Chapter 6
Multi-index and Multi-variable Mittag-Leffler Functions

6.1 The Four-Parametric Mittag-Leffler Function: The Luchko–Kilbas–Kiryakova Approach

6.1.1 Definition and Special Cases

Consider the function defined for $\alpha_1,\ \alpha_2 \in \mathbb{R}$ $(\alpha_1^2 + \alpha_2^2 \neq 0)$ and $\beta_1, \beta_2 \in \mathbb{C}$ by the series

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1)\Gamma(\alpha_2 k + \beta_2)} \quad (z \in \mathbb{C}). \tag{6.1.1}$$

Such a function with positive $\alpha_1 > 0$, $\alpha_2 > 0$ and real $\beta_1, \beta_2 \in \mathbb{R}$ was introduced by Dzherbashian [Dzh60]. When $\alpha_1 = \alpha,\ \beta_1 = \beta$ and $\alpha_2 = 0$, $\beta_2 = 1$, this function coincides with the Mittag-Leffler function (4.1.1):

$$E_{\alpha,\beta;0,1}(z) = E_{\alpha,\beta}(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z \in \mathbb{C}). \tag{6.1.2}$$

Therefore (6.1.1) is sometimes called the *generalized Mittag-Leffler function* or *four-parametric Mittag-Leffler function*.

Certain special functions of Bessel type are expressed in terms of $E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$:

The Bessel function of the first kind (see e.g., [ErdBat-2, n. 7.2.1-2], [NIST, p. 217, 219])

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu} E_{1,\nu+1;1,1}\left(-\frac{z^2}{4}\right). \tag{6.1.3}$$

The Struve function (see e.g., [ErdBat-2, n. 7.5.4], [NIST, p. 288])

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$$\mathbf{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} E_{1,\nu+3/2;1,3/2}\left(-\frac{z^2}{4}\right). \tag{6.1.4}$$

The Lommel function (see e.g., [ErdBat-2, n. 7.5.5])

$$S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{4}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)E_{1,\frac{\mu-\nu+1}{2};1,\frac{\mu+\nu+1}{2}}\left(-\frac{z^2}{4}\right). \tag{6.1.5}$$

The Bessel–Maitland function (see e.g., [Kir94, App. E, ii])

$$J_\nu^\mu(z) = E_{\mu,\nu+1;1,1}(-z). \tag{6.1.6}$$

The generalized Bessel–Maitland function (see e.g., [Kir94, App. E, ii])

$$J_{\nu,\lambda}^\mu(z) = \left(\frac{z}{2}\right)^{\nu+2\lambda} E_{\mu,\lambda+\nu+1;1,\lambda+1}(-z). \tag{6.1.7}$$

6.1.2 Basic Properties

First of all we prove that (6.1.1) is an entire function if $\alpha_1 + \alpha_2 > 0$.

Theorem 6.1 *Let $\alpha_1,\ \alpha_2 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{C}$ be such that $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\alpha_1 + \alpha_2 > 0$. Then $E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ is an entire function of $z \in \mathbb{C}$ of order*

$$\rho = \frac{1}{\alpha_1 + \alpha_2} \tag{6.1.8}$$

and type

$$\sigma = \left(\frac{\alpha_1+\alpha_2}{|\alpha_1|}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_1+\alpha_2}{|\alpha_2|}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}. \tag{6.1.9}$$

◃ Rewrite (6.1.1) as the power series

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \sum_{k=0}^{\infty} c_k z^k,\ \ c_k = \frac{1}{\Gamma(\alpha_1 k + \beta_1)\Gamma(\alpha_2 k + \beta_2)}. \tag{6.1.10}$$

Using Stirling's formula for the Gamma function we obtain

$$\frac{|c_k|}{|c_{k+1}|} \sim |\alpha_1|^{\alpha_1}|\alpha_2|^{\alpha_2}k^{\alpha_1+\alpha_2} \to +\infty\ (k \to \infty).$$

Thus, $E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ is an entire function of z when $\alpha_1 + \alpha_2 > 0$.

We use [Appendix B, formulas (B.5) and (B.6)] to evaluate the order ρ and the type σ of (6.3.1). For this we apply the asymptotic formula for the logarithm of the

Gamma function $\Gamma(z)$ at infinity [ErdBat-1, 1.18(1)]:

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2z) + O\left(\frac{1}{z}\right) \quad (|z| \to \infty, \quad |\arg z| < \pi). \tag{6.1.11}$$

Applying this formula and taking (6.1.10) into account, we deduce the asymptotic estimate

$$\log\left(\frac{1}{c_k}\right) \sim k\log(k)(\alpha_1 + \alpha_2) \quad (k \to \infty)$$

from which, in accordance with [Appendix B, (B.5)], we obtain (6.1.8).

Further, according to [Appendix A, (A.24)], we have

$$\begin{aligned} &\Gamma(\alpha_j k + \beta_j) \\ &= (2\pi)^{1/2}\left(\alpha_j k + \beta_j\right)^{\alpha_j k + \beta_j - \frac{1}{2}} \mathrm{e}^{-(\alpha_j k + \beta_j)}\left[1 + O\left(\frac{1}{k}\right)\right] \quad (k \to \infty) \end{aligned} \tag{6.1.12}$$

for $j = 1, 2$, and we obtain the asymptotic estimate

$$\Gamma(\alpha_1 k + \beta_1)\Gamma(\alpha_2 k + \beta_2) \sim 2\pi \prod_{j=1}^{2}(\alpha_j k)^{\alpha_j k + \beta_j - \frac{1}{2}} \mathrm{e}^{-\alpha_j k} \quad (k \to \infty). \tag{6.1.13}$$

From (6.1.10) and (6.1.13) we have

$$\begin{aligned} &\limsup_{k\to\infty}\left(k^{1/\rho}|c_k|^{1/k}\right) = \limsup_{k\to\infty} k^{1/\rho}\prod_{j=1}^{2}\left[(|\alpha_j|k)^{-\alpha_j}\mathrm{e}^{\alpha_j}\right] \\ &= \mathrm{e}^{\alpha_1+\alpha_2}\prod_{j=1}^{2}|\alpha_j|^{-\alpha_j} = \mathrm{e}^{1/\rho}\prod_{j=1}^{2}|\alpha_j|^{-\alpha_j}. \end{aligned}$$

Substituting this relation into [Appendix B, (B.6)] we obtain

$$\begin{aligned} \sigma &= \frac{1}{\rho}\left(\prod_{j=1}^{2}|\alpha_j|^{-\alpha_j}\right)^{\rho} = (\alpha_1 + \alpha_2)\left(|\alpha_1|^{-\alpha_1}|\alpha_2|^{-\alpha_2}\right)^{\frac{1}{\alpha_1+\alpha_2}} \\ &= \left(\frac{\alpha_1+\alpha_2}{|\alpha_1|}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1+\alpha_2}{|\alpha_2|}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}, \end{aligned}$$

which proves (6.1.9). ▹

Remark 6.2 For $\alpha_1 > 0$ and $\alpha_2 > 0$, relations (6.1.8) and (6.1.9) were proved by Dzherbashian [Dzh60].

6.1.3 Integral Representations and Asymptotics

The four-parametric Mittag-Leffler function has the Mellin–Barnes integral representation

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta_1-\alpha_1 s)\Gamma(\beta_2-\alpha_2 s)}(-z)^{-s}\mathrm{d}s, \tag{6.1.14}$$

where $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop, i.e. the contour which is situated in a horizontal strip, starting at $-\infty + i\varphi_1$ and ending at $-\infty + i\varphi_2$, with $-\infty < \varphi_1 < 0 < \varphi_2 < +\infty$. This contour separates poles of the Gamma functions $\Gamma(s)$ and $\Gamma(1-s)$.

By using (6.1.14) the function $E_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ can be extended to non-real values of the parameters. If the parameters $\alpha_1, \beta_1; \alpha_2, \beta_2$ are such that Re $(\alpha_1+\alpha_2) > 0$, then the integral (6.1.14) converges for all $z \neq 0$. This is a consequence of the following asymptotic formulas for the function $H(s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta_1-\alpha_1 s)\Gamma(\beta_2-\alpha_2 s)}$ in the integrand of (6.1.14), where $s = t + i\sigma$, $(t \to -\infty)$, and the properties of the Mellin–Barnes integral:

– for $\mathrm{Re}\,\alpha_1 > 0,\ \mathrm{Re}\,\alpha_2 > 0$

$$|H(s)| \sim M_1 \left(\frac{|t|}{\mathrm{e}}\right)^{\mathrm{Re}(\alpha_1+\alpha_2)t} \frac{[\mathrm{Re}(\alpha_1)^{\mathrm{Re}(\alpha_1)}\mathrm{Re}(\alpha_2)^{\mathrm{Re}(\alpha_2)}]^t}{|t|^{\sum_{j=1}^{2}[\mathrm{Re}(\beta_j)+\sigma\mathrm{Im}(\alpha_i)]^{-1}}}; \tag{6.1.15}$$

– for $\mathrm{Re}\,\alpha_1 < 0,\ \mathrm{Re}\,\alpha_2 > 0$

$$|H(s)| \sim M_2 \left(\frac{|t|}{\mathrm{e}}\right)^{\mathrm{Re}(\alpha_1+\alpha_2)t} \frac{[|\mathrm{Re}(\alpha_1)|^{\mathrm{Re}(\alpha_1)}\mathrm{Re}(\alpha_2)^{\mathrm{Re}(\alpha_2)}]^t}{|t|^{\sum_{i=1}^{2}[\mathrm{Re}(\beta_i)+\sigma\mathrm{Im}(\alpha_i)]^{-1}}}\mathrm{e}^{-\pi\mathrm{Im}(\alpha_1)t}; \tag{6.1.16}$$

– for $\mathrm{Re}\,\alpha_1 > 0,\ \mathrm{Re}\,\alpha_2 < 0$

$$|H(s)| \sim M_3 \left(\frac{|t|}{\mathrm{e}}\right)^{\mathrm{Re}(\alpha_1+\alpha_2)t} \frac{[\mathrm{Re}(\alpha_1)^{\mathrm{Re}(\alpha_1)}|\mathrm{Re}(\alpha_2)|^{\mathrm{Re}(\alpha_2)}]^t}{|t|^{\sum_{i=1}^{2}[\mathrm{Re}(\beta_i)+\sigma\mathrm{Im}(\alpha_i)]^{-1}}}\mathrm{e}^{-\pi\mathrm{Im}(\alpha_2)t}. \tag{6.1.17}$$

We do not present here exact asymptotic formulas for $E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ as $z \to \infty$. They can be considered as formulas for a special case of the generalized Wright function and H-function (see Sect. 6.1.5 below).

From the series representation of the four-parametric Mittag-Leffler function we derive a simple asymptotics at zero, valid in the case Re $\{\alpha_1+\alpha_2\} > 0$ for all $N \in \mathbb{N}$:

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \sum_{k=0}^{N} \frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\Gamma(\alpha_2 k+\beta_2)} + O\left(|z|^{N+1}\right),\ z \to 0. \tag{6.1.18}$$

The following integral representation of the four-parametric Mittag-Leffler function (see [RogKor10]) shows its tight connection to the generalized Wright function (see Chap. 7).

Let $0 < \alpha_j < 2, \beta_j \in \mathbb{C}, j = 1, 2$. Then the following representation of the four-parametric generalized Mittag-Leffler function $E_{\alpha_1,\beta_1;\,\alpha_2,\beta_2}(z)$ holds ([RogKor10]).

$$E_{\alpha_1,\beta_1;\,\alpha_2,\beta_2}(z) = \tag{6.1.19}$$

$$\begin{cases} I_0(z), & z \in G^{(-)}(\epsilon, \mu_2), \\ I_0(z) + \dfrac{z^{\frac{-\beta_2+1}{\alpha_2}}}{2\pi i \alpha_2}\phi\left(\dfrac{\alpha_1}{\alpha_2}, \dfrac{\beta_1\alpha_2 - \beta_2\alpha_1 + 1}{\alpha_2}; z^{\frac{1}{\alpha_2}}\right), & z \in G^{(\mp)}(\epsilon, \mu_1, \mu_2), \\ I_0(z) + \dfrac{z^{\frac{-\beta_2+1}{\alpha_2}}}{2\pi i \alpha_2}\phi\left(\dfrac{\alpha_1}{\alpha_2}, \dfrac{\beta_1\alpha_2 - \beta_2\alpha_1 + 1}{\alpha_2}; z^{\frac{1}{\alpha_2}}\right) & \\ \quad + \dfrac{z^{\frac{-\beta_1+1}{\alpha_1}}}{2\pi i \alpha_1}\phi\left(\dfrac{\alpha_2}{\alpha_1}, \dfrac{\beta_2\alpha_1 - \beta_1\alpha_2 + 1}{\alpha_1}; z^{\frac{1}{\alpha_1}}\right), & z \in G^{(+)}(\epsilon, \mu_1), \end{cases} \tag{6.1.20}$$

with

$$I_0(z) = \frac{-1}{4\pi^2\alpha_1\alpha_2}\left\{\int_{\gamma(\epsilon;\,\mu_1)} e^{\zeta_1{}^{1/\alpha_1}}\zeta_1^{\frac{(-\beta_1+1)}{\alpha_1}}\,d\zeta_1 \int_{\gamma(\epsilon;\,\mu_2)} \frac{e^{\zeta_2{}^{1/\alpha_2}}\zeta_2^{\frac{(-\beta_2+1)}{\alpha_2}}\,d\zeta_2}{\zeta_1\zeta_2 - z}\right\}, \tag{6.1.21}$$

$$\phi(\alpha, \beta; z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}, \tag{6.1.22}$$

where $\phi(\alpha, \beta; z)$ is the classical Wright function (see Appendix F), $\mu_j \in \left(\frac{\pi\alpha_j}{2}, \min\{\pi\alpha_j, \pi\}\right)$, $0 < \mu_1 < \mu_2 < 2$, and $\epsilon > 0$ is an arbitrary positive number.

Here $\gamma(\epsilon; \theta)$ $(\epsilon > 0,\ 0 < \theta \le \pi)$ is a contour with non-decreasing $\arg\zeta$ consisting of the following parts:

(1) the ray $\arg\zeta = -\theta,\ |\zeta| \ge \epsilon$;
(2) the arc $-\theta \le \arg\zeta \le \theta$ of the circle $|\zeta| = \epsilon$;
(3) the ray $\arg\zeta = \theta,\ |\zeta| \ge \epsilon$.

In the case $0 < \theta < \pi$ the complex ζ-plane is divided by the contour $\gamma(\epsilon; \theta)$ into two unbounded parts: the domain $G^{(-)}(\epsilon; \theta)$ to the left of the contour and the domain $G^{(+)}(\epsilon; \theta)$ to the right. If $\theta = \pi$, the contour $\gamma(\epsilon; \theta)$ consists of the circle $|\zeta| = \epsilon$ and of the cut $-\infty < \zeta \le -\epsilon$. In this case the domain $G^{(-)}(\epsilon; \theta)$ becomes the circle $|\zeta| < \epsilon$ and the domain $G^{(+)}(\epsilon; \theta)$ becomes the domain $\{\zeta : |\arg\zeta| < \pi,\ |\zeta| > \epsilon\}$.

For two different values of $\theta_1, \theta_2, 0 < \theta_1 < \theta_2 < \pi$ the union of the two unbounded domains between the curves $\gamma(\epsilon; \theta_1)$ and $\gamma(\epsilon; \theta_2)$ is denoted by $G^{(\mp)}(\epsilon; \theta_1, \theta_2)$.

6.1.4 Extended Four-Parametric Mittag-Leffler Functions

Let the contour $\mathcal{L}$ in the Mellin–Barnes integral

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta_1 - \alpha_1 s)\Gamma(\beta_2 - \alpha_2 s)} (-z)^{-s} \mathrm{d}s, \tag{6.1.23}$$

now coincide with the right loop $\mathcal{L}_{+\infty}$, i.e. with a curve starting at $+\infty + i\varphi_1$ and ending at $+\infty + i\varphi_2$ $(-\infty < \varphi_1 < \varphi_2 < +\infty)$, leaving the poles of $\Gamma(s)$ at the left and the poles of $\Gamma(1-s)$ at the right. Then this integral exists for all $z \neq 0$ whenever $\mathrm{Re}\,\{\alpha_1 + \alpha_2\} < 0$.

Thus the integral (6.1.23) possesses an extension to another set of parameters. It defines a new function which is called the *extended generalized Mittag-Leffler function* and is denoted $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ (see [KilKor05], [KilKor06a]).

Using the same approach as before, i.e. calculating the integral (6.1.23) by the Residue Theorem, one can obtain the following Laurent series representation of $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$:

$$\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \sum_{k=0}^{\infty} \frac{d_k}{z^{k+1}}, \tag{6.1.24}$$

where

$$d_k = -\frac{1}{\Gamma(-\alpha_1(k+1) - \beta_1)\Gamma(-\alpha_2(k+1) - \beta_2)}.$$

In the case $\mathrm{Re}\,\{\alpha_1 + \alpha_2\} < 0$ the series (6.1.24) is convergent for all $z \in \mathbb{C}, z \neq 0$. The function $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ has an asymptotics at $z \to 0$ similar to that of the standard four-parametric Mittag-Leffler function $E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$, $\mathrm{Re}\,\{\alpha_1 + \alpha_2\} > 0$ at $z \to \infty$. The asymptotics of $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ at $z \to \infty$ can be displayed in the form

$$\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \sum_{k=0}^{N} \frac{d_k}{z^{k+1}} + O\left(\frac{1}{|z|^{N+1}}\right), \ z \to \infty. \tag{6.1.25}$$

6.1.5 Relations to the Wright Function and the H-Function

For short, let us use the common notation $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ for the usual four-parametric Mittag-Leffler function and for its extension in this subsection. For real values of the parameters $\alpha_1, \alpha_2 \in \mathbb{R}$ and complex values of $\beta_1, \beta_2 \in \mathbb{C}$ the four-parametric Mittag-Leffler function $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ can be represented in terms of the generalized Wright function and the H–function.

These representations follow immediately from the Mellin–Barnes integral representation of the function $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ and the properties of the corresponding integrals.

