Electromagnetism and Generalized Continua

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Abstract In this series of lectures, after an introduction of the most basic elements and some historical perspective on the matter, an exposition of electromagnetic source terms to be taken into account in the Galilean invariant continuum thermodynamics of deformable continua is first given. The emphasis here is placed on the notions of ponderomotive force and couple, of which the latter already hints at some generalization of usual continua by the necessity to envisage nonsymmetric Cauchy stress tensors. Then the notion of magnetic continua endowed with a dynamic magnetic structure, such as in ferromagnets and antiferromagnets, is envisioned. The corresponding modelling can be made by exploiting different methods, among these a direct model of magneto-mechanical interactions in the manner of H. F. Tiersten but also an application of a generalized version of the principle of virtual power, or that of a Hamiltonian variational principle in the absence of dissipation. Such approaches allow one to exhibit a strict analogy with the equations that govern generalized, purely mechanical, continua such as micropolar or oriented media in the manner of the Cosserat brothers or Eringen, by introducing the notions of spin and couple stress. A parallel approach is given for electro-deformable media endowed with permanent electric polarization, e.g., ferroelectrics. Then analogies are established between the resonance couplings arising in certain structures (plates, shells) as shown by Mindlin in classical studies, and those existing for coupled magnetoelastic and electroelastic waves of different types. Finally, the contribution of these electromagnetic microstructures in the computation of configurational forces (e.g., driving forces acting on cracks) is shown to be quite similar to the terms due to a mechanical microstructure.

1 Introduction and Historical Perspective

An introduction to basic properties of electromagnetic properties is to be found in Chapter 1, pp. 1–61, of Maugin (1988).

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Placed in an electric field $\mathbf{E}$, a point-like electric charge (monopole, a scalar) q is subjected to a force

$$\mathbf{F}^e = q\,\mathbf{E}. \tag{1}$$

In the same condition an electric dipole $\mathbf{p}$ (a polar vector) is acted upon by a couple (axial vector)

$$\mathbf{C}^e = \mathbf{p} \times \mathbf{E}. \tag{2}$$

There exists, so far, no evidence for the existence of magnetic monopoles (magnetic charges). But a magnetic dipole $\mathbf{m}$ (an axial vector) placed in a magnetic field $\mathbf{H}$ is the object of a mechanical couple

$$\mathbf{C}^m = \mathbf{m} \times \mathbf{H}. \tag{3}$$

It is this concept that explains the alignment of the needle of a compass with the local earth magnetic field. Full alignment after a transient period nullifies the couple (3).

In a continuous body where one assumes a continuous distribution of electric charges, electric dipoles, magnetic dipoles, etc., the force expression per unit volume generalizing (1) is much more complicated and requires a specific derivation (see Section 2). But the generalizations of (2) and (3) are relatively simple. One defines the electric polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ per unit volume in such a way that (2) and (3) are replaced by

$$\mathbf{C}^m = \mathbf{P} \times \mathbf{E} + \mathbf{M} \times \mathbf{H}. \tag{4}$$

Here $\mathbf{P}$ and $\mathbf{M}$ are primarily determined by $\mathbf{E}$ and $\mathbf{H}$, i.e., we can write symbolically

$$\mathbf{P} = \mathbf{P}(\mathbf{E};.), \qquad \mathbf{M} = \mathbf{M}(\mathbf{H};.) \tag{5}$$

the missing arguments being temperature, strain, etc. Equations (5) are *constitutive equations* that characterize a specific material. The fields $\mathbf{P}$ and $\mathbf{M}$ are “material” fields (they vanish in a vacuum) and are classically defined per unit of matter. They introduce the difference between $\mathbf{E}$ and the electric displacement vector $\mathbf{D}$ on the one hand, and between $\mathbf{H}$ and the magnetic induction $\mathbf{B}$, on the other hand, so that — in so-called Lorentz–Heaviside electromagnetic units —

$$\mathbf{H} = \mathbf{B} - \mathbf{M}, \qquad \mathbf{D} = \mathbf{E} + \mathbf{P}. \tag{6}$$

This, in turn, means that $\mathbf{D}$ can be used instead of $\mathbf{E}$ in equation (4). Similarly, $\mathbf{B}$ can be used in place of $\mathbf{H}$ in that equation.

Equation (4) is important in the present context for the following reason. It has long been difficult to conceive of a purely mechanical means to produce a couple per unit volume. Accordingly, the possible existence of couples such as (4) has regularly been advanced as a justification to consider **nonsymmetric** stress tensors (that are classically shown to be symmetric in the absence of body couple) and this, as we know (cf. Maugin, 2010, 2011a), was the primary reason to introduce the first and simplest generalization of continuum mechanics.

In truth the interaction between electromagnetic behavior and microstructure and the mechanical, statical or dynamical, response of a material may be much more farfetched than simply through the couple (4).

First of all, while (4) may be referred to as the *ponderomotive couple* (the naming is misleading as it still smells of its "point-like" origin), the corresponding *ponderomotive force* (a force per unit volume of the material) may be much more involved than that given by (1). A justification for a rather reasonable form of this force will be given in Section 2.

Second, and that is most important as quite often these are the only remaining effects accounted for by engineers, there may exist couplings of *energetic origin* such as of piezoelectric, electrostrictive, electroelastic, piezomagnetic and magnetostrictive types. It might be a surprise to most readers to learn that *magnetostriction* (a longitudinal strain produced in a mechanically free body by a magnetic field in the length direction of the specimen, and that goes like the *square* of this field) was discovered by James Joule (of electric-conduction and thermodynamics fame) in 1842 in Nickel. This magneto-mechanical coupling exists in all ferromagnetic bodies to a greater or lesser extent. There is no symmetry restriction. The analogous electro-mechanical coupling is called *electrostriction* and also exists in all electro-deformable bodies to a greater or lesser extent for the same reason. Like magnetostriction it is an effect of the *second order* in the field. Because of its smallness the effect was in fact observed and modelled only in the 1920s. The energy electro-magneto-mechanical effects of the *first order* are of different nature and complexity for they necessarily involve appropriate symmetry properties of the considered material. Thus, (inverse) *piezoelectricity*, discovered in 1881 by the Curie brothers after their discovery of the "direct" piezoelectric effect (1880; appearance of electric charges at the boundary of a deformed body), provides a strain that is linear in the applied electric field but only for certain allowed material symmetries of crystals (see Katzir, 2003). It requires the absence of a center of symmetry to allow for a direct coupling between a stress (essentially a second-order tensor variable) and an electric field (essentially a polar vector). *Piezomagnetism*, the somewhat equivalent magneto-mechanical coupling first observed in Russia in 1960 (Borovik-Romanov), is even rarer in that the axial nature of a magnetic field requires a specific magnetic symmetry of the material (e.g., some antiferromagnetic fluorides). Higher order electro- and magneto-mechanical couplings of energy origin are simply referred to as *electroelastic couplings* and *magnetoelastic couplings* of higher order (see, e.g., Maugin et al., 1992). It must be noted that Pierre Curie paid special attention to the differences between polar vectors (e.g., electric polarization) and axial vectors (e.g., magnetization) as this plays a fundamental role in his well known statement of his principle of symmetry.

The just mentioned energetic couplings do not modify before hand the standard balance equations of mechanics (e.g., the balance of moment of momentum: *the stress remains symmetric* as it still is the derivative of the internal or free energy with respect to a symmetric strain). With much more drastic consequences is the fact that some electromagnetic materials are endowed with an electric or magnetic microstructure which, in spite of being of microscopic origin, does influence the mechanical behavior and, in particular, yields a non symmetric stress tensor in an equation of moment of momentum that acquires additional contributions. This is the case in materials exhibiting so-called ferroic states (for a classification of these, see Aizu, 1970). Here, because of the different vectorial natures of electric polarization and magnetization we must distinguish between the magnetic case and the electric case.

Examining first the electric case, we note that electric polarization is akin to a *mechanical displacement* up to an electric charge (cf. Maugin, 1988, chap. 1). That is why in some electric materials we can associate a kind of classical inertia with the *electric microstructure* provided by a network of permanent electric dipoles. That is, $\dot{\mathbf{p}}$ denoting the time derivative of an electric dipole density, we may have to consider an associated "kinetic energy" of the type

$$K(\dot{\mathbf{p}}) = \frac{1}{2}\, d_E\, \dot{\mathbf{p}} \cdot \dot{\mathbf{p}}, \tag{7}$$

where d_E is a kind of inertia to be evaluated from a microscopic model. Such an evaluation was given in a work by Pouget et al. (1986a,b) for ferroelectric materials of the molecular-group type (e.g., $NaNO_2$). An expression such as (7) is reminiscent of the kinetic energy formally associated with so-called directors in Ericksen's (1960) theory of "anisotropic" (microstructured, liquid-crystal) fluids. In addition, because of a prevalent electric ordering in the network of electric dipoles in ferroelectrics, the interaction between neighbouring electric dipoles leads to considering the presence of the gradient of electric polarization in the internal or free density energy of the material considered as a continuum, a kind of *weak nonlocality.*

In the magnetic case of ferroic states — in ferromagnetism, antiferromagnetism and ferrimagnetism — the magnetic dipole density is akin to an angular momentum — a spin, as of a particle in rotation about an axis; see the microscopic definition of magnetization in Maugin (1988). Accordigly, there exists the celebrated gyromagnetic relation of quantum-mechanical origin between a spin density and a magnetization density, e.g., per unit of mass of the material,

$$\mathbf{s} = \gamma^{-1}\,\mu, \tag{8}$$

where γ is the so-called gyromagnetic ratio and μ denotes the magnetization per unit mass (cf. Van Vleck, 1932).

The existence of the relation (8) makes that inertia in magnetic ferroic states is completely different in form from that in ferroelectrics — equation (7). As a matter of fact, since (8) obviously relates to rotational-precessional dynamical properties, there does not exist a closed form for the continuum magnetic kinetic energy in such bodies. The reason is that, magnetization having reached saturation in small domains of the material, we have the important identity

$$\dot{\mathbf{s}} \cdot \omega \equiv 0, \tag{9}$$

where ω is the precessional velocity of the magnetic spin. We say that $\dot{\mathbf{s}}$ is a *d'Alembertian inertia couple* (i.e., like a minute gyroscope, it does not produce any power in a real precessional velocity; see Tiersten, 1964). Only a quantum-mechanical formulation using the appropriate formalism (spinors, Pauli matrices) allows one to introduce an integrated form of the kinetic energy (cf. Nelson and Chen, 1994). Authors not aware of these facts were tempted to introduce a magnetization kinetic energy by analogy with that of electric dipoles (7) and Ericksen's (1960) director theory. This is the case of Lenz (1972) and more recent works that ignore the physical bases (e.g., De Simone and Podio-Guidugli, 1996). This is pitiful because it yields wrong results concerning wave propagation phenomena. But similarly to the ferroelectric case, the interaction between neighboring magnetic spins induces a dependency of the internal or free energy on the gradient of magnetization.

On account of the above-made remarks, we can cite a selection of those papers that have contributed much to the development of the equations, and their dynamical exploitation thereof, of deformable ferroelectrics and ferromagnets in a continuum landscape:

- For ferroelectrics and ionic crystals: Voigt (1928), Tiersten (1971), Mindlin (1972), Maugin (1976c), Maugin and Pouget (1980);
- For ferromagnets and their generalizations: Kittel (1958), Tiersten (1964, 1965), Brown (1966), Akhiezer et al. (1968), Maugin and Eringen (1972a), Maugin (1971, 1976a,b, 1988), Soumahoro and Pouget (1994), Maugin et al. (1992).

From the above exposed arguments, we deduce that *a microstructure related to electric polarization properties* will be primarily described dynamically by an equation that governs a "displacement", i.e., an equation that will resemble a standard equation of motion governing a *linear momentum* in continuum physics. In contrast, *a microstructure related to magnetic properties* will be described dynamically by an equation that will essentially be an equation governing a *moment of momentum.* As to the means of establishing such equations, we identify three basic ones:

(*i*) In the absence of dissipation but knowing the functional dependence of the energy density, a variational principle may be appropriate, but caution should be taken in the magnetic case because of the property (9) — see Tiersten (1965); Maugin and Eringen (1972a). For the electric case see Suhubi (1969).

(*ii*) For a general thermodynamical behavior and basing on an a priori statement of balance laws, one needs a model for introducing the balance equation related to the relevant electromagnetic microstructure. For this approach see Askar et al. (1970), Tiersten (1971), Mindlin (1972), Maugin (1976c) for electric properties; Tiersten (1964), Maugin and Eringen (1972a), Maugin (1976a, p. 1737), Maugin (1988, Figure 6.2.1), Eringen and Maugin (1990, chap. 9) for magnetic properties.

(*iii*) For an equivalent state of generality but a more formal structural algebraic approach, one may exploit a modern formulation of the principle of virtual power in which new electromagnetic degrees of freedom are granted a status equivalent to that of standard deformation processes; for this see Collet and Maugin (1974), Maugin and Pouget (1980), Soumahoro and Pouget (1994) for electric properties, and Maugin (1974, 1976a,b) for magnetic ones; Maugin (1980a) for a general comprehensive presentation.

Before proceeding to the construction of these equations, we need recall the expression of source terms due to electromagnetic fields that appear in standard balance laws of continuum mechanics. This itself is based on the consideration of microscopic fields and a meaningful modelling.

2 Electromagnetic Sources in Galilean Invariant Continuum Physics

2.1 Maxwell's Equations

In magnetized, electrically polarized, and electrically conducting matter, the celebrated set of Maxwell's equations *in a fixed laboratory frame* reads in full generality — and according to Heaviside's synthetic formulation — as

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \qquad \nabla \cdot \mathbf{B} = 0, \tag{10}$$

and

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c}\mathbf{J}, \qquad \nabla \cdot \mathbf{D} = q_f, \tag{11}$$

where c is the velocity of light in vacuum, $\mathbf{J}$ is the electric current vector, and q_f is the density of free electric charges. The first set (10) is valid everywhere and yields the notion of electromagnetic potentials. In order to close the system of field equations (10)–(11), we must be given electromagnetic constitutive equations (5) together with an equation for electric current, e.g., $\mathbf{J} = \mathbf{J}(\mathbf{E}, ..)$ and the relations (6).

Taking the divergence of $(11)_1$ and accounting for $(11)_2$, we obtain the law of *conservation of electric charge*:

$$\frac{\partial q_f}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{12}$$

a *strict* conservation law. Second, by a usual manipulation, one also deduces from (10)–(11) an *energy identity* called the "Poynting–Umov theorem", such that

$$\mathbf{H} \cdot \frac{\partial B}{\partial t} + \mathbf{E} \cdot \frac{\partial D}{\partial t} = -\mathbf{J} \cdot \mathbf{E} - \nabla \cdot \mathbf{S}, \qquad \mathbf{S} \equiv c\,\mathbf{E} \times \mathbf{H}, \tag{13}$$

without any hypothesis concerning the electromagnetic constitutive equations. Note that the Joule term $\mathbf{J}\cdot\mathbf{E}$ can be interpreted as a power expended by an electric force. Indeed, we can write as an example

$$\mathbf{J} \cdot \mathbf{E} = (q\,\mathbf{v}) \cdot \mathbf{E} = (q\,\mathbf{E}) \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v}, \tag{14}$$

where $\mathbf{f} = q\,\mathbf{E}$ is seen in statics, according to Lorentz, as the elementary mechanical force acting on a point particle of electric charge q in an electric field $\mathbf{E}$. For a particle moving at velocity $\dot{\mathbf{x}} = \mathbf{v}$, we have the *Lorentz force*

$$\mathbf{f} = q\,\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B} = q\,\widetilde{\mathbf{E}}, \qquad \widetilde{\mathbf{E}} = \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}, \tag{15}$$

where the electric field $\widetilde{\mathbf{E}}$ is called the *electromotive intensity*. In addition to this field, we will be led to introducing other fields in a so-called co-moving frame:

$$\widetilde{\mathbf{J}} = \mathbf{J} - q_f\,\mathbf{v}, \qquad \widetilde{\mathbf{M}} = \mathbf{M} + \frac{1}{c}\mathbf{v} \times \mathbf{P} \tag{16}$$

where the first is the conduction current per se, and the last is the magnetization per unit volume in this frame. The large classes of electromagnetic materials are defined thus:

- *Insulators*:
$$q_f = 0, \qquad \widetilde{\mathbf{J}} = \mathbf{0}; \tag{17}$$
- *Non-polarized materials* [Galilean approximation; compare to $(16)_2$, see Maugin (1988)]:
$$\widetilde{\mathbf{P}} \equiv \mathbf{P} = \mathbf{0}. \tag{18}$$
- *Non-magnetized materials*:
$$\widetilde{\mathbf{M}} = \mathbf{0}. \tag{19}$$
This reduces to $\mathbf{M} = \mathbf{0}$ in statics or if the material simultaneously is non polarized electrically.
- *Dielectric materials*: These are insulators in which, generally,
$$\mathbf{P} \neq \mathbf{0}. \tag{20}$$
- In *quasi-electrostatics* (true electro-magnetic effects discarded but the remaining fields are still time-dependent), (10) and (11) reduce to
$$\nabla \times \mathbf{E} = \mathbf{0}, \qquad \nabla \cdot \mathbf{D} = q_f, \qquad \mathbf{D} = \mathbf{E} + \mathbf{P}; \tag{21}$$
The first of these tells us that we can introduce a scalar electric potential φ such that $\mathbf{E} = -\nabla\varphi$.
- In *quasi-magnetostatics* (true electro-magnetic effects discarded but the remaining fields are still time-dependent), (10) and (11) reduce to
$$\nabla \times \mathbf{H} = \frac{1}{c}\mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \mathbf{H} = \mathbf{B} - \mathbf{M}. \tag{22}$$
- For *dielectrics* one has to take $q_f = 0$ in the second of (21), while for *insulators* one has to take $\mathbf{J} = \mathbf{0}$ in the first of (22). In this case $\nabla \times \mathbf{H} = \mathbf{0}$ and one can introduce a scalar magneto-static potential ϕ such that $\mathbf{H} = -\nabla\phi$.

