Chapter 4
G-Martingales and Jensen's Inequality

In this chapter, we introduce the notion of G-martingales and the related Jensen's inequality for a new type of G-convex functions. One essential difference from the classical situation is that here "M is a G-martingale" does not imply that "$-M$ is a G-martingale".

4.1 The Notion of G-Martingales

We now give the notion of G-martingales.

Definition 4.1.1 A process $(M_t)_{t\geq 0}$ is called a ***G-supermartingale*** (respectively, ***G-submartingale***) if for any $t \in [0, \infty)$, $M_t \in L_G^1(\Omega_t)$ and for any $s \in [0, t]$, we have

$$\hat{\mathbb{E}}[M_t|\Omega_s] \leq M_s \quad \text{(respectively, } \geq M_s).$$

$(M_t)_{t\geq 0}$ is called a ***G-martingale*** if it is both G-supermartingale and G-submartingale. If a G-martingale M satisfies also

$$\hat{\mathbb{E}}[-M_t|\Omega_s] = -M_s,$$

then it is called a ***symmetric G–martingale***.

Example 4.1.2 For any fixed $X \in L_G^1(\Omega)$, it is clear that $(\hat{\mathbb{E}}[X|\Omega_t])_{t\geq 0}$ is a G–martingale.

Example 4.1.3 For any fixed $\mathbf{a} \in \mathbb{R}^d$, it is easy to check that $(B_t^{\mathbf{a}})_{t\geq 0}$ and $(-B_t^{\mathbf{a}})_{t\geq 0}$ are G–martingales. The process $(\langle B^{\mathbf{a}}\rangle_t - \sigma^2_{\mathbf{aa}^T} t)_{t\geq 0}$ is a G-martingale since

S. Peng, *Nonlinear Expectations and Stochastic Calculus under Uncertainty*,
Probability Theory and Stochastic Modelling 95,
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$$\begin{aligned}\hat{\mathbb{E}}[\langle B^{\mathbf{a}}\rangle_t - \sigma^2_{\mathbf{aa}^T} t|\Omega_s] &= \hat{\mathbb{E}}[\langle B^{\mathbf{a}}\rangle_s - \sigma^2_{\mathbf{aa}^T} t + (\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)|\Omega_s] \\ &= \langle B^{\mathbf{a}}\rangle_s - \sigma^2_{\mathbf{aa}^T} t + \hat{\mathbb{E}}[\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s] \\ &= \langle B^{\mathbf{a}}\rangle_s - \sigma^2_{\mathbf{aa}^T} s.\end{aligned}$$

However, the processes $(-(\langle B^{\mathbf{a}}\rangle_t - \sigma^2_{\mathbf{aa}^T} t))_{t\geq 0}$ and $((B^{\mathbf{a}}_t)^2)_{t\geq 0}$ are G-submartingales, as seen from the relations

$$\begin{aligned}\hat{\mathbb{E}}[(B^{\mathbf{a}}_t)^2|\Omega_s] &= \hat{\mathbb{E}}[(B^{\mathbf{a}}_s)^2 + (B^{\mathbf{a}}_t - B^{\mathbf{a}}_s)^2 + 2B^{\mathbf{a}}_s(B^{\mathbf{a}}_t - B^{\mathbf{a}}_s)|\Omega_s] \\ &= (B^{\mathbf{a}}_s)^2 + \hat{\mathbb{E}}[(B^{\mathbf{a}}_t - B^{\mathbf{a}}_s)^2|\Omega_s] \\ &= (B^{\mathbf{a}}_s)^2 + \sigma^2_{\mathbf{aa}^T}(t-s) \geq (B^{\mathbf{a}}_s)^2.\end{aligned}$$

Similar reasoning shows that $((B^{\mathbf{a}}_t)^2 - \sigma^2_{\mathbf{aa}^T} t)_{t\geq 0}$ and $((B^{\mathbf{a}}_t)^2 - \langle B^{\mathbf{a}}\rangle_t)_{t\geq 0}$ are G-martingales.

In general, we have the following important property.

