1 Introduction

The quest for a fundamental theory of all elementary particles and their interactions has been one of the most fascinating scientific endeavors during the past century. One of the main guiding principles in the construction of the ever more refined theories of high energy physics has been the systematic use of symmetry principles. They form the basic language in terms of which the Poincaré invariant quantum field theory based on the gauge group $SU(3) \times SU(2) \times U(1)$ we now call the Standard Model (SM) is formulated.

Among the many interesting ideas for physics beyond the Standard Model, the most fruitful one has arguably been the introduction of the concept of *supersymmetry*, i.e., a symmetry between bosonic and fermionic degrees of freedom. Indeed, supersymmetric extensions of the Standard Model have been put forward as possible solutions to the hierarchy problem, though the simplest models are challenged by current Large Hadron Collider results; they improve the unification of the three Standard Model gauge couplings at the Grand Unified Theory (GUT) scale, $M_{\mathrm{GUT}} \cong 2 \times 10^{16}$GeV; and they might provide interesting dark matter candidates.

The one missing major player in these constructions, however, is the gravitational interaction with its elegant geometric description in terms of Einstein's general theory of relativity. Combining the principles of supersymmetry with gravity defines what is called *supergravity*, the topic of these lecture notes. As we now briefly explain, the idea of supergravity has many other intriguing applications in various areas of particle physics, cosmology, string theory, and mathematics that go far beyond the simple desire to marry supersymmetry with gravity and can serve as separate motivations to study supergravity theories.

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1.1 The Many Facets of Supergravity

Supergravity as a Gauge Theory of Supersymmetry

Supersymmetric field theories in rigid Minkowski spacetime feature supersymmetry as a global (or "rigid") symmetry. In view of the success of local gauge invariances in the Standard Model, it is natural to try to promote supersymmetry likewise to a local gauge symmetry. This leads directly to supergravity, and in fact this was the way supergravity was first constructed. Supergravity theories can hence equivalently be defined as the *gauge theories of supersymmetry*.

Just as for an ordinary Yang–Mills symmetry, the gauging of supersymmetry requires the introduction of a suitable gauge field that transforms into the spacetime derivative of the infinitesimal symmetry parameter:

$$\delta_{gauge}(\text{gauge field}) = \partial_\mu(\text{gauge parameter}) + \ldots \tag{1.1}$$

In supersymmetry, the symmetry parameter is a spinorial quantity, ϵ_α, with α being a spinor index. The supersymmetry gauge field is therefore not a vector field as in ordinary Yang–Mills theories but a vector-spinor field, $\psi_{\mu\alpha}$. On-shell (and for unbroken supersymmetry), this field propagates two helicity $\pm 3/2$ states, whereas off-shell it also contains states with helicity $\pm 1/2$ (just as an ordinary gauge field, A_μ, contains helicities ± 1 on-shell, but off-shell also two states with helicity 0).

We will see later that supersymmetry further implies that the graviton, $g_{\mu\nu}$, and the supersymmetry gauge field, $\psi_{\mu\alpha}$, sit in the same supersymmetry multiplet, the supergravity multiplet, so that local realizations of supersymmetry necessarily need to include gravity, which explains the name supergravity. Being the superpartner of the graviton, $\psi_{\mu\alpha}$ is called the *gravitino*.

Extended Supergravity and Unification

In extended supergravity theories with $\mathcal{N} \geq 2$ supersymmetries, the supermultiplet of the graviton contains $\mathcal{N}$ gravitini but also fields with spin ≤ 1. Extended supersymmetry can thus interpolate between the graviton and ordinary gauge fields, thereby leading to some sort of unification of the interactions mediated by these fields. For $\mathcal{N} > 2$, the supergravity multiplet also contains spin $1/2$ particles, and one may wonder whether sufficiently extended supergravity could even provide a unified theory of all interactions and matter particles. The attempts in this direction culminated in the construction of the maximally extended $\mathcal{N} = 8$ supergravity with the maximally possible compact gauge group SO(8) by de Wit and Nicolai in 1982 [1]. Unfortunately, this and other extended supergravity theories generally suffer from the problem that the corresponding gauge interactions are non-chiral, in contrast to what we see in the electroweak sector of the Standard Model. Moreover, the SO(8) gauge group studied in [1] is too small to accommodate the Standard Model gauge group SU(3) $\times$ SU(2) $\times$ U(1) as a subgroup, and the theory also has a very large negative cosmological constant. Although unification via extended supergravity theories has not met with phenomenological success, the study of many extended supergravity theories in the early 1980s has proven to be a very valuable resource for many modern theoretical developments, notably in string theory; see below.

