Bi-proportional Apportionments

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Abstract. An apportionment method is proposed that generalises Hamilton's method for matrices, optimising proportionality in both directions, both for rows and columns. The resulting matrix respects fixed totals for rows and columns even when such totals do not satisfy standard criteria (monotonicity, maximum or minimum Hare), for example following the allocation of majority prizes to parties or coalitions.

Optionally, if required, the result can also respect the minimum Hare quotae for rows and columns. The algorithm may easily be expressed on the basis of rules.

Keywords: Bi-proportional · Apportionment · Electoral system · Calculation software

1 Introduction

This paper proposes an apportionment method that generalises that of Hamilton [5] for matrices, optimising proportionality in both directions, both for rows and columns. The resulting matrix respects fixed totals for rows and columns even when such totals do not satisfy standard criteria (monotonicity, maximum or minimum Hare): for example, due to the allocation of majority prizes to parties or to coalitions.

Optionally, if required, the result can also respect the minimum Hare quotae for rows and columns.

Over the following three sections, we deal with the problem of representativity and its applications, and we explain how this work may be applied to such contexts. The method proposed is described in Sects. 5 and 6, with certain characteristics of the solution presented in Sects. 7 and 8. An application to a recent case is given in Sect. 9. References for software are provided in the Appendix.

2 Representativity

Let us consider a population in which a subdivision is defined according to whether a component belongs to given economic, professional, biological or geographical categories, and so on. The problem of representation consists in associating such a

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population with a set with a lower cardinality (which may or may not be a subset of the first) that is able to describe it on the basis of established criteria. Other examples might be representatives in a board meeting who own shares in certain goods, or political party representatives in a Parliament related to votes received in elections, and so on.

3 Apportionments

In those cases in which the above-mentioned sets are described by integer vectors, it is usual to speak of apportionments. With a view to differing objectives, various apportionment criteria and methods have been studied; see, for example, the article by Gambarelli and Stach [4]. We limit ourselves here to mentioning the main ones, inasmuch as they are relevant to this paper. For simplicity of exposition, we shall refer to votes and seats, but what follows may equally well be applied to other contexts.

3.1 Objectives

There are two principal, although mutually opposed, goals in apportionment: representativity and governability. The former demands a distribution of seats as proportional as possible to that of the votes, in order to minimise the distance (using a suitable measure) between the percentages of votes and seats. Contrary to this, the latter demands a distribution of seats such as to guarantee a majority of seats for a party or preconstituted coalition. Given that these two goals are generally opposed, the tendency is to identify systems targeted at intermediate goals.

3.2 Criteria

Symmetry is a common criterion for both stated objectives. The apportionment must not depend on the order in which parties are considered when the apportionment method is implemented[1].

In terms of representativity and a majority prize for a party, common criteria are: the same number of seats for the same number of votes and monotonicity (not having fewer seats for majority votes). Such criteria are not valid in the instance of a majority prize for a coalition, since the relative parties can obtain seats as prizes, which gives them an advantage with respect to the others.

Respect of Hare quotae is among the criteria related to the sole objective of representativity. The Hare quota is defined as the quota of seats proportional to that of votes. Respect of the Hare minimum consists of the provision that seats assigned to each party will not be fewer than the Hare quota rounded down. Respect of the Hare maximum consists of the provision that seats assigned to each party will not be greater in number than the Hare quota rounded up. For brevity, we omit other criteria that have been proposed.

[1] From Gambarelli and Palestini [3].

At first sight, all of the preceding criteria would appear obvious, but, in given situations, some of them cannot be applied, for example if an odd number of seats must be assigned in a system made up of an even number of parties, where each party has received the same number of votes, it is impossible to respect the criterion of "the same number of seats for the same votes" and symmetry.

3.3 Methods

Apportionment methods biased towards partial or total governability use thresholds for parties with low voting percentages and/or techniques that favour the parties most voted for (including various types of large divisors), until the majority prize is awarded to a party, coalition, or relative majority.

One of the most well-known apportionment methods aimed at representativity is Alexander Hamilton's proportional system [5]. It consists in initially assigning seats equal to minimum Hare quotae, with a subsequent assignation of residual seats to parties with higher decimal places in their Hare quotae.

In all apportionment techniques, in the event of more than one distribution, a final choice is made on the basis of exogenous methods (i.e., in the case of elections, sex and/or age of candidates, the drawing of lots, and so forth).