Let us present some formulas relating $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ to the generalized Wright function ${}_p\Psi_q$:

(1) If $\alpha_1 + \alpha_2 > 0$ and the contour of integration in (6.1.14) is chosen as $\mathcal{L} = \mathcal{L}_{-\infty}$, then

$$\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = {}_1\Psi_2\left[\begin{matrix}(1,1)\\(\beta_1,\alpha_1),(\beta_2,\alpha_2)\end{matrix}\middle| z\right]. \tag{6.1.26}$$

(2) If $\alpha_1 + \alpha_2 < 0$ and the contour of integration in (6.1.14) is chosen as $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = \frac{1}{z}\,{}_1\Psi_2\left[\begin{matrix}(1,1)\\(\beta_1-\alpha_1,-\alpha_1),(\beta_2-\alpha_2,-\alpha_2)\end{matrix}\middle| \frac{1}{z}\right]. \tag{6.1.27}$$

Analogously, one can obtain the following representation of $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ in terms of the H-function:

(1) If $\alpha_1 > 0$, $\alpha_2 > 0$ and the contour of integration in (6.1.14) is chosen as $\mathcal{L} = \mathcal{L}_{-\infty}$, then

$$\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = H^{1,1}_{1,3}\left[\begin{matrix}(0,1)\\(0,1),(1-\beta_1,\alpha_1),(1-\beta_2,\alpha_2)\end{matrix}\middle| z\right]. \tag{6.1.28}$$

(2) If $\alpha_1 > 0$, $\alpha_2 < 0$ and the contour of integration in (6.1.14) is chosen as $\mathcal{L} = \mathcal{L}_{-\infty}$ when $\alpha_1 + \alpha_2 > 0$ or $\mathcal{L} = \mathcal{L}_{+\infty}$ when $\alpha_1 + \alpha_2 < 0$, then

$$\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = H^{1,1}_{2,2}\left[\begin{matrix}(0,1),(\beta_2,-\alpha_2)\\(0,1),(1-\beta_1,\alpha_1)\end{matrix}\middle| x\right]. \tag{6.1.29}$$

(3) If $\alpha_1 < 0$, $\alpha_2 > 0$ and the contour of integration in (6.1.14) is chosen as $\mathcal{L} = \mathcal{L}_{-\infty}$ when $\alpha_1 + \alpha_2 > 0$ or $\mathcal{L} = \mathcal{L}_{+\infty}$ when $\alpha_1 + \alpha_2 < 0$, then

$$\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = H^{1,1}_{2,2}\left[\begin{matrix}(0,1),(\beta_1,-\alpha_1)\\(0,1),(1-\beta_2,\alpha_2)\end{matrix}\middle| x\right]. \tag{6.1.30}$$

(4) If $\alpha_1 < 0$, $\alpha_2 < 0$ and the contour of integration in (6.1.14) is chosen as $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) = H^{1,1}_{3,1}\left[\begin{matrix}(0,1),(\beta_1,-\alpha_1),(\beta_2,-\alpha_2)\\(0,1)\end{matrix}\middle| x\right]. \tag{6.1.31}$$

6.1.6 Integral Transforms of the Four-Parametric Mittag-Leffler Function

In order to present elements of the theory of integral transforms of the extended four-parametric Mittag-Leffler function we introduce a set of weighted Lebesgue spaces $\mathcal{L}_{\nu,r}(\mathbb{R}_+)$. These spaces are suitable for the above mentioned integral transforms since the latter are connected with the classical Mellin transform (see, e.g., [Mari83, p. 36–39]).

Let us denote by $\mathcal{L}_{\nu,r}(\mathbb{R}_+)$ $(1 \leq r \leq \infty, \nu \in \mathbb{R})$ the space of all Lebesgue measurable functions f such that $\|f\|_{\nu,r} < \infty$, where

$$\|f\|_{\nu,r} \equiv \left(\int_0^\infty |t^\nu f(t)|^r \frac{\mathrm{d}t}{t}\right)^{1/r} < \infty \ \ (1 \leq r < \infty);\ \ \|f\|_{\nu,\infty} \equiv \operatorname{ess\,sup}_{t>0} \|t^\nu f(t)\|. \tag{6.1.32}$$

In particular, for $\nu = 1/r$ the spaces $\mathcal{L}_{\nu,r}$ coincide with the classical spaces of r-summable functions: $\mathcal{L}_{1/r,r} = \mathcal{L}_r(\mathbb{R}_+)$ endowed with the norm

$$\|f\|_r = \left\{\int_0^\infty |f(t)|^r \,\mathrm{d}t\right\}^{1/r} < \infty \quad (1 \leq r < \infty).$$

For any function $f \in \mathcal{L}_{\nu,r}(\mathbb{R}_+)$ $(1 \leq r \leq 2)$ its Mellin transform $\mathcal{M}f$ is defined (see, e.g., [KilSai04, (3.2.5)]) by the equality

$$(\mathcal{M}f)(s) = \int_{-\infty}^{+\infty} f(\mathrm{e}^\tau)\,\mathrm{e}^{s\tau}\,\mathrm{d}\tau \quad (s = \nu + it;\ \nu, t \in \mathbb{R}). \tag{6.1.33}$$

If $f \in \mathcal{L}_{\nu,r} \bigcap \mathcal{L}_{\nu,1}$, then the transform (6.1.33) can be written in the form of the classical Mellin transform with $\operatorname{Re} s = \nu$ (see Appendix C):

$$(\mathcal{M}f)(s) = \int_0^{+\infty} f(t)\,t^{s-1}\,\mathrm{d}t\,. \tag{6.1.34}$$

An inverse Mellin transform in this case can be determined by the formula

$$f(t) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} (\mathcal{M}f)(s)t^{-s}\mathrm{d}s \quad (\nu = \operatorname{Re} s).$$

We have for the Mellin transform of the generalized hypergeometric Wright function

$$\mathcal{M}\left[{}_p\Psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\Big| t\right]\right](s) = \frac{\Gamma(s)\prod\limits_{i=1}^{p}\Gamma(a_i-\alpha_i s)}{\prod\limits_{j=1}^{q}\Gamma(b_j-\beta_j s)}, \tag{6.1.35}$$

$$\left(\alpha_i > 0,\ \beta_j > 0;\ i = 1,\ldots,p;\ j = 1,\ldots,q;\ 0 < \mathrm{Re}\, s < \min_{1\le i\le p}\left[\frac{\mathrm{Re}\,(a_i)}{\alpha_i}\right]\right),$$

and, in particular, for the Mellin transform of the classical Wright function

$$\mathcal{M}\left[\phi(\alpha,\beta;t)\right](s) = \frac{\Gamma(s)}{\Gamma(\beta-\alpha s)} \quad (\mathrm{Re}\, s > 0). \tag{6.1.36}$$

The Mellin transform of the H-function under certain assumptions on its parameters coincides with the function $\mathcal{H}^{m,n}_{p,q}(s)$ in the Mellin–Barnes integral representation of the H-function (see [PrBrMa-V3, 8.4.51.11], [KilSai04, Theorem 2.2]).

Let us introduce the following parameters characterizing the behavior of the H-function (see Appendix F)

$$\mathcal{H}^{m,n}_{p,q}(z) = \mathcal{H}^{m,n}_{p,q}\left[z\,\Big|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

$$\begin{aligned}
a^* &= \sum_{i=1}^{n}\alpha_i - \sum_{i=n+1}^{p}\alpha_i + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j,\\
\mu &= \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i + \frac{p-q}{2},\ \Delta = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i,\\
\alpha &= -\min_{1\le j\le m}\left[\frac{\mathrm{Re}\, b_j}{\beta_j}\right], \beta = \min_{1\le i\le n}\left[\frac{1-\mathrm{Re}\, a_i}{\alpha_i}\right].
\end{aligned} \tag{6.1.37}$$

Let $a^* \ge 0$, $s \in \mathbb{C}$ be such that

$$\alpha < \mathrm{Re}\, s < \beta \tag{6.1.38}$$

and for $a^* = 0$ assume the following additional inequality holds:

$$\Delta \mathrm{Re}\, s + \mathrm{Re}\,\mu < -1. \tag{6.1.39}$$

Then the Mellin transform of the H-function exists and satisfies the relation

$$\left(\mathcal{M}\, H^{m,n}_{p,q}\left[z\,\Big|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(s) = \mathcal{H}^{m,n}_{p,q}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\Big| s\right]. \tag{6.1.40}$$

Since the four-parametric Mittag-Leffler function is related to the generalized Wright function and to the H-function (see Sect. 6.1.5), then one can use (6.1.35) or (6.1.40) to define the Mellin transform of the function $E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ and of its extension $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$.

6.1.7 Integral Transforms with the Four-Parametric Mittag-Leffler Function in the Kernel

Integral transforms with the four-parametric Mittag-Leffler function in the kernel can be considered as a special case of the more general **H**-transform. Let us recall a few facts from the theory of the **H**-transform following [KilSai04]). The **H**-transform is introduced as a Mellin-type convolution with the H-function in the kernel:

$$(\mathbf{H}f)(x) = \int_0^\infty H^{m,n}_{p,q}\left[xt\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]f(t)\,\mathrm{d}t \qquad (x>0). \tag{6.1.41}$$

Let us recall some results on the **H**-transform in $\mathcal{L}_{\nu,2}$-type spaces following [KilSai04, Chap. 3] (elements of the so-called $\mathcal{L}_{\nu,2}$-theory of **H**-transforms). Here we use the notation (6.1.37) for the parameters a^*, μ, Δ, α, β. We also introduce a so-called exceptional set $\mathcal{E}_{\mathcal{H}}$ for the function $\mathcal{H}(s)$:

$$\mathcal{E}_{\mathcal{H}} = \{\nu\in\mathbb{R} : \alpha < 1-\nu < \beta \text{ and } \mathcal{H}(s) \text{ has zeros on } \mathrm{Re}\, s = 1-\nu\}\,. \tag{6.1.42}$$

Let

(i) $\alpha < 1-\nu < \beta$ and suppose one of the following conditions holds:
(ii) $a^* > 0$, or
(iii) $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}\,\mu \le 0$.

Then the following statements are satisfied:

(a) There exists an injective transform $\mathbf{H}^* \in [\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}]$ such that for any $f \in \mathcal{L}_{\nu,2}$ the Mellin transform satisfies the relation

$$(\mathcal{M}\mathbf{H}^* f)(s) = \mathcal{H}^{m,n}_{p,q}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right|s\left.\right](\mathcal{M}f)(1-s) \quad (\mathrm{Re}\, s = 1-\nu). \tag{6.1.43}$$

If $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}\,\mu = 0$, $\nu \notin \mathcal{E}_{\mathcal{H}}$, then $\mathbf{H}^*$ is bijective from $\mathcal{L}_{\nu,2}$ onto $\mathcal{L}_{1-\nu,2}$.

(b) For any $f, g \in \mathcal{L}_{\nu,2}$ the following equality holds:

$$\int_0^\infty f(x)(\mathbf{H}^* g)(x)\,\mathrm{d}x = \int_0^\infty (\mathbf{H}^* f)(x) g(x)\,\mathrm{d}x\,. \tag{6.1.44}$$

(c) Let $f \in \mathcal{L}_{\nu,2}$, $\lambda \in \mathbb{C}$ and $h > 0$. If $\operatorname{Re}\lambda > (1-\nu)h - 1$, then for almost all $x > 0$ the transform $\mathbf{H}^*$ can be represented in the form:

$$(\mathbf{H}^* f)(x) = hx^{1-(\lambda+1)/h}\frac{\mathrm{d}}{\mathrm{d}x}x^{(\lambda+1)/h} \times \int_0^\infty H^{m,n+1}_{p+1,q+1}\left[xt\,\middle|\,\begin{matrix}(-\lambda,h),(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},(-\lambda-1,h)\end{matrix}\right]f(t)\,\mathrm{d}t\,. \tag{6.1.45}$$

If $\operatorname{Re}\lambda < (1-\nu)h - 1$, then

$$(\mathbf{H}^* f)(x) = -hx^{1-(\lambda+1)/h}\frac{\mathrm{d}}{\mathrm{d}x}x^{(\lambda+1)/h} \times \int_0^\infty H^{m+1,n}_{p+1,q+1}\left[xt\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p},(-\lambda,h)\\(-\lambda-1,h),(b_j,\beta_j)_{1,q}\end{matrix}\right]f(t)\,\mathrm{d}t\,. \tag{6.1.46}$$

(d) The $\mathbf{H}^*$-transform does not depend on ν in the following sense: if two values of the parameter, say ν and $\widetilde{\nu}$, satisfy condition (i) and one of the conditions (ii) or (iii), and if the transforms $\mathbf{H}^*$ and $\widetilde{\mathbf{H}}^*$ are defined by the relation (6.1.43) in $\mathcal{L}_{\nu,2}$ and $\mathcal{L}_{\widetilde{\nu},2}$, respectively, then $\mathbf{H}^* f = \widetilde{\mathbf{H}}^* f$ for any $f \in \mathcal{L}_{\nu,2} \bigcap \mathcal{L}_{\widetilde{\nu},2}$.

(e) If either $a^* > 0$ or $a^* = 0$, and $\Delta(1-\nu) + \operatorname{Re}\mu < 0$, then for any $f \in \mathcal{L}_{\nu,2}$ we have $\mathbf{H}^* f = \mathbf{H} f$, i.e. $\mathbf{H}^*$ is defined by the equality (6.1.41).

An extended $\mathcal{L}_{\nu,r}$-theory (for any $1 \le r \le +\infty$) of the $\mathbf{H}$-transform is presented in [KilSai04].

The integral transform with the four-parametric Mittag-Leffler function in the kernel is defined for $\alpha_1, \alpha_2 \in \mathbb{R}$, $\beta_1, \beta_2 \in \mathbb{C}$ by the formula:

$$\left(\mathbf{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2} f\right)(x) = \int_0^\infty \mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(-xt)f(t)\mathrm{d}t \quad (x > 0), \tag{6.1.47}$$

where for $\alpha_1 + \alpha_2 > 0$ the kernel $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2} = E_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ (i.e. it is the four-parametric generalized Mittag-Leffler function defined by (6.1.1)), and for $\alpha_1 + \alpha_2 < 0$ the kernel $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ is the extended four-parametric generalized Mittag-Leffler function defined by (6.1.23).

The properties of this transform follow from its representation as a special case of the $\mathbf{H}$-transform.

(1) If $\alpha_1 > 0,\ \alpha_2 > 0$, then

$$\left(\mathbf{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2} f\right)(x) = \int_0^\infty H^{1,1}_{1,3}\left[xt\,\middle|\,\begin{matrix}(0,1)\\(0,1),(1-\beta_1,\alpha_1),(1-\beta_2,\alpha_2)\end{matrix}\right]f(t)\mathrm{d}t. \tag{6.1.48}$$

(2) If $\alpha_1 > 0,\ \alpha_2 < 0$, then

$$\left(\mathbf{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2} f\right)(x) = \int\limits_0^\infty H_{2,2}^{1,1}\left[xt \,\middle|\, \begin{matrix} (0,1),\,(\beta_2,-\alpha_2) \\ (0,1),\,(1-\beta_1,\alpha_1) \end{matrix}\right] f(t)\mathrm{d}t. \quad (6.1.49)$$

(3) If $\alpha_1 < 0,\ \alpha_2 > 0$, then

$$\left(\mathbf{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2} f\right)(x) = \int\limits_0^\infty H_{2,2}^{1,1}\left[xt \,\middle|\, \begin{matrix} (0,1),\,(\beta_1,-\alpha_1) \\ (0,1),\,(1-\beta_2,\alpha_2) \end{matrix}\right] f(t)\mathrm{d}t. \quad (6.1.50)$$

(4) If $\alpha_1 < 0,\ \alpha_2 < 0$, then

$$\left(\mathbf{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2} f\right)(x) = \int\limits_0^\infty H_{3,1}^{1,1}\left[xt \,\middle|\, \begin{matrix} (0,1),\,(\beta_1,-\alpha_1),\,(\beta_2,-\alpha_2) \\ (0,1) \end{matrix}\right] f(t)\mathrm{d}t. \quad (6.1.51)$$

Based on (6.1.48)–(6.1.51) and on the above presented elements of the $\mathcal{L}_{\nu,2}$-theory of the **H**-transform one can formulate the following results for the integral transforms with the four-parametric generalized Mittag-Leffler function in the kernel. Let us present these only in the case (1) (i.e. when $\alpha_1 > 0,\ \alpha_2 > 0$). All other cases can be considered analogously (see, e.g.. [KilKor06a], [KilKor06b]).

Let $\alpha_1 > 0,\ \alpha_2 > 0$. Then the parameters a^*, μ, Δ, α, β are related to the parameters of the four-parametric Mittag-Leffler function as follows:

$$a^* = 2 - \alpha_1 - \alpha_2,\ \ \Delta = \alpha_1 + \alpha_2,\ \ \mu = 1 - \beta_1 - \beta_2,\ \ \alpha = 0,\ \ \beta = 1.$$

Let $0 < \nu < 1$, $\alpha_1 > 0$, $\alpha_2 > 0$ and $\beta_1, \beta_2 \in \mathbb{C}$ be such that $\alpha_1 + \alpha_2 < 2$ or $\alpha_1 + \alpha_2 = 2$ and $3 - 2\nu \le \mathrm{Re}\,(\beta_1 + \beta_2)$. Then:

(a) There exists an injective mapping $\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2} \in \left[\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}\right]$ such that for any $f \in \mathcal{L}_{\nu,2}$ the following relation holds:

$$\left(\mathcal{M}\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2} f\right)(s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta_1 - \alpha_1 s)\Gamma(\beta_2 - \alpha_2 s)}(\mathcal{M}f)(1-s)\ \ (\mathrm{Re}\, s = 1-\nu). \quad (6.1.52)$$

If either $\alpha_1 + \alpha_2 < 2$ or $\alpha_1 + \alpha_2 = 2$ and $3 - 2\nu \le \mathrm{Re}\,(\beta_1 + \beta_2)$ and the additional conditions

$$s \ne \frac{\beta_1 + k}{\alpha_1},\ \ s \ne \frac{\beta_2 + l}{\alpha_2}\ \ (k, l = 0, 1, 2, \cdots),\ \text{for } \mathrm{Re}\, s = 1 - \nu, \quad (6.1.53)$$

are satisfied, then the operator $\mathbf{E}^*$ is bijective from $\mathcal{L}_{\nu,2}$ onto $\mathcal{L}_{1-\nu,2}$.

