Speed-Gradient Control of Mechanical Systems with Constraints

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Abstract State-of-the-art and some applications of the Speed-Gradient method to control of complex systems is presented. A universal speed-gradient method and speed-gradient method for control problems with phase constraints is proposed. Some analytical results are obtained. The application of proposed methods is illustrated by two examples: the selective energy control problem of two pendulums and the average energy control problem of quantum diatomic molecule. Computer simulation results confirm fast convergence rate of algorithms.

1 Problem Formulation

Consider a nonlinear time-varying system

$$\dot{x} = f(x, u, t), \quad x \in R^n, \quad u \in R^m, \quad t \in R, \quad x(0) = x_0, \tag{1}$$

with control goal

$$J\left(u(\cdot)\right) = \limsup_{t \to +\infty} Q\left(x(t, u(\cdot), x_0), t\right), \quad J\left(u(\cdot)\right) \to \min, \tag{2}$$

and constraints

$$\forall t \geqslant 0 : \quad B_k\left(x(t, u(\cdot), x_0), t\right) > 0, \quad k = 1, \dots, \mu, \tag{3}$$

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here x—state, u—control, t—time, x_0—initial condition, $Q(\cdot)$, $B_k(\cdot) : R^n \times R \to R$—some functions, $x(t, u(\cdot), x_0)$—solutions of the system (1) with control $u(\cdot)$ and initial condition x_0.

2 Speed-Gradient Method

In order to design control algorithm the scalar function $w(x, u, t)$ is calculated that is the speed of changing $Q(x, t)$ along trajectories $x(t)$ of (1)

$$w(x, u, t) = \frac{\partial Q(x, t)}{\partial t} + \frac{\partial Q}{\partial x} f(x, u, t). \tag{4}$$

Then it is needed to evaluate the gradient of $w(x, u, t)$ with respect to input variables

$$\nabla_u w(x, u, t) = \nabla_u \frac{\partial Q}{\partial x} f(x, u, t). \tag{5}$$

Finally the algorithm of changing $u(t)$ is determinated according to the differential equation (differential form)

$$\dot{u} = -\Gamma \nabla_u w(x, u, t), \quad u(0) = u_0 \tag{6}$$

or to the algebraic equation (finite form)

$$u = u_0 - \Gamma \nabla_u w(x, u, t), \tag{7}$$

where $\Gamma = \Gamma^T > 0$ is the positive definite gain matrix, u_0 is some initial value of control algorithm. It can be also introduced a speed-pseudogradient algorithm

$$u = u_0 - \Gamma \psi(x, u, t), \tag{8}$$

where $\psi(x, u, t)$ satisfies the pseudogradient condition

$$\psi(x, u, t)^T \nabla_u w(x, u, t) \geqslant 0. \tag{9}$$

The algorithm (6) is called speed-gradient algorithm [1], since it suggests to change $u(t)$ proportionally to the gradient of the speed of changing $Q\left(x(t), t\right)$.

2.1 *Universal Speed-Gradient Method*

Consider the Taylor approximation for $w(t) = w\left(x(t), u(t), t\right)$, where $\left(x(t), u(t)\right)$ is a trajectory of the system (1)

$$w(t+\tau) = w\left(x(t), u(t), t\right) + \\ + \left(\frac{\partial w}{\partial t}\left(x(t), u(t), t\right) + \frac{\partial w}{\partial x} f\left(x(t), u(t), t\right) + \frac{\partial w}{\partial u}\dot{u}\right)\tau + o(\tau). \tag{10}$$

If $w(x, u, t)$ is non-positive, then (with some additional assumptions) according to a La-Salle principle the control goal (2) is fulfilled.

Consider

$$\dot{u} = -\gamma(x, u, t)\psi(x, u, t), \tag{11}$$

where $\psi : R^n \times R^m \times R \to R^m$, $\gamma : R^n \times R^m \times R \to R$ and

$$\gamma(x, u, t)\frac{\partial w}{\partial u}\psi(x, u, t) > \frac{\partial w(x, u, t)}{\partial t} + \frac{\partial w}{\partial x} f(x, u, t). \tag{12}$$

For example it can be used $\psi(\cdot) = \nabla_u w(\cdot)$ and

$$\gamma(x, u, t) = \frac{\eta(x, u, t) + \lambda\sqrt{\eta(x, u, t)^2 + \zeta(x, u, t)^2}}{\zeta(x, u, t)}, \quad \lambda > 0, \tag{13}$$

with

$$\eta(x, u, t) = \frac{\partial w(x, u, t)}{\partial t} + \frac{\partial w}{\partial x} f(x, u, t), \quad \zeta(x, u, t) = \frac{\partial w}{\partial u}\psi(x, u, t). \tag{14}$$

For affine systems the same algorithm was proposed by Sontag in 1989 [2].