Proposition 4.1.4 *Let $M_0 \in \mathbb{R}$, $\varphi = (\varphi^j)_{j=1}^d \in M_G^2(0,T;\mathbb{R}^d)$ and $\eta = (\eta^{ij})_{i,j=1}^d \in M_G^1(0,T;\mathbb{S}(d))$ be given and let*

$$M_t = M_0 + \int_0^t \varphi^j_u dB^j_u + \int_0^t \eta^{ij}_u d\,\langle B\rangle^{ij}_u - \int_0^t 2G(\eta_u)du \;\; \textit{for } t \in [0,T].$$

Then M is a G-martingale. As before, we follow the Einstein convention: the above repeated indices i and j meaning the summation.

Proof Since $\hat{\mathbb{E}}[\int_s^t \varphi^j_u dB^j_u|\Omega_s] = \hat{\mathbb{E}}[-\int_s^t \varphi^j_u dB^j_u|\Omega_s] = 0$, we only need to prove that

$$\bar{M}_t = \int_0^t \eta^{ij}_u d\,\langle B\rangle^{ij}_u - \int_0^t 2G(\eta_u)du \;\; \text{for } t \in [0,T]$$

is a G-martingale. It suffices to consider the case of $\eta \in M_G^{1,0}(0,T;\mathbb{S}(d))$, i.e.,

$$\eta_t = \sum_{k=0}^{N-1} \eta_{t_k} I_{[t_k,t_{k+1})}(t), \quad 0 = t_0 < t_1 < \cdots < t_n = T.$$

We have, for $s \in [t_{N-1}, t_N]$,

$$\begin{aligned}\hat{\mathbb{E}}[\bar{M}_t|\Omega_s] &= \bar{M}_s + \hat{\mathbb{E}}[(\eta_{t_{N-1}}, \langle B\rangle_t - \langle B\rangle_s) - 2G(\eta_{t_{N-1}})(t-s)|\Omega_s] \\ &= \bar{M}_s + \hat{\mathbb{E}}[(A, \langle B\rangle_t - \langle B\rangle_s)]_{A=\eta_{t_{N-1}}} - 2G(\eta_{t_{N-1}})(t-s) \\ &= \bar{M}_s.\end{aligned}$$

We can repeat this procedure backwardly thus proving the result for $s \in [0, t_{N-1}]$. □

Corollary 4.1.5 *Let $\eta \in M_G^1(0,T)$. Then for any fixed $\mathbf{a} \in \mathbb{R}^d$, we have*

$$\sigma^2_{-\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}\left[\int_0^T |\eta_t|dt\right] \le \hat{\mathbb{E}}\left[\int_0^T |\eta_t|d\langle B^{\mathbf{a}}\rangle_t\right] \le \sigma^2_{\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}\left[\int_0^T |\eta_t|dt\right]. \qquad (4.1.1)$$

Proof Proposition 4.1.4 implies that, for any $\xi \in M_G^1(0,T)$,

$$\hat{\mathbb{E}}\left[\int_0^T \xi_t d\langle B^{\mathbf{a}}\rangle_t - \int_0^T 2G_{\mathbf{a}}(\xi_t)dt\right] = 0,$$

where $G_{\mathbf{a}}(\alpha) = \frac{1}{2}(\sigma^2_{\mathbf{a}\mathbf{a}^T}\alpha^+ - \sigma^2_{-\mathbf{a}\mathbf{a}^T}\alpha^-)$. Letting $\xi = |\eta|$ and $\xi = -|\eta|$, we get

$$\hat{\mathbb{E}}\left[\int_0^T |\eta_t|d\langle B^{\mathbf{a}}\rangle_t - \sigma^2_{\mathbf{a}\mathbf{a}^T}\int_0^T |\eta_t|dt\right] = 0,$$
$$\hat{\mathbb{E}}\left[-\int_0^T |\eta_t|d\langle B^{\mathbf{a}}\rangle_t + \sigma^2_{-\mathbf{a}\mathbf{a}^T}\int_0^T |\eta_t|dt\right] = 0.$$

Thus the result follows from the sub-additivity of G-expectation. □

Remark 4.1.6 If $\varphi \equiv 0$ in Proposition 4.1.4, then $M_t = \int_0^t \eta_u^{ij} d\langle B\rangle_u^{ij} - \int_0^t 2G(\eta_u)du$ is a G-martingale. This is a surprising result because M_t is a continuous and non-increasing process.

