheres on the axis z of the original Cartesian system of coordinates with the parameters r, φ, z and ρ, γ, ξ in the regions Ω_1 and Ω_2, respectively. These systems of coordinates provide the tangential stratification of the boundary for the chosen manifold with the edge. Applying transformation (3) to Eq. (2), we construct the outer form [4], which in the coordinates r, φ, z takes the form

$$\omega = g\left\langle \left[\frac{\partial\psi}{\partial\theta} - \psi\frac{\partial P_l^{|m|}(\cos\theta)}{\partial\theta}\left\{P_l^{|m|}(\cos\theta)\right\}^{-1}\right] r\sin\theta d\varphi\wedge dr - \right.$$
$$-\left[\frac{\partial\psi}{\partial\varphi} - im\psi\right] rd\theta\wedge dr +$$
$$\left. + \left[\frac{\partial\psi}{\partial r} - \psi\frac{\partial r^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\lambda r)}{\partial r}\left\{r^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\lambda r)\right\}^{-1}\right] r^2\sin\theta d\theta\wedge d\varphi\right\rangle, \tag{10}$$
$$g(\theta,\varphi,r) = Y_l^{m+}(\theta,\varphi)r^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\lambda r).$$

A similar form is obtained in terms of the system ρ, γ, ξ.

We implement the transition to the functional equation which can be represented in the form (4).

Let us designate the boundary of the body under consideration as $\partial\Omega_3 = \partial\Omega_{10} \cup \partial\Omega_{20}$ where $\partial\Omega_{10}$ represents the surface of the sphere of radius a of boundary $\partial\Omega_3$, while $\partial\Omega_{20}$ is the surface of the sphere of radius b of the indicated boundary.

Applying the conventional algorithm of construction of the pseudo-differential equations [1, 2] we use the introduced local spherical systems of coordinates for each boundary in Eq. (5) taking that

$$\int_{\partial\Omega_3}\omega = \int_{\partial\Omega_{10}}\omega_{2m-1} + \int_{\partial\Omega_{20}}\omega_{2m}\ , \quad m = 1,2,3,4\ , \quad \partial\Omega_{30} = \partial\Omega_2 - \partial\Omega_{20}, \tag{11}$$

where $\partial\Omega_{30}$ is the addition to the boundary $\partial\Omega_{20}$ in $\partial\Omega_2$.

It should be noted that in what follows we consider the outer and inner boundary-value problems for Eq. (3) in various regions which are formed by the boundaries $\partial\Omega_{10}$, $\partial\Omega_{20}$ and $\partial\Omega_{30}$. For the sake of brevity, the solutions on these boundaries are designated as ψ_1 and ψ_2, respectively, although their numerical values for various boundary-value problems are different.

Now, we construct the block element for the boundary-value problem in the region (9), i.e., in addition to the intersection $\Omega_1 \cap \Omega_2$ in Ω_2, where the inequality means that the centers of spheres are outside of the region Ω_4. For this case, the values of the outer form on the boundaries have the form

$$\omega_1 = g_1\left(r^{-\frac{1}{2}}H^{(1)}_{l+\frac{1}{2}}(\lambda r)\frac{\partial\psi_1}{\partial r} - \psi_1\frac{\partial r^{-\frac{1}{2}}H^{(1)}_{l+\frac{1}{2}}(\lambda r)}{\partial r}\right) a^2\sin\theta d\theta\wedge d\varphi,$$
$$\omega_2 = g_2\left(f_2^p\frac{\partial\psi_2}{\partial\rho} - \psi_2\frac{\partial f_2^p}{\partial\rho}\right) b^2\sin\gamma d\gamma\wedge d\sigma,$$

$$\omega_3 = g_1\left(f_1^r\frac{\partial\psi_1}{\partial r} - \psi_1\frac{\partial f_1^r}{\partial r}\right)a^2\sin\theta d\theta\wedge d\varphi,$$

$$\omega_4 = g_2\left(\rho^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\lambda\rho)\frac{\partial\psi_2}{\partial\rho} - \psi_2\frac{\partial\rho^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\lambda\rho)}{\partial\rho}\right)b^2\sin\gamma d\gamma\wedge d\sigma.$$

