Labeled Network Allocation Problems. An Application to Transport Systems

Encarnación Algaba[1](✉), Vito Fragnelli[2], Natividad Llorca[3], and Joaquin Sánchez-Soriano[3]

[1] Department of Applied Mathematics II and IMUS, University of Seville, Seville, Spain
ealgaba@us.es

[2] Department of Science and Innovative Technologies, University of Eastern Piedmont, Viale Teresa Michel 11, 15121 Alessandria, Italy
vito.fragnelli@uniupo.it

[3] Research Institute CIO and Department of Statistics, Mathematics and Computer Science, University Miguel Hernández of Elche, Elche, Spain
{nllorca,joaquin}@umh.es

Abstract. We deal with networks in which there are more than one arc connecting two nodes. These multiple arcs connecting two nodes are labeled in order to differentiate each other. Likewise, there is traffic or flow among the nodes of the network. The links can have different meanings as such roads, wire connections or social relationships; and the traffic can be for example passengers, information or commodities. When we consider that labels of a network are controlled or owned by different agents then we can analyze how the worth (cost, profit, revenues, power...) associated with the network can be allocated to the agents. The Shapley quota allocation mechanism is proposed and characterized by using reasonable properties. Finally, in order to illustrate the advantages of this approach and the Shapley quota allocation mechanism, an application to the case of the Metropolitan Consortium of Seville is outlined.

Keywords: Allocation mechanisms · Networks · Shapley quota allocation mechanism

1 Introduction and Literature Review

Networks are very often used to graphically represent many different situations from social relationships to real physical problems such as road maps. For example, in Operations Research and Management Science, due to the possibilities offered by such a representation, it is commonplace. We draw attention to the volumes by Ball et al. (1995a and 1995b) for a very informative survey. Networks also play an important role to analyze social and economic problems (see, for example, Megiddo 1978; Sharkey 1995; Slikker and van den Nouweland 2001;

N. T. Nguyen et al. (Eds.): TCCI XXXIV, LNCS 11890, pp. 90–108, 2019.
https://doi.org/10.1007/978-3-662-60555-4_7

Jackson and Zenou 2014; Algaba et al. 2017, 2018). Networks are also used to model the interaction between parts of information or computer systems (see Tardos (2004) and references herein). Therefore, network models are interesting enough to go further in their analysis.

In this paper, we consider networks in which there is one perfectly divisible unit of flow or traffic between different nodes of the network. Somehow, we have multi-flow networks because the flow between each pair of nodes can be considered different from others. Likewise, multiple arcs connecting two nodes are allowed, for this reason they are labeled in order to distinguish each other. The part of the unit of flow between two nodes can go throughout different routes and this is known due to different reasons, for example capacity conditions. In the network there are agents who control different sets of labels. Thus, the set of arcs of the network is partitioned among the agents. This network model is more general than the one introduced by Algaba et al. (2019a) to study the profit allocation problem in horizontal cooperation in public transport systems. We call *labeled networks* the situations described above and *labeled network allocation problem* the problem of allocating the worth associated with the network among the agents controlling its different components.

Moreover, allocation network problems start from an existing network, and often deal with the problem of allocating profits and/or costs for building and/or maintaining the network among the users. For example, Granot and Hojati (1990) study how to allocate the cost of constructing a communication network. They consider two possible situations and for both determine the nucleolus and the Shapley value. Tijs et al. (2006) study the Bird core correspondence for minimum cost spanning tree games. Koster et al. (2001) study the core of standard fixed tree games and prove that the core of these games coincides with the set of all weighted constrained egalitarian solutions. Bjorndal et al. (2004) study standard fixed tree games, for which each vertex unequal to the root is inhabited by exactly one player, and give an alternative proof of that their cores equal their corresponding sets of weighted Shapley values. Gupta et al. (2004) study how to define good cost-sharing mechanisms for single-source network design problems. Maschler et al. (2010) introduce a new algorithm to compute the nucleolus of standard tree games. Bergantiños et al. (2014) introduce an allocation rule to divide the cost of a network which connects the agents to a service provided from a source. Roughgarden and Schrijvers (2014) study network cost-sharing games in which the cost of each edge is shared using the Shapley value. They then study the equilibria of the associated non cooperative games. In all the previous papers, how to distribute the cost of building or maintaining a network that connects the agents to a source that provides a useful service is studied, while in this paper we study how to distribute the known flow that circulates through a network between the agents who control that network. On the other hand, there is a number of papers that study from a game theoretical point of view flow problems in networks. Kalai and Zemel (1982), Curiel et al. (1989) and Reijnierse et al. (1996) study the nonemptiness of the core of different simple flow games. Derks and Tijs (1985, 1986) study the case of multi-commodity flow situations and also study the nonemptiness of the core of associated games. In all these papers, how to distribute the maximum flow that can be obtained from cooperation between the agents that control the network is studied, while in this paper we study how to distribute the flow (or the profit/cost associated with this flow), which has effectively occurred, among the agents that control the network. In addition, we focus on the allocation problem rather than studying the game associated with this problem.

Finally, in order to illustrate how our network model can be applied to a real-life situation, we consider the case of the Metropolitan Consortium of Seville. In particular, a reduced and stylized situation from the real transport system is simulated. In this case, when the members of the consortium cooperate, 405 feasible routes connecting different points of the transport network in the city center are determined, but only 92 of these feasible routes are operated by a single company. Therefore, the advantage of cooperation is clear. For this problem, we propose the Shapley quota allocation which takes into consideration not only the number of routes, but also the traffic flow. Furthermore, this solution is compared with a proportional distribution based on the number of routes in which each company is involved. Some interesting examples which relate to this situation are the following. Fragnelli et al. (2000b) study how to share the profit of a shortest path situation and Sánchez-Soriano (2003 and 2006) proposes two solutions for the profit allocation problem arising from the classic transport problem, based on pairwise distributions. We can also find real-life applications to transport situations in which there is an underlying network. For example, Fragnelli et al. (2000a) and Norde et al. (2002) study how to allocate the cost of a railway line used by different trains, each of which has different needs and requirements; and Sánchez-Soriano et al. (2002) study how to share the cost of a public transport system for students in the area of Alicante.

The rest of the paper is organized as follows. In Sect. 2, we introduce the network allocation problem which we analyze in this paper. In Sect. 3, the Shapley quota allocation mechanism for labeled network problems is characterized by using reasonable properties related to the context of networks. In Sect. 4, the labeled network allocation problem is applied to the Metropolitan Consortium of Seville and the Shapley quota allocation is computed and commented. Section 5 concludes.