Remark 4.1.7 It is worth mentioning that for a G-martingale M, in general, $-M$ is not a G-martingale. Notice however, in Proposition 4.1.4 with $\eta \equiv 0$, the process $-M$ is still a G-martingale.

4.2 Heuristic Explanation of G-Martingale Representation

Proposition 4.1.4 tells us that a G-martingale contains a special additional term which is a decreasing martingale of the form

$$K_t = \int_0^t \eta_s d\langle B\rangle_s - \int_0^t G(\eta_s)ds.$$

In this section, we provide a formal proof to show that a G-martingale can be decomposed into a sum of a symmetric martingale and a decreasing martingale.

Let us consider a generator $G : \mathbb{S}(d) \mapsto \mathbb{R}$ satisfying the uniformly elliptic condition, i.e., there exists $\beta > 0$ such that, for each $A, \bar{A} \in \mathbb{S}(d)$ with $A \ge \bar{A}$,

$$G(A) - G(\bar{A}) \ge \beta \text{tr}[A - \bar{A}].$$

For $\xi = (\xi^j)_{j=1}^d \in M_G^2(0, T; \mathbb{R}^d)$ and $\eta = (\eta^{ij})_{i,j=1}^d \in M_G^1(0, T; \mathbb{S}(d))$, we use the following notations

$$\int_0^T \langle \xi_t, dB_t \rangle := \sum_{j=1}^d \int_0^T \xi_t^j dB_t^j; \ \int_0^T (\eta_t, d\langle B \rangle_t) := \sum_{i,j=1}^d \int_0^T \eta_t^{ij} d\langle B \rangle_t^{ij}.$$

Let us first consider a G-martingale $(M_t)_{t\in[0,T]}$ with terminal condition $M_T = \xi = \varphi(B_T - B_{t_1})$ for $0 \le t_1 \le T < \infty$.

Lemma 4.2.1 *Let $\xi = \varphi(B_T - B_{t_1})$, $\varphi \in C_{b.Lip}(\mathbb{R}^d)$. Then we have the following representation:*

$$\xi = \hat{\mathbb{E}}[\xi] + \int_{t_1}^T \langle \beta_t, dB_t \rangle + \int_{t_1}^T (\eta_t, d\langle B \rangle_t) - \int_{t_1}^T 2G(\eta_t)dt.$$

Proof We know that $u(t, x) = \hat{\mathbb{E}}[\varphi(x + B_T - B_t)]$ is the solution of the following PDE:

$$\partial_t u + G(D^2 u) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad u(T, x) = \varphi(x).$$

For any $\varepsilon > 0$, by the interior regularity of u (see Appendix C), we have

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T-\varepsilon]\times\mathbb{R}^d)} < \infty \text{ for some } \alpha \in (0, 1).$$

Applying G-Itô's formula to $u(t, B_t - B_{t_1})$ on $[t_1, T - \varepsilon]$, since $Du(t, x)$ is uniformly bounded, letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}
\xi &= \hat{\mathbb{E}}[\xi] + \int_{t_1}^T \partial_t u(t, B_t - B_{t_1})dt + \int_{t_1}^T \langle Du(t, B_t - B_{t_1}), dB_t \rangle \\
&\quad + \frac{1}{2} \int_{t_1}^T (D^2 u(t, B_t - B_{t_1}), d\langle B \rangle_t) \\
&= \hat{\mathbb{E}}[\xi] + \int_{t_1}^T \langle Du(t, B_t - B_{t_1}), dB_t \rangle + \frac{1}{2} \int_{t_1}^T (D^2 u(t, B_t - B_{t_1}), d\langle B \rangle_t) \\
&\quad - \int_{t_1}^T G(D^2 u(t, B_t - B_{t_1}))dt.
\end{aligned}$$

□

This method can be applied to treat a more general martingale $(M_t)_{0\le t\le T}$ with terminal condition

$$\begin{aligned}
M_T &= \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_N} - B_{t_{N-1}}), \\
\varphi &\in C_{b.Lip}(\mathbb{R}^{d\times N}), \ \ 0 \le t_1 < t_2 < \cdots < t_N = T < \infty.
\end{aligned} \tag{4.2.1}$$