(b) For any $f, g \in \mathcal{L}_{\nu,2}$ we have the integration by parts formula

$$\int_0^\infty f(x)\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2}g(x)\mathrm{d}x = \int_0^\infty \mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2}f(x)g(x)\mathrm{d}x. \tag{6.1.54}$$

(c) If $f \in \mathcal{L}_{\nu,2}$, $\lambda \in \mathbb{C}$, $h > 0$, then $\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2}f$ is represented in the form:

$$\left(\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2}f\right)(x) = hx^{1-(\lambda+1)/h}\frac{\mathrm{d}}{\mathrm{d}x}x^{(\lambda+1)/h} \times \int_0^\infty H^{1,2}_{2,4}\left[xt\,\middle|\, \begin{matrix}(-\lambda, h), (0,1)\\ (0,1), (1-\beta_1,\alpha_1), (1-\beta_2,\alpha_2), (-\lambda-1,h)\end{matrix}\right] f(t)\mathrm{d}t \tag{6.1.55}$$

when $\mathrm{Re}\,\lambda > (1-\nu)h - 1$, and in the form:

$$\left(\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2}f\right)(x) = -hx^{1-(\lambda+1)/h}\frac{\mathrm{d}}{\mathrm{d}x}x^{(\lambda+1)/h} \times \int_0^\infty H^{2,1}_{2,4}\left[xt\,\middle|\, \begin{matrix}(0,1), (-\lambda, h)\\ (-\lambda-1,h), (0,1), (1-\beta_1,\alpha_1), (1-\beta_2,\alpha_2)\end{matrix}\right] f(t)\mathrm{d}t \tag{6.1.56}$$

when $\mathrm{Re}\,\lambda < (1-\nu)h - 1$.

(d) The mapping $\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ does not depend on ν in the following sense: if $0 < \nu_1, \nu_2 < 1$ and the mappings $\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2;1}$, $\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2;2}$ are defined on the spaces $\mathcal{L}_{\nu_1,2}$, $\mathcal{L}_{\nu_2,2}$ respectively, then $\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2;1}f = \mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2;2}f$ for all $f \in \mathcal{L}_{\nu_1,2}\bigcap\mathcal{L}_{\nu_2,2}$.

(e) If $f \in \mathcal{L}_{\nu,2}$ and either $\alpha_1+\alpha_2 < 2$ or $\alpha_1+\alpha_2 = 2$ and $3-2\nu \le \mathrm{Re}\,(\beta_1+\beta_2)$, then for all $f \in \mathcal{L}_{\nu,2}$ we have $\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2}f = \mathbf{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}f$, i.e. the mapping $\mathbf{E}^*_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ is defined by the formula (6.1.48).

6.1.8 Relations to the Fractional Calculus

Let us present a number of (left- and right-sided) Riemann–Liouville fractional integration and differentiation formulas for the four-parametric Mittag-Leffler function. Both cases ($\alpha_1+\alpha_2 > 0$ and $\alpha_1+\alpha_2 < 0$) will be considered simultaneously (see [KiKoRo13]). For simplicity we use the notation $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}$ for the four-parametric Mittag-Leffler function in both cases.

Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_1, \beta_2 \in \mathbb{C}$, and let the contour of integration in (6.1.14) be chosen as $\mathcal{L} = \mathcal{L}_{-\infty}$ when $\alpha_1+\alpha_2 > 0$, and as $\mathcal{L} = \mathcal{L}_{+\infty}$ when $\alpha_1+\alpha_2 < 0$. Let the additional parameters $\gamma, \sigma, \lambda \in \mathbb{C}$ be such that $\mathrm{Re}\,\gamma > 0$, $\mathrm{Re}\,\sigma > 0$ and $\omega \in \mathbb{R}$, $(\omega \neq 0)$.

The left-sided Riemann–Liouville fractional integral of the four-parametric Mittag-Leffler function is given by the following formulas:

(a) If $\alpha_1 < 0$ and $\alpha_2 > 0$, then for $x > 0$

$$\left(I_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{\omega})\right)(x) = \begin{cases} x^{\sigma+\gamma-1} H_{3,3}^{1,2}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma,\omega),\ (\beta_1,-\alpha_1) \\ (0,1),\ (1-\sigma-\gamma,\omega),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega > 0), \\ x^{\sigma+\gamma-1} H_{3,3}^{2,1}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma+\gamma,-\omega),\ (\beta_1,-\alpha_1) \\ (0,1),\ (\sigma,-\omega),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega < 0). \end{cases}$$

(b) If $\alpha_1 < 0$ and $\alpha_2 < 0$, then for $x > 0$

$$\left(I_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{\omega})\right)(x) = \begin{cases} x^{\sigma+\gamma-1} H_{4,2}^{1,2}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma,\omega),\ (\beta_1,-\alpha_1),\ (\beta_2,-\alpha_2) \\ (0,1),\ (1-\sigma-\gamma,\omega) \end{matrix}\right] \\ \qquad (\omega > 0), \\ x^{\sigma+\gamma-1} H_{4,2}^{2,1}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma+\gamma,-\omega),\ (\beta_1,-\alpha_1),\ (\beta_2,-\alpha_2) \\ (0,1),\ (\sigma,-\omega) \end{matrix}\right] \\ \qquad (\omega < 0). \end{cases}$$

(c) If $\alpha_1 > 0$ and $\alpha_2 > 0$, then for $x > 0$

$$\left(I_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{\omega})\right)(x) = \begin{cases} x^{\sigma+\gamma-1} H_{2,4}^{1,2}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma,\omega) \\ (0,1),\ (1-\sigma-\gamma,\omega),\ (1-\beta_1,\alpha_1),\ (1-\beta_2,\alpha_2) \end{matrix}\right] \\ \qquad (\omega > 0), \\ x^{\sigma+\gamma-1} H_{2,4}^{2,1}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma+\gamma,-\omega) \\ (0,1),\ (\sigma,-\omega),\ (1-\beta_1,\alpha_1),\ (1-\beta_2,\alpha_2) \end{matrix}\right] \\ \qquad (\omega < 0). \end{cases}$$

The right-sided Riemann–Liouville fractional integral of the four-parametric Mittag-Leffler function is given by the following formulas:

(a) If $\alpha_1 < 0$ and $\alpha_2 > 0$, then for $x > 0$

$$\left(I_{-}^{\gamma} t^{-\sigma} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{-\omega})\right)(x) = \begin{cases} x^{\gamma-\sigma} H_{3,3}^{1,2}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma+\gamma,\omega),\ (\beta_1,-\alpha_1) \\ (0,1),\ (1-\sigma,\omega),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega > 0), \\ x^{\gamma-\sigma} H_{3,3}^{2,1}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma,-\omega),\ (\beta_1,-\alpha_1) \\ (0,1),\ (\sigma-\gamma,-\omega),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega < 0). \end{cases}$$

(b) If $\alpha_1 < 0$ and $\alpha_2 < 0$, then for $x > 0$

$$\left(I_{-}^{\gamma} t^{-\sigma} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{-\omega})\right)(x) =$$
$$\begin{cases} x^{\gamma-\sigma} H_{4,2}^{1,2}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma+\gamma,\omega), (\beta_1,-\alpha_1),\ (\beta_2,-\alpha_2) \\ (0,1),\ (1-\sigma,\omega) \end{matrix}\right] \\ \hfill (\omega > 0), \\ x^{\gamma-\sigma} H_{4,2}^{2,1}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma,-\omega),\quad (\beta_1,-\alpha_1),\ (\beta_2,-\alpha_2) \\ (0,1),\ (\sigma-\gamma,-\omega) \end{matrix}\right] \\ \hfill (\omega < 0). \end{cases}$$

(c) If $\alpha_1 > 0$ and $\alpha_2 > 0$, then for $x > 0$

$$\left(I_{-}^{\gamma} t^{-\sigma} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{-\omega})\right)(x) =$$
$$\begin{cases} x^{\gamma-\sigma} H_{2,4}^{1,2}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma+\gamma,\omega) \\ (0,1),\ (1-\sigma,\omega),\quad (1-\beta_1,\alpha_1),\ (1-\beta_2,\alpha_2) \end{matrix}\right] \\ \hfill (\omega > 0), \\ x^{\gamma-\sigma} H_{2,4}^{2,1}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma,-\omega) \\ (0,1),\ (\sigma-\gamma,-\omega), (1-\beta_1,\alpha_1),\ (1-\beta_2,\alpha_2) \end{matrix}\right] \\ \hfill (\omega < 0). \end{cases}$$

The left-sided Riemann–Liouville fractional derivative of the four-parametric Mittag-Leffler function is given by the following formulas:

(a) If $\alpha_1 < 0$ and $\alpha_2 > 0$, then for $x > 0$

$$\left(D_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{\omega})\right)(x) =$$
$$\begin{cases} x^{\sigma-\gamma-1} H_{3,3}^{2,1}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma,\omega),\quad (\beta_1,-\alpha_1) \\ (0,1),\ (1-\sigma+\gamma,\omega),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega > 0), \\ x^{\sigma-\gamma-1} H_{3,3}^{1,2}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma-\gamma,-\omega), (\beta_1,-\alpha_1) \\ (0,1),\ (\sigma,-\omega),\quad (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega < 0). \end{cases}$$

(b) If $\alpha_1 < 0$ and $\alpha_2 < 0$, then for $x > 0$

$$\left(D_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{\omega})\right)(x) =$$
$$\begin{cases} x^{\sigma-\gamma-1} H_{4,2}^{1,2}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma,\omega),\quad (\beta_1,-\alpha_1),\ (\beta_2,-\alpha_2) \\ (0,1),\ (1-\sigma+\gamma,\omega) \end{matrix}\right] \\ \hfill (\omega > 0), \\ x^{\sigma-\gamma-1} H_{4,2}^{2,1}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma-\gamma,-\omega),\ (\beta_1,-\alpha_1),\ (\beta_2,-\alpha_2) \\ (0,1),\ (\sigma,-\omega) \end{matrix}\right] \\ \hfill (\omega < 0). \end{cases}$$

(c) If $\alpha_1 > 0$ and $\alpha_2 > 0$, then for $x > 0$

$$\left(D_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{\omega})\right)(x) =$$
$$\begin{cases} x^{\sigma-\gamma-1} H_{2,4}^{1,2}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma,\omega) \\ (0,1),\ (1-\sigma+\gamma,\omega),\ (1-\beta_1,\alpha_1),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega > 0), \\ x^{\sigma-\gamma-1} H_{2,4}^{2,1}\left[-\lambda x^{\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma-\gamma,-\omega) \\ (0,1),\ (\sigma,-\omega),\ (1-\beta_1,\alpha_1),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega < 0). \end{cases}$$

The right-sided Riemann–Liouville fractional derivative of the four-parametric Mittag-Leffler function is given by the following formulas:

(a) If $\alpha_1 < 0$ and $\alpha_2 > 0$, then for $x > 0$

$$\left(D_{-}^{\gamma} t^{-\sigma} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(\lambda t^{-\omega})\right)(x) =$$
$$\begin{cases} x^{-\sigma-\gamma} H_{3,3}^{2,1}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma-\gamma,\omega),\ (\beta_1,-\alpha_1) \\ (0,1),\ (1-\sigma,\omega),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega > 0), \\ x^{-\sigma-\gamma} H_{3,3}^{1,2}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma,-\omega),\ (\beta_1,-\alpha_1) \\ (0,1),\ (\sigma+\gamma,-\omega),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega < 0). \end{cases}$$

(b) If $\alpha_1 < 0$ and $\alpha_2 < 0$, then for $x > 0$

$$\left(D_{-}^{\gamma} t^{-\sigma} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(-\lambda t^{-\omega})\right)(x) =$$
$$\begin{cases} x^{-\sigma-\gamma} H_{4,2}^{2,1}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma-\gamma,\omega),\ (\beta_1,-\alpha_1),\ (\beta_2,-\alpha_2) \\ (0,1),\ (1-\sigma,\omega) \end{matrix}\right] & (\omega > 0), \\ x^{-\sigma-\gamma} H_{4,2}^{1,2}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma,-\omega),\ (\beta_1,-\alpha_1),\ (\beta_2,-\alpha_2) \\ (0,1),\ (\sigma+\gamma,-\omega) \end{matrix}\right] & (\omega < 0). \end{cases}$$

(c) If $\alpha_1 > 0$ and $\alpha_2 > 0$, then for $x > 0$

$$\left(D_{-}^{\gamma} t^{-\sigma} \mathcal{E}_{\alpha_1,\beta_1;\ \alpha_2,\beta_2}(-\lambda t^{-\omega})\right)(x) =$$
$$\begin{cases} x^{-\sigma-\gamma} H_{2,4}^{2,1}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (1-\sigma-\gamma,\omega) \\ (0,1),\ (1-\sigma,\omega),\ (1-\beta_1,\alpha_1),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega > 0), \\ x^{-\sigma-\gamma} H_{2,4}^{1,2}\left[-\lambda x^{-\omega} \,\middle|\, \begin{matrix} (0,1),\ (\sigma,-\omega) \\ (0,1),\ (\sigma+\gamma,-\omega),\ (1-\beta_1,\alpha_1),\ (1-\beta_2,\alpha_2) \end{matrix}\right] & (\omega < 0). \end{cases}$$

6.2 The Four-Parametric Mittag-Leffler Function: A Generalization of the Prabhakar Function

In this section we mainly follow the article [SriTom09].

6.2.1 *Definition and General Properties*

A generalization of the Prabhakar function (5.1.1) is proposed in [SriTom09] in the following form

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} (z;\, \beta, \gamma \in \mathbb{C};\, \mathrm{Re}\,\alpha > \max\{0, \mathrm{Re}\,\kappa - 1\};\, \mathrm{Re}\,\kappa > 0), \tag{6.2.1}$$

where $(\gamma)_\delta$ with $\delta > 0$ is the generalized Pochhammer symbol $(\gamma)_\delta = \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)}$ (cf. Appendix A, Sect. A.1.5). The function (6.2.1) is sometimes called the four-parametric Mittag-Leffler function (*a four-parametric generalization of the Prabhakar function*). With $\kappa = q \in \mathbb{N}_0$, $\min\{\mathrm{Re}\,\beta, \mathrm{Re}\,\gamma\} > 0$, this definition coincides with the definition proposed in [ShuPra07].

Theorem 6.3 ([SriTom09, Thm. 1]) *The four-parametric Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ defined by (6.2.1) is an entire function in the complex z-plane of order ρ and type σ given by*

$$\rho = \frac{1}{\mathrm{Re}\,(\alpha-\kappa)+1}, \quad \sigma = \frac{1}{\rho}\left(\frac{(\mathrm{Re}\,\kappa)^{\mathrm{Re}\,\kappa}}{(\mathrm{Re}\,\alpha)^{\mathrm{Re}\,\alpha}}\right)^{\rho}. \tag{6.2.2}$$

Moreover, the power series in the defining equation (6.2.1) converges absolutely in the disc $|z| < \frac{(\mathrm{Re}\alpha)^{\mathrm{Re}\alpha}}{(\mathrm{Re}\kappa)^{\mathrm{Re}\kappa}}$ whenever

$$\mathrm{Re}\,\alpha = \mathrm{Re}\,\kappa - 1 > 0.$$

◁ The proof follows from the asymptotic properties of the Gamma function

$$\Gamma(z) = z^z e^{-z}\sqrt{\frac{2\pi}{z}}\left[1 + \frac{1}{12z} + \frac{1}{288z^2} + O\left(\frac{1}{z^3}\right)\right],$$

where

$$(z \to \infty,\ |\arg z| \le \pi - \varepsilon(0 < \varepsilon < \pi)),$$

and

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + \frac{(a-b)(a+b-1)}{2z} + O\left(\frac{1}{z^2}\right)\right],$$

where $a, b \in \mathbb{C}$ and $z \to \infty$ along any curve joining $z = 0$ and $z = \infty$ provided $z \neq -a, -a-1, \dots$ and $z \neq -b, -b-1, \dots$.