Equation (12) is fully discarded in the case of dielectrics. The Poynting–Umov theorem (13) takes the form
$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = -\nabla \cdot \mathbf{S} \tag{23}$$
in non-magnetized dielectrics in quasi-electrostatics, while it reduces to
$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\nabla \cdot \mathbf{S}, \tag{24}$$
in insulating non-polarized magnetic materials in quasi-magnetostatics.

These two equations show that without loss in generality we can define the Poynting–Umov energy flux as

$$\mathbf{S} = \varphi \frac{\partial \mathbf{D}}{\partial t} \quad \text{and} \quad \mathbf{S} = \phi \frac{\partial \mathbf{B}}{\partial t} \tag{25}$$

in the corresponding approximations because $\nabla \cdot \mathbf{D} = 0$ in dielectrics and $\nabla \cdot \mathbf{B} = 0$ always, respectively.

2.2 Ponderomotive Force and Couple in a Continuum

The generalization of the force expression (15) and the corresponding couple in a general electromagnetic continuum is a difficult matter that was pondered for a long time. Rather than postulating a form (on what bases?) we prefer to follow the line of H. A. Lorentz (1909, 1952) already followed by Dixon and Eringen (1965), Nelson (1979) and Maugin and Eringen (1977). This involves a type of homogenization (passing from the discrete to the continuum) introducing the approximations of multipoles, a truncation of these at a certain order, and effecting a volume or phase-space average. Lorentz's vision is essentially that of a free space containing charged point particles (Figure 1). We report here only the general traits of this derivation, and give the resulting expressions in the comprehensive form given by Maugin

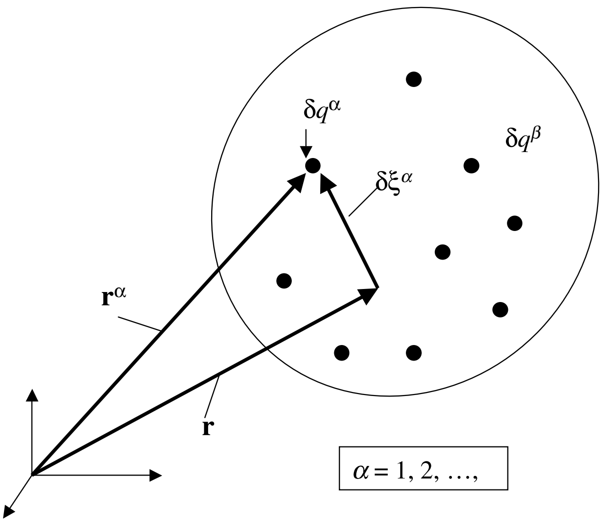

Figure 1. Stable group of elementary electric charges in the Lorentz's averaging approach.

and Eringen (1977). Each elementary electric charge δq_α, $\alpha = 1, 2, \ldots$ contained in a representative volume element ΔV is acted upon by a *Lorentz force* (compare (15))

$$\delta \mathbf{f}_\alpha = \delta q_\alpha \left(\mathbf{e}(\mathbf{r}_\alpha) + \frac{1}{c}\, \dot{\mathbf{x}}_\alpha \times \mathbf{b}(\mathbf{r}_\alpha) \right), \tag{26}$$

where $\mathbf{e}$ and $\mathbf{b}$ are the electric field and magnetic induction at the actual placement $\mathbf{r}_\alpha$ of the charge δq_α. The computation consists then in evaluating the quantities (here, for the sake of simplicity, we adopt a simple volume average, while De Groot and Suttorp (1972) use a relativistically invariant phase average):

$$\sum_{\alpha \in \Delta V} \delta \mathbf{f}_\alpha, \qquad \sum_{\alpha \in \Delta V} (\mathbf{r}_\alpha \times \delta \mathbf{f}_\alpha), \qquad \sum_{\alpha \in \Delta V} \delta \mathbf{f}_\alpha \cdot \dot{\mathbf{x}}_\alpha, \tag{27}$$

and then dividing by ΔV. On neglecting quadrupole contributions and higher order multipoles, lengthy calculations (cf. Maugin and Eringen, 1977) lead to electromagnetic source terms of force, couple **and** energy per unit continuum volume:

$$\mathbf{f}^{em} = q_f\, \widetilde{\mathbf{E}} + \frac{1}{c}\, (\widetilde{\mathbf{J}} + \mathbf{P}^*) \times \mathbf{B} + (\mathbf{P} \cdot \nabla)\, \mathbf{E} + (\nabla \mathbf{B}) \cdot \widetilde{\mathbf{M}}, \tag{28}$$

$$\mathbf{c}^{em} = \mathbf{r} \times \mathbf{f}^{em} + \tilde{\mathbf{c}}^{em}, \tag{29}$$

$$w^{em} = \mathbf{f}^{em} \cdot \mathbf{v} + \tilde{\mathbf{c}}^{em} \cdot \Omega + \rho\, h^{em}, \tag{30}$$

where $\mathbf{r}$ refers to the center of charges of the volume element, ρ is the matter density, and $\mathbf{v}$ is the physical velocity of the continuum, Ω is the vorticity $\Omega = (\nabla \times \mathbf{v})/2$, and we have set

$$q_f(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \delta q_\alpha, \tag{31}$$

$$\mathbf{P}(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \delta q_\alpha\, \xi_\alpha(\mathbf{x}, t), \tag{32}$$

$$\mathbf{M}(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \frac{1}{2\, c}\, \delta q_\alpha\, \xi_\alpha \times \dot{\xi}_\alpha, \tag{33}$$

where $\xi_\alpha = \mathbf{x}_\alpha(t) - \mathbf{x}$ are internal coordinates vectors in ΔV. Note the lack of symmetry between polarization and magnetization effects which clearly demonstrates the “displacement” nature of the polarization and the axial nature (involving a rotation) of the magnetization — cf. section 1 above.

Various contributions in (28) are easily identified. The first term is a continuum generalization of (1). The third term means that $\mathbf{P}$, essentially a continuum density of electric dipoles, is the object of a force when placed in a spatially nonuniform electric field. The last term is quite similar but it relates to a continuum density of magnetic dipoles that feels a nonuniform magnetic induction. As to the second term, it can be understood since a time varying electric polarization creates a displacement current according to Maxwell's equations. We have also defined the intrinsic electromagnetic sources of couple, energy and stress by (here tr = trace; subscript s stands for symmetrization)

$$\tilde{\mathbf{c}}^{em} = \mathbf{P} \times \widetilde{\mathbf{E}} + \widetilde{\mathbf{M}} \times \mathbf{B}, \tag{34}$$

$$\rho\, h^{em} = \widetilde{\mathbf{J}} \cdot \widetilde{\mathbf{E}} + \widetilde{\mathbf{E}} \cdot \mathbf{P}^{*} - \widetilde{\mathbf{M}} \cdot \mathbf{B}^{*} + \mathrm{tr}\left(\tilde{\mathbf{t}}^{em}\,(\nabla \mathbf{v})_S\right), \tag{35}$$

and

$$\tilde{\mathbf{t}}^{em} = \mathbf{P} \otimes \widetilde{\mathbf{E}} - \mathbf{B} \otimes \widetilde{\mathbf{M}} + \left(\widetilde{\mathbf{M}} \cdot \mathbf{B}\right)\mathbf{1}. \tag{36}$$

We note that $\tilde{\mathbf{c}}^{em}$ is the axial vector dual to the skewsymmetric part of this last tensor. Electromagnetic fields in a co-moving frame have already been defined while $\mathbf{E}$ and $\mathbf{B}$ are simple volume averages of $\mathbf{e}$ and $\mathbf{b}$. The first contribution in the *r-h-s* of (28) is none other than a "Lorentz force" per unit volume (compare (15)).

Finally, a right asterisk denotes a so-called convected time derivative such that

$$\mathbf{P}^{*} = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{v}) + \mathbf{v}\,(\nabla \cdot \mathbf{P}) = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} - (\mathbf{P} \cdot \nabla)\mathbf{v} + \mathbf{P}\,(\nabla \cdot \mathbf{v}). \tag{37}$$

In principle, the above obtained *source terms*, once their origin forgotten, have to be carried into the classical balance laws of a continuum (allowing for a possibly *non symmetric* Cauchy stress), leaving however the internal/free energy of the medium to depend on the electromagnetic fields. A remarkable fact is that in spite of their farfetched outlook, some may be given a form that reminds us of some standard expression (such as in (5)). For instance, Maugin and Eringen (1977) have shown that (30) can also be written as

$$\begin{aligned} w^{em} &= \mathbf{J} \cdot \mathbf{E} + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} + \nabla\big(\mathbf{v}\,(\mathbf{E} \cdot \mathbf{P})\big) \\ &= -\frac{\partial u^{em.f}}{\partial t} - \nabla \cdot \big(\mathbf{S} - \mathbf{v}\,(\mathbf{E} \cdot \mathbf{P})\big), \end{aligned} \tag{38}$$

in which we identify some of the terms in (13) or a possible direct combination with some of them. The volume energy density of *free* electromagnetic fields $u^{em.f}$ is given by

$$u^{em.f} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right). \tag{39}$$

Another interesting equivalent form of (38) is given by (cf. Maugin and Eringen, 1977, a superimposed dot denotes the material time derivative):

$$w^{em} = \mathbf{f}^{em} \cdot \mathbf{v} + \rho\, \dot{e}^{mag} + \rho\, \widetilde{\mathbf{E}} \cdot \dot{\pi} + \rho\, \mathbf{B} \cdot \dot{\mu} + \widetilde{\mathbf{J}} \cdot \widetilde{\mathbf{E}}, \tag{40}$$

where π and μ are the electric polarization and magnetization per unit mass

$$\pi = \mathbf{P}/\rho, \qquad \mu = \widetilde{\mathbf{M}}/\rho, \tag{41}$$

and

$$e^{mag} = -\mu \cdot \mathbf{B} \tag{42}$$

is the energy of magnetic doublets per unit mass.

Particular Cases

We have the following obvious reductions for the ponderomotive force and couple and the accompagning energy source:

In the quasi-electrostatics of dielectrics:

$$\begin{aligned} \mathbf{f}^{em} &= (\mathbf{P} \cdot \nabla)\, \mathbf{E} \equiv (\nabla \mathbf{E}) \cdot \mathbf{P}, \\ \tilde{\mathbf{c}}^{em} &= \mathbf{P} \times \mathbf{E} = \mathbf{P} \times \mathbf{D}, \end{aligned} \tag{43}$$

$$w^{e} = \mathbf{f}^{em} \cdot \mathbf{v} + \rho\, \mathbf{E} \cdot \dot{\pi}, \tag{44}$$

In the quasi-magnetostatics of insulators:

$$\begin{aligned} \mathbf{f}^{em} &= (\nabla \mathbf{B}) \cdot \mathbf{M} \equiv (\mathbf{M} \cdot \nabla)\, \mathbf{H} + \tfrac{1}{2}\, \nabla \mathbf{M}^2, \\ \tilde{\mathbf{c}}^{em} &= \mathbf{M} \times \mathbf{B} \equiv \mathbf{M} \times \mathbf{H}, \end{aligned} \tag{45}$$

$$w^{em} = \mathbf{f}^{em} \cdot \mathbf{v} + \rho\, \dot{e}^{mag} + \rho\, \mathbf{B} \cdot \dot{\mu}. \tag{46}$$

3 Deformable Magnetized Bodies with Magnetic Microstructure

3.1 Model of Interactions

For the sake of simplicity we consider the *quasi-magnetostatics* of non-electrically polarized *insulators*.

Using the standard notation of nonlinear continuum mechanics we may consider to start with the following *generalized motion* for deformable magnetized bodies of the ferroïc type:

$$\mathbf{x} = \overline{\mathbf{x}}(\mathbf{X}, t); \qquad \mu = \bar{\mu}(\mathbf{X}, t), \tag{47}$$

where the first of these denotes the classical finite deformation at Newtonian time t between the reference configuration K_R and the actual configuration K_t. Here $\mathbf{x}$ is the placement of Euclidean coordinates x_i, $i = 1, 2, 3$, and $\mathbf{X}$ denotes the material point of coordinates X_K, $K = 1, 2, 3$ in material space. The second of (47) means that the magnetization per unit mass here is considered as a primary quantity. The reason for this is that in ferroïc states in small regions of the bodies (so-called domains), one may have a nonvanishing magnetization in the absence of applied magnetic field. This in fact is the very definition of such a state. Borrowing the denomination introduced by Tiersten (1964), we may say that the first of (47) describes the time evolution of the **lattice continuum** or LC (standard matter in the macroscopic description), while — because of the relation (8) — the second of (47) provides the time evolution of the (magnetic or electronic) **spin continuum** or SC. These two continua should be treated on an equal footing in the vision of **generalized continua**. But they do not respond exactly to the same kind of loads while we must also envisage interactions between these two "continua".

In particular, the spin continuum cannot translate with respect to the lattice continuum. It, therefore, "expands" and "contracts" with the lattice continuum and, accordingly, its volumetric behavior is governed by the usual continuity equation. As usual, the lattice continuum is assumed to be able to respond to volume and surface forces, hence exhibits stresses, and to volume couples, so that stress is not expected to be symmetric. We assume that it is not equipped with any mechanism to respond to surface couples, so that it does not exhibit couple stresses of mechanical origin. The balance of linear (physical) momentum simply says that whatever force of magnetic origin — e.g., the reduced ponderomotive force in (45) — is applied to a point in the spin continuum, it is directly transferred to the lattice continuum at the same point. The spin continuum, by its very nature, can respond only to **couples**, which may be either of the volume or of the surface type. Accordingly, we consider that the ponderomotive couple (cf. Equations (45))

$$\mathbf{c}^{em} = \widetilde{\mathbf{M}} \times \mathbf{B} = \rho\,\mu \times \mathbf{B} \tag{48}$$

is directly applied to the spin continuum.

In so far as the interactions between lattice and spin continua are concerned, they must necessarily be of the couple type since the spin continuum is sensitive only to that type of interaction. Following Tiersten (1964), we naturally assume that this couple is due to a *local magnetic induction* $\mathbf{B}^L$ — to be given a constitutive equation —, so that we can apply the "recipe" (compare (48))

$$\mathbf{c}_{(LC/SC)} = \mathbf{M} \times \mathbf{B}^L = \rho\,\mu \times \mathbf{B}^L. \tag{49}$$

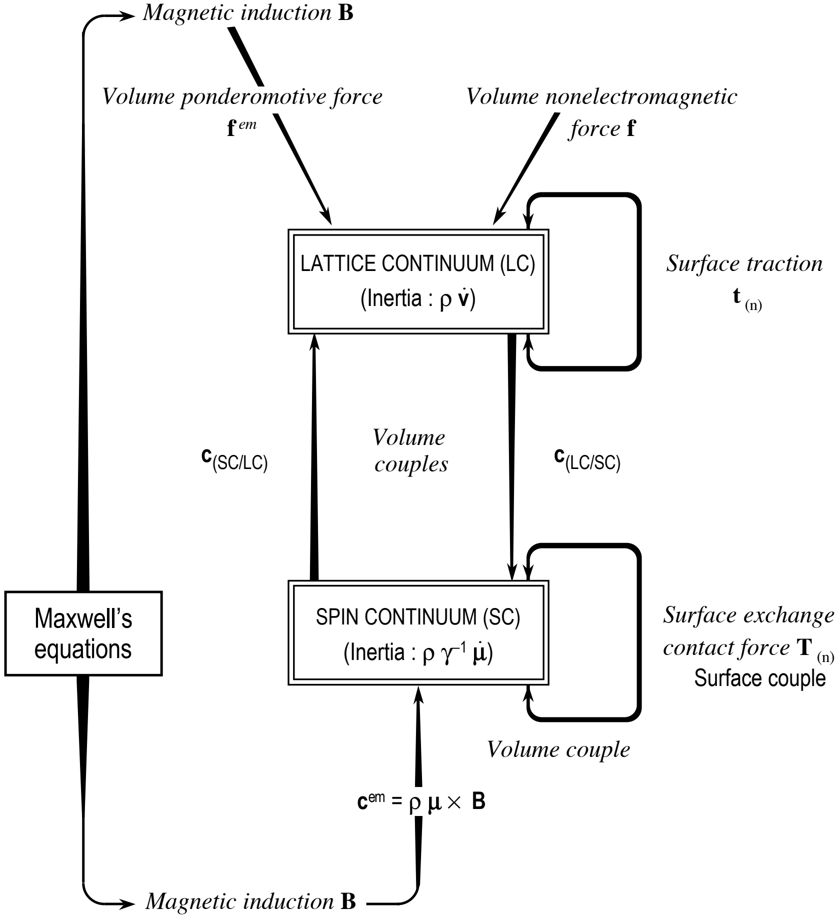

Figure 2. Interactions in deformable ferromagnets (after Maugin, 1979).

Angular momentum being conserved (exchanged) between the two continua, an equal and opposite couple (see Figure 2)

$$\mathbf{c}_{(SC/LC)} = -\mathbf{c}_{(LC/SC)} = \mathbf{B}^L \times \mathbf{M} = \rho \mathbf{B}^L \times \mu \tag{50}$$

is exerted on the unit volume of the lattice continuum.