Indeed, it suffices to consider the case

$$\xi = \hat{\mathbb{E}}[\xi] + \int_0^T \langle \beta_t, dB_t \rangle + \int_0^T (\eta_t, d\langle B \rangle_t) - \int_0^T 2G(\eta_t)dt.$$

For $\xi = \varphi(B_{t_1}, B_T - B_{t_1})$, we set, for each $(x, y) \in \mathbb{R}^{2d}$,

$$u(t, x, y) = \hat{\mathbb{E}}[\varphi(x, y + B_T - B_t)]; \ \varphi_1(x) = \hat{\mathbb{E}}[\varphi(x, B_T - B_{t_1})].$$

For $x \in \mathbb{R}^d$, we denote $\bar{\xi} = \varphi(x, B_T - B_{t_1})$. By Lemma 4.2.1, we have

$$\bar{\xi} = \varphi_1(x) + \int_{t_1}^T \langle D_y u(t, x, B_t - B_{t_1}), dB_t \rangle + \frac{1}{2} \int_{t_1}^T (D_y^2 u(t, x, B_t - B_{t_1}), d\langle B \rangle_t)$$
$$- \int_{t_1}^T G(D_y^2 u(t, x, B_t - B_{t_1}))dt.$$

Intuitively, we can replace x by B_{t_1}, apply Lemma 4.2.1 to $\varphi_1(B_{t_1})$ and conclude that

$$\xi = \varphi_1(B_{t_1}) + \int_{t_1}^T \langle D_y u(t, B_{t_1}, B_t - B_{t_1}), dB_t \rangle$$
$$+ \frac{1}{2} \int_{t_1}^T (D_y^2 u(t, B_{t_1}, B_t - B_{t_1}), d\langle B \rangle_t) - \int_{t_1}^T G(D_y^2 u(t, B_{t_1}, B_t - B_{t_1}))dt.$$

We repeat this procedure and show that the G-martingale $(M_t)_{t \in [0,T]}$ with terminal condition M_T given in (4.2.1) has the following representation:

$$M_t = \hat{\mathbb{E}}[M_T] + \int_0^t \langle \beta_s, dB_s \rangle + K_t$$

with $K_t = \int_0^t (\eta_s, d\langle B \rangle_s) - \int_0^t 2G(\eta_s)ds$ for $0 \le t \le T$.

Remark 4.2.2 Here there is a very interesting and challenging question: can we prove the above new G-martingale representation theorem for a general L_G^p-martingale? The answer of this question is provided in Theorem 7.1.1 of Chap. 7.

4.3 G-Convexity and Jensen's Inequality for G-Expectations

Here the question of interest is whether the well–known Jensen's inequality still holds for G-expectations.

First, we give a new notion of convexity.

Definition 4.3.1 A continuous function $h : \mathbb{R} \mapsto \mathbb{R}$ is called G ***–convex*** if for any bounded $\xi \in L_G^1(\Omega)$, the following Jensen's inequality holds:

$$\hat{\mathbb{E}}[h(\xi)] \geq h(\hat{\mathbb{E}}[\xi]).$$

In this section, we mainly consider C^2-functions.

Proposition 4.3.2 *Let $h \in C^2(\mathbb{R})$. Then the following statements are equivalent:*
(i) *The function h is G –convex.*
(ii) *For each bounded $\xi \in L_G^1(\Omega)$, the following Jensen's inequality holds:*

$$\hat{\mathbb{E}}[h(\xi)|\Omega_t] \geq h(\hat{\mathbb{E}}[\xi|\Omega_t]) \ \textit{for } t \geq 0.$$

(iii) *For each $\varphi \in C_b^2(\mathbb{R}^d)$, the following Jensen's inequality holds:*

$$\hat{\mathbb{E}}[h(\varphi(B_t))] \geq h(\hat{\mathbb{E}}[\varphi(B_t)]) \ \textit{for } t \geq 0.$$

(iv) *The following condition holds for each $(y, z, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$:*

$$G(h'(y)A + h''(y)zz^T) - h'(y)G(A) \geq 0. \tag{4.3.1}$$

To prove Proposition 4.3.2, we need the following lemmas.

