Chapter 1
Introduction

This book is devoted to an extended description of the properties of the Mittag-Leffler function, its numerous generalizations and their applications in different areas of modern science.

The function $E_\alpha(z)$ is named after the great Swedish mathematician **Gösta Magnus Mittag-Leffler** (1846–1927) who defined it by a power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C},\ \operatorname{Re}\alpha > 0, \tag{1.0.1}$$

and studied its properties in 1902–1905 in five subsequent notes [ML1, ML2, ML3, ML4, ML5-5] in connection with his summation method for divergent series.

This function provides a simple generalization of the exponential function because of the replacement of $k! = \Gamma(k+1)$ by $(\alpha k)! = \Gamma(\alpha k + 1)$ in the denominator of the power terms of the exponential series.

During the first half of the twentieth century the Mittag-Leffler function remained almost unknown to the majority of scientists. They unjustly ignored it in many treatises on special functions, including the most popular (Abramowitz and Stegun [AbrSte72] and its new version "NIST Handbook of Mathematical Functions" [NIST]). Furthermore, there appeared some relevant works where the authors arrived at series or integral representations of this function without recognizing it, e.g., (Gnedenko and Kovalenko [GneKov68]), (Balakrishnan [BalV85]) and (Sanz-Serna [San88]). A description of the most important properties of this function is present in the third volume [ErdBat-3] of the Handbook on Higher Transcendental Functions of the Bateman Project, (Erdelyi et al.). In it, the authors have included the Mittag-Leffler functions in their Chapter XVIII devoted to the so-called miscellaneous functions. The attribution of 'miscellaneous' to the Mittag-Leffler function is due to the fact that it was only later, in the sixties, that it was recognized to belong to

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a more general class of higher transcendental functions, known as Fox H-functions (see, e.g., [MatSax78, KilSai04, MaSaHa10]). In fact, this class was well-established only after the seminal paper by Fox [Fox61]. A more detailed account of the Mittag-Leffler function is given in the treatise on complex functions by Sansone and Gerretsen [SanGer60]. However, the most specialized treatise, where more details on the functions of Mittag-Leffler type are given, is surely the book by Dzherbashyan [Dzh66], in Russian. Unfortunately, no official English translation of this book is presently available. Nevertheless, Dzherbashyan has done a lot to popularize the Mittag-Leffler function from the point of view of its special role among entire functions of a complex variable, where this function can be considered as the simplest non-trivial generalization of the exponential function.

Successful applications of the Mittag-Leffler function and its generalizations, and their direct involvement in problems of physics, biology, chemistry, engineering and other applied sciences in recent decades has made them better known among scientists. A considerable literature is devoted to the investigation of the analyticity properties of these functions; in the references we quote several authors who, after Mittag-Leffler, have investigated such functions from a mathematical point of view. At last, the 2000 Mathematics Subject Classification has included these functions in item 33E12: "Mittag-Leffler functions and generalizations".

Starting from the classical paper of Hille and Tamarkin [HilTam30] in which the solution of Abel integral equation of the second kind

$$\phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int\limits_0^x \frac{\phi(t)}{(x-t)^{1-\alpha}} \mathrm{d}t = f(x), \ \ 0 < \alpha < 1, \quad 0 < x < 1, \tag{1.0.2}$$

is presented in terms of the Mittag-Leffler function, this function has become very important in the study of different types of integral equations. We should also mention the 1954 paper by Barret [Barr54], which was concerned with the general solution of the linear fractional differential equation with constant coefficients.

But the real importance of this function was recognized when its special role in fractional calculus was discovered (see, e.g., [SaKiMa93]). In recent times the attention of mathematicians and applied scientists towards the functions of Mittag-Leffler type has increased, overall because of their relation to the Fractional Calculus and its applications. Because the Fractional Calculus has attracted wide interest in different areas of applied sciences, we think that the Mittag-Leffler function is now beginning to leave behind its isolated life as Cinderella. We like to refer to the classical Mittag-Leffler function as the Queen Function of Fractional Calculus, and to consider all the related functions as her court.

A considerable literature is devoted to the investigation of the analytical properties of this function. In the references, in addition to purely mathematical investigations, we also mention several monographs, surveys and research articles dealing with different kinds of applications of the higher transcendental functions related to the Mittag-Leffler function. However, we have to point out once more that there exists no treatise specially devoted to the Mittag-Leffler function itself. In our opinion, it is now time for a book aimed at a wide audience. This book has to serve both as a textbook for beginners, describing the basic ideas and results in the area, and as a table-book for applied scientists in which they can find the most important facts for applications, and it should also be a good source for experts in Analysis and Applications, collecting deep results widely spread in the special literature. These ideas have been implemented into our plan for the present book. Because of the relevance of the Mittag-Leffler function to the theory and applications of Fractional Calculus we were invited to write a survey chapter for the first volume of the Handbook of Fractional Calculus with Applications [HAND1]. This chapter [GoMaRo19] presents in condensed form the main ideas of our book.