Better Behavior of Ultraviolet Divergencies

Another interesting motivation for studying supergravity is the *better behavior of ultraviolet divergencies* of its corresponding quantum theory in comparison with ordinary general relativity (GR). In the quantization of GR, one can remove the potential on-shell one-loop divergencies by field redefinitions. However, there is an ultraviolet (UV) divergence at two loops which cannot be removed [2, 3], and if matter fields are added, divergencies can appear already at one loop. However, if the matter content is consistent with supersymmetry, the situation improves again. It is even still under debate at the time of this writing whether maximally supersymmetric gravity in four dimensions is divergent or not. Recent calculations show that divergencies in $\mathscr{N} = 8$ supergravity do not show up before five loops [4] and current consensus is that the first counterterm appears at seven loops, though it is still possible that its coefficient is vanishing [5]. The study of the UV properties of $\mathscr{N} = 8$ supergravity and its connections with amplitudes in $\mathscr{N} = 4$ super Yang–Mills theory is presently a very active research area, and new techniques developed for their computation are now used in a much wider context.

Phenomenology

Whereas the lack of chiral gauge interactions precludes any direct use of extended supersymmetry and supergravity for phenomenological applications, $\mathscr{N} = 1$ supergravity theories are phenomenologically very interesting and could resolve some issues in globally supersymmetric extensions of the Standard Model. For instance:

- Supergravity can remove the large tree-level cosmological constant of spontaneously broken rigid supersymmetry.
- Supergravity suggests new mechanisms of supersymmetry breaking and their transmission to the Standard Model sector.
- When supersymmetry is broken in supergravity, the goldstino is “eaten” by the gravitino, which provides one way of explaining why the goldstino has not been seen.

Supergravity can also have important implications for the dark matter sector or early universe cosmology, as we will also explain later.

Supergravity as an Effective Theory of String Theory

While supergravity theories are in general not renormalizable, they do arise as the *low-energy effective actions of (super)string theory*, the best viable candidate for a consistent unified quantum theory of gravity and all gauge interactions. From this point of view, supergravity forms the interface between string theory and most of its potentially observable low-energy phenomena, with its non-renormalizability no longer being an issue. Moreover, even as the infrared limit of string theory, supergravity with all its non-linear interactions still captures some of the non-perturbative properties of string theory, which would hardly be accessible via the conventional world sheet conformal field theory approach. For instance, string theory is not only a theory of strings but also contains other extended objects such as D-branes or other solitonic p-branes. Some of these objects first arose as solutions of supergravity models, and the supergravity perspective has often given interesting insights into their physics. Similarly, various duality symmetries have been first, or better, understood by looking at the effective supergravity theories and their solutions. Often when different compactifications lead to the same low-energy spectra and vacua, one may uncover some new underlying symmetry of the higher dimensional theories.

The Gauge/Gravity Correspondence

One of the biggest revolutions in our understanding of string theory came from the observation that certain string models on curved spaces can be dual to non-gravitating (often conformal) gauge field theories: this is the gauge/gravity correspondence or anti-de Sitter/conformal field theory (AdS/CFT) correspondence [6, 7]. The weak coupling limit of string models, i.e., supergravity, is then an important tool to compute quantities that are related to the strong coupling regime in the dual field theory. A special role in this correspondence is played by *gauged supergravity*,[1] because global symmetries in the CFT become local in the corresponding supergravity model.

Geometry

Supergravity solutions that preserve some of the supersymmetry of the action are particularly interesting also from a mathematical point of view. For one thing this is due to the circumstance that supersymmetric solutions often satisfy *first-order* differential equations, which in general are much easier to solve than the usual second-order field equations of non-supersymmetric solutions. These first-order differential equations define Killing spinors, which prove to be a powerful concept in different areas of mathematics. The supersymmetric compactification backgrounds of string or M-theory, for example, define compactification manifolds with interesting mathematical structures such as restricted holonomy groups or special types of gauge bundles, which are consequences of the corresponding Killing spinor equations. But also the spaces of fluctuations about such compactification backgrounds carry non-trivial geometric structures that result in various interesting scalar field geometries or "moduli spaces" in the dimensionally reduced field theories, such as, e.g., Kähler, special Kähler, or quaternionic Kähler geometries, or various types of coset spaces (see later chapters). Various cascades of dimensional

[1] Gauged supergravity refers to supergravity theories that contain also non-trivial conventional gauge interactions, as we will see in Chap. 9.

reductions then reveal unexpected mathematical relations between different classes of restricted geometries that would have been difficult to obtain or even suspect otherwise.

Fake Supergravity

As a final motivation, we would like to mention the concept of *fake supergravity* [8, 9]. Fake supergravity describes classes of solutions (e.g., domain walls or black holes) of non-supersymmetric gravity theories for which the second-order field equations can be rewritten in a first-order form that resembles the Killing spinor equations of genuine supergravity theories. This is especially useful for the discussion of the stability of these gravity solutions, as it allows the use of the Nester–Witten argument [10] in a large class of theories without the usual consistency requirements and limitations imposed by actual supersymmetry. In this same framework, also cosmological solutions may be under better control [11].

1.2 Plan of the Lectures

Clearly, the present lecture notes cannot cover all the above topics and applications in full detail. Instead, their purpose is to give a survey of the basic ingredients needed to construct a supergravity action and to discuss its physical implications (mainly for particle physics), without introducing too much technical formalism or spending too much time on the many possible applications. Another focus will be on the differences between globally supersymmetric theories and supergravity, so that the reader may better understand what really needs supergravity and what can already be implemented in a globally supersymmetric theory without gravity.