Lemma 4.3.3 *Let $\Phi : \mathbb{R}^d \mapsto \mathbb{S}(d)$ be a continuous function with polynomial growth. Then*

$$\lim_{\delta \downarrow 0} \hat{\mathbb{E}}\left[\int_t^{t+\delta} (\Phi(B_s), d\langle B\rangle_s)\right]\delta^{-1} = 2\hat{\mathbb{E}}[G(\Phi(B_t))]. \tag{4.3.2}$$

Proof If Φ is a Lipschitz function, it is easy to show that

$$\hat{\mathbb{E}}\left[\left|\int_t^{t+\delta} (\Phi(B_s) - \Phi(B_t), d\langle B\rangle_s)\right|\right] \leq C_1\delta^{3/2},$$

where C_1 is a constant independent of δ. Thus

$$\begin{aligned}\lim_{\delta \downarrow 0} \delta^{-1}\hat{\mathbb{E}}\left[\int_t^{t+\delta} (\Phi(B_s), d\langle B\rangle_s)\right] &= \lim_{\delta \downarrow 0} \delta^{-1}\hat{\mathbb{E}}[(\Phi(B_t), \langle B\rangle_{t+\delta} - \langle B\rangle_s)] \\ &= 2\hat{\mathbb{E}}[G(\Phi(B_t))].\end{aligned}$$

Otherwise, we can choose a sequence of Lipschitz functions $\Phi_N : \mathbb{R}^d \to \mathbb{S}(d)$ such that

$$|\Phi_N(x) - \Phi(x)| \leq \frac{C_2}{N}(1 + |x|^k),$$

where C_2 and k are positive constants independent of N. It is see to show that

$$\hat{\mathbb{E}}\left[\left|\int_t^{t+\delta}(\Phi(B_s)-\Phi_N(B_s),d\langle B\rangle_s)\right|\right]\leq\frac{C}{N}\delta$$

and

$$\hat{\mathbb{E}}[|G(\Phi(B_t))-G(\Phi_N(B_t))|]\leq\frac{C}{N},$$

where C is a universal constant. Thus

$$\begin{aligned}&\left|\hat{\mathbb{E}}\left[\int_t^{t+\delta}(\Phi(B_s),d\langle B\rangle_s)\right]\delta^{-1}-2\hat{\mathbb{E}}\left[G(\Phi(B_t))\right]\right|\\ \leq&\left|\hat{\mathbb{E}}\left[\int_t^{t+\delta}(\Phi_N(B_s),d\langle B\rangle_s)\right]\delta^{-1}-2\hat{\mathbb{E}}\left[G(\Phi_N(B_t))\right]\right|+\frac{3C}{N}.\end{aligned}$$

Then we have

$$\limsup_{\delta\downarrow 0}\left|\hat{\mathbb{E}}\left[\int_t^{t+\delta}(\Phi(B_s),d\langle B\rangle_s)\right]\delta^{-1}-2\hat{\mathbb{E}}[G(\Phi(B_t))]\right|\leq\frac{3C}{N}.$$

Since N can be arbitrarily large, this completes the proof. □

Lemma 4.3.4 *Let Ψ be a C^2-function on $\mathbb{R}^d$ with $D^2\Psi$ satisfying a polynomial growth condition. Then we have*

$$\lim_{\delta\downarrow 0}\delta^{-1}(\hat{\mathbb{E}}[\Psi(B_\delta)]-\Psi(0))=G(D^2\Psi(0)).\qquad(4.3.3)$$

Proof Applying G-Itô's formula to $\Psi(B_\delta)$, we get

$$\Psi(B_\delta)=\Psi(0)+\int_0^\delta\langle D\Psi(B_s),dB_s\rangle+\tfrac{1}{2}\int_0^\delta(D^2\Psi(B_s),d\langle B\rangle_s).$$

Therefore

$$\hat{\mathbb{E}}[\Psi(B_\delta)]-\Psi(0)=\tfrac{1}{2}\hat{\mathbb{E}}\left[\int_0^\delta(D^2\Psi(B_s),d\langle B\rangle_s)\right].$$

By Lemma 4.3.3, we obtain the result. □

Lemma 4.3.5 *Let $h\in C^2(\mathbb{R})$ and satisfy (4.3.1). For any $\varphi\in C_{b.Lip}(\mathbb{R}^d)$, let $u(t,x)$ be the solution of the G-heat equation:*

$$\partial_t u-G(D^2u)=0\quad(t,x)\in[0,\infty)\times\mathbb{R}^d,\ u(0,x)=\varphi(x).\qquad(4.3.4)$$

Then $\tilde{u}(t,x):=h(u(t,x))$ is a viscosity subsolution of the G-heat Eq. (4.3.4) with initial condition $\tilde{u}(0,x)=h(\varphi(x))$.