Finally, in order to account for ferromagnetic (Heisenberg) exchange forces of quantum origin (interactions between neighbouring spins) that fall off rapidly with distance, we can represent these "forces" in a continuum description by a *contact action* in much the same manner as the stress vector for a Cauchy deformable continuum, except that this "surface exchange contact force" must also obey the "recipe" of a magnetic couple (the density ρ is included in $\mathbf{A}$):

$$\mathbf{c}_{(LC-surface)} = \mu \times \mathbf{A}, \tag{51}$$

where $\mathbf{A}$ is an axial vector that depends on the local unit normal $\mathbf{n}$ to the surface and can be written in the same way as the classical stress vector (Cauchy principle), i.e.,

$$\mathbf{A} = \mathbf{A}_{(n)} = \mathbf{A}(\mathbf{x}, t; \mathbf{n}) = \mathbf{n} \cdot \widehat{\mathbf{B}}(\mathbf{x}, t), \tag{52}$$

so that (51) yields the following surface couple density acting on the spin continuum:

$$\mathbf{c}_{(SC-surface)} = \mu \times \mathbf{A} = \mu \times (\mathbf{n} \cdot \widehat{\mathbf{B}}). \tag{53}$$

Because of this very expression we can surmise that only the portion of $\mathbf{A}$ orthogonal to $\mathbf{M}$ is effectively defined. Thus, without loss in generality we can set forth the following condition:

$$\mathbf{A} \cdot \mathbf{M} = \mathbf{n} \cdot \widehat{\mathbf{B}} \cdot \mathbf{M} = 0 \tag{54}$$

at any point at the surface of the body. A similar orthogonality condition can be imposed on $\mathbf{B}^L$ but at any point inside the body.

3.2 Statement of Global Balance Laws

Collecting now the various proposed expressions according to the scheme shown in Figure 2, we can write down the global balance laws at time t in K_t for a magnetic body of volume B_t and regular bounding surface ∂B_t (for the sake of simplicity we ignore any discontinuity surface within the body; for the equations at discontinuity surfaces, see Maugin (1988)):

Balance of mass for the combined continuum:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_B \rho \, \mathrm{d}v = 0; \tag{55}$$

Balance of linear momentum for the LC:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_B \rho \mathbf{v} \, \mathrm{d}v = \int_B (\mathbf{f} + \mathbf{f}^{em}) \, \mathrm{d}v + \int_{\partial B} \mathbf{t}_{(n)} \, \mathrm{d}a; \tag{56}$$

Balance of angular momentum for the LC:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_B (\mathbf{r} \times \rho \mathbf{v}) \, \mathrm{d}v = & \int_B (\mathbf{r} \times (\mathbf{f} + \mathbf{f}^{em}) + \mathbf{c}_{(SC/LC)}) \, \mathrm{d}v \\ & + \int_{\partial B} (\mathbf{r} \times \mathbf{t}_{(n)}) \, \mathrm{d}a; \end{aligned} \tag{57}$$

Balance of angular momentum for the SC:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \rho\,\gamma^{-1}\,\mu\,\mathrm{d}v = \int_B (\mathbf{c}^{em} + \mathbf{c}_{(LC/SC)})\,\mathrm{d}v + \int_{\partial B} \mu \times \mathbf{A}_{(n)}\,\mathrm{d}a; \tag{58}$$

First law of thermodynamics for the combined continuum:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_B \rho\left(\frac{1}{2}\mathbf{v}^2 + e\right)\,\mathrm{d}v &= \int_B (\mathbf{f}\cdot\mathbf{v} + w^{em} + \rho\,h)\,\mathrm{d}v \\ &\quad + \int_{\partial B}\left(\mathbf{t}_{(n)}\cdot\mathbf{v} + \mathbf{A}_{(n)}\cdot\dot{\mu} + q_{(n)}\right)\,\mathrm{d}a;\end{aligned} \tag{59}$$

Second law of thermodynamics for the combined continuum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \rho\,\eta\,\mathrm{d}v \geq \int_B \rho\,\theta^{-1}\,h\,\mathrm{d}v + \int_{\partial B}\theta^{-1}\,q_{(n)}\,\mathrm{d}a. \tag{60}$$

In these equations, $\mathbf{t}_{(n)}$ is the surface traction, $\mathbf{f}$ is a body mechanical force (e.g., gravity), e is the internal energy per unit mass, η is the entropy per unit mass, h is the body heat source, q is the heat influx. Classically (compare (52)),

$$\mathbf{t}_{(n)} = \mathbf{n}\cdot\mathbf{t}, \qquad q_{(n)} = -\mathbf{n}\cdot\mathbf{q}, \tag{61}$$

where $\mathbf{t}$ is the Cauchy stress and $\mathbf{q}$ is the heat-flux vector.

Standard localization of these global equations on account of the assumed continuity of all fields yields the following *local equations* at any point in B:

$$\dot{\rho} + \rho\,\nabla\cdot\mathbf{v} = 0, \tag{62}$$

$$\rho\,\dot{\mathbf{v}} = \mathrm{div}\;\mathbf{t} + \mathbf{f} + \mathbf{f}^{em}, \tag{63}$$

$$\varepsilon_{ijk}\,(t_{jk} + \rho\,B^L_j\,\mu_k) = 0, \tag{64}$$

$$\gamma^{-1}\,\dot{\mu}_i = \left[\mu\times\left(\mathbf{B} + \mathbf{B}^L + \rho^{-1}\,\mathrm{div}\;\widehat{\mathbf{B}}\right)\right]_i + \rho^{-1}\,\varepsilon_{ijk}\,\widehat{B}_{pk}\,\mu_{j,p}, \tag{65}$$

$$\begin{aligned}\rho\left(\dot{e} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{v}^2\right)\right) &= t_{ji}\,v_{i,j} + (t_{kj,k}\,v_j + \mathbf{f}\cdot\mathbf{v}) + \widehat{B}_{kj}\,\dot{\mu}_{j,k} \\ &\quad + \widehat{B}_{kj,k}\,\dot{\mu}_j + w^{em} + \rho\,h - \nabla\cdot\mathbf{q},\end{aligned} \tag{66}$$

and

$$\rho\,\dot{\eta} \geq \theta^{-1}\,\rho\,h - \theta^{-1}\,\nabla\cdot\mathbf{q} - \mathbf{q}\cdot\nabla(\theta^{-1}), \tag{67}$$

where the divergence of nonsymmetric tensors is to be taken on the first index, and a superimposed dot denotes the classical material time derivative.

Equations (66), (64), (65) and (67) are transformed thus. In (66), we must account for the kinetic energy theorem obtained by taking the inner product of the motion equation (63) by $\mathbf{v}$:

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathbf{v}^2\right) = (t_{kj,k}\, v_j + \mathbf{f}\cdot\mathbf{v}) + \mathbf{f}^{em}\cdot\mathbf{v}, \tag{68}$$

while w^{em} is given by the reduced form (46). Therefore, (66) reads

$$\begin{aligned} &\rho\,\dot{\hat{e}} = t_{ji}\, v_{i,j} + \rho\,\mathbf{B}\cdot\dot{\mu} + \widehat{B}_{kj}\,\dot{\mu}_{jk} + \widehat{B}_{kj,k}\,\dot{\mu}_j + \rho\, h - \nabla\cdot\mathbf{q}, \\ &\hat{e} = e - e^{mag} \equiv e + \mu\cdot\mathbf{B}. \end{aligned} \tag{69}$$

But if (8) and (9) hold good, $\dot{\mu}$ must be of the purely *precessional* form

$$\dot{\mu} = \omega\times\mu. \tag{70}$$

This corresponds to saturation of the magnetization in each magnetic domain. As a consequence the last contribution in (65) must vanish:

$$\widehat{B}_{k[j}\,\mu_{i],k} = 0. \tag{71}$$

On account of this we check that

$$(\mathrm{div}\,\widehat{\mathbf{B}})\cdot\dot{\mu} = -\rho\,(\mathbf{B}+\mathbf{B}^L)\cdot\dot{\mu}, \tag{72}$$

because

$$\omega = -\gamma\,\mathbf{B}^{eff}, \qquad \mathbf{B}^{eff} = \mathbf{B} + \mathbf{B}^L + \rho^{-1}\,\mathrm{div}\,\widehat{\mathbf{B}}. \tag{73}$$

This may be viewed as a continuum generalization of the celebrated Larmor precession equation $\omega_{\mathrm{Larmor}} = -\gamma\,\mathbf{B}$ for an isolated electron in a magnetic induction $\mathbf{B}$.

Finally, (69) transforms to the following form using an intrinsic notation (T = transpose):

$$\rho\,\dot{\hat{e}} = \mathrm{tr}\big[\mathbf{t}\,(\nabla\mathbf{v})^T\big] - \rho\,\mathbf{B}^L\cdot\dot{\mu} + \mathrm{tr}\big[\widehat{\mathbf{B}}\,(\nabla\dot{\mu})^T\big] - \nabla\cdot\mathbf{q} + \rho\, h. \tag{74}$$

We let the reader show that in the same conditions (67) provides the following *Clausius–Duhem inequality*:

$$-\rho\,(\dot{\widehat{\psi}} + \eta\,\dot{\theta}) + \mathrm{tr}\big[\mathbf{t}\,(\nabla\mathbf{v})^T\big] - \rho\,\mathbf{B}^L\cdot\dot{\mu} + \mathrm{tr}\big[\widehat{\mathbf{B}}\,(\nabla\dot{\mu})^T\big] - \theta^{-1}\,\mathbf{q}\cdot\nabla\theta \geq 0, \tag{75}$$

wherein the free energy density has been defined by

$$\widehat{\psi} = \hat{e} - \eta\,\theta. \tag{76}$$

Equation (75) nowadays plays an essential role in the construction of constitutive equations that we need for the set of quantities

$$\left\{\widehat{\psi}, \eta, \mathbf{t}, \mathbf{B}^L, \widehat{\mathbf{B}}, \mathbf{q}\right\}. \tag{77}$$

Equation (75) represents a constraint imposed by thermodynamic irreversibility (in particular in the so-called Coleman–Noll exploitation that we shall adopt here). In this formulation we usually look for the expression of so-called objective (or materially indifferent) entities. To that purpose we should rewrite (75) in terms of such quantities. This is achieved as follows. On the one hand we note from (64) that the skewsymmetric part of $\mathbf{t}$ is given by

$$t_{[ji]} = \rho\, \mu_{[j}\, B^L_{i]} \tag{78}$$

and we can write

$$\mathbf{t} = \mathbf{t}^S + \mathbf{t}^A, \quad \text{i.e.,} \quad t_{ji} = t_{(ji)} + t_{[ji]}. \tag{79}$$

We introduce the following objective time rates (Maugin, 1974):

$$D_{ij} = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \tag{80}$$

and

$$\widehat{m}_i = (D_J\, \mu)_i \equiv \dot{\mu}_i - \Omega_{ij}\, \mu_j, \qquad \widehat{M}_{ij} = (\dot{\mu}_i)_{,j} - \Omega_{ik}\, \mu_{k,j}, \tag{81}$$

with

$$\Omega_{ij} = \frac{1}{2}\,(v_{i,j} - v_{j,i}). \tag{82}$$

We let the reader prove that the quantities defined in (80) and (81) are indeed *objective*.

The first of (81) is none other than a so-called *Jaumann* derivative. The second of (81) is not exactly the Jaumann derivative of the gradient of μ, but it is closely related to it modulo a term involving the rate of strain (81). On account of these we show that

$$\begin{aligned} &\mathrm{tr}\big[\mathbf{t}\,(\nabla\mathbf{v})^T\big] - \rho\,\mathbf{B}^L\cdot\dot{\mu} + \mathrm{tr}\big[\widehat{\mathbf{B}}\,(\nabla\dot{\mu})^T\big] \\ &\equiv \mathrm{tr}\big(\mathbf{t}^S\,\mathbf{D}\big) - \rho\,\mathbf{B}^L\cdot\widehat{\mathbf{m}} + \mathrm{tr}\big(\widehat{\mathbf{B}}\,\widehat{\mathbf{M}}^T\big), \end{aligned} \tag{83}$$

whence the looked for reduced useful expression for (74) and (75).

In summary, for the present modeling the local field equations at any regular material point in the body B are provided by equations (62), (63), (70), (74) and the reduced form of Maxwell's equations

$$\nabla\times\mathbf{H} = \mathbf{0}, \quad \nabla\cdot\mathbf{B} = 0, \quad \mathbf{H} = \mathbf{B} - \rho\,\mu, \tag{84}$$

in which the Cauchy stress $\mathbf{t}$ is given by equations (78) and (79), and the effective magnetic induction $\mathbf{B}^{eff}$ is given by (73). Equation (74) will eventually provide the heat-propagation equation while the inequality (75) constrains the constitutive behavior. Interestingly enough, we note that the energy equation (59) does not contain any contribution due to the spin lattice by virtue of the d'Alembertian nature of this quantity (cf. (9)) — more on this point in the following two paragraphs.

3.3 Approach via the Principle of Virtual Power

In modern continuum mechanics, an elegant and powerful means of constructing field equations and associated natural boundary conditions is provided by an algebraically structured formulation of the (d'Alembert) principle of virtual power as exposed at length in Maugin (1980a). In this somewhat abstract formulation this principle is enunciated in the following form for global powers over the body B and its boundary ∂B: *The virtual power of inertial forces is, at each instant of time, balanced by the total virtual power of "internal forces" and that of externally applied forces both in the bulk and at the surface*, the word "force" being understood in a generalized manner. Inertial forces have an expression provided by physics, internal forces need to be given a constitutive equation, and external forces are prescribed in form and perhaps in value. In mathematical terms:

$$P^*_{inert}(B) = P^*_{int}(B) + P^*_{extern}(B, \partial B), \tag{85}$$

where an asterisk will denote the value of an expression is a so-called virtual velocity field (itself noted with an asterisk). In the present case, the generalized kinematical description of the model (47) from which the basic virtual velocity field is given by

$$v^* = \left\{ v^*_i, (\dot{\mu}_i)^* = (\omega^* \times \mu)_i \right\}, \tag{86}$$

where ω^* is a virtual precessional velocity of the SC. Thus

$$P^*_{inert}(B) = \int_B (\rho\, \dot{\mathbf{v}} \cdot \mathbf{v}^* + \gamma^{-1} \rho\, \dot{\mu} \cdot \omega^*)\, \mathrm{d}v, \tag{87}$$

where we clearly distinguish between real fields (no asterisks; actual solutions of a problem) and virtual ones (noted with an asterisk; at our disposal in this type of variational formulation). In particular, for real fields, because of (8) and (9), (86) yields

$$P_{inert}(B) = \frac{\mathrm{d}}{\mathrm{d}t} \int_B \left(\frac{1}{2} \rho\, \mathbf{v}^2 \right)\, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} K(B), \tag{88}$$

where $K(B)$ is the total kinetic energy of the traditional motion.

The total power of external forces is obviously given by the following expression:

$$P^*_{extern}(B, \partial B) = P^*(B) + P^*(\partial B) \tag{89}$$

wherein

$$P^*(B) = \int_B \left((\mathbf{f} + \mathbf{f}^{em}) \cdot \mathbf{v}^* + \rho\, \mathbf{B} \cdot (\dot{\mu})^* \right) \mathrm{d}v, \tag{90}$$

and

$$P^*(\partial B) = \int_{\partial B} \left((\mathbf{t}_{(n)} + \mathbf{t}^{em}_{(n)}) \cdot \mathbf{v}^* + \mathbf{A} \cdot (\dot{\mu})^* \right) \mathrm{d}a, \tag{91}$$

where $\mathbf{t}^{em}_{(n)}$ is an eventual magnetic surface traction related to the possible existence of a magnetic field outside B (see Maugin, 1988, chap. 6).

Finally, the global virtual power of internal forces is constructed as follows. First a "gradient order" is selected for the kinematics associated with internal forces. Generalizing classical continuum mechanics (which is a first order gradient theory of displacement) we consider a first-order gradient theory based on (86). That is,

$$V = \left\{ v_i,\ v_{i,j},\ \dot{\mu}_i,\ \dot{\mu}_{i,j} \right\}. \tag{92}$$

But internal forces must be *objective*, i.e., frame indifferent, or invariant under changes of observer in the actual configuration (superimposition of a rigid body motion of dimension 6). Accordingly, one must extract from the 24-dimensional space spanned by (92) a set of objective quantities, this set V_{obj}, a quotient space, being necessarily of dimension $24 - 6 = 18$. We have shown elsewhere (Maugin, 1980a) how to systematically construct such quotient spaces. In the present case a good set is given by

$$V_{obj} = \left\{ D_{ij},\ \widehat{m}_i,\ \widehat{M}_{ij} \right\}, \tag{93}$$

where it happens that the quantities thus formally introduced have already been defined in (80)–(82). Then the power P^*_{inter} is written as a continuous linear form on the set V^*_{obj}, introducing thus formally internal forces $\{\mathbf{t}^S, -\rho\, \mathbf{B}^L, \widehat{\mathbf{B}}\}$ as co-factors of the elements of V^*_{obj}. That is (signs are chosen for convenience),

$$P^*_{int}(B) = -\int_B \left(t^S_{ji}\, v^*_{i,j} - \rho\, B^L_i\, \widehat{m}^*_i + \widehat{B}_{ji}\, \widehat{M}^*_{ij} \right) \mathrm{d}v. \tag{94}$$

Collecting the various contributions and assuming that the obtained global expression is valid for any element of volume and surface and any virtual velocity field (86) we obtain the local equations

$$\rho\, \dot{\mathbf{v}} = \operatorname{div} \mathbf{t} + \mathbf{f} + \mathbf{f}^{em} \quad \text{in } B, \tag{95}$$

and

$$\gamma^{-1}\,\dot{\mu} = -(\mathbf{B}^{\mathit{eff}} \times \mu) \quad \text{in } B, \tag{96}$$

with the nonsymmetric stress $\mathbf{t}$ given by

$$t_{ji} = t^{S}_{ji} - \rho\, B^{L}_{[j}\, \mu_{i]} \tag{97}$$

on account of the constraint (71), and $\mathbf{B}^{\mathit{eff}}$ given by the second of (73). Simultaneously, we obtain the natural boundary conditions (not given here — see Maugin, 1988, chap. 6 —) for $\mathbf{t}$ and $\mathbf{A}$.