More explicitly, the outline of the lecture notes is as follows: In the first part, which consists of Chaps. 1–4, we work our way toward the construction of the simplest possible supergravity theory in four dimensions, namely, pure 4D, $\mathcal{N} = 1$ supergravity. To this end, we first introduce our spinor conventions in the remainder of Chap. 1 and then explain, in Chap. 2, how the requirement of local supersymmetry naturally leads to the inclusion of the graviton supermultiplet. In Chap. 2, we also motivate the form of the supergravity action and the supersymmetry transformation laws. In order to write these down properly, we review spinors in curved spacetime and the vierbein formulation of general relativity in Chap. 3. Chapter 4 then is devoted to the detailed discussion of pure 4D, $\mathcal{N} = 1$ supergravity with and without a cosmological constant and gives the complete proof of its invariance under local supersymmetry. In these first chapters, we work out all the details and often show pedantically all the steps necessary for this construction. The serious reader is recommended to go through this material in full detail, as it greatly contributes to developing a good intuition that will also be helpful for the remaining chapters.

In the second part of the book, we describe how the minimal supergravity sector discussed in Part I can be coupled to matter multiplets and what the implications of these matter couplings for phenomenology are. To this end, we first discuss matter couplings in global supersymmetry in Chap. 5 and then explain, in Chap. 6, the differences that arise when supergravity and local supersymmetry are introduced. Various consequences of supergravity for the phenomenology of particle physics and cosmology are explored in Chap. 7. In these chapters, we emphasize in particular the differences between rigid and local supersymmetry, both at a formal/mathematical and at a physical level.

In the third part of the book, the presentation will be more formal, and we will discuss a number of more advanced topics. In Chap. 8, we introduce the concept of electric-magnetic duality and show how its interplay with the R-symmetry group essentially fixes the geometrical structures encountered in models with extended supersymmetry, detailing especially the case of $\mathcal{N} = 2$ theories. We then give, in Chap. 9, a brief but modern introduction to gauged supergravity models, which are playing a prominent role in many interesting recent developments in string theory. Special emphasis will be given here to the case of $\mathcal{N} = 8$ gauged supergravity. Finally, we conclude with some remarks on higher-dimensional theories and the relations between these models and four-dimensional gauged supergravities in Chap. 10.

We used a number of references throughout the lectures, mainly to point to some additional sources of information on the discussed topics. There are obviously already many very good reviews on various aspects of supergravity theories, which we used as inspiration to prepare these lectures; some of them are [12–29]. Also, very good complementary recent references on supergravity are [30, 31].

1.3 A Quick Guide Through Our Spinor Conventions

Before we embark on our journey through the world of supergravity theories, we briefly summarize the 4D spinor conventions used in this book.[2] For a generic treatment of spinors in any spacetime dimension, see appendix 10.A. Very good references are also [18, 32].

Assuming an orthonormal set of basis vectors, e_a, of Minkowski space ($a, b, \ldots = 0, 1, 2, 3$), the Minkowski metric with our choice of signature is

$$\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1). \tag{1.2}$$

The generators of Lorentz transformations are denoted by $M_{ab} = -M_{ba}$ when they are taken as anti-Hermitian generators or as $\mathscr{M}_{ab} = -i M_{ab}$ when they are taken as Hermitian generators and satisfy the Lorentz algebra $\mathfrak{so}(1, 3)$,

$$[M_{ab}, M_{cd}] = -2\,\eta_{c[a} M_{b]d} + 2\,\eta_{d[a} M_{b]c} \tag{1.3}$$

[2] Throughout the book, it is understood that $c = 1$ and $\hbar = 1$.

In these notes, symmetrization () and antisymmetrization [] are always taken with weight one, i.e., $(ab) = 1/2(ab + ba)$, $[ab] = 1/2(ab - ba)$, etc.

A spinor representation is a representation of the above Lorentz algebra that does not integrate to an ordinary (i.e., "single-valued") representation of the corresponding Lorentz *group*. Instead, it gives rise only to a "double-valued" representation of the Lorentz group in the sense that spatial rotations by 2π give minus the identity.

Mathematically, this is possible because the Lorentz group[3] is not simply connected but contains closed loops that cannot be continuously contracted to a point. The universal covering group of the Lorentz group is a group that is locally isomorphic to the Lorentz group but with a different global structure such that all closed curves are fully contractible. Spinor representations are then equivalently described as single-valued representations of this universal covering group that project to double-valued representations of the Lorentz group itself. The universal covering group of the 4D Lorentz group happens to be isomorphic to $SL(2, \mathbb{C})$, the group of unimodular complex (2×2) matrices.

In four dimensions, there are two commonly used notations for spinor representations: the two-component spinor notation and the four-component spinor notation. The two-component spinor notation is based on the abovementioned accidental isomorphism between the universal cover of the 4D Lorentz group and $SL(2, \mathbb{C})$, which does not have a direct analogue in a generic spacetime dimension. The four-component spinor notation, by contrast, readily generalizes to any spacetime dimension and is based on the notion of Clifford algebras and gamma matrices.