Proof For each $\varepsilon > 0$, we denote by u_ε the solution of the following PDE:

$$\partial_t u_\varepsilon - G_\varepsilon(D^2 u_\varepsilon) = 0 \quad (t, x) \in [0, \infty) \times \mathbb{R}^d, \; u_\varepsilon(0, x) = \varphi(x),$$

where $G_\varepsilon(A) := G(A) + \varepsilon \text{tr}[A]$. Since G_ε satisfies the uniformly elliptic condition, by Appendix C, we have $u_\varepsilon \in C^{1,2}((0, \infty) \times \mathbb{R}^d)$. By simple calculation, we have

$$\partial_t h(u_\varepsilon) = h'(u_\varepsilon)\partial_t u_\varepsilon = h'(u_\varepsilon)G_\varepsilon(D^2 u_\varepsilon)$$

and

$$\partial_t h(u_\varepsilon) - G_\varepsilon(D^2 h(u_\varepsilon)) = f_\varepsilon(t, x), \; h(u_\varepsilon(0, x)) = h(\varphi(x)),$$

where

$$f_\varepsilon(t, x) = h'(u_\varepsilon)G(D^2 u_\varepsilon) - G(D^2 h(u_\varepsilon)) - \varepsilon h''(u_\varepsilon)|Du_\varepsilon|^2.$$

Since h satisfies (4.3.1), it follows that $f_\varepsilon \leq -\varepsilon h''(u_\varepsilon)|Du_\varepsilon|^2$. We can also deduce that $|Du_\varepsilon|$ is uniformly bounded by the Lipschitz constant of φ. It is easy to show that u_ε uniformly converges to u as $\varepsilon \to 0$. Thus $h(u_\varepsilon)$ uniformly converges to $h(u)$ and $h''(u_\varepsilon)$ is uniformly bounded. Then we get

$$\partial_t h(u_\varepsilon) - G_\varepsilon(D^2 h(u_\varepsilon)) \leq C\varepsilon, \; h(u_\varepsilon(0, x)) = h(\varphi(x)),$$

where C is a constant independent of ε. By Appendix C, we conclude that $h(u)$ is a viscosity subsolution. □

Proof of Proposition 4.3.2 Obviously **(ii)** $\Longrightarrow$**(i)**$\Longrightarrow$**(iii)** . We now show **(iii)**$\Longrightarrow$**(ii)**. For $\xi \in L_G^1(\Omega)$ of the form

$$\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}),$$

where $\varphi \in C_b^2(\mathbb{R}^{d\times n})$, $0 \leq t_1 \leq \cdots \leq t_n < \infty$, by the definitions of $\hat{\mathbb{E}}[\cdot]$ and $\hat{\mathbb{E}}[\cdot|\Omega_t]$, we have

$$\hat{\mathbb{E}}[h(\xi)|\Omega_t] \geq h(\hat{\mathbb{E}}[\xi|\Omega_t]), \; t \geq 0.$$

This Jensen's inequality can be extended to hold under the norm $||\cdot|| = \hat{\mathbb{E}}[|\cdot|]$, to each $\xi \in L_G^1(\Omega)$ satisfying $h(\xi) \in L_G^1(\Omega)$.