The control algorithm (11) with the inequality (12) we named the "Universal speed-gradient method".

Theorem 1. *Let the following assumptions be valid:*

1. *$w\left(x_*(t), u_*(t), t\right)$ is a twice continuously differentiable function along the trajectories $\left(x_*(t), u_*(t)\right)$ of system (1), (11);*
2. *The function $Q(x, t)$ is nonnegative, uniformly continuous in any set of the form $\{(x, t) : ||x|| < \beta, t \geqslant 0\}$ and radially unbounded;*
3. *For initial condition the inequality $w\left(x(0), u(0), 0\right) \leqslant 0$ is true;*
4. *Inequality (12) is true for all $(x, u, t) : w(x, u, t) = 0,\ Q(x, t) \neq 0$;*
5. *Control (11) is a continuous in (x, u) function;*

then any solution $\left(x(t), u(t)\right)$ of (1), (11) is bounded and $\lim\limits_{t\to+\infty} \frac{d}{dt} Q\left(x_*(t), t\right) = 0$.

Proof. The Taylor approximation (10) is true according to assumption 1. According to assumptions 3, 4 inequality $\dot{Q}\left(x(t), t\right) < 0$ is true for all $(x, u, t) : Q(x, t) \neq 0$. Consequently, from assumption 2 follows that any solution $\left(x(t), u(t)\right)$ of (1) and (11) is bounded and $\lim\limits_{t\to+\infty} \frac{d}{dt} Q\left(x_*(t), t\right) = 0$. □

2.2 Universal Speed-Gradient Method with Constraints

Consider the derivative of constraints (3) along the trajectories of the system (1)

$$\frac{\partial B_k}{\partial t}(x,u,t) + \frac{\partial B_k}{\partial x} f(x,u,t) + \frac{\partial B_k}{\partial u}\dot{u} > 0\ , \quad k = 1,\ldots,\mu. \tag{15}$$

According to (11) consider the following inequalities

$$\gamma(x,u,t)\frac{\partial B_k}{\partial u}\psi(x,u,t) > -\frac{\partial B_k}{\partial t}(x,u,t) - \frac{\partial B_k}{\partial x} f(x,u,t), \quad k = 1,\ldots,\mu. \tag{16}$$

The control algorithm (11) with the inequalities (12) and (16) we named the "Universal speed-gradient method with constraints".

Theorem 2. *Let the following assumptions be valid:*

1. $w\left(x_*(t),u_*(t),t\right)$ *is a twice continuously differentiable function along the trajectories* $\left(x_*(t),u_*(t)\right)$ *of system* (1), (11)*;*
2. *The function* $Q(x,t)$ *is nonnegative, uniformly continuous in any set of the form* $\{(x,t) : ||x|| < \beta, t \geqslant 0\}$ *and radially unbounded;*
3. *For initial condition inequalities* $w\left(x(0),u(0),0\right) \leqslant 0$ *and* $g\left(x(0),u(0),0\right) > 0$ *are true;*
4. *Inequalities* (12)*,* (16) *are true for all* $(x,u,t) : w(x,u,t) = 0$*,* $Q(x,t) \neq 0$*;*
5. *Control* (11) *is a continuous in* (x,u) *function;*

then any solution $\left(x(t),u(t)\right)$ *of* (1)*,* (11) *is bounded, the constraint* (3) *fulfilled and* $\lim_{t\to+\infty} \frac{d}{dt}Q\left(x_*(t),t\right) = 0$.

Proof. The Taylor approximation (10) is true according to assumption 1. According to assumptions 3, 4 inequality $\dot{Q}\left(x(t),t\right) < 0$ is true for all $(x,u,t) : Q(x,t) \neq 0$. Consequently, from assumption 2 follows that any solution $\left(x(t),u(t)\right)$ of (1) and (11) is bounded and $\lim_{t\to+\infty} \frac{d}{dt}Q\left(x_*(t),t\right) = 0$. From the assumptions 3, 4 follows that constraints are fulfilled for any solution $\left(x(t),u(t)\right)$ of (1) and (11). □

3 Two Pendulums Example

Consider the system of two pendulums (Fig. 1) with a single control input

$$\begin{cases} \dot{q}_k = \dfrac{1}{ml^2}p_k\ , \\ \dot{p}_k = -mgl\sin q_k + ul\cos q_k\ , \quad k = 1,2, \end{cases} \tag{17}$$

