Chapter 5
Stochastic Differential Equations

In this chapter, we consider the stochastic differential equations and backward stochastic differential equations driven by G-Brownian motion. The conditions and proofs of existence and uniqueness of a stochastic differential equation is similar to the classical situation. However the corresponding problems for backward stochastic differential equations are not that easy, many are still open. We only give partial results to this direction.

5.1 Stochastic Differential Equations

In this chapter, we denote by $\bar{M}_G^p(0,T;\mathbb{R}^n)$, $p \geq 1$, the completion of $M_G^{p,0}(0,T;\mathbb{R}^n)$ under the norm $(\int_0^T \hat{\mathbb{E}}[|\eta_t|^p]dt)^{1/p}$. It is not hard to prove that $\bar{M}_G^p(0,T;\mathbb{R}^n) \subseteq M_G^p(0,T;\mathbb{R}^n)$. We consider all the problems in the space $\bar{M}_G^p(0,T;\mathbb{R}^n)$. The following lemma is useful in our future discussion.

Lemma 5.1.1 *Suppose that $\varphi \in M_G^2(0,T)$. Then for $\mathbf{a} \in \mathbb{R}^d$, it holds that*

$$\eta_t := \int_0^t \varphi_s dB_s^{\mathbf{a}} \in \bar{M}_G^2(0,T).$$

Proof Choosing a sequence of processes $\varphi^n \in M_G^{2,0}(0,T)$ such that

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\varphi_s - \varphi_s^n|^2 ds\right] = 0.$$

Then for each integer n, it is easy to check that the process $\eta_t^n = \int_0^t \varphi_s^n dB_s^{\mathbf{a}}$ belongs to the space $\bar{M}_G^2(0,T)$.

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On the other hand, it follows from the property of G-Itô integral that

$$\int_0^T \hat{\mathbb{E}}[|\eta_t - \eta_t^n|^2]dt = \sigma^2_{\mathbf{aa}^T} \int_0^T \hat{\mathbb{E}}\left[\int_0^t |\varphi_s - \varphi_s^n|^2 ds\right] dt \le \sigma^2_{\mathbf{aa}^T} T \hat{\mathbb{E}}\left[\int_0^T |\varphi_s - \varphi_s^n|^2 ds\right],$$

which implies the desired result. □

Now we consider the following SDE driven by a d-dimensional G-Brownian motion:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t h_{ij}(s, X_s)d\langle B\rangle_s^{ij} + \int_0^t \sigma_j(s, X_s)dB_s^j, \; t \in [0, T], \tag{5.1.1}$$

where the initial condition $X_0 \in \mathbb{R}^n$ is a given constant, b, h_{ij}, σ_j are given functions satisfying $b(\cdot, x)$, $h_{ij}(\cdot, x)$, $\sigma_j(\cdot, x) \in M_G^2(0, T; \mathbb{R}^n)$ for each $x \in \mathbb{R}^n$ and the Lipschitz condition, i.e., $|\phi(t, x) - \phi(t, x')| \le K|x - x'|$, for each $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $\phi = b$, h_{ij} and σ_j, respectively. Here the horizon $[0, T]$ can be arbitrarily large. The solution is a process $(X_t)_{t\in[0,T]} \in \bar{M}_G^2(0, T; \mathbb{R}^n)$ satisfying the SDE (5.1.1).

We first introduce the following mapping on a fixed interval $[0, T]$:

$$\Lambda_\cdot : \bar{M}_G^2(0, T; \mathbb{R}^n) \mapsto \bar{M}_G^2(0, T; \mathbb{R}^n)$$

by setting Λ_t, $t \in [0, T]$, with

$$\Lambda_t(Y) = X_0 + \int_0^t b(s, Y_s)ds + \int_0^t h_{ij}(s, Y_s)d\langle B\rangle_s^{ij} + \int_0^t \sigma_j(s, Y_s)dB_s^j.$$

From Lemma 5.1.1 and Exercise 5.4.2 of this chapter, we see that the mapping Λ is well-defined.

We immediately have the following lemma, whose proof is left to the reader.

Lemma 5.1.2 *For any $Y, Y' \in \bar{M}_G^2(0, T; \mathbb{R}^n)$, we have the following estimate:*

$$\hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|^2] \le C \int_0^t \hat{\mathbb{E}}[|Y_s - Y_s'|^2]ds, \; t \in [0, T], \tag{5.1.2}$$

where the constant C depends only on the Lipschitz constant K.