4 Bi-apportionment

The problem of apportionment increases when the distribution must be made on bi-dimensional partitions, that is, the initial set is made up of votes obtained by various parties in various districts and the final set is made up of seats to be assigned to parties, with reference to pre-established totals for each district. In such cases, the problem is to transform a matrix of known integer elements (votes) into a matrix of unknown integer elements (seats), for which total seats have normally been given for each district (usually, in proportion to their population) and for each party (according to voting results and electoral regulations). Such problems are designated as bi-apportionment. For simplicity of exposition, in the course of this paper matrix rows will be called districts and columns will be called parties.

4.1 Infringement of Totals

As mentioned, district and party totals are given. Following this, seats are assigned within the matrix, bearing in mind both vectors of totals. Problems may arise from this process. Let us take, for instance, a matrix of votes and arrange the columns so that total votes for parties are in a non-decreasing order. Having determined total seats using a suitable apportionment method that respects monotonicity, there is a possibility that a distribution of seats within the matrix may not be found, such that it respects the monotonicity of all districts. Analogous problems may arise with regard to Hare quotae and other criteria.

4.2 Bi-apportionment in Multi-chamber Systems and Further Infringement of Totals

The problem of seat totals that fail to meet standard criteria may also arise from bi-apportionment in multi-chamber systems. Let us consider, for example, a two-chamber system in which there is a wish to award a national majority prize to the same party in both chambers to guarantee maximum governability. The party to be awarded the prize could be the one that has the largest total of votes related to both chambers. However, the party concerned may not have a relative majority (in terms of votes) in both chambers in which case the prize could infringe monotonicity at a national level, as well as other criteria, such as Hare quotae (maximum for the winning party, minimum for the others). This might happen even more in the case of national majority prizes involving a coalition. Analogous examples may be given for systems with more than two chambers.

The infringement of monotonicity and Hare quotae may also occur with regard to totals for districts, inasmuch as seats are assigned to the district on the basis of related population size, although the percentage of voters in a district may be different to the population of the district itself.

4.3 Bi-apportionment Methods

Having established totals for rows and columns, the problem remains of allocating seats within the matrix, respecting such totals. Various methods have been proposed to obtain seat matrices that are as proportional as possible (according to certain metrics) to those of votes. For further information on this, we refer the reader to the work by Demange [1]. However, such methods encounter difficulties, in sofar as for the most part they seek proportionality in a single direction, that is in regard to the rows, or to the columns, of the matrix. In such cases, they result in distorted effects, whose remedy, at times, may even involve a modification of the totals. By way of an example, in order to square the accounts during the Italian elections on February 24–25, 2013, an additional seat was assigned to Trentino-Alto-Adige and Sardegna (Sardinia), and one seat fewer to Friuli-Venezia-Giulia and Molise. Analogously, in previous legislation (2008), an additional seat was allocated to Veneto 1 and Piemonte (Piedmont) 2, and one seat fewer to Sicilia (Sicily) 1 and Trentino-Alto-Adige. In legislation preceding this (2006), an additional seat was assigned to Trentino-Alto-Adige and one seat fewer to Molise[2].

A method that respects line and column totals was introduced by Pukelsheim [7] (see also Pukelsheim et al. [8]). For an explanation of this method, see Sect. 9.

A general procedure was proposed by Gambarelli [2] and extended by Gambarelli and Palestini [3] to the bi-apportionment case. It involves a preliminary choice in the order of criteria to be respected (at a local level and/or at the level of totals). The process advances with progressively narrowing limits to the set of possible solutions, eliminating those that do not respect the first criterion, then those that remain that do

[2] For more information about calculation methods used for these elections, please refer to the law in force at the time of the elections [11].

not respect the second, and so on, skipping those criteria that would leave the set of remaining solutions empty. The disadvantage of this method is the computational complexity.

4.4 Our Intentions in This Paper

Our current proposal consists of an application of Gambarelli's and Palestini's procedure [3], prioritising criteria that determine totals and, following this, assigning seats at a local level, respecting, as the user prefers, first the minimum Hare quotae in the two directions and then the maximum proportionality in terms of Hamilton.

5 The Proposed Method

We shall now describe our method in simple terms.

5.1 Procedure

Once a table of total votes and seats to be assigned to each district has been determined (row totals), a preferred method is used to calculate the total numbers of seats to be assigned to parties (column totals).

A table of row Hare quotae is then created, each element of this being the product of votes obtained by a specific party in a specific district with regard to the total seats in that district, divided by the total votes in that district.

