Chapter 8
Some Further Results of Itô's Calculus

In this chapter, we use the quasi-surely analysis theory to develop Itô's integrals without the quasi-continuity condition. This allows us to define Itô's integral on stopping time interval. In particular, this new formulation can be applied to obtain Itô's formula for a general $C^{1,2}$-function, thus extending previously available results.

8.1 A Generalized Itô's Integral

Recall that $B_b(\Omega)$ is the space of all bounded and Borel measurable real functions defined on $\Omega = C_0^d(\mathbb{R}^+)$. We denote by $L_*^p(\Omega)$ the completion of $B_b(\Omega)$ under the natural norm $\|X\|_p := \hat{\mathbb{E}}[|X|^p]^{1/p}$. Similarly, we can define $L_*^p(\Omega_T)$ for any fixed $T \geq 0$. For any fixed $\mathbf{a} \in \mathbb{R}^d$, we still use the notation $B_t^{\mathbf{a}} := \langle \mathbf{a}, B_t \rangle$. Then we introduce the following properties, which are important in our stochastic calculus.

Proposition 8.1.1 *For any* $0 \leq t < T$, $\xi \in L_*^2(\Omega_t)$, *we have*

$$\hat{\mathbb{E}}[\xi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] = 0.$$

Proof For a fixed $P \in \mathcal{P}$, $B^{\mathbf{a}}$ is a martingale on $(\Omega, \mathcal{F}_t, P)$. Then we have

$$E_P[\xi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] = 0,$$

which completes the proof. □

Proposition 8.1.2 *For any* $0 \leq t \leq T$ *and* $\xi \in B_b(\Omega_t)$, *we have*

$$\hat{\mathbb{E}}[\xi^2(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 - \sigma^2_{\mathbf{a}\mathbf{a}^T}\xi^2(T-t)] \leq 0. \tag{8.1.1}$$

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Proof If $\xi \in C_b(\Omega_t)$, then we get that $\hat{\mathbb{E}}[\xi^2(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 - \xi^2\sigma_{\mathbf{aa}^T}^2(T-t)] = 0$. Thus (8.1.1) holds for $\xi \in C_b(\Omega_t)$. This implies that, for a fixed $P \in \mathcal{P}$,

$$E_P[\xi^2(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 - \xi^2\sigma_{\mathbf{aa}^T}^2(T-t)] \leq 0. \tag{8.1.2}$$

If we take $\xi \in B_b(\Omega_t)$, we can find a sequence $\{\xi_n\}_{n=1}^{\infty}$ in $C_b(\Omega_t)$, such that $\xi_n \to \xi$ in $L^p(\Omega, \mathcal{F}_t, P)$, for some $p > 2$. Thus we conclude that

$$E_P[\xi_n^2(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 - \xi_n^2\sigma_{\mathbf{aa}^T}^2(T-t)] \leq 0.$$

Then letting $n \to \infty$, we obtain (8.1.2) for $\xi \in B_b(\Omega_t)$. □

In what follows, we use the notation $L_*^p(\Omega)$, instead of $L_G^p(\Omega)$, to generalize Itô's integral on a larger space of stochastic processes $M_*^2(0,T)$ defined as follows. For fixed $p \geq 1$ and $T \in \mathbb{R}_+$, we first consider the following simple type of processes:

$$M_{b,0}(0,T) = \Big\{\eta : \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t),$$
$$\forall N > 0,\ \ 0 = t_0 < \cdots < t_N = T,\ \ \xi_j(\omega) \in B_b(\Omega_{t_j}),\ j = 0, \cdots, N-1\Big\}.$$

Definition 8.1.3 For an element $\eta \in M_{b,0}(0,T)$ with $\eta_t = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{1}_{[t_j,t_j+1)}(t)$, the related Bochner integral is

$$\int_0^T \eta_t(\omega)dt = \sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j).$$

For any $\eta \in M_{b,0}(0,T)$ we set

$$\tilde{\mathbb{E}}_T[\eta] := \tfrac{1}{T}\hat{\mathbb{E}}\left[\int_0^T \eta_t dt\right] = \frac{1}{T}\hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} \xi_j(\omega)(t_{j+1} - t_j)\right].$$

Then $\tilde{\mathbb{E}} : M_{b,0}(0,T) \mapsto \mathbb{R}$ forms a sublinear expectation. We can introduce a natural norm $\|\eta\|_{M^p(0,T)} = \left\{\hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p dt\right]\right\}^{1/p}$.

Definition 8.1.4 For any $p \geq 1$, we denote by $M_*^p(0,T)$ the completion of $M_{b,0}(0,T)$ under the norm

$$||\eta||_{M^p(0,T)} = \left\{\hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p dt\right]\right\}^{1/p}.$$

We have $M_*^p(0,T) \supset M_*^q(0,T)$, for $p \le q$. The following process

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t),\ \xi_j \in L_*^p(\Omega_{t_j}),\ j = 1, \cdots, N$$

is also in $M_*^p(0,T)$.

Definition 8.1.5 For any $\eta \in M_{b,0}(0,T)$ of the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t),$$

we define Itô's integral

$$I(\eta) = \int_0^T \eta_s dB_s^{\mathbf{a}} := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}}^{\mathbf{a}} - B_{t_j}^{\mathbf{a}}).$$

Lemma 8.1.6 *The mapping $I : M_{b,0}(0,T) \mapsto L_*^2(\Omega_T)$ is a linear continuous mapping and thus can be continuously extended to $I : M_*^2(0,T) \mapsto L_*^2(\Omega_T)$. Moreover, we have*

$$\hat{\mathbb{E}}\left[\int_0^T \eta_s dB_s^{\mathbf{a}}\right] = 0, \tag{8.1.3}$$

$$\hat{\mathbb{E}}\left[(\int_0^T \eta_s dB_s^{\mathbf{a}})^2\right] \le \sigma_{\mathbf{aa}^T}^2 \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^2 dt\right]. \tag{8.1.4}$$