To determine the radius of convergence R of the power series $\sum\limits_{n=0}^{\infty} c_n z^n$ one can use the Cauchy–Hadamard formula

$$R = \limsup_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|,$$

and for the order ρ and the type σ of an entire function the following standard formulas (see [Lev56])

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log 1/|c_n|}, \quad \mathrm{e}\rho\sigma = \limsup_{n\to\infty} n|c_n|^{\frac{\rho}{n}}.$$

▷

A number of further properties of the four-parametric Mittag-Leffler function follows from its relation with the Fox–Wright function

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) = \frac{1}{\Gamma(\gamma)} {}_1W_1(z) \left[\begin{matrix} (\gamma, \kappa) \\ (\beta, \alpha) \end{matrix} \middle| z \right], \tag{6.2.3}$$

and with the Fox H-function (see [AgMiNi15])

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) = \frac{1}{\Gamma(\gamma)} H_{2,2}^{1,2}(z) \left[z \middle| \begin{matrix} (1-\gamma, \kappa), (0, 1) \\ (0, 1), (1-\beta, \alpha) \end{matrix} \right]. \tag{6.2.4}$$

6.2.2 The Four-Parametric Mittag-Leffler Function of a Real Variable

Following [SriTom09] one can introduce an integral operator with the four-parametric Mittag-Leffler function in the kernel

$$\left(\mathcal{E}_{a+;\alpha,\beta}^{\omega;\gamma,\kappa} \varphi \right)(x) := \int\limits_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,\kappa}(\omega(x-t)^{\alpha-1}) \varphi(t) \mathrm{d}t. \tag{6.2.5}$$

It is well-defined for the following values of parameters:

$$\gamma, \omega \in \mathbb{C}; \ \operatorname{Re}\alpha > \max\{0, \operatorname{Re}\kappa - 1\}, \ \min\{\operatorname{Re}\beta, \operatorname{Re}\kappa\} > 0.$$

Moreover, this operator is bounded in the Lebesgue space L_1 on any finite interval $[a, b], b > a$:

$$\|\mathcal{E}_{a+;\alpha,\beta}^{\omega;\gamma,\kappa} \varphi\|_1 \leq C_1 \|\varphi\|_1,$$

where

$$C_1 = (b-a)^{\operatorname{Re}\beta} \sum_{n=0}^{\infty} \frac{|(\gamma)_{\kappa n}|}{(n\operatorname{Re}\alpha + \operatorname{Re}\beta)|\Gamma(\alpha n + \beta)|} \frac{|\omega(b-a)^{\operatorname{Re}\alpha}|^n}{n!}.$$

The following theorem describes the action of the Riemann–Liouville fractional integral I^{μ}_{a+} and derivative D^{μ}_{a+} as well as the generalized Riemann–Liouville fractional derivative (the Hilfer fractional derivative)

$$\left(D^{\mu,\nu}_{a+}\phi\right)(x) := \left(I^{\nu(1-\mu)}_{a+} \frac{\mathrm{d}}{\mathrm{d}x} \left(I^{(1-\nu)(1-\mu)}_{a+}\phi\right)\right)(x) \tag{6.2.6}$$

on the four-parametric Mittag-Leffler function $E^{\gamma,\kappa}_{\alpha,\beta}(t)$

Theorem 6.4 ([SriTom09, Thm 3]) *Let* $x > a$, $a \in \mathbb{R}_+$, $0 < \mu < 1$, $0 \le \nu \le 1$, *and*

$$\operatorname{Re}\alpha > \max\{0, \operatorname{Re}\kappa - 1\},\ \min\{\operatorname{Re}\beta, \operatorname{Re}\kappa, \operatorname{Re}\lambda\} > 0,\ \gamma, \omega \in \mathbb{C}.$$

Then the following relations hold:

$$\left(I^{\lambda}_{a+}\left((t-a)^{\beta-1} E^{\gamma,\kappa}_{\alpha,\beta}(\omega(t-a)^{\alpha})\right)\right)(x) = (x-a)^{\beta+\lambda-1} E^{\gamma,\kappa}_{\alpha,\beta+\lambda}(\omega(t-a)^{\alpha}), \tag{6.2.7}$$

$$\left(D^{\lambda}_{a+}\left((t-a)^{\beta-1} E^{\gamma,\kappa}_{\alpha,\beta}(\omega(t-a)^{\alpha})\right)\right)(x) = (x-a)^{\beta-\lambda-1} E^{\gamma,\kappa}_{\alpha,\beta-\lambda}(\omega(t-a)^{\alpha}), \tag{6.2.8}$$

$$\left(D^{\mu,\nu}_{a+}\left((t-a)^{\beta-1} E^{\gamma,\kappa}_{\alpha,\beta}(\omega(t-a)^{\alpha})\right)\right)(x) = (x-a)^{\beta-\mu-1} E^{\gamma,\kappa}_{\alpha,\beta-\mu}(\omega(t-a)^{\alpha}). \tag{6.2.9}$$

The Laplace transform of the four-parametric Mittag-Leffler function is given by the following formula, which can be obtained using a term-by-term transformation of the corresponding power series

$$\mathcal{L}\left[x^{a-1} E^{\gamma,\kappa}_{\alpha,\beta}(\omega x^b)\right](s) = \frac{s^{-a}}{\Gamma(\gamma)} {}_2W_1\left[\begin{matrix}(a,b),(\gamma,\kappa)\\(\beta,\alpha)\end{matrix}\,\middle|\,\frac{\omega}{s^b}\right]. \tag{6.2.10}$$

6.3 Mittag-Leffler Functions with 2*n* Parameters

6.3.1 Definition and Basic Properties

Consider the function defined for $\alpha_i \in \mathbb{R}$ $(\alpha_1^2 + \cdots + \alpha_n^2 \neq 0)$ and $\beta_i \in \mathbb{C}$ $(i = 1, \cdots, n \in \mathbb{N})$ by

$$E\left((\alpha,\beta)_n;z\right)=\sum_{k=0}^{\infty}\frac{z^k}{\prod_{j=1}^{n}\Gamma(\alpha_j k+\beta_j)}\quad (z\in\mathbb{C}). \tag{6.3.1}$$

When $n=1$, (6.3.1) coincides with the Mittag-Leffler function (4.1.1):

$$E\left((\alpha,\beta)_1;z\right)=E_{\alpha,\beta}(z)\equiv\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\quad (z\in\mathbb{C}), \tag{6.3.2}$$

and, for $n=2$, with the four-parametric function (6.1.1):

$$E\left((\alpha,\beta)_2;z\right)=E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)\equiv\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\Gamma(\alpha_2 k+\beta_2)}\quad (z\in\mathbb{C}). \tag{6.3.3}$$

First of all we prove that (6.3.1) under the condition $\alpha_1+\alpha_2+\cdots+\alpha_n>0$ is an entire function.

Theorem 6.5 *Let $n\in\mathbb{N}$ and $\alpha_i\in\mathbb{R}$, $\beta_i\in\mathbb{C}$ $(i=1,2,\cdots,n)$ be such that*

$$\alpha_1^2+\cdots+\alpha_n^2\neq 0,\quad \alpha_1+\alpha_2+\cdots+\alpha_n>0. \tag{6.3.4}$$

Then $E\left((\alpha,\beta)_n;z\right)$ is an entire function of $z\in\mathbb{C}$ of order

$$\rho=\frac{1}{(\alpha_1+\alpha_2+\cdots+\alpha_n)} \tag{6.3.5}$$

and type

$$\sigma=\prod_{i=1}^{n}\left(\frac{\alpha_1+\cdots+\alpha_n}{|\alpha_i|}\right)^{\frac{\alpha_i}{\alpha_1+\cdots+\alpha_n}}. \tag{6.3.6}$$

◁ Rewrite (6.3.1) as the power series

$$E\left((\alpha,\beta)_n;z\right)=\sum_{k=0}^{\infty}c_k z^k,\quad c_k=\left[\prod_{j=1}^{n}\Gamma(\alpha_j k+\beta_j)\right]^{-1}. \tag{6.3.7}$$

According to the asymptotic property (A.27) we have

$$\frac{|c_k|}{|c_{k+1}|}\sim\prod_{j=1}^{n}|\alpha_j k|^{\alpha_j}=\prod_{j=1}^{n}|\alpha_j|^{\alpha_j}k^{\alpha_1+\alpha_2+\cdots+\alpha_n}\to+\infty\ (k\to\infty).$$

Then, if $\alpha_1+\alpha_2\cdots+\alpha_n>0$, we see that $R=\infty$, where R is the radius of convergence of the power series in (6.3.7). This means that $E\left((\alpha,\beta)_n;z\right)$ is an entire function of z.

We use [Appendix B, (B.5) and (B.6)] to evaluate the order ρ and the type σ of (6.3.1). Applying Stirling's formula for the Gamma function $\Gamma(z)$ at infinity and taking (6.3.7) into account, we have

$$\log\left(\frac{1}{c_k}\right) = \log\left[\prod_{j=1}^{n} \Gamma(\alpha_j k + \beta_j)\right]$$

$$= \sum_{j=1}^{n}\left(\alpha_j k + \beta_j - \frac{1}{2}\right)\log(\alpha_j k) - \sum_{j=1}^{n}\left(\alpha_j k\right) + \frac{n}{2}\log(2\pi) + O\left(\frac{1}{k}\right) \quad (k \to \infty).$$

Hence the following asymptotic estimate holds:

$$\log\left(\frac{1}{c_k}\right) \sim k\log(k)(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \quad (k \to \infty). \tag{6.3.8}$$

Thus, in accordance with [Appendix B, (B.5)], we obtain (6.3.5).

Further, according to (6.1.12) we obtain the asymptotic estimate

$$\prod_{j=1}^{n} \Gamma(\alpha_j k + \beta_j) \sim (2\pi)^{n/2} \prod_{j=1}^{n} (\alpha_j k)^{\alpha_j k + \beta_j - \frac{1}{2}} \mathrm{e}^{-\alpha_j k} \quad (k \to \infty). \tag{6.3.9}$$

By (6.3.7) and (6.3.9) we have

$$\limsup_{k\to\infty}\left(k^{1/\rho}|c_k|^{1/k}\right) = \limsup_{k\to\infty} k^{1/\rho}\prod_{j=1}^{n}\left[(|\alpha_j|k)^{-\alpha_j}\mathrm{e}^{\alpha_j}\right]$$

$$= \mathrm{e}^{\alpha_1+\alpha_2+\cdots+\alpha_n}\prod_{j=1}^{n}|\alpha_j|^{-\alpha_j} = \mathrm{e}^{1/\rho}\prod_{j=1}^{n}|\alpha_j|^{-\alpha_j}.$$

Substituting this relation into [Appendix B, (B.6)] we have

$$\sigma = \frac{1}{\rho}\left(\prod_{j=1}^{n}|\alpha_j|^{-\alpha_j}\right)^{\rho} = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)\left(\prod_{j=1}^{n}|\alpha_j|^{-\alpha_j}\right)^{\frac{1}{\alpha_1+\cdots+\alpha_n}}$$

$$= \prod_{j=1}^{n}\left(\frac{\alpha_1 + \cdots + \alpha_n}{|\alpha_j|}\right)^{\frac{\alpha_j}{\alpha_1+\cdots+\alpha_n}},$$

which proves (6.3.6). ▹

Remark 6.6 In the general case $\alpha_1 + \cdots \alpha_n > 0$ the relations (6.3.5) and (6.3.6) have been proved by Kilbas and Koroleva [KilKor05] (and also in a paper by

Rogosin, Kilbas and Koroleva [KiKoRo13]), while in the particular case $\alpha_j > 0$ ($j = 1, \ldots, n$), in the works of Kiryakova, as [Kir99], [Kir00]. Note that if $n > 1$ the type σ in (6.3.6) is greater than 1 (Th.1, Kiryakova [Kir10b]).

Remark 6.7 When $n = 1, \alpha_1 = \alpha > 0$ and $\beta_1 = \beta \in \mathbb{C}$, relations (6.3.5) and (6.3.6) yield the known order and type of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ in (4.1.1) [Sect. 4.1]:

$$\rho = \frac{1}{\alpha}, \quad \sigma = 1. \tag{6.3.10}$$

Remark 6.8 When $n = 2$, $\alpha_j \in \mathbb{R}$, $\beta_j \in \mathbb{C}$ ($j = 1, 2$) with $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\alpha_1 + \alpha_2 > 0$, formulas (6.3.5) and (6.3.6) coincide with (6.1.8) and (6.1.9), respectively.

6.3.2 Representations in Terms of Hypergeometric Functions

We consider the generalized Mittag-Leffler function $E\left((\alpha, \beta)_n; z\right)$ in (6.3.1) under the conditions of Theorem 6.5. First we give a representation of $E\left((\alpha, \beta)_n; z\right)$ in terms of the generalized Wright hypergeometric function ${}_p\Psi_q(z)$ defined in Appendix F, (F.2.6)]. By (A.17), $(1)_k = k! = \Gamma(k+1)$ ($k \in \mathbb{N}_0$) and we can rewrite (6.3.1) in the form

$$E\left((\alpha, \beta)_n; z\right) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\prod_{j=1}^{n} \Gamma(\beta_j + \alpha_j k)} \frac{z^k}{k!} \quad (z \in \mathbb{C}). \tag{6.3.11}$$

This yields the following representation of $E\left((\alpha, \beta)_n; z\right)$ via the generalized Wright hypergeometric function ${}_1\Psi_n(z)$:

$$E\left((\alpha, \beta)_n; z\right) = {}_1\Psi_n\left[\begin{array}{c} (1, 1) \\ (\beta_1, \alpha_1), \cdots, (\beta_n, \alpha_n) \end{array} \middle| z\right] \quad (z \in \mathbb{C}). \tag{6.3.12}$$

Next we consider the generalized Mittag-Leffler function (6.3.1) with $n \geq 2$ and $\alpha_j = m_j \in \mathbb{N}$ ($j = 1, \cdots, n$):

$$\begin{aligned} E\left((m, \beta)_n; z\right) &= \sum_{k=0}^{\infty} \frac{z^k}{\prod_{j=1}^{n} \Gamma(m_j k + \beta_j)} \\ &= \sum_{k=0}^{\infty} \frac{(1)_k}{\prod_{j=1}^{n} \Gamma(m_j k + \beta_j)} \frac{z^k}{k!} \quad (z \in \mathbb{C}). \end{aligned} \tag{6.3.13}$$

According to (A.14) with $z = k + \frac{\beta_j}{m_i}$, $m = m_j$ ($j = 1, \cdots, n$) and (A.17) we have

$$\begin{aligned}\Gamma(m_jk+\beta_j) &= \Gamma\left[m_j\left(k+\frac{\beta_j}{m_j}\right)\right]\\ &= (2\pi)^{(1-m_j)/2} m_j^{m_jk+\beta_j-\frac{1}{2}} \prod_{s=0}^{m_j-1} \Gamma\left(\frac{\beta_j+s}{m_j}+k\right)\\ &= (2\pi)^{(1-m_j)/2} m_j^{m_jk+\beta_j-\frac{1}{2}} \prod_{s=0}^{m_j-1} \Gamma\left(\frac{\beta_j+s}{m_j}\right)\left(\frac{\beta_j+s}{m_j}\right)_k\\ &= m_j^{m_ik}\left[(2\pi)^{(1-m_j)/2} m_j^{\beta_j-\frac{1}{2}} \prod_{s=0}^{m_j-1} \Gamma\left(\frac{\beta_j+s}{m_j}\right)\right]\prod_{s=0}^{m_j-1}\left(\frac{\beta_j+s}{m_j}\right)_k.\end{aligned}$$

Then applying (A.14) with $z = \frac{\beta_j}{m_i}$, $m = m_j$, we get

$$\Gamma(m_jk+\beta_j) = m_j^{m_jk}\Gamma(\beta_j)\prod_{s=0}^{m_j-1}\left(\frac{\beta_j+s}{m_j}\right)_k.$$

Hence

$$E\left((m,\beta)_n;z\right) = \frac{1}{\prod_{j=1}^{n}\Gamma(\beta_j)}\sum_{k=0}^{\infty}\frac{(1)_k}{\prod_{j=1}^{n}\prod_{s=0}^{m_j-1}\left(\frac{\beta_j+s}{m_j}\right)_k}\left(\frac{z}{\prod_{j=1}^{n} m_j^{m_j}}\right)^k\frac{1}{k!}.$$

Therefore, we obtain the following representation of the 2n-parametric Mittag-Leffler function via a generalized hypergeometric function in the case of positive integer first parameters $\alpha_j = m_j \in \mathbb{N}$ $(j = 1, \cdots, n)$

$$E\left((m,\beta)_n;z\right) = \frac{1}{\prod_{j=1}^{n}\Gamma(\beta_j)} \tag{6.3.14}$$

$$\times {}_1F_{m_1+\ldots+m_n}\left(1;\frac{\beta_1}{m_1},\ldots,\frac{\beta_1+m_1-1}{m_1},\ldots,\frac{\beta_n}{m_n},\ldots,\frac{\beta_n+m_n-1}{m_n};\frac{z}{\prod_{j=1}^{n} m_j^{m_j}}\right).$$

6.3.3 Integral Representations and Asymptotics

The $2n$-parametric Mittag-Leffler function can be introduced either in the form of a series (6.3.1) or in the form of a Mellin–Barnes integral

$$E_{(\alpha,\,\beta)_n}(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)}{\prod_{j=1}^{n}\Gamma(\beta_j-\alpha_j s)}(-z)^{-s}\mathrm{d}s \quad (z\neq 0). \tag{6.3.15}$$

For $\mathrm{Re}\,\alpha_1+\ldots+\alpha_n>0$ one can choose the left loop $\mathcal{L}_{-\infty}$ as a contour of integration in (6.3.15). Calculating this integral by using Residue Theory we immediately obtain the series representation (6.3.1).

If $\alpha_j>0$; $\beta_j\in\mathbb{R}$ $(j=1,\ldots,n)$, then the $2n$-parametric Mittag-Leffler function $E((\alpha,\,\beta)_n;z)$ is an entire function of the complex variable $z\in\mathbb{C}$ of finite order, see Theorem 6.5.

This result gives an upper bound for the growth of the $2n$-parametric Mittag-Leffler function at infinity, namely, for any positive $\varepsilon>0$ there exists a positive r_ε such that

$$\left|E_{(\alpha,\,\beta)_n}(z)\right| < \exp\{(\sigma+\varepsilon)|z|^{\rho}\},\ \forall z,\,|z|>r_\varepsilon. \tag{6.3.16}$$

More precisely, the asymptotic behavior of the function $E_{(\alpha,\,\beta)_n}(z)$ can be described using the representation of the latter in terms of the H-function with special values of parameters (see Sect. 6.3.7 below) and asymptotic results for the H-function (see [KilSai04]).

6.3.4 Extension of the 2n-Parametric Mittag-Leffler Function

An extension of the $2n$-parametric Mittag-Leffler function is given by the representation

$$\mathcal{E}((\alpha,\,\beta)_n;z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)}{\prod_{j=1}^{n}\Gamma(\beta_j-\alpha_j s)}(-z)^{-s}\mathrm{d}s \quad (z\neq 0), \tag{6.3.17}$$

where the right loop $\mathcal{L}=\mathcal{L}_{+\infty}$ is chosen as the contour of integration $\mathcal{L}$.

By using Stirling's asymptotic formula for the Gamma function

$$|\Gamma(x+iy)| = (2\pi)^{1/2}|x|^{x-1/2}\mathrm{e}^{-x-\pi[1-\mathrm{sign}(x)]y/2}\ \ (x,\,y\in\mathbb{R};\,|x|\to\infty), \tag{6.3.18}$$

one can show directly that with the above choice of the integration contour the integral (6.3.17) is convergent for all values of parameters $\alpha_1,\ldots,\alpha_n\in\mathbb{C}$, $\beta_1,\ldots,\beta_n\in\mathbb{C}$ such that $\mathrm{Re}\,\alpha_1+\ldots+\alpha_n<0$ (cf., e.g., [KilKor05]).