2 The Labeled Network Allocation Problem

We consider networks in which there is one perfectly divisible unit of flow or traffic between different nodes of the network. Somehow we have multi-flow networks because the flow between each pair of nodes can be considered different from others. Likewise, multiple arcs connecting two nodes are allowed, for this reason they are labeled in order to distinguish each other. The part of the unit of flow between two nodes can go throughout different routes and this is known due to different reasons, for example capacity conditions. In the network there are agents who control different sets of labels. Thus, the set of arcs of the network is partitioned among the agents. The worth obtained by a subset of agents is the part of the unit of flow that they can obtain by using only their arcs.

Formally, a *labeled graph* is described by the 3-tuple $\mathcal{G} = (V, L, A)$, where V is a finite set of nodes; L is a finite set of labels; and $A \subset V \times V \times L$ is a finite set of labeled directed arcs connecting nodes of V, such that $(i, j, l) \in A$ means that i is the initial node, j is the end node and l is the label. Moreover, we assume that $(i, i, l) \notin A, \forall\, i \in V, \forall\, l \in L$, i.e., loops are not allowed.

A *labeled route* connecting two nodes $i, j \in V$ in a labeled graph (V, L, A) is a sequence of labeled arcs $\{(i, i_1, l_1), (i_1, i_2, l_2), ..., (i_{k-1}, j, l_k)\} \subseteq A$.

Let $\mathcal{R}$ be a set of feasible labeled routes connecting two nodes of V. $\mathcal{R}$ could be a proper subset of the set of all possible labeled routes connecting two nodes of V, $\mathcal{R}(A)$. In this situation, some labeled routes would have been discarded because they are useless or impossible. Let f be a function describing how a (perfectly divisible) unit of flow is distributed throughout all labeled routes, i.e., $f : \mathcal{R}(A) \to [0, 1]$ such that

- $f(r) = 0$, if $r \notin \mathcal{R}$, and $f(r) \geq 0$, if $r \in \mathcal{R}$.
- $\sum_{r \in \mathcal{R}} f(r) = 1$.

Therefore, we assume that the distribution of the unit of flow throughout the graph is perfectly determined. This could occur due to different reasons, for example, capacity constraints, ex-post observation of the traffic, preferences of individuals using the network, a centralized management of the network controlling the traffic throughout the graph, an exogenous condition, etc. Likewise, this function f can be derived from an origin-destination (OD) matrix and it can measure the probability of a particular labeled route to be used in the network.

A labeled network arises when we consider a labeled graph and the flow throughout it. Therefore, a *labeled network* is described by the 3-tuple $\mathcal{N} = (\mathcal{G}, \mathcal{R}, f)$, where $\mathcal{G} = (V, L, A)$ is a labeled graph, $\mathcal{R}$ is a set of feasible labeled routes and f is a distribution of one unit of flow among all feasible labeled routes.

We now provide a simple example to illustrate the different elements of a labeled network.

Example 1. *Consider a simple network as depicted in Fig. 1 with 5 nodes, three labels R (continuous line), B (dotted line) and G (dashed line) and the arcs always go from i to j such that i < j.*

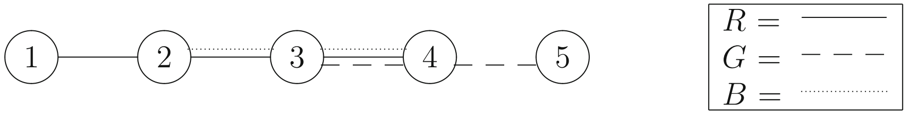

Fig. 1. A simple network

The possible origin-destination (OD) pairs are 1–2, 1–3, 1–4, 1–5, 2–3, 2–4, 2–5, 3–4, 3–5 and 4–5.

We can consider the following set of non-exhaustive labeled routes:

OD	$Feasible\ labeledroute$	$\#labeledroute$
$1 \to 2$	$(1,2,R)$	1
$1 \to 3$	$(1,2,R)(2,3,R)$	2
	$(1,2,R)(2,3,B)$	3
$1 \to 4$	$(1,2,R)(2,3,R)(3,4,R)$	4
	$(1,2,R)(2,3,B)(3,4,B)$	5
	$(1,2,R)(2,3,R)(3,4,G)$	6
	$(1,2,R)(2,3,B)(3,4,G)$	7
$1 \to 5$	$(1,2,R)(2,3,R)(3,4,R)(4,5,G)$	8
	$(1,2,R)(2,3,B)(3,4,B)(4,5,G)$	9
$2 \to 3$	$(2,3,R)$	10
	$(2,3,B)$	11
$2 \to 4$	$(2,3,R)(3,4,R)$	12
	$(2,3,B)(3,4,B)$	13
	$(2,3,R)(3,4,B)$	14
	$(2,3,R)(3,4,G)$	15
	$(2,3,B)(3,4,G)$	16
$2 \to 5$	$(2,3,R)(3,4,G)(4,5,G)$	17
	$(2,3,B)(3,4,G)(4,5,G)$	18
	$(2,3,R)(3,4,B)(4,5,G)$	19
$3 \to 4$	$(3,4,R)$	20
	$(3,4,B)$	21
	$(3,4,G)$	22
$3 \to 5$	$(3,4,G)(4,5,G)$	23
	$(3,4,R)(4,5,G)$	24
	$(3,4,B)(4,5,G)$	25
$4 \to 5$	$(4,5,G)$	26

Of course, we may have a more detailed representation, increasing the number of feasible labeled routes; e.g., considering the feasible labeled route labeled 14 in which the flow goes from node 2 to node 4, using first the arc $(2,3)$ *labeled with R and then the arc* $(3,4)$ *labeled with B, it is possible to add another feasible labeled route using first the arc* $(2,3)$ *labeled with B and then the arc* $(3,4)$ *labeled with R. On the other hand, we may reduce the number of feasible labeled routes reducing the number of changes; e.g., referring to the OD* $2 \to 4$*, we may consider only the feasible labeled routes labeled 12 and 13, supposing that when flow goes on arcs with a particular label it does not change if it is not strictly needed.*

Now, we can consider that the unit of flow is distributed among the feasible labeled routes as follows:

#labeled route	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(r)$	0.03	0.04	0.04	0.03	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.04	0.04

#labeled route	14	15	16	17	18	19	20	21	22	23	24	25	26
$f(r)$	0.04	0.04	0.04	0.05	0.05	0.05	0.04	0.04	0.04	0.05	0.05	0.05	0.05

In view of function f, we can say that a 5% of the flow goes along labeled route 26, i.e., between nodes 4 and 5, or that labeled routes from 12 to 16 that connect nodes 2 and 4 are equally used.

