Chapter 3
G-Brownian Motion and Itô's Calculus

The aim of this chapter is to introduce the concept of G-Brownian motion, study its properties and construct Itô's integral with respect to G-Brownian motion. We emphasize here that this G-Brownian motion $B_t, t \geq 0$ is consistent with the classical one. In fact once its mean uncertainty and variance uncertainty vanish, namely

$$\hat{\mathbb{E}}[B_1] = -\hat{\mathbb{E}}[-B_1] \quad \text{and} \quad \hat{\mathbb{E}}[B_1^2] = -\hat{\mathbb{E}}[-B_1^2],$$

then B becomes a classical Brownian motion. This G-Brownian motion also has independent and stable increments. G-Brownian motion has a very rich and interesting new structure which non-trivially generalizes the classical one. Thus we can develop the related stochastic calculus, especially Itô's integrals and the related quadratic variation process. A very interesting feature of the G-Brownian motion is that its quadratic process also has independent increments which are identically distributed. The corresponding G-Itô's formula is also presented.

We emphasize that the above construction of G-Brownian motion and the establishment of the corresponding stochastic analysis of generalized Itô's type, from this chapter to Chap. 5, have been rigorously realized without firstly constructing a probability space or its generalization, whereas its special situation of linear expectation corresponds in fact to the classical Brownian motion under a Wiener probability measure space. This is an important advantage of the expectation-based framework. The corresponding path-wise analysis of G-Brownian motion functional will be established in Chap. 6, after the introduction of the corresponding G-capacity. We can see that all results obtained in this chapter to Chap. 5 still hold true in G-capacity surely analysis.

3.1 Brownian Motion on a Sublinear Expectation Space

Definition 3.1.1 Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. $(X_t)_{t\geq 0}$ is called a d-dimensional **stochastic process** if for each $t \geq 0$, X_t is a d-dimensional random vector in $\mathcal{H}$.

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We now give the definition of Brownian motion on sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

Definition 3.1.2 A d-dimensional stochastic process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a ***G*-Brownian motion** if the following properties are satisfied:

(i) $B_0(\omega)= 0$;

(ii) For each $t, s \geq 0$, $B_{t+s} - B_t$ and B_s are identically distributed and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

(iii) $\lim_{t\downarrow 0} \hat{\mathbb{E}}[|B_t|^3]t^{-1} = 0$.

Moreover, if $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0$, then $(B_t)_{t\geq 0}$ is called a **symmetric *G*-Brownian motion**.

In the sublinear expectation space, symmetric G-Brownian motion is an important case of Brownian motion. From now on up to Sect. 3.6, we will study its properties, which are needed in stochastic analysis of G-Brownian motion. The following theorem gives a characterization of the symmetric Brownian motion.

Theorem 3.1.3 *Let $(B_t)_{t\geq 0}$ be a given $\mathbb{R}^d$–valued symmetric G-Brownian motion on a sublinear expectation $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then, for each fixed $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, the function*

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + B_t)], \quad (t, x) \in [0, \infty) \times \mathbb{R}^d$$

is the viscosity solution of the following parabolic PDE:

$$\partial_t u - G(D^2 u) = 0, \quad u|_{t=0} = \varphi. \tag{3.1.1}$$

where

$$G(A) = \frac{1}{2}\hat{\mathbb{E}}[\langle AB_1, B_1\rangle], \ A \in \mathbb{S}(d). \tag{3.1.2}$$

In particular, B_1 is G-normally distributed and $B_t \overset{d}{=} \sqrt{t}B_1$.

Proof We only need to prove that u is the viscosity solution. We first show that

$$\hat{\mathbb{E}}[\langle AB_t, B_t\rangle] = 2G(A)t, \quad A \in \mathbb{S}(d).$$

For each given $A \in \mathbb{S}(d)$, we set $b(t) =\hat{\mathbb{E}}[\langle AB_t, B_t\rangle]$. Then $b(0) = 0$ and $|b(t)| \leq |A|(\hat{\mathbb{E}}[|B_t|^3])^{2/3} \to 0$ as $t \to 0$. Note that $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0$, we have for each $t, s \geq 0$,

$$\begin{aligned} b(t + s) &= \hat{\mathbb{E}}[\langle AB_{t+s}, B_{t+s}\rangle] = \hat{\mathbb{E}}[\langle A(B_{t+s} - B_s + B_s), B_{t+s} - B_s + B_s\rangle] \\ &= \hat{\mathbb{E}}[\langle A(B_{t+s} - B_s), (B_{t+s} - B_s)\rangle + \langle AB_s, B_s\rangle + 2\langle A(B_{t+s} - B_s), B_s\rangle] \\ &= b(t) + b(s), \end{aligned}$$

thus $b(t) = b(1)t = 2G(A)t$.

Then we show that u is Lipschitz in x and $\frac{1}{2}$-Hölder continuous in t. In fact, for each fixed t, $u(t,\cdot) \in C_{b.Lip}(\mathbb{R}^d)$ since

$$\begin{aligned}|u(t,x)-u(t,y)| &= |\hat{\mathbb{E}}[\varphi(x+B_t)] - \hat{\mathbb{E}}[\varphi(y+B_t)]| \\ &\leq \hat{\mathbb{E}}[|\varphi(x+B_t)-\varphi(y+B_t)|] \\ &\leq C|x-y|,\end{aligned}$$

where C is the Lipschitz constant of φ.

For each $\delta \in [0,t]$, since $B_t - B_\delta$ is independent from B_δ, we also have

$$\begin{aligned}u(t,x) &= \hat{\mathbb{E}}[\varphi(x+B_\delta+(B_t-B_\delta)] \\ &= \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(y+(B_t-B_\delta))]_{y=x+B_\delta}],\end{aligned}$$

hence

$$u(t,x) = \hat{\mathbb{E}}[u(t-\delta, x+B_\delta)]. \tag{3.1.3}$$

Thus

$$\begin{aligned}|u(t,x)-u(t-\delta,x)| &= |\hat{\mathbb{E}}[u(t-\delta,x+B_\delta)-u(t-\delta,x)]| \\ &\leq \hat{\mathbb{E}}[|u(t-\delta,x+B_\delta)-u(t-\delta,x)|] \\ &\leq \hat{\mathbb{E}}[C|B_\delta|] \leq C\sqrt{2G(I)}\sqrt{\delta}.\end{aligned}$$

To show that u is a viscosity solution of (3.1.1), we fix $(t,x) \in (0,\infty)\times\mathbb{R}^d$ and let $v \in C_b^{2,3}([0,\infty)\times\mathbb{R}^d)$ be such that $v \geq u$ and $v(t,x) = u(t,x)$. From (3.1.3) we have

$$v(t,x) = \hat{\mathbb{E}}[u(t-\delta,x+B_\delta)] \leq \hat{\mathbb{E}}[v(t-\delta,x+B_\delta)].$$

Therefore by Taylor's expansion,

$$\begin{aligned}0 &\leq \hat{\mathbb{E}}[v(t-\delta,x+B_\delta)-v(t,x)] \\ &= \hat{\mathbb{E}}[v(t-\delta,x+B_\delta)-v(t,x+B_\delta)+(v(t,x+B_\delta)-v(t,x))] \\ &= \hat{\mathbb{E}}[-\partial_t v(t,x)\delta + \langle Dv(t,x), B_\delta\rangle + \tfrac{1}{2}\langle D^2v(t,x)B_\delta, B_\delta\rangle + I_\delta] \\ &\leq -\partial_t v(t,x)\delta + \tfrac{1}{2}\hat{\mathbb{E}}[\langle D^2v(t,x)B_\delta, B_\delta\rangle] + \hat{\mathbb{E}}[I_\delta] \\ &= -\partial_t v(t,x)\delta + G(D^2v(t,x))\delta + \hat{\mathbb{E}}[I_\delta],\end{aligned}$$

where

$$I_\delta = \int_0^1 -[\partial_t v(t-\beta\delta, x+B_\delta) - \partial_t v(t,x)]\delta d\beta$$

$$+\int_0^1\int_0^1 \langle (D^2v(t,x+\alpha\beta B_\delta)-D^2v(t,x))B_\delta, B_\delta\rangle \alpha d\beta d\alpha.$$

In view of condition (iii) in Definition 3.1.2, we can check that $\lim_{\delta\downarrow 0}\delta^{-1}\hat{\mathbb{E}}[|I_\delta|]=0$, from which we get $\partial_t v(t,x)-G(D^2v(t,x))\leq 0$, hence u is a viscosity subsolution of (3.1.1). We can analogously prove that u is a viscosity supersolution. Thus u is a viscosity solution. □

For simplicity, symmetric Brownian motion is also called ***G*-Brownian motion**, associated with the generator G given by (3.1.2).

Remark 3.1.4 We can prove that, for each $t_0>0$, $(B_{t+t_0}-B_{t_0})_{t\geq 0}$ is a G-Brownian motion. For each $\lambda>0$, $(\lambda^{-\frac{1}{2}}B_{\lambda t})_{t\geq 0}$ is also a symmetric G-Brownian motion. This is the scaling property of G-Brownian motion, which is the same as that for the classical Brownian motion.

In the rest of this book we will use the notation

$$B_t^{\mathbf{a}}=\langle \mathbf{a}, B_t\rangle \quad \text{for each } \mathbf{a}=(a_1,\cdots,a_d)^T\in\mathbb{R}^d.$$

By the above definition we have the following proposition which is important in stochastic calculus.

Proposition 3.1.5 *Let $(B_t)_{t\geq 0}$ be a d-dimensional G-Brownian motion on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$. Then $(B_t^{\mathbf{a}})_{t\geq 0}$ is a 1-dimensional $G_{\mathbf{a}}$-Brownian motion for each $\mathbf{a}\in\mathbb{R}^d$, where $G_{\mathbf{a}}(\alpha)=\frac{1}{2}(\sigma^2_{\mathbf{aa}^T}\alpha^+-\sigma^2_{-\mathbf{aa}^T}\alpha^-)$, $\sigma^2_{\mathbf{aa}^T}=2G(\mathbf{aa}^T)=\hat{\mathbb{E}}[\langle\mathbf{a},B_1\rangle^2]$, $\sigma^2_{-\mathbf{aa}^T}=-2G(-\mathbf{aa}^T)=-\hat{\mathbb{E}}[-\langle\mathbf{a},B_1\rangle^2]$.*

In particular, for each $t,s\geq 0$, $B^{\mathbf{a}}_{t+s}-B^{\mathbf{a}}_t\overset{d}{=}N(\{0\}\times[s\sigma^2_{-\mathbf{aa}^T},s\sigma^2_{\mathbf{aa}^T}])$.

Proposition 3.1.6 *For each convex function $\varphi\in C_{l.Lip}(\mathbb{R})$, we have*

$$\hat{\mathbb{E}}[\varphi(B^{\mathbf{a}}_{t+s}-B^{\mathbf{a}}_t)]=\frac{1}{\sqrt{2\pi s\sigma^2_{\mathbf{aa}^T}}}\int_{-\infty}^{\infty}\varphi(x)\exp(-\frac{x^2}{2s\sigma^2_{\mathbf{aa}^T}})dx.$$

For each concave function $\varphi\in C_{l.Lip}(\mathbb{R})$ and $\sigma^2_{-\mathbf{aa}^T}>0$, we have

$$\hat{\mathbb{E}}[\varphi(B^{\mathbf{a}}_{t+s}-B^{\mathbf{a}}_t)]=\frac{1}{\sqrt{2\pi s\sigma^2_{-\mathbf{aa}^T}}}\int_{-\infty}^{\infty}\varphi(x)\exp(-\frac{x^2}{2s\sigma^2_{-\mathbf{aa}^T}})dx.$$

In particular, the following relations are true:

$$\hat{\mathbb{E}}[(B^{\mathbf{a}}_t-B^{\mathbf{a}}_s)^2]=\sigma^2_{\mathbf{aa}^T}(t-s),\quad \hat{\mathbb{E}}[(B^{\mathbf{a}}_t-B^{\mathbf{a}}_s)^4]=3\sigma^4_{\mathbf{aa}^T}(t-s)^2,$$
$$\hat{\mathbb{E}}[-(B^{\mathbf{a}}_t-B^{\mathbf{a}}_s)^2]=-\sigma^2_{-\mathbf{aa}^T}(t-s),\quad \hat{\mathbb{E}}[-(B^{\mathbf{a}}_t-B^{\mathbf{a}}_s)^4]=-3\sigma^4_{-\mathbf{aa}^T}(t-s)^2.$$

3.2 Existence of G-Brownian Motion

In the rest of this book, we use the notation $\Omega = C_0^d(\mathbb{R}^+)$ for the space of all $\mathbb{R}^d$–valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^{(1)}, \omega^{(2)}) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^{(1)} - \omega_t^{(2)}|) \wedge 1], \quad \omega^{(1)}, \omega^{(2)} \in \Omega.$$

For each fixed $T \in [0, \infty)$, we set $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$. We will consider the canonical process $B_t(\omega) = \omega_t$, $t \in [0, \infty)$, for $\omega \in \Omega$.

For each fixed $T \in [0, \infty)$, we set also

$$Lip(\Omega_T) := \{\varphi(B_{t_1\wedge T}, \cdots, B_{t_n\wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, \infty), \ \varphi \in C_{l.Lip}(\mathbb{R}^{d\times n})\ \}.$$

It is clear that $Lip(\Omega_t)\subseteq Lip(\Omega_T)$, for $t \le T$. We set

$$Lip(\Omega) := \bigcup_{n=1}^{\infty} Lip(\Omega_n).$$

Remark 3.2.1 It is clear that $C_{l.Lip}(\mathbb{R}^{d\times n})$, $Lip(\Omega_T)$ and $Lip(\Omega)$ are vector lattices. Moreover, note that $\varphi, \psi \in C_{l.Lip}(\mathbb{R}^{d\times n})$ implies $\varphi \cdot \psi \in C_{l.Lip}(\mathbb{R}^{d\times n})$, then $X, Y \in Lip(\Omega_T)$ implies $X \cdot Y \in Lip(\Omega_T)$. In particular, for each $t \in [0, \infty)$, $B_t \in Lip(\Omega)$.

Let $G(\cdot) : \mathbb{S}(d) \mapsto \mathbb{R}$ be a given monotone and sublinear function. By Theorem 1.2.1 in Chap. 1, there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{S}_+(d)$ such that

$$G(A) = \frac{1}{2} \sup_{B\in\Sigma} (A, B), \quad A \in \mathbb{S}(d).$$

By Sect. 2.3 in Chap. 2, we know that the G-normal distribution $N(\{0\} \times \Sigma)$ exists.

Let us construct a sublinear expectation on $(\Omega, Lip(\Omega))$ such that the canonical process $(B_t)_{t\ge 0}$ is a G-Brownian motion. For this, we first construct a sequence of d-dimensional random vectors $(\xi_i)_{i=1}^{\infty}$ on a sublinear expectation space $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$ such that ξ_i is G-normally distributed and ξ_{i+1} is independent from $(\xi_1, \cdots, \xi_i)$ for each $i = 1, 2, \cdots$.