The automorphism requirement results in the following pseudo-differential equations:

$$\mathbf{B}_{21}^{-1}(\theta,\varphi)\Bigg[\left(r^{-\frac{1}{2}}H_{l+\frac{1}{2}}^{(1)}(kr)\right)^{-1}\times$$

$$\times\left\{\int_0^{2\pi}\int_{\pi}^{\pi-\theta_0} g_1\left\langle r^{-\frac{1}{2}}H_{l+\frac{1}{2}}^{(1)}(kr)\frac{\partial\psi_1}{\partial r} - \psi_1\frac{\partial r^{-\frac{1}{2}}H_{l+\frac{1}{2}}^{(1)}(kr)}{\partial r}\right\rangle a^2\sin\theta d\theta d\varphi+\right.$$

$$\left.+\int_0^{2\pi}\int_{\gamma_0}^{\pi} g_2(\gamma,\sigma,l,m)\left\langle f_2(\rho,\gamma,l,k)\frac{\partial\psi_2}{\partial\rho} - \psi_2\frac{f_2(\rho,\gamma,l,k)}{\partial\rho}\right\rangle b^2\sin\gamma d\gamma d\sigma\right\}\Bigg]=0,$$

$$r=a\,,\quad \rho=b\,,\qquad \theta,\varphi\in\partial\Omega_{10},$$

$$\mathbf{B}_{21}^{-1}(\gamma,\sigma)\Bigg[\left(\rho^{-\frac{1}{2}}J_{l+\frac{1}{2}}(kr)\right)^{-1}\times$$

$$\times\left\{\int_0^{2\pi}\int_{\gamma_0}^{\pi} g_1\left\langle \rho^{-\frac{1}{2}}J_{l+\frac{1}{2}}(kr)\frac{\partial\psi_2}{\partial\rho} - \psi_2\frac{\partial r^{-\frac{1}{2}}J_{l+\frac{1}{2}}(kr)}{\partial\rho}\right\rangle b^2\sin\gamma d\gamma d\sigma+\right.$$

$$\left.+\int_0^{2\pi}\int_{\pi}^{\pi-\theta_0} g_1(\theta,\varphi,s,n)\left\langle f_1(r,\theta,s,k)\frac{\partial\psi_1}{\partial r} - \psi_1\frac{f_1(r,\theta,s,k)}{\partial r}\right\rangle a^2\sin\theta d\theta d\varphi\right\}\Bigg]=0,$$

$$\gamma,\sigma\in\partial\Omega_{20},$$

$$\theta_0 = \arccos\frac{b^2-a^2-h^2}{2ah},\quad \gamma_0 = \arccos\frac{a^2-b^2-h^2}{2bh}. \tag{12}$$

The pseudo-differential equations taken in one of these forms enable us to formulate an arbitrary number of possible boundary-value problems for the Helmholtz equation. The above form is intended for the both Dirichlet and Neumann boundary-value problems.

In order to couple two block elements and create a new join block element it is necessary to put the boundary conditions (7) and (8) of the first block element into the pseudo-differential equations (12) of the second block element on the boundary of their contact $\partial\Omega_{10}$. This boundary contains parameter a in Eq. (12). The boundary conditions obtained from these equations are introduced in the representation of outer forms. After this procedure, the general representation of solutions in all considered cases is given by the relation

$$\psi(r,\theta,\varphi) = \mathbf{B}_3^{-1}(r,\theta,\varphi)K^{-1}(\lambda,k)\int_{\partial\Omega}\omega\ , \quad r,\theta,\varphi \in \Omega_k\ , \quad \partial\Omega = \partial\Omega_0 \cup \Omega_{10} \cup \Omega_{20}. \tag{13}$$