Proof It suffices to prove (8.1.3) and (8.1.4) for any $\eta \in M_{b,0}(0,T)$. From Proposition 8.1.1, for any j,

$$\hat{\mathbb{E}}[\xi_j(B_{t_{j+1}}^{\mathbf{a}} - B_{t_j}^{\mathbf{a}})] = \hat{\mathbb{E}}[-\xi_j(B_{t_{j+1}}^{\mathbf{a}} - B_{t_j}^{\mathbf{a}})] = 0.$$

Thus we obtain (8.1.3):

$$\begin{aligned}\hat{\mathbb{E}}\left[\int_0^T \eta_s dB_s^{\mathbf{a}}\right] &= \hat{\mathbb{E}}\left[\int_0^{t_{N-1}} \eta_s dB_s^{\mathbf{a}} + \xi_{N-1}(B_{t_N}^{\mathbf{a}} - B_{t_{N-1}}^{\mathbf{a}})\right] \\ &= \hat{\mathbb{E}}\left[\int_0^{t_{N-1}} \eta_s dB_s^{\mathbf{a}}\right] = \cdots = \hat{\mathbb{E}}[\xi_0(B_{t_1}^{\mathbf{a}} - B_{t_0}^{\mathbf{a}})] = 0.\end{aligned}$$

We now prove (8.1.4). By a similar analysis as in Lemma 3.3.4 of Chap. 3, we derive that

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t^{\mathbf{a}}\right)^2\right] = \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \xi_i^2 (B_{t_{i+1}}^{\mathbf{a}} - B_{t_i}^{\mathbf{a}})^2\right].$$

Then from Proposition 8.1.2, we obtain that

$$\hat{\mathbb{E}}\left[\xi_j^2 (B_{t_{j+1}}^{\mathbf{a}} - B_{t_j}^{\mathbf{a}})^2 - \sigma_{\mathbf{aa}^T}^2 \xi_j^2 (t_{j+1} - t_j)\right] \leq 0.$$

Thus

$$\begin{aligned}
\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t^{\mathbf{a}}\right)^2\right] &= \hat{\mathbb{E}}[\sum_{i=0}^{N-1} \xi_i^2 (B_{t_N}^{\mathbf{a}} - B_{t_{N-1}}^{\mathbf{a}})^2] \\
&\leq \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \xi_i^2 [(B_{t_N}^{\mathbf{a}} - B_{t_{N-1}}^{\mathbf{a}})^2 - \sigma_{\mathbf{aa}^T}^2 (t_{i+1} - t_i)]\right] + \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \sigma_{\mathbf{aa}^T}^2 \xi_i^2 (t_{i+1} - t_i)\right] \\
&\leq \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \sigma_{\mathbf{aa}^T}^2 \xi_i^2 (t_{i+1} - t_i)\right] = \sigma_{\mathbf{aa}^T}^2 \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^2 dt\right],
\end{aligned}$$

which is the desired result. □

The following proposition can be verified directly by the definition of Itô's integral with respect to G-Brownian motion.

Proposition 8.1.7 *Let $\eta, \theta \in M_*^2(0, T)$. Then for any $0 \leq s \leq r \leq t \leq T$, we have:*

(i) $\int_s^t \eta_u dB_u^{\mathbf{a}} = \int_s^r \eta_u dB_u^{\mathbf{a}} + \int_r^t \eta_u dB_u^{\mathbf{a}}$;
(ii) $\int_s^t (\alpha\eta_u + \theta_u) dB_u^{\mathbf{a}} = \alpha \int_s^t \eta_u dB_u^{\mathbf{a}} + \int_s^t \theta_u dB_u^{\mathbf{a}}$, *where* $\alpha \in B_b(\Omega_s)$.

Proposition 8.1.8 *For any $\eta \in M_*^2(0, T)$, we have*

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\int_0^t \eta_s dB_s^{\mathbf{a}}\right|^2\right] \leq 4\sigma_{\mathbf{aa}^T}^2 \hat{\mathbb{E}}\left[\int_0^T \eta_s^2 ds\right]. \tag{8.1.5}$$

Proof Since for any $\alpha \in B_b(\Omega_t)$, we have

$$\hat{\mathbb{E}}\left[\alpha \int_t^T \eta_s dB_s^{\mathbf{a}}\right] = 0.$$

Then, for a fixed $P \in \mathcal{P}$, the process $\int_0^\cdot \eta_s dB_s^{\mathbf{a}}$ is a martingale on $(\Omega, \mathcal{F}_t, P)$. It follows from the classical Doob's maximal inequality (see Appendix B) that

$$E_P\left[\sup_{0\le t\le T}\left|\int_0^t \eta_s dB_s^{\mathbf{a}}\right|^2\right] \le 4E_P\left[\left|\int_0^T \eta_s dB_s^{\mathbf{a}}\right|^2\right] \le 4\hat{\mathbb{E}}\left[\left|\int_0^T \eta_s dB_s^{\mathbf{a}}\right|^2\right]$$
$$\le 4\sigma^2_{\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}\left[\int_0^T \eta_s^2 ds\right].$$

Thus (8.1.5) holds. □

Proposition 8.1.9 *For any $\eta \in M_*^2(0,T)$ and $0 \le t \le T$, the integral $\int_0^t \eta_s dB_s^{\mathbf{a}}$ is continuous q.s., i.e., $\int_0^t \eta_s dB_s^{\mathbf{a}}$ has a modification whose paths are continuous in t.*

Proof The claim is true for $\eta \in M_{b,0}(0,T)$ since $(B_t^{\mathbf{a}})_{t\ge0}$ is a continuous process. In the case of $\eta \in M_*^2(0,T)$, there exists $\eta^{(n)} \in M_{b,0}(0,T)$, such that $\hat{\mathbb{E}}[\int_0^T(\eta_s - \eta_s^{(n)})^2 ds] \to 0$, as $n\to\infty$. By Proposition 8.1.8, we have

$$\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t (\eta_s-\eta_s^{(n)})dB_s^{\mathbf{a}}\right|^2\right] \le 4\sigma^2_{\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}\left[\int_0^T(\eta_s-\eta_s^{(n)})^2 ds\right] \to 0, \quad \text{as} \quad n\to\infty.$$