We now construct a sublinear expectation $\hat{\mathbb{E}}$ defined on $Lip(\Omega)$ via the following procedure: for each $X \in Lip(\Omega)$ with

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$$

for some $\varphi \in C_{l.Lip}(\mathbb{R}^{d\times n})$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$, we set

$$\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] := \widetilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

The related conditional expectation of $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}[X|\Omega_{t_j}] = \hat{\mathbb{E}}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})|\Omega_{t_j}] := \psi(B_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}}),$$

where

$$\psi(x_1, \cdots, x_j) = \widetilde{\mathbb{E}}[\varphi(x_1, \cdots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

$\hat{\mathbb{E}}[\cdot]$ consistently defines a sublinear expectation on $Lip(\Omega)$ and $(B_t)_{t\geq 0}$ is a G-Brownian motion. Since $Lip(\Omega_T)\subseteq Lip(\Omega)$, $\hat{\mathbb{E}}[\cdot]$ is also a sublinear expectation on $Lip(\Omega_T)$.

Definition 3.2.2 The sublinear expectation $\hat{\mathbb{E}}[\cdot]$: $Lip(\Omega) \mapsto \mathbb{R}$ defined through the above procedure is called a G–**expectation**. The corresponding canonical process $(B_t)_{t\geq 0}$ on the sublinear expectation space $(\Omega, Lip(\Omega), \hat{\mathbb{E}})$ is called a G–Brownian motion.

In the rest of the book, when we talk about G–Brownian motion, we mean that the canonical process $(B_t)_{t\geq 0}$ is under G-expectation.

Proposition 3.2.3 *We list the properties of* $\hat{\mathbb{E}}[\cdot|\Omega_t]$ *that hold for each* $X, Y \in Lip(\Omega)$*:*

(i) *If* $X \geq Y$*, then* $\hat{\mathbb{E}}[X|\Omega_t] \geq \hat{\mathbb{E}}[Y|\Omega_t]$.
(ii) $\hat{\mathbb{E}}[\eta|\Omega_t] = \eta$, *for each* $t \in [0, \infty)$ *and* $\eta \in Lip(\Omega_t)$.
(iii) $\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t]$.
(iv) $\hat{\mathbb{E}}[\eta X|\Omega_t] = \eta^+\hat{\mathbb{E}}[X|\Omega_t] + \eta^-\hat{\mathbb{E}}[-X|\Omega_t]$, *for each* $\eta \in Lip(\Omega_t)$.
(v) $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\Omega_t]|\Omega_s] = \hat{\mathbb{E}}[X|\Omega_{t\wedge s}]$, *in particular,* $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\Omega_t]] = \hat{\mathbb{E}}[X]$.

For each $X \in Lip(\Omega^t)$, $\hat{\mathbb{E}}[X|\Omega_t] = \hat{\mathbb{E}}[X]$, *where* $Lip(\Omega^t)$ *is the linear space of random variables with the form*

$$\varphi(B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \cdots, B_{t_{n+1}} - B_{t_n}),$$
$$n = 1, 2, \cdots, \ \varphi \in C_{l.Lip}(\mathbb{R}^{d\times n}), \ t_1, \cdots, t_n, t_{n+1} \in [t, \infty).$$

Remark 3.2.4 Properties (ii) and (iii) imply

$$\hat{\mathbb{E}}[X + \eta|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \eta \ \text{ for } \eta \in Lip(\Omega_t).$$

We now consider the completion of sublinear expectation space $(\Omega, Lip(\Omega), \hat{\mathbb{E}})$. For $p \geq 1$, we denote by

$$L_G^p(\Omega) := \{ \textit{the completion of the space } Lip(\Omega) \textit{ under the norm } \|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}\}.$$

Similarly, we can define $L_G^p(\Omega_T)$, $L_G^p(\Omega_T^t)$ and $L_G^p(\Omega^t)$. It is clear that for each $0 \leq t \leq T < \infty$, $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$.

According to Sect. 1.4 in Chap. 1, $\hat{\mathbb{E}}[\cdot]$ can be continuously extended to a sublinear expectation on $(\Omega, L_G^1(\Omega))$ and still denoted by $\hat{\mathbb{E}}[\cdot]$. We now consider the extension of conditional expectations. For each fixed $t \leq T$, the conditional G-expectation $\hat{\mathbb{E}}[\cdot|\Omega_t] : Lip(\Omega_T) \mapsto Lip(\Omega_t)$ is a continuous mapping under $\|\cdot\|$. Indeed, we have

$$\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t] \leq \hat{\mathbb{E}}[|X - Y||\Omega_t],$$

then

$$|\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t]| \leq \hat{\mathbb{E}}[|X - Y||\Omega_t].$$

We thus obtain

$$\left\|\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t]\right\| \leq \|X - Y\|.$$

It follows that $\hat{\mathbb{E}}[\cdot|\Omega_t]$ can also be extended as a continuous mapping

$$\hat{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega_T) \mapsto L_G^1(\Omega_t).$$

If the above T is not fixed, then we can obtain $\hat{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega) \mapsto L_G^1(\Omega_t)$.

Remark 3.2.5 Proposition 3.2.3 also holds for $X, Y \in L_G^1(\Omega)$. But in (iv), $\eta \in L_G^1(\Omega_t)$ should be bounded, since $X, Y \in L_G^1(\Omega)$ does not imply that $X \cdot Y \in L_G^1(\Omega)$.

In particular, we have the following independence property:

$$\hat{\mathbb{E}}[X|\Omega_t] = \hat{\mathbb{E}}[X], \quad \forall X \in L_G^1(\Omega^t).$$

We give the following definition similar to the classical one:

Definition 3.2.6 An n-dimensional random vector $Y \in (L_G^1(\Omega))^n$ is said to be independent from Ω_t for some given t if for each $\varphi \in C_{b.Lip}(\mathbb{R}^n)$ we have

$$\hat{\mathbb{E}}[\varphi(Y)|\Omega_t] = \hat{\mathbb{E}}[\varphi(Y)].$$

Remark 3.2.7 Just as in the classical situation, the increments of G–Brownian motion $(B_{t+s} - B_t)_{s\geq 0}$ are independent from Ω_t, for each $t \geq 0$.

Example 3.2.8 For each fixed $\mathbf{a} \in \mathbb{R}^d$ and for each $0 \leq s \leq t$, we have

$$\begin{aligned}
&\hat{\mathbb{E}}[B_t^{\mathbf{a}} - B_s^{\mathbf{a}}|\Omega_s] = 0, \quad \hat{\mathbb{E}}[-(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})|\Omega_s] = 0,\\
&\hat{\mathbb{E}}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^2|\Omega_s] = \sigma^2_{\mathbf{aa}^T}(t-s), \quad \hat{\mathbb{E}}[-(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^2|\Omega_s] = -\sigma^2_{-\mathbf{aa}^T}(t-s),\\
&\hat{\mathbb{E}}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^4|\Omega_s] = 3\sigma^4_{\mathbf{aa}^T}(t-s)^2, \quad \hat{\mathbb{E}}[-(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^4|\Omega_s] = -3\sigma^4_{-\mathbf{aa}^T}(t-s)^2,
\end{aligned}$$

where $\sigma^2_{\mathbf{aa}^T} = 2G(\mathbf{aa}^T)$ and $\sigma^2_{-\mathbf{aa}^T} = -2G(-\mathbf{aa}^T)$.

The following property is very useful.

Proposition 3.2.9 *Let* $X, Y \in L_G^1(\Omega)$ *be such that* $\hat{\mathbb{E}}[Y|\Omega_t] = -\hat{\mathbb{E}}[-Y|\Omega_t]$, *for some* $t \in [0, T]$. *Then we have*

$$\hat{\mathbb{E}}[X + Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t].$$

In particular, if $\hat{\mathbb{E}}[Y|\Omega_t] = \hat{\mathbb{E}}[-Y|\Omega_t] = 0$, *then* $\hat{\mathbb{E}}[X + Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t]$.

Proof This follows from the following two inequalities:

$$\begin{aligned}
&\hat{\mathbb{E}}[X + Y|\Omega_t] \le \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t], \\
&\hat{\mathbb{E}}[X + Y|\Omega_t] \ge \hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[-Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t].
\end{aligned}$$

□

Example 3.2.10 For each $\mathbf{a} \in \mathbb{R}^d$, $0 \le t \le T$, $X \in L_G^1(\Omega_t)$ and $\varphi \in C_{l.Lip}(\mathbb{R})$, we have

$$\begin{aligned}
\hat{\mathbb{E}}[X\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] &= X^+\hat{\mathbb{E}}[\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] + X^-\hat{\mathbb{E}}[-\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] \\
&= X^+\hat{\mathbb{E}}[\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] + X^-\hat{\mathbb{E}}[-\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})].
\end{aligned}$$

In particular,

$$\hat{\mathbb{E}}[X(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] = X^+\hat{\mathbb{E}}[(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] + X^-\hat{\mathbb{E}}[-(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] = 0.$$

This, together with Proposition 3.2.9, yields

$$\hat{\mathbb{E}}[Y + X(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] = \hat{\mathbb{E}}[Y|\Omega_t], \quad Y \in L_G^1(\Omega).$$

We also have

$$\begin{aligned}
\hat{\mathbb{E}}[X(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2|\Omega_t] &= X^+\hat{\mathbb{E}}[(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2] + X^-\hat{\mathbb{E}}[-(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2] \\
&= [X^+\sigma^2_{\mathbf{a}\mathbf{a}^T} - X^-\sigma^2_{-\mathbf{a}\mathbf{a}^T}](T - t).
\end{aligned}$$

For $n \in \mathbb{N}$,

$$\begin{aligned}
\hat{\mathbb{E}}[X(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^{2n-1}|\Omega_t] &= X^+\hat{\mathbb{E}}[(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^{2n-1}] + X^-\hat{\mathbb{E}}[-(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^{2n-1}] \\
&= |X|\hat{\mathbb{E}}[(B_{T-t}^{\mathbf{a}})^{2n-1}].
\end{aligned}$$

Example 3.2.11 Since

$$\hat{\mathbb{E}}[2B_s^{\mathbf{a}}(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})|\Omega_s] = \hat{\mathbb{E}}[-2B_s^{\mathbf{a}}(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})|\Omega_s] = 0,$$

we have

$$\begin{aligned}\hat{\mathbb{E}}[(B_t^{\mathbf{a}})^2-(B_s^{\mathbf{a}})^2|\Omega_s] &= \hat{\mathbb{E}}[(B_t^{\mathbf{a}}-B_s^{\mathbf{a}}+B_s^{\mathbf{a}})^2-(B_s^{\mathbf{a}})^2|\Omega_s]\\ &= \hat{\mathbb{E}}[(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})^2+2(B_t^{\mathbf{a}}-B_s^{\mathbf{a}})B_s^{\mathbf{a}}|\Omega_s]\\ &= \sigma^2_{\mathbf{a}\mathbf{a}^T}(t-s).\end{aligned}$$

3.3 Itô's Integral with Respect to G-Brownian Motion

For $T\in\mathbb{R}^+$, a partition π_T of $[0,T]$ is a finite ordered subset $\pi_T=\{t_0,t_1,\cdots,t_N\}$ such that $0=t_0<t_1<\cdots<t_N=T$. Set

$$\mu(\pi_T):=\max\{|t_{i+1}-t_i|\ :\ i=0,1,\cdots,N-1\}.$$

We use $\pi_T^N=\{t_0^N,t_1^N,\cdots,t_N^N\}$ to denote a sequence of partitions of $[0,T]$ such that $\lim_{N\to\infty}\mu(\pi_T^N)=0$.

Let $p\geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T=\{t_0,\cdots,t_N\}$ of $[0,T]$ we set

$$\eta_t(\omega)=\sum_{k=0}^{N-1}\xi_k(\omega)\mathbf{1}_{[t_k,t_{k+1})}(t),$$

where $\xi_k\in L_G^p(\Omega_{t_k})$, $k=0,1,2,\cdots,N-1$ are given. The collection of these processes is denoted by $M_G^{p,0}(0,T)$.

Definition 3.3.1 For an $\eta\in M_G^{p,0}(0,T)$ with $\eta_t(\omega)=\sum_{k=0}^{N-1}\xi_k(\omega)\mathbf{1}_{[t_k,t_{k+1})}(t)$, the related **Bochner integral** is

$$\int_0^T\eta_t(\omega)dt:=\sum_{k=0}^{N-1}\xi_k(\omega)(t_{k+1}-t_k).$$

For each $\eta\in M_G^{p,0}(0,T)$, we set

$$\widetilde{\mathbb{E}}_T[\eta]:=\frac{1}{T}\hat{\mathbb{E}}\left[\int_0^T\eta_t dt\right]=\frac{1}{T}\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}\xi_k(\omega)(t_{k+1}-t_k)\right].$$

It is easy to check that $\widetilde{\mathbb{E}}_T:M_G^{p,0}(0,T)\mapsto\mathbb{R}$ forms a sublinear expectation. We then can introduce a natural norm $\|\cdot\|_{M_G^p}$, under which, $M_G^{p,0}(0,T)$ can be extended to $M_G^p(0,T)$ which is a Banach space.

Definition 3.3.2 For each $p \geq 1$, we denote by $M_G^p(0, T)$ the completion of $M_G^{p,0}(0, T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} := \left\{\hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p dt\right]\right\}^{1/p}.$$

It is clear that $M_G^p(0, T) \supset M_G^q(0, T)$ for $1 \leq p \leq q$. We also denote by $M_G^p(0, T; \mathbb{R}^d)$ the space of all d-dimensional stochastic processes $\eta_t = (\eta_t^1, \cdots, \eta_t^d)$, $t \geq 0$ such that $\eta_t^i \in M_G^p(0, T)$, $i = 1, 2, \cdots, d$.

We now give the definition of Itô's integral. For simplicity, we first introduce Itô's integral with respect to 1-dimensional G–Brownian motion.

Let $(B_t)_{t\geq 0}$ be a 1-dimensional G–Brownian motion with $G(\alpha) = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$.

Definition 3.3.3 For each $\eta \in M_G^{2,0}(0, T)$ of the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j \mathbf{1}_{[t_j,t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

Lemma 3.3.4 *The mapping* $I : M_G^{2,0}(0, T) \mapsto L_G^2(\Omega_T)$ *is a continuous linear mapping and thus can be continuously extended to* $I : M_G^2(0, T) \mapsto L_G^2(\Omega_T)$. *In particular, we have*

$$\hat{\mathbb{E}}\left[\int_0^T \eta_t dB_t\right] = 0, \tag{3.3.1}$$

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t\right)^2\right] \leq \overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T \eta_t^2 dt\right]. \tag{3.3.2}$$

Proof From Example 3.2.10, for each j,

$$\hat{\mathbb{E}}[\xi_j (B_{t_{j+1}} - B_{t_j})|\Omega_{t_j}] = \hat{\mathbb{E}}[-\xi_j (B_{t_{j+1}} - B_{t_j})|\Omega_{t_j}] = 0.$$

We have

$$\begin{aligned}\hat{\mathbb{E}}\left[\int_0^T \eta_t dB_t\right] &= \hat{\mathbb{E}}\left[\int_0^{t_{N-1}} \eta_t dB_t + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})\right] \\ &= \hat{\mathbb{E}}\left[\int_0^{t_{N-1}} \eta_t dB_t + \hat{\mathbb{E}}[\xi_{N-1}(B_{t_N} - B_{t_{N-1}})|\Omega_{t_{N-1}}]\right]\end{aligned}$$

$$= \hat{\mathbb{E}}\left[\int_0^{t_{N-1}} \eta_t dB_t\right].$$

Then we can repeat this procedure to obtain (3.3.1).