It is readily checked that equations (95) and (96), together with (97) and (73) are identical to the equations deduced in the foregoing paragraph. Pursuing along the same line, and considering the principle (85) for real velocity fields, on account of (88) we obtain the *global equation of kinetic energy* in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}K(B) = P_{int}(B) + P_{extern}(B, \partial B). \tag{98}$$

This is to be combined with the global statement of the first law of thermodynamics (59) to deduce the global form of the *internal-energy theorem.* By localization this will yield (74) with the already transformed expression involving the objective internal forces. The exploitation of the inequality (75) is unchanged.

The present formulation (introduced in Maugin, 1974) — formal as it is — has certain advantages, one of which being the account of the d'Alembert-inertia couple in the expression (86). But more interestingly, it provides a direct modelling of more general ferroïc cases such as in ferrimagnets and antiferromagnets (see Paragraph 3.5).

3.4 Hamiltonian Variational Formulation

The above-given formulation is valid for both deformable solid and fluid behaviors and also in the presence of dissipative processes such as *viscosity* (via $\mathbf{D}$) and *spin-lattice relaxation* (via $\widehat{\mathbf{m}}$). In the absence of dissipative processes and for an a priori known behavior — e.g., elasticity — it is possible to approach the present theory via a Hamiltonian variational principle. Such an approach to *elastic ferromagnets* is to be found in Tiersten (1965), Brown (1966), and Maugin and Eringen (1972a). We base the present exposition on the latter. Again, we must account for the d'alembertian nature of the magnetic spin inertia and, therefore, introduce an *already varied term* for this effect. That is, we shall write down the *variational formulation* as follows:

$$\delta W_{spin} + \delta A + \delta W_{data} + \delta W_{constr} = 0. \tag{99}$$

Here δW_{spin} is the mentioned already varied term

$$\delta W_{spin} = \int_t \mathrm{d}t \int_{B_R} \rho_R \, \gamma^{-1} \, \dot{\mu} \cdot \delta\Theta \, \mathrm{d}v_R, \tag{100}$$

the scalar A is the *action* such that

$$A = \int_t \mathrm{d}t \int_{B_R} L \, \mathrm{d}v_R, \tag{101}$$

$$L = \frac{1}{2} \rho_R \, \mathbf{v}^2 - \Psi(\mathbf{F} = \nabla_R \, \overline{\mathbf{x}}, \ \mu, \ \nabla_R \, \bar{\mu}), \tag{102}$$

δW_{data} accounts for the external loads in such a way that

$$\begin{aligned} \delta W_{data} = & \int_t \mathrm{d}t \int_{B_R} \big((\mathbf{f} + \mathbf{f}^{em}) \cdot \delta \mathbf{x} + \rho_R \, \mathbf{B} \cdot \delta\mu \big) \, \mathrm{d}v_R \\ & + \int_t \mathrm{d}t \int_{\partial B_R} \big((\mathbf{t}_{(n)} + \mathbf{t}_{(n)}^{em}) \cdot \delta \mathbf{x} + \mathbf{A} \cdot \delta\mu \big) \, \mathrm{d}a_R, \end{aligned} \tag{103}$$

and δW_{constr} is possibly introduced to account, via the introduction of appropriate Lagrange multipliers, for the constraints provided by the constancy of the modulus of μ i.e., $\mu_i \, \mu_i = \mu_S^2 = const.$, and the derived relation $(\nabla_R \mu) \cdot \mu = 0$ for its spatial uniformity within a domain, where ∇_R denotes the material gradient that commutes with the partial time derivative. But these are accounted for systematically in the variation of the other terms. Here the variations are Lagrangian (taken at fixed material coordinates so that they commute with the material gradient), ρ_R is the matter density in the reference configuration K_R of the body of volume B_R and regular boundary ∂B_R. The Lagrangian variation $\delta\mu$ respects the constraint $\mu \cdot \delta\mu = 0$ and therefore is such that it involves the infinitesimal (vectorial) angular variation $\delta\Theta$ through the relation $\delta\mu = \delta\Theta \times \mu$. The Lagrangian density L involves the standard kinetic energy per unit material volume and a magneto-elastic energy Ψ that accounts for a first-order gradient theory with respect to the two basic elements of the generalized motion. We let the reader exploit the variational formulation (99) for arbitrary variations $(\delta\mathbf{x}, \delta\Theta)$ as the above expressions are given only for the sake of comparison with the previous formulation exploiting the principle of virtual power. In particular, we note the introduction of the term (100) that compares to the spin contribution in $P^*_{inertia}$. Just the same, we emphasize the similarity between the expression of δW_{data} and that of P^*_{extern}. We also note the following that is of interest compared to the Cosserats'work of 1909 on "Cosserat" continua. This is the possible exploitation of the so-called

Euclidean invariance — a first attempt to use group theory in continuum mechanics taken over by Toupin (1964) and Maugin (1970). This consists in applying (99) with special variations corresponding to *pure spatial translations*:

$$\delta \mathbf{x} = \varepsilon \, \mathbf{d}, \qquad \delta \mu = 0 \tag{104}$$

and then simultaneous *infinitesimal rotation* of both LC and SC such as

$$\delta x_i = \varepsilon \, \varpi_{ij} \, x_j, \qquad \delta \mu_i = \varepsilon \, \varpi_{ij} \, \mu_j, \tag{105}$$

where ε is an infinitesimally small parameter, $\mathbf{d}$ is a fixed finite vector, and $\varpi_{ij} = -\varpi_{ji}$ is a fixed skewsymmetric tensor. Application of (104) and (105) directly yields the local balance of linear and angular momentum in the form of the theory of Cosserat continua (Maugin, 1971; Maugin and Eringen, 1972a). We shall return to this point in Paragraph 3.6.

3.5 Ferrimagnetic and Antiferromagnetic Materials

The magnetic description considered in the foregoing paragraphs often is insufficient and not realistic enough for many magnetic materials such as ferrites. Louis Néel (Nobel prize in physics 1970 for this matter) introduced in the early 1940s a model in which the most general description of the magnetization field in a magnetically ordered crystal below its magnetic-phase-transition temperature consists in the vectorial resultant of the sum of n magnetization fields μ_α, $\alpha = 1, 2, \ldots, n$ per unit mass — referred to as *magnetic sub-lattices* — arising at each point from n different ionic species having different spectroscopic splitting factors, thus various gyromagnetic ratios γ_α, so that the total magnetic spin per unit mass is not necessarily aligned with the total magnetization. This model proved to be efficient in accounting for the unusual magnetic properties (e.g., susceptibility) of ferrites — iron oxides — for which Néel coined the behavior name *ferrimagnetism.* Simple *antiferromagnetism* is the special case for which only two magnetic sub-lattices subsist, of equal magnitude and opposite direction, allowing for the absence of global magnetization in the absence of applied magnetic field. But the magnetic response is quite different from that of classical ferromagnetism when a magnetic field is applied (see Eringen and Maugin, 1990, Vol. I, pp. 110–111). The resulting dynamics is also much more involved yielding a multiplicity of magnon branches in the case of ferrimagnetism.

A rational modelling of deformable ferrimagnetic bodies in the spirit of the model of Paragraph 3.1 would be somewhat messy, although a scheme generalizing that of Figure 2 can easily be drawn for antiferromagnetic deformable bodies equipped with two co-existing interacting magnetic sub-

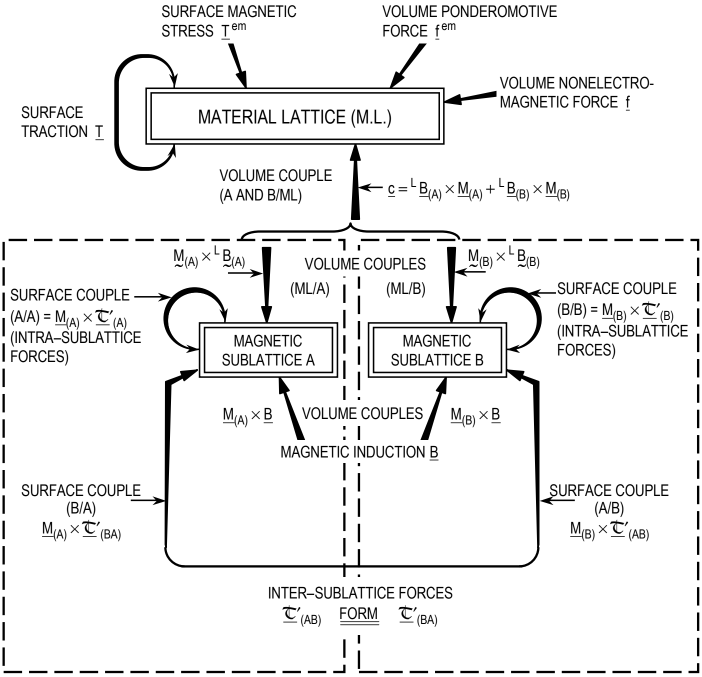

Figure 3. Interactions in a deformable antiferromagnet (after Maugin, 1976a,b, JMP).

lattices (see Figure 3 — from Maugin, 1976a). It is much more convenient and safe to exploit a modelling generalizing that of the principle of virtual power in the manner of Paragraph 3.3 in which it is "sufficient" to enlarge the initial set by replacing μ by the series of $\mu_\alpha{}'s$ and constructing the various sets of velocities for the corresponding power of internal forces once a gradient order has been selected (first-order is sufficient). A new type of interactions will appear, that between different magnetic sub-lattices, having in turn strong consequences for the coupling between elastic waves (phonons) and magnetic oscillations (magnons) — see Eringen and Maugin, 1990, Vol. II, pp. 492–493. Such a modelling was initially proposed by Maugin (1976a,b) and its dynamical consequences examined in detail by Sioké-Rainaldy and Maugin (1983); Maugin and Sioké-Rainaldy (1983).

3.6 Analogy with Cosserat Continua

Returning of the ferromagnetic case, we note that the spin-precession equation (96) deals with axial vectors. Accordingly, we can introduce dual skewsymmetric tensors by applying the alternation symbol ε_{kli} to its i-component. On account of the well known formula

$$\varepsilon_{kli}\,\varepsilon_{ipq} = \delta_{kp}\,\delta_{lq} - \delta_{kq}\,\delta_{lp}, \tag{106}$$

this operation results in the equation

$$\frac{1}{2}\,\rho\,\gamma^{-1}\,\varepsilon_{kli}\,\dot{\mu}_i = \rho\,\mu_{[k}\,B_{l]} + \rho\,\mu_{[k}\,B^L_{l]} + \left(\mu_{[k}\,\widehat{B}_{ml]}\right)_{,m} - \mu_{[k,m}\,\widehat{B}_{ml]}. \tag{107}$$

But we note that the last term in the right-hand of this equation vanishes identically because of the constraint (71), while the resulting penultimate term is none other that the skew part of the Cauchy stress according to (97), and the first term is none other than the ponderomotive couple written as a skew tensor (dual of the axial vector $\mathbf{c}^{em}$). Thus equation (107) reads

$$\rho\,\dot{S}_{kl} = M_{pkl,p} + t_{[kl]} + C_{kl}, \tag{108}$$

wherein

$$S_{kl} = \frac{1}{2}\,\gamma^{-1}\,\varepsilon_{kli}\,\mu_i, \quad M_{pkl} = \mu_{[k}\,\widehat{B}_{pl]}, \quad C_{kl} = C^{em}_{kl} = \varepsilon_{kli}\,\tilde{c}^{em}_i. \tag{109}$$

Equation (108) is in the canonical form of the balance equation of angular momentum in *Cosserat or micropolar continua* — compare Eringen (1999) and the present Appendix A — except that all contributions have a magnetic origin, the gyromagnetic relation for the inertial term S_{kl}, Heisenberg exchange forces for the couple stress tensor M_{pkl}, the applied coupled C_{ij}, and the skew part of the nonsymmetric Cauchy stress. Equation (108) and the accompanying boundary condition were deduced by the author in his PhD thesis (Maugin, 1971). A formally similar result can easily be obtained for the model of ferrimagnetic deformable bodies mentioned in the foregoing paragraph with the appropriate summation over the various magnetic sub-lattices. In the case of an exploitation of a Hamiltonian variational formulation, application of the rotational part (105) of the Euclidean invariance directly yields (108) (Maugin, 1971).

3.7 Reduction to a Model without Microstructure (Paramagnetic and Soft-Ferromagnetic Bodies)

When true ferromagnetic effects (gyromagnetic effect, Heisenberg exchange forces) are discarded, equation (108), reduces to

$$t_{[kl]} = C^{em}_{[kl]} = -M_{[k}\,B_{l]} = B_{[k}\,M_{l]}. \tag{110}$$

This applies to the simpler cases of nonlinear paramagnetic and soft-ferromagnetic bodies. The resulting theory applies, in particular, to magnetoelastic polymers as recently developed (on this subject, one can refer to the Udine course of 2009; Ogden and Steigmann, 2010, Editors; and also, Eringen and Maugin, 1990, chap. 8, Vol. I). Whenever field and magnetization are aligned (case of magnetically isotropic bodies) or in a purely linear theory in which one discards the right-hand side of (110) as being second order in the fields, the skew part of the stress is zero. The only remaining magneto-mechanical coupling in the first case remains magnetostriction for any symmetry, while in the second case, only piezomagnetism may exist, under severe symmetry conditions however.

4 Deformable Dielectrics with Electric-Polarization Microstructure

4.1 Model of Interactions

We consider the case of the quasi-electrostatics of deformable dielectrics for the sake of simplicity. We can envisage a generalized motion described by the functions

$$\mathbf{x} = \overline{\mathbf{x}}(\mathbf{X}, t), \qquad \boldsymbol{\pi} = \overline{\boldsymbol{\pi}}(\mathbf{X}, t), \tag{111}$$

where $\boldsymbol{\pi}$ is an electric polarization (polar vector) per unit mass in the deformed configuration. The second function defines a **polarization continuum**, PC. According to the contents of Section 1, PC responds to electric fields only. Accounting for the inertia introduced in Section 1, we are tempted to write down a balance equation for PC for the whole body B in a more or less standard form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_B \rho \, d_E \, \dot{\pi} \, \mathrm{d}v = \int_B \rho \, (\mathbf{E} + \mathbf{E}^L) \, \mathrm{d}v + \int_{\partial B} \mathbf{A} \, \mathrm{d}a, \tag{112}$$

where $\mathbf{E}$ is the Maxwellian electrostatic field, $\mathbf{E}^L$ is a quantity akin to an electric field and due to the possible interaction with the lattice continuum LC, whose deformation is described by the first of (111). Finally, $\mathbf{A}$, also akin to an electric field or a surface electric polarization, accounts in the form of a contact action for interactions between neighbouring electric dipoles. Applying to this the Cauchy principle, we can introduce a second order — nonsymmetric — tensor $\widehat{\mathbf{E}}$ such that

$$\mathbf{A} = \mathbf{n} \cdot \widehat{\mathbf{E}} \quad \text{at } \partial B. \tag{113}$$

Localization of (112) therefore yields the balance equation

$$d_E \, \ddot{\pi} = \mathbf{E} + \mathbf{E}^L + \rho^{-1} \operatorname{div} \widehat{\mathbf{E}} \quad \text{in } B. \tag{114}$$

Although considered by some authors (e.g., Maugin, 1976c), it is difficult to grant a true physical meaning to the balance (112) which strongly resembles the balance law postulated, with the same degree of arbitrariness, in anisotropic fluids (nematic liquid crystals) by Ericksen (1960). One possibility of interpretation is that (114) is a standard equation of motion for a unit (hypothetical) electric charge (but the medium considered is a dielectric free of charges). Indeed, as exemplified by equation (1) the product of a charge and an electric field is a classical force. In this interpretation LC and PC may be viewed as two interpenetrating continua. Such an a priori interpretation was advanced by Tiersten (1971). As to the surface condition (113) we can write it more explicitly as

$$\rho^{-1}\,\mathbf{n}\cdot\widehat{\mathbf{E}} = \pi_S, \tag{115}$$

where π_S is a density of surface electric polarization (a polar vector).