We now prove that the SDE (5.1.1) has a unique solution. We multiply on both sides of (5.1.2) by e^{-2Ct} and integrate them on $[0, T]$, thus deriving

$$\begin{aligned}\int_0^T \hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|^2]e^{-2Ct}dt &\le C \int_0^T e^{-2Ct} \int_0^t \hat{\mathbb{E}}[|Y_s - Y_s'|^2]dsdt \\ &= C \int_0^T \int_s^T e^{-2Ct}dt \hat{\mathbb{E}}[|Y_s - Y_s'|^2]ds \\ &= \frac{1}{2}\int_0^T (e^{-2Cs} - e^{-2CT})\hat{\mathbb{E}}[|Y_s - Y_s'|^2]ds.\end{aligned}$$

We then have

$$\int_0^T \hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|^2]e^{-2Ct}dt \leq \frac{1}{2}\int_0^T \hat{\mathbb{E}}[|Y_t - Y_t'|^2]e^{-2Ct}dt. \qquad (5.1.3)$$

Note that the following two norms are equivalent in the space $\bar{M}_G^2(0, T; \mathbb{R}^n)$:

$$\left(\int_0^T \hat{\mathbb{E}}[|Y_t|^2]dt\right)^{1/2} \sim \left(\int_0^T \hat{\mathbb{E}}[|Y_t|^2]e^{-2Ct}dt\right)^{1/2}.$$

From (5.1.3) we obtain that $\Lambda(Y)$ is a contraction mapping. Consequently, we have the following theorem.

Theorem 5.1.3 *There exists a unique solution $(X_t)_{0\leq t\leq T} \in \bar{M}_G^2(0, T; \mathbb{R}^n)$ of the stochastic differential equation* (5.1.1).

We now consider a particular but important case of a linear SDE. For simplicity, assume that $d = 1$, $n = 1$. and let

$$X_t = X_0 + \int_0^t (b_s X_s + \tilde{b}_s)ds + \int_0^t (h_s X_s + \tilde{h}_s)d\langle B\rangle_s + \int_0^t (\sigma_s X_s + \tilde{\sigma}_s)dB_s,\ t \in [0, T]. \qquad (5.1.4)$$

Here $X_0 \in \mathbb{R}$ is given, $b_., h_., \sigma_.$ are given bounded processes in $M_G^2(0, T; \mathbb{R})$ and $\tilde{b}_., \tilde{h}_., \tilde{\sigma}_.$ are given processes in $M_G^2(0, T; \mathbb{R})$. It follows from Theorem 5.1.3 that the linear SDE (5.1.4) has a unique solution.

Remark 5.1.4 The solution of the linear SDE (5.1.4) is

$$X_t = \Gamma_t^{-1}(X_0 + \int_0^t \tilde{b}_s\Gamma_s ds + \int_0^t (\tilde{h}_s - \sigma_s\tilde{\sigma}_s)\Gamma_s d\langle B\rangle_s + \int_0^t \tilde{\sigma}_s\Gamma_s dB_s),\ t \in [0, T],$$

where $\Gamma_t = \exp(-\int_0^t b_s ds - \int_0^t (h_s - \frac{1}{2}\sigma_s^2)d\langle B\rangle_s - \int_0^t \sigma_s dB_s)$.

In particular, if $b_., h_., \sigma_.$ are constants and $\tilde{b}_., \tilde{h}_., \tilde{\sigma}_.$ are zero, then X is a geometric G-Brownian motion.

Definition 5.1.5 We say that $(X_t)_{t\geq 0}$ is a **geometric G-Brownian motion** if

$$X_t = \exp(\alpha t + \beta\langle B\rangle_t + \gamma B_t), \qquad (5.1.5)$$

where α, β, γ are constants.