Under these conditions (the choice of contour and assumption on the parameters) the integral (6.3.17) can be calculated by using Residue Theory. This gives the following Laurent series representation of the extended 2n-parametric Mittag-Leffler function: let $\alpha_j, \beta_j \in \mathbb{C}$ ($j = 1...n$), $z \in \mathbb{C}$ ($z \neq 0$) with $\mathrm{Re}\,\alpha_1 + \ldots + \alpha_n < 0$ and $\mathcal{L} = \mathcal{L}_{+\infty}$, then the function $\mathcal{E}((\alpha, \beta)_n; z)$ has the Laurent series representation

$$\mathcal{E}((\alpha, \beta)_n; z) = \sum_{k=0}^{\infty} \frac{d_k}{z^{k+1}}, \quad d_k = \prod_{j=1}^{n} \frac{1}{\Gamma(-\alpha_j k - \alpha_j + \beta_j)}. \tag{6.3.19}$$

The series in (6.3.19) is convergent for all $z \in \mathbb{C} \setminus \{0\}$. Convergence again follows from the asymptotic properties of the Gamma function, which yield the relation

$$\frac{|d_k|}{|d_{k+1}|} \sim \prod_{j=1}^{n} \left[|\alpha_j|^{-\mathrm{Re}\,(\alpha_j)} \mathrm{e}^{\mathrm{Im}\,(\alpha_j) \arg(-\alpha_j k)}\right] k^{-\sum_{j=1}^{n} \mathrm{Re}\,(\alpha_j)} \quad (k \to \infty).$$

By using the series representation of the extended 2n-parametric Mittag-Leffler function it is not hard to obtain an asymptotic formula for $z \to \infty$. Namely, if $\alpha_j, \beta_j \in \mathbb{C}$ ($j = 1, \ldots, n$), $z \in \mathbb{C}$ ($z \neq 0$) and $\mathrm{Re}\,\alpha_1 + \ldots + \alpha_n) < 0$, with contour of integration in (6.3.17) chosen as $\mathcal{L} = \mathcal{L}_{+\infty}$, then for any $N \in \mathbb{N}$ we have for $z \to \infty$ the asymptotic representation

$$\mathcal{E}((\alpha, \beta)_n; z) = \sum_{k=0}^{N} \frac{1}{\prod_{j=1}^{n} \Gamma(-\alpha_j k - \alpha_j + \beta_j) z^{k+1}} \left[1 + O\left(\frac{1}{z}\right)\right] \quad (z \to \infty).$$

The main term of this asymptotics is equal to

$$\mathcal{E}((\alpha, \beta)_n; z) = \prod_{j=1}^{n} \frac{1}{\Gamma(-\alpha_j + \beta_j)} \left[1 + O\left(\frac{1}{z}\right)\right] \quad (z \to \infty).$$

The asymptotics at $z \to 0$ is more complicated. It can be derived by using the relations of the extended 2n-parametric Mittag-Leffler function with the generalized Wright function and the H-functions (see Sect. 6.3.5 below) and the asymptotics of the latter presented in [KilSai04].

Another possible way to get the asymptotics of $\mathcal{E}((\alpha, \beta)_n; z)$ for $z \to 0$ is to use the following. If $\alpha_j, \beta_j \in \mathbb{C}$ ($j = 1...n$), $z \in \mathbb{C}$ ($z \neq 0$), $\mathrm{Re}\,\alpha_1 + ... + \alpha_n < 0$, $\mathcal{L} = \mathcal{L}_{+\infty}$, then the extended 2$n$-parametric Mittag-Leffler function can be presented in terms of the "usual" 2n-parametric Mittag-Leffler function:

$$\mathcal{E}((\alpha, \beta)_n; z) = \frac{1}{z} E\left((-\alpha, \beta - \alpha)_n; \frac{1}{z}\right). \tag{6.3.20}$$

6.3.5 Relations to the Wright Function and to the H-Function

In this section we present some formulas representing the $2n$-parametric Mitttag-Leffler function $E((\alpha, \beta)_n; z)$ and its extension $\mathcal{E}((\alpha, \beta)_n; z)$ in terms of the generalized Wright function ${}_p\Psi_q$ and the H-function.

For short, we use the same notation $\mathcal{E}((\alpha, \beta)_n; z)$ for the $2n$-parametric Mitttag-Leffler function and for its extension. These functions differ in values of the parameters α_i and in the choice of the contour of integration $\mathcal{L}$ in their Mellin–Barnes integral representation.

For real values of the parameters $\alpha_j \in \mathbb{R}$ and complex $\beta_j \in \mathbb{C}$ $(j = 1, \dots, n)$ the following representations hold:

(1) if $\sum_{j=1}^{n} \alpha_j > 0$, $\mathcal{L} = \mathcal{L}_{-\infty}$, then

$$\mathcal{E}((\alpha, \beta)_n; z) = {}_1\Psi_n \left[\begin{matrix} (1, 1) \\ (\beta_1, \alpha_1), \dots, (\beta_n, \alpha_n) \end{matrix} \middle| z \right]; \tag{6.3.21}$$

(2) if $\sum_{j=1}^{n} \alpha_j < 0$, $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$\mathcal{E}((\alpha, \beta)_n; z) = \frac{1}{z}\, {}_1\Psi_2 \left[\begin{matrix} (1, 1) \\ (\beta_1 - \alpha_1, -\alpha_1), \dots, (\beta_n - \alpha_n, -\alpha_n) \end{matrix} \middle| \frac{1}{z} \right]. \tag{6.3.22}$$

The above representations can be obtained by comparing the series representation of the corresponding functions. In the case (6.3.22) one can also use the relation (6.3.20).

In the same manner one can obtain the following representations of the $2n$-parametric Mittag-Leffler function and its extension in terms of the H-function:

(1) if $\alpha_j > 0$ $(j = 1, \dots, n)$, and $\mathcal{L} = \mathcal{L}_{-\infty}$, then

$$\mathcal{E}((\alpha, \beta)_n; z) = H^{1,1}_{1,n+1} \left[\begin{matrix} (0, 1) \\ (0, 1)(1 - \beta_1, \alpha_1), \dots, (1 - \beta_n, \alpha_n) \end{matrix} \middle| z \right]; \tag{6.3.23}$$

(2) if $\alpha_j > 0$ $(j = 1, \dots, p,\ p < n)$, $\alpha_j < 0$ $(j = p + 1, \dots, n)$, and either $\sum_{j=1}^{n} \alpha_j > 0$, $\mathcal{L} = \mathcal{L}_{-\infty}$, or $\sum_{j=1}^{n} \alpha_j < 0$, $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$\mathcal{E}((\alpha, \beta)_n; z) = H^{1,1}_{n-p+1,p+1} \left[\begin{matrix} (0, 1)(\beta_{p+1}, -\alpha_{p+1}) \dots (\beta_n, -\alpha_n) \\ (0, 1)(1 - \beta_1, \alpha_1), \dots, (1 - \beta_p, \alpha_p) \end{matrix} \middle| z \right]; \tag{6.3.24}$$

(3) if $\alpha_j < 0$ $(j = 1, \dots, p,\ p < n)$, $\alpha_j > 0$ $(j = p + 1, \dots, n)$, and either $\sum_{j=1}^{n} \alpha_j > 0$, $\mathcal{L} = \mathcal{L}_{-\infty}$, or $\sum_{j=1}^{n} \alpha_j < 0$, $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$\mathcal{E}((\alpha,\ \beta)_n; z) = H^{1,1}_{p+1,n-p+1}\left[\begin{array}{l}(0,1)(\beta_1,-\alpha_1)\dots(\beta_p,-\alpha_p)\\(0,1)(1-\beta_{p+1},\alpha_{p+1}),\dots,(1-\beta_n,\alpha_n)\end{array}\Bigg| z\right]; \tag{6.3.25}$$

(4) if $\alpha_j < 0$, $(j = 1,\dots,n)$ and $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$\mathcal{E}((\alpha,\ \beta)_n; z) = H^{1,1}_{n+1,1}\left[\begin{array}{l}(0,1)(\beta_1,-\alpha_1),\dots,(\beta_n,-\alpha_n)\\(0,1)\end{array}\Bigg| z\right]. \tag{6.3.26}$$

6.3.6 Integral Transforms with the Multi-parametric Mittag-Leffler Functions

Here we consider only the case when the parameters α_i in the definition of the $2n$-parametric Mittag-Leffler function and its extension are real numbers.

Since the $2n$-parametric Mittag-Leffler function is related to the generalized Wright function and to the H-function with special values of parameters (see Sect. 6.3.5), one can use (6.1.35) or (6.1.40) to define the Mellin transform of the function $E((\alpha,\ \beta)_n; z)$ and of its extension $\mathcal{E}((\alpha,\ \beta)_n; z)$.

Now we present a few results on integral transforms with the $2n$-parametric function in the kernel. The transforms are defined by the formula

$$\left(\mathbf{E}(\alpha,\ \beta)_n f\right)(x) = \int\limits_0^\infty \mathcal{E}((\alpha,\ \beta)_n; -xt) f(t)\mathrm{d}t \quad (x > 0), \tag{6.3.27}$$

with the $2n$-parametric Mittag-Leffler function in the kernel. These transforms are special cases of more general **H**-transforms (see Sect. 6.1.7). This can be seen from the definition of the **H**-transforms (6.1.32) and the following formulas which relate $\mathbf{E}(\alpha,\ \beta)_n$-transforms to **H**-transforms under different assumptions on the parameters.

(1) Let $\alpha_j > 0$ $(j = 1,\dots,n)$, $\mathcal{L} = \mathcal{L}_{-\infty}$, then

$$(\mathbf{E}(\alpha,\ \beta)_n f)(x) = \tag{6.3.28}$$
$$\int\limits_0^\infty H^{1,1}_{1,n+1}\left[xt\Bigg|\begin{array}{l}(0,1)\\(0,1),(1-\beta_1,\alpha_1),\dots,(1-\beta_n,\alpha_n)\end{array}\right] f(t)\mathrm{d}t.$$

(2) Let $\alpha_j > 0$ $(j = 1,\dots,p,\ p < n)$, $\alpha_j < 0$ $(j = p+1,\dots,n)$ and either $\sum\limits_{j=1}^n \alpha_j > 0$, $\mathcal{L} = \mathcal{L}_{-\infty}$ or $\sum\limits_{j=1}^n \alpha_j < 0$, $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$(\mathbf{E}(\alpha,\ \beta)_n f)(x) = \tag{6.3.29}$$
$$\int_0^\infty H^{1,1}_{n-p+1,p+1}\left[xt\left|\begin{matrix}(0,1),(\beta_{p+1},-\alpha_{p+1}),\ldots,(\beta_n,-\alpha_n)\\(0,1),(1-\beta_1,\alpha_1),\ldots,(1-\beta_p,\alpha_p)\end{matrix}\right.\right]f(t)\mathrm{d}t.$$

(3) Let $\alpha_j < 0$ $(j = 1, \ldots, p,\ p < n)$, $\alpha_j > 0$ $(j = p+1, \ldots, n)$ and either $\sum_{j=1}^{n} \alpha_j > 0$, $\mathcal{L} = \mathcal{L}_{-\infty}$, or $\sum_{j=1}^{n} \alpha_j < 0$, $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$(\mathbf{E}(\alpha,\ \beta)_n f) = \tag{6.3.30}$$
$$\int_0^\infty H^{1,1}_{p+1,n-p+1}\left[xt\left|\begin{matrix}(0,1),(\beta_1,-\alpha_1),\ldots,(\beta_p,-\alpha_p)\\(0,1),(1-\beta_{p+1},\alpha_{p+1}),\ldots,(1-\beta_n,\alpha_n)\end{matrix}\right.\right]f(t)\mathrm{d}t. \tag{6.3.31}$$

(4) Let $\alpha_j < 0$,$(j = 1, \ldots, n)$ and $\mathcal{L} = \mathcal{L}_{+\infty}$, then

$$(\mathbf{E}(\alpha,\ \beta)_n f)(x) = \int_0^\infty H^{1,1}_{n+1,1}\left[xt\left|\begin{matrix}(0,1),(\beta_1,-\alpha_1),\ldots,(\beta_n,-\alpha_n)\\(0,1)\end{matrix}\right.\right]f(t)\mathrm{d}t. \tag{6.3.32}$$

Convergence of the integrals depends on the values of some constants (as defined in formula (F.4.9), Appendix F). The constant a^* takes different values in the above cases:

1) $a^* = 2 - \sum_{j=1}^{n} \alpha_j$; (2) $a^* = 2 - \sum_{j=1}^{p} \alpha_j + \sum_{j=p+1}^{n} \alpha_j$;
3) $a^* = 2 + \sum_{j=1}^{p} \alpha_j - \sum_{j=p+1}^{n} \alpha_j$; 4) $a^* = 2 + \sum_{j=1}^{n} \alpha_j$;

and the constants $\Delta, \mu, \alpha, \beta$ take the same values in all four cases:

$$\Delta = \sum_{j=1}^{n} \alpha_j;\ \mu = \frac{n}{2} - \sum_{j=1}^{n} \beta_j;\ \alpha = 0;\ \beta = 1.$$

We present results on $\mathbf{E}(\alpha,\ \beta)_n$-transforms for two essentially different cases, namely for the case when all α_j are positive, and for the case when some of them are negative.

A. Let $0 < \nu < 1, \alpha_j > 0\ (j = 1, \ldots, n), \beta_j \in \mathbb{C}\ (j = 1, \ldots, n)$ be such that either $0 < \sum_{j=1}^{n} \alpha_j < 2$ or $\sum_{j=1}^{n} \alpha_j = 2$ and $2\nu + \sum_{j=1}^{n} \mathrm{Re}\, \beta_j \geq 2 + \frac{n}{2}$.

(a) There exists an injective mapping (transform) $\mathbf{E}^*(\alpha,\ \beta)_n \in \left[\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}\right]$ such that the equality

$$\left(\mathcal{M}\mathbf{E}^*(\alpha,\ \beta)_n f\right)(s) = \frac{\Gamma(s)\Gamma(1-s)}{\prod\limits_{j=1}^{n}\Gamma(\beta_j - \alpha_j s)}\left(\mathcal{M}f\right)(1-s), \quad (\mathrm{Re}\, s = 1-\nu) \tag{6.3.33}$$

holds for any $f \in \mathcal{L}_{\nu,2}$.

If $\sum\limits_{j=1}^{n}\alpha_j = 2,\ 2\nu + \sum\limits_{j=1}^{n}\mathrm{Re}\,\beta_j = 2 + \frac{n}{2}$ and

$$s \neq \frac{\beta_1 + k}{\alpha_1}, \ldots,\ s \neq \frac{\beta_n + l}{\alpha_n} \ \ (k, l = 0, 1, 2, \cdots) \text{ for } \mathrm{Re}\, s = 1-\nu, \tag{6.3.34}$$

then the mapping $\mathbf{E}^*(\alpha,\ \beta)_n$ is bijective from $\mathcal{L}_{\nu,2}$ onto $\mathcal{L}_{1-\nu,2}$.

(b) For any $f, g \in \mathcal{L}_{\nu,2}$ the following integration by parts formula holds:

$$\int\limits_0^\infty f(x)\left(\mathbf{E}^*(\alpha,\ \beta)_n g\right)(x)\mathrm{d}x = \int\limits_0^\infty \left(\mathbf{E}^*(\alpha,\ \beta)_n f\right)(x) g(x)\mathrm{d}x. \tag{6.3.35}$$

(c) If $f \in \mathcal{L}_{\nu,2}$, $\lambda \in \mathbb{C}$, $h > 0$, then the value $\mathbf{E}^*(\alpha,\ \beta)_n f$ can be represented in the form:

$$\left(\mathbf{E}^*(\alpha,\ \beta)_n f\right)(x) = h x^{1-(\lambda+1)/h}\frac{\mathrm{d}}{\mathrm{d}x}x^{(\lambda+1)/h} \tag{6.3.36}$$

$$\times \int\limits_0^\infty H^{1,2}_{2,n+2}\left[xt \,\middle|\, \begin{matrix}(-\lambda, h), (0, 1)\\ (0,1), (1-\beta_1, \alpha_1), \ldots, (1-\beta_n, \alpha_n), (-\lambda-1, h)\end{matrix}\right] f(t)\mathrm{d}t, \tag{6.3.37}$$

when $\mathrm{Re}\,(\lambda) > (1-\nu)h - 1$, or

$$\left(\mathbf{E}^*(\alpha,\ \beta)_n f\right)(x) = -h x^{1-(\lambda+1)/h}\frac{\mathrm{d}}{\mathrm{d}x}x^{(\lambda+1)/h} \tag{6.3.38}$$

$$\times \int\limits_0^\infty H^{2,1}_{2,n+2}\left[xt \,\middle|\, \begin{matrix}(0, 1), (-\lambda, h)\\ (-\lambda-1, h), (0,1), (1-\beta_1, \alpha_1), \ldots, (1-\beta_n, \alpha_n)\end{matrix}\right] f(t)\mathrm{d}t, \tag{6.3.39}$$

when $\mathrm{Re}\,(\lambda) < (1-\nu)h - 1$.

(d) The mapping $\mathbf{E}^*(\alpha,\ \beta)_n$ does not depend on ν in the following sense: if two values of the parameter $0 < \nu_1, \nu_2 < 1$ and the corresponding mappings $\mathbf{E}^*(\alpha,\ \beta)_{n;1}$, $\mathbf{E}^*(\alpha,\ \beta)_{n;2}$ are defined on the spaces $\mathcal{L}_{\nu_1,2}$, $\mathcal{L}_{\nu_2,2}$, respectively, then $\mathbf{E}^*(\alpha,\ \beta)_{n;1} f = \mathbf{E}^*(\alpha,\ \beta)_{n;2} f$ for all $f \in \mathcal{L}_{\nu_1,2} \bigcap \mathcal{L}_{\nu_2,2}$.

(e) If $f \in \mathcal{L}_{\nu,2}$ and either $0 < \sum_{j=1}^{n} \alpha_j < 2$ or $\sum_{j=1}^{n} \alpha_j = 2$ and $2\nu + \sum_{j=1}^{n} \mathrm{Re}\,\beta_j \geq 2 + \frac{n}{2}$, then the mapping (transform) $\mathbf{E}^*(\alpha, \beta)_n$ coincides with the transform $\mathbf{E}(\alpha, \beta)_n$ given by the formula (6.3.27), i.e. $\mathbf{E}^*(\alpha, \beta)_n f = \mathbf{E}(\alpha, \beta)_n f, \forall f \in \mathcal{L}_{\nu,2}$.

B. Let $0 < \nu < 1$, $\alpha_j > 0$ $(j = 1, \ldots, jp, j\ p < n)$ and $\alpha_j < 0$ $(j = p+1, j \ldots, n)$, $\beta_j \in \mathbb{C}$ $(j = 1, \ldots, n)$, be such that either $2 - \sum_{j=1}^{p} \alpha_j + \sum_{j=p+1}^{n} \alpha_j > 0$ or $2 - \sum_{j=1}^{p} \alpha_i + \sum_{j=p+1}^{n} \alpha_j = 0$ and $(1-\nu) \sum_{j=1}^{n} \alpha_j + \frac{n}{2} \leq \sum_{j=1}^{n} \beta_i$.