Now, let us consider that there are several agents controlling different arcs of the network and labels are used to identify the agents who control the arcs of the network. In this sense, in Example 1 we would have up to three agents corresponding to the three labels R, B and G. For example, in a transport network these labels can represent different companies providing the passenger transport service between different cities or stops within the same city; or in a computer network these labels can represent links belonging to different Internet Service Providers (ISPs). If we are interested in knowing the relevance or contribution of each of the agents involved in the network, one possibility is to allocate the worth associated with the network in a *fair* way, i.e. determining which part of the worth associated with the network can be reasonably or in a fair way attributed to each agent. Formally, we introduce the following definition of labeled network allocation problem.

Definition 1. *Let $\mathcal{N} = (\mathcal{G}, \mathcal{R}, f)$, where $\mathcal{G} = (V, L, A)$, be a labeled network. A* labeled network allocation problem *associated with $\mathcal{N}$ is given by the 3-tuple $(\mathcal{N}, N, \mathcal{L})$ where:*

- *$\mathcal{N}$ is the labeled network.*
- *$N = \{1, 2, \ldots, n\}$ is the set of agents who control different arcs of the network.*
- *$\mathcal{L} = \{L_1, L_2, ..., L_n\}$ is a partition of the set of labels L, such that each agent $i \in N$ controls the subset L_i of labels.*

If each agent controls exactly one label, then the problem is called *simple labeled network allocation problem.*

We denote by $L(r)$ the set of different labels in labeled route r.

Example 2. *In order to illustrate the labeled network allocation problem, we refer to the situation in Example 1 where we consider that there are two agents, $\{A1, A2\}$ controlling labels $\{R, G\}$ and $\{B\}$ respectively. The next table shows the labeled routes controlled by each agent.*

Agent i	*Feasible labeled route $r \in \mathcal{R} : L(r) \subseteq L_i$*
$A1$	$1, 2, 4, 6, 8, 10, 12, 15, 17, 20, 22, 23, 24, 26$
$A2$	$11, 13, 21$

However, labeled routes $\{3, 5, 7, 9, 14, 16, 18, 19, 25\}$ need both agents to be completely controlled.

This labeled network allocation problem is not simple, because agent A1 controls two labels.

We are now interested in how to allocate the unit of flow among all agents controlling the different arcs of the network. This allocation give us the quotas or proportions of the unit of flow which are assigned or attributed to each agent.

These quotas will also measure, in some way, the contribution of each agent to the network.

Let $\mathcal{LNA}^N$ be the set of all labeled network allocation problems with set of agents N, a *flow quota allocation mechanism* for $\mathcal{LNA}^N$ is a function $\gamma : \mathcal{LNA}^N \to \mathbb{R}^N$ such that

1. $\gamma_i(\mathcal{N}, N, \mathcal{L}) \geq 0, \ \forall \ i \in N$,
2. $\sum_{i \in N} \gamma_i(\mathcal{N}, N, \mathcal{L}) = 1$.

Considering this definition, we can introduce many different quota allocation mechanisms. A simple possibility is the following. The flow of each feasible labeled route is divided equally among all agents involved. Summing up the results for agents, we obtain the amount of flow assigned to each agent. It is intuitive because the agents that are not involved in the flow of a given feasible labeled route do not take part in the division of the flow and those that are needed for determining the labeled route are rewarded equally, as each of them is equally important for that particular feasible labeled route. This procedure can be related to the well-known Shapley value (Shapley 1953) (see Algaba et al. 2019b), in the same way as in Algaba et al. (2019a).

Let $(\mathcal{N}, N, \mathcal{L})$ be a labeled network allocation problem, the *Shapley quota allocation mechanism* is defined for each $i \in N$ as follows:

$$\phi_i(\mathcal{N}, N, \mathcal{L}) = \sum_{r \in \mathcal{R}} \frac{f(r)}{\sum_{j \in N} \delta_j(L(r))} \delta_i(L(r)),$$

where $\delta_i(L(r)) = 1$, if $L(r) \cap L_i \neq \varnothing$, and $\delta_i(L(r)) = 0$, otherwise.

This flow quota allocation mechanism distributes the total flow 1 by labeled routes and within each labeled route equally among all agents involved in it. Furthermore, this allocation can be seen as a mixture of proportional allocation (of common revenue to the paths) and equal sharing (of path-revenue to providers). An alternative could be a proportional allocation of common revenue to the paths and proportional sharing of path-revenue to the arcs of each provider involved. Thus, we can define the *doubly proportional quota allocation mechanism* as follows:

$$\psi_i(\mathcal{N}, N, \mathcal{L}) = \sum_{r \in \mathcal{R}} \frac{f(r)}{\sum_{j \in N} \varepsilon_j(r)} \varepsilon_i(r), \ \ i \in N,$$

where $\varepsilon_i(r) = |\{(i_k, j_k, l_k) \in r : l_k \in L_i\}|$, i.e. the number of labeled arcs of route r whose labels belong to L_i.

Finally, we could also take into account different costs per route, different profits per route or different ticket prices by simply multiplying the function $f(r)$ in the numerator of the quota allocation mechanism by the corresponding cost, profit or ticket price.

3 Properties and Characterization

In this section, we deal with the problem of providing a set of properties that allow for the characterization of the Shapley quota allocation. In this context, we consider efficient solutions in order to allocate the full flow. The characterization of a solution is important because each proposed mechanism will give a different allocation of the flow. Consequently, it is not possible for all agents to agree on any solution. The axiomatic approach allows us to switch from a choice based on the amount each agent receives (or pays) to a choice focused on the fairness of the solution. Another positive aspect for studying the properties of a solution is that they can be used to explain the advantages of a solution more convincingly and so make it easier for the agents to accept. Some properties are the following:

- *No flow controlled property* (NFC): Let γ be a flow quota allocation mechanism defined on $\mathcal{LNA}^N$. It is said to satisfy the no flow controlled property, if for all $i \in N$, such that for all $r \in \mathcal{R}$, $L(r) \cap L_i = \varnothing$, then $\gamma_i = 0$.
- *Equal treatment of equals property* (ETE): Let γ be a flow quota allocation mechanism defined on $\mathcal{LNA}^N$. It is said to satisfy the equal treatment of equals property, if for all $i, j \in N$, such that for all $r \in \mathcal{R}$, $L(r) \cap L_i \neq \varnothing$ if and only if $L(r) \cap L_j \neq \varnothing$, then $\gamma_i = \gamma_j$.