We now give the proof of (3.3.2). First, from Example 3.2.10, we have

$$\begin{aligned}
\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t\right)^2\right] &= \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t dB_t + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})\right)^2\right] \\
&= \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t dB_t\right)^2 + \xi_{N-1}^2(B_{t_N} - B_{t_{N-1}})^2\right. \\
&\quad \left. +2\left(\int_0^{t_{N-1}} \eta_t dB_t\right)\xi_{N-1}(B_{t_N} - B_{t_{N-1}})\right] \\
&= \hat{\mathbb{E}}\left[\left(\int_0^{t_{N-1}} \eta_t dB_t\right)^2 + \xi_{N-1}^2(B_{t_N} - B_{t_{N-1}})^2\right] \\
&= \cdots = \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \xi_i^2(B_{t_{i+1}} - B_{t_i})^2\right]
\end{aligned}$$

Then, for each $i = 0, 1, \cdots, N-1$, the following relations hold:

$$\begin{aligned}
&\hat{\mathbb{E}}[\xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] \\
=&\hat{\mathbb{E}}[\hat{\mathbb{E}}[\xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \overline{\sigma}^2\xi_i^2(t_{i+1} - t_j)|\Omega_{t_i}]] \\
=&\hat{\mathbb{E}}[\overline{\sigma}^2\xi_i^2(t_{i+1} - t_i) - \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] = 0.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t\right)^2\right] = \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \xi_i^2(B_{t_{i+1}} - B_{t_i})^2\right] \\
\leq&\hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \sum_{i=0}^{N-1} \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)\right] + \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)\right] \\
\leq&\sum_{i=0}^{N-1} \hat{\mathbb{E}}[\xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] + \hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)\right] \\
=&\hat{\mathbb{E}}\left[\sum_{i=0}^{N-1} \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)\right] = \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T \eta_t^2 dt\right].
\end{aligned}$$

□

Definition 3.3.5 For a fixed $\eta \in M_G^2(0, T)$, we define the stochastic integral

$$\int_0^T \eta_t dB_t := I(\eta).$$

It is clear that (3.3.1) and (3.3.2) still hold for $\eta \in M_G^2(0, T)$.

We list below the main properties of Itô's integral with respect to G-Brownian motion. We denote, for some $0 \leq s \leq t \leq T$,

$$\int_s^t \eta_u dB_u := \int_0^T \mathbf{1}_{[s,t]}(u)\eta_u dB_u.$$

Proposition 3.3.6 *Let $\eta, \theta \in M_G^2(0, T)$ and let $0 \leq s \leq r \leq t \leq T$. Then we have*

(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.

(ii) $\int_s^t (\alpha\eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, *if α is bounded and in $L_G^1(\Omega_s)$.*

(iii) $\hat{\mathbb{E}}\left[X + \int_r^T \eta_u dB_u \middle| \Omega_s\right] = \hat{\mathbb{E}}[X|\Omega_s]$ *for all $X \in L_G^1(\Omega)$.*

We now consider the multi-dimensional case. Let $G(\cdot) : \mathbb{S}(d) \mapsto \mathbb{R}$ be a given monotone and sublinear function and let $(B_t)_{t\geq 0}$ be a d-dimensional G-Brownian motion. For each fixed $\mathbf{a} \in \mathbb{R}^d$, we still use $B_t^{\mathbf{a}} := \langle \mathbf{a}, B_t \rangle$. Then $(B_t^{\mathbf{a}})_{t\geq 0}$ is a 1-dimensional $G_{\mathbf{a}}$-Brownian motion with $G_{\mathbf{a}}(\alpha) = \frac{1}{2}(\sigma^2_{\mathbf{aa}^T}\alpha^+ - \sigma^2_{-\mathbf{aa}^T}\alpha^-)$, where $\sigma^2_{\mathbf{aa}^T} = 2G(\mathbf{aa}^T)$ and $\sigma^2_{-\mathbf{aa}^T} = -2G(-\mathbf{aa}^T)$. Similarly to the 1-dimensional case, we can define Itô's integral by

$$I(\eta) := \int_0^T \eta_t dB_t^{\mathbf{a}}, \quad \text{for } \eta \in M_G^2(0, T).$$

We still have, for each $\eta \in M_G^2(0, T)$,

$$\hat{\mathbb{E}}\left[\int_0^T \eta_t dB_t^{\mathbf{a}}\right] = 0,$$

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t^{\mathbf{a}}\right)^2\right] \leq \sigma^2_{\mathbf{aa}^T}\hat{\mathbb{E}}\left[\int_0^T \eta_t^2 dt\right].$$

Furthermore, Proposition 3.3.6 still holds for the integral with respect to $B_t^{\mathbf{a}}$.

3.4 Quadratic Variation Process of *G*-Brownian Motion

We first consider the quadratic variation process of 1-dimensional G-Brownian motion $(B_t)_{t\geq 0}$ with $B_1 \stackrel{d}{=} N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$. Let π_t^N, $N = 1, 2, \cdots$, be a sequence of partitions of $[0, t]$. We start with the obvious relations

$$
\begin{aligned}
B_t^2 &= \sum_{j=0}^{N-1}(B_{t_{j+1}^N}^2 - B_{t_j^N}^2) \\
&= \sum_{j=0}^{N-1} 2B_{t_j^N}(B_{t_{j+1}^N} - B_{t_j^N}) + \sum_{j=0}^{N-1}(B_{t_{j+1}^N} - B_{t_j^N})^2.
\end{aligned}
$$

As $\mu(\pi_t^N) \to 0$, the first term of the right side converges to $2\int_0^t B_s dB_s$ in $L_G^2(\Omega)$. The second term must be convergent. We denote its limit by $\langle B\rangle_t$, i.e.,

$$
\langle B\rangle_t := \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1}(B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2\int_0^t B_s dB_s. \tag{3.4.1}
$$

By the above construction, $(\langle B\rangle_t)_{t\geq 0}$ is an increasing process with $\langle B\rangle_0 = 0$. We call it the **quadratic variation process** of the G-Brownian motion B. It characterizes the part of statistic uncertainty of G-Brownian motion. It is important to keep in mind that $\langle B\rangle_t$ is not a deterministic process unless $\underline{\sigma} = \overline{\sigma}$, i.e., when $(B_t)_{t\geq 0}$ is a classical Brownian motion. In fact, the following lemma is true.

Lemma 3.4.1 *For each* $0 \leq s \leq t < \infty$, *we have*

$$
\hat{\mathbb{E}}[\langle B\rangle_t - \langle B\rangle_s | \Omega_s] = \overline{\sigma}^2(t-s), \tag{3.4.2}
$$

$$
\hat{\mathbb{E}}[-(\langle B\rangle_t - \langle B\rangle_s) | \Omega_s] = -\underline{\sigma}^2(t-s). \tag{3.4.3}
$$

Proof By the definition of $\langle B\rangle$ and Proposition 3.3.6 (iii),

$$
\begin{aligned}
\hat{\mathbb{E}}[\langle B\rangle_t - \langle B\rangle_s | \Omega_s] &= \hat{\mathbb{E}}\left[B_t^2 - B_s^2 - 2\int_s^t B_u dB_u | \Omega_s\right] \\
&= \hat{\mathbb{E}}[B_t^2 - B_s^2 | \Omega_s] = \overline{\sigma}^2(t-s).
\end{aligned}
$$

The last step follows from Example 3.2.11. We then have (3.4.2). The equality (3.4.3) can be proved analogously in view of the the relation $\hat{\mathbb{E}}[-(B_t^2 - B_s^2)|\Omega_s] = -\underline{\sigma}^2(t-s)$. □

Here is a very interesting property of the quadratic variation process $\langle B\rangle$, just like for the G–Brownian motion B itself: the increment $\langle B\rangle_{s+t} - \langle B\rangle_s$ is independent from Ω_s and identically distributed with $\langle B\rangle_t$. We formulate this as a lemma.

Lemma 3.4.2 *For each fixed* $s, t \geq 0$, $\langle B\rangle_{s+t} - \langle B\rangle_s$ *is identically distributed with* $\langle B\rangle_t$ *and is independent from* Ω_s, *for any* $s \geq 0$.

Proof The claims follow directly from

$$\begin{aligned}\langle B\rangle_{s+t}-\langle B\rangle_s &= B_{s+t}^2-2\int_0^{s+t}B_r dB_r-\left(B_s^2-2\int_0^s B_r dB_r\right)\\ &=(B_{s+t}-B_s)^2-2\int_s^{s+t}(B_r-B_s)d(B_r-B_s)\\ &=\langle B^s\rangle_t,\end{aligned}$$

where $\langle B^s\rangle$ is the quadratic variation process of the G-Brownian motion $B_t^s=B_{s+t}-B_s,\ t\geq 0$. □

We now define the integral of a process $\eta\in M_G^{1,0}(0,T)$ with respect to $\langle B\rangle$. We start with the mapping:

$$Q_{0,T}(\eta)=\int_0^T\eta_t d\langle B\rangle_t:=\sum_{j=0}^{N-1}\xi_j(\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j}):M_G^{1,0}(0,T)\mapsto L_G^1(\Omega_T).$$

Lemma 3.4.3 *For each $\eta\in M_G^{1,0}(0,T)$,*

$$\hat{\mathbb{E}}[|Q_{0,T}(\eta)|]\leq\overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T|\eta_t|dt\right]. \tag{3.4.4}$$

Thus $Q_{0,T}:M_G^{1,0}(0,T)\mapsto L_G^1(\Omega_T)$ is a continuous linear mapping. Consequently, $Q_{0,T}$ can be uniquely extended to $M_G^1(0,T)$. We still denote this mapping by

$$\int_0^T\eta_t d\langle B\rangle_t:=Q_{0,T}(\eta)\ \textit{for}\ \eta\in M_G^1(0,T).$$

As before, the following relation holds:

$$\hat{\mathbb{E}}\left[\left|\int_0^T\eta_t d\langle B\rangle_t\right|\right]\leq\overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T|\eta_t|dt\right]\ \textit{for}\ \eta\in M_G^1(0,T). \tag{3.4.5}$$

Proof First, for each $j=1,\cdots,N-1$, we have

$$\begin{aligned}&\hat{\mathbb{E}}[|\xi_j|(\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j})-\overline{\sigma}^2|\xi_j|(t_{j+1}-t_j)]\\ =&\hat{\mathbb{E}}[\hat{\mathbb{E}}[|\xi_j|(\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j})|\Omega_{t_j}]-\overline{\sigma}^2|\xi_j|(t_{j+1}-t_j)]\\ =&\hat{\mathbb{E}}[|\xi_j|\overline{\sigma}^2(t_{j+1}-t_j)-\overline{\sigma}^2|\xi_j|(t_{j+1}-t_j)]=0.\end{aligned}$$

Then (3.4.4) can be checked as follows:

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\sum_{j=0}^{N-1}\xi_j(\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j})\right|\right] \le \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1}|\xi_j|(\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j})\right] \\
\le&\hat{\mathbb{E}}\left[\sum_{j=0}^{N-1}|\xi_j|[(\langle B\rangle_{t_{j+1}}-\langle B\rangle_{t_j})-\overline{\sigma}^2(t_{j+1}-t_j)]\right]+\hat{\mathbb{E}}\left[\overline{\sigma}^2\sum_{j=0}^{N-1}|\xi_j|(t_{j+1}-t_j)\right] \\
=&\hat{\mathbb{E}}\left[\overline{\sigma}^2\sum_{j=0}^{N-1}|\xi_j|(t_{j+1}-t_j)\right]=\overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T|\eta_t|dt\right].
\end{aligned}$$

□

Proposition 3.4.4 *Let* $0\le s\le t$, $\xi\in L_G^2(\Omega_s)$, $X\in L_G^1(\Omega)$. *Then*

$$\begin{aligned}
\hat{\mathbb{E}}[X+\xi(B_t^2-B_s^2)] &= \hat{\mathbb{E}}[X+\xi(B_t-B_s)^2] \\
&= \hat{\mathbb{E}}[X+\xi(\langle B\rangle_t-\langle B\rangle_s)].
\end{aligned}$$

Proof By (3.4.1) and Proposition 3.3.6 (iii), we have

$$\begin{aligned}
\hat{\mathbb{E}}[X+\xi(B_t^2-B_s^2)] &= \hat{\mathbb{E}}\left[X+\xi(\langle B\rangle_t-\langle B\rangle_s+2\int_s^t B_u dB_u)\right] \\
&= \hat{\mathbb{E}}[X+\xi(\langle B\rangle_t-\langle B\rangle_s)].
\end{aligned}$$

We also have

$$\begin{aligned}
\hat{\mathbb{E}}[X+\xi(B_t^2-B_s^2)] &= \hat{\mathbb{E}}[X+\xi((B_t-B_s)^2+2(B_t-B_s)B_s)] \\
&= \hat{\mathbb{E}}[X+\xi(B_t-B_s)^2].
\end{aligned}$$

□

We have the following isometry.

