Lightweight engineering design of nonlinear dynamic systems with gradient-based structural design optimization

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Abstract. Reducing the weight of a system leads to lower forces being exerted, which in turn allows for lower requirements and an even lighter system. This "virtuous circle of lightweight engineering design" can especially be present when designing dynamic systems. Design optimization is a tool to enable and exploit this favorable phenomenon. This work introduces a unified approach to reap the benefits of optimally designed lightweight systems in structural dynamics and multibody dynamics. An efficient gradient-based optimization framework has been implemented and this is explained and demonstrated. The centerpiece of this optimization methodology is the design sensitivity analysis applied to the time integration with a nonlinear solver. A semi-analytical approach is chosen to balance computational effort and implementation effort, where the sensitivities are derived via direct differentiation with numerical differences for the sensitivities of the system parameters. Nomenclature is introduced to simplify these equations for a more lucid description showing the intrinsic equivalence of the solving routines of structural dynamics and multibody dynamics. The method is shown on the practical example for the optimal design of a hydraulic engineering mechanism.

Keywords: Lightweight engineering design · Design optimization · Sensitivity analysis · Structural dynamics · Multibody dynamics

Nomenclature

∇	sensitivities w.r.t. x (total derivative)
∂	partial derivatives w.r.t. x
$\frac{\partial(\cdot)}{\partial(\cdot)}$	partial derivative
$(\cdot)$	scalar
$\underline{(\cdot)}$	vector
$\underline{\underline{(\cdot)}}$	matrix
$\underline{\underline{\underline{(\cdot)}}}$	3D matrix
$\underline{\underline{\underline{\underline{(\cdot)}}}}$	4D matrix
F_c	right-hand side of acceleration constraints
F_{ext}	external force
F_{pseudo}	pseudo load
F_R	residual force
F_v	quadratic velocity term
d	damping
$\underline{\underline{\underline{\underline{e}}}}$	4D identity matrix
i	time step index
J	Jacobian w.r.t. q
$\dot{\mathrm{J}}$	Jacobian w.r.t. $\dot{q}$

S. Pfingstl et al. (eds.), *Proceedings of the Munich Symposium on Lightweight Design 2020*, https://doi.org/10.1007/978-3-662-63143-0_5

$\ddot{\mathrm{J}}$	Jacobian w.r.t. $\ddot{q}$	$\dot{q}$	velocity
$\overset{\lambda}{\mathrm{J}}$	Jacobian w.r.t. λ	$\dot{q}_{\mathrm{pred}}$	predicted velocity
$\overset{\nabla}{\mathrm{J}}$	Jacobian w.r.t. ∇q	$\ddot{q}$	acceleration
$\overset{\nabla}{\mathrm{J}}$	Jacobian w.r.t. $\nabla \dot{q}$	t	time
$\overset{\nabla}{\mathrm{J}}$	Jacobian w.r.t. $\nabla \ddot{q}$	x	design variables
$\overset{\nabla\lambda}{\mathrm{J}}$	Jacobian w.r.t. $\nabla \lambda$	Δ	finite change
k	stiffness	λ	Lagrangian multipliers
m	mass	Φ	kinematic constraints
q	position	β, γ	integration constants
q_{pred}	predicted position		

1 The Virtuous Circle of Lightweight Engineering Design

The lightweight engineering design philosophy is exemplified by the Virtuous Circle of Lightweight Engineering Design. With less structural mass, the structural requirements, motorization requirements or both are reduced and therefore the structural mass can in turn be reduced again. This can continue until some theoretical minimum is reached. The virtuous circle design philosophy is magnified when looking at dynamic machines. This stands in stark contrast to the vicious circle of excess structural weight. Here extra structural mass increases the structural or motorization requirements, which in turn results in more weight, ending in a concept which no longer is able to fulfill design requirements..

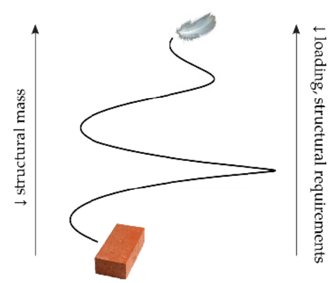


Fig. 1: Virtuous Circle of Lightweight Engineering Design

Efficient structural design optimization is an effective tool in entering the Virtuous Circle of Lightweight Engineering Design. Gradient-based optimization using analytical sensitivity analysis is especially efficient. Although structural design optimization has been brought to maturity for linear elasto-static structures [1, 9, 17], questions remain in areas of multiphysics, dynamics and nonlinearity. Structural dynamics and multibody dynamics are briefly introduced and the analytical design sensitivities using direct differentiation derived. Structural dynamics covers structural analysis under dynamic loading and behavior. Mechanisms and mechanical systems consisting of multiple bodies connected by joints are modeled and analyzed with multibody dynamics.

2 Sensitivity analysis of nonlinear dynamic systems

The centerpiece of efficient gradient-based design optimization is the sensitivity analysis. The categorization of sensitivity analysis is shown in fig. 2. The semi-analytical approach combines aspects of numerical and analytical sensitivities, using the analytical derivative of the governing equation with numerical sensitivities of the system parameters, i.e. the system matrices, e.g. mass and stiffness matrices. In this work a semi-analytical approach will be introduced unifying the calculation between structural dynamics and rigid multibody dynamics and their sensitivities. In doing so, the following sections builds and expands on the work of [1, 16].