The meaning of the no flow controlled property (NFC) is that agents which do not control any flow, will not be relevant to cooperation. The equal treatment of equals property (ETE) means that those agents which are symmetric, with respect to the number of labeled routes they participate in, must receive the same. Both properties seem reasonable and fair in the context of this problem.

An interesting question is how to merge two different labeled networks when both have the same set of labels and agents controlling the same labels. They could have different nodes, arcs and feasible labeled routes, but the merging of the two systems should provide a new labeled network involving all the structural elements of both. Additionally, one important aspect is that each system can have a different weight, relevance or size in terms of flow, so we should take this into account when merging both systems.

Let $(\mathcal{N}, N, \mathcal{L})$ and $(\mathcal{N}', N, \mathcal{L})$ be two labeled network allocation problems with set of agents N, and labeled networks $\mathcal{N} = (\mathcal{G}, \mathcal{R}, f)$ and $\mathcal{N}' = (\mathcal{G}', \mathcal{R}', f')$, such that $\mathcal{G} = (V, L, A)$ and $\mathcal{G}' = (V', L, A')$, and with relative weights w and w' ($w + w' = 1; w, w' > 0$), then a *merging* of both networks

$$(\mathcal{N}'', N, \mathcal{L}) \equiv (\mathcal{N}, N, \mathcal{L}) \oplus (\mathcal{N}', N, \mathcal{L})$$

is defined as follows:

- $\mathcal{G}'' = (V \cup V', L, A \cup A')$
- $\mathcal{R}' \cup \mathcal{R} \subseteq \mathcal{R}''$
- $f'' = wf + w'f'$ is given by

$$f''(r) = \begin{cases} wf(r) + w'f'(r), & \text{if } r \in \mathcal{R} \cap \mathcal{R}' \\ wf(r), & \text{if } r \in \mathcal{R} - \mathcal{R}' \\ w'f'(r), & \text{if } r \in \mathcal{R}' - \mathcal{R} \\ 0, & \text{if } r \in \mathcal{R}'' - (\mathcal{R} \cap \mathcal{R}') \end{cases} \qquad \forall\, r \in \mathcal{R}''$$

It is not difficult to check that all elements are well-defined[1]. Likewise, $f''(r) = 0, \quad \forall r \in \mathcal{R}''(A'') \backslash \mathcal{R}''$. Therefore, the definition of f'' implies that new possible labeled routes do not generate new flow when merging the labeled networks.

The interpretation of the merging operation is that we construct a new labeled network whose graph structure consists of all the structural elements of both graphs and the weights are used to redefine the distribution of the unit of flow adapted to the new structure.

We now introduce the following property for solutions in labeled networks with set of agents N.

- *Weighted merging property* (WM): Let $(\mathcal{N}, N, \mathcal{L})$ and $(\mathcal{N}', N, \mathcal{L})$ be two labeled network allocation problems with set of agents N, and labeled networks $\mathcal{N} = (\mathcal{G}, \mathcal{R}, f)$ and $\mathcal{N}' = (\mathcal{G}', \mathcal{R}', f')$, such that $\mathcal{G} = (V, L, A)$ and $\mathcal{G}' = (V', L, A')$, and with relative weights w and w'. And let γ be a flow quota allocation mechanism defined on $\mathcal{LNA}^N$. It is said to satisfy the weighted merging property, if the following holds

$$\gamma((\mathcal{N}, N, \mathcal{L}) \oplus (\mathcal{N}', N, \mathcal{L})) = w\gamma(\mathcal{N}, N, \mathcal{L}) + w'\gamma(\mathcal{N}', N, \mathcal{L}).$$

Proposition 1. *The Shapley quota allocation mechanism satisfies NFC, ETE and WM.*

Proof. It is straightforward to prove that the Shapley quota allocation mechanism satisfies NFC and ETE. Let $(\mathcal{N}, N, \mathcal{L})$ and $(\mathcal{N}', N, \mathcal{L})$ be two labeled network allocation problems with set of agents N, and labeled networks $\mathcal{N} = (\mathcal{G}, \mathcal{R}, f)$ and $\mathcal{N}' = (\mathcal{G}', \mathcal{R}', f')$, such that $\mathcal{G} = (V, L, A)$ and $\mathcal{G}' = (V', L, A')$, and with relative weights w and w'. On the one hand, we have for every $i \in N$

$$\phi_i(\mathcal{N}, N, \mathcal{L}) = \sum_{r \in \mathcal{R}} \frac{f(r)}{\sum_{j \in N} \delta_j(L(r))} \delta_i(L(r)),$$

and

$$\phi_i(\mathcal{N}', N, \mathcal{L}) = \sum_{r \in \mathcal{R}'} \frac{f'(r)}{\sum_{j \in N} \delta_j(L(r))} \delta_i(L(r)).$$

On the other hand, for every $(\mathcal{N}'', N, \mathcal{L}) \equiv (\mathcal{N}, N, \mathcal{L}) \oplus (\mathcal{N}', N, \mathcal{L})$, we have that

$$\phi_i(\mathcal{N}'', N, \mathcal{L}) = \sum_{r \in \mathcal{R}''} \frac{f''(r)}{\sum_{j \in N} \delta_j(L(r))} \delta_i(L(r))$$

[1] The difference of two sets A and B in the definition of $f''(r)$ is as follows: $A - B = A \setminus (A \cap B)$.

$$= \sum_{r\in\mathcal{R}\cap\mathcal{R}'} \frac{wf(r)+w'f'(r)}{\sum_{j\in N}\delta_j(L(r))}\delta_i(L(r)) + \sum_{r\in\mathcal{R}-\mathcal{R}'} \frac{wf(r)}{\sum_{j\in N}\delta_j(L(r))}\delta_i(L(r))$$
$$+ \sum_{r\in\mathcal{R}'-\mathcal{R}} \frac{w'f'(r)}{\sum_{j\in N}\delta_j(L(r))}\delta_i(L(r))$$
$$= \sum_{r\in\mathcal{R}} \frac{wf(r)}{\sum_{j\in N}\delta_j(L(r))}\delta_i(L(r)) + \sum_{r\in\mathcal{R}'} \frac{w'f'(r)}{\sum_{j\in N}\delta_j(L(r))}\delta_i(L(r))$$
$$= w\sum_{r\in\mathcal{R}} \frac{f(r)}{\sum_{j\in N}\delta_j(L(r))}\delta_i(L(r)) + w'\sum_{r\in\mathcal{R}'} \frac{f'(r)}{\sum_{j\in N}\delta_j(L(r))}\delta_i(L(r))$$
$$= w\phi_i(\mathcal{N},N,\mathcal{L}) + w'\phi_i(\mathcal{N}',N,\mathcal{L}).$$

Therefore, the statement holds. □

Theorem 1. *The Shapley quota allocation mechanism is the unique flow quota allocation mechanism satisfying NFC, ETE and WM on $\mathcal{LNA}^N$.*

Proof. Let $(\mathcal{N},N,\mathcal{L})$ be a labeled network allocation problem with set of agents N, labeled networks $\mathcal{N}=(\mathcal{G},\mathcal{R},f)$, such that $\mathcal{G}=(V,L,A)$ and let γ be a quota solution satisfying NFC, ETE and WM on $\mathcal{LNA}^N$.