Proposition 3.4.5 *Let* $\eta\in M_G^2(0,T)$. *Then*

$$\hat{\mathbb{E}}\left[\left(\int_0^T\eta_t dB_t\right)^2\right]=\hat{\mathbb{E}}\left[\int_0^T\eta_t^2 d\langle B\rangle_t\right]. \tag{3.4.6}$$

Proof For any process $\eta\in M_G^{2,0}(0,T)$ of the form

$$\eta_t(\omega)=\sum_{j=0}^{N-1}\xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t)$$

we have $\int_0^T \eta_t dB_t = \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j})$. From Proposition 3.3.6, we get

$$\hat{\mathbb{E}}[X + 2\xi_j(B_{t_{j+1}} - B_{t_j})\xi_i(B_{t_{i+1}} - B_{t_i})] = \hat{\mathbb{E}}[X], \quad \text{for all } X \in L_G^1(\Omega), i \neq j.$$

Thus

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t\right)^2\right] = \hat{\mathbb{E}}[(\sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}))^2] = \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} \xi_j^2(B_{t_{j+1}} - B_{t_j})^2\right].$$

From this and Proposition 3.4.4, it follows that

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t\right)^2\right] = \hat{\mathbb{E}}\left[\sum_{j=0}^{N-1} \xi_j^2(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j})\right] = \hat{\mathbb{E}}\left[\int_0^T \eta_t^2 d\langle B\rangle_t\right].$$

This shows that (3.4.6) holds for $\eta \in M_G^{2,0}(0,T)$. We can continuously extend the above equality to the case $\eta \in M_G^2(0,T)$ and get (3.4.6). □

We now consider the multi-dimensional case. Let $(B_t)_{t\geq 0}$ be a d-dimensional G-Brownian motion. For each fixed $\mathbf{a} \in \mathbb{R}^d$, $(B_t^{\mathbf{a}})_{t\geq 0}$ is a 1-dimensional $G_{\mathbf{a}}$-Brownian motion. Similar to the 1-dimensional case, we can define

$$\langle B^{\mathbf{a}}\rangle_t := \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1} (B^{\mathbf{a}}_{t^N_{j+1}} - B^{\mathbf{a}}_{t^N_j})^2 = (B_t^{\mathbf{a}})^2 - 2\int_0^t B_s^{\mathbf{a}} dB_s^{\mathbf{a}},$$

where $\langle B^{\mathbf{a}}\rangle$ is called the **quadratic variation process** of $B^{\mathbf{a}}$. The above results, see Lemma 3.4.3 and Proposition 3.4.5, also hold for $\langle B^{\mathbf{a}}\rangle$. In particular,

$$\hat{\mathbb{E}}\left[\left|\int_0^T \eta_t d\langle B^{\mathbf{a}}\rangle_t\right|\right] \leq \sigma^2_{\mathbf{a}\mathbf{a}^T} \hat{\mathbb{E}}\left[\int_0^T |\eta_t| dt\right], \quad \text{for all } \eta \in M_G^1(0,T)$$

and

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t^{\mathbf{a}}\right)^2\right] = \hat{\mathbb{E}}\left[\int_0^T \eta_t^2 d\langle B^{\mathbf{a}}\rangle_t\right] \quad \text{for all } \eta \in M_G^2(0,T).$$

Let $\mathbf{a} = (a_1, \cdots, a_d)^T$ and $\bar{\mathbf{a}} = (\bar{a}_1, \cdots, \bar{a}_d)^T$ be two given vectors in $\mathbb{R}^d$. We then have their quadratic variation processes of $\langle B^{\mathbf{a}}\rangle$ and $\langle B^{\bar{\mathbf{a}}}\rangle$. We can define their **mutual variation process** by

$$\begin{aligned}\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t &:= \frac{1}{4}[\langle B^{\mathbf{a}} + B^{\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}} - B^{\bar{\mathbf{a}}}\rangle_t] \\ &= \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t].\end{aligned}$$

Since $\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle = \langle B^{\bar{\mathbf{a}}-\mathbf{a}}\rangle = \langle -B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle$, we see that $\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t = \langle B^{\bar{\mathbf{a}}}, B^{\mathbf{a}}\rangle_t$. In particular, we have $\langle B^{\mathbf{a}}, B^{\mathbf{a}}\rangle = \langle B^{\mathbf{a}}\rangle$. Let π_t^N, $N = 1, 2, \cdots$, be a sequence of partitions of $[0, t]$. We observe that

$$\sum_{k=0}^{N-1}(B^{\mathbf{a}}_{t^N_{k+1}} - B^{\mathbf{a}}_{t^N_k})(B^{\bar{\mathbf{a}}}_{t^N_{k+1}} - B^{\bar{\mathbf{a}}}_{t^N_k}) = \frac{1}{4}\sum_{k=0}^{N-1}[(B^{\mathbf{a}+\bar{\mathbf{a}}}_{t_{k+1}} - B^{\mathbf{a}+\bar{\mathbf{a}}}_{t_k})^2 - (B^{\mathbf{a}-\bar{\mathbf{a}}}_{t_{k+1}} - B^{\mathbf{a}-\bar{\mathbf{a}}}_{t_k})^2].$$

As $\mu(\pi_t^N) \to 0$ we obtain

$$\lim_{N\to\infty}\sum_{k=0}^{N-1}(B^{\mathbf{a}}_{t^N_{k+1}} - B^{\mathbf{a}}_{t^N_k})(B^{\bar{\mathbf{a}}}_{t^N_{k+1}} - B^{\bar{\mathbf{a}}}_{t^N_k}) = \langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t.$$

We also have

$$\begin{aligned}\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t &= \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t] \\ &= \frac{1}{4}\left[(B^{\mathbf{a}+\bar{\mathbf{a}}}_t)^2 - 2\int_0^t B^{\mathbf{a}+\bar{\mathbf{a}}}_s dB^{\mathbf{a}+\bar{\mathbf{a}}}_s - (B^{\mathbf{a}-\bar{\mathbf{a}}}_t)^2 + 2\int_0^t B^{\mathbf{a}-\bar{\mathbf{a}}}_s dB^{\mathbf{a}-\bar{\mathbf{a}}}_s\right] \\ &= B^{\mathbf{a}}_t B^{\bar{\mathbf{a}}}_t - \int_0^t B^{\mathbf{a}}_s dB^{\bar{\mathbf{a}}}_s - \int_0^t B^{\bar{\mathbf{a}}}_s dB^{\mathbf{a}}_s.\end{aligned}$$

Now for each $\eta \in M_G^1(0, T)$, we can consistently define

$$\int_0^T \eta_t d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t := \frac{1}{4}\int_0^T \eta_t d\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \frac{1}{4}\int_0^T \eta_t d\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t.$$

Lemma 3.4.6 *Let $\eta^N \in M_G^{2,0}(0, T)$, $N = 1, 2, \cdots$, be of the form*

$$\eta_t^N(\omega) = \sum_{k=0}^{N-1}\xi_k^N(\omega)\mathbf{1}_{[t_k^N, t_{k+1}^N)}(t)$$

with $\mu(\pi_T^N) \to 0$ and $\eta^N \to \eta$ in $M_G^2(0, T)$, as $N \to \infty$. Then we have the following convergence in $L_G^2(\Omega_T)$:

$$\sum_{k=0}^{N-1}\xi_k^N(B^{\mathbf{a}}_{t^N_{k+1}} - B^{\mathbf{a}}_{t^N_k})(B^{\bar{\mathbf{a}}}_{t^N_{k+1}} - B^{\bar{\mathbf{a}}}_{t^N_k}) \to \int_0^T \eta_t d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t.$$

Proof Since

$$\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_{t_{k+1}^N} - \langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_{t_k^N} = (B^{\mathbf{a}}_{t_{k+1}^N} - B^{\mathbf{a}}_{t_k^N})(B^{\bar{\mathbf{a}}}_{t_{k+1}^N} - B^{\bar{\mathbf{a}}}_{t_k^N}) \\ - \int_{t_k^N}^{t_{k+1}^N} (B^{\mathbf{a}}_s - B^{\mathbf{a}}_{t_k^N}) dB^{\bar{\mathbf{a}}}_s - \int_{t_k^N}^{t_{k+1}^N} (B^{\bar{\mathbf{a}}}_s - B^{\bar{\mathbf{a}}}_{t_k^N}) dB^{\mathbf{a}}_s,$$

we only need to show the convergence

$$\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} (\xi_k^N)^2 \left(\int_{t_k^N}^{t_{k+1}^N} (B^{\mathbf{a}}_s - B^{\mathbf{a}}_{t_k^N}) dB^{\bar{\mathbf{a}}}_s\right)^2\right] \to 0.$$

For each $k = 1, \cdots, N-1$, denoting $c = \overline{\sigma}^2_{\mathbf{a}\mathbf{a}^T}\overline{\sigma}^2_{\bar{\mathbf{a}}\bar{\mathbf{a}}^T}/2$, we have,

$$\begin{aligned}
&\hat{\mathbb{E}}\left[(\xi_k^N)^2 \left(\int_{t_k^N}^{t_{k+1}^N} (B^{\mathbf{a}}_s - B^{\mathbf{a}}_{t_k^N}) dB^{\bar{\mathbf{a}}}_s\right)^2 - c(\xi_k^N)^2 (t_{k+1}^N - t_k^N)^2\right] \\
=&\hat{\mathbb{E}}\left[\hat{\mathbb{E}}\left[(\xi_k^N)^2 \left(\int_{t_k^N}^{t_{k+1}^N} (B^{\mathbf{a}}_s - B^{\mathbf{a}}_{t_k^N}) dB^{\bar{\mathbf{a}}}_s\right)^2 |\Omega_{t_k^N}\right] - c(\xi_k^N)^2 (t_{k+1}^N - t_k^N)^2\right] \\
\leq&\hat{\mathbb{E}}[c(\xi_k^N)^2 (t_{k+1}^N - t_k^N)^2 - c(\xi_k^N)^2 (t_{k+1}^N - t_k^N)^2] \to 0, \quad \text{as } N \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} (\xi_k^N)^2 \left(\int_{t_k^N}^{t_{k+1}^N} (B^{\mathbf{a}}_s - B^{\mathbf{a}}_{t_k^N}) dB^{\bar{\mathbf{a}}}_s\right)^2\right] \\
\leq&\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} (\xi_k^N)^2 \left[\left(\int_{t_k^N}^{t_{k+1}^N} (B^{\mathbf{a}}_s - B^{\mathbf{a}}_{t_k^N}) dB^{\bar{\mathbf{a}}}_s\right)^2 - c(t_{k+1}^N - t_k^N)^2\right]\right] \\
&+ \hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} c(\xi_k^N)^2 (t_{k+1}^N - t_k^N)^2\right] \\
\leq&\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1} c(\xi_k^N)^2 (t_{k+1}^N - t_k^N)^2\right] \leq c\mu(\pi_T^N)\hat{\mathbb{E}}\left[\int_0^T |\eta_t^N|^2 dt\right] \to 0, \quad \text{as} \quad N \to \infty.
\end{aligned}$$

□

3.5 Distribution of the Quadratic Variation Process $\langle B\rangle$

In this section, we first consider the 1-dimensional G-Brownian motion $(B_t)_{t\geq 0}$ with $B_1 \overset{d}{=} N(\{0\}\times[\underline{\sigma}^2,\overline{\sigma}^2])$.

The quadratic variation process $\langle B\rangle$ of the **G**-Brownian motion B is a very interesting process. We have seen that the **G**-Brownian motion B is a typical process with variance uncertainty but without mean-uncertainty. In fact, all distributional uncertainty of the G-Brownian motion B is concentrated in $\langle B\rangle$. Moreover, $\langle B\rangle$ itself is a typical process with mean-uncertainty. This fact will be applied later to measure the mean-uncertainty of risk positions.

Lemma 3.5.1 *We have the following upper bound:*

$$\hat{\mathbb{E}}[\langle B\rangle_t^2] \leq 10\overline{\sigma}^4 t^2. \tag{3.5.1}$$

Proof Indeed,

$$\begin{aligned}
\hat{\mathbb{E}}[\langle B\rangle_t^2] &= \hat{\mathbb{E}}\left[\left({B_t}^2 - 2\int_0^t B_u dB_u\right)^2\right] \\
&\leq 2\hat{\mathbb{E}}[B_t^4] + 8\hat{\mathbb{E}}\left[\left(\int_0^t B_u dB_u\right)^2\right] \\
&\leq 6\overline{\sigma}^4 t^2 + 8\overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^t {B_u}^2 du\right] \\
&\leq 6\overline{\sigma}^4 t^2 + 8\overline{\sigma}^2\int_0^t \hat{\mathbb{E}}[{B_u}^2]du \\
&= 10\overline{\sigma}^4 t^2.
\end{aligned}$$

□

Proposition 3.5.2 *Let $(b_t)_{t\geq 0}$ be a d-dimensional Brownian motion on a sublinear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ satisfying:*

(i) $b_0 = 0$*;*
(ii) For each $t, s \geq 0$, $b_{t+s} - b_t$ is identically distributed with b_s and independent from $(b_{t_1}, b_{t_2},\cdots,b_{t_n})$ for all $0\leq t_1,\cdots,t_n\leq t$;
(iii) $\lim_{t\downarrow 0} t^{-1}\hat{\mathbb{E}}[|b_t|^2] = 0$.

Then b_t is maximally distributed in the sense that:

$$\hat{\mathbb{E}}[\varphi(b_t)] = \max_{v\in\Gamma}\varphi(vt),$$

where Γ is the bounded closed and convex subset in $\mathbb{R}^d$ satisfying

$$\max_{v\in\Gamma}(p,v)=\hat{\mathbb{E}}[\langle p,b_1\rangle],\quad p\in\mathbb{R}^d.$$

In particular, if b is 1-dimensional ($d=1$), then $\Gamma=[\underline{\mu},\overline{\mu}]$, with $\overline{\mu}=\hat{\mathbb{E}}[b_1]$ and $\underline{\mu}=-\hat{\mathbb{E}}[-b_1]$.

Remark 3.5.3 Observe that for a symmetric G-Brownian motion B defined in Definition 3.1.2, the assumption corresponding to (iii) is: $\lim_{t\downarrow 0}\hat{\mathbb{E}}[|B_t|^3]t^{-1}=0$.

Proof We only give a proof for the case $d=1$ (see the proof of Theorem 3.8.2 for a more general situation). We first show that

$$\hat{\mathbb{E}}[pb_t]=g(p)t,\ \ p\in\mathbb{R}.$$

We set $\varphi(t):=\hat{\mathbb{E}}[b_t]$. Then $\varphi(0)=0$ and $\lim_{t\downarrow 0}\varphi(t)=0$. For each $t,s\geq 0$,

$$\begin{aligned}\varphi(t+s)=\hat{\mathbb{E}}[b_{t+s}]&=\hat{\mathbb{E}}[(b_{t+s}-b_s)+b_s]\\&=\varphi(t)+\varphi(s).\end{aligned}$$

Hence $\varphi(t)$ is linear and uniformly continuous in t, which means that $\hat{\mathbb{E}}[b_t]=\overline{\mu}t$. Similarly we obtain that $-\hat{\mathbb{E}}[-b_t]=\underline{\mu}t$.

We now prove that b_t is $N([\underline{\mu}t,\overline{\mu}t]\times\{0\})$-distributed. By Exercise 2.5.3 in Chap. 2, we just need to show that for each fixed $\varphi\in C_{b.Lip}(\mathbb{R})$, the function

$$u(t,x):=\hat{\mathbb{E}}[\varphi(x+b_t)],\ \ (t,x)\in[0,\infty)\times\mathbb{R}$$

is a viscosity solution of the following parabolic PDE:

$$\partial_t u-g(\partial_x u)=0,\quad u|_{t=0}=\varphi \tag{3.5.2}$$

with $g(a)=\overline{\mu}a^+-\underline{\mu}a^-$.