5.2 Backward Stochastic Differential Equations (BSDE)

We consider the following type of BSDE:

$$Y_t = \hat{\mathbb{E}}\left[\xi + \int_t^T f(s, Y_s)ds + \int_t^T h_{ij}(s, Y_s)d\langle B\rangle_s^{ij} \,\Big|\Omega_t\right], \quad t \in [0, T], \tag{5.2.1}$$

where $\xi \in L_G^1(\Omega_T; \mathbb{R}^n)$, f, h_{ij} are given functions such that $f(\cdot, y)$, $h_{ij}(\cdot, y) \in M_G^1(0, T; \mathbb{R}^n)$ for each $y \in \mathbb{R}^n$ and these functions satisfy the Lipschitz condition, i.e.,

$$|\phi(t, y) - \phi(t, y')| \leq K|y - y'|, \quad \text{for each } \ t \in [0, T], \quad y, y' \in \mathbb{R}^n, \quad \phi = f \ \text{ and } h_{ij}.$$

The solution is a process $(Y_t)_{0\leq t\leq T} \in \bar{M}_G^1(0, T; \mathbb{R}^n)$ satisfying the above BSDE.

We first introduce the following mapping on a fixed interval $[0, T]$:

$$\Lambda_\cdot : \bar{M}_G^1(0, T; \mathbb{R}^n) \to \bar{M}_G^1(0, T; \mathbb{R}^n)$$

by setting Λ_t, $t \in [0, T]$ as follows:

$$\Lambda_t(Y) = \hat{\mathbb{E}}\left[\xi + \int_t^T f(s, Y_s)ds + \int_t^T h_{ij}(s, Y_s)d\langle B\rangle_s^{ij} \,\Big|\Omega_t\right],$$

which is well-defined by Lemma 5.1.1 and Exercises 5.4.2, 5.4.5.

We immediately derive a useful property of Λ_t.

Lemma 5.2.1 *For any $Y, Y' \in \bar{M}_G^1(0, T; \mathbb{R}^n)$, we have the following estimate:*

$$\hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|] \leq C \int_t^T \hat{\mathbb{E}}[|Y_s - Y_s'|]ds, \ t \in [0, T], \tag{5.2.2}$$

where the constant C depends only on the Lipschitz constant K.

Now we are going to prove that the BSDE (5.2.1) has a unique solution. We multiplying on both sides of (5.2.2) by e^{2Ct}, and integrate them on $[0, T]$. We find

$$\begin{aligned}
\int_0^T \hat{\mathbb{E}}[|\Lambda_t(Y) - \Lambda_t(Y')|]e^{2Ct}dt &\leq C \int_0^T \int_t^T \hat{\mathbb{E}}[|Y_s - Y_s'|]e^{2Ct}dsdt \\
&= C \int_0^T \hat{\mathbb{E}}[|Y_s - Y_s'|] \int_0^s e^{2Ct}dtds \\
&= \frac{1}{2} \int_0^T \hat{\mathbb{E}}[|Y_s - Y_s'|](e^{2Cs} - 1)ds \\
&\leq \frac{1}{2} \int_0^T \hat{\mathbb{E}}[|Y_s - Y_s'|]e^{2Cs}ds.
\end{aligned} \tag{5.2.3}$$

We observe that the following two norms in the space $\bar{M}_G^1(0, T; \mathbb{R}^n)$ are equivalent:

$$\int_0^T \hat{\mathbb{E}}[|Y_t|]dt \sim \int_0^T \hat{\mathbb{E}}[|Y_t|]e^{2Ct}dt.$$

From (5.2.3), we can obtain that $\Lambda(Y)$ is a contraction mapping. Consequently, we have proved the following theorem.

Theorem 5.2.2 *There exists a unique solution $(Y_t)_{t\in[0,T]} \in \bar{M}_G^1(0, T; \mathbb{R}^n)$ of the backward stochastic differential equation* (5.2.1).

Let $Y^{(v)}$, $v = 1, 2$, be the solutions of the following BSDE:

$$Y_t^{(v)} = \hat{\mathbb{E}}\left[\xi^{(v)} + \int_t^T (f(s, Y_s^{(v)}) + \varphi_s^{(v)})ds + \int_t^T (h_{ij}(s, Y_s^{(v)}) + \psi_s^{ij,(v)})d\langle B\rangle_s^{ij} \Big|\Omega_t\right].$$

Then the following estimate holds.

Proposition 5.2.3 *We have*

$$\hat{\mathbb{E}}\left[|Y_t^{(1)} - Y_t^{(2)}|\right] \leq Ce^{C(T-t)}\hat{\mathbb{E}}[|\xi^{(1)} - \xi^{(2)}| + \int_t^T |\varphi_s^{(1)} - \varphi_s^{(2)}| + |\psi_s^{ij,(1)} - \psi_s^{ij,(2)}|ds], \tag{5.2.4}$$

where the constant C depends only on the Lipschitz constant K.