The global balances of linear and angular momenta for the lattice continuum naturally read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \rho\,\mathbf{v}\,\mathrm{d}v = \int_B (\mathbf{f}+\mathbf{f}^{em})\,\mathrm{d}v + \int_{\partial B}\mathbf{t}_{(n)}\,\mathrm{d}a; \tag{116}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_B (\mathbf{r}\times\rho\,\mathbf{v})\,\mathrm{d}v = &\int_B \big(\mathbf{r}\times(\mathbf{f}+\mathbf{f}^{em}) + \mathbf{c}_{(PC/LC)}\big)\,\mathrm{d}v\\ &+\int_{\partial B}(\mathbf{r}\times\mathbf{t}_{(n)})\,\mathrm{d}a.\end{aligned} \tag{117}$$

Then, artificial as this may look, the balance of angular momentum for the PC is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \rho\,\pi\times\dot{\pi}\,\mathrm{d}v = \int_B \big(\mathbf{c}^{em}+\mathbf{c}_{(LC/PC)}\big)\,\mathrm{d}v + \int_{\partial B}\pi\times\mathbf{A}_{(n)}\,\mathrm{d}a. \tag{118}$$

This is complemented by the first law of thermodynamics for the combined continuum:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_B \rho\left(\frac{1}{2}\mathbf{v}^2+e\right)\mathrm{d}v = &\int_B \big(\mathbf{f}\cdot\mathbf{v}+w^{em}+\rho\,h\big)\,\mathrm{d}v\\ &+\int_{\partial B}\big(\mathbf{t}_{(n)}\cdot\mathbf{v}+\mathbf{A}_{(n)}\cdot\dot{\pi}+q_{(n)}\big)\,\mathrm{d}a,\end{aligned} \tag{119}$$

and the second law of thermodynamics for the combined continuum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \rho\,\eta\,\mathrm{d}v \geq \int_B \rho\,\theta^{-1}\,h\,\mathrm{d}v + \int_{\partial B}\theta^{-1}\,q_{(n)}\,\mathrm{d}a. \tag{120}$$

In these equations,

$$\mathbf{c}^{em} = \rho\,\pi \times \mathbf{E}, \quad \mathbf{c}_{LC/PC} = \rho\,\pi \times \mathbf{E}^L, \quad \mathbf{A}_{(n)} = \mathbf{n}\cdot\widehat{\mathbf{E}}. \tag{121}$$

On account of (114) and the local form of (116) the local forms of (117) and (118) are easily established as

$$t_{[ji]} = \rho\,E^L_{[j}\,\pi_{i]} - \widehat{E}_{p[j}\,\pi_{i],p} \tag{122}$$

and

$$\rho\,\frac{\mathrm{d}}{\mathrm{d}t}(\pi \times \dot{\pi})_i = c_i^{em} + c_i^L + \varepsilon_{ijk}\left(\pi_j\,\widehat{E}_{pk}\right)_{,p}. \tag{123}$$

The rest of this approach consists in expressing (119) and (120). In introducing objective time rates such as (80) and (compare (81))

$$\begin{aligned} \hat{p}_i &= (D_J\,\pi)_i \equiv \dot{\pi}_i - \Omega_{ij}\,\pi_j, \\ \widehat{\Pi}_{ij} &= (\dot{\pi}_i)_{,j} - \Omega_{ik}\,\pi_{k,j} \end{aligned} \tag{124}$$

we can show that (119) and (120) lead to the following local forms of the energy equation and of the Clausius–Duhem inequality:

$$\rho\,\dot{e} = \mathrm{tr}(\mathbf{t}^S\,\mathbf{D}) - \rho\,\mathbf{E}^L\cdot\hat{\mathbf{p}} + \mathrm{tr}\left(\widehat{\mathbf{E}}^L\,\widehat{\Pi}^T\right) - \nabla\cdot\mathbf{q} + \rho\,h \tag{125}$$

and

$$-\rho\left(\dot{\psi} + \eta\,\dot{\theta}\right) + \mathrm{tr}(\mathbf{t}^S\,\mathbf{D}) - \rho\,\mathbf{E}^L\cdot\hat{\mathbf{p}} + \mathrm{tr}\left(\widehat{\mathbf{E}}\,\widehat{\Pi}^T\right) - \theta^{-1}\,\mathbf{q}\cdot\nabla\theta \geq 0, \tag{126}$$

with

$$t_{ji} = t^S_{ji} + t_{[ji]}, \qquad t_{[ji]} = \rho\,E^L_{[j}\,\pi_{i]} - \widehat{E}_{p[j}\,\pi_{i],p}. \tag{127}$$

Together with the Maxwell's electrostatic equations for dielectrics,

$$\nabla\times\mathbf{E} = \mathbf{0}, \qquad \nabla\cdot\mathbf{D} = 0, \qquad \mathbf{D} = \mathbf{E} + \rho\,\pi, \tag{128}$$

this concludes the formal construction of the theory before establishing constitutive equations constrained by the inequality (126).

4.2 Approach via the Principle of Virtual Power

It is now clear that an approach exploiting directly the principle of virtual power for the present theory will be very much like what was achieved in Paragraph 3.4 for ferromagnets except for the essential difference regarding the inertial force of the polarization lattice PC. That is, we shall a priori write

$$P^*_{inert}(B) = \int_B \left(\rho\,\dot{\mathbf{v}}\cdot\mathbf{v}^* + \rho\,d_E\ddot{\pi}_i\,\dot{\pi}^*_i\right)\mathrm{d}v, \tag{129}$$

where we clearly distinguish between real fields (actual solutions of a problem) and virtual ones (at our disposal in this type of variational formulation). In particular, for real fields, this yields

$$P_{inert}(B) = \frac{\mathrm{d}}{\mathrm{d}t}\int_B \left(\frac{1}{2}\,\rho\,\mathbf{v}^2 + \frac{1}{2}\,\rho\,d_E\,\dot{\pi}^2\right)\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}K(B). \tag{130}$$

The other global virtual powers are directly written down as

$$P^*(B) = \int_B \left((\mathbf{f} + \mathbf{f}^{em})\cdot\mathbf{v}^* + \rho\,\mathbf{E}\cdot(\dot{\pi})^*\right)\,\mathrm{d}v, \tag{131}$$

$$P^*(\partial B) = \int_{\partial B} \left((\mathbf{t}_{(n)} + \mathbf{t}^{em}_{(n)})\cdot\mathbf{v}^* + \rho\,\pi_S\cdot(\dot{\pi})^*\right)\,\mathrm{d}a, \tag{132}$$

and

$$P^*_{int}(B) = -\int_B \left(t^S_{ji}\,v^*_{i,j} - \rho\,E^L_i\,\hat{p}^*_i + \widehat{E}_{ji}\,\widehat{\Pi}^*_{ij}\right)\,\mathrm{d}v. \tag{133}$$

From the standard application of the principle of virtual power for any volume and surface elements and for arbitrary members of the set $\{\mathbf{v}^*, (\dot{\pi})^*\}$, one deduces the local equations of linear momentum of the LC and the governing equation (114) of the PC, together with the accompanying natural boundary conditions. Then equations (115) and (126) follow in the usual way, using the result (130).

4.3 Hamiltonian Variational Principle

Again this strategy applies when one knows a priori the functional dependence of the internal energy. There is no special problem with the kinetic energy that is given by the expression appearing in (130). For instance, for a first-order gradient theory of electroelasticity one would consider a Lagrangian density per unit volume of the reference configuration K_R

$$L = \frac{1}{2}\,\rho_R\,\mathbf{v}^2 + \frac{1}{2}\,\rho_R\,\dot{\pi}^2 - \Psi(\mathbf{F} = \nabla_R\,\overline{\mathbf{x}},\ \pi,\ \nabla_R\,\overline{\pi}), \tag{134}$$

but one must add to this the electrostatic energy including both free-field and electric-dipole energies:

$$e^{elec} = \frac{1}{2}\,\mathbf{E}^2 + \rho_R\,\pi\cdot\mathbf{E}. \tag{135}$$

This formulation applies to both elastic ferroelectrics (theory of Pouget and Maugin) and elastic ionic crystals (theory of Mindlin). Suhubi (1969) gave such a formulation for Mindlin's theory of elastic dielectrics with polarization gradients. The true aficionados will find in Maugin and Eringen (1972b) a relativistically invariant variational formulation containing simultaneously both ferromagnetic and ferroelectric descriptions (see also Maugin, 1978). Similarly, he will find in Collet and Maugin (1974) and Maugin (1980a), a formulation using the principle of virtual power for these two descriptions simultaneously.

4.4 Antiferroelectric Materials

It is easily imagined that a theory of deformable antiferroelectrics (e.g., lead zirconate or sodium niobate) in which an antiparallel arrangement of permanent electric dipoles can be devised by analogy with the theory of antiferromagnetics, i.e., by considering the polarization density π as arising from the vectorial sum of two opposite polarization sub-lattices of equal magnitude. Such a model was constructed by Soumahoro and Pouget (1994) who also studied in detail its dynamical consequences.

4.5 Analogy with Cosserat Continua

Applying the alternation symbol to equation (123) and using the identity (106) or, equivalently, taking the tensor product of (114) and then the skew part of the result we obtain

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}(d_E\, \dot{\pi}_{[i}\, \pi_{j]}) = E_{[i}\, P_{j]} + \left(\rho\, E^L_{[i}\, \pi_{j]} - \widehat{E}_{k[i}\, \pi_{j],k}\right) + \left(\widehat{E}_{k[i}\, \pi_{j]}\right)_{,k}, \qquad (136)$$

or

$$\rho\, \dot{S}_{ij} = C^{em}_{ij} + t_{[ji]} + M_{kij,k}, \qquad (137)$$

where we accounted for (127) and we set

$$S_{ij} = d_E\, \dot{\pi}_{[i}\, \pi_{j]}, \qquad C^{em}_{ij} = E_{[i}\, P_{j]}, \qquad M_{kij} = \widehat{E}_{k[i}\, \pi_{j]}. \qquad (138)$$

Simultaneously, (115) yields the associated natural boundary condition at ∂B:

$$n_k\, M_{kij} = M_{(n)ij} \equiv \pi_{S[i}\, P_{j]}. \qquad (139)$$

Equations (137) and (139) are in the canonical form of the local balance of angular momentum for a Cosserat or micropolar continuum in Eringen's classification, but all terms have an electric origin. These equations were obtained by the author (Maugin, 1971, 1980a).

4.6 Reduction to a Model without Microstructure

When pure ferroelectric features are ignored or neglecting polarization inertia and polarization-gradient effects in Mindlin's theory, equation (137) reduces to

$$t_{[ij]} = E_{[i}\, P_{j]}. \tag{140}$$

This corresponds to the classical theory of nonlinear dielectrics as originally built by Toupin (1956, 1963) and Eringen (1963) — see Eringen and Maugin, 1990, Vol. 1, chap. 7). This nonlinear theory in finite strains applies in particular to electroelastic polymers. The skewsymmetric part of the stress vanishes when polarization and electric fields are aligned. This occurs in isotropic bodies. Still the ponderomotive force is present. However, if quadratic effects in the electric field are discarded altogether, corresponding to a fully linear theory, then both ponderomotive force and couple disappear leaving for only possible electromechanical couplings piezoelectricity, material symmetry permitting (no center of symmetry).

4.7 Remark on Electric Quadrupoles

The microscopic electric description considered in Section 2 and at the beginning of this section views electric macroscopoic polarization as a polar vector. Its thermodynamical dual is akin to an electric field ($\mathbf{E}^L$). Its gradient has for thermodynamical dual $\widehat{\mathbf{E}}$ (dimensionally, an elecric field multiplied by a length). However, another view consists, while making the construct recalled in Section 2, to consider macroscopic electric polarization as made of an electric dipole density $\overline{\mathbf{P}}$, per se, and an electric quadrupole density, $\overline{\mathbf{Q}}$, so that $\mathbf{P} = \overline{\mathbf{P}} - \operatorname{div} \overline{\mathbf{Q}}$ (a natural outcome of the Lorentz modelling), and then considering $\overline{\mathbf{P}}$ and $\overline{\mathbf{Q}}$ as independent electric independent variables. The thermodynamical dual of $\overline{\mathbf{Q}}$ will then be a gradient of electric field. Such a description, envisaged by the author in the early 1970s, also yields an electric continuum endowed with a microstructure involving couple stresses. For instance, in quasi-electrostatics, we would have instead of (28), (34), (36) and (44),

$$\begin{aligned} f_i^{em} &= \overline{P}_j\, E_{i,j} + \overline{Q}_{jp}\, E_{i,jp}, \\ c_i^{em} &= \varepsilon_{ijk}\left(\overline{P}_j\, E_k + \overline{Q}_{mj}\, E_{k,m}\right) \end{aligned} \tag{141}$$

and

$$\begin{aligned} t_{ji}^{em} &= D_j\, E_i + \overline{Q}_{jk}\, E_{i,k} - \frac{1}{2}\, \mathbf{E}^2\, \delta_{ji}, \\ w^{em} &= f_j^{em}\, v_j + \rho\, E_i\, \dot{\overline{\pi}}_i + \rho\, E_{i,j}\, \mathrm{d}\big(\overline{Q}_{ji}/\rho\big)/\mathrm{d}t. \end{aligned} \tag{142}$$

More on these in the form of problems in pp. 87–89 in Eringen and Maugin, 1990, Vol. 1.

5 Dynamical Couplings between Deformation and Electromagnetic Microstructure

5.1 Introductory Note: Resonance Coupling between Wave Modes

A. General Features

In linear or linearized dynamical theories of continua expressed by a system of partial differential equations with derivatives in terms of space and time, one is often interested in knowing the possible *travelling wave modes*, functions of a space-time phase variable $\varphi = \mathbf{k} \cdot \mathbf{x} - \omega\, t$, where $\mathbf{k} = k\,\mathbf{m}$ is the wave vector, k is the wave number, $\mathbf{m}$ denotes the director cosines, and ω is the circular frequency. In substituting for trigonometric functions of φ or exponential functions of $i\,\varphi$ in the system of field equations one is led for nonzero amplitudes to a relation between the components of $\mathbf{k}$ and ω known as the *dispersion relation* written as

$$D(k, \omega) = 0. \tag{143}$$

The quantity $v_\varphi = \omega / k$ is called the phase velocity while the quantity $\mathbf{v}_g = \partial \omega / \partial \mathbf{k}$ is the group velocity. The wavelength is defined as $\lambda = 2\,\pi / k$. When v_φ does not depend on λ or k, the studied system is said to be *nondispersive*. Then the corresponding group velocity equals the phase velocity for any ω and k (or λ). If this is not the case, then the system is said to be *dispersive*. In the latter case the Fourier components of a non-monochromatic signal travel at different velocities – causing the *dispersion* of the signal –, all this independently of the amplitude (that does not depend on these properties for a linear system). Dispersive systems are characterized by systems of partial differential equations that do not admit a polynomial of differentiation that is homogeneous (a homogeneous polynomial of differentiation has all terms with space-time partial derivatives of the same order). Thus the classical *d'Alembert equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2\, \frac{\partial^2 u}{\partial x^2} = 0, \tag{144}$$

where c is a constant provides a *nondispersive* system, while the *Klein–Gordon equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2\, \frac{\partial^2 u}{\partial x^2} + m\, u = 0 \tag{145}$$

where m is also a constant, yields a *dispersive* system. The so-called *sine-Gordon equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2\, \frac{\partial^2 u}{\partial x^2} + m\, \sin u = 0, \tag{146}$$

is both *dispersive* **and** *nonlinear* (since obviously trigonometric functions are not linear functions of their argument). An equation such as the *Boussinesq equation* of crystal physics,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} - c^2 \delta^2 \frac{\partial^4 u}{\partial x^4} - c^2 \beta \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} = 0 \tag{147}$$

where δ is a characteristic length and β a nondimensional nonlinearity parameter, also is both *dispersive* **and** *nonlinear*. We shall have the opportunity later on to deal with equations such as (146) or (147). For the time being we are concerned with equations of the simpler types (144) and (145).

B. Resonance Coupling between Modes

We are interested in the following exemplary situation. Consider a linear physical system in which three bulk modes, A_α, $\alpha = 1, 2, 3$, of the plane time-harmonic type may propagate:

$$A_\alpha = \widehat{A}_\alpha \exp[i\,(\mathbf{k} \cdot \mathbf{x} - \omega\, t)]. \tag{148}$$

Assume that the system of considered field equations is such that with trial solutions (148) it yields a dispersion relation of the type (cf. Maugin, 1980b)

$$D(\omega, k) = [\omega - \omega_3(k)] \left([\omega - \omega_1(k)]\, [\omega - \omega_2(k)] - \varepsilon\, \varpi^2 \right) = 0 \tag{149}$$

where the $\omega_\alpha{}'s$ are known functions of k and of material parameters, ϖ is a characteristic (eventually wave number dependent) angular frequency, and ε is an infinitesimally small parameter. The relations

$$D_\alpha(\omega, k) = \omega - \omega_\alpha(k) = 0, \qquad \alpha = 1, 2, 3, \tag{150}$$

are the dispersion relations for *uncoupled* modes.

Of course, equation (149) tells us that the component A_3 is not coupled with the other two, and its associated dispersion relation is influenced only by material parameters that may appear in $\omega_3(k)$. The remaining two solutions of (149) are coupled via ε. Let (ω_0, k_0) denote the intersection point of the two curves $D_\alpha(\omega, k) = 0$, $\alpha = 1, 2$, in the positive quarter of the (ω, k) plane. In the neighbourhood of this critical point C, which is called a *crossover region* for the coupled modes, we have

$$\omega_\alpha(k) \cong \omega_0 + v_\alpha(k - k_0), \qquad \omega_0 = \omega_1(k_0) = \omega_2(k_0), \tag{151}$$

and the $v_\alpha{}'s$ are the group velocities of the uncoupled modes of C.