Let $|\mathcal{R}|=1$, i.e., $\mathcal{R}$ contains a single labeled route r and $f(r)=1$. Since γ satisfies NFC we have that for all i such that $L(r)\cap L_i=\varnothing$, $\gamma_i(\mathcal{N},N,\mathcal{L})=0$. Now as γ satisfies ETE, we obtain that for all i such that $L(r)\cap L_i\neq\varnothing$, $\gamma_i(\mathcal{N},N,\mathcal{L})=\frac{1}{\sum_{j\in N}\delta_j(L(r))}$. Now by definition of the Shapley quota allocation mechanism ϕ, we have that

$$\phi_i(\mathcal{N},N,\mathcal{L}) = \begin{cases}\frac{1}{\sum_{j\in N}\delta_j(L(r))}, & \text{if } L(r)\cap L_i\neq\emptyset\\ 0, & \text{if } L(r)\cap L_i=\emptyset\end{cases}, \quad \forall\, i\in N.$$

Therefore, $\gamma=\phi$.

Let us suppose by induction that $\gamma=\phi$ for every $(\mathcal{N},N,\mathcal{L})\in\mathcal{LNA}^N$ such that $|\mathcal{R}|\leq m-1, m>1$, and let us consider $(N,\mathcal{G},\mathcal{R},f)\in\mathcal{LN}^N$ such that $|\mathcal{R}|=m>1$.

We choose one labeled route $r\in\mathcal{R}$, and we construct the two following labeled networks with set of agents N:

- $\mathcal{N}^1=(\mathcal{G}^1,\mathcal{R}^1,f^1)$:
  1. $\mathcal{G}^1=\mathcal{G}$.
  2. $\mathcal{R}^1=\mathcal{R}-\{r\}$.
  3. $f^1(s)=\dfrac{f(s)}{1-f(r)},\ \forall\, s\in\mathcal{R}^1$.

  Since $|\mathcal{R}|\geq 2$, $p^1(s)$ is well-defined because $0<f(r)<1$.
- $\mathcal{N}^2=(\mathcal{G}^2,\mathcal{R}^2,f^2)$:
  1. $\mathcal{G}^2=\mathcal{G}$.
  2. $\mathcal{R}^2=\{r\}$.
  3. $f^2(r)=1,\ r\in\mathcal{R}^2$.

If we consider the merging of $(\mathcal{N}^1, N, \mathcal{L})$ and $(\mathcal{N}^2, N, \mathcal{L})$ with $\mathcal{R} = \mathcal{R}^1 \cup \mathcal{R}^2$ and relative weights $1 - f(r)$ and $f(r)$ respectively, then applying the definition of merging of two labeled networks we obtain that

$$(\mathcal{N}, N, \mathcal{L}) \equiv (\mathcal{N}^1, N, \mathcal{L}) \oplus (\mathcal{N}^2, N, \mathcal{L}).$$

Now, by the induction hypothesis, we have that

$$\gamma(\mathcal{N}^1, N, \mathcal{L}) = \phi(\mathcal{N}^1, N, \mathcal{L}),$$

$$\gamma(\mathcal{N}^2, N, \mathcal{L}) = \phi(\mathcal{N}^2, N, \mathcal{L}).$$

By Proposition 1, ϕ satisfies WM and by hypothesis γ satisfies WM, then we have the following chain of equalities:

$$\begin{aligned}\gamma(\mathcal{N}, N, \mathcal{L}) &= \gamma((\mathcal{N}^1, N, \mathcal{L}) \oplus (\mathcal{N}^2, N, \mathcal{L})) = (1 - f(r))\gamma(\mathcal{N}^1, N, \mathcal{L}) + f(r)\gamma(\mathcal{N}^2, N, \mathcal{L}) \\ &= (1 - f(r))\phi(\mathcal{N}^1, N, \mathcal{L}) + f(r)\phi(\mathcal{N}^2, N, \mathcal{L}) = \phi((\mathcal{N}^1, N, \mathcal{L}) \oplus (\mathcal{N}^2, N, \mathcal{L})) \\ &= \phi(\mathcal{N}, N, \mathcal{L}).\end{aligned}$$

Therefore $\gamma = \phi$, and the result follows. □

Theorem 2. *The properties NFC, ETE and WM are logically independent.*

Proof. (1) The egalitarian solution satisfies ETE and WM but not NFC. Indeed, the egalitarian solution is defined as follows:

$$\varepsilon_i(\mathcal{N}, N, \mathcal{L}) = \frac{1}{|N|}, \quad \forall\, i \in N.$$

It can immediately be proved that the egalitarian solution does not satisfy NFC. Furthermore, it trivially satisfies ETE because all agents receive the same. Since the egalitarian solution is a constant solution on $\mathcal{LNA}^N$, it also satisfies WM.

(2) Let us consider the following version of the egalitarian solution:

$$\alpha_i(\mathcal{N}, N, \mathcal{L}) = \begin{cases} \frac{1}{|K|}, & \text{if } i \in K \subseteq N \\ 0, & \text{otherwise} \end{cases}, \quad \forall\, i \in N,$$

where $K = \{i \in N : \exists\, r \in \mathcal{R} \text{ s.t. } L(r) \cap L_i \neq \emptyset\}$. It is easy to prove that this solution satisfies NFC and ETE. However, it does not satisfy WM. Let us consider the following two labeled networks with set of agents $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{L} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$:

- $\mathcal{N}^1 = (\mathcal{G}, \mathcal{R}^1, f^1) : \mathcal{R}^1 = \{r\}; L(r) = \{1, 2, 3\}; f^1(r) = 1,$
- $\mathcal{N}^2 = (\mathcal{G}, \mathcal{R}^2, f^2) : \mathcal{R}^2 = \{s\}; L(s) = \{4, 5\}; f^2(s) = 1,$

with relative weights equal to $\frac{1}{2}$. Then we have that

$$\alpha((\mathcal{N}^1, N, \mathcal{L}) \oplus (\mathcal{N}^2, N, \mathcal{L})) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right),$$

while $\alpha(\mathcal{N}^1, N, \mathcal{L}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right)$ and $\alpha(\mathcal{N}^2, N, \mathcal{L}) = \left(0, 0, 0, \frac{1}{2}, \frac{1}{2}\right)$. Thus, $\frac{1}{2}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right) + \frac{1}{2}\left(0, 0, 0, \frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}\right)$. Hence α does not satisfies WM.