Proof As in the proof of Lemma 5.2.1, we have

$$\hat{\mathbb{E}}[|Y_t^{(1)} - Y_t^{(2)}|] \leq C\left(\int_t^T \hat{\mathbb{E}}[|Y_s^{(1)} - Y_s^{(2)}|]ds + \hat{\mathbb{E}}[|\xi^{(1)} - \xi^{(2)}| \right.$$
$$\left. + \int_t^T |\varphi_s^{(1)} - \varphi_s^{(2)}| + |\psi_s^{ij,(1)} - \psi_s^{ij,(2)}|ds]\right).$$

By applying the Gronwall inequality (see Exercise 5.4.4), we obtain the statement.

Remark 5.2.4 In particular, if $\xi^{(2)} = 0, \varphi_s^{(2)} = -f(s, 0), \psi_s^{ij,(2)} = -h_{ij}(s, 0), \xi^{(1)} = \xi$, $\varphi_s^{(1)} = 0$, $\psi_s^{ij,(1)} = 0$, we obtain the estimate of the solution of the BSDE. Let Y be the solution of the BSDE (5.2.1). Then

$$\hat{\mathbb{E}}[|Y_t|] \leq Ce^{C(T-t)}\hat{\mathbb{E}}\left[|\xi| + \int_t^T |f(s, 0)| + |h_{ij}(s, 0)|ds\right], \tag{5.2.5}$$

where the constant C depends only on the Lipschitz constant K.

5.3 Nonlinear Feynman-Kac Formula

Consider the following SDE:

$$\begin{cases} dX_s^{t,\xi} = b(X_s^{t,\xi})ds + h_{ij}(X_s^{t,\xi})d\langle B\rangle_s^{ij} + \sigma_j(X_s^{t,\xi})dB_s^j, \ s\in[t,T], \\ X_t^{t,\xi} = \xi, \end{cases} \tag{5.3.1}$$

where $\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$ and $b, h_{ij}, \sigma_j : \mathbb{R}^n \mapsto \mathbb{R}^n$ are given Lipschitz functions, i.e., $|\phi(x)-\phi(x')| \le K|x-x'|$, for all $x, x' \in \mathbb{R}^n$, $\phi = b$, h_{ij} and σ_j.

We then consider the associated BSDE:

$$Y_s^{t,\xi} = \hat{\mathbb{E}}\left[\Phi(X_T^{t,\xi}) + \int_s^T f(X_r^{t,\xi}, Y_r^{t,\xi})dr + \int_s^T g_{ij}(X_r^{t,\xi}, Y_r^{t,\xi})d\left\langle B^i, B^j\right\rangle_r \Big|\Omega_s\right], \tag{5.3.2}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a given Lipschitz function and $f, g_{ij} : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ are given Lipschitz functions, i.e., $|\phi(x,y)-\phi(x',y')| \le K(|x-x'|+|y-y'|)$, for each x, $x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $\phi = f$ and g_{ij}.

We have the following estimates:

Proposition 5.3.1 *For each $\xi, \xi' \in L_G^2(\Omega_t;\mathbb{R}^n)$, we have, for each $s\in[t,T]$,*

$$\hat{\mathbb{E}}[|X_s^{t,\xi} - X_s^{t,\xi'}|^2|\Omega_t] \le C|\xi-\xi'|^2 \tag{5.3.3}$$

and

$$\hat{\mathbb{E}}[|X_s^{t,\xi}|^2|\Omega_t] \le C(1+|\xi|^2), \tag{5.3.4}$$

where the constant C depends only on the Lipschitz constant K.

Proof It is easy to see that

$$\hat{\mathbb{E}}[|X_s^{t,\xi} - X_s^{t,\xi'}|^2\big|\Omega_t] \le C_1(|\xi-\xi'|^2 + \int_t^s \hat{\mathbb{E}}[|X_r^{t,\xi} - X_r^{t,\xi'}|^2|\Omega_t]dr).$$

By the Gronwall inequality, we obtain (5.3.3), namely

$$\hat{\mathbb{E}}[|X_s^{t,\xi} - X_s^{t,\xi'}|^2|\Omega_t] \le C_1 e^{C_1 T}|\xi-\xi'|^2.$$

Similarly, we derive (5.3.4). □

Corollary 5.3.2 *For any $\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$, we have*

$$\hat{\mathbb{E}}[|X_{t+\delta}^{t,\xi} - \xi|^2|\Omega_t] \le C(1+|\xi|^2)\delta \ \textit{for}\ \delta\in[0,T-t], \tag{5.3.5}$$

where the constant C depends only on the Lipschitz constant K.