In the neighbourhood of C the remaining factor in (149) yields two approximate coupled solutions

$$\omega_{\pm}(k)-\omega_0=\frac{1}{2}\Big((v_1+v_2)\,(k-k_0)\pm\big((k-k_0)^2\,(v_1-v_2)^2+4\,\varepsilon\,\varpi^2\big)^{1/2}\Big) \quad (152)$$

or

$$k_{\pm}(\omega)-k_0=\frac{1}{2\,v_1\,v_2}\Big((v_1+v_2)\,(\omega-\omega_0) \pm\big((\omega-\omega_0)^2\,(v_1-v_2)^2+4\,\varepsilon\,\varpi^2\,v_1\,v_2\big)^{1/2}\Big). \quad (153)$$

Depending on the sign of ε and of the product $v_1\,v_2$, four different pictures of the crossover region can be sketched out. But we single out the case $\varepsilon>0$, $v_1\,v_2>0$, both $v_\alpha>0$, that is typical of what happens in problems of mechanics. The overall behavior of the remaining two coupled solutions therefore is as follows. For $k\in[0,+\infty)=\mathbb{R}^+$, we have

$$\begin{aligned}\omega_I(k)&=f_I\big(\omega_1(k),\omega_2(k);\varepsilon\big),\\ \omega_{II}(k)&=f_{II}\big(\omega_1(k),\omega_2(k);\varepsilon\big),\end{aligned} \quad (154)$$

with

$$\omega_I\cong\omega_1,\quad \omega_{II}\cong\omega_2,\quad \text{for } k\ll k_0, \quad (155)$$

and the reverse situation for $k\gg k_0$, while in the neighbourhood of point C,

$$\omega_I\cong\omega_0+\omega_+,\qquad \omega_{II}\cong\omega_0+\omega_-.$$

We see that the critical point C, in fact, no longer belongs to the coupled dispersion diagram. It is said that a *repulsion* of the dispersion curves has occurred at point C. This repulsion has the essential property to be such that

$$\frac{\Delta\omega}{\varpi}=\left|\frac{\omega_I^2-\omega_{II}^2}{\varpi\,(\omega_I+\omega_{II})}\right|\Bigg|_{k=k_0}\cong\left|\frac{\omega_I^2-\omega_{II}^2}{2\,\omega_0\,\varpi}\right|(k_0)=\mathcal{O}(\sqrt{\varepsilon}). \quad (156)$$

Simultaneously, a *resonance effect* takes place in the crossover region since it can be shown that the amplitudes satisfy a relation of the type

$$\left|\frac{\widehat{A}_1}{\widehat{A}_2}\right|\propto\sqrt{\varepsilon}\,\varpi\,\big|\omega(k)-\omega_2(k)\big|^{-1}, \quad (157)$$

which blows up for $\omega(k)$ approaching $\omega_2(k)$ at C in the absence of damping. Furthermore, in following continuously one of the coupled dispersion curves with increasing k we observe an *energy conversion* from one type of oscillation to the other type. This is typical of some wave systems in the mechanics of deformable solids as studied in depth by R. D. Mindlin (e.g., in a three-term model of rectangular rod or in the dynamics of plates; cf. Mindlin, 1955, 1960; Graff, 1975).

Two essential remarks are in order concerning this resonance phenomenon:

- First, in the case where a dissipative process is associated with each of the $A_\alpha{}'s$, the resonance effect is smoothed out (it presents a maximum instead of a divergence), and the $\omega's$ becoming complex, a relaxation time accompanies each real-frequency solution and an interchange of relaxation is observed in the cross over region.
- Second, in the case of surface waves the requirement that the amplitudes decrease with depth in the substrate implies that the allowed domain of dispersion may be reduced and that some of the coupled branches (154) in fact are not attainable.

The subsequent sections illustrate this phenomenon in the models of coupled fields sketched out in previous sections.

5.2 The Case of Magnetoelasticity in Ferromagnets

A. Magnon

Magnons are the quasi-particles quantum mechanically associated with spin waves that are oscillations in the ordered array of magnetic spins such as described by equations (70) in the absence of couplings with elasticity. The corresponding frequency mode is shown to be parabolic ($\omega \approx k^2$) with a cut-off defined by the initial non-zero static magnetization M_0, i.e., typically for such a mode

$$\omega_S(k) = \omega_M\,(\alpha\,k^2 + \beta), \qquad \omega_M = \gamma\,M_0, \tag{158}$$

where α and β are reduced exchange and magnetic anisotropy constants.

In the magnetoelastic case the mode (158) will couple with elastic modes that typically have a linear dispersion relation (subscript P for "phonon" = elastic vibrations)

$$\omega_P(k) = c_T\,k. \tag{159}$$

The coupling between these two via *magnetostriction* (or piezomagnetism induced by magnetostriction in the presence of a bias magnetization) is of the resonance type discussed in the preceding paragraph.

B. Bulk Magnetoelastic Modes

It is not the purpose here to establish in detail the coupling between "magnons" and "phonons" on the basis of the coupled continuum equations recalled in Paragraph 3.2. The reader will find this dealt with at length in Maugin (1988, chap. 6), and Eringen and Maugin (1990, chap. 9). We shall rather exhibit exemplary pictures of this dynamical magnetoelastic (or "magnetoacoustic") coupling. What in fact happens is the resonance coupling between one elastic transverse mode and the spin mode while the other elastic transverse mode is practically uncoupled, and we are in the dispersion situation sketched out in equation (149), so that the comments in Paragraph 3.1 apply. This is true for a propagation at zero angle with respect to the direction of the initial magnetization (see Figure 4). For a nonzero angle an additional coupling with the elastic longitudinal mode occurs (see Figure 5). Figures 6 and 7 show numerically computed coupled dispersion relations for two good deformable ferromagnets, Cobalt and Yttrium-Iron-Garnet (YIG).

These figures also exhibit a so-called magnetoelastic **Faraday effect** (the fact that the polarization plane of magnetoelastic waves rotates as right-polarized and left-polarized waves propagate at different speeds; Part (c) in these figures). Furthermore, if viscosity and spin-relaxation are taken into account, according to the modelling developed by the author, an exchange of relaxation between modes take place in the cross-over region (Part (d) in

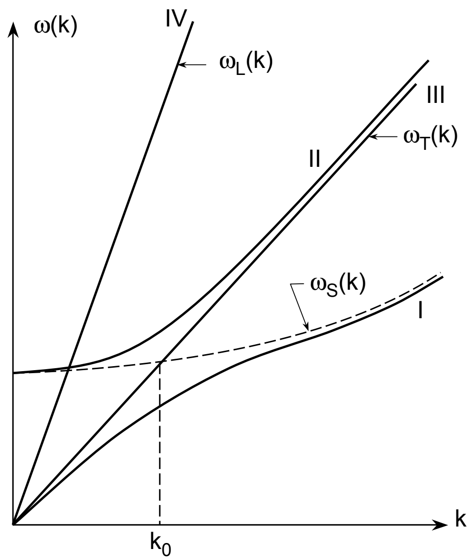

Figure 4. Dispersion diagram for coupled magnetoelastic waves in ferromagnets (after Maugin, 1981, IJES).

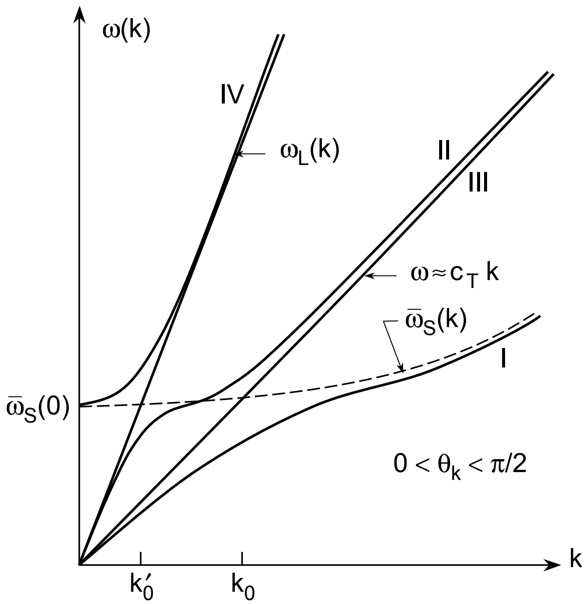

Figure 5. Dispersion diagram for coupled magnetoelastic waves in ferromagnets (after Maugin, 1981, IJES).

these figures).

C. Surface Magnetoelastic Modes

This case is much more subtle because we must account for the condition of existence of surface waves while the symmetry between right and left propagation is broken (so called "non-reciprocity" of propagation to the right and the left). With an initial setting of the static magnetization orthogonal the sagittal plane, and neglect of the curvature (magnetization gradients) of the spin-wave mode, one obtains dispersion curves such as sketched out in Figure 8. Here hatched regions are forbidden (surface modes with amplitude attenuation with depth do not exist), the low branches are limited to small wave numbers, while for forward travelling waves the upper branch tends to a shear-horizontal elastic mode, which is not the case for the backward travelling mode where the dynamic upper branch tends towards a so-called "magnetostatic" mode (as obtained in a nondeformable half space).

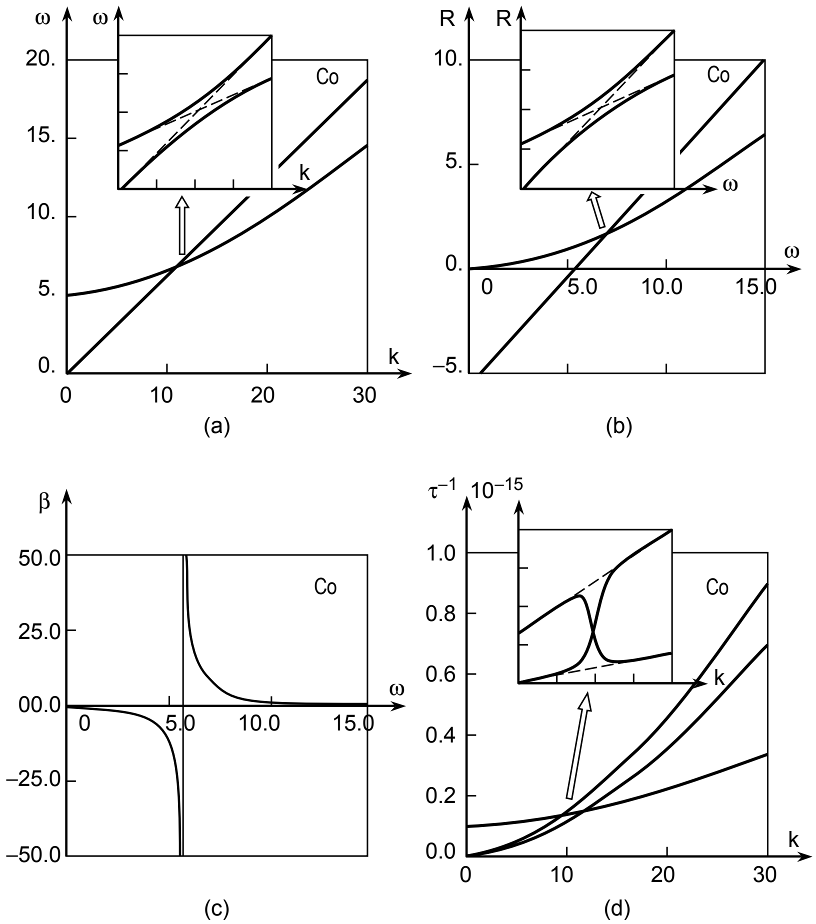

Figure 6. Magnetoaoustic resonance in Cobalt: (a) dimensionless real dispersion relation for coupled magnons and transverse phonons (ω versus k); (b) dimensionless real dispersion relation ($R = k^2$ versus ω); (c) magnetoacoustic Faraday effect; (d) exchange of relaxation between modes. (after Maugin and Pouget, 1981, IJES).

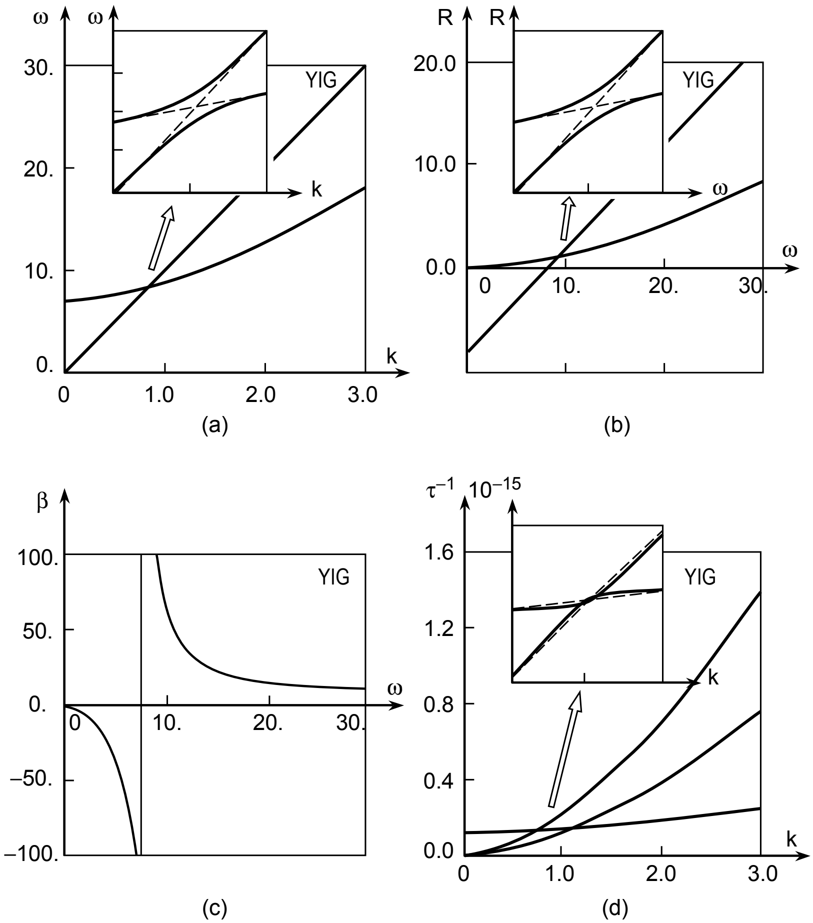

Figure 7. Magnetacoustic resonance in Yttrium-Iron-Garnet (YIG): (a) dimensionless real dispersion relation for coupled magnons and transverse phonons (ω versus k); (b) dimensionless real dispersion relation ($R = k^2$ versus ω); (c) magneoacoustic Faraday effect; (d) exchange of relaxation between modes. (after Maugin and Pouget, 1981, IJES).

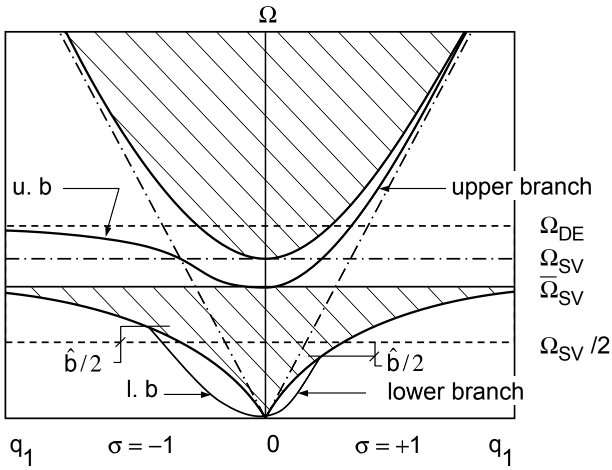

Figure 8. Dispersion curves for coupled surface magnetoelastic waves for an orthogonal setting of the bias magnetic field (exchange forces neglected): Difference between right and left propagation (after Maugin and Hakmi, 1985, JASA).

D. Case of Elastic Antiferromagnets

In the case of ferrimagnets, the multiplicity of magnetic lattices has for consequence the existence of several magnon branches, two in the case of antiferromagnets equipped with two magnetic sublattices. The magnetoelastic couplings are exhibited in the model sketched out in Figure 3. Examples of possible couplings between transverse elastic modes and two (lower and upper) spin-wave branches are shown in Figure 9 after the author and co-workers. The coupling scheme becomes complicated but reminds us exactly of what happens in the pure mechanical wave modes in some structures according to Mindlin. Figure 10 reproduces experimental results of such couplings in antiferromagnetic $FeCl_2$ (energy versus reduced wave number).