Let us show **(iii)**$\Longrightarrow$**(iv)**: for each $\varphi \in C_b^2(\mathbb{R}^d)$, we have $\hat{\mathbb{E}}[h(\varphi(B_t))] \geq h(\hat{\mathbb{E}}[\varphi(B_t)])$ for $t \geq 0$. By Lemma 4.3.4, we know that

$$\lim_{\delta\downarrow 0}(\hat{\mathbb{E}}[\varphi(B_\delta)] - \varphi(0))\delta^{-1} = G(D^2\varphi(0))$$

and

$$\lim_{\delta\downarrow 0}(\hat{\mathbb{E}}[h(\varphi(B_\delta))] - h(\varphi(0)))\delta^{-1} = G(D^2 h(\varphi)(0)).$$

Thus we obtain

$$G(D^2h(\varphi)(0)) \geq h'(\varphi(0))G(D^2\varphi(0)).$$

For each $(y, z, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$, we can choose $\varphi \in C_b^2(\mathbb{R}^d)$ such that

$$(\varphi(0), D\varphi(0), D^2\varphi(0)) = (y, z, A).$$

Thus we obtain **(iv)**.

Finally, **(iv)**$\Longrightarrow$**(iii)**: for each $\varphi \in C_b^2(\mathbb{R}^d)$, $u(t, x) = \hat{\mathbb{E}}[\varphi(x + B_t)]$ (respectively, $\bar{u}(t, x) = \hat{\mathbb{E}}[h(\varphi(x + B_t))]$) solves the G-heat Eq. (4.3.4). By Lemma 4.3.5, $h(u)$ is a viscosity subsolution of the G-heat Eq. (4.3.4). It follows from the maximum principle that $h(u(t, x)) \leq \bar{u}(t, x)$. In particular, **(iii)** holds. □

Remark 4.3.6 In fact, **(i)**$\Longleftrightarrow$**(ii)** $\Longleftrightarrow$**(iii)** still hold without assuming that $h \in C^2(\mathbb{R})$.

Proposition 4.3.7 *Let h be a G-convex function and $X \in L_G^1(\Omega)$ be bounded. Then the process $Y_t = h(\hat{\mathbb{E}}[X|\Omega_t])$, $t \geq 0$, is a G-submartingale.*

Proof For each $s \leq t$,

$$\hat{\mathbb{E}}[Y_t|\Omega_s] = \hat{\mathbb{E}}[h(\hat{\mathbb{E}}[X|\Omega_t])|\Omega_s] \geq h(\hat{\mathbb{E}}[X|\Omega_s]) = Y_s.$$ □

4.4 Exercises

Exercise 4.4.1 (a) Let $(M_t)_{t\geq 0}$ be a G-supermartingale. Show that the process $(-M_t)_{t\geq 0}$ is a G-submartingale.

(b) Find a G-submartingale $(M_t)_{t\geq 0}$ such that $(-M_t)_{t\geq 0}$ is not a G-supermartingale.

Exercise 4.4.2 (a) Assume that $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be two G -supermartingales. Prove that their sum $(M_t + N_t)_{t\geq 0}$ is a G -supermartingale.

(b) Assume that $(M_t)_{t\geq 0}$ and $(-M_t)_{t\geq 0}$ are two G-martingales. For each G-submartingale $(N_t)_{t\geq 0}$, prove that $(M_t + N_t)_{t\geq 0}$ is a G-submartingale.

Exercise 4.4.3 Suppose that G satisfies the uniformly elliptic condition and $h \in C^2(\mathbb{R})$. Show that h is G-convex if and only if h is convex.

Notes and Comments

The material in this chapter is mainly from Peng [140].

Peng [130] introduced a filtration consistent (or time consistent, or dynamic) nonlinear expectation, called g-expectation, via BSDE, developed further in (1999)

[132] for some basic properties of the g-martingale such as nonlinear Doob-Meyer decomposition theorem. See also Briand et al. [20] , Chen et al. [29], Chen and Peng [30, 31], Coquet, Hu, Mémin and Peng [35, 36], Peng [132, 135], Peng and Xu [148], Rosazza [152]. These works lead to a conjecture that all properties obtained for g-martingales must have their counterparts for G-martingale. However this conjecture is still far from being complete.

The problem of G-martingale representation has been proposed by Peng [140]. In Sect. 4.2, we only state a result with very regular random variables. Some very interesting developments to this important problem will be provided in Chap. 7.

Under the framework of g-expectation, Chen, Kulperger and Jiang [29], Hu [86], Jiang and Chen [97] investigate the Jensen's inequality for g-expectation. Jia and Peng [95] introduced the notion of g-convex function and obtained many interesting properties. Certainly, a G-convex function concerns fully nonlinear situations.