In this book, we will use the four-component spinor formalism, because it is more common in the supergravity literature. The two-component spinor notation, on the other hand, is frequently used in texts on global supersymmetry, so we briefly include it here as well to facilitate the translation of one formalism into the other.

1.3.1 Two-Component Spinors

The group $SL(2, \mathbb{C})$ has two equivalence classes of irreducible two-dimensional complex representations, corresponding to the defining representation of $SL(2, \mathbb{C})$ and its complex conjugate representation. These two representations are the minimal spinor representations of the 4D Lorentz group and are often denoted by $(1/2, 0)$ and $(0, 1/2)$ or by dotted and undotted two-component spinors, λ_A and $\chi^*_{\dot{A}}$. The Lorentz

[3] The full isometry group $O(1, 3)$ of 4D Minkowski spacetime decomposes into four disconnected components. In this book, we mean by Lorentz group only the component, $SO(1, 3)_0$, of Lorentz transformations that are continuously connected to the identity element. This subgroup is often called the "proper orthochronous Lorentz group".

group generators, M_{ab}, act on these via representation matrices that can be chosen as

$$(1/2, 0): \qquad \rho(M_{ab}) = -\frac{1}{4}(\sigma_a \overline{\sigma}_b - \sigma_b \overline{\sigma}_a) \tag{1.4}$$

$$(0, 1/2): \quad \rho^*(M_{ab}) = e\left[-\frac{1}{4}(\overline{\sigma}_a \sigma_b - \overline{\sigma}_b \sigma_a)\right]e^{-1}, \tag{1.5}$$

where $\sigma_i = -\overline{\sigma}_i$ ($i = 1, 2, 3$) are the Pauli matrices, $\sigma_0 = \overline{\sigma}_0 = \mathbb{1}_2$, and

$$e \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.6}$$

That (1.5) is indeed the complex conjugate of (1.4) follows from the identity $e^{-1}\sigma^\mu e = \overline{\sigma}^{\mu *} = \overline{\sigma}^{\mu T}$.

The matrix e is an invariant of $SL(2, \mathbb{C})$ in the sense that

$$A^T e A = e \quad \forall A \in SL(2, \mathbb{C}), \tag{1.7}$$

which implies that quantities such as $\lambda^T e \omega$ are Lorentz invariant products of two spinors λ_A and ω_A. This is often written as $\lambda_A \omega^A$ or $\lambda^A \omega_A$ using suitable raising and lowering conventions for spinor indices with the matrix e.

All finite-dimensional irreducible representations of $SL(2, \mathbb{C})$ can be obtained from symmetrized tensor products of these elementary building blocks, often denoted by $(n/2, m/2)$, where n and m count the $(1/2, 0)$ and $(0, 1/2)$ factors, respectively. For $(m + n)$ =even/odd, these describe single-/double-valued representations of the Lorentz group, corresponding to bosons/fermions.

1.3.2 Four-Component Spinors

The above two-component spinor formalism has many nice features and, as already mentioned, is frequently used in the literature on global supersymmetry. In these lectures, however, we will use four-component spinors instead, as these are more standard in the four-dimensional supergravity and particle phenomenology literature and generalize easily to other spacetime dimensions.

More precisely, we will use four-component *Majorana* spinors to describe all fermions in 4D. Majorana spinors satisfy a reality condition (see below) and naturally describe fermions that are gauge invariant or transform in real representations of the gauge group. This is true in particular for the gauge-invariant gravitino,[4] the superpartner of the graviton, but also for the gaugini, the superpartners of the

[4] As we will discuss later, an exception to this gauge invariance of the gravitino arises in the presence of Fayet–Iliopoulos terms in $\mathcal{N} = 1$ supergravity. These terms require a gauging of

gauge bosons, which (before spontaneous symmetry breaking) always transform in the adjoint and hence a real representation of the gauge group. Fermions that sit in $\mathcal{N} = 1$ chiral multiplets, on the other hand, may have chiral gauge interactions and hence may sit in complex gauge group representations. While these fermions can be nicely described by two-component spinors, they may equally well be described in terms of the chiral components of different sets of four-component Majorana spinors, as we will do in this book.

1.3.2.1 Dirac Spinors

A general four-component Dirac spinor, ψ_α ($\alpha = 1, \ldots, 4$), has four complex components and corresponds to a direct sum of two two-component spinors of the form $(1/2, 0) \oplus (0, 1/2)$, i.e., it combines one undotted two-component spinor, λ_A, and one dotted spinor, $\chi^*_{\dot{A}}$. A very convenient way to do so is to define

$$\psi = \begin{pmatrix} e \cdot \chi^* \\ \lambda \end{pmatrix}, \tag{1.8}$$

so that the Lorentz generators are represented by the manifestly reducible matrices

$$\Sigma_{ab} = \begin{pmatrix} e\rho^*(M_{ab})e^{-1} & 0 \\ 0 & \rho(M_{ab}) \end{pmatrix}. \tag{1.9}$$

The (4×4) matrices Σ_{ab} can conveniently be expressed in terms of Dirac gamma matrices,

$$\gamma_a = \begin{pmatrix} 0 & i\overline{\sigma}_a \\ i\sigma_a & 0 \end{pmatrix}, \tag{1.10}$$

so that

$$\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]. \tag{1.11}$$

The gamma matrices defined above satisfy the Clifford algebra

$$\{\gamma_a, \gamma_b\} = 2\,\eta_{ab}\,\mathbb{1}_4. \tag{1.12}$$

As one easily verifies, these Clifford algebra relations already imply the Lorentz algebra commutation relations for the matrices Σ_{ab} if one uses (1.11) as their definition. From this point of view, the particular expression (1.9) is just a special case that corresponds to the particular representation (1.10) of the Clifford algebra

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the R-symmetry group, under which also the gravitino is charged, but these theories are often anomalous.