Then choosing a subsequence if necessary, we can find a set $\hat{\Omega} \subset \Omega$ with $\hat{c}(\hat{\Omega}^c) = 0$ so that, for any $\omega \in \hat{\Omega}$ the sequence of processes $\int_0^\cdot \eta_s^{(n)} dB_s^{\mathbf{a}}(\omega)$ uniformly converges to $\int_0^\cdot \eta_s dB_s^{\mathbf{a}}(\omega)$ on $[0,T]$. Thus for any $\omega \in \hat{\Omega}$, we get that $\int_0^\cdot \eta_s dB_s^{\mathbf{a}}(\omega)$ is continuous in t. For any $(\omega,t) \in [0,T]\times\Omega$, we take the process

$$J_t(\omega) = \begin{cases} \int_0^t \eta_s dB_s^{\mathbf{a}}(\omega), & \omega \in \hat{\Omega}; \\ 0, & \text{otherwise}, \end{cases}$$

as the desired t-continuous modification. This completes the proof. □

We now define the integral of a process $\eta \in M_*^1(0,T)$ with respect to $\langle B^{\mathbf{a}}\rangle$. We also define a mapping:

$$Q_{0,T}(\eta) = \int_0^T \eta_t d\left\langle B^{\mathbf{a}}\right\rangle_t := \sum_{j=0}^{N-1}\xi_j(\left\langle B^{\mathbf{a}}\right\rangle_{t_{j+1}} - \left\langle B^{\mathbf{a}}\right\rangle_{t_j}) : M_b^{1,0}(0,T) \to L_*^1(\Omega_T).$$

Proposition 8.1.10 *The mapping $Q_{0,T} : M_b^{1,0}(0,T) \mapsto L_*^1(\Omega_T)$ is a continuous linear mapping and $Q_{0,T}$ can be uniquely extended to $M_*^1(0,T)$. Moreover, we have*

$$\hat{\mathbb{E}}\left[\left|\int_0^T \eta_t d\left\langle B^{\mathbf{a}}\right\rangle_t\right|\right] \le \sigma^2_{\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}\left[\int_0^T |\eta_t| dt\right] \quad \textit{for any} \quad \eta \in M_*^1(0,T). \tag{8.1.6}$$

Proof From the relation

$$\sigma^2_{-\mathbf{aa}^T}(t-s) \le \langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s \le \sigma^2_{\mathbf{aa}^T}(t-s)$$

it follows that

$$\hat{\mathbb{E}}\left[|\xi_j|(\langle B^{\mathbf{a}}\rangle_{t_{j+1}} - \langle B^{\mathbf{a}}\rangle_{t_j}) - \sigma^2_{\mathbf{aa}^T}|\xi_j|(t_{j+1}-t_j)\right] \le 0, \quad \text{for any} \quad j = 1, \cdots, N-1.$$

Therefore, we deduce the following chain of inequalities:

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\sum_{j=0}^{N-1} \xi_j(\langle B^{\mathbf{a}}\rangle_{t_{j+1}} - \langle B^{\mathbf{a}}\rangle_{t_j})\right|\right] \le \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} |\xi_j|\,\langle B^{\mathbf{a}}\rangle_{t_{j+1}} - \langle B^{\mathbf{a}}\rangle_{t_j}\right] \\
\le&\hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} |\xi_j|[(\langle B^{\mathbf{a}}\rangle_{t_{j+1}} - \langle B^{\mathbf{a}}\rangle_{t_j}) - \sigma^2_{\mathbf{aa}^T}(t_{j+1}-t_j)]\right] + \hat{\mathbb{E}}\left[\sigma^2_{\mathbf{aa}^T}\sum_{j=0}^{N-1}|\xi_j|(t_{j+1}-t_j)\right] \\
\le& \sum_{j=0}^{N-1}\hat{\mathbb{E}}[|\xi_j|[(\langle B^{\mathbf{a}}\rangle_{t_{j+1}} - \langle B^{\mathbf{a}}\rangle_{t_j}) - \sigma^2_{\mathbf{aa}^T}(t_{j+1}-t_j)]] + \hat{\mathbb{E}}\left[\sigma^2_{\mathbf{aa}^T}\sum_{j=0}^{N-1}|\xi_j|(t_{j+1}-t_j)\right] \\
\le&\hat{\mathbb{E}}\left[\sigma^2_{\mathbf{aa}^T}\sum_{j=0}^{N-1}|\xi_j|(t_{j+1}-t_j)\right] = \sigma^2_{\mathbf{aa}^T}\hat{\mathbb{E}}\left[\int_0^T |\eta_t|dt\right].
\end{aligned}$$

This completes the proof. □

From the above Proposition 8.1.9, we obtain that $\langle B^{\mathbf{a}}\rangle_t$ is continuous in t q.s.. Then for any $\eta \in M^1_*(0,T)$ and $0 \le t \le T$, the integral $\int_0^t \eta_s d\langle B^{\mathbf{a}}\rangle_s$ also has a t-continuous modification. In the sequel, we always consider the t-continuous modification of Itô's integral. Moreover, Itô's integral with respect to $\langle B^i, B^j\rangle = \langle B\rangle^{ij}$ can be similarly defined. This is left as an exercise for the readers.

Lemma 8.1.11 *Let $\eta \in M^2_b(0,T)$. Then η is Itô-integrable for every $P \in P$. Moreover,*

$$\int_0^T \eta_s dB^{\mathbf{a}}_s = \int_0^T \eta_s d_P B^{\mathbf{a}}_s, \quad P\text{-a.s.},$$

where the right hand side is the usual Itô integral.

We leave the proof of this lemma to readers as an exercise.