Analogously, a table is created for column Hare quotae, each element of which is the product of the votes obtained by the specific party in a specific district with regard to the total seats for the party, divided by the total votes for that party.

Then:

- if maximum preference is for the option of optimising proportionality, all table cells for seats under construction are zeroed;
- otherwise, each position is initially given a number of seats corresponding to the minimum between Hare quotae for row and column.

A reference matrix is then built, which, in each position, contains the maximum between Hare quotae for row and column.

After this, the following is applied:

LOOP.

- No further seats are awarded to districts and parties that have achieved relative totals.
- A seat is awarded in a position with the greatest difference between the element in the reference table and the number of seats assigned at present (all things being equal; see below).
- One seat is subtracted from the row and column total for seats still to be assigned, corresponding to that position.
- The loop cycle continues until there are no more seats to be assigned.

All things being equal, a comparison is made between the maximum differences obtained during the next step (and possibly in those following on from it) according to possible positions for a seat, and the position is chosen that shows the greatest difference. The exploration of the possible solutions does not influence the complexity of the implementation since, after a few steps, the same situation is always achieved.

The assignment of each seat does not preclude the allocation of subsequent seats because, at each step there is always one (or more) maximum values to which the next seat will be assigned. For this reason, the algorithm supplies one or more equivalent solutions.

6 An Example

Let us consider a parliament made up of three parties, A, B and C, and with two districts I and II, to which 4 and 6 seats are assigned respectively. Let us suppose that the votes received by the parties in an election are those given in Table 1 and that a national majority prize is awarded such as to give the party with a relative majority more than 50% of the seats, allocating the remaining seats to the other parties in proportion to the votes received, with numeric rounding following Hamilton. Seat totals are given in Table 2.

Table 1. Votes.

Parties/Districts	A	B	C	Totals
I	6	22	12	40
II	10	26	24	60
Totals	16	48	36	100

Table 2. Seat totals.

Parties/Districts	A	B	C	Totals
I				4
II				6
Totals	1	6	3	10

Our procedure begins with the calculation of line Hare quotae (Table 3). Therefore, C's Hare in the first district (=1.2) is obtained by dividing the votes received by C in that district (=12) by the total votes in that district (=40) and multiplying the result by the total number of seats in that district (=4).

Table 3. Line Hare quotae.

Parties/Districts	A	B	C	Totals
I	0.6	2.2	1.2	4
II	1.0	2.6	2.4	6

Column Hare quotae are calculated analogously (Table 4).

Table 4. Column Hare quotae.

Parties/Districts	A	B	C
I	0.38	2.75	1
II	0.62	3.25	2
Totals	1	6	3

At this point, we build a reference table (Table 5).

Table 5. Reference table.

Parties/Districts	A	B	C
I	0.6	2.75	1.2
II	1	3.25	2.4

Let us suppose that the option chosen is to favour respect of minimum Hare quotae. In this case, the matrix for seats is initialised as in Table 6.

Table 6. The initial matrix in the case of the option for favouring minimum Hare quotae.

Parties/Districts	A	B	C	To be added
I	0	2	1	1
II	0	2	2	2
To be added	1	2	0	3

In Table 7, we give the matrix of differences between the reference table (Table 5) and that for seats currently assigned (Table 6). Party C is removed from the count as all its seats have been allocated.

Table 7. The differences between Tables 5 and Table 6.

Parties/Districts	A	B	C
I	0.6	0.75	–
II	1	1.25	–

The largest element for such differences matrix is 1.25, which corresponds to party B in the second district. In this position, we therefore add a seat and update the number of seats still to be assigned. The result is given in Table 8.

Table 8. The matrix of seats provisionally assigned during the second step.

Parties/Districts	A	B	C	To be added
I	0	2	1	1
II	0	3	2	1
To be added	1	1	0	2

The new table of differences is given in Table 9.

Table 9. Differences related to the third step.

Parties/Districts	A	B	C
I	0.6	0.75	–
II	1	0.25	–

The next seat is assigned to party B in the first district because the largest element for differences matrix is 0.75. The last seat available is assigned to party A in the second district. Table 10 shows the final results.

Table 10. The solution in the case of a preference for minimum Hare quotae.

Parties/Districts	A	B	C
I	0	3	1
II	1	3	2

If, instead, the option had been to favour proportionality, the starting matrix would have contained only zeroes. In this case, too, acceptable results would have been those given below (Table 11).