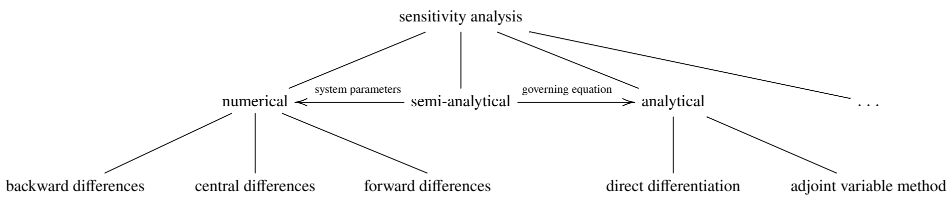


Fig. 2: Categorization for types of design sensitivity analysis

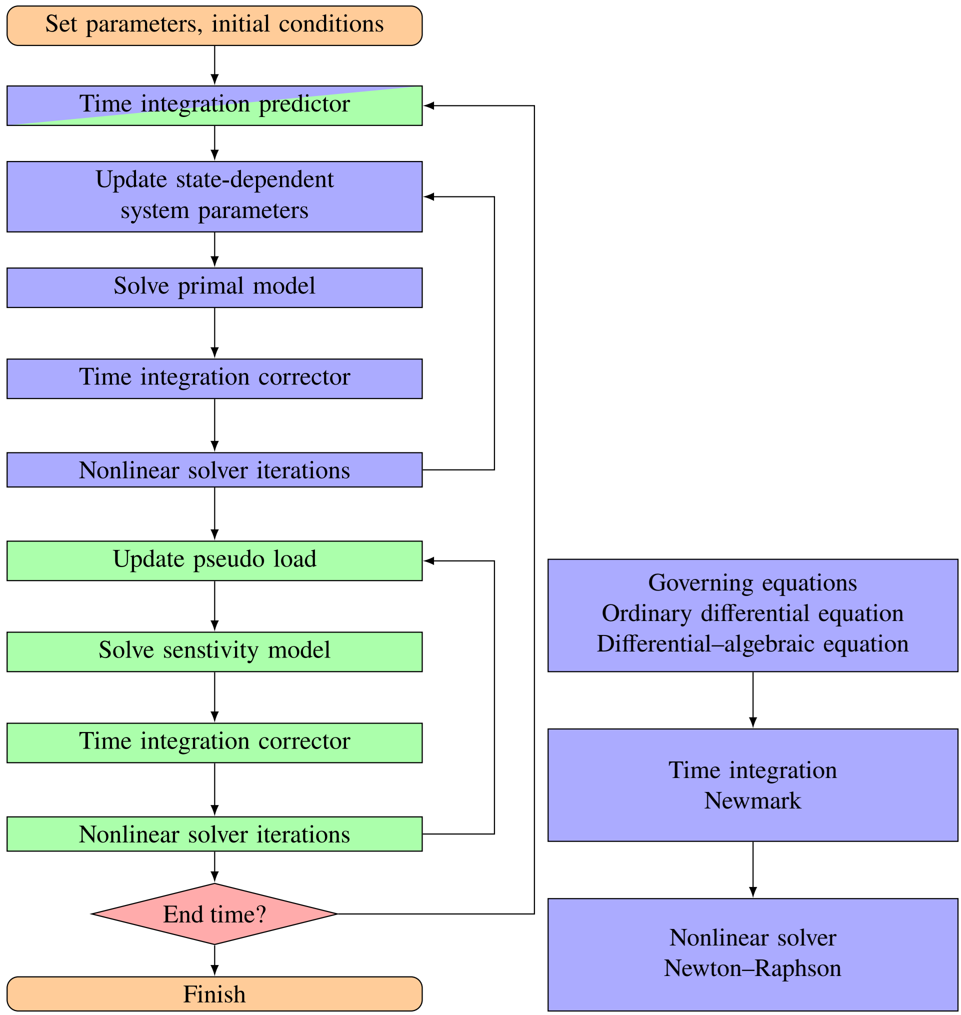
For transient analysis studied here, numerical time integration is needed with an integrated linear solver, which is called at every time step. In turn, sensitivity analysis must also be performed in this same fashion. It is most effective from an implementation view to use the same solving routine for both primal and sensitivity analysis. The unified code accepts matrices of higher dimensions to work properly. The flowchart and main building blocks are shown in fig. 3, which will be explained below for both structural dynamics and rigid multibody dynamics.

This will be shown using direct differentiation, which is best used (i.e. more efficient) when the number of optimization functions considered (i.e. sum of the number of objectives and number of constraints) is higher than the number of design variables. The adjoint variable method is more efficient when the number of design variables exceeds the number of optimization functions. This rule of thumb, though, does not consider the implementation complexity and effort.

3 Structural dynamics

3.1 Governing equation

In this section, the governing equation for structural dynamics will be introduced. In the second subsection, the sensitivity analysis will be derived and the nomenclature used throughout is introduced.