Lemma 8.1.12 (Generalized Burkholder-Davis-Gundy (BDG) inequality) *For any $\eta \in M^2_*(0,T)$ and $p > 0$, there exist constants c_p and C_p with $0 < c_p < C_p < \infty$, depending only on p, such that*

$$\sigma^p_{-\mathbf{aa}^T}c_p\hat{\mathbb{E}}\left[\left(\int_0^T |\eta_s|^2 ds\right)^{p/2}\right] \le \hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\left|\int_0^t \eta_s dB^{\mathbf{a}}_s\right|^p\right] \le \sigma^p_{\mathbf{aa}^T}C_p\hat{\mathbb{E}}\left[\left(\int_0^T |\eta_s|^2 ds\right)^{p/2}\right].$$

Proof Observe that, under any $P \in \mathcal{P}$, $B^{\mathbf{a}}$ is a P-martingale with

$$\sigma^2_{-\mathbf{a}\mathbf{a}^T} dt \leq d\langle B^{\mathbf{a}}\rangle_t \leq \sigma^2_{\mathbf{a}\mathbf{a}^T} dt.$$

The proof is then a simple application of the classical BDG inequality. □

8.2 Itô's Integral for Locally Integrable Processes

So far we have considered Itô's integral $\int_0^T \eta_t dB_t^{\mathbf{a}}$ where η in $M_*^2(0,T)$. In this section we continue our study of Itô's integrals for a type of locally integrable processes.

We first give some properties of $M_*^p(0,T)$.

Lemma 8.2.1 *For any $p \geq 1$ and $X \in M_*^p(0,T)$, the following relation holds:*

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left[\int_0^T |X_t|^p \mathbf{1}_{\{|X_t|>n\}} dt\right] = 0. \tag{8.2.1}$$

Proof The proof is similar to that of Proposition 6.1.22 in Chap. 6. □

Corollary 8.2.2 *For any $\eta \in M_*^2(0,T)$, let $\eta_s^{(n)} = (-n) \vee (\eta_s \wedge n)$, then, as $n \to \infty$, we have $\int_0^t \eta_s^{(n)} dB_s^{\mathbf{a}} \to \int_0^t \eta_s dB_s^{\mathbf{a}}$ in $L_*^2(0,T)$ for any $t \leq T$.*

Proposition 8.2.3 *Let $X \in M_*^p(0,T)$. Then for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all $\eta \in M_*^p(0,T)$ satisfying $\hat{\mathbb{E}}\left[\int_0^T |\eta_t| dt\right] \leq \delta$ and $|\eta_t(\omega)| \leq 1$, we have $\hat{\mathbb{E}}\left[\int_0^T |X_t|^p |\eta_t| dt\right] \leq \varepsilon$.*

Proof For any $\varepsilon > 0$, according to Lemma 8.2.1, there exists a number $N > 0$ such that $\hat{\mathbb{E}}\left[\int_0^T |X|^p \mathbf{1}_{\{|X|>N\}}\right] \leq \varepsilon/2$. Take $\delta = \varepsilon/2N^p$. Then we derive that

$$\begin{aligned}\hat{\mathbb{E}}\left[\int_0^T |X_t|^p |\eta_t| dt\right] &\leq \hat{\mathbb{E}}\left[\int_0^T |X_t|^p |\eta_t| \mathbf{1}_{\{|X_t|>N\}} dt\right] + \hat{\mathbb{E}}\left[\int_0^T |X_t|^p |\eta_t| \mathbf{1}_{\{|X_t|\leq N\}} dt\right] \\ &\leq \hat{\mathbb{E}}\left[\int_0^T |X_t|^p \mathbf{1}_{\{|X_t|>N\}} dt\right] + N^p \hat{\mathbb{E}}\left[\int_0^T |\eta_t| dt\right] \leq \varepsilon,\end{aligned}$$

which is the desired result. □

Lemma 8.2.4 *If $p \geq 1$ and $X, \eta \in M_*^p(0,T)$ are such that η is bounded, then the product $X\eta \in M_*^p(0,T)$.*

Proof We can find $X^{(n)}, \eta^{(n)} \in M_{b,0}(0,T)$ for $n = 1, 2, \ldots$, such that $\eta^{(n)}$ is uniformly bounded and

$$\|X - X^{(n)}\|_{M^p(0,T)} \to 0, \quad \|\eta - \eta^{(n)}\|_{M^p(0,T)} \to 0, \quad \text{as} \quad n \to \infty.$$

Then we obtain that

$$\hat{\mathbb{E}}\left[\int_0^T |X_t\eta_t - X_t^{(n)}\eta_t^{(n)}|^p dt\right] \le 2^{p-1}\left(\hat{\mathbb{E}}\left[\int_0^T |X_t|^p|\eta_t - \eta_t^{(n)}|^p dt\right] + \hat{\mathbb{E}}\left[\int_0^T |X_t - X_t^{(n)}|^p|\eta_t^{(n)}|^p dt\right]\right).$$

By Proposition 8.2.3, the first term on the right-hand side tends to 0 as $n \to \infty$. Since $\eta^{(n)}$ is uniformly bounded, the second term also tends to 0. □

Now we are going to study Itô's integrals on an interval $[0, \tau]$, where τ is a stopping time relative to the G-Brownian pathes.

Definition 8.2.5 A stopping time τ relative to the filtration $(\mathcal{F}_t)$ is a map on Ω with values in $[0, T]$ such that $\{\tau \le t\} \in \mathcal{F}_t$, for every $t \in [0, T]$.