We first notice that u is Lipschitz in x and $\frac{1}{2}$-Hölder continuous in t. Indeed, for each fixed t, $u(t,\cdot)\in C_{b.Lip}(\mathbb{R})$ since

$$\begin{aligned}|\hat{\mathbb{E}}[\varphi(x+b_t)]-\hat{\mathbb{E}}[\varphi(y+b_t)]|&\leq\hat{\mathbb{E}}[|\varphi(x+b_t)-\varphi(y+b_t)|]\\&\leq C|x-y|.\end{aligned}$$

For each $\delta\in[0,t]$, since b_t-b_δ is independent from b_δ, we have

$$\begin{aligned}u(t,x)&=\hat{\mathbb{E}}[\varphi(x+b_\delta+(b_t-b_\delta)]\\&=\hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(y+(b_t-b_\delta))]_{y=x+b_\delta}],\end{aligned}$$

hence

$$u(t,x)=\hat{\mathbb{E}}[u(t-\delta,x+b_\delta)]. \tag{3.5.3}$$

Thus

$$\begin{aligned}|u(t,x)-u(t-\delta,x)| &= |\hat{\mathbb{E}}[u(t-\delta,x+b_\delta)-u(t-\delta,x)]|\\ &\le \hat{\mathbb{E}}[|u(t-\delta,x+b_\delta)-u(t-\delta,x)|]\\ &\le \hat{\mathbb{E}}[C|b_\delta|]\le C_1\sqrt{\delta}.\end{aligned}$$

To prove that u is a viscosity solution of the PDE (3.5.2), we fix a point $(t,x)\in(0,\infty)\times\mathbb{R}$ and let $v\in C_b^{2,2}([0,\infty)\times\mathbb{R})$ be such that $v\ge u$ and $v(t,x)=u(t,x)$. From (3.5.3), we find that

$$v(t,x)=\hat{\mathbb{E}}[u(t-\delta,x+b_\delta)]\le\hat{\mathbb{E}}[v(t-\delta,x+b_\delta)].$$

Therefore, by Taylor's expansion,

$$\begin{aligned}0 &\le \hat{\mathbb{E}}[v(t-\delta,x+b_\delta)-v(t,x)]\\ &= \hat{\mathbb{E}}[v(t-\delta,x+b_\delta)-v(t,x+b_\delta)+(v(t,x+b_\delta)-v(t,x))]\\ &= \hat{\mathbb{E}}[-\partial_t v(t,x)\delta+\partial_x v(t,x)b_\delta+I_\delta]\\ &\le -\partial_t v(t,x)\delta+\hat{\mathbb{E}}[\partial_x v(t,x)b_\delta]+\hat{\mathbb{E}}[I_\delta]\\ &= -\partial_t v(t,x)\delta+g(\partial_x v(t,x))\delta+\hat{\mathbb{E}}[I_\delta],\end{aligned}$$

where

$$\begin{aligned}I_\delta=\delta\int_0^1&[-\partial_t v(t-\beta\delta,x+b_\delta)+\partial_t v(t,x)]d\beta\\ &+b_\delta\int_0^1[\partial_x v(t,x+\beta b_\delta)-\partial_x v(t,x)]d\beta.\end{aligned}$$

From the assumption that $\lim_{t\downarrow 0}t^{-1}\hat{\mathbb{E}}[b_t^2]=0$, we can check that

$$\lim_{\delta\downarrow 0}\delta^{-1}\hat{\mathbb{E}}[|I_\delta|]=0,$$

which implies that $\partial_t v(t,x)-g(\partial_x v(t,x))\le 0$. Hence u is a viscosity subsolution of (3.5.2). We can analogously prove that u is also a viscosity supersolution. It follows that b_t is $N([\underline{\mu}t,\overline{\mu}t]\times\{0\})$-distributed. □

It is clear that $\langle B\rangle$ satisfies all the conditions in Proposition 3.5.2, which leads immediately to another statement.

Theorem 3.5.4 *The process $\langle B\rangle_t$ is $N([\underline{\sigma}^2 t,\overline{\sigma}^2 t]\times\{0\})$-distributed, i.e.,*

$$\hat{\mathbb{E}}[\varphi(\langle B\rangle_t)]=\sup_{\underline{\sigma}^2\le v\le\overline{\sigma}^2}\varphi(vt),\quad \textit{for each}\ \ \varphi\in C_{l.Lip}(\mathbb{R}). \tag{3.5.4}$$

Corollary 3.5.5 *For each* $0 \le t \le T < \infty$*, we have*

$$\underline{\sigma}^2(T-t) \le \langle B\rangle_T - \langle B\rangle_t \le \overline{\sigma}^2(T-t) \;\; in\ L_G^1(\Omega).$$

Proof It is a direct consequence of the relations

$$\hat{\mathbb{E}}[(\langle B\rangle_T - \langle B\rangle_t - \overline{\sigma}^2(T-t))^+] = \sup_{\underline{\sigma}^2\le v\le\overline{\sigma}^2} (v-\overline{\sigma}^2)^+(T-t) = 0$$

and

$$\hat{\mathbb{E}}[(\langle B\rangle_T - \langle B\rangle_t - \underline{\sigma}^2(T-t))^-] = \sup_{\underline{\sigma}^2\le v\le\overline{\sigma}^2} (v-\underline{\sigma}^2)^-(T-t) = 0.$$

□

Corollary 3.5.6 *We have, for each* $t, s \ge 0$*,* $n \in \mathbb{N}$*,*

$$\hat{\mathbb{E}}[(\langle B\rangle_{t+s} - \langle B\rangle_s)^n|\Omega_s] = \hat{\mathbb{E}}[\langle B\rangle_t^n] = \overline{\sigma}^{2n}t^n \tag{3.5.5}$$

and

$$\hat{\mathbb{E}}[-(\langle B\rangle_{t+s} - \langle B\rangle_s)^n|\Omega_s] = \hat{\mathbb{E}}[-\langle B\rangle_t^n] = -\underline{\sigma}^{2n}t^n. \tag{3.5.6}$$

We now consider the multi-dimensional case. For notational simplicity, we write by $B^i := B^{\mathbf{e}_i}$ for the i-th coordinate of the G-Brownian motion B, under a given orthonormal basis $(\mathbf{e}_1, \cdots, \mathbf{e}_d)$ in the space $\mathbb{R}^d$. We denote

$$\langle B\rangle_t^{ij} := \langle B^i, B^j\rangle_t, \quad \langle B\rangle_t := (\langle B\rangle_t^{ij})_{i,j=1}^d.$$

Then $\langle B\rangle_t$, $t \ge 0$, is an $\mathbb{S}(d)$-valued process. Since

$$\hat{\mathbb{E}}[\langle AB_t, B_t\rangle] = 2G(A)\cdot t \;\; \text{for } A \in \mathbb{S}(d),$$

we have

$$\begin{aligned}\hat{\mathbb{E}}[(\langle B\rangle_t, A)] &= \hat{\mathbb{E}}\left[\sum_{i,j=1}^d a_{ij}\,\langle B\rangle_t^{ij}\right]\\ &= \hat{\mathbb{E}}\left[\sum_{i,j=1}^d a_{ij}(B_t^iB_t^j - \int_0^t B_s^i dB_s^j - \int_0^t B_s^j dB_s^i)\right]\\ &= \hat{\mathbb{E}}\left[\sum_{i,j=1}^d a_{ij}B_t^iB_t^j\right] = 2G(A)\cdot t \quad \text{for all} \quad A\in\mathbb{S}(d),\end{aligned}$$

where $A = (a_{ij})_{i,j=1}^d$.

Now we set, for each $\varphi \in C_{l.Lip}(\mathbb{S}(d))$,

$$v(t,x) := \hat{\mathbb{E}}[\varphi(x+\langle B\rangle_t)], \quad (t,x)\in[0,\infty)\times\mathbb{S}(d).$$

Let $\Gamma \subset \mathbb{S}_+(d)$ be the bounded, convex and closed subset such that

$$G(A) = \frac{1}{2}\sup_{B\in\Gamma}(A,B), \qquad A\in\mathbb{S}(d).$$

Proposition 3.5.7 *The function v solves the following first order PDE:*

$$\partial_t v - 2G(Dv) = 0, \ \ v|_{t=0} = \varphi,$$

where $Dv = (\partial_{x_{ij}} v)_{i,j=1}^d$. *We also have*

$$v(t,x) = \sup_{\gamma\in\Gamma}\varphi(x+t\gamma).$$

Sketch of the Proof. We start with the relation

$$\begin{aligned} v(t+\delta,x) &= \hat{\mathbb{E}}[\varphi(x+\langle B\rangle_\delta+\langle B\rangle_{t+\delta}-\langle B\rangle_\delta)] \\ &= \hat{\mathbb{E}}[v(t,x+\langle B\rangle_\delta)]. \end{aligned}$$

The rest of the proof is similar to that in the 1-dimensional case. □

Corollary 3.5.8 *The following inclusion is true.*

$$\langle B\rangle_t \in t\Gamma := \{t\times\gamma : \gamma\in\Gamma\}.$$

This is equivalent to $d_{t\Gamma}(\langle B\rangle_t) = 0$, *where* $d_U(x) = \inf\{\sqrt{(x-y,x-y)} : y\in U\}$.

Proof Since

$$\hat{\mathbb{E}}[d_{t\Gamma}(\langle B\rangle_t)] = \sup_{\gamma\in\Gamma} d_{t\Gamma}(t\gamma) = 0,$$

it follows that $d_{t\Gamma}(\langle B\rangle_t) = 0$. □

3.6 Itô's Formula

In Theorem 3.6.5 of this section, we provide Itô's formula for a "G-Itô process" X. Let us begin with considering a sufficiently regular function Φ.

Lemma 3.6.1 *Let* $\Phi \in C^2(\mathbb{R}^n)$ *with* $\partial_{x^\nu}\Phi,\ \partial^2_{x^\mu x^\nu}\Phi \in C_{b.Lip}(\mathbb{R}^n)$ *for* $\mu,\nu = 1,\cdots,n$. *Let* $s\in[0,T]$ *be fixed and let* $X = (X^1,\cdots,X^n)^T$ *be an n–dimensional process on* $[s,T]$ *of the form*

$$X_t^\nu = X_s^\nu + \alpha^\nu (t-s) + \eta^{\nu ij}(\langle B\rangle_t^{ij} - \langle B\rangle_s^{ij}) + \beta^{\nu j}(B_t^j - B_s^j).$$

Here, for $\nu = 1, \cdots, n$, $i, j = 1, \cdots, d$, α^ν, $\eta^{\nu ij}$ and $\beta^{\nu j}$ are bounded elements in $L_G^2(\Omega_s)$ and $X_s = (X_s^1, \cdots, X_s^n)^T$ is a given random vector in $L_G^2(\Omega_s)$. Then we have, in $L_G^2(\Omega_t)$,

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)\beta^{\nu j} dB_u^j + \int_s^t \partial_{x^\nu}\Phi(X_u)\alpha^\nu du \quad (3.6.1)$$
$$+ \int_s^t [\partial_{x^\nu}\Phi(X_u)\eta^{\nu ij} + \tfrac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta^{\mu i}\beta^{\nu j}] d\langle B\rangle_u^{ij}.$$

Here we adopt the Einstein convention, i.e., the above repeated indices μ, ν, i and j mean the summation.

Proof For any positive integer N, we set $\delta = (t-s)/N$ and take the partition

$$\pi_{[s,t]}^N = \{t_0^N, t_1^N, \cdots, t_N^N\} = \{s, s+\delta, \cdots, s+N\delta = t\}.$$

We have

$$\Phi(X_t) - \Phi(X_s) = \sum_{k=0}^{N-1}[\Phi(X_{t_{k+1}^N}) - \Phi(X_{t_k^N})] \quad (3.6.2)$$
$$= \sum_{k=0}^{N-1}\{\partial_{x^\nu}\Phi(X_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N})$$
$$+ \frac{1}{2}[\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})(X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N}) + \rho_k^N]\},$$

where

$$\rho_k^N = [\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N} + \theta_k(X_{t_{k+1}^N} - X_{t_k^N})) - \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})](X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N})$$

with $\theta_k \in [0, 1]$. The next is to derive that

$$\hat{\mathbb{E}}[|\rho_k^N|^2] = \hat{\mathbb{E}}[|[\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N} + \theta_k(X_{t_{k+1}^N} - X_{t_k^N})) - \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})]$$
$$\times (X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N})|^2]$$
$$\le c\hat{\mathbb{E}}[|X_{t_{k+1}^N} - X_{t_k^N}|^6] \le C[\delta^6 + \delta^3],$$

where c is the Lipschitz constant of $\{\partial^2_{x^\mu x^\nu}\Phi\}_{\mu,\nu=1}^n$ and C is a constant independent of k. Thus

$$\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1}\rho_k^N\right|^2\right] \le N\sum_{k=0}^{N-1}\hat{\mathbb{E}}\left[\left|\rho_k^N\right|^2\right] \to 0, \quad \text{as } N\to\infty.$$

The remaining terms in the summation in the right hand side of (3.6.2) are $\xi_t^N + \zeta_t^N$ with

$$\xi_t^N = \sum_{k=0}^{N-1}\{\partial_{x^\nu}\Phi(X_{t_k^N})[\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^{\nu ij}(\langle B\rangle_{t_{k+1}^N}^{ij} - \langle B\rangle_{t_k^N}^{ij}) \\ + \beta^{\nu j}(B_{t_{k+1}^N}^j - B_{t_k^N}^j)] + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\beta^{\mu i}\beta^{\nu j}(B_{t_{k+1}^N}^i - B_{t_k^N}^i)(B_{t_{k+1}^N}^j - B_{t_k^N}^j)\}$$

and

$$\zeta_t^N = \frac{1}{2}\sum_{k=0}^{N-1}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\{[\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^{\mu ij}(\langle B\rangle_{t_{k+1}^N}^{ij} - \langle B\rangle_{t_k^N}^{ij})] \\ \times[\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^{\nu lm}(\langle B\rangle_{t_{k+1}^N}^{lm} - \langle B\rangle_{t_k^N}^{lm})] \\ + 2[\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^{\mu ij}(\langle B\rangle_{t_{k+1}^N}^{ij} - \langle B\rangle_{t_k^N}^{ij})]\beta^{\nu l}(B_{t_{k+1}^N}^l - B_{t_k^N}^l)\}.$$

We observe that, for each $u \in [t_k^N, t_{k+1}^N)$,

$$\hat{\mathbb{E}}[|\partial_{x^\nu}\Phi(X_u) - \sum_{k=0}^{N-1}\partial_{x^\nu}\Phi(X_{t_k^N})\mathbf{1}_{[t_k^N,t_{k+1}^N)}(u)|^2] \\ = \hat{\mathbb{E}}[|\partial_{x^\nu}\Phi(X_u) - \partial_{x^\nu}\Phi(X_{t_k^N})|^2] \\ \le c^2\hat{\mathbb{E}}[|X_u - X_{t_k^N}|^2] \le C[\delta + \delta^2],$$

where c is the Lipschitz constant of $\{\partial_{x^\nu}\Phi\}_{\nu=1}^n$ and C is a constant independent of k. Hence $\sum_{k=0}^{N-1}\partial_{x^\nu}\Phi(X_{t_k^N})\mathbf{1}_{[t_k^N,t_{k+1}^N)}(\cdot)$ converges to $\partial_{x^\nu}\Phi(X_\cdot)$ in $M_G^2(0,T)$. Similarly, as $N\to\infty$,

$$\sum_{k=0}^{N-1}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\mathbf{1}_{[t_k^N,t_{k+1}^N)}(\cdot) \to \partial^2_{x^\mu x^\nu}\Phi(X_\cdot) \quad \textit{in } M_G^2(0,T).$$

From Lemma 3.4.6 and by the definitions of integration with respect to dt, dB_t and $d\langle B\rangle_t$, the limit of ξ_t^N in $L_G^2(\Omega_t)$ is just the right hand side of (3.6.1). The next remark also leads to $\zeta_t^N \to 0$ in $L_G^2(\Omega_t)$. This completes the proof. □

Remark 3.6.2 To show that $\zeta_t^N \to 0$ in $L_G^2(\Omega_t)$, we use the following estimates: for each $\psi_\cdot^N = \sum_{k=0}^{N-1}\xi_{t_k}^N\mathbf{1}_{[t_k^N,t_{k+1}^N)}(\cdot) \in M_G^{2,0}(0,T)$ with $\pi_T^N = \{t_0^N,\cdots,t_N^N\}$ such that

$$\lim_{N\to\infty}\mu(\pi_T^N)=0 \quad \text{and} \quad \hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_{t_k}^N|^2(t_{k+1}^N-t_k^N)\right]\leq C,$$

for all $N=1,2,\cdots$, we have

$$\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1}\xi_k^N(t_{k+1}^N-t_k^N)^2\right|^2\right]\to 0, \quad \text{as} \quad N\to\infty.$$