Proof It is easy to see that

$$\hat{\mathbb{E}}[|X_{t+\delta}^{t,\xi} - \xi|^2 \big| \Omega_t] \leq C_1 \int_t^{t+\delta} \left(1 + \hat{\mathbb{E}}[|X_s^{t,\xi}|^2 \big| \Omega_t]\right) ds.$$

Then the result follows from Proposition 5.3.1. □

Proposition 5.3.3 *For each $\xi, \xi' \in L_G^2(\Omega_t; \mathbb{R}^n)$, we have*

$$|Y_t^{t,\xi} - Y_t^{t,\xi'}| \leq C|\xi - \xi'| \tag{5.3.6}$$

and

$$|Y_t^{t,\xi}| \leq C(1 + |\xi|), \tag{5.3.7}$$

where the constant C depends only on the Lipschitz constant K.

Proof For each $s \in [0, T]$, it is easy to check that

$$|Y_s^{t,\xi} - Y_s^{t,\xi'}| \leq C_1 \hat{\mathbb{E}}\left[|X_T^{t,\xi} - X_T^{t,\xi'}| + \int_s^T (|X_r^{t,\xi} - X_r^{t,\xi'}| + |Y_r^{t,\xi} - Y_r^{t,\xi'}|) dr |\Omega_s\right].$$

Since

$$\hat{\mathbb{E}}[|X_s^{t,\xi} - X_s^{t,\xi'}||\Omega_t] \leq \left(\hat{\mathbb{E}}[|X_s^{t,\xi} - X_s^{t,\xi'}|^2|\Omega_t]\right)^{1/2},$$

we have

$$\hat{\mathbb{E}}[|Y_s^{t,\xi} - Y_s^{t,\xi'}||\Omega_t] \leq C_2(|\xi - \xi'| + \int_s^T \hat{\mathbb{E}}[|Y_r^{t,\xi} - Y_r^{t,\xi'}||\Omega_t] dr).$$

By the Gronwall inequality, we obtain (5.3.6). Similarly we derive (5.3.7). □

We are more interested in the case when $\xi = x \in \mathbb{R}^n$. Define

$$u(t, x) := Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^n. \tag{5.3.8}$$

By Proposition 5.3.3, we immediately have the following estimates:

$$|u(t, x) - u(t, x')| \leq C|x - x'|, \tag{5.3.9}$$

$$|u(t, x)| \leq C(1 + |x|), \tag{5.3.10}$$

where the constant C depends only on the Lipschitz constant K.

Remark 5.3.4 It is important to note that $u(t, x)$ is a deterministic function of (t, x), because $X_s^{t,x}$ and $Y_s^{t,x}$ are independent from Ω_t.

Theorem 5.3.5 *For any $\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$, we have*

$$u(t,\xi) = Y_t^{t,\xi}. \tag{5.3.11}$$

Proof Without loss of generality, suppose that $n = 1$.

First, we assume that $\xi \in Lip(\Omega_T)$ is bounded by some constant ρ. Thus for each integer $N > 0$, we can choose a simple function

$$\eta^N = \sum_{i=-N}^{N} x_i \mathbf{I}_{A_i}(\xi)$$

with $x_i = \frac{i\rho}{N}$, $A_i = [\frac{i\rho}{N}, \frac{(i+1)\rho}{N})$ for $i = -N, \ldots, N-1$ and $x_N = \rho$, $A_N = \{\rho\}$. From the definition of u, we conclude that

$$\begin{aligned}|Y_t^{t,\xi} - u(t,\eta^N)| = |Y_t^{t,\xi} - \sum_{i=-N}^{N} u(t,x_i)\mathbf{I}_{A_i}(\xi)| &= |Y_t^{t,\xi} - \sum_{i=-N}^{N} Y_t^{t,x_i}\mathbf{I}_{A_i}(\xi)| \\ &= \sum_{i=-N}^{N} |Y_t^{t,\xi} - Y_t^{t,x_i}|\mathbf{I}_{A_i}(\xi).\end{aligned}$$