Table 11. The solution in the case of a preference for the option of optimizing proportionality.

Parties/Districts	A	B	C
I	0	3	1
II	1	3	2

By way of contrast, in Table 12 we give the solutions we would have obtained using the Pukelsheim method adopted for elections in the Zurich District on February 12, 2006. Seat distribution coincides with the two solutions produced by means of the method we propose.

Table 12. The solution obtained with the Pukelsheim method (data calculated using BAZI software) [6].

Parties/Districts	A	B	C
I	0	3	1
II	1	3	2

7 Characteristics of the Solution

By construction, this method respects the following conditions:

- Monotonicity, defined by the assignment of priority to the position with greater Hare quota;
- Respect of row and column totals, due to the blocking of the assignment of seats;
- Minimum Hare quotae (row and column), since it uses them as a basis for initiating the assignation loop for remaining seats. Therefore, it is impossible to assign fewer seats than those corresponding to the minimum Hare quotae;
- Maximum Hare quotae (row and column), since the maximum number of seats that can be assigned corresponds to the maximum Hare.

The Pukelsheim method does not always guarantee respect of all these conditions (as we shall see in Sect. 9).

8 The Use of This Method in Italian Legislation

The method we propose assigns, at each step, a seat to the position in which there is the greatest difference between the reference table element and the number of seats currently assigned. A requirement for this system is that all parties are represented in all districts. In certain electoral systems, such as the Italian one, a party may choose not to be represented in all electoral districts, but only in some of them. In this case, to avoid seats being assigned in districts where the party is not represented, we have added a further control phase to the loop described in Sect. 5.1. This phase checks the remaining availability of seats to be assigned to each party only for those districts in which the party is a candidate. If this availability is equal to the number of total seats still to be assigned to the party at national level, the system assigns seats due to the given party directly, then it continues the assignation loop for seats for other parties/districts yet to be assigned.

9 A Comparison Between the New Method and Pukelsheim's Method

The laws adopted in the Italian electoral system in recent years have led to a non-respect of the row totals (*Mattarellum and Porcellum*) and those of the column (*Rosatellum*).

As has been shown in Sect. 4.3, this has resulted in district totals being adjusted to balance the figures. The procedure we propose, as with that proposed by Pukelsheim, does not lead to such distortions.

In Tables 13 and 14 we give votes and seats assigned during the elections on February 24–25, 2013, using the regulations in force. Complete data for the votes and seats assigned are to be found on the website for the Ministero dell'Interno "Archivio storico delle Elezioni" (Ministry of the Interior "Historical Archive of Elections"). Indicated in the same table ("diff") are the distortions introduced into district totals to guarantee that party totals balanced at a national level.

Table 13. Resulting votes to the Italian Chamber of Deputies during the political elections on February 24–25, 2013, following regulations in force [9].