(a) Flowchart (b) Generalized building blocks

Fig. 3: Solving procedure

3.1.1 Primary analysis The governing equation for structural dynamics is described here by the force equilibrium equation,

$$\underline{F}_R\,(t) = \underline{\underline{m}}\left(\underline{q}, t\right)\underline{\ddot{q}}\,(t) + \underline{\underline{d}}\left(\underline{q}, t\right)\underline{\dot{q}}\,(t) + \underline{\underline{k}}\left(\underline{q}, t\right)\underline{q}\,(t) - \underline{F}_{\text{ext}}\left(\underline{q}, t\right) = \underline{0}, \tag{1}$$

where $\underline{\underline{m}}$ is the mass matrix, $\underline{\underline{d}}$ is the damping matrix, $\underline{\underline{k}}$ is the stiffness matrix, t is time, $\underline{q}$ is the position vector, $\underline{\dot{q}}$ is the velocity vector, $\underline{\ddot{q}}$ is the acceleration vector, $\underline{F}$ is the external load vector and $\underline{F}_R$ is the residual force vector, which is zero when in equilibrium. The mass, damping, stiffness and external force are collectively referred to as the system parameters, while the acceleration, velocity and position are collectively the state variables. Here the solving of the governing equation is referred to as the primal analysis. We will use a compact notation in which the dependence is not explicitly shown,

$$\underline{F}_R = \underline{\underline{m}}\,\underline{\ddot{q}} + \underline{\underline{d}}\,\underline{\dot{q}} + \underline{\underline{k}}\,\underline{q} - \underline{F}_{\text{ext}} = \underline{0}. \tag{2}$$

3.1.2 Sensitivity analysis The direct differentiation of governing equation (2) results in

$$\underline{\underline{\nabla F_R}} = \underline{\underline{\underline{\nabla m}}}\,\underline{\ddot{q}} + \underline{\underline{m}}\,\underline{\underline{\nabla \ddot{q}}} + \underline{\underline{\underline{\nabla d}}}\,\underline{\dot{q}} + \underline{\underline{d}}\,\underline{\underline{\nabla \dot{q}}} + \underline{\underline{\underline{\nabla k}}}\,\underline{q} + \underline{\underline{k}}\,\underline{\underline{\nabla q}} - \underline{\underline{\nabla F_{\text{ext}}}} = \underline{\underline{0}}, \tag{3}$$

where ∇ is defined by partial derivative of some function $(\cdot)$ with respect to the vector of design variables $\underline{x}$,

$$\underline{\underline{\nabla\,(\cdot)}} = \nabla_{\underline{x}}\underline{(\cdot)} = \frac{\mathrm{d}\underline{(\cdot)}}{\mathrm{d}\underline{x}} = \begin{bmatrix} \frac{\mathrm{d}(\cdot)_1}{\mathrm{d}x_1} & \frac{\mathrm{d}(\cdot)_1}{\mathrm{d}x_2} & \cdots & \frac{\mathrm{d}(\cdot)_1}{\mathrm{d}x_m} \\ \frac{\mathrm{d}(\cdot)_2}{\mathrm{d}x_1} & \frac{\mathrm{d}(\cdot)_n}{\mathrm{d}x_2} & \cdots & \frac{\mathrm{d}(\cdot)_2}{\mathrm{d}x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}(\cdot)_n}{\mathrm{d}x_1} & \frac{\mathrm{d}(\cdot)_n}{\mathrm{d}x_2} & \cdots & \frac{\mathrm{d}(\cdot)_n}{\mathrm{d}x_m} \end{bmatrix}. \tag{4}$$

Applying the chain rule on each term and rearranging, leaves the same form as the primal governing equation,

$$\underline{\underline{\nabla F}}_R = \underline{\underline{m}}\,\underline{\underline{\nabla \ddot{q}}} + \underline{\underline{d}}\,\underline{\underline{\nabla \dot{q}}} + \underline{\underline{k}}\,\underline{\underline{\nabla q}} - \underline{\underline{F}}_{\text{pseudo}} = \underline{\underline{0}}, \tag{5}$$

where the pseudo load is defined by

$$\underline{\underline{F}}_{\text{pseudo}} = \left(\underline{\underline{\partial F}}_{\text{ext}} + \underline{\underline{\mathrm{J}}}_{F_{\text{ext}}}\,\underline{\underline{\partial q}}\right) - \left(\underline{\underline{\underline{\partial m}}} + \underline{\underline{\mathrm{J}}}_m\,\underline{\underline{\partial q}}\right)\underline{\ddot{q}} - \left(\underline{\underline{\underline{\partial d}}} + \underline{\underline{\mathrm{J}}}_d\,\underline{\underline{\partial q}}\right)\underline{\dot{q}} - \left(\underline{\underline{\underline{\partial k}}} + \underline{\underline{\mathrm{J}}}_k\,\underline{\underline{\partial q}}\right)\underline{q}. \tag{6}$$

The following nomenclature to show partials of the partial derivatives with respect to the design variables (to contrast them from total partial derivatives $\underline{\underline{\nabla\,(\cdot)}} = \frac{\mathrm{d}\underline{(\cdot)}}{\mathrm{d}\underline{x}}$):

$$\underline{\underline{\partial\,(\cdot)}} = \partial_{\underline{x}}\underline{(\cdot)} = \frac{\partial\underline{(\cdot)}}{\partial\underline{x}}. \tag{7}$$