Moreover, for any fixed $\mathbf{a},\bar{\mathbf{a}}\in\mathbb{R}^d$,

$$\begin{aligned}\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1}\xi_k^N(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})^2\right|^2\right]&\leq C\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_k^N|^2(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})^3\right]\\&\leq C\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_k^N|^2\sigma_{\mathbf{aa}^T}^6(t_{k+1}^N-t_k^N)^3\right]\to 0,\end{aligned}$$

$$\begin{aligned}&\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1}\xi_k^N(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})(t_{k+1}^N-t_k^N)\right|^2\right]\\\leq&C\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_k^N|^2(t_{k+1}^N-t_k^N)(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})^2\right]\\\leq&C\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_k^N|^2\sigma_{\mathbf{aa}^T}^4(t_{k+1}^N-t_k^N)^3\right]\to 0,\end{aligned}$$

as well as

$$\begin{aligned}&\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1}\xi_k^N(t_{k+1}^N-t_k^N)(B_{t_{k+1}^N}^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}})\right|^2\right]\\\leq&C\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_k^N|^2(t_{k+1}^N-t_k^N)|B_{t_{k+1}^N}^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}}|^2\right]\\\leq&C\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_k^N|^2\sigma_{\mathbf{aa}^T}^2(t_{k+1}^N-t_k^N)^2\right]\to 0\end{aligned}$$

and

$$\hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1}\xi_k^N(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})(B^{\bar{\mathbf{a}}}_{t_{k+1}^N}-B^{\bar{\mathbf{a}}}_{t_k^N})\right|^2\right]$$
$$\leq C\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_k^N|^2(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})|B^{\bar{\mathbf{a}}}_{t_{k+1}^N}-B^{\bar{\mathbf{a}}}_{t_k^N}|^2\right]$$
$$\leq C\hat{\mathbb{E}}\left[\sum_{k=0}^{N-1}|\xi_k^N|^2\sigma^2_{\mathbf{a}\mathbf{a}^T}\sigma^2_{\bar{\mathbf{a}}\bar{\mathbf{a}}^T}(t_{k+1}^N-t_k^N)^2\right]\to 0.$$

□

We are going now to derive a general form of Itô's formula. We start with

$$X_t^\nu=X_0^\nu+\int_0^t\alpha_s^\nu ds+\int_0^t\eta_s^{\nu ij}d\langle B\rangle_s^{ij}+\int_0^t\beta_s^{\nu j}dB_s^j,\quad \nu=1,\cdots,n,\quad i,j=1,\cdots,d.$$

Proposition 3.6.3 *Let $\Phi\in C^2(\mathbb{R}^n)$ with $\partial_{x^\nu}\Phi,\ \partial^2_{x^\mu x^\nu}\Phi\in C_{b.Lip}(\mathbb{R}^n)$ for $\mu,\nu=1,\cdots,n$. Let α^ν, $\beta^{\nu j}$ and $\eta^{\nu ij}$, $\nu=1,\cdots,n$, $i,j=1,\cdots,d$ be bounded processes in $M_G^2(0,T)$. Then for each $t\geq 0$ we have, in $L_G^2(\Omega_t)$, that*

$$\Phi(X_t)-\Phi(X_s)=\int_s^t\partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j}dB_u^j+\int_s^t\partial_{x^\nu}\Phi(X_u)\alpha_u^\nu du \tag{3.6.3}$$
$$+\int_s^t[\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij}+\tfrac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j}]d\langle B\rangle_u^{ij}.$$

Proof We first consider the case of α, η and β being step processes of the form

$$\eta_t(\omega)=\sum_{k=0}^{N-1}\xi_k(\omega)\mathbf{1}_{[t_k,t_{k+1})}(t).$$

From Lemma 3.6.1, it is clear that (3.6.3) holds true. Now let

$$X_t^{\nu,N}=X_0^\nu+\int_0^t\alpha_s^{\nu,N}ds+\int_0^t\eta_s^{\nu ij,N}d\langle B\rangle_s^{ij}+\int_0^t\beta_s^{\nu j,N}dB_s^j,$$

where α^N, η^N and β^N are uniformly bounded step processes that converge to α, η and β in $M_G^2(0,T)$ as $N\to\infty$, respectively. From Lemma 3.6.1,

$$\Phi(X_t^N) - \Phi(X_s^N) = \int_s^t \partial_{x^\nu}\Phi(X_u^N)\beta_u^{\nu j,N} dB_u^j + \int_s^t \partial_{x^\nu}\Phi(X_u^N)\alpha_u^{\nu,N} du \qquad (3.6.4)$$
$$+ \int_s^t [\partial_{x^\nu}\Phi(X_u^N)\eta_u^{\nu ij,N} + \tfrac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u^N)\beta_u^{\mu i,N}\beta_u^{\nu j,N}] d\langle B\rangle_u^{ij}.$$

Since

$$\hat{\mathbb{E}}[|X_t^{\nu,N} - X_t^\nu|^2]$$
$$\le C\hat{\mathbb{E}}\left[\int_0^T [(\alpha_s^{\nu,N} - \alpha_s^\nu)^2 + |\eta_s^{\nu,N} - \eta_s^\nu|^2 + |\beta_s^{\nu,N} - \beta_s^\nu|^2] ds\right],$$

where C is a constant independent of N. It follows that, in the space $M_G^2(0,T)$,

$$\partial_{x^\nu}\Phi(X_\cdot^N)\eta_\cdot^{\nu ij,N} \to \partial_{x^\nu}\Phi(X_\cdot)\eta_\cdot^{\nu ij},$$
$$\partial^2_{x^\mu x^\nu}\Phi(X_\cdot^N)\beta_\cdot^{\mu i,N}\beta_\cdot^{\nu j,N} \to \partial^2_{x^\mu x^\nu}\Phi(X_\cdot)\beta_\cdot^{\mu i}\beta_\cdot^{\nu j},$$
$$\partial_{x^\nu}\Phi(X_\cdot^N)\alpha_\cdot^{\nu,N} \to \partial_{x^\nu}\Phi(X_\cdot)\alpha_\cdot^{\nu},$$
$$\partial_{x^\nu}\Phi(X_\cdot^N)\beta_\cdot^{\nu j,N} \to \partial_{x^\nu}\Phi(X_\cdot)\beta_\cdot^{\nu j}.$$

Therefore, passing to the limit as $N \to \infty$ in both sides of (3.6.4), we get (3.6.3). □

In order to derive Itô's formula for a general function Φ, we first establish a useful inequality. For the G-expectation $\hat{\mathbb{E}}$, we have the following representation (see Chap. 6):

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathcal{P}} E_P[X] \quad \text{for } X \in L_G^1(\Omega), \qquad (3.6.5)$$

where $\mathcal{P}$ is a weakly compact family of probability measures on $(\Omega, \mathcal{B}(\Omega))$.

Proposition 3.6.4 *Let $\beta \in M_G^p(0,T)$ with $p \ge 2$ and let $\mathbf{a} \in \mathbb{R}^d$ be fixed. Then we have $\int_0^T \beta_t dB_t^{\mathbf{a}} \in L_G^p(\Omega_T)$ and*

$$\hat{\mathbb{E}}\left[\left|\int_0^T \beta_t dB_t^{\mathbf{a}}\right|^p\right] \le C_p\hat{\mathbb{E}}\left[\left|\int_0^T \beta_t^2 d\langle B^{\mathbf{a}}\rangle_t\right|^{p/2}\right]. \qquad (3.6.6)$$

Proof It suffices to consider the case where β is a step process of the form

$$\beta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)\mathbf{1}_{[t_k,t_{k+1})}(t).$$

For each $\xi \in Lip(\Omega_t)$ with $t \in [0,T]$, we have

$$\hat{\mathbb{E}}\left[\xi\int_t^T \beta_s dB_s^{\mathbf{a}}\right] = 0.$$

From this we can easily get $E_P[\xi \int_t^T \beta_s dB_s^{\mathbf{a}}] = 0$ for each $P \in \mathcal{P}$, which implies that $(\int_0^t \beta_s dB_s^{\mathbf{a}})_{t\in 0,T]}$ is a P-martingale. Similarly we can prove that

$$M_t := \left(\int_0^t \beta_s dB_s^{\mathbf{a}}\right)^2 - \int_0^t \beta_s^2 d\langle B^{\mathbf{a}}\rangle_s, \quad t \in [0, T],$$

is a P-martingale for each $P \in \mathcal{P}$. By the Burkholder-Davis-Gundy inequalities, we have

$$E_P\left[\left|\int_0^T \beta_t dB_t^{\mathbf{a}}\right|^p\right] \le C_p E_P\left[\left|\int_0^T \beta_t^2 d\langle B^{\mathbf{a}}\rangle_t\right|^{p/2}\right] \le C_p \hat{\mathbb{E}}\left[\left|\int_0^T \beta_t^2 d\langle B^{\mathbf{a}}\rangle_t\right|^{p/2}\right],$$

where C_p is a universal constant independent of P. Thus we get (3.6.6). □

We now give the general G-Itô's formula.

Theorem 3.6.5 *Let Φ be a C^2-function on $\mathbb{R}^n$ such that $\partial^2_{x^\mu x^\nu}\Phi$ satisfies polynomial growth condition for $\mu, \nu = 1, \cdots, n$. Let α^ν, $\beta^{\nu j}$ and $\eta^{\nu ij}$, $\nu = 1, \cdots, n$, $i, j = 1, \cdots, d$ be bounded processes in $M_G^2(0, T)$. Then for each $t \ge 0$ we have in $L_G^2(\Omega_t)$*

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j} dB_u^j + \int_s^t \partial_{x^\nu}\Phi(X_u)\alpha_u^\nu du \qquad (3.6.7)$$
$$+ \int_s^t \left[\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij} + \tfrac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j}\right] d\langle B\rangle_u^{ij}.$$

Proof By the assumptions on Φ, we can choose a sequence of functions $\Phi_N \in C_0^2(\mathbb{R}^n)$ such that

$$|\Phi_N(x) - \Phi(x)| + |\partial_{x^\nu}\Phi_N(x) - \partial_{x^\nu}\Phi(x)| + |\partial^2_{x^\mu x^\nu}\Phi_N(x) - \partial^2_{x^\mu x^\nu}\Phi(x)| \le \frac{C_1}{N}(1 + |x|^k),$$

where C_1 and k are positive constants independent of N. Obviously, Φ_N satisfies the conditions in Proposition 3.6.3, therefore,

$$\Phi_N(X_t) - \Phi_N(X_s) = \int_s^t \partial_{x^\nu}\Phi_N(X_u)\beta_u^{\nu j} dB_u^j + \int_s^t \partial_{x^\nu}\Phi_N(X_u)\alpha_u^\nu du \qquad (3.6.8)$$
$$+ \int_s^t \left[\partial_{x^\nu}\Phi_N(X_u)\eta_u^{\nu ij} + \tfrac{1}{2}\partial^2_{x^\mu x^\nu}\Phi_N(X_u)\beta_u^{\mu i}\beta_u^{\nu j}\right] d\langle B\rangle_u^{ij}.$$

For each fixed $T > 0$, by Proposition 3.6.4, there exists a constant C_2 such that

$$\hat{\mathbb{E}}[|X_t|^{2k}] \le C_2 \quad \text{for } t \in [0, T].$$

Thus we can show that $\Phi_N(X_t) \to \Phi(X_t)$ as $N \to \infty$ in $L_G^2(\Omega_t)$ and, in $M_G^2(0, T)$,

$$\begin{aligned}
\partial_{x^\nu}\Phi_N(X_\cdot)\eta_\cdot^{\nu ij} &\to \partial_{x^\nu}\Phi(X_\cdot)\eta_\cdot^{\nu ij}, \\
\partial^2_{x^\mu x^\nu}\Phi_N(X_\cdot)\beta_\cdot^{\mu i}\beta_\cdot^{\nu j} &\to \partial^2_{x^\mu x^\nu}\Phi(X_\cdot)\beta_\cdot^{\mu i}\beta_\cdot^{\nu j}, \\
\partial_{x^\nu}\Phi_N(X_\cdot)\alpha_\cdot^{\nu} &\to \partial_{x^\nu}\Phi(X_\cdot)\alpha_\cdot^{\nu}, \\
\partial_{x^\nu}\Phi_N(X_\cdot)\beta_\cdot^{\nu j} &\to \partial_{x^\nu}\Phi(X_\cdot)\beta_\cdot^{\nu j}.
\end{aligned}$$

We then can pass to limit as $N \to \infty$ in both sides of (3.6.8) to get (3.6.7). □

Corollary 3.6.6 *Let Φ be a polynomial and $\mathbf{a}, \mathbf{a}^\nu \in \mathbb{R}^d$ be fixed for $\nu = 1, \cdots, n$. Then we have*

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)dB_u^{\mathbf{a}^\nu} + \frac{1}{2}\int_s^t \partial^2_{x^\mu x^\nu}\Phi(X_u)d\left\langle B^{\mathbf{a}^\mu}, B^{\mathbf{a}^\nu}\right\rangle_u,$$

where $X_t = (B_t^{\mathbf{a}^1}, \cdots, B_t^{\mathbf{a}^n})^T$. In particular, we have, for $k = 2, 3, \cdots$,

$$(B_t^{\mathbf{a}})^k = k\int_0^t (B_s^{\mathbf{a}})^{k-1}dB_s^{\mathbf{a}} + \frac{k(k-1)}{2}\int_0^t (B_s^{\mathbf{a}})^{k-2}d\langle B^{\mathbf{a}}\rangle_s.$$

If the sublinear expectation $\hat{\mathbb{E}}$ becomes a linear expectation, then the above G-Itô's formula is the classical one.

3.7 Brownian Motion Without Symmetric Condition

In this section, we consider the Brownian motion B without the symmetric condition $\hat{\mathbb{E}}[B_t] = -\hat{\mathbb{E}}[-B_t]$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. The following theorem gives a characterization of the Brownian motion without the symmetric condition.