Then it follows from Proposition 5.3.3 that

$$|Y_t^{t,\xi} - u(t,\eta^N)| \le C\sum_{i=-N}^{N} |\xi - x_i|\mathbf{I}_{A_i}(\xi) \le C\frac{\rho}{N}.$$

Noting that

$$|u(t,\xi) - u(t,\eta^N)| \le C|\xi - \eta^N| \le C\frac{\rho}{N},$$

we get $\hat{\mathbb{E}}[|Y_t^{t,\xi} - u(t,\xi)|] \le 2C\frac{\rho}{N}$. Since N can be arbitrarily large, we obtain $Y_t^{t,\xi} = u(t,\xi)$.

In the general case, by Exercise 3.10.4 in Chap. 3, we can find a sequence of bounded random variables $\xi_k \in Lip(\Omega_T)$ such that

$$\lim_{k\to\infty} \hat{\mathbb{E}}[|\xi - \xi_k|^2] = 0.$$

Consequently, applying Proposition 5.3.3 again yields that

$$\lim_{k\to\infty} \hat{\mathbb{E}}[|Y_t^{t,\xi} - Y_t^{t,\xi_k}|^2] \le C\lim_{k\to\infty} \hat{\mathbb{E}}[|\xi - \xi_k|^2] = 0,$$

which together with $Y_t^{t,\xi_k} = u(t,\xi_k)$ imply the desired result. □

Proposition 5.3.6 *We have, for* $\delta \in [0, T-t]$,

$$u(t,x) = \hat{\mathbb{E}}\left[u(t+\delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} f(X_r^{t,x}, Y_r^{t,x})dr + \int_t^{t+\delta} g_{ij}(X_r^{t,x}, Y_r^{t,x})d\langle B\rangle_r^{ij}\right]. \tag{5.3.12}$$

Proof Since $X_s^{t,x} = X_s^{t+\delta, X_{t+\delta}^{t,x}}$ for $s \in [t+\delta, T]$, we get $Y_{t+\delta}^{t,x} = Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x}}$. By Theorem 5.3.5, we have $Y_{t+\delta}^{t,x} = u(t+\delta, X_{t+\delta}^{t,x})$, which implies the result. □

For any $A \in \mathbb{S}(n)$, $p \in \mathbb{R}^n$, $r \in \mathbb{R}$, we set

$$F(A, p, r, x) := G(B(A, p, r, x)) + \langle p, b(x)\rangle + f(x, r),$$

where $B(A, p, r, x)$ is a $d \times d$ symmetric matrix with

$$B_{ij}(A, p, r, x) := \langle A\sigma_i(x), \sigma_j(x)\rangle + \langle p, h_{ij}(x) + h_{ji}(x)\rangle + g_{ij}(x, r) + g_{ji}(x, r).$$

Theorem 5.3.7 *The function* $u(t,x)$ *is the unique viscosity solution of the following PDE:*

$$\begin{cases} \partial_t u + F(D^2u, Du, u, x) = 0, \\ u(T, x) = \Phi(x). \end{cases} \tag{5.3.13}$$

Proof We first show that u is a continuous function. By (5.3.9) we know that u is a Lipschitz function in x. It follows from (5.2.5) and (5.3.4) that

$$\hat{\mathbb{E}}[|Y_s^{t,x}|] \leq C(1+|x|), \quad \text{for } s \in [t, T].$$

In view of (5.3.5) and (5.3.12), we get $|u(t,x) - u(t+\delta, x)| \leq C(1+|x|)(\delta^{1/2} + \delta)$ for $\delta \in [0, T-t]$. Thus u is $\frac{1}{2}$-Hölder continuous in t, which implies that u is a continuous function. We can also show (see Exercise 5.4.8), that for each $p \geq 2$,

$$\hat{\mathbb{E}}[|X_{t+\delta}^{t,x} - x|^p] \leq C(1+|x|^p)\delta^{p/2}. \tag{5.3.14}$$