Lemma 8.2.6 *For any stopping time τ and any $X \in M_*^p(0, T)$, we have $\mathbf{1}_{[0,\tau]}(\cdot)X \in M_*^p(0, T)$.*

Proof Related to the given stopping time τ, we consider the following sequence:

$$\tau_n = \sum_{k=0}^{2^n-1} \frac{(k+1)T}{2^n}\mathbf{1}_{[\frac{kT}{2^n} \le \tau < \frac{(k+1)T}{2^n})} + T\mathbf{1}_{[\tau \ge T]}.$$

It is clear that $2^{-n} \ge \tau_n - \tau \ge 0$. It follows from Lemma 8.2.4 that any element of the sequence $\{1_{[0,\tau_n]}X\}_{n=1}^\infty$ is in $M_*^p(0, T)$. Note that, for $m \ge n$, we have

$$\begin{aligned}\hat{\mathbb{E}}\left[\int_0^T |\mathbf{1}_{[0,\tau_n]}(t) - \mathbf{1}_{[0,\tau_m]}(t)|dt\right] &\le \hat{\mathbb{E}}\left[\int_0^T |\mathbf{1}_{[0,\tau_n]}(t) - \mathbf{1}_{[0,\tau]}(t)|dt\right] \\ &= \hat{\mathbb{E}}[\tau_n - \tau] \le 2^{-n}T.\end{aligned}$$

Then applying Proposition 8.2.3, we derive that $\mathbf{1}_{[0,\tau]}X \in M_*^p(0, T)$ and the proof is complete. □

Lemma 8.2.7 *For any stopping time τ and any $\eta \in M_*^2(0, T)$, we have*

$$\int_0^{t\wedge\tau} \eta_s dB_s^{\mathbf{a}}(\omega) = \int_0^t \mathbf{1}_{[0,\tau]}(s)\eta_s dB_s^{\mathbf{a}}(\omega), \text{ for all } t \in [0, T] \text{ q.s.} \tag{8.2.2}$$

Proof For any $n \in \mathbb{N}$, let

$$\tau_n := \sum_{k=1}^{[t\cdot 2^n]} \frac{k}{2^n}\mathbf{1}_{[\frac{(k-1)t}{2^n} \le \tau < \frac{kt}{2^n})} + t\mathbf{1}_{[\tau \ge t]} = \sum_{k=1}^{2^n} \mathbf{1}_{A_n^k} t_n^k.$$

Here $t_n^k = k2^{-n}t$, $A_n^k = [t_n^{k-1} < t \wedge \tau \le t_n^k]$, for $k < 2^n$, and $A_n^{2^n} = [\tau \ge t]$. We see that $\{\tau_n\}_{n=1}^\infty$ is a decreasing sequence of stopping times which converges to $t \wedge \tau$.

We first show that

$$\int_{\tau_n}^{t} \eta_s dB_s^{\mathbf{a}} = \int_0^t \mathbf{1}_{[\tau_n,t]}(s)\eta_s dB_s^{\mathbf{a}}, \quad \text{q.s.} \tag{8.2.3}$$

By Proposition 8.1.7 we have

$$\begin{aligned}\int_{\tau_n}^{t} \eta_s dB_s^{\mathbf{a}} &= \int_{\sum_{k=1}^{2^n} \mathbf{1}_{A_n^k} t_n^k}^{t} \eta_s dB_s^{\mathbf{a}} = \sum_{k=1}^{2^n} \mathbf{1}_{A_n^k} \int_{t_n^k}^{t} \eta_s dB_s^{\mathbf{a}} \\ &= \sum_{k=1}^{2^n} \int_{t_n^k}^{t} \mathbf{1}_{A_n^k} \eta_s dB_s^{\mathbf{a}} = \int_0^t \sum_{k=1}^{2^n} \mathbf{1}_{[t_n^k,t]}(s) \mathbf{1}_{A_n^k} \eta_s dB_s^{\mathbf{a}},\end{aligned}$$

from which (8.2.3) follows. Hence we obtain that

$$\int_0^{\tau_n} \eta_s dB_s^{\mathbf{a}} = \int_0^t \mathbf{1}_{[0,\tau_n]}(s)\eta_s dB_s^{\mathbf{a}}, \text{ q.s.}$$

Observe now that $0 \le \tau_n - \tau_m \le \tau_n - t \wedge \tau \le 2^{-n}t$, for $n \le m$. Then Proposition 8.2.3 yields that $\mathbf{1}_{[0,\tau_n]}\eta$ converges in $M_*^2(0,T)$ to $\mathbf{1}_{[0,\tau\wedge t]}\eta$ as $n \to \infty$, which implies that $\mathbf{1}_{[0,\tau\wedge t]}\eta \in M_*^2(0,T)$. Consequently,

$$\lim_{n\to\infty} \int_0^{\tau_n} \eta_s dB_s^{\mathbf{a}} = \int_0^t \mathbf{1}_{[0,\tau]}(s)\eta_s dB_s^{\mathbf{a}}, \quad \text{q.s.}$$

Note that $\int_0^t \eta_s dB_s^{\mathbf{a}}$ is continuous in t, hence (8.2.2) is proved. □

The space of processes $M_*^p(0,T)$ can be further enlarged as follows.

Definition 8.2.8 For fixed $p \ge 1$, a stochastic process η is said to be in $M_w^p(0,T)$, if it is associated with a sequence of increasing stopping times $\{\sigma_m\}_{m\in\mathbb{N}}$, such that:

(i) For any $m \in \mathbb{N}$, the process $\left(\eta_t \mathbf{1}_{[0,\sigma_m]}(t)\right)_{t\in[0,T]} \in M_*^p(0,T)$;
(ii) If $\Omega^{(m)} := \{\omega \in \Omega : \sigma_m(\omega) \wedge T = T\}$ and $\hat{\Omega} := \lim_{m\to\infty} \Omega^{(m)}$, then $\hat{c}(\hat{\Omega}^c) = 0$.

Remark 8.2.9 Suppose there is another sequence of stopping times $\{\tau_m\}_{m=1}^{\infty}$ that satisfies the second condition in Definition 8.2.8. Then the sequence $\{\tau_m \wedge \sigma_m\}_{m\in\mathbb{N}}$ also satisfies this condition. Moreover, by Lemma 8.2.6, we know that for any $m \in \mathbb{N}$, $\eta\mathbf{1}_{[0,\tau_m\wedge\sigma_m]} \in M_*^p(0,T)$. This property allows to associate the same sequence of stopping times with several different processes in $M_w^p(0,T)$.