The Jacobians of the system variable are defined here by,

$$\underline{\underline{\mathrm{J}}}_{(\cdot)} = \mathrm{J}\left(\underline{(\cdot)}\right) = \frac{\partial \underline{(\cdot)}}{\partial \underline{q}}, \tag{8}$$

$$\underline{\underline{\dot{\mathrm{J}}}}_{(\cdot)} = \dot{\mathrm{J}}\left(\underline{(\cdot)}\right) = \frac{\partial \underline{(\cdot)}}{\partial \underline{\dot{q}}}, \tag{9}$$

$$\underline{\underline{\ddot{\mathrm{J}}}}_{(\cdot)} = \ddot{\mathrm{J}}\left(\underline{(\cdot)}\right) = \frac{\partial \underline{(\cdot)}}{\partial \underline{\ddot{q}}}. \tag{10}$$

For linear structural dynamics, the system parameters are independent of position and the total derivative is equal to the partial derivative, e.g. $\underline{\underline{\nabla m}} = \underline{\underline{\partial m}}$. The partials can be directly implemented for a fully analytical sensitivity analysis or calculated numerically, as the case is here, for a semi-analytical method.

3.2 Time integration

3.2.1 Primary analysis To solve this dynamic problem, the time integration will be shown here with Newmark-β method [14] with accelerations as the primary variable. This is carried out by solving eq. 2 for the accelerations at each time step (here i) using the predicted displacements and velocities,

$${}^{i}\underline{q}_{\text{pred}} = {}^{i-1}\underline{q} + \Delta t\, {}^{i-1}\underline{\dot{q}} + \left(\frac{1}{2} - \beta\right) \Delta t^{2}\, {}^{i-1}\underline{\ddot{q}}, \tag{11}$$

$${}^{i}\underline{\dot{q}}_{\text{pred}} = {}^{i-1}\underline{\dot{q}} + (1-\gamma)\, \Delta t\, {}^{i-1}\underline{\ddot{q}}, \tag{12}$$

where the left superscript i is the current time step, the left superscript $i-1$ is the previous time step, Δt is the time increment, while β and γ are the time integration constants. After updating the system parameters and calculating the acceleration, the effective mass and effective force are assembled,

$$\underline{\underline{m}}_{\text{eff}} = \underline{\underline{m}} + \gamma \Delta t \underline{\underline{d}} + \beta \Delta t^{2} \underline{\underline{k}}, \tag{13}$$

$$\underline{F}_{\text{eff}} = \underline{F}_{\text{ext}} - \underline{\underline{d}}\, {}^{i}\underline{\dot{q}}_{\text{pred}} - \underline{\underline{k}}\, {}^{i}\underline{q}_{\text{pred}}. \tag{14}$$

and then the following equation is solved for acceleration,

$$\underline{\underline{m}}_{\text{eff}}\, {}^{i}\underline{\ddot{q}} = \underline{F}_{\text{eff}}. \tag{15}$$

This acceleration is used to correct the predicted state variables,

$${}^{i}\underline{q} = {}^{i}\underline{q}_{\text{pred}} + \beta \Delta t^{2}\, {}^{i}\underline{\ddot{q}}, \tag{16}$$

$${}^{i}\underline{\dot{q}} = {}^{i}\underline{\dot{q}}_{\text{pred}} + \gamma \Delta t\, {}^{i}\underline{\ddot{q}}. \tag{17}$$

3.2.2 Sensitivity analysis The sensitivity analysis within the time integration is carried out completely analogously to the primary analysis, though using matrix-valued terms for the sensitivities instead of vectors of the system responses. For the prediction this becomes

$$
\begin{aligned}
{}^{i}\underline{\underline{\nabla q}}_{\text{pred}} &= {}^{i-1}\underline{\underline{\nabla q}} + \Delta t\, {}^{i-1}\underline{\underline{\nabla \dot{q}}} + \left(\frac{1}{2} - \beta\right) \Delta t^{2}\, {}^{i-1}\underline{\underline{\nabla \ddot{q}}}, &(18)\\
{}^{i}\underline{\underline{\nabla \dot{q}}}_{\text{pred}} &= {}^{i-1}\underline{\underline{\nabla \dot{q}}} + (1-\gamma)\, \Delta t\, {}^{i-1}\underline{\underline{\nabla \ddot{q}}}, &(19)
\end{aligned}
$$

resulting in the following effective mass sensitivity and effective force sensitivity:

$$
\begin{aligned}
\underline{\underline{\underline{\nabla m}}}_{\text{eff}} &= \underline{\underline{\underline{\nabla m}}} + \gamma \Delta t \underline{\underline{\underline{\nabla d}}} + \beta \Delta t^{2} \underline{\underline{\underline{\nabla k}}}, &(20)\\
\underline{\underline{\nabla F}}_{\text{eff}} &= \underline{\underline{\nabla F}}_{\text{ext}} - \underline{\underline{\underline{\nabla d}}}\, {}^{i}\underline{\dot{q}}_{\text{pred}} - \underline{\underline{d}}\, {}^{i}\underline{\underline{\nabla \dot{q}}}_{\text{pred}} - \underline{\underline{\underline{\nabla k}}}\, {}^{i}\underline{q}_{\text{pred}} - \underline{\underline{k}}\, {}^{i}\underline{\underline{\nabla q}}_{\text{pred}}. &(21)
\end{aligned}
$$

Then the following equation is solved for acceleration sensitivity:

$$
\underline{\underline{\underline{\nabla m}}}_{\text{eff}}\, {}^{i}\underline{\underline{\nabla \ddot{q}}} = \underline{\underline{\nabla F}}_{\text{eff}}. \tag{22}
$$