Theorem 3.7.1 *Let $(B_t)_{t\geq 0}$ be a given $\mathbb{R}^d$–valued Brownian motion on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then, for each fixed $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, the function defined by,*

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + B_t)], \quad (t, x) \in [0, \infty) \times \mathbb{R}^d$$

is the unique viscosity solution of the following parabolic PDE:

$$\partial_t u - G(Du, D^2u) = 0, \quad u|_{t=0} = \varphi, \tag{3.7.1}$$

where

$$G(p, A) = \lim_{\delta\downarrow 0}\hat{\mathbb{E}}[\langle p, B_\delta\rangle + \tfrac{1}{2}\langle AB_\delta, B_\delta\rangle]\delta^{-1} \text{ for } (p, A) \in \mathbb{R}^d \times \mathbb{S}(d). \tag{3.7.2}$$

Proof We first prove that $\lim_{\delta\downarrow 0}\hat{\mathbb{E}}[\langle p, B_\delta\rangle + \frac{1}{2}\langle AB_\delta, B_\delta\rangle]\delta^{-1}$ exists. For each fixed $(p, A) \in \mathbb{R}^d \times \mathbb{S}(d)$, we set

$$f(t) := \hat{\mathbb{E}}[\langle p, B_t\rangle + \tfrac{1}{2}\langle AB_t, B_t\rangle].$$

Since

$$|f(t+h) - f(t)| \le \hat{\mathbb{E}}[(|p| + 2|A||B_t|)|B_{t+h} - B_t| + |A||B_{t+h} - B_t|^2] \to 0, \quad \text{as } h \to 0,$$

we get that $f(t)$ is a continuous function. Observe that

$$\hat{\mathbb{E}}[\langle q, B_t\rangle] = \hat{\mathbb{E}}[\langle q, B_1\rangle]t, \quad \text{for } q \in \mathbb{R}^d.$$

Thus for each $t, s > 0$,

$$|f(t+s) - f(t) - f(s)| \le C\hat{\mathbb{E}}[|B_t|]s,$$

where $C = |A|\hat{\mathbb{E}}[|B_1|]$. By (iii), there exists a constant $\delta_0 > 0$ such that $\hat{\mathbb{E}}[|B_t|^3] \le t$ for $t \le \delta_0$. Thus for each fixed $t > 0$ and $N \in \mathbb{N}$ such that $Nt \le \delta_0$, we have

$$|f(Nt) - Nf(t)| \le \frac{3}{4}C(Nt)^{4/3}.$$

From this and the continuity of f, it is easy to show that $\lim_{t\downarrow 0} f(t)t^{-1}$ exists. Thus we can get $G(p, A)$ for each $(p, A) \in \mathbb{R}^d \times \mathbb{S}(d)$. It is also easy to check that G is a continuous sublinear function monotone in $A \in \mathbb{S}(d)$.

Then we prove that u is Lipschitz in x and $\frac{1}{2}$-Hölder continuous in t. In fact, for each fixed t, $u(t, \cdot) \in C_{b.Lip}(\mathbb{R}^d)$ since

$$\begin{aligned}|\hat{\mathbb{E}}[\varphi(x + B_t)] - \hat{\mathbb{E}}[\varphi(y + B_t)]| &\le \hat{\mathbb{E}}[|\varphi(x + B_t) - \varphi(y + B_t)|]\\ &\le C|x - y|.\end{aligned}$$

For each $\delta \in [0, t]$, since $B_t - B_\delta$ is independent from B_δ,

$$\begin{aligned}u(t, x) &= \hat{\mathbb{E}}[\varphi(x + B_\delta + (B_t - B_\delta)]\\ &= \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(y + (B_t - B_\delta))]_{y=x+B_\delta}].\end{aligned}$$

Hence

$$u(t, x) = \hat{\mathbb{E}}[u(t - \delta, x + B_\delta)]. \tag{3.7.3}$$

Thus

$$\begin{aligned}|u(t,x)-u(t-\delta,x)| &= |\hat{\mathbb{E}}[u(t-\delta,x+B_\delta)-u(t-\delta,x)]|\\ &\le \hat{\mathbb{E}}[|u(t-\delta,x+B_\delta)-u(t-\delta,x)|]\\ &\le \hat{\mathbb{E}}[C|B_\delta|] \le C\sqrt{G(0,I)+1}\sqrt{\delta}.\end{aligned}$$

To prove that u is a viscosity solution of (3.7.1), we fix a pair $(t,x)\in(0,\infty)\times\mathbb{R}^d$ and let $v\in C_b^{2,3}([0,\infty)\times\mathbb{R}^d)$ be such that $v\ge u$ and $v(t,x)=u(t,x)$. From (3.7.3), we have

$$v(t,x)=\hat{\mathbb{E}}[u(t-\delta,x+B_\delta)]\le\hat{\mathbb{E}}[v(t-\delta,x+B_\delta)].$$

Therefore, by Taylor's expansion,

$$\begin{aligned}0 &\le \hat{\mathbb{E}}[v(t-\delta,x+B_\delta)-v(t,x)]\\ &= \hat{\mathbb{E}}[v(t-\delta,x+B_\delta)-v(t,x+B_\delta)+(v(t,x+B_\delta)-v(t,x))]\\ &= \hat{\mathbb{E}}[-\partial_t v(t,x)\delta+\langle Dv(t,x),B_\delta\rangle+\tfrac{1}{2}\langle D^2v(t,x)B_\delta,B_\delta\rangle+I_\delta]\\ &\le -\partial_t v(t,x)\delta+\hat{\mathbb{E}}[\langle Dv(t,x),B_\delta\rangle+\tfrac{1}{2}\langle D^2v(t,x)B_\delta,B_\delta\rangle]+\hat{\mathbb{E}}[I_\delta],\end{aligned}$$

where

$$\begin{aligned}I_\delta &= \int_0^1 -[\partial_t v(t-\beta\delta,x+B_\delta)-\partial_t v(t,x)]\delta d\beta\\ &\quad+\int_0^1\int_0^1\langle(D^2v(t,x+\alpha\beta B_\delta)-D^2v(t,x))B_\delta,B_\delta\rangle\alpha d\beta d\alpha.\end{aligned}$$

By condition (iii) in Definition 3.1.2, we can check that $\lim_{\delta\downarrow 0}\hat{\mathbb{E}}[|I_\delta|]\delta^{-1}=0$, which implies that $\partial_t v(t,x)-G(Dv(t,x),D^2v(t,x))\le 0$. Hence u is a viscosity subsolution of (3.7.1). We can analogously show that u is also a viscosity supersolution. Thus u is a viscosity solution. □

In many situations we are interested in a $2d$-dimensional Brownian motion $(B_t,b_t)_{t\ge0}$ such that $\hat{\mathbb{E}}[B_t]=-\hat{\mathbb{E}}[-B_t]=0$ and $\hat{\mathbb{E}}[|b_t|^2]/t\to 0$, as $t\downarrow 0$. In this case B is in fact a symmetric Brownian motion. Moreover, the process $(b_t)_{t\ge0}$ satisfies the properties in the Proposition 3.5.2. We define $u(t,x,y)=\hat{\mathbb{E}}[\varphi(x+B_t,y+b_t)]$. By Theorem 3.7.1 it follows that u is the solution of the PDE

$$\partial_t u=G(D_y u,D^2_{xx}u),\quad u|_{t=0}=\varphi\in C_{b.Lip}(\mathbb{R}^{2d}),$$

where G is a sublinear function of $(p,A)\in\mathbb{R}^d\times\mathbb{S}(d)$, defined by

$$G(p,A):=\hat{\mathbb{E}}[\langle p,b_1\rangle+\langle AB_1,B_1\rangle].$$

3.8 G-Brownian Motion Under (Not Necessarily Sublinear) Nonlinear Expectation

Let $\widetilde{\mathbb{E}}$ be a nonlinear expectation and $\hat{\mathbb{E}}$ be a sublinear expectation defined on $(\Omega, \mathcal{H})$ such that $\widetilde{\mathbb{E}}$ is dominated by $\hat{\mathbb{E}}$, namely

$$\widetilde{\mathbb{E}}[X] - \widetilde{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y], \quad X, Y \in \mathcal{H}.$$

We can also define a Brownian motion on the nonlinear expectation space $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$. We emphasize that here the nonlinear expectation $\widetilde{\mathbb{E}}$ is not necessarily sublinear.

Definition 3.8.1 A d-dimensional process $(B_t)_{t\geq 0}$ on nonlinear expectation space $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$ is called a Brownian motion if the following properties are satisfied:

(i) $B_0(\omega) = 0$;

(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is identically distributed with B_s and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$;

(iii) $\lim_{t\downarrow 0} t^{-1}\hat{\mathbb{E}}[|B_t|^3] = 0$.

The following theorem gives a characterization of the nonlinear Brownian motion, and provides us with a new generator $\widetilde{G}$ associated with this more general nonlinear Brownian motion.

Theorem 3.8.2 *Let $(B_t, b_t)_{t\geq 0}$ be a given $\mathbb{R}^{2d}$–valued Brownian motion, both on $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$ and $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0$ and $\lim_{t\to 0} \hat{\mathbb{E}}[|b_t|^2]/t = 0$. Assume that $\widetilde{\mathbb{E}}$ is dominated by $\hat{\mathbb{E}}$. Then, for each fixed $\varphi \in C_{b.Lip}(\mathbb{R}^{2d})$, the function*

$$\tilde{u}(t, x, y) := \widetilde{\mathbb{E}}[\varphi(x + B_t, y + b_t)], \quad (t, x, y) \in [0, \infty) \times \mathbb{R}^{2d}$$

is a viscosity solution of the following parabolic PDE:

$$\partial_t \tilde{u} - \widetilde{G}(D_y \tilde{u}, D_x^2 \tilde{u}) = 0, \quad \tilde{u}|_{t=0} = \varphi. \tag{3.8.1}$$

where

$$\widetilde{G}(p, A) = \widetilde{\mathbb{E}}[\langle p, b_1\rangle + \tfrac{1}{2}\langle AB_1, B_1\rangle], \ (p, A) \in \mathbb{R}^d \times \mathbb{S}(d). \tag{3.8.2}$$

Remark 3.8.3 Let

$$G(p, A) := \hat{\mathbb{E}}[\langle p, b_1\rangle + \tfrac{1}{2}\langle AB_1, B_1\rangle], \quad (p, A) \in \mathbb{R}^d \times \mathbb{S}(d). \tag{3.8.3}$$

Then the function $\widetilde{G}$ is dominated by the sublinear function G in the following sense:

$$\widetilde{G}(p, A) - \widetilde{G}(p', A') \leq G(p - p', A - A'), \ \ (p, A), \ (p', A') \in \mathbb{R}^d \times \mathbb{S}(d). \tag{3.8.4}$$

Conversely, once we have two functions G and $\widetilde{G}$ defined on $(\mathbb{R}^d, \mathbb{S}(d))$ such that G is a sublinear function and monotone in $A \in \mathbb{S}(d)$, and that $\widetilde{G}$ is dominated by G, we can construct a Brownian motion $(B_t, b_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that a nonlinear expectation $\widetilde{\mathbb{E}}$ is well-defined on $(\Omega, \mathcal{H})$ and is dominated by $\hat{\mathbb{E}}$. Moreover, under $\widetilde{\mathbb{E}}$, $(B_t, b_t)_{t\geq 0}$ is also a $\mathbb{R}^{2d}$-valued Brownian motion in the sense of Definition 3.8.1 and relations (3.8.2) and (3.8.3) are satisfied.

Proof of Theorem 3.8.2 We set

$$f(t) = f_{A,t}(t) := \widetilde{\mathbb{E}}[\langle p, b_t\rangle + \tfrac{1}{2}\langle AB_t, B_t\rangle],\ t \geq 0.$$

Since

$$\begin{aligned}|f(t+h) - f(t)| \leq &\hat{\mathbb{E}}[|p||b_{t+h} - b_t| + (|p| + 2|A||B_t|)|B_{t+h} - B_t| \\ &+ |A||B_{t+h} - B_t|^2] \to 0, \quad \text{as } h \to 0,\end{aligned}$$

we get that $f(t)$ is a continuous function. Since $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0$, it follows from Proposition 3.8 that $\widetilde{\mathbb{E}}[X + \langle p, B_t\rangle] = \widetilde{\mathbb{E}}[X]$ for each $X \in \mathcal{H}$ and $p \in \mathbb{R}^d$. Thus

$$\begin{aligned}f(t+h) &= \widetilde{\mathbb{E}}[\langle p, b_{t+h} - b_t\rangle + \langle p, b_t\rangle \\ &\quad + \tfrac{1}{2}\langle AB_{t+h} - B_t, B_{t+h} - B_t\rangle + \tfrac{1}{2}\langle AB_t, B_t\rangle] \\ &= \widetilde{\mathbb{E}}[\langle p, b_h\rangle + \tfrac{1}{2}\langle AB_h, B_h\rangle] + \widetilde{\mathbb{E}}[\langle p, b_t\rangle + \tfrac{1}{2}\langle AB_t, B_t\rangle] \\ &= f(t) + f(h).\end{aligned}$$

It then follows that $f(t) = f(1)t = \widetilde{G}(A, p)t$. We now prove that the function $\tilde{u}$ is Lipschitz in x and uniformly continuous in t. Indeed, for each fixed t, $\tilde{u}(t, \cdot) \in C_{b.Lip}(\mathbb{R}^d)$ since

$$\begin{aligned}&|\widetilde{\mathbb{E}}[\varphi(x + B_t, y + b_t)] - \widetilde{\mathbb{E}}[\varphi(x' + B_t, y' + b_t)]| \\ &\leq \hat{\mathbb{E}}[|\varphi(x + B_t, y + b_t) - \varphi(x' + B_t, y' + b_t)|] \leq C(|x - x'| + |y - y'|).\end{aligned}$$

For each $\delta \in [0, t]$, since $(B_t - B_\delta, b_t - b_\delta)$ is independent from (B_δ, b_δ),

$$\begin{aligned}\tilde{u}(t, x, y) &= \widetilde{\mathbb{E}}[\varphi(x + B_\delta + (B_t - B_\delta), y + b_\delta + (b_t - b_\delta)] \\ &= \widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[\varphi(\bar{x} + (B_t - B_\delta), \bar{y} + (b_t - b_\delta))]_{\bar{x}=x+B_\delta, \bar{y}=y+b_\delta}].\end{aligned}$$

Hence

$$\tilde{u}(t, x, y) = \widetilde{\mathbb{E}}[\tilde{u}(t - \delta, x + B_\delta, y + b_\delta)]. \tag{3.8.5}$$

Thus

$$
\begin{aligned}
|\tilde{u}(t,x,y)-\tilde{u}(t-\delta,x,y)| &= |\tilde{\mathbb{E}}[\tilde{u}(t-\delta,x+B_\delta,y+b_\delta)-\tilde{u}(t-\delta,x,y)]| \\
&\le \hat{\mathbb{E}}[|\tilde{u}(t-\delta,x+B_\delta,y+b_\delta)-\tilde{u}(t-\delta,x,y)|] \\
&\le C\hat{\mathbb{E}}[|B_\delta|+|b_\delta|].
\end{aligned}
$$

It follows from condition (iii) in Definition 3.8.1 that $\tilde{u}(t,x,y)$ is continuous in t uniformly in $(t,x,y)\in[0,\infty)\times\mathbb{R}^{2d}$.