(a) There exists an injective mapping (transform) $\mathbf{E}^*(\alpha, \beta)_n \in [\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}]$ such that the equality (6.3.33) holds for any $f \in \mathcal{L}_{\nu,2}$.
If $2 - \sum_{j=1}^{p} \alpha_j + \sum_{j=p+1}^{n} \alpha_j = 0$, $(1-\nu) \sum_{j=1}^{n} \alpha_j + \frac{n}{2} = \sum_{j=1}^{n} \beta_j$ and the parameter s (which determines the line of integration for the inverse Mellin transform in (6.3.33)) satisfies (6.3.34), then the mapping $\mathbf{E}^*(\alpha, \beta)_n$ is bijective from $\mathcal{L}_{\nu,2}$ onto $\mathcal{L}_{1-\nu,2}$.

(b) For any $f, g \in \mathcal{L}_{\nu,2}$ the integration by parts formula (6.3.35) is satisfied.
(c) If $f \in \mathcal{L}_{\nu,2}$, $\lambda \in \mathbb{C}$, $h > 0$, then the value $\mathbf{E}^*(\alpha, \beta)_n f$ can be represented in the form:

$$\left(\mathbf{E}^*(\alpha, \beta)_n f\right)(x) = h x^{1-(\lambda+1)/h} \frac{\mathrm{d}}{\mathrm{d}x} x^{(\lambda+1)/h} \tag{6.3.40}$$
$$\times \int_0^\infty H^{1,2}_{n-p+2,p+2}\left[xt \,\middle|\, \begin{matrix} (-\lambda, h), (0,1), (\beta_{p+1}, -\alpha_{p+1}), \ldots, (\beta_n, -\alpha_n) \\ (0,1), (1-\beta_1, \alpha_1), \ldots, (1-\beta_p, \alpha_p), (-\lambda-1, h) \end{matrix}\right] f(t)\mathrm{d}t,$$

when $\mathrm{Re}\,(\lambda) > (1-\nu)h - 1$, or

$$\left(\mathbf{E}^*(\alpha, \beta)_n f\right)(x) = -h x^{1-(\lambda+1)/h} \frac{\mathrm{d}}{\mathrm{d}x} x^{(\lambda+1)/h} \tag{6.3.41}$$
$$\times \int_0^\infty H^{2,1}_{n-p+2,p+2}\left[xt \,\middle|\, \begin{matrix} (0,1), (\beta_{p+1}, -\alpha_{p+1}), \ldots, (\beta_n, -\alpha_n), (-\lambda, h) \\ (-\lambda-1, h), (0,1), (1-\beta_1, \alpha_1), \ldots, (1-\beta_p, \alpha_p) \end{matrix}\right] f(t)\mathrm{d}t,$$

when $\mathrm{Re}\,(\lambda) < (1-\nu)h - 1$.

(d) The mapping $\mathbf{E}^*(\alpha, \beta)_n$ does not depend on ν in the following sense: if $0 < \nu_1, \nu_2 < 1$ and the mappings $\mathbf{E}^*(\alpha, \beta)_{n;1}$, $\mathbf{E}^*(\alpha, \beta)_{n;2}$ are defined on the spaces $\mathcal{L}_{\nu_1,2}$, $\mathcal{L}_{\nu_2,2}$, respectively, then $\mathbf{E}^*(\alpha, \beta)_{n;1} f = \mathbf{E}^*(\alpha, \beta)_{n;2} f$ for all $f \in \mathcal{L}_{\nu_1,2} \bigcap \mathcal{L}_{\nu_2,2}$.

(e) If $f \in \mathcal{L}_{\nu,2}$ and either $2 - \sum\limits_{i=1}^{p} \alpha_i + \sum\limits_{i=p+1}^{n} \alpha_i > 0$ or $2 - \sum\limits_{i=1}^{p} \alpha_i + \sum\limits_{i=p+1}^{n} \alpha_i = 0$ and $(1-\nu)\sum\limits_{i=1}^{n} \alpha_i + \frac{n}{2} \le \sum\limits_{i=1}^{n} \beta_i$, then the mapping (transform) $\mathbf{E}^*(\alpha,\ \beta)_n$ coincides with the transform $\mathbf{E}(\alpha,\ \beta)_n$ given by the formula (6.3.27), i.e. $\mathbf{E}^*(\alpha,\ \beta)_n f = \mathbf{E}(\alpha,\ \beta)_n f, \forall f \in \mathcal{L}_{\nu,2}$.

6.3.7 Relations to the Fractional Calculus

In this subsection we present a few formulas relating the $2n$-parametric Mittag-Leffler function (with different values of the parameters α_j) to the left- and right-sided Riemann–Liouville fractional integral and derivative. For short, we use the same notation $\mathcal{E}((\alpha,\ \beta)_n; z)$ for the $2n$-parametric Mitttag-Leffler function and for its extension. These functions differ in values of the parameters α_i and in the choice of the contour of integration $\mathcal{L}$ in their Mellin–Barnes integral representation. The results in this subsection are obtained (see [KiKoRo13]) by using known formulas for the fractional integration and differentiation of power-type functions (see [SaKiMa93, (2.44) and formula 1 in Table 9.3]).

Let $\alpha_j \in \mathbb{R}, \alpha_j \neq 0$ $(j = 1, \ldots, n)$, $\alpha_1 < 0, \ldots, \alpha_l < 0, \alpha_{l+1} > 0, \ldots, \alpha_n > 0$ $(1 \le l \le n)$ and let the contour $\mathcal{L}$ be given by one of the following:

$$\mathcal{L} = \mathcal{L}_{-\infty} \text{ if } \alpha_1 + \ldots + \alpha_n > 0 \text{ or } \mathcal{L} = \mathcal{L}_{+\infty} \text{ if } \alpha_1 + \ldots + \alpha_n < 0.$$

Let $\gamma, \sigma, \lambda \in \mathbb{C}$ be such that $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}\,(\sigma) > 0$ and $\omega \in \mathbb{R}$, $(\omega \neq 0)$. Then the following assertions are true.

A. Calculation of the left-sided Riemann–Liouville fractional integral.

(a) If $\omega > 0$, then for $x > 0$

$$\left(I_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}((\alpha, \beta)_n; \lambda t^{\omega})\right)(x) = x^{\sigma+\gamma-1} H_{2+l,2+n-l}^{1,2}\left[-\lambda x^{\omega} \left| \begin{matrix} (0,1),\ (1-\sigma,\omega), & (\beta_j, -\alpha_j)_{1,l} \\ (0,1),\ (1-\sigma-\gamma,\omega), & (1-\beta_j, \alpha_j)_{l+1,n} \end{matrix} \right.\right]. \tag{6.3.42}$$

(b) If $\omega < 0$, then for $x > 0$

$$\left(I_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}((\alpha, \beta)_n; \lambda t^{\omega})\right)(x) = x^{\sigma+\gamma-1} H_{2+l,2+n-l}^{2,1}\left[-\lambda x^{\omega} \left| \begin{matrix} (0,1),\ (\gamma+\sigma,-\omega), & (\beta_j, -\alpha_j)_{1,l} \\ (0,1),\ (\sigma,-\omega), & (1-\beta_j, \alpha_j)_{l+1,n} \end{matrix} \right.\right]. \tag{6.3.43}$$

B. Calculation of the right-sided Liouville fractional integral.

(a) If $\omega > 0$, then for $x > 0$

$$\left(I_{-}^{\gamma} t^{-\sigma}\mathcal{E}((\alpha,\beta)_n;\lambda t^{-\omega})\right)(x) \tag{6.3.44}$$
$$= x^{\gamma-\sigma} H^{1,2}_{2+l,2+n-l}\left[-\lambda x^{\omega}\left|\begin{matrix}(0,1),\ (1-\sigma+\gamma,\omega), (\beta_j,-\alpha_j)_{1,l}\\ (0,1),\ (1-\sigma,\omega),\qquad (1-\beta_j,\alpha_j)_{l+1,n}\end{matrix}\right.\right].$$

(b) If $\omega < 0$, then for $x > 0$

$$\left(I_{-}^{\gamma} t^{-\sigma}\mathcal{E}((\alpha,\beta)_n;\lambda t^{-\omega})\right)(x) \tag{6.3.45}$$
$$= x^{\gamma-\sigma} H^{2,1}_{2+l,2+n-l}\left[-\lambda x^{-\omega}\left|\begin{matrix}(0,1),\ (\sigma,-\omega),\qquad (\beta_j,-\alpha_j)_{1,l}\\ (0,1),\ (\sigma-\gamma,-\omega), (1-\beta_j,\alpha_j)_{l+1,n}\end{matrix}\right.\right].$$

C. Calculation of the left-sided Riemann–Liouville fractional derivative.

(a) If $\omega > 0$, then for $x > 0$

$$\left(D_{0+}^{\gamma} t^{\sigma-1}\mathcal{E}((\alpha,\beta)_n;\lambda t^{\omega})\right)(x) \tag{6.3.46}$$
$$= x^{\sigma-\gamma-1} H^{1,2}_{2+l,2+n-l}\left[-\lambda x^{\omega}\left|\begin{matrix}(0,1),\ (1-\sigma,\omega),\qquad (\beta_j,-\alpha_j)_{1,l}\\ (0,1),\ (1-\sigma+\gamma,\omega), (1-\beta_j,\alpha_j)_{l+1,n}\end{matrix}\right.\right].$$

(b) If $\omega < 0$, then for $x > 0$

$$\left(D_{0+}^{\gamma} t^{\sigma-1}\mathcal{E}((\alpha,\beta)_n;\lambda t^{\omega})\right)(x) \tag{6.3.47}$$
$$= x^{\sigma-\gamma-1} H^{2,1}_{2+l,2+n-l}\left[-\lambda x^{\omega}\left|\begin{matrix}(0,1),\ (\gamma-\sigma,-\omega), (\beta_j,-\alpha_j)_{1,l}\\ (0,1),\ (\sigma,-\omega),\qquad (1-\beta_j,\alpha_j)_{l+1,n}\end{matrix}\right.\right].$$

D. Calculation of the right-sided Liouville fractional derivative.

(a) If $\omega > 0$, then for $x > 0$

$$\left(D_{-}^{\gamma} t^{-\sigma}\mathcal{E}((\alpha,\beta)_n;-\lambda t^{-\omega})\right)(x) \tag{6.3.48}$$
$$= x^{-\sigma-\gamma} H^{2,1}_{2+l,2+n-l}\left[-\lambda x^{-\omega}\left|\begin{matrix}(0,1),\ (1-\sigma-\gamma,\omega), (\beta_j,-\alpha_j)_{1,l}\\ (0,1),\ (1-\sigma,\omega),\qquad (1-\beta_j,\alpha_j)_{l+1,n}\end{matrix}\right.\right].$$

(b) If $\omega < 0$ then for $x > 0$

$$\left(D_{-}^{\gamma} t^{-\sigma}\mathcal{E}((\alpha,\beta)_n;\lambda t^{-\omega})\right)(x) \tag{6.3.49}$$
$$= x^{-\sigma-\gamma} H^{1,2}_{2+l,2+n-l}\left[-\lambda x^{-\omega}\left|\begin{matrix}(0,1),\ (\sigma,-\omega),\qquad (\beta_j,-\alpha_j)_{1,l}\\ (0,1),\ (\sigma+\gamma,-\omega), (1-\beta_j,\alpha_j)_{l+1,n}\end{matrix}\right.\right].$$

6.4 Mittag-Leffler Functions of Several Variables

In this section we present a few results on Mittag-Leffler function of several variables (see, e.g., [Lav18])

$$E_{(\alpha)_n,\beta}(z_1, z_2, \ldots, z_n) = \sum_{m_1,m_2,\ldots,m_n \ge 0} \frac{z_1^{m_1} z^{m_2} \cdots z_n^{m_n}}{\Gamma(\alpha_1 m_1 + \alpha_2 m_2 + \ldots + \alpha_n m_n + \beta)}$$

$$=: \sum_{\mathbf{m}=\mathbf{0}}^{\infty} \frac{\mathbf{z}^{\mathbf{m}}}{\Gamma(\langle, \mathbf{m}\rangle + \beta)}, \quad = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n, \operatorname{Re}\alpha_j > 0, \ \beta \in \mathbb{C}. \quad (6.4.1)$$

Other forms of the Mittag-Leffler type functions of several variables can be found too (see, e.g., [GaMaKa13], [Dua18], [Mam18] and references therein). There is a natural interest in studying the properties of this class of functions as it is related to the presentation of solutions of systems of linear fractional differential equations (in particular, of incommensurate orders).

In order to avoid additional technical details we focus here only on the case of two variables. The definition of the Mittag-Leffler function of two complex variables is similar the above presented for n variables (to avoid additional indexing we use the following variable names: $x = z_1$, $y = z_2$, $\alpha = \alpha_1$, $\beta = \alpha_2$, $\gamma = \beta$, $n = m_1$, $m = m_2$)

$$E_{\alpha,\beta;\gamma}(x, y) = \sum_{n,m\ge 0} \frac{x^n y^m}{\Gamma(\alpha n + \beta m + \gamma)}, \ \alpha, \beta, \gamma \in \mathbb{C}, \ \operatorname{Re}\alpha, \operatorname{Re}\beta > 0. \quad (6.4.2)$$

It is straightforward to check that, under above conditions, $E_{\alpha,\beta;\gamma}(x, y)$ is an entire function of two complex variables $(x, y) \in \mathbb{C}^2$.

6.4.1 Integral Representations

For applications it is interesting to describe the behavior of the function (6.4.2) for large values of arguments. For this we use the known results in the case of the Mittag-Leffler function of one variable. First, we find the integral representations of the considered function $E_{\alpha,\beta;\gamma}(x, y)$. Let us recall the definition of the Hankel path (see Sect. 3.4). For fixed $\theta \in (0, \pi)$, $\varepsilon > 0$ it is denoted by $\omega(\varepsilon, \theta)$. The path oriented by non-decreasing $\arg\zeta$ consists of two rays $S_\theta := \{\zeta \in \mathbb{C} : \arg\zeta = \theta, |\zeta| > \varepsilon\}$, $S_{-\theta} := \{\arg\zeta = -\theta, |\zeta| > \varepsilon\}$ and a part of the circle $C_\varepsilon(\theta) := \{\zeta \in \mathbb{C} : |\zeta| = \varepsilon, -\theta \le \arg\zeta \le \theta\}$. When $\theta = \pi$ the rays $S_{\pm\theta}$ degenerate into parts of the sides of negative semi-axes. This path divides the complex plane into two domains $\Omega^{(-)}(\varepsilon; \theta)$ and $\Omega^{(+)}(\varepsilon; \theta)$ which are situated, respectively, to the left and to the right of $\omega(\theta, \varepsilon)$ with respect to the orientation on it.

Below we derive integral representations of $E_{\alpha,\beta;\gamma}(x, y)$ in four different domains in $\mathbb{C}^2$, namely $\Omega^{(-)}(\varepsilon_\alpha; \theta_\alpha) \times \Omega^{(-)}(\varepsilon_\beta; \theta_\beta)$, $\Omega^{(+)}(\varepsilon_\alpha; \theta_\alpha) \times \Omega^{(-)}(\varepsilon_\beta; \theta_\beta)$, $\Omega^{(-)}(\varepsilon_\alpha; \theta_\alpha) \times \Omega^{(+)}(\varepsilon_\beta; \theta_\beta)$, and $\Omega^{(+)}(\varepsilon_\alpha; \theta_\alpha) \times \Omega^{(+)}(\varepsilon_\beta; \theta_\beta)$. For this we use two representations of the reciprocal to the Gamma functions appearing in definition (6.4.2) (see Sect. 3.4 of this book).

$$\frac{1}{\Gamma(\alpha n+\beta m+\gamma)}=\frac{1}{2\pi i\alpha}\int\limits_{\omega(\varepsilon,\theta_\alpha)} e^{\zeta^{\frac{1}{\alpha}}}\zeta^{\frac{-\alpha n-\beta m-\gamma+1}{\alpha}}d\zeta, \qquad (6.4.3)$$

$$\frac{1}{\Gamma(\alpha n+\beta m+\gamma)}=\frac{1}{2\pi i\beta}\int\limits_{\omega(\varepsilon,\theta_\beta)} e^{\zeta^{\frac{1}{\beta}}}\zeta^{\frac{-\alpha n-\beta m-\gamma+1}{\alpha}}d\zeta. \qquad (6.4.4)$$

In the first integral (6.4.3) we have inequalities for θ_α (see, e.g., Sect. 3.4)

$$\frac{\pi\alpha}{2}<\theta_\alpha\le\min\{\pi;\pi\alpha\},$$

and in the second integral (6.4.4) we have analogous inequalities for θ_β (see, e.g., Sect. 3.4)

$$\frac{\pi\beta}{2}<\theta_\beta\le\min\{\pi;\pi\beta\}.$$

In order to satisfy both sets of inequalities we put $\theta_\alpha=\frac{\theta}{\beta}$, $\theta_\beta=\frac{\theta}{\alpha}$ and fix θ such that

$$\frac{\pi\alpha\beta}{2}<\theta\le\min\{\pi,,\pi\alpha,\pi\beta,\pi\alpha\beta\}, \qquad (6.4.5)$$

and write $\epsilon_\alpha:=\varepsilon^{1/\beta}$, $\epsilon_\beta:=\varepsilon^{1/\alpha}$. In all cases we also suppose that α,β are "small", i.e.

$$0<\alpha,\ \beta<2,\ \alpha\beta<2. \qquad (6.4.6)$$

Note that it follows from (6.4.6) that the left-hand side is smaller than the right-hand side in (6.4.5).