The predicted sensitivity values are then corrected, giving

$$
\begin{aligned}
{}^{i}\underline{\underline{\nabla q}} &= {}^{i}\underline{\underline{\nabla q}}_{\text{pred}} + \beta \Delta t^{2}\, {}^{i}\underline{\underline{\nabla \ddot{q}}}, &(23)\\
{}^{i}\underline{\underline{\nabla \dot{q}}} &= {}^{i}\underline{\underline{\nabla \dot{q}}}_{\text{pred}} + \gamma \Delta t\, {}^{i}\underline{\underline{\nabla \ddot{q}}}. &(24)
\end{aligned}
$$

3.3 Nonlinear solver

3.3.1 Primary analysis In the general form, the system parameters (mass, damping, stiffness and force) are dependent of the state variables (acceleration, velocity and position), therefore a nonlinear solver is needed. The methodology here is general and this specific case will be shown with Newton–Raphson method,

$$
\frac{\partial \underline{F}_{R}}{\partial \underline{\ddot{q}}} \Delta \underline{\ddot{q}} + \underline{F}_{R} = 0, \tag{25}
$$

which requires then the Jacobian of the residual with respect to the accelerations, denoted here as $\underline{\underline{\ddot{\mathrm{J}}}}_{F_R}$.

3.3.2 Sensitivity analysis For the sensitivity analysis, the Newton–Raphson step is

$$
\frac{\partial \underline{\underline{\nabla F}}_{R}}{\partial \underline{\underline{\nabla \ddot{q}}}} \Delta \underline{\underline{\nabla \ddot{q}}} = \underline{\underline{\nabla F}}_{R}, \tag{26}
$$

where the Jacobian of the residual sensitivity with respect to the acceleration sensitivity is compactly denoted by $\underline{\underline{\underline{\overset{\ddot{\nabla}}{\mathrm{J}}}}}_{\nabla F_R}$. Via expansion of the partial derivatives, it is found that this value is the Jacobian for the primary analysis times the four-dimensional identity matrix

$$
\underline{\underline{\underline{\overset{\ddot{\nabla}}{\mathrm{J}}}}}_{\nabla F_R} = \underline{\underline{\ddot{\mathrm{J}}}}_{F_R} \underline{\underline{\underline{e}}}, \tag{27}
$$

allowing the use with a conventional solver

$$
\underline{\underline{\ddot{\mathrm{J}}}}_{F_R}\, \Delta \underline{\underline{\nabla \ddot{q}}} = -\underline{\underline{\nabla F}}_{R}. \tag{28}
$$

4 Rigid multibody dynamics

Rigid multibody dynamics is the analysis method for problems of several bodies connected via kinematic constraints. Previous studies of sensitivity analysis with multibody analysis include [10, 2, 19]. Building upon these, the derivation is shown below categorized into the building blocks introduced above, the critical primal equations and their derivatives for the sensitivity analysis.

4.1 Governing equation

4.1.1 Primary analysis Analogously to sensitivity analysis with structural dynamics, we find the design sensitivities for multibody dynamics. The governing equations are shown here using the Lagrangian multiplier form leading to a system of differential–algebraic equations (DAE). In its most generic form, the index-3 DAE with holonomic constraints, is written as follows:

$$\underline{\underline{m}}\left(\underline{q},t\right)\,\underline{\ddot{q}}+\underline{\underline{\mathrm{J}}}_{\Phi}^{T}\,\underline{\lambda}=\underline{F}_{\text{ext}}\left(\underline{q},\underline{\dot{q}},t\right)+\underline{F}_{v}\left(\underline{q},\underline{\dot{q}},t\right), \tag{29}$$

$$\underline{\Phi}\left(\underline{q},t\right)=\underline{0}. \tag{30}$$

The multibody problem is implemented here using an index-1 differential algebraic equation to avoid numerical issues associated with the index-3 formulation. As such the constraint function is differentiated twice with respect to time, resulting in

$$\underline{\underline{m}}\,\underline{\ddot{q}}+\underline{\underline{\mathrm{J}}}_{\Phi}^{T}\,\underline{\lambda}=\underline{F}_{\text{ext}}+\underline{F}_{v}, \tag{31}$$

$$\underline{\underline{\mathrm{J}}}_{\Phi}\,\underline{\ddot{q}}=\underbrace{-\frac{\partial\underline{\dot{\Phi}}}{\partial t}-\underline{\underline{\mathrm{J}}}_{\dot{\Phi}}\,\underline{\dot{q}}}_{\underline{F}_{c}}, \tag{32}$$

where the velocity-based forces are put together. Rewriting this equation in matrix form gives

$$\underline{F}_{R}=\begin{bmatrix}\underline{\underline{m}} & \underline{\underline{\mathrm{J}}}_{\Phi}^{T}\\ \underline{\underline{\mathrm{J}}}_{\Phi} & \underline{\underline{0}}\end{bmatrix}\begin{bmatrix}\underline{\ddot{q}}\\ \underline{\lambda}\end{bmatrix}-\begin{bmatrix}\underline{F}_{\text{ext}}+\underline{F}_{v}\\ \underline{F}_{c}\end{bmatrix}=\underline{0}. \tag{33}$$