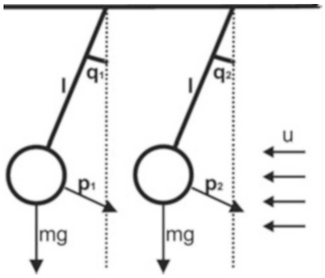

Fig. 1 Two pendulums with a single control input u

the Hamiltonians of pendulums are the following

$$H_0^k(p_k, q_k) = \frac{1}{ml^2} p_k^2 + mgl(1 - \cos q_k), \qquad k = 1, 2. \tag{18}$$

Consider the control goal

$$\lim_{t \to +\infty} H_0^1 \left(p_1(t), q_1(t)\right) = E_1. \tag{19}$$

with phase constraints

$$H_0^2 \left(p_2(t), q_2(t)\right) < E_2 , \quad t \geqslant 0. \tag{20}$$

According to the Speed-gradient approach we obtained the following control function

$$u(p, q) = -\Gamma \left(\frac{p_1 \cos q_1}{ml} \left(H_0^1(p_1, q_1) - E_1\right) + \alpha \frac{p_2 \cos q_2}{ml \left(H_0^2(p_2, q_2) - E_2\right)^2} \right). \tag{21}$$

To demonstrate the ability of the controller to achieve the control goal and to fulfill the phase constraints we carried out computer simulation. The following value of system parameters and initial conditions were chosen: $m = 1$, $l = 1$, $g = 10$, $q_1(0) = 0$, $q_2(0) = 0.05$, $p_1(0) = 0$, $p_2(0) = 0$. Energy goal value for the first pendulum was taken $E_1 = 8$, energy constraint for the second one was taken $E_2 = 5$. Algorithm parameters were: $\Gamma = 0.015$, $\alpha = 10$. Time for simulating was 80 s. Simulations shows that proposed algorithm solve the control problem: energy of the first pendulum converged to the goal value E_1 and the energy of the second was constrained by E_2. The simulating results are presented in (Figs. 2 and 3). The complete analysis of this control system was presented in [3].

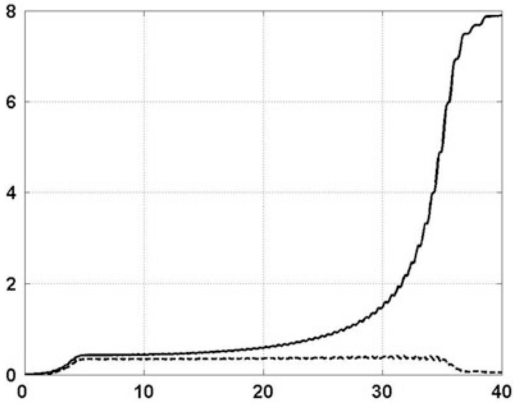

Fig. 2 Energy of pendulums. *Solid line* corresponds to the energy of the first pendulum $H_0^1\left(q_1(t), p_1(t)\right)$, *dash line*—to the energy of the second one $H_0^2\left(q_2(t), p_2(t)\right)$

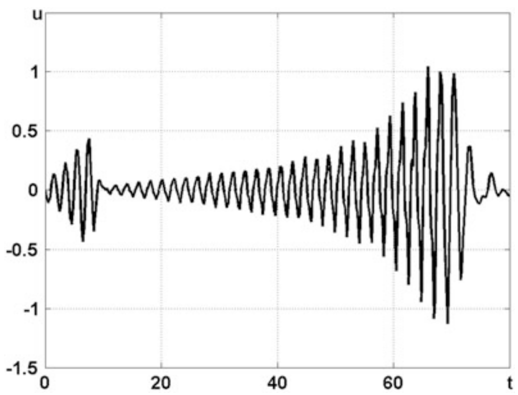

Fig. 3 Control function: $u(t) = u\left(q_1(t), q_2(t), p_1(t), p_2(t)\right)$

4 Molecular Example

Consider a quantum model for diatomic molecule, described by the Schrodinger equation with control [4–6]

$$i\hbar \frac{\partial \Psi(r,t)}{\partial t} = H_0 \Psi(r,t) + f(u) H_1 \Psi(r,t), \tag{22}$$

where

$$H_0 = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial r^2} + V(r)\ , \quad H_1 = A\mu(r), \tag{23}$$

and Morse potential

$$V(r) = D\left(exp\left(-\alpha \frac{r - r_0}{r_0}\right) - 1\right)^2 - D, \tag{24}$$

here $i = \sqrt{-1}$, $\hbar = 1$—Planck constant, $\Psi(t,r)$—wave function, r—distance between nuclei of the molecule, M—reduce mass of the molecule, α—parameter of a nonlinearity, $\mu(r)$—molecular dipole momentum, D—dissociation energy, r_0—distance of equilibrium, u—control function of electromagnetic field, $f(\cdot)$—some function. All parameters are in the atomic Hartree unit system.