(3) Let us consider the solution defined, for each $i \in N$, as follows:

$$\varphi_i(\mathcal{N}, N, \mathcal{L}) = \sum_{r \in \mathcal{R}} \frac{i \cdot f(r)}{\sum_{j \in N} j \cdot \delta_j(L(r))} \, \delta_i(L(r)).$$

This solution satisfies NFC and also WM but not ETE. □

4 A Stylized Application to the Metropolitan Consortium of Seville

The purpose of this section is to illustrate how labeled networks can describe real-life situations such as a public transport system. Furthermore, the Shapley quota allocation mechanism is not difficult to compute and is easy to apply, because we only need to know the labeled routes and the distribution of one unit of flow among all labeled routes. Of course, in real-life situations both elements should be updated from time to time.

In particular, we apply labeled networks and the Shapley quota allocation mechanism to the Transport Consortium of Seville in which collaborate several transport companies to offer a better service to the passengers, in particular the Consortium offers travel tickets which can be used in whatever of the transport companies including transfers between the different companies. This transport system covers six zones (A,...,F) and connects different points of Seville and its metropolitan area. This network uses three modes: trains, metro and buses. In fact, there are 5 lines operated by trains, 3 by underground and 64 by buses which correspond to seven different companies. The complete map can be found at the web page www.consorciotransportes-sevilla.com (see also Fig. 2). We should mention that the urban buses are not included in the transport consortium of Seville.

A public transport system as described above can be modeled by means of a simple labeled network allocation problem as follows:

- The labeled graph $\mathcal{G}$:
  - $V = \{$ the set of all stops $\}$.
  - $L = \{$ the set of all companies operating in the transport system $\}$.
  - $A = \{$ each connection between two consecutive stops operated by each transport company $\}$.

- The set of agents $N = \{$ the set of all companies operating in the transport system $\}$.
- The partition of the labels, $\mathcal{L} = \{\{\text{company } 1\}, \{\text{company } 2\}, \ldots, \{\text{company } n\}\}$.
- The feasible labeled routes $\mathcal{R} = \{$ the set of all labeled routes used by passengers $\}$.
- The distribution of one unit of the flow $f(r) =$ proportion of all passengers using *labeled route* r.

If we consider that the price of one ticket is constant, which is true within the same zone, then we can consider that this price is exactly 1, and f measures the proportion of profit derived of the use of each labeled route. Thus, the application of a flow quota allocation mechanism provides the proportion of the ticket price that corresponds to each agent when they cooperate.

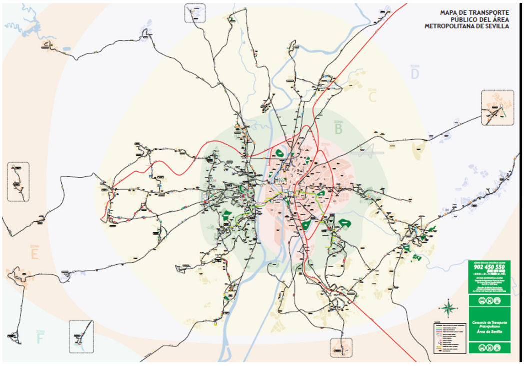

Fig. 2. Map of the metropolitan consortium of Seville

The real network is too large to be illustrated in the paper, so we have decided to give our attention to a limited problem in which we consider only zone A which corresponds to the city center of Seville. Moreover, we aggregate different stops, if they serve the same area and different lines, if they have common labeled routes and common stops in the area under consideration. We do not consider those companies that do not operate at all in the area or if the service they provide is limited to few stops. This leads us to consider three bus companies, Carjema (B1), Casal (B2), and Los Amarillos (B3); underground, Metro (M), and trains, Renfe (T). The resulting network has 15 nodes and 15 edges some of which are connected by different companies (see Fig. 3). All these simplifications do not affect the way in which the Shapley quota allocation mechanism would be applied, therefore, this example adequately illustrates how this mechanism would be calculated in a real-life situation. In fact, the computational complexity of the problem lies in the algorithm for determining all possible routes and not

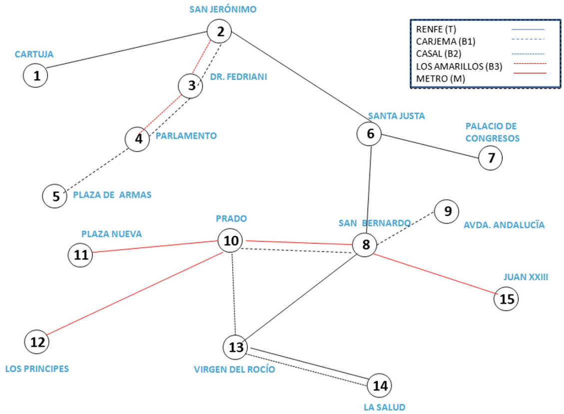

Fig. 3. Simplified map of the Zone A of the metropolitan network of Seville

in the application of the quota allocation mechanism. But the first is not the subject of this paper.

Starting from the simplified network, we compute the feasible labeled routes according to the following hypotheses: a passenger enters the origin node and takes the first public transport available traveling towards the destination. The passenger remains on this public transport as long as possible. When the passenger leaves that public transport s/he then takes again the first public transport available going to the destination, and so on until reaching the final destination. Following this procedure, we obtained a total of 405 feasible labeled routes (see Table 1). The number of labeled routes operated by each company are shown in Table 2

For instance, let us consider a passenger who wants to go from Plaza de Armas (node 5) to San Jerónimo (node 2). We assume that there is just one feasible labeled route, because in Plaza de Armas the only transport is by Casal company (B2) bus that goes directly to the final destination of San Jerónimo. In other words, we do not consider as feasible the labeled route that uses the B2 bus from Plaza de Armas to Parlamento (node 4) and the Carjema company (B1) bus from Parlamento to San Jerónimo, and the labeled route that uses the B2 bus from Plaza de Armas to Dr. Fedriani (node 3) and B1 bus from Dr. Fedriani to San Jerónimo. On the other hand, for the path in the opposite direction, from San Jerónimo to Plaza de Armas, the direct labeled route with B2 bus and the labeled route that uses B1 bus from San Jerónimo to Parlamento and B2 bus from Parlamento to Plaza de Armas, both are feasible, depending on which bus arrives first at San Jerónimo, but we do not consider as feasible the labeled route that uses the B1 bus from San Jerónimo to Dr. Fedriani and the B2 bus from Dr. Fedriani to Plaza de Armas because Dr. Fedriani is not the last possible stop for bus B1.