4.1.2 Sensitivity analysis The sensitivity analysis of the governing equation (33) is again carried out with direct differentiation, resulting in

$$\underline{\underline{\nabla F}}_{R}=\begin{bmatrix}\underline{\underline{m}} & \underline{\underline{\mathrm{J}}}_{\Phi}^{T}\\ \underline{\underline{\mathrm{J}}}_{\Phi} & \underline{\underline{0}}\end{bmatrix}\begin{bmatrix}\underline{\underline{\nabla\ddot{q}}}\\ \underline{\underline{\nabla\lambda}}\end{bmatrix}-\underline{\underline{F}}_{\text{pseudo}}=\underline{\underline{0}}, \tag{34}$$

where the pseudo load is a function of the total derivatives of the system parameters. It is important to note that this, as with structural dynamics, has the same form as the primal problem, enabling the use of the same solving procedure.

4.2 Time integration

The time integration and its sensitivity follows the pattern introduced above for structural dynamics. For rigid multibody dynamics, the updates are carried out from the accelerations $\underline{\ddot{q}}$ and is unnecessary for the Lagrangian multipliers from the kinematic constraint equations $\underline{\lambda}$. For the sensitivity analysis, the acceleration sensitivities $\underline{\underline{\nabla \ddot{q}}}$ are used to update velocity and position sensitivities, as with structural dynamics, see § 3.2.

4.3 Nonlinear solver

4.3.1 Primary analysis The nonlinear solver is applied to the sensitivity analysis in multibody dynamics analogously to structural dynamics. The primary variables, i.e. those for which we are solving for, are acceleration and the Lagrangian multiplier of the kinematic constraints $\left[\,\underline{\ddot{q}}\ \underline{\lambda}\,\right]^T$, giving

$$\left[\overset{\ddot{}}{\underline{\underline{\mathrm{J}}}}_{F_R}\ \overset{\lambda}{\underline{\underline{\mathrm{J}}}}_{F_R}\right]\begin{bmatrix}\Delta\underline{\ddot{q}}\\ \Delta\underline{\lambda}\end{bmatrix} + \underline{F}_R = \underline{0}, \tag{35}$$

where

$$\overset{\lambda}{\underline{\underline{\mathrm{J}}}}_{F_R} = \frac{\partial \underline{F}_R}{\partial \underline{\lambda}}. \tag{36}$$

4.3.2 Sensitivity analysis For the sensitivity analysis, the Newton–Raphson step is

$$\left[\overset{\ddot{\nabla}}{\underline{\underline{\underline{\mathrm{J}}}}}_{\nabla F_R}\ \overset{\nabla\lambda}{\underline{\underline{\underline{\mathrm{J}}}}}_{\nabla F_R}\right]\begin{bmatrix}\Delta\underline{\underline{\nabla\ddot{q}}}\\ \Delta\underline{\underline{\nabla\lambda}}\end{bmatrix} + \underline{\underline{\nabla F}}_R = \underline{0}, \tag{37}$$

where

$$\overset{\nabla\lambda}{\underline{\underline{\underline{\mathrm{J}}}}}_{\nabla F_R} = \frac{\partial \underline{\underline{\nabla F}}_R}{\partial \underline{\underline{\nabla\lambda}}}. \tag{38}$$

As with structural dynamics, the expansion of the partial derivatives results in the Jacobian for the primary analysis times the four-dimensional identity matrix,

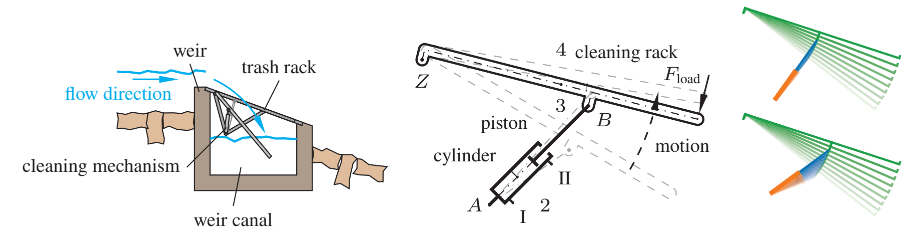
$$\overset{\ddot{\nabla}}{\underline{\underline{\underline{\mathrm{J}}}}}_{\nabla F_R} = \overset{\ddot{}}{\underline{\underline{\mathrm{J}}}}_{F_R}\underline{\underline{\underline{e}}}, \tag{39}$$

$$\overset{\nabla\lambda}{\underline{\underline{\underline{\mathrm{J}}}}}_{\nabla F_R} = \overset{\lambda}{\underline{\underline{\mathrm{J}}}}_{F_R}\underline{\underline{\underline{e}}}. \tag{40}$$

With this, the same solving routine can be used and the necessary Jacobians have already been calculated in the primal step.

5 Numerical example – Optimal design of a hydropower intake rack cleaning mechanism

Multibody dynamics is applied to the Tyrolean weir cleaning mechanism shown in fig. 4. A Tyrolean weir is a water intake system for hydroelectric power plants developed for rivers with steep slope and high sediment transport. Water flows over a weir into a canal, where it is collected and forwarded to the pipeline and the turbine. A trash rack prevents particles, e.g. stones and branches, from entering at the intake. The cleaning mechanism is installed to clear the rack of particles that block water from the intake. The mechanism consists of a hydraulically driven cleaning rack that forces particles from the trash rack and these are then washed away by the water flow.