The problem is to design the control function $u(t)$ to stabilize the average energy on the goal value:

$$\lim_{t\to+\infty} \phi(t)^* H_0 \phi(t) = E_* \; . \tag{25}$$

All the following calculations are made for a finite-level approximation obtained by a Bubnov-Galerkin method.

According to a speed-gradient method the following goal function is introduced

$$Q(\phi) = (\phi^* H_0 \phi - E_*)^2 \tag{26}$$

and $w(\phi, u, t)$ is calculated

$$w(\phi, u, t) = \dot{Q}(\phi) = 2\frac{i}{\hbar}(\phi^* H_0 \phi - E_*)\phi^*[H_1, H_0]\phi f(u) \; . \tag{27}$$

A Tailor approximation for $w(t) = w\left(\phi(t), u(t), t\right)$ is the following

$$w(t+\tau) = g_0(\phi, u) + \left(g_1(\phi, u) + g_2(\phi, u)\dot{u}\right)\tau + o(\tau) \; , \tag{28}$$

where

$$g_0(\phi, u) = \frac{i}{\hbar}(\phi^* H_0 \phi - E_*)\phi^*[H_1, H_0]\phi f(u) \; , \tag{29}$$

$$g_1(\phi, u) = \frac{i}{\hbar} f(u) \frac{d}{dt}\left((\phi^* H_0 \phi - E_*)\phi^*[H_1, H_0]\phi\right) \; , \tag{30}$$

$$g_2(\phi, u) = \frac{i}{\hbar}(\phi^* H_0 \phi - E_*)\phi^*[H_1, H_0]\phi \frac{d}{du} f(u) \; . \tag{31}$$

According to a universal speed-gradient method the following algorithm was obtained

$$\dot{u} = sat\left(\frac{-g_2(\phi, u)^2 - g_1(\phi, u)}{g_2(\phi, u)}\right), \qquad u(0) = 0. \tag{32}$$

For computer simulation we used the parameters of iodine molecule $J^{127}J^{127}$: $M = 114{,}842$, $\alpha = 4.954$, $D = 0.0572$, $r_0 = 5.0366$, with control function in the following form $f(u) = 0.02\sin(u)$. Computer simulations shows that the energy converged to the goal value. The simulation results presented in Figs. 4 and 5.

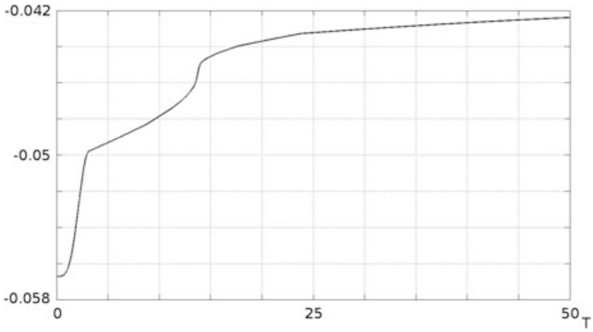

Fig. 4 Average energy: $\phi(t)^* H_0 \phi(t)$, duration: 50 fs (10^{-15} s), goal energy value is $E_* = 0.043$

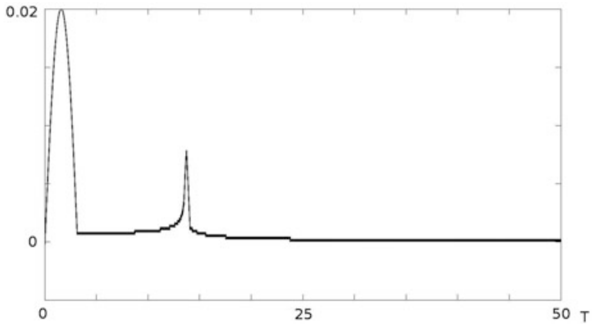

Fig. 5 Control function $f\big(u(t)\big) = A \sin\big(u(t)\big)$, duration: 50 fs ($10^{-15}$ s)

5 Conclusion

A new version of speed-gradient method is proposed that generates "universal" control algorithms both for differential and for finite form. Efficiency of this "universal" method is illustrated by computer simulation for energy control of quantum diatomic molecule. A speed-gradient method for control problems with phase constraints is also proposed and its efficiency is illustrated by computer simulation for selective energy control of two pendulums.

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