For given $\eta \in M_w^2(0,T)$ associated with $\{\sigma_m\}_{m\in\mathbb{N}}$, we consider, for any $m \in \mathbb{N}$, the t-continuous modification of the process $\left(\int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}}\right)_{0\le t\le T}$. For any m, $n \in \mathbb{N}$ with $n > m$, by Lemma 8.2.7 we can find a polar set $\hat{A}_{m,n}$, such that for all $\omega \in (\hat{A}_{m,n})^c$, the following equalities hold:

$$\begin{aligned}\int_0^{t\wedge\sigma_m} \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}}(\omega) &= \int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}}(\omega)\\ &= \int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s)\mathbf{1}_{[0,\sigma_n]}(s) dB_s^{\mathbf{a}}(\omega) \qquad (8.2.4)\\ &= \int_0^{t\wedge\sigma_m} \eta_s \mathbf{1}_{[0,\sigma_n]}(s) dB_s^{\mathbf{a}}(\omega),\ \ 0 \le t \le T.\end{aligned}$$

Define the polar set

$$\hat{A} := \bigcup_{m=1}^{\infty} \bigcup_{n=m+1}^{\infty} \hat{A}_{m,n}.$$

For any $m \in \mathbb{N}$ and any $(\omega, t) \in \Omega \times [0, T]$, we set

$$X_t^{(m)}(\omega) := \begin{cases} \int_0^t \eta_s \mathbf{1}_{[0,\sigma_m]}(s) dB_s^{\mathbf{a}}(\omega), & \omega \in \hat{A}^c \cap \hat{\Omega}; \\ 0, & \text{otherwise.} \end{cases}$$

From (8.2.4), for any $m, n \in \mathbb{N}$ with $n > m$, $X^{(n)}(\omega) \equiv X^{(m)}(\omega)$ on $[0, \sigma_m(\omega) \wedge T]$ for any $\omega \in \hat{A}^c \cap \hat{\Omega}$ and $X^{(n)}(\omega) \equiv X^{(m)}(\omega)$ on $[0, T]$ for all other ω. Note that for $\omega \in \hat{A}^c \cap \hat{\Omega}$, we can find $m \in \mathbb{N}$, such that $\sigma_m(\omega) \wedge T = T$. Consequently, for any $\omega \in \Omega$, $\lim_{m\to\infty} X_t^{(m)}(\omega)$ exists for any t. From Lemma 8.2.7, it is not difficult to verify that choosing a different sequence of stopping times will only alter this limitation on the polar set. The details are left to the reader. Thus, the following definition is well posed.

Definition 8.2.10 Giving $\eta \in M_w^2([0, T])$, for any $(\omega, t) \in \Omega \times [0, T]$, we define

$$\int_0^t \eta_s dB_s^{\mathbf{a}}(\omega) := \lim_{m\to\infty} X_t^{(m)}(\omega). \qquad (8.2.5)$$

For any $\omega \in \Omega$ and $t \in [0, \sigma_m]$, $\int_0^t \eta_s dB_s^{\mathbf{a}}(\omega) = X_t^{(m)}(\omega)$, $0 \le t \le T$. Since each of the processes $\{X_t^{(m)}\}_{0\le t\le T}$ has t-continuous paths, we conclude that the paths of $\left(\int_0^t \eta_s dB_s^{\mathbf{a}}\right)_{0\le t\le T}$ are also t-continuous. The following theorem is an direct consequence of the above discussion.

Theorem 8.2.11 *Assume that $\eta \in M_w^2([0, T])$. Then the stochastic process $\int_0^{\cdot} \eta_s dB_s^{\mathbf{a}}$ is a well-defined continuous process on* $[0, T]$.

For any $\eta \in M_w^1(0, T)$, the integrals $\int_0^t \eta_s d\langle B^{\mathbf{a}}\rangle_s$ and $\int_0^t \eta_s d\langle B\rangle_s^{ij}$ are both well-defined continuous stochastic processes on $[0, T]$ by a similar analysis.

8.3 Itô's Formula for General C^2 Functions

The objective of this section is to give a very general form of Itô's formula with respect to G-Brownian motion, which is comparable with that from the classical Itô's calculus.

Consider the following G-Itô diffusion process:

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^{\nu ij} d\langle B\rangle_s^{ij} + \int_0^t \beta_s^{\nu j} dB_s^j.$$

Lemma 8.3.1 *Suppose that $\Phi \in C^2(\mathbb{R}^n)$ and that all first and second order derivatives of Φ are in $C_{b,Lip}(\mathbb{R}^n)$. Let α^ν, $\beta^{\nu j}$ and $\eta^{\nu ij}$, $\nu = 1, \cdots, n$, $i, j = 1, \cdots, d$, be bounded processes in $M_*^2(0, T)$. Then for any $t \geq 0$, we have in $L_*^2(\Omega_t)$,*

$$\begin{aligned}\Phi(X_t) - \Phi(X_0) = &\int_0^t \partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j} dB_u^j + \int_0^t \partial_{x^\nu}\Phi(X_u)\alpha_u^\nu du \\ &+ \int_0^t [\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j}] d\langle B\rangle_u^{ij}.\end{aligned} \quad (8.3.1)$$

The proof is parellel to that of Proposition 6.3, in Chap. 3. The details are left as an exercise for the readers.