(a) Cross section of Tyrolean weir (b) Multibody system (c) Optimal designs

Fig. 4: Tyrolean weir cleaning mechanism

Design optimization of a Tyrolean weir cleaning mechanism is shown in [8] where the mechanism is modeled as planar multibody system and the sensitivities are computed with a reduced time integration method. Here, the mechanism is modeled in three-dimensional space and the sensitivity analysis is performed with the direct differentiation method shown in § 4. The Newmark-β method is used for the time integration and Newton–Raphson iterations are performed for both the primal and sensitivity analyses.

The mechanism consists of three moving bodies: the cylinder, the piston rod and the cleaning rack. These bodies are connected by three revolute joints and one prismatic joint. For the design optimization, three geometric and one hydraulic design variables are used to define the mechanism. The objective function is a multi-objective function with the minimization of the maximum force inside the joints that is approximated by the Kreiselmeier–Steinhauser function [11] and the pump flow rate multiplied by a weighting factor. Lower forces cause lower strains and stresses inside the components and enables to reduce the weight. Therefore, the energy consumption and the costs of the mechanism can be reduced. The optimization is limited by three constraint functions. The first constraint function defines an opening angle of at least $20\,^\circ$ between the fully open cleaning rack and the fixed trash rack to allow the water to enter easily into the plant. The second constraint function defines a maximum cleaning time of $8\,\mathrm{s}$ in order to keep the energy losses caused by the cleaning process low. The third constraint function limits the stroke of the hydraulic cylinder to a maximum value of $500\,\mathrm{mm}$, which was given by a limited budget for the hydraulic cylinder. The mathematical formulation of the optimization formulation is given by

$$\begin{aligned} &\min_{\mathrm{x}\epsilon\mathcal{X}} \mathcal{F}_{\text{KS}}\left(\underline{F}_{\text{res}}\right) + 0.01\,\dot{V}_S \\ \text{such that}\quad & \underline{\mathrm{g}}\left(\underline{\mathrm{x}}\right) = \begin{cases} \frac{-20\,^\circ}{\theta^{(4)}(t_0)} - 1 \\ \frac{r_{hc}(t_F)}{500\,\text{mm}} - 1 \\ \frac{t_F}{8\,\text{s}} - 1 \end{cases} \\ \text{where}\quad & \underline{\mathrm{x}} = \left[\theta_{\overline{\text{AZ}}}\ \ell_{\overline{\text{AZ}}}\ \ell_{\overline{\text{BZ}}}\ \dot{V}_S\right]^T \\ \text{governed by}\quad & \underline{\mathrm{F}}_{\text{R}} = \begin{bmatrix} \underline{\underline{m}} & \underline{\underline{\mathrm{J}}}_\Phi^T & \underline{\underline{0}} \\ \underline{\underline{\mathrm{J}}}_\Phi & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{e}} \end{bmatrix} \begin{bmatrix} \underline{\ddot{q}} \\ \underline{\lambda} \\ \underline{\dot{p}} \end{bmatrix} - \begin{bmatrix} \underline{F}_{\text{ext}} + \underline{F}_v \\ \underline{F}_c \\ \underline{h} \end{bmatrix} = \underline{0}. \end{aligned} \tag{41}$$

The semi-analytic sensitivity analysis was used in the design optimization. Tab. 1 and fig. (5) compare the introduced semi-analytic method and the numerical sensitivity method with forward differences using different values for the perturbation $\varDelta$x. The sensitivities of the objective as well as of constraints 1 and 2 show convergence of the numerical sensitivities to the semi-analytical value with ever smaller perturbations. Conversely, this is not the case with the sensitivity of the third constraint (time of operation). Therefore, no perturbation value gives satisfactory results. Further, the direct sensitivity method drastically reduced the computation effort from $n_x + 1$ system evaluations (ca. 25 min per evaluation) to two system evaluations. Thus the theoretical speedup was achieved.

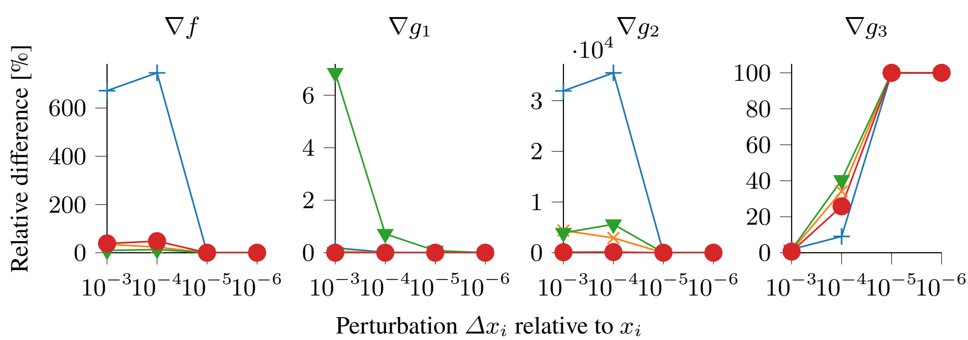


Fig. 5: Relative difference of numerical sensitivities with respect to semi-analytical sensitivities

Multiple optimization runs with the second-order algorithm NLPQLP [4, 15] from different start designs were performed. The optimizations converged to two different optima, which turned out to be equal from a qualitative point of view. The found optima are shown in fig. 4c. They can be considered as a mirrored design were the values of the lengths $\ell_{\overline{\text{AZ}}}$ and $\ell_{\overline{\text{BZ}}}$ are interchanged, leading to the same maximum joint force without violating the constraint functions.