Now for fixed $(t,x) \in (0,T) \times \mathbb{R}^n$, let $\psi \in C_{l,Lip}^{2,3}([0,T] \times \mathbb{R}^n)$ be such that $\psi \geq u$ and $\psi(t,x) = u(t,x)$. By (5.3.12), (5.3.14) and Taylor's expansion, it follows that, for $\delta \in (0, T-t)$,

$$\begin{aligned} 0 &\leq \hat{\mathbb{E}}\Bigg[\psi(t+\delta, X_{t+\delta}^{t,x}) - \psi(t,x) + \int_t^{t+\delta} f(X_r^{t,x}, Y_r^{t,x})dr \\ &\quad + \int_t^{t+\delta} g_{ij}(X_r^{t,x}, Y_r^{t,x})d\left\langle B^i, B^j\right\rangle_r\Bigg] \\ &\leq \frac{1}{2}\hat{\mathbb{E}}[(B(D^2\psi(t,x), D\psi(t,x), \psi(t,x), x), \langle B\rangle_{t+\delta} - \langle B\rangle_t)] \\ &\quad + (\partial_t\psi(t,x) + \langle D\psi(t,x), b(x)\rangle + f(x, \psi(t,x)))\delta + C(1+|x|^m)\delta^{3/2} \end{aligned}$$

$$\leq (\partial_t \psi(t,x) + F(D^2\psi(t,x), D\psi(t,x), \psi(t,x), x))\delta + C(1+|x|^m)\delta^{3/2},$$

where m is some constant depending on the function ψ. Consequently, it is easy to check that

$$\partial_t \psi(t,x) + F(D^2\psi(t,x), D\psi(t,x), \psi(t,x), x) \geq 0.$$

This implies that u is a viscosity subsolution of (5.3.13). Similarly we can show that u is also a viscosity supersolution of (5.3.13). The uniqueness is from Theorem C.2.9 (in Appendix C). □

Example 5.3.8 Let $B = (B^1, B^2)$ be a 2-dimensional G-Brownian motion with

$$G(A) = G_1(a_{11}) + G_2(a_{22}),$$

where

$$G_i(a) = \frac{1}{2}(\overline{\sigma}_i^2 a^+ - \underline{\sigma}_i^2 a^-), \quad i = 1, 2.$$

In this case, we consider the following 1-dimensional SDE:

$$dX_s^{t,x} = \mu X_s^{t,x} ds + \nu X_s^{t,x} d\left\langle B^1 \right\rangle_s + \sigma X_s^{t,x} dB_s^2, \quad X_t^{t,x} = x,$$

where μ, ν and σ are constants.

The corresponding function u is defined by

$$u(t,x) := \hat{\mathbb{E}}[\varphi(X_T^{t,x})].$$

Then

$$u(t,x) = \hat{\mathbb{E}}[u(t+\delta, X_{t+\delta}^{t,x})]$$

and u is the viscosity solution of the following PDE:

$$\partial_t u + \mu x \partial_x u + 2G_1(\nu x \partial_x u) + \sigma^2 x^2 G_2(\partial_{xx}^2 u) = 0, \; u(T,x) = \varphi(x).$$

5.4 Exercises

Exercise 5.4.1 Prove that $\bar{M}_G^p(0,T;\mathbb{R}^n) \subseteq M_G^p(0,T;\mathbb{R}^n)$.

Exercise 5.4.2 Show that $b(s, Y_s) \in M_G^p(0,T;\mathbb{R}^n)$ for each $Y \in M_G^p(0,T;\mathbb{R}^n)$, where b is given by Eq. (5.1.1).

Exercise 5.4.3 Complete the proof of Lemma 5.1.2.

Exercise 5.4.4 (The Gronwall inequality) Let $u(t)$ be a Lebesgue integrable function in $[0,T]$ such that

$$u(t) \leq C + A \int_0^t u(s)ds \quad \text{for } 0 \leq t \leq T,$$

where $C > 0$ and $A > 0$ are constants. Prove that $u(t) \leq Ce^{At}$ for $0 \leq t \leq T$.

Exercise 5.4.5 For any $\xi \in L_G^1(\Omega_T; \mathbb{R}^n)$, show that the process $(\hat{\mathbb{E}}[\xi|\Omega_t])_{t\in[0,T]}$ belongs to $\bar{M}_G^1(0, T; \mathbb{R}^n)$.

Exercise 5.4.6 Complete the proof of Lemma 5.2.1.

Exercise 5.4.7 Suppose that ξ, f and h_{ij} are all deterministic functions. Solve the BSDE (5.2.1).