Parties/Districts	CDE	PD	SEL	SVP	FDI	PDL	LN	SC	UDC	M5S	Total votes
Piemonte 1	3.787	358.768	49.562	0	26.839	237.410	43.966	139.753	13.976	393.079	1.267.140
Piemonte 2	2.790	285.095	26.624	0	39.091	269.174	78.400	130.870	16.763	313.573	1.162.380
Lombardia 1	4.373	638.627	68.974	0	35.074	476.981	200.214	258.452	20.414	472.154	2.175.263
Lombardia 2	5.037	580.837	47.076	0	37.335	518.705	442.669	274.783	31.985	462.797	2.401.224
Lombardia 3	2.901	248.016	19.056	0	17.493	196.392	98.120	78.271	11.764	191.195	863.208
Trentino A.A.	0	101.216	23.061	146.800	0	66.128	25.350	79.549	4.803	88.632	535.539
Veneto 1	3.388	363.768	29.962	0	29.948	344.649	194.033	178.631	29.683	458.082	1.632.144
Veneto 2	2.252	264.398	23.083	0	14.491	204.791	115.977	118.225	14.928	317.636	1.075.781
Friuli Ven.Giu.	2.346	178.001	17.880	0	12.920	134.118	48.310	77.557	11.633	196.037	678.802
Liguria	2.353	258.766	29.386	0	13.411	174.568	21.862	78.409	10.556	300.080	889.391
Emilia Roma.	6.062	989.810	77.312	0	35.990	434.534	69.108	211.777	29.568	658.475	2.512.636
Toscana	6.882	831.464	84.033	0	40.139	388.046	16.213	153.551	25.673	532.699	2.078.700
Umbria	1.512	168.726	16.772	0	14.563	102.329	3.081	41.366	6.796	142.959	498.104
Marche	3.572	256.886	27.744	0	19.993	162.480	6.405	78.210	16.737	298.114	870.141
Lazio 1	7.009	656.650	101.017	0	62.794	498.904	3.006	170.925	31.385	689.613	2.221.303
Lazio 2	2.514	196.186	26.762	0	28.750	257.799	2.869	53.660	18.425	240.880	827.845
Abruzzo	4.492	175.857	23.817	0	27.677	185.537	1.407	49.777	13.654	232.627	714.845
Molise	1.264	42.499	10.428	0	11.168	39.588	343	15.968	3.278	52.059	176.595
Campania 1	10.025	329.616	52.057	0	32.226	449.811	3.188	98.260	38.120	349.682	1.362.985
Campania 2	13.611	323.557	47.256	0	57.140	415.312	5.636	101.960	69.758	311.766	1.345.996
Puglia	32.054	407.279	144.465	0	34.264	637.815	1.578	172.307	45.567	562.398	2.037.727
Basilicata	8.009	79.631	18.357	0	7.397	59.171	382	24.569	7.960	75.260	280.736
Calabria	16.489	209.379	39.129	0	12.724	222.671	2.344	51.726	38.335	232.811	825.608
Sicilia 1	6.550	218.665	24.149	0	15.303	306.846	2.001	60.671	31.608	404.944	1.070.737
Sicilia 2	12.526	249.059	27.171	0	23.800	359.474	2.742	68.724	39.256	438.613	1.221.365
Sardegna	5.530	233.278	34.098	0	16.235	188.901	1.330	55.891	25.696	275.241	836.200
Totals votes	167.328	8.646.034	1.089.231	146.800	666.765	7.332.134	1.390.534	2.823.842	608.321	8.691.406	31.562.395

Table 14. Seats assigned to the Italian Chamber of Deputies during the political elections on February 24–25, 2013, following regulations in force.

Parties/Districts	CDE	PD	SEL	SVP	FDI	PDL	LN	SC	UDC	M5S	Total seats	Seats predicted	Diff.
Piemonte 1	0	11	2	0	0	3	1	2	0	4	23	23	–
Piemonte 2	0	10	1	0	1	3	1	2	0	4	22	22	–
Lombardia 1	0	21	2	0	1	5	2	3	0	6	40	40	–
Lombardia 2	0	20	2	0	0	7	6	4	0	6	45	45	–
Lombardia 3	0	8	1	0	1	2	1	1	0	2	16	16	–
Trentino A.A.	0	3	1	5	0	1	0	1	0	1	12	11	+1
Veneto 1	0	13	1	0	0	5	3	2	1	6	31	31	–
Veneto 2	0	9	1	0	0	2	2	2	0	4	20	20	–
Friuli Ven.Giu.	0	6	1	0	0	1	1	1	0	2	12	13	−1
Liguria	0	9	1	0	0	2	0	1	0	3	16	16	–
Emilia Roma.	0	28	2	0	0	5	1	2	0	7	45	45	–
Toscana	1	23	2	0	1	4	0	2	0	5	38	38	–
Umbria	0	5	0	0	0	1	0	1	0	2	9	9	–
Marche	0	9	1	0	0	2	0	1	0	3	16	16	–

(*continued*)

Table 14. (*continued*)

Parties/Districts	CDE	PD	SEL	SVP	FDI	PDL	LN	SC	UDC	M5S	Total seats	Seats predicted	Diff.
Lazio 1	0	21	3	0	1	6	0	2	1	8	42	42	–
Lazio 2	0	7	1	0	1	3	0	1	0	3	16	16	–
Abruzzo	0	6	1	0	0	3	0	1	0	3	14	14	–
Molise	0	2	0	0	0	0	0	0	0	0	2	3	−1
Campania 1	1	14	2	0	1	7	0	1	1	5	32	32	–
Campania 2	0	12	2	0	1	6	0	2	1	4	28	28	–
Puglia	1	15	5	0	1	9	0	2	1	8	42	42	–
Basilicata	0	3	1	0	0	1	0	0	0	1	6	6	–
Calabria	1	9	1	0	0	4	0	0	1	4	20	20	–
Sicilia 1	0	10	1	0	0	6	0	1	1	6	25	25	–
Sicilia 2	1	10	1	0	0	6	0	1	1	7	27	27	–
Sardegna	1	8	1	0	0	3	0	1	0	4	18	17	+1
Totals	6	292	37	5	9	97	18	37	8	108	617	617	–

In the following tables we give seat distribution as it would have been using the new method proposed (Table 15) and using the Pukelsheim method, adopted during elections in the Zurich District (Table 16), with a subsequent comparison of the two results obtained (Table 17).