3 Conclusions

By extending the presented way of coupling the block elements it is possible to cover the arbitrary convex region with smooth boundary by the constructed blocks. This method is used to consider the outer boundary problems when the restricted convex region with smooth boundary is cut from the space. It should be taking into account that the calculation of integral (13) demands the special selection of the way for its integration. The principle of radiation of the energy must be applied. For the outer problems, the contour of integration in Eq. (12) should be deformed in an appropriate way for the operator $\mathbf{B}_3^{-1}(r,\varphi,z)$ when going around the material pole. The described results are transferred by the presented algorithm to the cases of sets of partial differential equations, for example, in the theory of elasticity, which proves to be more complicated [5] only from a technical perspective.

This approach can be applied for the arbitrary block structures. By block structures, we mean materials occupying bounded, semi-bounded, or unbounded domains which are called contacting blocks. It is assumed that each block in a block structure has its own specific behavior in response to physical fields of a various nature. It is also assumed that these fields are described by boundary-value problems for systems of coupled partial differential equations with constant coefficients. Media of this type are typical of the earth's crust, structural materials under complex physical-mechanical conditions, non-material structures of various types and electronic materials. A similar structure is also typical for various materials, including those created by combining only nanoscale components or macro- and nanoscale components.

The absence of considerable constraints on boundary value problems describing the properties of individual blocks suggests that these block structures can have a wide variety of properties. In the general case, the concept of a block requires the boundary of the domain a boundary value problem including multiply connected domains to be unchanged and piecewise smooth. Each block can be bounded or unbounded and can involve coupled processes related to solid and fluid mechanics and electromagnetic, diffusion, thermal, acoustic and other processes.

Block structures are more general objects than piecewise homogeneous structures in which the physical parameters of the medium are assumed to change in jumps in the transition from one block to another with the preservation of the medium material. The latter property means that certain coefficients in the differential equations of a boundary value problem undergo jump variations in the transition from one block to another with the type of the boundary value problem being preserved.

Block structures have a wider range of properties than piecewise homogeneous structures. This follows from the variety of blocks' properties, their shapes, and the character of interblock interactions and also results from the interaction of physical fields, some of which are produced or transformed by blocks.

Short information about application of the block element method is presented in what follows, cf. [6].

1. Reduction of the differential equation to a functional equation by applying the Fourier transforms. The three-dimensional Fourier transform is applied to the system of the differential equation to reduce it to a functional equation. The components of the vector of exterior forms are introduced, namely vectors of an arbitrary coordinate system lying in the coverings of the tangent bundle of the body surface are introduced. In a Cartesian coordinate system, we used the tangent vectors of an arbitrary element of a covering.
2. Fulfillment of prescribed boundary conditions. To achieve this, the solution and its normal derivatives on boundary taken from the boundary conditions are introduced into the representations of the exterior forms. The tangent derivatives are not taken into account. The exterior forms contain the solution and its derivatives on boundary. The functions or normal derivatives on the boundary are found by fitting and inverting the nonsingular matrix from boundary conditions and are introduced into the corresponding representations of exterior forms. The remaining functions or normal derivatives have to be found from the pseudo-differential equations obtained by transformations of the functional equations. The following steps are to be performed to determine the remaining unknowns in the representation of the solution.
3. Factorization of the matrix function coefficient in the functional equation. This representation implies that the elements of coefficient in the functional equation are rational functions only with singularities in the restricted region. For the factorization to be realized we need to apply the method suggested in [6].
4. Reduction of the functional equation to a system of pseudo-differential equations. In order to obtain the required pseudo-differential equations, it is necessary to require that the corresponding Leray's residue forms vanish. Calculating these residue forms in the neighborhood of the local coordinate system we obtain the sought-for relationships.
5. Derivation of a representation of the solution to the boundary-value problem. Introducing the determined components of solution of the pseudo-differential equations into the vector of exterior forms and applying the three-dimensional Fourier transform of the solution we obtain the boundary value problem.

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