Let us start with the derivation of the integral representation in the first domain. Let $y\in\Omega^{(-)}(\varepsilon_\beta;\theta_\beta)$, $x\in\mathbb{C}$, $|x|<\varepsilon_\alpha$. Then

$$\sup_{\zeta\in\omega(\theta_\beta,\varepsilon_\beta)}|x\zeta^{-\alpha/\beta}|<1.$$

This allows us to reduce to the one-dimensional case, due to the identity

$$E_{\alpha,\beta;\gamma}(x,y)=\sum_{n=0}^{\infty}x^n\sum_{m=0}^{\infty}\frac{y^m}{\Gamma(\beta m+(\alpha n+\gamma))}=\sum_{n=0}^{\infty}x^nE_{\beta,\alpha n+\gamma}(y),$$

and the corresponding integral representation for $E_{\beta,\alpha n+\gamma}(y)$:

$$E_{\alpha,\beta;\gamma}(x,y)=\sum_{n=0}^{\infty}x^n\frac{1}{2\pi i\beta}\int\limits_{\omega(\varepsilon_\beta,\theta_\beta)}\frac{e^{\zeta^{1/\beta}}\zeta^{\frac{1-\alpha n-\gamma}{\beta}}}{\zeta-y}d\zeta$$

$$= \frac{1}{2\pi i\beta} \int_{\omega(\varepsilon_\beta,\theta_\beta)} \frac{e^{\zeta^{1/\beta}} \zeta^{\frac{1-\gamma}{\beta}} d\zeta}{\zeta - y} \sum_{n=0}^{\infty} \left(x\zeta^{-\alpha/\beta}\right)^n = \frac{1}{2\pi i\beta} \int_{\omega(\varepsilon_\beta,\theta_\beta)} \frac{e^{\zeta^{1/\beta}} \zeta^{\frac{1+\alpha-\gamma}{\beta}} d\zeta}{(\zeta - y)(\zeta^{\alpha/\beta} - x)}.$$

By changing variables $\zeta = \xi^{1/\alpha}$ we arrive at the following representation

$$E_{\alpha,\beta;\gamma}(x,y) = \frac{1}{2\pi i\alpha\beta} \int_{\omega(\varepsilon,\theta)} \frac{e^{\xi^{\frac{1}{\alpha\beta}}} \xi^{\frac{\alpha+\beta-\gamma}{\alpha\beta}-1} d\xi}{(\xi^{1/\beta} - x)(\xi^{1/\alpha} - y)}, \quad |x| < \varepsilon_\alpha, \ y \in \Omega^{(-)}(\varepsilon_\beta;\theta_\beta). \tag{6.4.7}$$

Since the circle $x \in \mathbb{C}$, $|x| < \varepsilon_\alpha$, is contained in the domain of analyticity of the right-hand side of (6.4.7), by the Principle of Analytic Continuation formula (6.4.7) is valid in $\Omega^{(-)}(\varepsilon_\alpha;\theta_\alpha) \times \Omega^{(-)}(\varepsilon_\beta;\theta_\beta)$.

In order to prove the formula for the domain $\Omega^{(-)}(\varepsilon_\alpha;\theta_\alpha) \times \Omega^{(+)}(\varepsilon_\beta;\theta_\beta)$ we take $\varepsilon^1 > \varepsilon$. Then by the previous case we obtain for $y \in \Omega^{(-)}(\varepsilon^1_\beta;\theta_\beta)$, $y < \varepsilon^1_\beta$ and $x \in \Omega^{(-)}(\varepsilon_\alpha;\theta_\alpha)$ the following representation

$$E_{\alpha,\beta;\gamma}(x,y) = \frac{1}{2\pi i\beta} \int_{\omega(\varepsilon^1_\beta,\theta_\beta)} \frac{e^{\zeta^{1/\beta}} \zeta^{\frac{1+\alpha-\gamma}{\beta}} d\zeta}{(\zeta - y)(\zeta^{\alpha/\beta} - x)}. \tag{6.4.8}$$

On the other hand, for each $\varepsilon_\beta < |y| < \varepsilon^1_\beta$, $|\arg y| < \theta_\beta$, we have by the Cauchy theorem

$$E_{\alpha,\beta;\gamma}(x,y) = \frac{1}{2\pi i\beta} \int_{\omega(\varepsilon^1_\beta,\theta_\beta)-\omega(\varepsilon_\beta,\theta_\beta)} \frac{e^{\zeta^{1/\beta}} \zeta^{\frac{1+\alpha-\gamma}{\beta}} d\zeta}{(\zeta - y)(\zeta^{\alpha/\beta} - x)} = \frac{1}{\beta} \frac{e^{y^{1/\beta}} y^{\frac{1+\alpha-\gamma}{\beta}}}{y^{\alpha/\beta} - x}. \tag{6.4.9}$$

By adding the difference between the right-hand side and the middle integral in the last formula to the right-hand side of (6.4.8) and performing the change of variables in the integral term we arrive at the following representation

$$E_{\alpha,\beta;\gamma}(x,y) = \frac{1}{\beta} \frac{e^{y^{1/\beta}} y^{\frac{1+\alpha-\gamma}{\beta}}}{y^{\alpha/\beta} - x} + \frac{1}{2\pi i\alpha\beta} \int_{\omega(\varepsilon,\theta)} \frac{e^{\xi^{\frac{1}{\alpha\beta}}} \xi^{\frac{\alpha+\beta-\gamma}{\alpha\beta}-1} d\xi}{(\xi^{1/\beta} - x)(\xi^{1/\alpha} - y)} \tag{6.4.10}$$

valid for all $(x,y) \in \Omega^{(-)}(\varepsilon_\alpha;\theta_\alpha) \times \Omega^{(+)}(\varepsilon_\beta;\theta_\beta)$. The result in the domain $\Omega^{(+)}(\varepsilon_\alpha;\theta_\alpha) \times \Omega^{(-)}(\varepsilon_\beta;\theta_\beta)$ has a symmetric form, namely

$$E_{\alpha,\beta;\gamma}(x,y) = \frac{1}{\alpha} \frac{e^{y^{1/\alpha}} x^{\frac{1+\beta-\gamma}{\alpha}}}{x^{\beta/\alpha} - y} + \frac{1}{2\pi i\alpha\beta} \int_{\omega(\varepsilon,\theta)} \frac{e^{\xi^{\frac{1}{\alpha\beta}}} \xi^{\frac{\alpha+\beta-\gamma}{\alpha\beta}-1} d\xi}{(\xi^{1/\beta} - x)(\xi^{1/\alpha} - y)}. \tag{6.4.11}$$

Finally, by unifying the argument of the previous cases we obtain an integral representation of the Mittag-Leffler function of two variables in the domain $\Omega^{(+)}(\varepsilon_\alpha;\theta_\alpha)\times\Omega^{(+)}(\varepsilon_\beta;\theta_\beta)$:

$$E_{\alpha,\beta;\gamma}(x,y)=\frac{1}{\alpha}\frac{e^{x^{1/\alpha}}x^{\frac{1+\beta-\gamma}{\alpha}}}{x^{\beta/\alpha}-y}+\frac{1}{\beta}\frac{e^{y^{1/\beta}}y^{\frac{1+\alpha-\gamma}{\beta}}}{y^{\alpha/\beta}-x}+\frac{1}{2\pi i\alpha\beta}\int\limits_{\omega(\varepsilon,\theta)}\frac{e^{\xi^{\frac{1}{\alpha\beta}}}\xi^{\frac{\alpha+\beta-\gamma}{\alpha\beta}-1}d\xi}{(\xi^{1/\beta}-x)(\xi^{1/\alpha}-y)}. \tag{6.4.12}$$

By assumption, each of the points x and y lies on the right-hand side of the Hankel contours $\omega(\varepsilon_\alpha;\theta_\alpha)$ and $\omega(\varepsilon_\beta;\theta_\beta)$, respectively. Note that the parameters in the definition of the above paths depend on a certain number $\varepsilon>0$. Now choose ε^1 ($\varepsilon^1>\varepsilon$) such that one of the coordinates is to the right of the contour (say y) and the other coordinate to its left (i.e. x). This means $(x,y)\in\Omega^{(-)}(\varepsilon^1_\alpha;\theta_\alpha)\times\Omega^{(+)}(\varepsilon^1_\beta;\theta_\beta)$. In this case we have representation (6.4.11) with ε replaced by ε^1 in the integral. This integral can be rewritten as

$$\frac{1}{2\pi i\alpha}\int\limits_{\omega(\varepsilon^1_\alpha,\theta_\alpha)}\frac{e^{\zeta^{1/\alpha}}\zeta^{\frac{1+\beta-\gamma}{\alpha}}d\zeta}{(\zeta-x)(\zeta^{\beta/\alpha}-y)}.$$

For each $\varepsilon_\alpha<|x|<\varepsilon^1_\alpha$, $|\arg x|<\theta_\alpha$, we have by the Cauchy theorem

$$\frac{1}{2\pi i\beta}\int\limits_{\omega(\varepsilon^1_\alpha,\theta_\alpha)-\omega(\varepsilon_\alpha,\theta_\alpha)}\frac{e^{\zeta^{1/\alpha}}\zeta^{\frac{1+\beta-\gamma}{\alpha}}d\zeta}{(\zeta-x)(\zeta^{\beta/\alpha}-x)}=\frac{1}{\alpha}\frac{e^{x^{1/\alpha}}x^{\frac{1+\beta-\gamma}{\alpha}}}{x^{\beta/\alpha}-y}. \tag{6.4.13}$$

As before, this immediately yields the desired formula (6.4.12), valid in the domain $\Omega^{(+)}(\varepsilon_\alpha;\theta_\alpha)\times\Omega^{(+)}(\varepsilon_\beta;\theta_\beta)$.

6.4.2 Asymptotic Behavior for Large Values of Arguments

Here we describe the asymptotic behavior of the Mittag-Leffler function $E_{\alpha,\beta;\gamma}(x,y)$ of two complex variables x and y for large values of $|x|$ and $|y|$. The result follows from the above integral representations and the standard techniques for the description of the asymptotics of the corresponding integrals presented in Sects. 3.4 and 4.4.

Theorem 6.9 ([Lav18, Thm. 3.1]) *Let* $0<\alpha,\beta<2$, $\alpha\beta<2$ *and the angle* θ *be chosen as*

$$\frac{\pi\alpha\beta}{2}<\theta\le\min\{\pi,\pi\alpha,\pi\beta,\pi\alpha\beta\}.$$

Then, for all pairs of positive integers $\mathbf{p} = (p_\alpha, p_\beta)$, $p_\alpha, p_\beta > 1$, *the following asymptotic formulas for the function* $E_{\alpha,\beta;\gamma}(x, y)$ *hold as* $|x| \to \infty, |y| \to \infty$.

(i) If $|\arg x| < \frac{\theta}{\beta}$, $|\arg y| < \frac{\theta}{\alpha}$, *then*

$$E_{\alpha,\beta;\gamma}(x, y) = \frac{1}{\alpha} \frac{e^{x^{1/\alpha}} x^{\frac{1+\beta-\gamma}{\alpha}}}{x^{\beta/\alpha} - y} + \frac{1}{\beta} \frac{e^{y^{1/\beta}} y^{\frac{1+\alpha-\gamma}{\beta}}}{y^{\alpha/\beta} - x} \tag{6.4.14}$$

$$+ \sum_{n=1}^{p_\alpha} \sum_{m=1}^{p_\beta} \frac{x^{-n} y^{-m}}{\Gamma(\gamma - \alpha n - \beta m} + o\left(|xy|^{-1}|x|^{-p_\alpha}\right) + o\left(|xy|^{-1}|y|^{-p_\beta}\right);$$

(ii) If $|\arg x| < \frac{\theta}{\beta}$, $\frac{\theta}{\alpha} < |\arg y| \le \pi$, *then*

$$E_{\alpha,\beta;\gamma}(x, y) = \frac{1}{\alpha} \frac{e^{x^{1/\alpha}} x^{\frac{1+\beta-\gamma}{\alpha}}}{x^{\beta/\alpha} - y} \tag{6.4.15}$$

$$+ \sum_{n=1}^{p_\alpha} \sum_{m=1}^{p_\beta} \frac{x^{-n} y^{-m}}{\Gamma(\gamma - \alpha n - \beta m} + o\left(|xy|^{-1}|x|^{-p_\alpha}\right) + o\left(|xy|^{-1}|y|^{-p_\beta}\right);$$

(iii) If $\frac{\theta}{\beta} < |\arg x| \le \pi$, $|\arg y| < \frac{\theta}{\alpha}$, *then*

$$E_{\alpha,\beta;\gamma}(x, y) = \frac{1}{\beta} \frac{e^{y^{1/\beta}} y^{\frac{1+\alpha-\gamma}{\beta}}}{y^{\alpha/\beta} - x} \tag{6.4.16}$$

$$+ \sum_{n=1}^{p_\alpha} \sum_{m=1}^{p_\beta} \frac{x^{-n} y^{-m}}{\Gamma(\gamma - \alpha n - \beta m} + o\left(|xy|^{-1}|x|^{-p_\alpha}\right) + o\left(|xy|^{-1}|y|^{-p_\beta}\right);$$

(iv) If $\frac{\theta}{\beta} < |\arg x| \le \pi$, $\frac{\theta}{\alpha} < |\arg y| \le \pi$, *then*

$$E_{\alpha,\beta;\gamma}(x, y) = \sum_{n=1}^{p_\alpha} \sum_{m=1}^{p_\beta} \frac{x^{-n} y^{-m}}{\Gamma(\gamma - \alpha n - \beta m} + o\left(|xy|^{-1}|x|^{-p_\alpha}\right) + o\left(|xy|^{-1}|y|^{-p_\beta}\right). \tag{6.4.17}$$

The result of the theorem is obtained by expanding and further estimating the kernel in the integral terms. The complete proof is presented in [Lav18] (cf. [GoLoLu02]).

6.5 Mittag-Leffler Functions with Matrix Arguments

In this section the problem of defining and evaluating Mittag-Leffler functions with matrix arguments is discussed. The idea to generalize a given function of a scalar variable to matrix arguments goes back to the work of Cayley (1858) and nowadays this topic attracts the attention of researchers due to its applications to numerical solutions of fractional multiterm differential equations and fractional partial differential equations, in control theory and so on.

Since Mittag-Leffler functions are entire, it is not a problem to introduce the formal definition

$$E_{\alpha,\beta}(A) = \sum_{j=0}^{\infty} \frac{A^j}{\Gamma(\alpha j + \beta)}, \tag{6.5.1}$$

which is valid for any $n \times n$ square matrix A. This series representation is suitable for defining the value of a Mittag-Leffler function with matrix argument but not for practical and computational needs since the main issues related to the slow convergence of (6.5.1) and the possible numerical cancellation in summing terms with alternate signs are amplified by the presence of the matrix argument.

The *Jordan canonical form* provides an alternative way to introduce a function with matrix argument which (if suitably modified) can also be exploited for computational purposes.

If the $n \times n$ matrix A has s distinct eigenvalues λ_k, $k = 1, \ldots, s$, each with geometric multiplicity m_k (namely the smallest integer such that $(A - \lambda_k I)^{m_k} = 0$), the Jordan canonical form of A is

$$A = Z \begin{pmatrix} J_1 & & & \\ & J_2 & \ddots & \\ & & \ddots & \\ & & & J_s \end{pmatrix} Z^{-1}, \quad J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \in \mathbb{C}^{m_k \times m_k}.$$

Based on the Jordan canonical form it is possible to define the extension of a Mittag-Leffler function to a matrix argument according to

$$E_{\alpha,\beta}(A) = Z \begin{pmatrix} E_{\alpha,\beta}(J_1) & & & \\ & E_{\alpha,\beta}(J_2) & & \\ & & \ddots & \\ & & & E_{\alpha,\beta}(J_s) \end{pmatrix} Z^{-1}$$

with each Jordan block J_k, $k = 1, \ldots, s$, being mapped to

$$E_{\alpha,\beta}(J_k) = \begin{pmatrix} E_{\alpha,\beta}(\lambda_k) & E^{[1]}_{\alpha,\beta}(\lambda_k) & E^{[2]}_{\alpha,\beta}(\lambda_k) & \dots & E^{[m_k-1]}_{\alpha,\beta}(\lambda_k) \\ & E_{\alpha,\beta}(\lambda_k) & E^{[2]}_{\alpha,\beta}(\lambda_k) & \dots & E^{[m_k-2]}_{\alpha,\beta}(\lambda_k) \\ & & \ddots & \ddots & \vdots \\ & & & E_{\alpha,\beta}(\lambda_k) & E^{[1]}_{\alpha,\beta}(\lambda_k) \\ & & & & E_{\alpha,\beta}(\lambda_k) \end{pmatrix},$$

where, for compactness, we denote by $E^{[k]}_{\alpha,\beta}(z)$ the k-th term in the Taylor expansion of $E_{\alpha,\beta}(z)$, for which we incidentally note its relationship with the Prabhakar function $E^{k}_{\alpha,\beta}(z)$ since

$$E^{[k]}_{\alpha,\beta}(z) = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z^k} E_{\alpha,\beta}(z) = E^{k+1}_{\alpha,\alpha k+\beta}(z).$$

It is clear that the evaluation of a Mittag-Leffler function with matrix arguments reduces to the evaluation of derivatives of the scalar function in the spectrum of the matrix. We refer to Sect. 4.3 for a more detailed discussion about derivatives of Mittag-Leffler functions.

From the practical point of view, however, evaluating the Jordan canonical form is an ill-conditioned problem and, except for matrices with favorable properties, in most cases it cannot be used in practice. A more efficient strategy considers the Schur–Parlett algorithm [DavHig03], which is based on the Schur decomposition of the matrix argument combined with Parlett recurrence to evaluate the matrix function of the triangular factors. In this case extensive computation of derivatives of scalar Mittag-Leffler functions is required. This problem has been extensively discussed in [GarPop18], where a series of applications to fractional calculus are also illustrated.

The numerical experiments presented in [GarPop18] have shown that combining the Schur–Parlett algorithm with techniques for the evaluation of derivatives of Mittag-Leffler functions makes it possible to evaluate the matrix Mittag-Leffler functions with high accuracy, in some cases very close to machine precision. A Matlab code for evaluating Mittag-Leffler functions with matrix arguments is freely available in the file exchange service of the Mathworks website.[1]

6.6 Historical and Bibliographical Notes

In recent decades, starting from the eighties in the last century, we have observed a rapidly increasing interest in the classical Mittag-Leffler function and its generalizations. This interest mainly stems from their use in the explicit solution of certain classes of fractional differential equations (especially those modelling processes of fractional relaxation, oscillation, diffusion and waves). This topic is under develop-

[1] www.mathworks.com/matlabcentral/fileexchange/66272-mittag-leffler-function-with-matrix-arguments.

ment and scientists are looking for further applications of the results presented in this chapter and their generalizations.