(1.12). This representation (1.10) is called the *Weyl representation*, and its main virtue is that the Lorentz generators assume the block diagonal form (1.9), making their reducibility manifest.[5] However, there are infinitely many other irreducible representations of (1.12) that one could also use; apart from the Weyl representation, the most common representations are the Dirac representation and the Majorana representation.[6] Fortunately, all these representations are equivalent to the Weyl representation, and we rarely will make use of particular representations. For convenience, however, we will always assume "friendly" representations whose defining properties are

$$\gamma_0^\dagger = -\gamma_0 \tag{1.13}$$

$$\gamma_i^\dagger = +\gamma_i \tag{1.14}$$

$$\gamma_a^T = \pm\gamma_a, \tag{1.15}$$

where the last equation means that each gamma matrix is either symmetric or antisymmetric. The Weyl, Dirac, and Majorana representations are obviously all friendly representations.

An important object in the following will be the completely antisymmetrized products of several gamma matrices,

$$\gamma_{a_1\ldots a_p} \equiv \gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_p]}, \tag{1.16}$$

where, as usual, the antisymmetrization involves a prefactor $1/p!$, so that, e.g., $\gamma_{ab} = 1/2[\gamma_a, \gamma_b]$, etc. Note that the Clifford relation (1.12) implies

$$\gamma_{a_1a_2\ldots a_p} = \begin{cases} \gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_p} & \text{if all } a_i \text{ are different} \\ 0 \text{ otherwise} & \end{cases} \tag{1.17}$$

The 16 matrices $\gamma_M = \{\mathbb{1}_4, \gamma_a, \gamma_{ab}, \gamma_{abc}, \gamma_{abcd}\}$ are linearly independent and form a basis of the complex (4×4) matrices.

As we have seen, an irreducible representation of the Clifford algebra (1.12) induces a spinor representation (1.11) of the Lorentz group. This is true in any spacetime dimension, which makes the Clifford algebra approach to spinor representations so useful for higher-dimensional supergravity theories. As was stressed in footnote 5, however, the Lorentz algebra representations obtained in

[5] Note that the Weyl representation is *irreducible* as a representation of the *Clifford algebra* (1.12). It is only the induced representation Σ_{ab} of the *Lorentz algebra* that is reducible.

[6] The Dirac representation corresponds to $\gamma_0 = i\sigma_3 \otimes \mathbb{1}_2 \equiv \begin{pmatrix} i\mathbb{1}_2 & 0 \\ 0 & -i\mathbb{1}_2 \end{pmatrix}$, $\gamma_i = \sigma_2 \otimes \sigma_i$, and the Majorana representation is given by $\gamma_0 = i\sigma_2 \otimes \sigma_3$, $\gamma_1 = -\sigma_1 \otimes \mathbb{1}_2$, $\gamma_2 = \sigma_2 \otimes \sigma_2$, $\gamma_3 = \sigma_3 \otimes \mathbb{1}_2$, but they are not really needed for this book.

this way are in general not irreducible, even if the underlying Clifford algebra representation is irreducible.

For supersymmetry, it is convenient to work with minimal representations of the Lorentz group, as these typically correspond also to the minimal amount of supersymmetry one can have in the respective spacetime dimension. There are essentially two ways to reduce the number of degrees of freedom of a Clifford algebra spinor so as to obtain irreducible representations (irreps) of the corresponding Lorentz algebra. One possibility is to impose a chirality condition, which leads to Weyl spinors, and the other one is to impose a reality condition, which leads to Majorana spinors.[7] This is not always possible in every spacetime dimension: The Weyl condition can only be imposed in even dimensions, whereas the possibility to impose a Majorana condition shows a somewhat more complicated dependence on the spacetime dimension, as we will see in Chap. 10. Moreover, the Weyl and Majorana condition can often not be imposed simultaneously. For the moment, we restrict ourselves to four spacetime dimensions, where one can impose a Weyl *or* a Majorana condition, but not both of them at the same time.