Table 15. The results that would have been produced for the Italian Chamber of Deputies during the political elections on February 24–25, 2013, if the method proposed in this paper had been used instead of the regulations in force.

Parties/Districts	CDE	PD	SEL	SVP	FDI	PDL	LN	SC	UDC	M5S	Total seats	Seats predicted	Diff.
Piemonte 1	0	12	1	0	0	3	0	2	0	5	23	23	–
Piemonte 2	0	10	2	0	1	3	1	2	0	3	22	22	–
Lombardia 1	0	20	1	0	0	7	2	4	0	6	40	40	–
Lombardia 2	0	19	1	0	0	7	8	4	0	6	45	45	–
Lombardia 3	0	9	1	0	0	2	1	1	0	2	16	16	–
Trentino A.A.	0	3	1	5	0	0	0	1	0	1	11	11	–
Veneto 1	0	12	1	0	1	4	3	3	1	6	31	31	–
Veneto 2	0	10	1	0	0	2	2	2	0	3	20	20	–
Friuli Ven.Giu.	0	7	1	0	0	1	1	1	0	2	13	13	–
Liguria	0	9	1	0	0	2	0	1	0	3	16	16	–
Emilia Roma.	0	29	1	0	0	5	0	2	0	8	45	45	–
Toscana	0	24	1	0	0	5	0	2	0	6	38	38	–
Umbria	0	6	1	0	0	1	0	0	0	1	9	9	–
Marche	0	9	1	0	0	2	0	1	0	3	16	16	–
Lazio 1	0	21	2	0	0	7	0	2	0	10	42	42	–

(*continued*)

Table 15. (*continued*)

Parties/Districts	CDE	PD	SEL	SVP	FDI	PDL	LN	SC	UDC	M5S	Total seats	Seats predicted	Diff.
Lazio 2	0	8	2	0	1	3	0	0	0	2	16	16	–
Abruzzo	0	8	1	0	1	2	0	0	0	2	14	14	–
Molise	0	2	1	0	0	0	0	0	0	0	3	3	–
Campania 1	1	11	2	0	1	8	0	2	1	6	32	32	–
Campania 2	1	12	2	0	1	6	0	1	1	4	28	28	–
Puglia	1	13	4	0	0	11	0	3	1	9	42	42	–
Basilicata	1	4	1	0	0	0	0	0	0	0	6	6	–
Calabria	1	9	2	0	0	3	0	1	1	3	20	20	–
Sicilia 1	0	8	2	0	1	5	0	1	1	7	25	25	–
Sicilia 2	1	9	1	0	1	6	0	1	1	7	27	27	–
Sardegna	0	8	2	0	1	2	0	0	1	3	17	17	–
Totals	6	292	37	5	9	97	18	37	8	108	617	617	–

As mentioned above, Pukelsheim used a method quite similar to ours. According to this method, we start by calculating all the Hare quotae and the decision on the seat to be assigned is taken step by step, in the course of the process, and not on the basis of the maximum Hare quota (as in our case), but on the average of the row and column Hare quotae for each element.

Table 16. The results that would have been produced for the Italian Chamber of Deputies if, instead of the regulations in force, the Pukelsheim method had been employed, as adopted for the election in the Zurich District (data calculated using BAZI software) [6].