Lemma 8.3.2 *Suppose that $\Phi \in C^2(\mathbb{R}^n)$ and all first and second order derivatives of Φ are in $C_{b,Lip}(\mathbb{R}^n)$. Let α^ν, $\beta^{\nu j}$ be in $M_*^1(0, T)$ and $\eta^{\nu ij}$ belong to $M_*^2(0, T)$ for $\nu = 1, \cdots, n$, $i, j = 1, \cdots, d$. Then for any $t \geq 0$, relation (8.3.1) holds in $L_*^1(\Omega_t)$.*

Proof For simplicity, we only deal with the case $n = d = 1$. Let $\alpha^{(k)}$, $\beta^{(k)}$ and $\eta^{(k)}$ be bounded processes such that, as $k \to \infty$,

$$\alpha^{(k)} \to \alpha, \ \ \eta^{(k)} \to \eta \text{ in } M_*^1(0, T) \ \text{ and } \ \beta^{(k)} \to \beta \text{ in } M_*^2(0, T)$$

and let

$$X_t^{(k)} = X_0 + \int_0^t \alpha_s^{(k)} ds + \int_0^t \eta_s^{(k)} d\langle B\rangle_s + \int_0^t \beta_s^{(k)} dB_s.$$

Then applying Hölder's inequality and BDG inequality yields that

$$\lim_{k\to\infty} \hat{\mathbb{E}}[\sup_{0\leq t\leq T} |X_t^{(k)} - X_t|] = 0 \ \text{ and } \ \lim_{k\to\infty} \hat{\mathbb{E}}[\sup_{0\leq t\leq T} |\Phi(X_t^{(k)}) - \Phi(X_t)|] = 0.$$

Note that

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\int_0^T |\partial_x \Phi(X_t^{(k)})\beta_t^{(k)} - \partial_x \Phi(X_t)\beta_t|^2 dt\right] \\
&\quad \le 2\hat{\mathbb{E}}\left[\int_0^T |\partial_x \Phi(X_t^{(k)})\beta_t^{(k)} - \partial_x \Phi(X_t^{(k)})\beta_t|^2 dt\right] \\
&\qquad + 2\hat{\mathbb{E}}\left[\int_0^T |\partial_x \Phi(X_t^{(k)})\beta_t - \partial_x \Phi(X_t)\beta_t|^2 dt\right] \\
&\quad \le 2C^2\hat{\mathbb{E}}\left[\int_0^T |\beta_t^{(k)} - \beta_t|^2 dt\right] + 2\hat{\mathbb{E}}\left[\int_0^T |\beta_t|^2 |\partial_x \Phi(X_t^{(k)}) - \partial_x \Phi(X_t)|^2 dt\right],
\end{aligned}$$

where C is the upper bound of $\partial_x \Phi$. Since $\sup_{0\le t\le T} |\partial_x \Phi(X_t^{(k)}) - \partial_x \Phi(X_t)|^2 \le 4C^2$, we conclude that

$$\hat{\mathbb{E}}\left[\int_0^T |\partial_x \Phi(X_t^{(k)}) - \partial_x \Phi(X_t)|^2 dt\right] \to 0, \text{ as } k \to \infty.$$

Thus we can apply Proposition 8.2.3 to prove that, in $M_*^2(0, T)$, as $k \to \infty$,

$$\begin{aligned}
&\partial_x \Phi(X^{(k)})\beta^{(k)} \to \partial_x \Phi(X)\beta, \quad \partial_x \Phi(X^{(k)})\alpha^{(k)} \to \partial_x \Phi(X)\alpha, \\
&\quad \partial_x \Phi(X^{(k)})\eta^{(k)} \to \partial_x \Phi(X)\eta, \quad \partial_{xx}^2 \Phi(X^{(k)})(\beta^{(k)})^2 \to \partial_{xx}^2 \Phi(X)\beta^2.
\end{aligned}$$

However, from the above lemma we have

$$\begin{aligned}
\Phi(X_t^{(k)}) - \Phi(X_0^{(k)}) &= \int_0^t \partial_x \Phi(X_u^{(k)})\beta_u^{(k)} dB_u + \int_0^t \partial_x \Phi(X_u^{(k)})\alpha_u^{(k)} du \\
&\quad + \int_0^t [\partial_x \Phi(X_u^{(k)})\eta_u^{(k)} + \tfrac{1}{2}\partial_{xx}^2 \Phi(X_u^{(k)})(\beta_u^{(k)})^2] d\langle B\rangle_u.
\end{aligned}$$

Therefore passing to the limit on both sides of this equality, we obtain the desired result. □

Lemma 8.3.3 *Let X be given as in Lemma 8.3.2 and let $\Phi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ be such that Φ, $\partial_t \Phi$, $\partial_x \Phi$ and $\partial_{xx}^2 \Phi$ are bounded and uniformly continuous on $[0, T] \times \mathbb{R}^n$. Then we have the following relation in $L_*^1(\Omega_t)$:*

$$\begin{aligned}
\Phi(t, X_t) - \Phi(0, X_0) &= \int_0^t \partial_{x^\nu} \Phi(u, X_u)\beta_u^{\nu j} dB_u^j + \int_0^t [\partial_t \Phi(u, X_u) + \partial_{x^\nu} \Phi(u, X_u)\alpha_u^\nu] du \\
&\quad + \int_0^t [\partial_{x^\nu} \Phi(u, X_u)\eta_u^{\nu ij} + \tfrac{1}{2}\partial_{x^\mu x^\nu}^2 \Phi(u, X_u)\beta_u^{\mu i}\beta_u^{\nu j}] d\langle B\rangle_u^{ij}.
\end{aligned}$$

Proof Choose a sequence of functions $\{\Phi_k\}_{k=1}^\infty$ such that, Φ_k and all its first order and second order derivatives are in $C_{b,Lip}([0, T] \times \mathbb{R}^n)$. Moreover, as $n \to \infty$, Φ_n,

$\partial_t \Phi_n$, $\partial_x \Phi_n$ and $\partial^2_{xx} \Phi_n$ converge respectively to Φ, $\partial_t \Phi$, $\partial_x \Phi$ and $\partial^2_{xx} \Phi$ uniformly on $[0, T] \times \mathbb{R}$. Then we use the above Itô's formula to $\Phi_k(X^0_t, X_t)$, with $Y_t = (X^0_t, X_t)$, where $X^0_t \equiv t$:

$$\Phi_k(t, X_t) - \Phi_k(0, X_0) = \int_0^t \partial_{x^\nu} \Phi_k(u, X_u)\beta^{\nu j}_u dB^j_u + \int_0^t [\partial_t \Phi_k(u, X_u) + \partial_{x^\nu} \Phi_k(u, X_u)\alpha^\nu_u]du$$
$$+ \int_0^t [\partial_{x^\nu} \Phi_k(u, X_u)\eta^{\nu ij}_u + \tfrac{1}{2}\partial^2_{x^\mu x^\nu} \Phi_k(u, X_u)\beta^{\mu i}_u \beta^{\nu j}_u] d\langle B\rangle^{ij}_u .$$