For $\alpha_1,\ \alpha_2 \in \mathbb{R}\,(\alpha_1^2 + \alpha_2^2 \neq 0)$ and $\beta_1, \beta_2 \in \mathbb{C}$ the four-parametric Mittag-Leffler function is defined by the series

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1)\Gamma(\alpha_2 k + \beta_2)} \quad (z \in \mathbb{C}). \tag{6.6.1}$$

For positive $\alpha_1 > 0,\ \alpha_2 > 0$ and real $\beta_1, \beta_2 \in \mathbb{R}$ it was introduced by Djrbashian [Dzh60]. When $\alpha_1 = \alpha,\ \beta_1 = \beta$ and $\alpha_2 = 0, \beta_2 = 1$, it coincides with the Mittag-Leffler function $E_{\alpha,\beta}(z)$:

$$E_{\alpha,\beta;0,1}(z) = E_{\alpha,\beta}(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z \in \mathbb{C}). \tag{6.6.2}$$

Generalizing the four-parametric Mittag-Leffler function, Al-Bassam and Luchko [Al-BLuc95] introduced the Mittag-Leffler type function

$$E((\alpha,\ \beta)_n; z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{j=1}^{n} \Gamma(\alpha_j k + \beta_j)} \quad (n \in \mathbb{N}) \tag{6.6.3}$$

with $2n$ real parameters $\alpha_j > 0;\ \beta_j \in \mathbb{R}\ (j = 1, ..., n)$ and with complex $z \in \mathbb{C}$. In [Al-BLuc95] an explicit solution to a Cauchy type problem for a fractional differential equation is given in terms of (6.6.3). The theory of this class of functions was developed in a series of articles by Kiryakova et al. [Kir99], [Kir00], [Kir08], [Kir10a], [Kir10b].

Among the results dealing with multi-index Mittag-Leffler functions we point out those which show their relation to a general class of special functions, namely to Fox's H-function. Representations of the multi-index Mittag-Leffler functions as special cases of the H-function and the generalized Wright function are obtained in [AlKiKa02], [Kir10b]. Relations of such multi-index functions to the Erdelyi–Kober (E-K) operators of fractional integration are discussed. The novel Mittag-Leffler functions are also used as generating functions of a class of so-called Gelfond–Leontiev (G-L) operators of generalized differentiation and integration. Laplace-type integral transforms corresponding to these G-L operators are considered too. The multi-index Mittag-Leffler functions (6.6.3) can be regarded as "fractional index" analogues of the hyper-Bessel functions, and the multiple Borel–Dzrbashian integral transforms (being H-transforms) as "fractional index" analogues of the Obrechkoff transforms (being G-transforms).

In a more precise terminology, these are Gelfond–Leontiev (G-L) operators of generalized differentiation and integration with respect to the entire function, a multi-index generalization of the Mittag-Leffler function. Fractional multi-order integral equations

$$y(z) - \lambda \mathfrak{L} y(z) = f(z) \tag{6.6.4}$$

and initial value problems for the corresponding fractional multi-order differential equations

$$\mathfrak{D} y(z) - \lambda y(z) = f(z) \tag{6.6.5}$$

are considered. From the known solution of the Volterra-type integral equation with m-fold integration, via a Poisson-type integral transformation $\mathcal{P}$ as a transformation (transmutation) operator, the corresponding solution of the integral equation (6.6.4) is found. Then a solution of the fractional multi-order differential equation (6.6.5) comes out, in an explicit form, as a series of integrals involving Fox's H-functions. For each particularly chosen right-hand side function $f(z)$, such a solution can be evaluated as an H-function. Special cases of the equations considered here lead to solutions in terms of the Mittag-Leffler, Bessel, Struve, Lommel and hyper-Bessel functions, and some other known generalized hypergeometric functions.

In [Kir10b] (see also [Kir10a]) a brief description of recent results by Kiryakova et al. on an important class of "Special Functions of Fractional Calculus" is presented. These functions became important in solutions of fractional order (or multi-order) differential and integral equations, control systems and refined mathematical models of various physical, chemical, economical, management and bioengineering phenomena. The notion "Special Functions of Fractional Calculus" essentially means the Wright generalized hypergeometric function ${}_p\Psi_q$, as a special case of the Fox H-function.

A generalization of the Prabhakar type function was given by Shukla and Prajapati [ShuPra07]:

$$E_{\alpha;\beta}^{\gamma,\kappa}(z) = E(\alpha, \beta; \gamma, \kappa; z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{\kappa n} z^n}{\Gamma(\alpha n + \beta)} \quad (n \in \mathbb{N}), \tag{6.6.6}$$

where the generalized Pochhammer symbol is defined by

$$(\gamma)_{\kappa n} = \frac{\Gamma(\gamma + \kappa n)}{\Gamma(\gamma)}.$$

In [SriTom09] the existence of the function (6.6.6) for a wider set of parameters was shown, and its relation to the fractional calculus operators was described (see also [AgMiNi15], [GaShMa15]). Definition (6.6.6) was combined with (6.6.3) in [SaxNis10] (see also [Sax-et-al10]). As a result, the following definition of the generalized multi-index Mittag-Leffler function appears:

$$E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(z) = E_{\gamma,\kappa}((\alpha_j,\beta_j)^m_{j=1}; z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n} z^n}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \quad (m \in \mathbb{N}). \tag{6.6.7}$$

A four-parametric generalization of the Mittag-Leffler function similar to (6.6.6) (a so-called k-Mittag-Leffler function) was proposed in [DorCer12]

$$E^{\gamma}_{k,\alpha,\beta} := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta) n!} z^n, \tag{6.6.8}$$

with Pochhammer k-symbol

$$(z)_{n,k} := z(z+k)\ldots(z+(n-1)k)$$

and k-Gamma function

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt = k^{1-\frac{z}{k}} \Gamma\left(\frac{z}{k}\right), \quad (z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)},$$

appearing in the definition. A generalization of the function (6.6.8) (a $(p-k)$-Mittag-Leffler function) was proposed and studied in [CeLuDo18]:

$$ {}_pE^{\gamma}_{k,\alpha,\beta} := \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n + \beta) n!} z^n, \tag{6.6.9}$$

$$ {}_p(z)_{n,k} := \frac{zp}{k}\left(\frac{zp}{k} + p\right)\left(\frac{zp}{k} + 2p\right)\ldots\left(\frac{zp}{k} + (n-1)p\right),$$

$$ {}_p\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{p}} dt = \frac{p^{\frac{z}{k}}}{k} \Gamma\left(\frac{z}{k}\right), \quad {}_p(z)_{n,k} = \frac{{}_p\Gamma_k(z+nk)}{{}_p\Gamma_k(z)}.$$

Generalizations of the Mittag-Leffler function involving the Beta function and generalized Beta function were defined and studied in [OzaYlm14], [MiPaJo16].

Chudasama and Dave proposed a unification of the Mittag-Leffler and Wright functions in the following form, with conditions on the parameters ($\mathrm{Re}(\alpha\delta) \geq 0$, $\mathrm{Re}(\beta\delta + \sigma\gamma - \delta/2 - r + 1) > 0$, $\alpha, \sigma \neq 0$, $\mu \in \mathbb{C}$)

$$E^{\sigma,\nu,\gamma}_{\alpha,\beta,\delta}(\mu, r; z) = \sum_{k=0}^{\infty} \frac{(\mu)_{rk}}{\Gamma^{\delta k}(\alpha k + \beta)\Gamma^{\gamma}(\sigma k + \nu)} \frac{z^k}{k!}. \tag{6.6.10}$$

On the basis of the above described results a special H-transform was constructed in [Al-MKiVu02] (see also [KilSai04]). This transform turns out to exhibit many properties similar to the Laplace transform. Moreover, the inverse transform and the operational calculus, which is based on it, are related to the recently introduced multi-index Mittag-Leffler function. Some basic operational properties, complex and real inversion formulas, as well as a convolution theorem, have been derived.

Further generalizations of the Mittag-Leffler functions have been proposed recently in [Pan-K11], [Pan-K12], [Pan-K13].

The $3m$-parametric Mittag-Leffler functions generalizing the Prabhakar three parametric Mittag-Leffler function are introduced by the relation

$$E_{(\alpha_j),(\beta_j)}^{(\gamma_j),m} = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k \dots (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1) \dots \Gamma(\alpha_m k + \beta_m)} \frac{z^k}{k!}, \tag{6.6.11}$$

where $(\gamma)_k$ is the Pochhammer symbol, $\alpha_j, \beta_j, \gamma_j \in \mathbb{C}$, $j = 1, \dots, m$, $\mathrm{Re}\,\alpha_j > 0$. These are entire functions for which the order and the type have been calculated. Representations of the $3m$-parametric Mittag-Leffler functions as generalized Wright functions and Fox H-functions have been obtained. Special cases of novel special functions have been discussed. Composition formulas with Riemann–Liouville fractional integrals and derivatives have been given. Analogues of the Cauchy–Hadamard, Abel, Tauber and Hardy–Littlewood theorems for the three multi-index Mittag-Leffler functions have also been presented.

Pathway type fractional integration of the $3m$-parametric Mittag-Leffler functions is performed in [JaAgKi17].

Two important families of special functions, namely the Bessel functions and Mittag-Leffler functions, and their multi-parametric generalizations are discussed in [Pan-K16]. The following main problems related to the classical and generalized functions of Bessel and Mittag-Leffler type are studied: integral representations and convergence, asymptotic behavior, Tauberian type theorems, completeness of systems of these functions, representations in terms of the generalized Wright function, the Meijer G- and the Fox H-functions with special values of parameters. Special attention is paid to the relations of these functions to the problems of Fractional Calculus.

The extension of the Mittag-Leffler function to a wider set of parameters by using Mellin–Barnes integrals was realized in a series of papers [KilKor05]–[KilKor06c] (see also the paper [Han-et-al09]). The method of extension of different special functions having a representation via a Mellin–Barnes integral has been developed recently.

First of all we have to mention the paper [Han-et-al09]. In this paper the Mittag-Leffler function $E_{\alpha,\beta}(z)$ for negative values of the parameter α is introduced. This definition is based on an analytic continuation of the integral representation

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\mathrm{Ha}} \frac{t^{\alpha-\beta} \mathrm{e}^t}{t^\alpha - z} \mathrm{d}t, \quad z \in \mathbb{C}, \tag{6.6.12}$$

where the path of integration Ha is the Hankel path, a loop starting and ending at $-\infty$, and encircling the disk $|t| \leq |z|^{1/\alpha}$ counterclockwise in the positive sense: $-\pi < \arg t \leq \pi$ on Ha. The integral representation of $E_{\alpha,\beta}(z)$ given in Eq. (6.6.12) can be shown to satisfy the criteria for analytic continuation by noting that for the domain $\alpha > 0$, Eq. (6.6.12) is equivalent to the infinite series representation for the Mittag-Leffler function. This is accomplished by expanding the integrand in Eq. (6.6.12) in powers of z and integrating term-by-term, making use of Hankel's contour integral for the reciprocal of the Gamma function (see, e.g., [NIST]).

To find a defining equation for $E_{-\alpha,\beta}(z)$, the integral representation of the Mittag-Leffler function is rewritten as

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int\limits_{\mathrm{Ha}} \frac{\mathrm{e}^t}{t^\beta - z t^{-\alpha+\beta}} \mathrm{d}t, \quad z \in \mathbb{C}. \tag{6.6.13}$$

By expanding a part of the integrand in Eq. (6.6.13) into partial fractions

$$\frac{1}{t^\beta - z t^{-\alpha+\beta}} = \frac{1}{t^\beta} - \frac{1}{t^\beta - z^{-1} t^{\alpha+\beta}},$$

substituting it into (6.6.13) we get another representation

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int\limits_{\mathrm{Ha}} \frac{\mathrm{e}^t}{t^\beta} \mathrm{d}t - \frac{1}{2\pi i} \int\limits_{\mathrm{Ha}} \frac{\mathrm{e}^t}{t^\beta - z^{-1} t^{\alpha+\beta}} \mathrm{d}t, \quad z \in \mathbb{C} \setminus \{0\}. \tag{6.6.14}$$

Thus we arrive at the following definition of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ for negative values of the parameter α:

$$E_{-\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} - E_{\alpha,\beta}\left(\frac{1}{z}\right). \tag{6.6.15}$$

General properties of $E_{-\alpha,\beta}(z)$ were discussed and many of the common relationships between Mittag-Leffler functions of negative α were compared with their analogous relationships for positive α. A special case of (6.6.15), namely the function $E_{-\alpha}(z)$, has found application in the analysis of the transient kinetics of a two-state model for anomalous diffusion (see [Shu01]). The Mittag-Leffler functions with negative α and the results of this work are likely to become increasingly important as fractional-order differential equations find more applications.

This method of extension was also applied recently in [Kil-et-al12] for the generalized hypergeometric functions. This paper is devoted to the study of a certain function ${}_p\mathcal{F}_q[z] \equiv {}_p\mathcal{F}_q\left[a_1, \cdots, a_p; b_1, \cdots, b_q; z\right]$ (with complex $z \neq 0$ and complex parameters a_j $(j = 1, \cdots p)$ and b_j $(j = 1, \cdots, q)$), represented by the Mellin–Barnes integral. Such a function is an extension of the classical generalized hypergeometric function ${}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z]$ defined for all complex $z \in \mathbb{C}$ when $p < q + 1$ and for $|z| < 1$ when $p = q + 1$. Conditions are given for the existence of ${}_p\mathcal{F}_q[z]$ and of its representations by the Meijer G-function and the H-function. Such an approach allows us to give meaning to the function ${}_p\mathcal{F}_q[z]$ for all ranges of parameters when $p < q + 1$, $p = q + 1$ and $p > q + 1$. The series representations and the asymptotic expansions of ${}_p\mathcal{F}_q[z]$ at infinity and at the origin are established. Special cases have been considered.

In Sect. 6.4 we mainly follow the article [Lav18]. Several other attempts to consider Mittag-Leffler functions and their generalizations as functions of several complex variables have to be mentioned: [SaKaSa11], [GaMaKa13], [Dua18], [Mam18] along with the book [SrGuGo82] devoted to the multivariable analog of the Fox H-function. We also have to mention here the article [YuZha06] in which an $(n+1)$-variable analog of the Mittag-Leffler function is introduced and studied

$$\varepsilon(t, y; \alpha, \beta, \gamma) := t^{\beta-1} E_{\alpha,\beta}(-D|\mathbf{y}|^{\gamma} t^{\alpha}),$$

where $t > 0$ is a time variable, $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, α, β, γ are arbitrary real parameters and D is a physical constant. This function is used in the study of the diffusion-wave equation in $(n+1)$ variables.

Section 6.5 presents in a condensed way the results from [GarPop18], which is devoted to the numerical evaluation of Mittag-Leffler functions with a matrix argument. The corresponding routine implemented in Matlab is also mentioned there. The evaluation of matrix Mittag-Leffler functions is closely related to the evaluation of exponential functions, a problem which has been deeply investigated due its applications to the solution of ordinary differential equations. Several methods have been proposed for matrix exponentials and a comparative discussion is available in the famous review paper by Moler and Van Loan [MolvLoa78] and in its 2003 extension [MolvLoa03]. Unfortunately, not all the methods presented in these two papers can be applied to Mittag-Leffler functions, often due to the absence of the semigroup property, which is exploited in several methods for the computation of the exponential.

The method described in Sect. 6.5 is however based on the work in [DavHig03] which is successive to the two reviews by Moler and Van Loan. Although it exploits some ideas (such as the Schur decomposition) already discussed in these papers, it exploits the more sophisticated Schur–Parlett algorithm, which is presently one of the most powerful methods for matrix computations.

6.7 Exercises

6.7.1 Let I_{0+}^{γ} be the left-sided Riemann–Liouville fractional integral and $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ be either the four-parametric Mittag-Leffler function or its extension. In the case $\alpha_1 > 0$, $\alpha_2 < 0$ calculate the following compositions

$$a)\ \left(I_{0+}^{\gamma} t^{\sigma-1} \mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(\lambda t^{\omega})\right)(x)\ (\omega, \lambda > 0,\ 0 < x \le d < +\infty);$$

$$b) \ \left(I^{\gamma}_{0+}t^{-\sigma}\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(\lambda t^{-\omega})\right)(x) \ (\omega, \lambda > 0, \ 0 < x \le d < +\infty).$$

6.7.2 Let D^{γ}_{0+} be the left-sided Riemann–Liouville fractional derivative and $\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z)$ be either the four-parametric Mittag-Leffler function or its extension. In the case $\alpha_1 > 0$, $\alpha_2 < 0$ calculate the following compositions

$$(a) \ \left(D^{\gamma}_{0+}t^{\sigma-1}\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(\lambda t^{\omega})\right)(x) \ (\omega, \lambda > 0, \ 0 < x \le d < +\infty);$$

$$(b) \ \left(D^{\gamma}_{0+}t^{-\sigma}\mathcal{E}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(\lambda t^{-\omega})\right)(x) \ (\omega, \lambda > 0, \ 0 < x \le d < +\infty).$$

6.7.3 In the case of positive integer $\alpha_1 = m_1$ and $\alpha_2 = m_2$ represent the four-parametric Mittag-Leffler function $E_{m_1,\beta_1;m_2,\beta_2}(z)$ in term of a generalized hypergeometric function ${}_pF_q$ with appropriate p, q.

6.7.4 [KirLuc10, p. 601]. Prove that the Laplace transform of a hyper-Bessel type generalized hypergeometric function ${}_0\Psi_m$ is related to the $2n$-parametric Mittag-Leffler function as follows

$$\left(\mathcal{L}_0\Psi_m\left[\begin{matrix} --- \\ (\beta_1,\alpha_1),\ldots,(\beta_n,\alpha_n)\end{matrix}\right]\right)(s) = \frac{1}{s}E_{(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)}\left(\frac{1}{s}\right).$$

6.7.5 [KirLuc10, p. 603–604]. Let $I^{(\gamma_i),(\delta_i)}_{(\beta_i),n}f(z) = \left[\prod_{i=1}^{n} I^{(\gamma_i),(\delta_i)}_{(\beta_i),n}\right]f(z)$ be the generalized fractional integral of multi-order, where

$$I^{\gamma,\delta}_{\beta}f(z) = \frac{1}{\Gamma(\delta)}\int_0^1 (1-\sigma)^{\delta-1}\sigma^{\gamma}f(z)\sigma^{1/\beta}\mathrm{d}\sigma \ (\delta, \beta > 0, \gamma \in \mathbb{R})$$

is the Erdelyi–Kober fractional integral.

Prove the following formulas

$$(\lambda z)\left(I^{(\beta_i-1),(\alpha_i)}_{(1/\alpha_i),n}E_{(\alpha_i),(\beta_i)}\right)(\lambda z) = E_{(\alpha_i),(\beta_i)}(\lambda z) - \frac{1}{\prod_{i=1}^{n}\Gamma(\beta_i)},$$

$$\left(D^{(\beta_i-1-\alpha_i),(\alpha_i)}_{(1/\alpha_i),n}E_{(\alpha_i),(\beta_i)}\right)(\lambda z) = (\lambda z)E_{(\alpha_i),(\beta_i)}(\lambda z) + \frac{1}{\prod_{i=1}^{n}\Gamma(\beta_i-\alpha_i)}.$$