1.3.2.2 The Weyl Condition

The Weyl condition projects out the part of a spinor that has a particular handedness. It is imposed with the γ_5 matrix:

$$\gamma_5 \equiv \gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = +i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{1.18}$$

The Clifford algebra (1.12) implies

$$(\gamma_5)^2 = \mathbb{1}_4 \tag{1.19}$$

$$\{\gamma_5, \gamma_a\} = 0 \Rightarrow [\gamma_5, \Sigma_{ab}] = 0 \tag{1.20}$$

so that the chirality projectors

$$P_L \equiv \frac{1}{2}\left(1 + \gamma^5\right), \quad P_R \equiv \frac{1}{2}\left(1 - \gamma^5\right), \tag{1.21}$$

can be used to define left- and right-handed spinors.

$$\psi_L \equiv P_L\psi, \qquad \psi_R \equiv P_R\psi. \qquad \text{(Weyl condition)} \tag{1.22}$$

[7] Note that imposing a Weyl or Majorana condition in 4D does not necessitate the use of the Weyl or the Majorana representation of the gamma matrices. The Weyl condition just takes on a particularly simple form in the Weyl representation, and the Majorana condition leads to a particularly simple result in the Majorana representation. We usually do not make use of these simplified forms and write down the conditions in a covariant way.

Because of (1.20), this projection is consistent with Lorentz covariance, and left- and right-handed spinors form separate representations of the Lorentz group. In the Weyl representation, $\gamma_5 = \sigma_3 \otimes \mathbb{1}_2$, i.e., the left- and right-handed spinors, are simply the upper or lower two components of ψ as in (1.8). Note that $\gamma^5 \psi_L = \psi_L$, while $\gamma^5 \psi_R = -\psi_R$.

In a friendly representation, $\gamma_0 \gamma_a^\dagger \gamma_0 = \gamma_a \Rightarrow \Sigma_{ab}^\dagger \gamma_0 = -\gamma_0 \Sigma_{ab}$, and the *Dirac conjugate* of a general four-component spinor is defined by

$$\overline{\psi} \equiv i\psi^\dagger \gamma^0 = -i\psi^\dagger \gamma_0 \tag{1.23}$$

so that bilinears such as $\overline{\psi}\chi$ are Lorentz invariant.

In a friendly representation, we also have that $\gamma_5^\dagger = \gamma_5$, and hence $P_{L,R}^\dagger = P_{L,R}$. Moreover $P_L \gamma_0 = \gamma_0 P_R$ and $P_R \gamma_0 = \gamma_0 P_L$, and this implies

$$\overline{\psi_R} = \overline{\psi} P_L, \text{ and } \overline{\psi_L} = \overline{\psi} P_R, \tag{1.24}$$

where

$$\overline{\psi_R} \equiv \overline{(P_R \psi)} = -i(P_R \psi)^\dagger \gamma_0. \tag{1.25}$$

Because of this we will often write

$$\overline{\psi}_L \equiv \overline{\psi} P_L = \overline{\psi_R}, \qquad \overline{\psi}_R \equiv \overline{\psi} P_R = \overline{\psi_L}. \tag{1.26}$$

The matrix γ_5 also enters a number of very useful duality relations between the antisymmetrized products of gamma matrices, as the reader is asked to verify in Exercise 1.1.

1.3.2.3 The Majorana Condition

The Majorana condition is a reality condition that can be written as

$$\psi^* = B\psi, \qquad \text{(Majorana condition)} \tag{1.27}$$

with some matrix B. This condition is self-consistent (i.e., $\psi^{**} = \psi$) and Lorentz covariant if B satisfies

$$B^* B = \mathbb{1}_4, \qquad \gamma_a^* = B\gamma_a B^{-1} \Rightarrow \Sigma_{ab}^* B = B\Sigma_{ab}. \tag{1.28}$$

In the Weyl basis (1.10), one can choose

$$B = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}, \tag{1.29}$$

so that a Majorana spinor would be of the form (1.8), but with $\lambda = \chi$. From this we see that a Majorana spinor describes the same number of independent degrees of freedom as a Weyl spinor.

Another equivalent, but for many purposes more convenient, way to write the Majorana condition is via the charge conjugation matrix, C, which satisfies

$$C^T = -C, \qquad \gamma_a^T = -C\gamma_a C^{-1}. \tag{1.30}$$

In a friendly representation, one can moreover choose C such that it also satisfies

$$C^{-1} = -C = C^\dagger. \tag{1.31}$$

In terms of C, the charge conjugate spinor is defined as

$$\psi^c = C\overline{\psi}^T = iC\gamma^{0T}\psi^* \tag{1.32}$$

and a *Majorana spinor* is defined as

$$\psi^c = \psi. \qquad \text{(alternative form of Majorana condition)} \tag{1.33}$$

This is equivalent to (1.27) if we identify

$$B = (i\,C\gamma_0^T)^{-1}, \tag{1.34}$$

so that, in terms of B, charge conjugation reads

$$\psi^c = B^{-1}\psi^*. \tag{1.35}$$

The advantage of C is that for a Majorana spinor the Dirac conjugate can be written as

$$\overline{\psi} = \psi^T C. \tag{1.36}$$

Notice that using (1.30) one finds the symmetry properties

$$\begin{aligned} &C^T = -C, \qquad (C\gamma^{abc})^T = -(C\gamma^{abc}), \;\; (C\gamma^{abcd})^T = -(C\gamma^{abcd}), \\ &(C\gamma^a)^T = (C\gamma^a), \;\; (C\gamma^{ab})^T = (C\gamma^{ab}). \end{aligned} \tag{1.37}$$