Parties/Districts	CDE	PD	SEL	SVP	FDI	PDL	LN	SC	UDC	M5S	Total seats	Seats predicted	Diff.
Piemonte 1	0	12	1	0	0	3	1	2	0	4	23	23	–
Piemonte 2	0	10	1	0	1	3	1	2	0	4	22	22	–
Lombardia 1	0	21	2	0	1	6	2	3	0	5	40	40	–
Lombardia 2	0	20	1	0	1	6	6	4	1	6	45	45	–
Lombardia 3	0	9	1	0	0	2	1	1	0	2	16	16	–
Trentino A.A.	0	2	1	5	0	1	0	1	0	1	11	11	–
Veneto 1	0	13	1	0	1	4	3	2	1	6	31	31	–
Veneto 2	0	9	1	0	0	2	2	2	0	4	20	20	–
Friuli Ven.Giu.	0	6	1	0	0	2	1	1	0	2	13	13	–
Liguria	0	9	1	0	0	2	0	1	0	3	16	16	–
Emilia Roma.	0	28	2	0	0	5	1	2	0	7	45	45	–
Toscana	0	24	2	0	1	4	0	2	0	5	38	38	–
Umbria	0	5	1	0	0	1	0	0	0	2	9	9	–
Marche	0	8	1	0	0	2	0	1	0	4	16	16	–
Lazio 1	0	22	3	0	1	6	0	2	0	8	42	42	–
Lazio 2	0	7	1	0	0	4	0	1	0	3	16	16	–

(*continued*)

Table 16. (*continued*)

Abruzzo	0	6	1	0	0	3	0	1	0	3	14	14	–
Molise	0	1	0	0	0	1	0	0	0	1	3	3	–
Campania 1	1	13	2	0	1	7	0	2	1	5	32	32	–
Campania 2	1	12	2	0	1	6	0	1	1	4	28	28	–
Puglia	2	15	5	0	1	9	0	2	1	7	42	42	–
Basilicata	0	3	1	0	0	1	0	0	0	1	6	6	–
Calabria	1	9	2	0	0	3	0	1	1	3	20	20	–
Sicilia 1	0	10	1	0	0	5	0	1	1	7	25	25	–
Sicilia 2	1	10	1	0	0	6	0	1	1	7	27	27	–
Sardegna	0	8	1	0	0	3	0	1	0	4	17	17	–
Totals	6	292	37	5	9	97	18	37	8	108	617	617	–

The advantage of our method lies in approaching the solution through a global MaxMin principle that can exclude extreme situations. Furthermore, our method has two possible variants, one of which guarantees respect of the minimum Hares unlike Pukelsheim's method.

Moreover, the algorithmic nature of Pukelsheim's method means they are difficult to translate into legislative terms, so that the Swiss Cantons that adopted them (such as the Canton in which Zurich is located) were obliged to refer directly to the software in related legislation.

Table 17. Differences between the two methods.

Parties/Districts	CDE	PD	SEL	SVP	FDI	PDL	LN	SC	UDC	M5S
Piemonte 1	0	0	0	0	0	0	**−1**	0	0	**+1**
Piemonte 2	0	0	**+1**	0	0	0	0	0	0	**−1**
Lombardia 1	0	**−1**	**−1**	0	**−1**	**+1**	0	**+1**	0	**+1**
Lombardia 2	0	**−1**	0	0	**−1**	**+1**	**+2**	0	**−1**	0
Lombardia 3	0	0	0	0	0	0	0	0	0	0
Trentino A.A.	0	**+1**	0	0	0	**−1**	0	0	0	0
Veneto 1	0	**−1**	0	0	0	0	0	**+1**	0	0
Veneto 2	0	**+1**	0	0	0	0	0	0	0	**−1**
Friuli Ven.Giu.	0	**+1**	0	0	0	**−1**	0	0	0	0
Liguria	0	0	0	0	0	0	0	0	0	0
Emilia Roma.	0	**+1**	**−1**	0	0	0	**−1**	0	0	**+1**
Toscana	0	0	**−1**	0	**−1**	**+1**	0	0	0	**+1**
Umbria	0	**+1**	0	0	0	0	0	0	0	**−1**
Marche	0	**+1**	0	0	0	0	0	0	0	**−1**
Lazio 1	0	**−1**	**−1**	0	**−1**	**+1**	0	0	0	**+2**
Lazio 2	0	**+1**	**+1**	0	**+1**	**−1**	0	**−1**	0	**−1**
Abruzzo	0	**+2**	0	0	**+1**	**−1**	0	**−1**	0	**−1**
Molise	0	**+1**	**+1**	0	0	**−1**	0	0	0	**−1**

(*continued*)

Table 17. (*continued*)

Parties/Districts	CDE	PD	SEL	SVP	FDI	PDL	LN	SC	UDC	M5S
Campania 1	0	**−2**	0	0	0	**+1**	0	0	0	**+1**
Campania 2	0	0	0	0	0	0	0	0	0	0
Puglia	**−1**	**−2**	**−1**	0	**−1**	**+2**	0	**+1**	0	**+2**
Basilicata	**+1**	**+1**	0	0	0	**−1**	0	0	0	**−1**
Calabria	0	0	0	0	0	0	0	0	0	0
Sicilia 1	0	**−2**	**+1**	0	**+1**	0	0	0	0	0
Sicilia 2	0	**−1**	0	0	**+1**	0	0	0	0	0
Sardegna	0	0	**+1**	0	**+1**	**−1**	0	**−1**	**+1**	**−1**

As may be seen from Tables 15 and 16, both solutions resolve distortions for the total of seats assigned to various districts, while Table 17 shows that differences in distribution between the two systems are significantly reduced (a maximum variation of two seats in a smaller number of cases).