The book has the following structure. It can be formally considered as consisting of four main parts. The first part (INTRODUCTION AND HISTORY) consists of two chapters. The second part (THEORY) presents different aspects of the theory of the Mittag-Leffler function and its generalizations, in particular those arising in applied models. This part is divided into five chapters. The third part (APPLICATIONS) deals with different kinds of applications involving the Mittag-Leffler function and its generalizations. This part is divided into three chapters. Since the variety of models related to the Mittag-Leffler function is very large and rapidly growing, we mainly focus on how to use this function in different situations. We also separate theoretical applications dealing mainly with the solution of certain equations in terms of the Mittag-Leffler function from the more "practical" applications related to its use in modelling. Most of the auxiliary facts are collected in the fourth part consisting of six APPENDICES. The role of the appendices is multi-fold. First, we present those results which are helpful in reading the main text. Secondly, we discuss in part the machinery which can be omitted at the first reading of the corresponding chapter. Lastly, the appendices partly play the role of a handbook on some auxiliary subjects related to the Mittag-Leffler function. In this sense these appendices can be used to further develop the ideas contained in our book and in the references mentioned in it.

Each structural part of the book (either chapter or appendix) ends with a special section "Historical and Bibliographical Notes". We hope that these sections will help the readers to understand the features of the Mittag-Leffler function more deeply. We also hope that acquaintance with the book will give the readers new practical instruments for their research. In addition, since one of the aims of the book is to attract students, we present at the end of each chapter and each appendix a collection of exercises connected with different aspects of the theory and applications. Special attention is paid to the list of references which we have tried to make as complete as possible. Only seldom does the main text give references to the literature, the references are mainly deferred to the notes sections at the end of chapters and appendices. The bibliography contains a remarkably large number of references to articles and books not mentioned in the text, since they have attracted the author's attention over the last few decades and cover topics more or less related to this monograph. In the second edition we have significantly updated the bibliography. The interested reader will hopefully take advantage of this bibliography, enlarging and improving the scope of the monograph itself and developing new results.

Chapter 2 has in a sense a historical nature. We present here a few bibliographical notes about the creator of this book's subject, G.M. Mittag-Leffler. The contents of his pioneering works on the considered function is given here together with a brief description of the further development of the theory of the Mittag-Leffler function and its generalizations.

Chapter 3 is devoted to the classical Mittag-Leffler function (1.0.1). We collect here the main results on the function which were discovered during the century following Mittag-Leffler's definition. These are of an analytic nature, comprising rules of composition and asymptotic properties, and its character as an entire function of a complex variable. Special attention is paid to integral transforms related to the Mittag-Leffler function because of their importance in the solution of integral and differential equations. We point out its role in the Fractional Calculus and its place among the whole collection of higher transcendental functions.

In Chap. 4 we discuss questions similar to those of Chap. 3. This chapter deals with the simplest (and for applications most important) generalizations of the Mittag-Leffler function, namely the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C},\ \mathrm{Re}\,\alpha > 0, \tag{1.0.3}$$

which was deeply investigated independently by Humbert and Agarwal in 1953 [Hum53, Aga53, HumAga53] and by Dzherbashyan in 1954 [Dzh54a, Dzh54b, Dzh54c] (but formally appeared first in the paper by Wiman [Wim05a]).

Chapter 4 presents the theory of two types of three-parametric Mittag-Leffler function. First of all it is the three-parametric Mittag-Leffler function (or Prabhakar function) introduced by Prabhakar [Pra71]

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\Gamma(\alpha k + \beta)} z^k, \alpha, \beta, \gamma \in \mathbb{C},\ \mathrm{Re}\,\alpha, \gamma > 0, \tag{1.0.4}$$

where $(\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$ is the Pochhammer symbol (see (A.17) in Appendix A). This function is now widely used for different applied problems. Another type of three-parametric Mittag-Leffler function is not as well-known as the Prabhakar function (1.0.4). It was introduced and studied by Kilbas and Saigo [KilSai95b] in connection with the solution of a new type of fractional differential equation. This function (the Kilbas–Saigo function) is defined as follows

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k \ (z \in \mathbb{C}), \tag{1.0.5}$$

where

$$c_0 = 1, \quad c_k = \prod_{i=1}^{k-1} \frac{\Gamma(\alpha[im+l]+1)}{\Gamma(\alpha[im+l+1]+1)} \ (k = 1, 2, \cdots), \ \alpha \in \mathbb{C}, \ \mathrm{Re}\,\alpha > 0. \tag{1.0.6}$$

Some basic results on this function are also included in Chap. 5. In the second edition we also include in this chapter another three-parametric generalization of the Mittag-Leffler function, namely, the Le Roy type function

$$F_{\alpha,\beta}^{(\gamma)} := \sum_{k=1}^{\infty} \frac{z^k}{(\Gamma(\alpha k + \beta))^{\gamma}}, \tag{1.0.7}$$

which is a function of a different nature than the other functions in this chapter. The Le Roy type function is a simple generalization of the Le Roy function [LeR00], which appeared as a competitor of the Mittag-Leffler function in the study of divergent series.