E. Magnetoacoustic Solitons

Solitons is the name given to strongly localized dynamical solutions that propagate undeformed and interact just like elastic particles during collisions. They exist because of a strict compensation between nonlinearity (that tends to make a signal getting more and more steep like in the formation of a shock wave) and dispersion (that tends to spread out a sig-

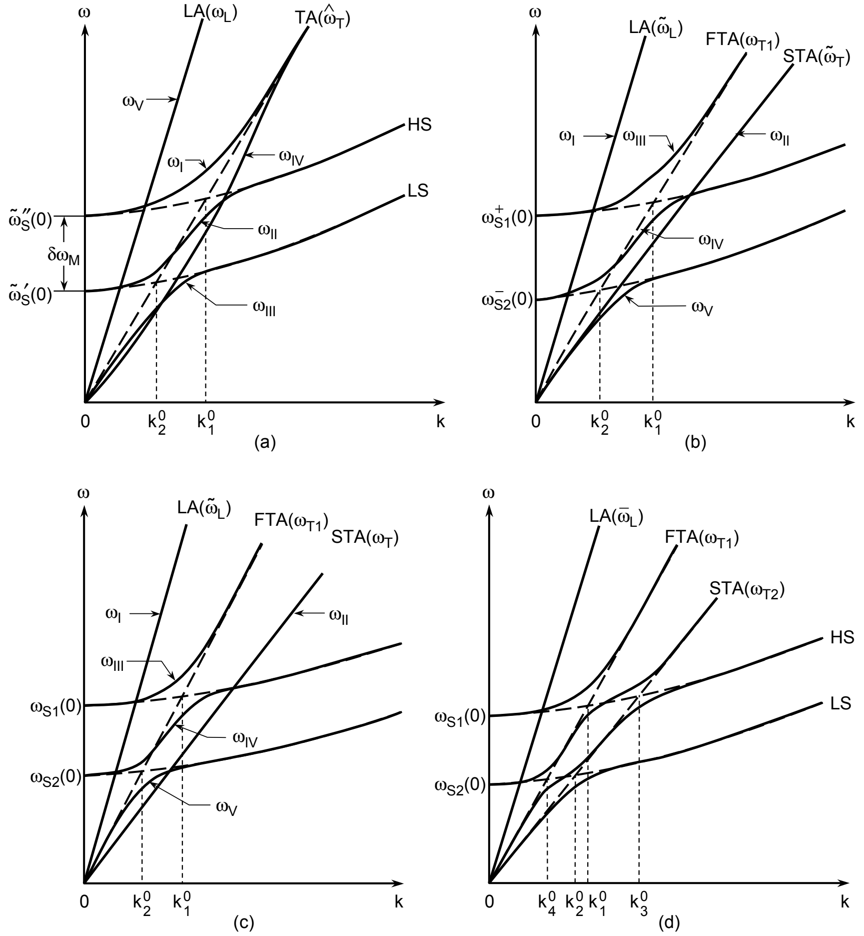

Figure 9. Qualitative dispersion relation for coupled magnetoelastic waves in simple antiferromagnets (after Maugin and Sioké-Rainaldy, 1983, 1985, JAP): (a) Longitudinal setting for a moderate bias field for nonzero global magnetization initially; (b) orthogonal setting for a moderate bias field, same initial configuration; (c) orthogonal setting for a strong bias field with nonzero global magnetization initially; (d) longitudinal setting for a strong bias field.

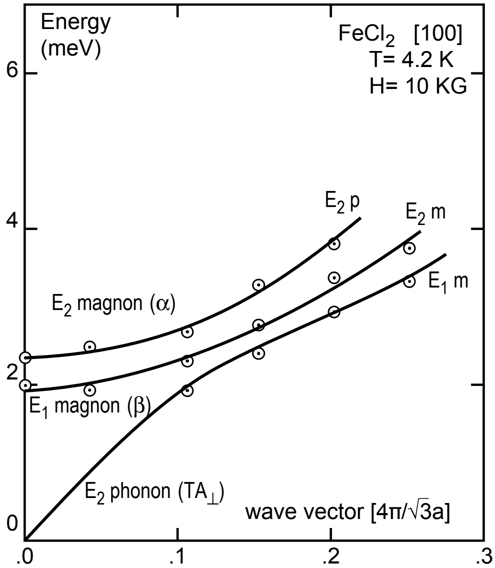

Figure 10. Real dispersion relation in antiferromagnetic $FeCl_2$ (experimental results of Ziebeck and Houmann; after Maugin, 1980b).

nal by making the various Fourier components of the signal to travel at different speeds). Equations (146) and (147) exhibit such solutions.

In the foregoing paragraphs only small amplitude oscillations of both the mechanical displacement, and the magnetic spin orientation were considered. But we may also contemplate large angular deviations of the magnetic spin allowed by the essentially non-linear (gyroscopic-like) spin equation, still coupled via magnetostriction with small elastic strains. This situation in which the simultaneous presence of nonlinearity and dispersion favours the existence of **solitons** was first studied by the author and Miled (1986a). They have shown that the coupled system of Section 3.2 can yield a system of partial differential equations now called the *sine-Gordon-D'Alembert* system that coupled via magnetostriction an equation such as (146) with a linear elastic mode. That is, in appropriate units,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin\phi = -\beta \frac{\partial u}{\partial x} (\cos\phi), \tag{160}$$

$$\frac{\partial^2 u}{\partial t^2} - c_T^2 \frac{\partial^2 u}{\partial x^2} = \beta \frac{\partial}{\partial x} (\sin\phi), \tag{161}$$

where u is a transverse elastic displacement, ϕ is the only remaining angle describing the spin precession, and β stands for the magnetostriction coupling. System (160)–(161) exhibits soliton-like solutions which are not exactly soliton solutions in the mathematical sense because it exhibits some radiations due to the essentially linear equation (161). Such solutions represent the dynamics of magnetoelastic domain walls of the Néel type (in-plane 180 degrees rotation of the local magnetization through the wall) in thin magnetic films. Remarkably enough, a similar nonlinear dispersive wave system was exhibited in the purely mechanical cases of micropolar elasticity — in the sense of Eringen — (Maugin and Miled, 1986b; also Eringen, 1999) and elastic media endowed with a microstructure described by a set of rigid directors – i.e., oriented media in the sense of Duhem & Ericksen – (Pouget and Maugin, 1989).

5.3 The Case of Electroelasticity in Ferroelectrics

A. Polaritons

Polaritons refer to the waves that result from a coupling between oscillations in the system of ordered electric dipoles — governed by equation of (114) — and the full set of Maxwell electromagnetic equations. The polarization mode is slightly dispersive and presents a cut-off, while the electromagnetic one is characterized by its high (light) velocity. This results in Figure 11a in a large split between the two modes 2 and 3.

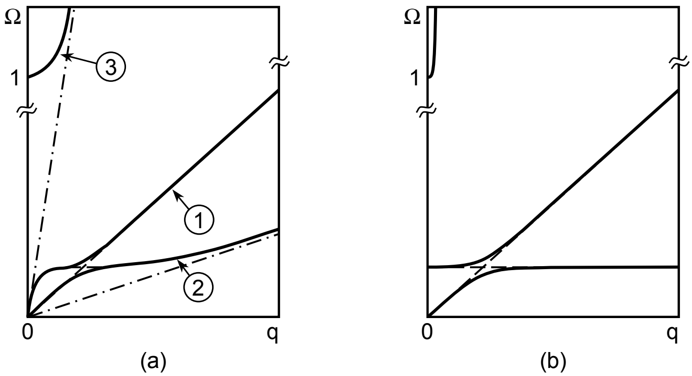

Figure 11. Dispersion relation curves for coupled transverse modes in elastic ferroelectrics: (a) mixed acoustic-polariton branches; (b) schematic view with Maxwell electrostatic equations and neglect of polarization gradients (after Maugin, 1988, p. 531).

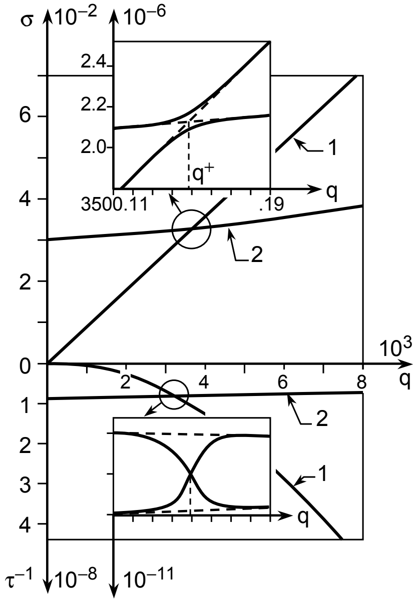

Figure 12. Numerical dispersion curves for $BaTiO_3$ (mixed transverse acoustic and polariton modes in presence of dissipation (after Pouget and Maugin, 1980, JASA).

B. Bulk and Surface Coupled Mode

For our purpose, however, the coupling with the electromagnetic mode can be discarded (quasi-electrostatic hypothesis) and we retain only the possible resonance coupling between an elastic mode and a practically flat polarization mode such as shown in Figure 11b. Such coupled modes calculated on the basis of the model developed by Maugin and Pouget (1980) are sketched in Figure 12 for a good ferroelectric material such as $BaTiO_3$ (after Pouget and Maugin, 1980). The eventual exchange of relaxation accompanying this coupling is also shown in the bottom part of the figure. Coupled surface wave modes localized in the vicinity of a limiting plane surface and characterized by a decrease of amplitude with depth can also be placed in evidence. Figure 13 shows the corresponding coupling between the polarization mode (c) and a transverse elastic mode (a) for a wave of the Rayleigh surface type (elastic displacement polarized in the sagittal plane) with a hatched forbidden dispersion zone (after Pouget and Maugin, 1981).

Finally, quite similar to the magnetoelastic solitons of the previous para-

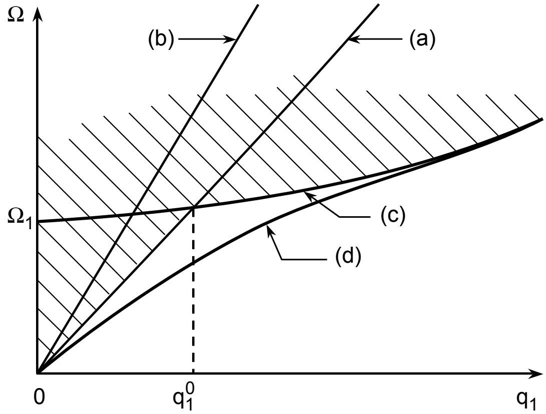

Figure 13. Piezoelectric Rayleigh waves in elastic ferroelectrics: qualitative sketch of the dispersion relation (hatched region is forbidden) (after Maugin, 1988, p. 552).

graph, **electroelastic solitons** can be shown to exist for ferroelectrics of the molecular group type (e.g., $NaCl_2$). They are governed by a nonlinear dispersive system of equations of the type of (160)–(161); cf. Pouget and Maugin, 1984.

6 Configurational Forces in Presence of an Electromagnetic Microstructure

6.1 Definition

The theory of *configurational forces* has recently become a rapidly developing active chapter of continuum thermomechanics (see the book Maugin, 2011b for an overview for professionals). We remind the reader that configurational forces are those forces of thermodynamical nature that are associated with changes of the reference configuration while traditional Newtonian-Eulerian forces are those that appear in the actual configuration of the body (standard applied forces or couples, Cauchy stress). Accordingly, configurational forces are the driving forces behind the evolution of structural defects on, and topological changes of, the material manifold. The theories of fracture, dislocation and disclination motions, structural changes such as in plasticity, damage and material growth, and the progress of phase-transition fronts belong in the general theory of configurational forces (Maugin, 2011b). In the absence of dissipation — a case which is sufficient in the present context — the relevant thermomechanical equations of the theory of configurational forces are those equations of conservation that follow from the application of Noether's theorem in a Hamiltonian-Lagrangian variational formulation. They are the local conservation equations of energy and material momentum that account for the invariance of the considered physical system under changes of time and space parametrization (the system is then said to be homogeneous in time and material space). These two equations can also be obtained by direct manipulation of the field equations by appropriate multiplication by time derivative and spatial gradient of the fields and rearrangement of the results on account of the known constitutive equations. We shall only give a flavour of the matter by recalling first what happens for a pure mechanical system, e.g., the case of micropolar solids in small deformation and small micro-rotations.

6.2 Reminder of a Purely Mechanical Case

In the dynamical case of small deformation and small micro-rotation the standard balance laws of linear and angular momenta are given in Cartesian tensorial components by (cf. present Appendix and Eringen, 1968; no applied force and couple for the sake of simplicity):

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\sigma_{ji} = 0, \tag{162}$$

and

$$\rho_0 \, j_{ij} \frac{\partial^2 \phi_j}{\partial t^2} - \frac{\partial m_{ji}}{\partial x_j} - \varepsilon_{ipq}\,\sigma_{pq} = 0. \tag{163}$$

Isotropic microinertia, $\mathbf{j} = I\,\mathbf{1}$, i.e., $j_{ij} = I\,\delta_{ij}$ is often assumed for the sake of simplicity or as an evident conclusion from a true micro-analysis. In terms of the displacement of components u_i and the microrotation of components ϕ_i, the relevant measures of generalized deformations for elastic solids are defined by

$$\begin{aligned} &\mathbf{e} := (\nabla \mathbf{u})^T + \text{dual}\ \phi = \{e_{ji} = u_{i,j} - \varepsilon_{jik}\,\phi_k\}, \\ &\gamma := \nabla\phi = \{\gamma_{ji} = \phi_{i,j}\}, \end{aligned} \tag{164}$$

and the constitutive equations of interest are given by

$$\sigma = \frac{\partial \widehat{W}}{\partial \mathbf{e}}, \quad \mathbf{m} = \frac{\partial \widehat{W}}{\partial \gamma}, \quad S = -\frac{\partial \widehat{W}}{\partial \theta}; \quad W = \widehat{W}(\mathbf{e}, \gamma, \theta; \mathbf{x}) \tag{165}$$

for the usual (but here nonsymmetric) *Cauchy stress* tensor σ and the *couple-stress* tensor $\mathbf{m}$. Here θ is the thermodynamical temperature and

S is the entropy per unit volume. The presence of $\mathbf{x}$ among the dependence of the free energy W indicates a possible material inhomogeneity via the constitutive equations. Such a dependence is also a priori considered for the density ρ_0 in the reference configuration. Let us define a "Lagrangian" density per unit volume by

$$\widehat{L} = K - W, \tag{166}$$

with

$$K = \frac{1}{2}\,\rho_0\left(\dot{u}_i\,\dot{u}_i + \dot{\phi}_i\,j_{ij}\,\dot{\phi}_j\right), \qquad W = \widehat{W}. \tag{167}$$

We encourage the reader to prove the following two equations by multiplying equations (162) and (163), respectively by $\dot{u}_i$ and $\dot{\phi}_i$, adding the two results, and rearranging terms on account of (165), and performing similar operations but by applying $u_{i,k}$ and $\phi_{i,k}$ to (162) and (163), adding the two results, and accounting for (165). This results in obtaining the local canonical balance equations of *energy* and *momentum* in the following form (e = internal energy per unit mass, $\mathbf{q}$ is a possible heat flux)

$$\frac{\partial}{\partial t}\left(\rho_0\left(e + \frac{1}{2}\,\dot{\mathbf{u}}^2 + \frac{1}{2}\,\dot{\phi}_i\,j_{ij}\,\dot{\phi}_j\right)\right) - \frac{\partial}{\partial x_j}\left(\sigma_{ji}\,\dot{u}_i + m_{ji}\,\dot{\phi}_i - q_j\right) = 0, \tag{168}$$

and

$$\frac{\partial}{\partial t}P_i^{tot.f} - \frac{\partial}{\partial x_j}b_{ji} = f_i^{inh} + f_i^{th}, \tag{169}$$

wherein

$$P_i^{tot.f} = -\rho_0\left(\dot{u}_j\,u_{j,i} + \dot{\phi}_k\,j_{kj}\,\phi_{j,i}\right),\ f_i^{inh} = \left(\frac{\partial \widehat{L}}{\partial x_i}\right)_{expl},\ f_i^{th} = S\,\frac{\partial \theta}{\partial x_i}, \tag{170}$$

and

$$\begin{aligned} \mathbf{b} &= -L\,\mathbf{1} - \sigma\cdot(\nabla\mathbf{u})^T - m\cdot(\nabla\phi)^T \\ \text{or}\ \ b_{ji} &= -(L\,\delta_{ji} + \sigma_{jk}\,u_{k,i} + m_{jk}\,\phi_{k,i}). \end{aligned} \tag{171}$$

The nonsymmetric stress tensor $\mathbf{b}$ is referred to as the Eshelby stress tensor; it is a true fully material stress tensor on the material manifold in the original finite-strain formulation.

Here the notation used in $(170)_2$ means (assuming j_{ij} is the same at all material points)

$$\left(\frac{\partial \widehat{L}}{\partial x_k}\right)_{expl} = \frac{1}{2}\,\frac{\partial \rho_0}{\partial x_k}\,(\dot{u}_i\,\dot{u}_i + \dot{\phi}_i\,j_{ij}\,\dot{\phi}_j) - \frac{\partial}{\partial x_k}\widehat{W}\bigg|_{fixed\ fields}. \tag{172}$$

In the absence of thermal effects and for materially homogeneous bodies, equations (168) and (169) reduce to the following strict conservation laws:

$$\frac{\partial}{\partial t} H - \frac{\partial}{\partial x_j} \left(\sigma_{ji}\, \dot{u}_i + m_{ji}\, \dot{\phi}_i\right) = 0, \tag{173}$$

and

$$\frac{\partial}{\partial t} P_i^{tot.f} - \frac{\partial}{\partial x_j} b_{ji} = 0, \tag{174}$$

wherein

$$H = K + W, \qquad W = \widehat{W}(e_{ij}, \gamma_{ij}). \tag{175}$$

Equations (173) and (174) and the associated jump relations across a discontinuity surface provide the basis for most calculations of driving forces on defects. In particular, for a straight through crack in the direction x_1 (see Figure 14) the celebrated J-integral of fracture theory for a quasi-static progress is given by

$$J = \oint_{\Gamma} \left(W\, n_1 - n_j \left(\sigma_{ji}\, \frac{\partial u_i}{\partial x_1} + m_{ji}\, \frac{\partial \phi_i}{\partial x_1} \right) \right) \, \mathrm{d}\Gamma, \tag{176}$$

where Γ is a circuit in the (x_1, x_2)-plane starting form the bottom stress-free face of the crack and ending on its tip stress-free face (hence in a counterclockwise circuit with unit outward normal of components n_i; component n_1 along the direction x_1). Formula (176), generalizing the standard (no microstructure) elastic case of Rice (1968), was first given by Atkinson and Leppington (1974), and reformulated since then by various authors (see

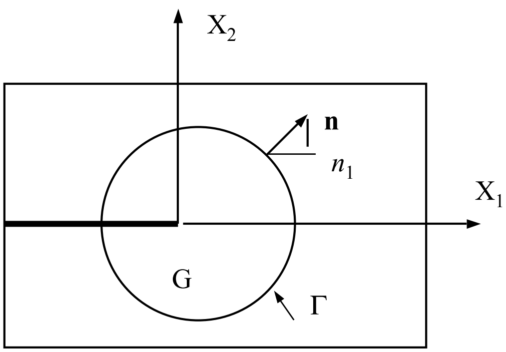

Figure 14. Straight through crack with integration contour for the J-integral.