However, if we carry out a check with regard to minimum Hare quotae, we discover that, while our system respects them in all distribution cells, the Pukelsheim method does not assign a sufficient number of seats to comply with the minimum Hare. In the example given of elections for the Italian Chamber of Deputies on February 24–25, 2013, there are 3 cases in which the minimum Hare is violated:

(1) The PDL in Toscana has a column Hare of 5.13[3]. Our method assigns 5 seats (equal to the truncated minimum Hare), while the Pukelsheim method assigns only 4 seats;
(2) The M5S in Emilia Romagna has a column Hare of 8.18[4]. Our method assigns 8 seats (equal to the truncated minimum Hare), while the Pukelsheim method assigns only 7 seats;
(3) The M5S in Toscana has a column Hare of 6.62[5]. Our method assigns 6 seats (equal to the truncated minimum Hare), while the Pukelsheim method assigns only 5 seats.

10 Changes Introduced by the Italian Electoral Law 2017

To solve the distortion problem of the total seats by districts, the new Italian electoral law (November 3, 2017) introduced a cross-district compensation, in order to reassign some seats where the previously analyzed errors occur. The use of this compensation,

[3] Hare 5.13 obtained by dividing the 388,046 district votes by the 7,332,134 national votes and multiplying by 97 total seats to be assigned to the party at a national level.

[4] Hare 8.18 obtained by dividing the 658,475 district votes by the 8,691,406 national votes and multiplying by 108 total seats to be assigned to the party at a national level.

[5] Hare 6.62 obtained by dividing the 532,699 district votes by the 8,691,406 national votes and multiplying by 108 total seats to be assigned to the party at a national level.

however, creates a new problem, that of non-compliance with monotonicity in the allocation of seats for each district.

The election result for the Chamber of Deputies of March 4, 2018, shows an example of non-compliance with monotonicity in the Molise district (Table 18), where the only available seat was assigned to the fourth ranked party instead of the first.

Table 18. Allocation of seats for the Italian Chamber of Deputies for the Molise district during the political elections on March 4, 2018 [10]

Party	Votes	Decimal part of the attribution quotient	Assigned Seats	Seat Compensation	Final Seats
Movimento 5 Stelle	78.079	0,472833	1	**−1**	
Forza Italia, Lega, Fratelli d'Italia, UDC	51.992	0,314854			
PD, +Europa, SVT-PATT, Civica Popolare, Italia Europa Insieme	28.568	0,173003			
Liberi e Uguali	6.491	0,039308		**1**	1
Totals	165.130				

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Appendix A: Legislation

A draft legislative rule, referring to the case of the minimum Hare quotae, could be the following:

- Table of the row Hare quotae is determined, each element of which is the product of the votes obtained by that party in that district for the total seats in that district, divided by the total votes in that district;
- Table of column Hare quotae is determined, each element of which is the product of the votes obtained by that party in that district for the total seats of that party, divided by the total votes in that party;
- An *assigned seats matrix* is prepared, initially assigning the minimum value truncated, between the row Hare and the column Hare for each position;
- The *reference matrix* is then prepared, assigning the maximum value between the row Hare and the column Hare for each position;

- At each step, the method assigns a seat to the position in which there is the greatest difference between the "*reference matrix*" element and the "*assigned seats matrix*" element. The assigned seat is updated in the "*assigned seats matrix*";
- The method checks the residual availability of seats that may be assigned to each party only for those seats for which a given party is represented and when residual availability is equal to the remaining number of seats to be assigned to the given party at a national level;
- The system continues in the assignment loop until all the seats have been allocated.

Appendix B: Automatic Calculation Software

The latest version of the bi-proportional apportionment software described in this paper is available at the following web address:
http://dinamico2.unibg.it/dmsia/staff/gampubl.html#software.

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