It follows that, as $k \to \infty$, the following uniform convergences:

$$|\partial_{x_\nu} \Phi_k(u, X_u) - \partial_{x_\nu} \Phi(u, X_u)| \to 0, \ |\partial^2_{x_\mu x_\nu} \Phi_k(u, X_u) - \partial^2_{x_\mu x_\nu} \Phi_k(u, X_u)| \to 0,$$
$$|\partial_t \Phi_k(u, X_u) - \partial_t \Phi(u, X_u)| \to 0.$$

Sending $k \to \infty$, we arrive at the desired result. □

Theorem 8.3.4 *Suppose $\Phi \in C^{1,2}([0, T] \times \mathbb{R}^n)$. Let α^ν, $\eta^{\nu ij}$ be in $M^1_w(0, T)$ and $\beta^{\nu j}$ be in $M^2_w(0, T)$ associated with a common stopping time sequence $\{\sigma_m\}_{m=1}^\infty$. Then for any $t \geq 0$, we have q.s.*

$$\Phi(t, X_t) - \Phi(0, X_0) = \int_0^t \partial_{x^\nu} \Phi(u, X_u)\beta^{\nu j}_u dB^j_u + \int_0^t [\partial_t \Phi(u, X_u) + \partial_{x^\nu} \Phi(u, X_u)\alpha^\nu_u]du$$
$$+ \int_0^t [\partial_{x^\nu} \Phi(u, X_u)\eta^{\nu ij}_u + \tfrac{1}{2}\partial^2_{x^\mu x^\nu} \Phi(u, X_u)\beta^{\mu i}_u \beta^{\nu j}_u] d\langle B\rangle^{ij}_u .$$

Proof For simplicity, we only deal with the case $n = d = 1$. We set, for $k = 1, 2, \cdots$,

$$\tau_k := \inf\{t \geq 0| \ \ |X_t - X_0| > k\} \wedge \sigma_k.$$

Let Φ_k be a $C^{1,2}$-function on $[0, T] \times \mathbb{R}^n$ such that Φ_k, $\partial_t \Phi_k$, $\partial_{x_i} \Phi_k$ and $\partial^2_{x_i x_j} \Phi_k$ are uniformly bounded continuous functions satisfying $\Phi_k = \Phi$, for $|x| \leq 2k, t \in [0, T]$. It is clear that the process $\mathbf{1}_{[0,\tau_k]}\beta$ is in $M^2_w(0, T)$, while $\mathbf{1}_{[0,\tau_k]}\alpha$ and $\mathbf{1}_{[0,\tau_k]}\eta$ are in $M^1_w(0, T)$ and they are all associated to the same sequence of stopping times $\{\tau_k\}_{k=1}^\infty$. We also have

$$X_{t\wedge\tau_k} = X_0 + \int_0^t \alpha_s \mathbf{1}_{[0,\tau_k]} ds + \int_0^t \eta_s \mathbf{1}_{[0,\tau_k]} d\langle B\rangle_s + \int_0^t \beta_s \mathbf{1}_{[0,\tau_k]} dB_s$$

Then we can apply Lemma 8.3.3 to $\Phi_k(s, X_{s\wedge\tau_k})$, $s \in [0, t]$, to obtain

$$\Phi(t, X_{t\wedge\tau_k}) - \Phi(0, X_0)$$
$$= \int_0^t \partial_x \Phi(u, X_u)\beta_u \mathbf{1}_{[0,\tau_k]} dB_u + \int_0^t [\partial_t \Phi(u, X_u) + \partial_x \Phi(u, X_u)\alpha_u]\mathbf{1}_{[0,\tau_k]} du$$
$$+ \int_0^t [\partial_x \Phi(u, X_u)\eta_u \mathbf{1}_{[0,\tau_k]} + \tfrac{1}{2}\partial^2_{xx} \Phi(u, X_u)|\beta_u|^2 \mathbf{1}_{[0,\tau_k]}] d\langle B\rangle_u.$$

Letting $k \to \infty$ and noticing that X_t is continuous in t, we get the desired result. □

Example 8.3.5 For given $\varphi \in C^2(\mathbb{R})$, we have

$$\Phi(B_t) - \Phi(B_0) = \int_0^t \Phi_x(B_s)dB_s + \frac{1}{2}\int_0^t \Phi_{xx}(B_s)d\langle B\rangle_s .$$

This generalizes the previous results to more general situations.

Notes and Comments

The results in this chapter were mainly obtained by Li and Peng [110, 2011]. Li and Lin [109, 2013] found a point of incompleteness and proposed to use a more essential condition (namely, Condition (ii) in Definition 8.2.8) to replace the original one which was $\int_0^T |\eta_t|^p dt < \infty$, q.s.

A difficulty hidden behind is that the G-expectation theory is mainly based on the space of random variables $X = X(\omega)$ which are quasi-continuous with respect to the G-capacity $\hat{c}$. It is not yet clear that the martingale properties still hold for random variables without quasi-continuity condition.

There are still several interesting and fundamentally important issues on G-expectation theory and its applications. It is known that stopping times play a fundamental role in classical stochastic analysis. However, it is often nontrivial to directly apply stopping time techniques in a G-expectation space. The reason is that the stopped process may not belong to the class of processes which are meaningful in the G-framework. Song [160] considered the properties of hitting times for G- martingale and, moreover the stopped processes. He proved that the stopped processes for G-martingales are still G-martingales and that the hitting times for symmetric G-martingales with strictly increasing quadratic variation processes are quasi-continuous. Hu and Peng [82] introduced a suitable definition of stopping times and obtained the optional stopping theorem.