Exercise 5.4.8 For each $\xi \in L_G^p(\Omega_t; \mathbb{R}^n)$ with $p \geq 2$, show that SDE (5.3.1) has a unique solution in $\bar{M}_G^p(t, T; \mathbb{R}^n)$. Further, show that the following estimates hold:

$$\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi} - X_{t+\delta}^{t,\xi'}|^p] \leq C|\xi - \xi'|^p,$$

$$\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi}|^p] \leq C(1 + |\xi|^p),$$

$$\hat{\mathbb{E}}_t[\sup_{s\in[t,t+\delta]} |X_s^{t,\xi} - \xi|^p] \leq C(1 + |\xi|^p)\delta^{p/2},$$

where the constant C depends on L, G, p, n and T.

Exercise 5.4.9 Let $\widetilde{\mathbb{E}}$ be a nonlinear expectation dominated by G-expectation, where $\widetilde{G} : \mathbb{S}(d) \mapsto \mathbb{R}$ is dominated by G and $\widetilde{G}(0) = 0$. Then we replace the G-expectation $\hat{\mathbb{E}}$ by $\widetilde{\mathbb{E}}$ in BSDEs (5.2.1) and (5.3.2). Show that

(i) the BSDE (5.2.1) admits a unique solution $Y \in \bar{M}_G^1(0, T)$.
(ii) u is the unique viscosity solution of the PDE (5.3.13) corresponding to $\widetilde{G}$.

Notes and Comments

The material in this chapter is mainly from Peng [140].

There are many excellent books on Itô's stochastic calculus and stochastic differential equations based by Itô's original paper [92]. The ideas of that notes were further developed to build the nonlinear martingale theory. For the corresponding classical Brownian motion framework under a probability measure space, readers are referred to Chung and Williams [34], Dellacherie and Meyer [43], He, Wang and Yan [74], Itô and McKean [93], Ikeda and Watanabe [90], Kallenberg [100], Karatzas and Shreve [101], Øksendal [122], Protter [150], Revuz and Yor [151] and Yong and Zhou [177].

Linear backward stochastic differential equations (BSDEs) were first introduced by Bismut in [17, 19]. Bensoussan developed this approach in [12, 13]. The existence and uniqueness theorem of a general nonlinear BSDE, was obtained in 1990 in Pardoux and Peng [124]. Here we present a version of a proof based on El Karoui, Peng and Quenez [58], which is an excellent survey paper on BSDE theory and its applications, especially in finance. Comparison theorem of BSDEs was obtained in Peng [128] for the case when g is a C^1-function and then in [58] when g is Lipschitz. Nonlinear Feynman-Kac formula for BSDE was introduced by Peng [127, 129]. Here we obtain the corresponding Feynman-Kac formula for a fully nonlinear PDE, within the framework of G-expectation. We also refer to Yong and Zhou [177], as well as Peng [131] (in 1997, in Chinese) and [133] and more resent monographs of Crepey [40], Pardoux and Rascanu [125] and Zhang [179] for systematic presentations of BSDE theory and its applications.

In the framework of fully nonlinear expectation, typically G-expectation, a challenging problem is to prove the well-posedness of a BSDE which is general enough to contain the above 'classical' BSDE as a special case. By applying and developing methods of quasi-surely analysis and aggregations, Soner et al. [156–158], introduced a weak formulation and then proved the existence and uniqueness of weak solution 2nd order BSDE (2BSDE). We also refer to Zhang [179] a systematic presentation. Then, by using a totally different approach of G-martingale representation and a type of Galerkin approximation, Hu et al. [79] proved the existence and uniqueness of solution of BSDE driven by G-Brownian motions (G-BSDE). As in the classical situation, G-BSDE is a natural generalization of representation of G-martingale. The assumption for the well-posedness of 2BSDEs is weaker than that of G-BSDE, whereas the solution (Y, Z, K) obtained by GBSDE is quasi-surely continuous which is in general smoother than that of 2BSDE. A very interesting problem is how to combine the advantages of both methods.

Then Hu and Wang [84] considered ergodic G-BSDEs, see also [77]. In [75], Hu, Lin and Soumana Hima studied G-BSDEs under quadratic assumptions on coefficients. In [111], Li, Peng and Soumana Hima investigated the existence and uniqueness theorem for reflected G-BSDEs. Furthermore, Cao and Tang [25] dealed with reflected Quadratic BSDEs driven by G-Brownian Motions.