Maugin, 2011b, p. 268). Equation (176) is obtained by integrating (174) in a domain of the (x_1, x_2)-plane encircling the crack tip, ignoring the inertia terms, invoking the divergence theorem, and projecting the result onto the x_1 direction. In the same approximation, the expression of the accompanying energy-release rate is obtained by performing similar manipulations on equation (173).

In a similar manner, the configurational force driving a phase-transition front Σ of unit oriented normal $\mathbf{n}$ is obtained by the formula

$$f_\Sigma = \mathbf{n} \cdot [\mathbf{b}] \cdot \mathbf{n}, \tag{177}$$

where $\mathbf{b}$ is reduced to its quasi-static part and the brackets denote the jump of the enclosed quantity. The details of the proof of (177) are given in Maugin (2011b, pp. 264–267) in the case of finite fields.

6.3 The Ferroelectric Case

It is clear that the ferroelectric elastic case sketched out in Section 4 is very similar to the micropolar case recalled in Paragraph 6.2. In particular, in parallel with (166) and (167) we have the following expressions:

$$\widehat{L} = K - W, \tag{178}$$

with

$$K = \frac{1}{2}\rho_0\,(\dot{u}_i\,\dot{u}_i + \dot{\pi}_i\,d_E\,\dot{\pi}_i), \qquad W = \widehat{W}, \tag{179}$$

but here

$$\widehat{W} = \overline{W}(e_{ij} = u_{(i,j)}, \pi_i, \pi_{i,j}) - \left(\frac{1}{2}\,\mathbf{E}^2 + \rho_0\,\pi_i\,E_i\right), \tag{180}$$

where E_i are the components of the Maxwellian quasi-static electric field $\mathbf{E}$, and we consider only small strains and electric fields in the absence of thermal and other dissipative effects, and of material inhomogeneities, the whole in the quasi-electrostatics of dielectrics. The relevant constitutive equations are given by (see, e.g., Maugin, 1988, chap. 7)

$$\begin{aligned} \sigma_{ji} &= \frac{\partial \overline{W}}{\partial e_{ij}}, \\ E_i^L &= -\rho_0^{-1}\,\frac{\partial \overline{W}}{\partial \pi_i}, \\ \widehat{E}_{ji} &= \frac{\partial \overline{W}}{\partial \pi_{i,j}}, \end{aligned} \tag{181}$$

while the field equations are the equation of motion, the equation governing the electric polarization, and the remaining Maxwell's equations for dielectrics. That is,

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ji}}{\partial x_j} + f_i^{em}, \qquad f_i^{em} = \rho_0 \, \pi_j \frac{\partial E_i}{\partial x_j}, \tag{182}$$

$$d_E \frac{\partial^2 \pi_i}{\partial t^2} = E_i + E_i^L + \rho_0^{-1} \frac{\partial \widehat{E}_{ji}}{\partial x_j}, \tag{183}$$

and

$$\begin{aligned} &\nabla \cdot \mathbf{D} = 0, \\ &\nabla \times \mathbf{E} = \mathbf{0} \Rightarrow \mathbf{E} = -\nabla \phi, \\ &D_i = E_i + \rho_0 \, \pi_i. \end{aligned} \tag{184}$$

Here ϕ is the electrostatic potential not to be mistaken for the microrotation of the previous paragraph.

Because the conservation laws of energy and material momentum are canonical (i.e., their formal expression is independent of the true physical interpretation of the variables; see Maugin, 2011b) we could write them down at once by analogy with equations (168) through (171). The thermomechanics of the corresponding configurational forces was given by Restuccia and Maugin (2008) in the finite-strain framework. We could exploit Noether's theorem — as there is no dissipation — since all equations are derivable from a unique Hamiltonian-Lagrangian variational principle. More naively, for example, in order to obtain the conservation equation of canonical (material) momentum, we can simply combine the three co-vectorial equations obtained by applying the field $u_{i,k}$, $\pi_{i,k}$ and $\nabla\phi$, respectively to (182), (183) and (184) and accounting for (179) through (181), to arrive at the equation

$$\frac{\partial}{\partial t} P_i^{tot.f} - \frac{\partial}{\partial x_j} b_{ji} = 0 \tag{185}$$

wherein

$$P_i^{tot.f} = -\rho_0 \, (\dot{u}_j \, u_{j,i} + d_E \, \dot{\pi}_j \, \pi_{j,i}), \tag{186}$$

and

$$b_{ji} = -\big(L \, \delta_{ji} + \sigma_{jk} \, u_{k,i} + D_j \, \phi_{,i} + \widehat{E}_{jk} \, \pi_{k,i}\big). \tag{187}$$

Just the same as (174), equation (185) provides the basis for the construction of configurational forces acting on defects in elastic ferroelectrics. In particular, in quasi-statics, the integral of (185) in the proper plane (see Figure 14) yields the generalized J-integral useful in fracture studies:

$$J = \oint_\Gamma \left(W \, n_1 - n_j \left(\sigma_{ji} \frac{\partial u_i}{\partial x_1} + D_j \frac{\partial \phi}{\partial x_1} + \widehat{E}_{ji} \frac{\partial \pi_i}{\partial x_1} \right) \right) \mathrm{d}\Gamma. \tag{188}$$

If the body is piezoelectric but not ferroelectric, then $d_E = 0$, $\widehat{E}_{ij} = 0$, and equation (183) reduces to

$$\mathbf{E} + \mathbf{E}^L = 0, \tag{189}$$

while

$$\overline{W} = \overline{W}(e_{ij}, \pi_i), \qquad E_i^L = -\rho_0^{-1} \frac{\partial \overline{W}}{\partial \pi_i}. \tag{190}$$

We can perform a partial Legendre transformation on W such as, on account of (189):

$$\begin{aligned} E_i &= \frac{\partial \overline{W}}{\partial P_i}, \\ P_i &= \rho_0 \, \pi_i = -\frac{\partial \widetilde{W}}{\partial E_i}, \\ \overline{W}(e_{ij}, P_i) - \widetilde{W}(e_{ij}, E_j) &= E_k \, P_k = \rho_0 \, \pi_k \, E_k. \end{aligned} \tag{191}$$

Thus, in this approximation,

$$D_i = E_i + P_i = -\frac{\partial \breve{W}}{\partial E_i}, \qquad \breve{W}(e_{ji}, E_i) = -\frac{1}{2} \, \mathbf{E}^2 + \widetilde{W}(e_{ij}, E_j). \tag{192}$$

This is the standard theory of electroelasticity of piezoelectrics for which (188) reduces to the known formula (given by several authors, see Maugin, 2011b, sec. 11.9)

$$J = \oint_\Gamma \left(W \, n_1 - n_j \left(\sigma_{ji} \, \frac{\partial u_i}{\partial x_1} + D_j \, \frac{\partial \phi}{\partial x_1} \right) \right) \, \mathrm{d}\Gamma. \tag{193}$$

For the formulation of the driving force acting on phase-transition fronts, we refer the reader to our book (Maugin, 2011b, sec. 11.9).

6.4 The Ferromagnetic Case

This case is peculiar in the dynamic framework because of the special nature of the magnetic spin for which there is no kinetic energy expressed in the traditional form. First of all it can be shown (Maugin, 2011b, p. 370) that the local energy equation (66) for small strains can be rewritten in a more traditional (canonical) form as

$$\frac{\partial}{\partial t} \left(\rho_0 \left(\frac{1}{2} \, \dot{\mathbf{u}}^2 + e - \mathbf{B} \cdot \mu \right) \right) - \frac{\partial}{\partial x_j} \left(\sigma_{ji} \, \dot{u}_i + \widehat{B}_{ji} \, \dot{\mu}_i - q_j \right) = 0, \tag{194}$$

where e is the internal energy per unit mass and q_j stands for the heat flux. In the absence of thermal and dissipative processes this is rewritten as

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \, \rho_0 \, \dot{\mathbf{u}}^2 + W - \rho_0 \, B_j \, \mu_j \right) - \frac{\partial}{\partial x_j} \left(\sigma_{ji} \, \dot{u}_i + \widehat{B}_{ji} \, \dot{\mu}_i \right) = 0, \tag{195}$$

where there is no apparent kinetic energy for the magnetic spin, while the volume energy W remains a function of the set of variables $(e_{ji}, \mu_i, \mu_{i,j})$. The associated canonical conservation law of (material) momentum was formulated by Fomethe and Maugin (1996). In small strains and materially homogeneous materials it reads

$$\frac{\partial}{\partial t} P_i^{mech} - \frac{\partial}{\partial x_j} b_{ji} = f_i^{ferro}, \tag{196}$$

wherein

$$P_i^{mech} = -\rho_0 \, \dot{u}_k \, u_{k,i}, \tag{197}$$

$$b_{ji} = -\big(L \, \delta_{ji} + \sigma_{jk} \, u_{k,i} + \widehat{B}_{jk} \, \mu_{k,i}\big), \tag{198}$$

$$L = \frac{1}{2} \rho_0 \, \dot{\mathbf{u}}^2 + \rho_0 \, B_j \, \mu_j - W(e_{ji}, \mu_i, \mu_{i,j}), \tag{199}$$

and

$$f_i^{ferro} = -\gamma^{-1} \, \omega_k \, \mu_{k,i}, \tag{200}$$

where ω_i are the components of the precessional velocity of the magnetic spin, i.e.,

$$\omega_i = -\gamma \left(B_i + B_i^L + \rho_0^{-1} \, \widehat{B}_{ji,j}\right). \tag{201}$$

The generally nonvanishing right-hand side in (196) is the print left by the peculiar gyroscopic nature of the magnetic spin. Whatever we do, we cannot incorporate it in any of the two terms in the left-hand side. This “material” force (200) can also be written in the self-speaking form

$$f_i^{ferro} = \rho_0 \, \frac{\delta L}{\delta \mu_k} \, \mu_{k,i}, \tag{202}$$

where the Euler–Lagrange variational derivative is given by

$$\frac{\delta L}{\delta \mu_k} = \frac{\partial L}{\partial \mu_k} - \frac{\partial}{\partial x_p} \frac{\partial L}{\partial \mu_{k,p}} = B_k - \frac{\delta W}{\delta \mu_k}. \tag{203}$$

It is verified that the “ferromagnetic” material force (200) or (202) has no dissipative contents by computing the (identically nil) power that it expends in a material velocity field on the material manifold by virtue of equation (9), in full agreement with (195). Thus it will not contribute any term in the energy-release rate that we could deduce from (196) in a study of fracture. We refer the reader to Chapter 11 of Maugin (2011b) for further applications to fracture and the progress of phase-transition fronts and magnetic domain walls.

7 Conclusive Remark

In the early developments of three-dimensional generalized continuum mechanics the difficulty of physically realizing a volume density of distributed mechanical couples was noticed. An early justification for considering magnetized and electrically polarized materials was the possibility to induce such couples by electromagnetic means: the general non-alignment of magnetic field and magnetization in the magnetic case, that of electric field and electric dipoles in the electric case. What we have shown in the foregoing sections is that a further electromagnetic microstructure, whether magnetic or electric in nature, induces the presence of other fields that theoretically exist in purely mechanical theories, those of hyperstress and intrinsic spin. As shown in this set of lectures, a consequence of the presence of such fields makes that some of the results and expressions of pure continuum mechanics are translated into new electromagnetically-based quantities. This is true in most of the applications such as coupled-wave propagation, driving forces on cracks and phase-transformation. This does not come as a surprise in the field-theoretical approach presented in these lectures where many of the expressions are indeed canonical, and thus formally independent of the precise physical meaning of the involved fields.

A Reminder of Basic Equations of Generalized Mechanical Continua

Here we remind the reader of the basic local equations of balance of now standard generalized continua (see, e.g., Eringen, 1999; Maugin, 2011a).

Let σ the Cauchy stress of continuum mechanics, i.e., the stress tensor of Cartesian tensor components σ_{ji} in the actual configuration of a body B at Newtonian time t. In classical continuum mechanics (no applied couple, no internal structure) this is symmetric satisfying the two local balance equations of linear momentum and moment of momentum:

$$\frac{\partial}{\partial t}(\rho_0\,\mathbf{v}) - \operatorname{div}\sigma = 0 \qquad \text{or} \qquad \frac{\partial}{\partial t}(\rho_0\,\dot{u}_i) - \sigma_{ji,j} = 0, \tag{204}$$

and

$$\sigma = \sigma^T \quad \text{or} \quad \sigma_{ji} = \sigma_{ij} \iff \sigma_{[ji]} = 0, \tag{205}$$

in the absence of body force. Here ρ_0 is the constant matter density (for a homogeneous body) and $v_i := \dot{u}_i$ denotes the velocity.

In the most popular (purely mechanical) generalized continuum mechanics, Equation (205) generalizes to the following ones (written in quasi-statics for the sake of simplicity):

Micromorphic Bodies (Eringen, Mindlin; 1964):

$$\begin{aligned}&\mu_{kij,k} + \sigma_{ji} - s_{ji} + l_{ij} = 0,\\ &\sigma_{ji} = \sigma_{(ji)} + \sigma_{[ji]},\ s_{[ji]} = 0,\\ &l_{ji} = C_{ji} + l_{(ji)}.\end{aligned} \tag{206}$$

Micropolar Bodies (Cosserat brothers, etc.):

$$\mu_{k[ji],k} + \sigma_{[ji]} + C_{ij} = 0 \tag{207}$$

or

$$m_{ji,j} + \varepsilon_{ikj}\, \sigma_{kj} + C_i = 0. \tag{208}$$

Bodies with Microstretch (Eringen, 1969):

$$\mu_{klm} = \frac{1}{3}\, m_k\, \delta_{lm} - \frac{1}{2}\, \varepsilon_{lmr}\, m_{kr} \tag{209}$$

so that

$$\begin{aligned}m_{kl,k} + \varepsilon_{lmn}\, \sigma_{mn} + C_l &= 0,\\ m_{k,k} + \sigma - s + l &= 0.\end{aligned} \tag{210}$$

Dilatational Elasticity (Cowin and Nunziato, 1983):

$$m_{k,k} + \sigma - s + l = 0. \tag{211}$$

In these equations given in Cartesian components in order to avoid any misunderstanding (note that the divergence is always taken on the first index of the tensorial object to which it applies), μ_{kij} is a new internal force having the nature of a third-order tensor. It has no specific symmetry in Equation (206) and it may be referred to as a *hyperstress*. In the case of Equations (207) this quantity μ_{kij} is skewsymmetric in its last two indices and a dual second order tensor — called a *couple stress* — of components m_{ji} can be introduced having *axial* nature with respect to its second index. The fields s_{ji} and l_{ij} are, respectively, a symmetric second order tensor and a general second order tensor. The former is an *intrinsic interaction stress*, while the latter refers to an external source of *both* stress and couple according to the last of Equations (206). Only the skew part of the later remains in the special case of micropolar materials (Equations (207) in which C_i represents the components of an *applied couple*, an axial vector associated with the skewsymmetric tensor C_{ji}). The latter can be of electromagnetic origin, and more rarely of pure mechanical origin. Equations (209) and (210) represent a kind of intermediate case between micromorphic and micropolar materials. The case of dilatational elasticity in Equation (211) appears as a further reduction of that in Equation (210). This can be useful in describing the mechanical behavior of media exhibiting a distribution of holes or cavities in evolution.

In a fully dynamical case, a dynamic (inertia) term is present in the right-hand side of equations $(206)_1$, $(207)_1$, $(210)_1$ and (211). For instance, in the case of $(207)_1$, its dynamical generalization reads

$$\mu_{k[ji],k} + \sigma_{[ji]} + C_{ij} = \rho_0\, S_{ji} \tag{212}$$

or

$$m_{ji,j} + \varepsilon_{ikj}\, \sigma_{kj} + C_i = \rho_0\, S_i, \tag{213}$$

where the intrinsic spin tensor of components S_{ji} (and its dual axial vector S_k) are given by

$$S_{ji} = -\varepsilon_{jik}\, S_k, \qquad S_k = \frac{\mathrm{d}}{\mathrm{d}t}(j_{kp}\, \nu_p), \tag{214}$$

where j_{kp} stands for a symmetric tensor of rotational (or micro) "inertia" and ν_p denotes the vector components of an intrinsic rotational velocity (generally different from standard vorticity $\omega_i = \frac{1}{2}\,(\nabla\times\mathbf{v})_i$). Eringen (1966) has shown that j_{kp} satisfies a "balance law of micro-inertia" in perfect analogy with the standard conservation of mass for the macroscopic motion. For the purpose of analogy with electromagnetically micro-structured media, we note that the spin tensor that will appear in the right-hand side of equation $(206)_1$ would be defined microscopically by an average of the type

$$S_{ji} \equiv \langle \ddot{\xi}_j\, \xi_i \rangle, \tag{215}$$

where ξ_j refers to internal coordinates in a micro-element defined at point $\mathbf{X}$ in the body. Only the skewsymmetric (i.e., antisymmetric) part of this tensor is involved in a *micropolar* body so that we have the reduction

$$S_{ji} = \langle \ddot{\xi}_{[j}\, \xi_{i]} \rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\xi}_{[j}\, \xi_{i]}\big) \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{\xi}_{[j}\, \xi_{i]} \rangle. \tag{216}$$

This is to be compared to expressions such as that in the left-hand side of (123).

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