For *anti*-commuting Majorana spinors, this then implies

$$\overline{\psi}_1 M \psi_2 = \begin{cases} +\overline{\psi}_2 M \psi_1 & \text{for } M = \mathbb{1}_4, \gamma_{abc}, \gamma_{abcd} \\ -\overline{\psi}_2 M \psi_1 & \text{for } M = \gamma_a, \gamma_{ab} \end{cases} \tag{1.38}$$

*Unless stated otherwise, we will, in the following, always use anti-commuting Majorana spinors but often also take in addition the chiral projections ψ_L and ψ_R of these Majorana spinors, which therefore are **not** independent.*

More specifically, we have, in our conventions,

$$(\psi_L)^c = \psi_R, \qquad (\psi_R)^c = \psi_L. \tag{1.39}$$

To show this, we use

$$B^{-1}\gamma_5^* = -\gamma_5 B^{-1} \Leftrightarrow B^{-1}P_L^* = P_R B^{-1} \tag{1.40}$$

so that

$$(\psi_L)^c = (P_L\psi)^c = B^{-1}P_L^*\psi^* = P_R B^{-1}\psi^* = P_R\psi^c = P_R\psi = \psi_R. \tag{1.41}$$

Note that (1.39) implies that ψ_R is no longer a Majorana spinor, because that would require $(\psi_R)^c$ being equal to ψ_R. Thus, in 4D, a four-component spinor cannot be simultaneously chiral and Majorana. Nevertheless, it makes sense to talk about the projection ψ_R or ψ_L of a given Majorana spinor ψ.

From the definition of γ_5 and (1.30), one also gets

$$C\gamma_5 = \gamma_5^T C \Leftrightarrow CP_L = P_L^T C \tag{1.42}$$

so that for a Majorana spinor ψ,

$$\overline{\psi}_L = \overline{\psi}P_L = \psi^T CP_L = \psi^T P_L^T C = (\psi_L)^T C \tag{1.43}$$

even though ψ_L is not Majorana. From this we can obtain more symmetry properties for the chiral projections that are very similar to those for the Majorana spinors themselves,

$$\begin{aligned}
&\overline{\chi}_L\psi_L = \chi_L^T C\psi_L = -\psi_L^T C^T\chi_L = \overline{\psi}_L\chi_L,\\
&\overline{\chi}_L\gamma^a\psi_R = -\overline{\psi}_R\gamma^a\chi_L, \quad \overline{\chi}_L\gamma^{ab}\psi_L = -\overline{\psi}_L\gamma^{ab}\chi_L\\
&\overline{\chi}_L\gamma^{abc}\psi_R = \overline{\psi}_R\gamma^{abc}\chi_L.
\end{aligned} \tag{1.44}$$

We also note that under charge conjugation

$$(\gamma_a)^c = \gamma_a, \quad (\gamma_5)^c = -\gamma_5, \tag{1.45}$$

in the sense that $(\gamma^a\psi)^c = \gamma^a\psi^c$, etc.

In all the subsequent formulae, the Hermitian conjugate, $+h.c.$, of a field operator is denoted with a superscript *, whereas the superscript † is reserved for matrix expressions when in addition to Hermitian conjugation also a transposition of the matrix is involved. On ordinary complex numbers and classical fields, the Hermitian conjugation acts as complex conjugation, where, however, the order of anti-commuting spinor fields is exchanged to mimic the effect of Hermitian conjugation on the corresponding quantum fields. This results in a minus sign when the original spinor order is restored. Fortunately, the effect of this Hermitian conjugation can simply be obtained by writing down the charge conjugate expression with all the rules obtained so far, including (1.45), but without exchanging the order of the spinors.

As an example, we show $(\overline{\psi}_L\gamma^a\chi_R)^* = (\overline{\psi}_L\gamma^a\chi_R)^c = \overline{\psi}_R\gamma^a\chi_L$:

$$\begin{aligned}(\overline{\psi}_L\gamma^a\chi_R)^* &= (\psi_L^T C\gamma^a\chi_R)^* = -\psi_L^\dagger C^*\gamma^{a*}\chi_R^* = -\psi_L^\dagger C^*\gamma^{a*}B(\chi_R)^c \\ &= -\psi_L^\dagger C^* B\gamma^a\chi_L. \end{aligned} \tag{1.46}$$

Inserting (1.34), $C^* = C$, and $(\gamma^{0T})^{-1} = -C(\gamma^0)^{-1}C^{-1} = -C^{-1}(\gamma^0)^{-1}C$, this becomes

$$\begin{aligned}(\overline{\psi}_L\gamma^a\chi_R)^* &= -i\psi_L^\dagger(\gamma^0)^{-1}\gamma^a\chi_L = \overline{\psi_L}\ \gamma^a\chi_L = \overline{\psi}_R\gamma^a\chi_L = (\overline{\psi}_L)^c(\gamma^a)^c(\chi_R)^c \\ &= (\overline{\psi}_L\gamma^a\chi_R)^c, \end{aligned} \tag{1.47}$$

where in the second equation we used $(\gamma^0)^{-1} = -\gamma^0$, which follows from $(\gamma^0)^2 = -\mathbb{1}_4$.