Table 1: Comparison of semi-analytical and numerical sensitivity analysis calculated at $\underline{x} = [-1.22173,\ 1698.25,\ 1450.59,\ 493913]^T$

	semi-analytical	numerical			
Δx	–	$\underline{x} \cdot 10^{-3}$	$\underline{x} \cdot 10^{-4}$	$\underline{x} \cdot 10^{-5}$	$\underline{x} \cdot 10^{-6}$
t_{calc} [hh:mm:ss][1]	00 : 50 : 01	02 : 04 : 10	02 : 05 : 14	02 : 04 : 19	02 : 07 : 53
$\underline{\nabla f}$	$\begin{bmatrix} 1259.48 \\ -9.68942 \\ -19.7972 \\ 0.0161638 \end{bmatrix}$	$\begin{bmatrix} -7204.76 \\ -6.28831 \\ -17.9277 \\ 0.00995745 \end{bmatrix}$	$\begin{bmatrix} -8135.57 \\ -7.42888 \\ -17.161 \\ 0.00841248 \end{bmatrix}$	$\begin{bmatrix} 1267.79 \\ -9.68495 \\ -19.8033 \\ 0.0161657 \end{bmatrix}$	$\begin{bmatrix} 1265.29 \\ -9.68196 \\ -19.801 \\ 0.0161629 \end{bmatrix}$
$\underline{\nabla g_1}$	$\begin{bmatrix} 1.50778 \\ -0.000539986 \\ -1.255 \cdot 10^{-5} \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 1.50501 \\ -0.000540256 \\ -1.341 \cdot 10^{-5} \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 1.50751 \\ -0.000540013 \\ -1.264 \cdot 10^{-5} \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 1.50776 \\ -0.000539989 \\ -1.256 \cdot 10^{-5} \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 1.50778 \\ -0.000539986 \\ -1.255 \cdot 10^{-05} \\ 0.0 \end{bmatrix}$
$\underline{\nabla g_2}$	$\begin{bmatrix} 0.00861134 \\ 2.509 \cdot 10^{-5} \\ -1.53 \cdot 10^{-5} \\ 2.003 \cdot 10^{-6} \end{bmatrix}$	$\begin{bmatrix} -2.74135 \\ 0.00112417 \\ 0.000585078 \\ -1.415 \cdot 10^{-8} \end{bmatrix}$	$\begin{bmatrix} -3.04703 \\ 0.000757833 \\ 0.000842538 \\ -5.163 \cdot 10^{-7} \end{bmatrix}$	$\begin{bmatrix} 0.00861104 \\ 2.509 \cdot 10^{-5} \\ -1.531 \cdot 10^{-5} \\ 2.003 \cdot 10^{-06} \end{bmatrix}$	$\begin{bmatrix} 0.00861358 \\ 2.509 \cdot 10^{-05} \\ -1.531 \cdot 10^{-05} \\ 2.003 \cdot 10^{-06} \end{bmatrix}$
$\underline{\nabla g_3}$	$\begin{bmatrix} -2.81437 \\ 0.00111765 \\ 0.000613756 \\ -2.012 \cdot 10^{-6} \end{bmatrix}$	$\begin{bmatrix} -2.76248 \\ 0.00110408 \\ 0.000603204 \\ -2.025 \cdot 10^{-6} \end{bmatrix}$	$\begin{bmatrix} -3.06942 \\ 0.00073605 \\ 0.000861719 \\ -2.531 \cdot 10^{-6} \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$

[1]PC with Intel® Core™ i7-8700 CPU @ 3.2 GHz × 12, 31.3 GiB

6 Conclusion

Cast within the Virtuous Circle of Lightweight Engineering Design, a unified approach, including a simplified nomenclature, was introduced for the sensitivity analysis of structural dynamics and rigid multibody dynamics. The analysis has been divided in three elementary building blocks and the respective equations are derived and implemented. These are then integrated into a gradient-based design optimization framework, and the optimization results are shown. The advantages in speedup and accuracy are all shown.

Although successful in the example shown, this method has the limitations common with gradient-based optimization. Nonsmooth, discontinuous and bifurcated optimization functions may cause convergence problems, especially when in the objective function. This can partially be avoided by using the lightweight engineering design formulation where the mass is the objective function, which a smooth and continuous function. Ill-conditioned optimization problems can be successfully handled with non-gradient algorithms (see e.g. [6, 5, 12, 13]) and approximation methods (see e.g. [3, 18, 7]).

The logical extension of this work is to derive and implement the equations for the adjoint variable method. While direct differentiation has advantages in its implementation and generality, the adjoint variable method has the edge of computational effort when the number of design variables is larger than the number of optimization functions (number of objectives plus the number of constraints). This becomes more apparent with great numbers of design variables as the case with shape and topology optimization.

Acknowledgments

This work is supported by the project CRC 2017 TN2091 doloMULTI *Design of Lightweight Optimized structures and systems under MULTIdisciplinary considerations through integration of MULTIbody dynamics in a MULTIphysics framework* funded by the Free University of Bozen-Bolzano.

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