Table 1. # of labeled routes for each pair O-D

OD	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	–	1	2	2	2	1	1	1	1	2	2	2	1	1	1
2	1	–	2	2	2	1	1	1	1	2	2	2	1	1	1
3	2	2	–	2	2	2	2	2	2	4	3	3	2	2	2
4	2	2	2	–	1	2	2	2	2	4	3	3	2	2	2
5	1	1	1	1	–	1	1	1	1	2	1	1	1	1	1
6	1	1	2	2	2	–	1	1	1	2	2	2	1	1	1
7	1	1	2	2	2	1	–	1	1	2	2	2	1	1	1
8	1	1	2	2	2	1	1	–	1	2	2	2	3	3	1
9	1	1	2	2	2	1	1	1	–	1	1	1	2	2	1
10	2	2	4	4	4	2	2	2	2	–	1	1	3	3	2
11	1	1	2	2	2	1	1	1	1	1	–	1	2	2	1
12	1	1	2	2	2	1	1	1	1	1	1	–	2	2	1
13	3	3	6	6	6	3	3	3	3	3	3	3	–	2	3
14	4	4	7	7	8	4	4	4	4	4	3	4	2	–	4
15	1	1	2	2	2	1	1	1	1	1	1	1	2	2	–

Table 2. The labeled routes and the companies

Companies	M	T	B1	B2	B3
# of labeled routes involving each company	180	315	89	203	88
# of labeled routes operated by a unique company	20	42	6	18	6

After the previous modifications to the network, we have simulated the O-D matrix, taking into consideration the population distribution in the metropolitan area and available data of passengers in 2014.

Based on data provided by the Metropolitan Consortium of Seville, we have assigned a weight to each node that represents its relevance in the traffic (see Table 3) and consider an average of 13.500.000 passengers per year. We have simulated the O-D matrix, where the average of passengers in each O-D has been calculated proportional to the products of the assigned weights to each node in the pair O-D. For each pair O-D, we apply a normal distribution with the average previously obtained and a relative standard deviation (coefficient of variation) of 7.5%. Next, we have generated random numbers for the matrix O-D. Using these numbers and the normal distribution we have obtained a traffic flow throughout the transport network (see Table 4). We would like to emphasize that we have implemented a program on spreadsheets to simulate the traffic of passengers taking as input the stop weights, the relative standard deviation and the average of total passengers in order to calculate the Shapley quota allocation mechanism and a proportional solution according to the number of

Table 3. Weights of the nodes

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Weight	1	3	3	3	10	8	2	8	2	10	5	5	8	4	2

Table 4. Number of passengers for each pair O-D

OD	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	–	7904	8771	8176	29245	22734	5617	20740	5306	25433	13508	14606	20544	9116	5135
2	8145	–	23641	25489	85749	64302	14529	61720	13393	75368	36430	39825	66047	32446	16654
3	7102	19765	–	22741	84691	69579	16014	62701	17586	82785	38789	48299	65673	30181	15968
4	8969	24321	23690	–	81794	56529	17777	71428	16933	84858	43108	41902	65342	28700	17068
5	25786	89615	89606	76253	–	222254	57499	218904	49359	274017	138958	145169	218926	103733	46576
6	20761	64905	64244	66932	221379	–	41907	157638	47267	214889	115142	112964	158628	80630	43869
7	5110	16674	18278	16230	54424	43022	–	40096	10441	60367	30912	31284	36590	21881	10431
8	22536	55140	64589	62622	237452	173367	42101	–	44490	249182	108662	120792	175477	81645	40405
9	6408	16724	17832	16488	52697	42572	12117	50059	–	61691	30058	27738	44548	21937	10662
10	26936	67826	78332	91698	305185	207276	59092	207887	58424	–	150398	138593	214263	106047	55600
11	12315	41216	44813	42168	144143	107359	28061	116516	30423	118116	–	63994	107589	54928	30030
12	15915	37891	38309	40060	138808	103039	25341	106975	29109	128213	65672	–	103663	48972	28294
13	23586	61210	57843	59137	214729	190465	45843	193877	34290	202610	108369	109267	–	89055	45269
14	11006	31567	33569	30413	95339	86084	24804	94003	21720	127600	60320	54098	89252	–	24125
15	5139	16814	15119	16717	51685	48327	10831	42964	9987	58279	26595	27805	37667	21015	–

labeled routes. We would like to point out that if a real traffic matrix is available then it can be easily imported.

In our problem the number of labeled routes is fixed. Notice that this is not a strong assumption since the licenses are conceded to the companies for a long period of time. Although the computation of the number of feasible labeled routes is an NP-hard problem, this is initially solved by the metropolitan consortium and it is beyond the scope of this paper.

Once simulation is applied to the previous data, the Shapley quota allocation mechanism is obtained together with a proportional solution based on the number of labeled routes in which every company operates (see Table 2). These solutions, as a percentage of the total price of the ticket, are reported in Tables 5 and 6.

Table 5. The Shapley quota allocation mechanism

M	T	B1	B2	B3
24.98	37.82	5.34	25.17	6.69

In Tables 5 and 6, we can observe that the results are quite different. For instance, Company B1 would obtain a quota of 5.34% with the Shapley quota allocation mechanism and 10.17% considering the proportional solution. We

Table 6. Proportional quota according to the number of labeled routes

M	T	B1	B2	B3
20.57	36.00	10.17	23.20	10.06

highlight that, whereas the proportional quota only takes into account the number of labeled routes, the Shapley quota allocation mechanism is based not only on the number of labeled routes, but also on the traffic flow in the network. In this case, this solution is also easily computed which is a relevant advantage when working with examples from the real world and has a clear interpretation. Note that we have assumed in this case, that if there is more than one labeled route connecting the same pair OD, we consider them equally probable. However, different probabilities could have been considered as shown in the theoretical part. In that case, only one additional input should be added to the program which can easily be implemented.

5 Conclusions

In this paper, we have introduced a class of network problems, the labeled network allocation problems. Furthermore, we have studied and characterized the so-called Shapley quota allocation mechanism, which is based on the structure of the Shapley value but directly used with the labeled network.