By introducing additional parameters one can discover new interesting properties of these functions (discussed in Chaps. 3–5) and extend their range of applicability. This is exactly the case with the generalizations described in this chapter. Together with some appendices, the above mentioned chapters constitute a short course on the Mittag-Leffler function and its generalizations. This course is self-contained and requires only a basic knowledge of Real and Complex Analysis.

Chapter 6 is rooted deeper mathematically. The reader can find here a number of modern generalizations. The ideas leading to them are described in detail. The main focus is on four-parametric Mittag-Leffler functions (Dzherbashyan [Dzh60]) and $2n$-parametric Mittag-Leffler functions (Al-Bassam and Luchko [Al-BLuc95] and Kiryakova [Kir99]). Experts in higher transcendental functions and their applications will find here many interesting results, obtained recently by various authors. These generalizations will all be labelled by the name Mittag-Leffler, in spite of the fact that some of them can be considered for many values of parameters as particular cases of the general class of Fox H-functions. These H-functions offer a powerful tool for formally solving many problems, however by inserting relevant parameters one often arrives at functions whose behavior is easier to handle. This is the case for the Mittag-Leffler functions, and so these functions are often more appropriate for applied scientists who prefer direct work to a detour through a wide field of generalities.

In the second edition we have decided to give a much wider presentation (new Chap. 7) of the classical Wright function

$$\phi(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}, \ \alpha > -1, \beta \in \mathbb{C}. \tag{1.0.8}$$

This function is closely related to the Mittag-Leffler function (especially to the four-parametric Mittag-Leffler function, see, e.g., [GoLuMa99, RogKor10]) and is of great importance for Fractional Calculus too. In spite of this similarity, some properties of the Wright function are not completely analogous to those of the Mittag-Leffler function. Thus, we found it important to discuss here the properties of the Wright function in detail.

The last three chapters deal with applications of the functions treated in the preceding chapters. We start (Chap. 8) with the "formal" (or mathematical) applications of Mittag-Leffler functions. The title of the chapter is "Applications to Fractional Order Equations". By fractional order equations we mean either integral equations with weak singularities or differential equations with ordinary or partial fractional derivatives. The collection of such equations involving Mittag-Leffler functions in their analysis or in their explicit solution is fairly big. Of course, we should note that a large number of fractional order equations arise in certain applied problems. We would like to separate the questions of mathematical analysis (solvability, asymptotics of solutions, their explicit presentation etc.) from the motivation and description of those models in which such equations arise. In Chap. 8 we focus on the development of a special "fractional" technique and give the reader an idea of how this technique can be applied in practice.

Further applications are presented in the two subsequent chapters devoted to mathematical modelling of special processes of interest in the applied sciences. Chapter 9 deals mainly with the role of Mittag-Leffler functions in discovering and analyzing deterministic models based on certain equations of fractional order. Special attention is paid to fractional relaxation and oscillation phenomena, to fractional diffusion and diffusive wave phenomena, to fractional models in dielectrics, models of particle motion in a viscous fluid, and to hereditary phenomena in visco-elasticity and hydrodynamics. These are models in physics, chemistry, biology etc., which by adopting a macroscopic viewpoint can be described without using probabilistic ideas and machinery.

In contrast, in Chap. 10 we describe the role of Mittag-Leffler functions in models involving randomness. We explain here the key role of probability distributions of Mittag-Leffler type which enter into a variety of stochastic processes, including fractional Poisson processes and the transition from continuous time random walk to fractional diffusion.

Our six appendices can be divided into two groups. First of all we present here some basic facts from certain areas of analysis. Such appendices are useful additions to the course of lectures which can be extracted from Chaps. 3–5. The second type of appendices constitute those which can help the reader to understand modern results in the areas in which the Mittag-Leffler function is essential and important. They serve to make the book self-contained.

The book is addressed to a wide audience. Special attention is paid to those topics which are accessible for students in Mathematics, Physics, Chemistry, Biology and Mathematical Economics. Also in our audience are experts in the theory of the Mittag-Leffler function and its applications. We hope that they will find the technical parts of the book and the historical and bibliographical remarks to be a source of new ideas. Lastly, we have to note that our main goal, which we always had in mind during the writing of the book, was to make it useful for people working in different areas of applications (even those far from pure mathematics).