In the following, we will often need to rewrite three or four Fermi terms to complete our analysis of the supersymmetry properties of an action, and hence Fierz identities will be extremely useful. We list here the main ones for two spinors:

$$\psi_R\overline{\chi}_R = -\frac{1}{2}\overline{\chi}_R\psi_R\ P_R + \frac{1}{8}\overline{\chi}_R\gamma_{ab}\psi_R\ \gamma^{ab}\ P_R, \tag{1.48}$$

$$\psi_R\overline{\chi}_L = -\frac{1}{2}\overline{\chi}_L\gamma^a\psi_R\ \gamma_a\ P_L, \tag{1.49}$$

where for the sake of clarity we explicitly left the projectors on the right hand side.

In the rest of these lectures, we will also often make use of spinor one-forms $\psi = dx^\mu\psi_\mu$. Exchanging such spinor one-forms then leads to an additional minus sign from the anti-commutativity of the wedge product, and hence we have

$$\psi_R \wedge \overline{\psi}_R = -\frac{1}{8}\overline{\psi}_R \wedge \gamma_{ab}\psi_R\ \gamma^{ab}\ P_R, \tag{1.50}$$

$$\psi_R \wedge \overline{\psi}_L = \frac{1}{2}\overline{\psi}_L \wedge \gamma^a\psi_R\ \gamma_a\ P_L, \tag{1.51}$$

where now $\overline{\psi}_L \wedge \psi_L = 0$ because of (1.44) and the wedge product. A crucial consequence is the *cyclic identity*:

$$\gamma^a \psi_L \wedge \overline{\psi}_L \wedge \gamma_a \psi_R = 0. \tag{1.52}$$

1.3.3 Susy Algebra in Four Dimensions

Using the conventions described so far, the $\mathcal{N} = 1$ supersymmetry algebra in four dimensions has the following form:

$$\begin{aligned}
\{Q, \bar{Q}\} &= -2i\gamma^a \mathscr{P}_a, \\
[\mathscr{P}_a, Q] &= 0, \\
[\mathscr{M}_{ab}, Q] &= \frac{i}{2}\gamma_{ab} Q, \\
[R, Q] &= i\,\gamma_5 Q, \\
[\mathscr{P}_a, \mathscr{P}_b] &= 0, \\
[\mathscr{P}_a, \mathscr{M}_{bc}] &= -2i\,\eta_{a[b}\mathscr{P}_{c]}, \\
[\mathscr{M}_{ab}, \mathscr{M}_{cd}] &= 2i\,\eta_{c[a}\mathscr{M}_{b]d} - 2i\,\eta_{d[a}\mathscr{M}_{b]c}.
\end{aligned} \tag{1.53}$$

Here, Q is the supersymmetry generator described by a Majorana spinor; $\mathscr{M}_{ab}$ and $\mathscr{P}_a$ denote, respectively, the usual generators of Lorentz transformations and spacetime translations; and R is the U(1) internal R-symmetry generator. In (1.53), we used Hermitian generators $\mathscr{P}_a$ and $\mathscr{M}_{ab}$. Just as in (1.3), we will sometimes also use their anti-Hermitian counterparts,

$$P_a = i\,\mathscr{P}_a, \qquad M_{ab} = i\,\mathscr{M}_{ab} \tag{1.54}$$

when this is more convenient. Note that, for simplicity, we have not included internal bosonic symmetry generators other than the R-symmetry, as they commute with the above generators.

Exercises

1.1. Using the totally antisymmetric epsilon tensor with

$$\epsilon_{0123} = 1, \tag{1.55}$$

check the duality relations

$$\begin{aligned} \gamma^{abc} &= i\,\epsilon^{abcd}\gamma_d\gamma_5, \qquad & i\,\gamma_a\gamma_5 &= \frac{1}{3!}\epsilon_{abcd}\gamma^{bcd}, \\ \gamma^{abcd} &= -i\,\epsilon^{abcd}\gamma_5, \qquad & i\,\gamma_5 &= \frac{1}{4!}\epsilon_{abcd}\gamma^{abcd}, \\ \gamma^{ab} &= \frac{i}{2}\epsilon^{abcd}\gamma_{cd}\gamma_5. \end{aligned} \tag{1.56}$$

1.2. Verify that $M_{ab} = \frac{1}{2}\gamma_{ab}$ satisfies $[M_{ab}, M_{cd}] = -2\,\eta_{c[a}M_{b]d} + 2\,\eta_{d[a}M_{b]c}$.

1.3. Using just the Clifford algebra (1.12) in an *arbitrary* representation and the definition (1.11) of Σ_{ab}, compute the rotation matrix $R(\theta) = e^{\theta\Sigma^{12}}$ for a rotation in the $(1, 2)$-plane by a finite angle θ and read off from your result that $R(2\pi) = -\mathbb{1}$, i.e., that Σ_{ab} is indeed a spinor representation.

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