Finally, we have illustrated the application of labeled network models to real-life situations by using a stylized example of the Metropolitan Consortium of Seville.

Acknowledgements. First of all, the authors thank two anonymous referees for her/his helpful comments and suggestions to improve the contents of the paper. Financial support from the Ministerio de Ciencia, Innovación y Universidades (MCIU), the Agencia Estatal de Investigación (AEI) and the Fondo Europeo de Desarrollo Regional (FEDER) under the project PGC2018-097965-B-I00 is gratefully acknowledged.

References

Algaba, E., Fragnelli, V., Llorca, N., Sánchez-Soriano, J.: Horizontal cooperation in a multimodal public transport system: the profit allocation problem. Eur. J. Oper. Res. **275**, 659–665 (2019a)

Algaba, E., Fragnelli, V., Sánchez-Soriano, J.: Handbook of the Shapley Value, CRC Press, Taylor & Francis Group, Boca Raton (2019b). ISBN 9780815374688

Algaba, E., van den Brink, R., Dietz, C.: Power measures and solutions for games under precedence constraints. J. Optim. Theory Appl. **172**, 1008–1022 (2017)

Algaba, E., van den Brink, R., Dietz, C.: Network structures with hierarchy and communication. J. Optim. Theory Appl. **179**, 265–282 (2018)

Ball, M.O., Magnanti, T.L., Monma, C.L., Nemhauser, G.L.: Network models. In: Handbooks in Operations Research and Management Science, vol. 7. North-Holland, Amsterdam (1995a)

Ball, M.O., Magnanti, T.L., Monma, C.L., Nemhauser, G.L.: Network routing. In: Handbooks in Operations Research and Management Science, vol. 8. North-Holland, Amsterdam (1995b)

Bergantiños, G., Gómez-Rúa, M., Llorca, N., Pulido, M., Sánchez-Soriano, J.: A new rule for source connection problems. Eur. J. Oper. Res. **234**, 780–788 (2014)

Bjørndal, E., Koster, M., Tijs, S.: Weighted allocation rules for standard fixed tree games. Math. Methods Oper. Res. **59**, 249–270 (2004)

Curiel, I., Derks, J., Tijs, S.: On balanced games and games with committee control. OR Spektrum **11**, 83–88 (1989)

Derks, J.J.M., Tijs, S.H.: Stable outcomes for multi-commodity flow games. Methods Oper. Res. **50**, 493–504 (1985)

Derks, J.J.M., Tijs, S.H.: Totally balanced multi-commodity games and flow games. Methods Oper. Res. **54**, 335–347 (1986)

Fragnelli, V., García-Jurado, I., Norde, H., Patrone, F., Tijs, S.: How to share railway infrastructure costs? In: Patrone, F., Tijs, S. (eds.) Game Practice: Contributions from Applied Game Theory (García-Jurado I, pp. 91–101. Kluwer, Amsterdam (2000a)

Fragnelli, V., García-Jurado, I., Méndez-Naya, L.: On shortest path games. Math. Methods Oper. Res. **52**, 251–264 (2000b)

Granot, D., Hojati, M.: On cost allocation in communication networks. Networks **20**, 209–229 (1990)

Gupta, A., Srinivasan, A., Tardos, É.: Cost-sharing mechanisms for network design. In: Jansen, K., Khanna, S., Rolim, J.D.P., Ron, D. (eds.) APPROX/RANDOM -2004. LNCS, vol. 3122, pp. 139–150. Springer, Heidelberg (2004). https://doi.org/10.1007/978-3-540-27821-4_13

Jackson, M., Zenou, Y.: Games on networks. In: Zamir, S., Young, P. (eds.) Handbook of Game Theory, vol. 4, Chapter 3, 89 pages. North Holland (Elsevier), Amsterdam (2014)

Kalai, E., Zemel, E.: Totally balanced games and games of flow. Math. Oper. Res. **7**, 476–478 (1982)

Koster, M., Molina, E., Sprumont, Y., Tijs, S.: Sharing the cost of a network: core and core allocations. Int. J. Game Theory **30**, 567–599 (2001)

Maschler, M., Potters, J., Reijnierse, H.: The nucleolus of a standard tree game revisited: a study of its monotonicity and computational properties. Int. J. Game Theory **39**, 89–104 (2010)

Megiddo, N.: Cost allocation for steiner trees. Networks **8**, 1–6 (1978)

Norde, H., Fragnelli, V., García-Jurado, I., Patrone, F., Tijs, S.: Balancedness of infrastructure cost games. Eur. J. Oper. Res. **136**, 635–654 (2002)

Reijnierse, J.H., Maschler, M.B., Potters, J.A.M., Tijs, S.H.: Simple flow games. Games Econ. Behav. **16**, 238–260 (1996)

Roughgarden, T., Schrijvers, O.: Network cost-sharing without anonymity. In: Lavi, R. (ed.) SAGT 2014. LNCS, vol. 8768, pp. 134–145. Springer, Heidelberg (2014). https://doi.org/10.1007/978-3-662-44803-8_12

Sánchez-Soriano, J.: The pairwise egalitarian solution. Eur. J. Oper. Res. **150**, 220–231 (2003)

Sánchez-Soriano, J.: Pairwise solutions and the core of transportation situations. Eur. J. Oper. Res. **175**, 101–110 (2006)

Sánchez-Soriano, J., Llorca, N., Meca, A., Molina, E., Pulido, M.: An integrated transport system for Alacant's students. UNIVERCITY, Ann. Oper. Res. **109**, 41–60 (2002)

Shapley, L.S.: A value for n-person games. Ann. Math. Stud. **28**, 307–317 (1953)

Sharkey, W.W.: Network models in economics. In: Ball, M.O., Magnanti, T.L., Monma, C.L., Nemhauser, G.L. (eds.) Handbooks in Operations Research and Management Science, vol. 8, pp. 713–765 (1995)

Slikker, M., van den Nouweland, A.: Social and Economic Networks in Cooperative Game Theory. Kluwer Academic Publisher, Norwell (2001)

Tardos, E.: Network games. In: STOC 2004 Proceedings of the Thirty-Sixth Annual ACM Symposium on Theory of Computing, pp. 341–342 (2004)

Tijs, S., Moretti, S., Branzei, R., Norde, H.: The bird core for minimum cost spanning tree problems revisited: monotonicity and additivity aspects. In: Seeger, A. (ed.) Recent Advances in Optimization. Lecture Notes in Economics and Mathematical Systems, vol. 563, pp. 305–322. Springer, Heidelberg (2006). https://doi.org/10.1007/3-540-28258-0_19