To prove that $\tilde{u}$ is a viscosity solution of (3.8.1), we fix $(t,x,y)\in(0,\infty)\times\mathbb{R}^{2d}$ and let $v\in C_b^{2,3}([0,\infty)\times\mathbb{R}^{2d})$ be such that $v\ge u$ and $v(t,x,y)=\tilde{u}(t,x,y)$. From (3.8.5), we have

$$
v(t,x,y)=\widetilde{\mathbb{E}}[\tilde{u}(t-\delta,x+B_\delta,y+b_\delta)]\le\widetilde{\mathbb{E}}[v(t-\delta,x+B_\delta,y+b_\delta)].
$$

Therefore, by Taylor's expansion,

$$
\begin{aligned}
0 &\le \widetilde{\mathbb{E}}[v(t-\delta,x+B_\delta,y+b_\delta)-v(t,x,y)] \\
&= \widetilde{\mathbb{E}}[v(t-\delta,x+B_\delta,y+b_\delta)-v(t,x+B_\delta,y+b_\delta)+v(t,x+B_\delta,y+b_\delta)-v(t,x,y)] \\
&= \widetilde{\mathbb{E}}[-\partial_t v(t,x,y)\delta+\langle D_y v(t,x,y),b_\delta\rangle+\langle\partial_x v(t,x,y),B_\delta\rangle+\tfrac{1}{2}\langle D_{xx}^2 v(t,x,y)B_\delta,B_\delta\rangle+I_\delta] \\
&\le -\partial_t v(t,x,y)\delta+\widetilde{\mathbb{E}}[\langle D_y v(t,x,y),b_\delta\rangle+\tfrac{1}{2}\langle D_{xx}^2 v(t,x,y)B_\delta,B_\delta\rangle]+\hat{\mathbb{E}}[I_\delta].
\end{aligned}
$$

Here

$$
\begin{aligned}
I_\delta &= \int_0^1 -[\partial_t v(t-\delta\gamma,x+B_\delta,y+b_\delta)-\partial_t v(t,x,y)]\delta d\gamma \\
&+\int_0^1 \langle\partial_y v(t,x+\gamma B_\delta,y+\gamma b_\delta)-\partial_y v(t,x,y),b_\delta\rangle d\gamma \\
&+\int_0^1 \langle\partial_x v(t,x,y+\gamma b_\delta)-\partial_x v(t,x,y),B_\delta\rangle d\gamma \\
&+\int_0^1\int_0^1 \langle(D_{xx}^2 v(t,x+\alpha\gamma B_\delta,y+\gamma b_\delta)-D_{xx}^2 v(t,x,y))B_\delta,B_\delta\rangle\gamma d\gamma d\alpha.
\end{aligned}
$$

We use assumption (iii) to check that $\lim_{\delta\downarrow 0}\hat{\mathbb{E}}[|I_\delta|]\delta^{-1}=0$. This implies that $\partial_t v(t,x)-\widetilde{G}(Dv(t,x),D^2v(t,x))\le 0$, hence u is a viscosity subsolution of (3.8.1). We can analogously prove that $\tilde{u}$ is a viscosity supersolution. Thus $\tilde{u}$ is a viscosity solution. □

3.9 Construction of Brownian Motions on a Nonlinear Expectation Space

Let $G : \mathbb{R}^d \times \mathbb{S}(d) \mapsto \mathbb{R}$ be a given continuous sublinear function monotone in $A \in \mathbb{S}(d)$. By Theorem 1.2.1 in Chap. 1, there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{R}^d \times \mathbb{S}_+(d)$ such that

$$G(p, A) = \sup_{(q,B)\in\Sigma} [\tfrac{1}{2}\mathrm{tr}[AB] + \langle p, q\rangle] \quad \text{for } (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

By the results in Chap. 2, we know that there exists a pair of d-dimensional random vectors (X, Y) which is G-distributed.

Let $\widetilde{G}(\cdot) : \mathbb{R}^d \times \mathbb{S}(d) \mapsto \mathbb{R}$ be a given function dominated by G in the sense of (3.8.4). The construction of a $\mathbb{R}^{2d}$-dimensional Brownian motion $(B_t, b_t)_{t\geq 0}$ under a nonlinear expectation $\widetilde{\mathbb{E}}$, dominated by a sublinear expectation $\hat{\mathbb{E}}$ is based on a similar approach introduced in Sect. 3.2. In fact, we will see that by our construction $(B_t, b_t)_{t\geq 0}$ is also a Brownian motion under the sublinear expectation $\hat{\mathbb{E}}$.

We denote by $\Omega = C_0^{2d}(\mathbb{R}^+)$ the space of all $\mathbb{R}^{2d}$–valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$. For each fixed $T \in [0, \infty)$, we set $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$. We will consider the canonical process $(B_t, b_t)(\omega) = \omega_t$, $t \in [0, \infty)$, for $\omega \in \Omega$. We also follow Sect. 3.2 to introduce the spaces of random variables $Lip(\Omega_T)$ and $Lip(\Omega)$ so that to define the expectations $\hat{\mathbb{E}}$ and $\widetilde{\mathbb{E}}$ on $(\Omega, Lip(\Omega))$.

For this purpose we first construct a sequence of $2d$-dimensional random vectors $(X_i, \eta_i)_{i=1}^{\infty}$ on a sublinear expectation space $(\overline{\Omega}, \overline{\mathcal{H}}, \overline{\mathbb{E}})$ such that (X_i, η_i) is G-distributed and (X_{i+1}, η_{i+1}) is independent from $((X_1, \eta_1), \cdots, (X_i, \eta_i))$ for each $i = 1, 2, \cdots$. By the definition of G-distribution, the function

$$u(t, x, y) := \overline{\mathbb{E}}[\varphi(x + \sqrt{t}X_1, y + t\eta_1)], \quad t \geq 0, \quad x, y \in \mathbb{R}^d$$

is the viscosity solution of the following parabolic PDE, which is the same as Eq. (2.2.6) in Chap. 2:

$$\partial_t u - G(D_y u, D_{xx}^2 u) = 0, \quad u|_{t=0} = \varphi \in C_{l.Lip}(\mathbb{R}^{2d}).$$

We also consider another PDE (see Theorem C.3.5 of Appendix C for the existence and uniqueness):

$$\partial_t \tilde{u} - \widetilde{G}(D_y \tilde{u}, D_{xx}^2 \tilde{u}) = 0, \quad \tilde{u}|_{t=0} = \varphi \in C_{l.Lip}(\mathbb{R}^{2d}),$$

and denote $\widetilde{P}_t[\varphi](x, y) = \tilde{u}(t, x, y)$. Then it follows from Theorem C.3.5 in Appendix C, that, for each $\varphi, \psi \in C_{l.Lip}(\mathbb{R}^{2d})$,

$$\widetilde{P}_t[\varphi](x, y) - \widetilde{P}_t[\psi](x, y) \leq \overline{\mathbb{E}}[(\varphi - \psi)(x + \sqrt{t}X_1, y + t\eta_1)].$$

We now introduce a sublinear expectation $\hat{\mathbb{E}}$ and a nonlinear expectation $\widetilde{\mathbb{E}}$ both defined on $Lip(\Omega)$ via the following procedure: for each $X \in Lip(\Omega)$ with

$$X = \varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}})$$

for $\varphi \in C_{l.Lip}(\mathbb{R}^{2d\times n})$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$, we define

$$\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}})]$$

$$:= \overline{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}X_1, (t_1 - t_0)\eta_1, \cdots, \sqrt{t_n - t_{n-1}}X_n, (t_n - t_{n-1})\eta_n)].$$

Then we define

$$\widetilde{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}})] = \varphi_n,$$

where φ_n is obtained iteratively as follows:

$$\begin{aligned}
\varphi_1(x_1, y_1, \cdots, x_{n-1}, y_{n-1}) &= \widetilde{P}_{t_n - t_{n-1}}[\varphi(x_1, y_1, \cdots, x_{n-1}, y_{n-1}, \cdot)](0, 0),\\
&\ \ \vdots\\
\varphi_{n-1}(x_1, y_1) &= \widetilde{P}_{t_2 - t_1}[\varphi_{n-2}(x_1, y_1, \cdot)](0, 0),\\
\varphi_n &= \widetilde{P}_{t_1}[\varphi_{n-1}(\cdot)](0, 0).
\end{aligned}$$

The related conditional expectation of $X = \varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}})$ under Ω_{t_j} is defined by

$$\begin{aligned}
\hat{\mathbb{E}}[X|\Omega_{t_j}] &= \hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}}) \,|\Omega_{t_j}] \\
&:= \psi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_j} - B_{t_{j-1}}, b_{t_j} - b_{t_{j-1}}),
\end{aligned} \tag{3.9.1}$$

where

$$\psi(x_1, \cdots, x_j) = \overline{\mathbb{E}}[\varphi(x_1, \cdots, x_j, \sqrt{t_{j+1} - t_j}X_{j+1}, (t_1 - t_0)\eta_{j+1}, \cdots, \sqrt{t_n - t_{n-1}}X_n, (t_1 - t_0)\eta_n)].$$

Similarly,

$$\widetilde{\mathbb{E}}[X|\Omega_{t_j}] = \varphi_{n-j}(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_j} - B_{t_{j-1}}, b_{t_j} - b_{t_{j-1}}).$$

It is easy to check that $\hat{\mathbb{E}}[\cdot]$ (resp., $\widetilde{\mathbb{E}}$) consistently defines a sublinear (resp. nonlinear) expectation on $(\Omega, Lip(\Omega))$. Moreover $(B_t, b_t)_{t\geq 0}$ is a Brownian motion under both $\hat{\mathbb{E}}$ and $\widetilde{\mathbb{E}}$.

Proposition 3.9.1 *Let us list the properties of* $\widetilde{\mathbb{E}}[\cdot|\Omega_t]$ *that hold for each* $X, Y \in Lip(\Omega)$*:*

(i) *If* $X \geq Y$*, then* $\widetilde{\mathbb{E}}[X|\Omega_t] \geq \widetilde{\mathbb{E}}[Y|\Omega_t]$.
(ii) $\widetilde{\mathbb{E}}[X+\eta|\Omega_t] = \widetilde{\mathbb{E}}[X|\Omega_t] + \eta$, *for each* $t \geq 0$ *and* $\eta \in Lip(\Omega_t)$.
(iii) $\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X-Y|\Omega_t]$.
(iv) $\widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[X|\Omega_t]|\Omega_s] = \widetilde{\mathbb{E}}[X|\Omega_{t\wedge s}]$, *in particular,* $\widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[X|\Omega_t]] = \widetilde{\mathbb{E}}[X]$.
(v) *For each* $X \in Lip(\Omega^t)$, $\widetilde{\mathbb{E}}[X|\Omega_t] = \widetilde{\mathbb{E}}[X]$, *where* $Lip(\Omega^t)$ *is the linear space of random variables of the form*

$$\varphi(B_{t_2}-B_{t_1}, b_{t_2}-b_{t_1}, \cdots, B_{t_{n+1}}-B_{t_n}, b_{t_{n+1}}-b_{t_n}),$$
$$n = 1, 2, \cdots, \ \varphi \in C_{l.Lip}(\mathbb{R}^{d\times n}), \ t_1, \cdots, t_n, t_{n+1} \in [t, \infty).$$

Since $\hat{\mathbb{E}}$ can be considered as a special nonlinear expectation of $\widetilde{\mathbb{E}}$ which is dominated by itself, it follows that $\hat{\mathbb{E}}[\cdot|\Omega_t]$ also satisfies the above properties (i)–(v).

Proposition 3.9.2 *The conditional sublinear expectation* $\hat{\mathbb{E}}\,[\cdot|\Omega_t]$ *satisfies (i)–(v). Moreover* $\hat{\mathbb{E}}[\cdot|\Omega_t]$ *itself is sublinear, i.e.,*
(vi) $\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X-Y|\Omega_t]$, .
(vii) $\hat{\mathbb{E}}[\eta X|\Omega_t] = \eta^+\hat{\mathbb{E}}[X|\Omega_t] + \eta^-\hat{\mathbb{E}}[-X|\Omega_t]$ *for each* $\eta \in Lip(\Omega_t)$.

We now consider the completion of sublinear expectation space $(\Omega, Lip(\Omega), \hat{\mathbb{E}})$. Denote by $L_G^p(\Omega)$, $p \geq 1$, the completion of $Lip(\Omega)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. Similarly, we can define $L_G^p(\Omega_T)$, $L_G^p(\Omega_T^t)$ and $L_G^p(\Omega^t)$. It is clear that for each $0 \leq t \leq T < \infty$, $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$.

According to Sect. 1.4 in Chap. 1, the expectation $\hat{\mathbb{E}}[\cdot]$ can be continuously extended to $(\Omega, L_G^1(\Omega))$. Moreover, since the nonlinear expectation $\widetilde{\mathbb{E}}$ is dominated by $\hat{\mathbb{E}}$, it can also be continuously extended to $(\Omega, L_G^1(\Omega))$. $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$ is a sublinear expectation space while $(\Omega, L_G^1(\Omega), \widetilde{\mathbb{E}})$ is a nonlinear expectation space. We refer to Definition 1.4.4 in Chap. 1.

The next is to look for the extension of conditional expectation. For each fixed $t \leq T$, the conditional expectation $\widetilde{\mathbb{E}}[\cdot|\Omega_t] : Lip(\Omega_T) \mapsto Lip(\Omega_t)$ is a continuous mapping under $\|\cdot\|$. Indeed, we have

$$\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X-Y|\Omega_t] \leq \hat{\mathbb{E}}[|X-Y||\Omega_t],$$

then

$$|\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t]| \leq \hat{\mathbb{E}}[|X-Y||\Omega_t].$$

We thus obtain

$$\left\|\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t]\right\| \leq \|X-Y\|.$$

It follows that $\widetilde{\mathbb{E}}[\cdot|\Omega_t]$ can also be extended as a continuous mapping

$$\widetilde{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega_T) \mapsto L_G^1(\Omega_t).$$

If the parameter T is not fixed, then we can obtain $\widetilde{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega) \mapsto L_G^1(\Omega_t)$.

Remark 3.9.3 Propositions 3.9.1 and 3.9.2 also hold for $X, Y \in L_G^1(\Omega)$. However, in (iv), $\eta \in L_G^1(\Omega_t)$ should be bounded, since $X, Y \in L_G^1(\Omega)$ does not imply that $X \cdot Y \in L_G^1(\Omega)$.

In particular, we have the following independence:

$$\widetilde{\mathbb{E}}[X|\Omega_t] = \widetilde{\mathbb{E}}[X], \quad \forall X \in L_G^1(\Omega^t).$$

We give the following definition similar to the classical one:

Definition 3.9.4 An n-dimensional random vector $Y \in (L_G^1(\Omega))^n$ is said to be independent from Ω_t for some given t if for each $\varphi \in C_{b.Lip}(\mathbb{R}^n)$ we have

$$\widetilde{\mathbb{E}}[\varphi(Y)|\Omega_t] = \widetilde{\mathbb{E}}[\varphi(Y)].$$

3.10 Exercises

Exercise 3.10.1 Let $(B_t)_{t\geq 0}$ be a 1-dimensional G-Brownian motion, such that its value at $t = 1$ is $B_1 \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$. Prove that for each $m \in \mathbb{N}$,

$$\hat{\mathbb{E}}[|B_t|^m] = \begin{cases} 2(m-1)!!\overline{\sigma}^m t^{\frac{m}{2}}/\sqrt{2\pi}, & \text{if } m \text{ is odd}, \\ (m-1)!!\overline{\sigma}^m t^{\frac{m}{2}}, & \text{if } m \text{ is even}. \end{cases}$$

Exercise 3.10.2 Show that if $X \in Lip(\Omega_T)$ and $\hat{\mathbb{E}}[X] = -\hat{\mathbb{E}}[-X]$, then $\hat{\mathbb{E}}[X] = E_P[X]$, where P is a Wiener measure on Ω.

Exercise 3.10.3 For each $s, t \geq 0$, we set $B_t^s := B_{t+s} - B_s$. Let $\eta = (\eta_{ij})_{i,j=1}^d \in L_G^1(\Omega_s; \mathbb{S}(d))$. Prove that

$$\hat{\mathbb{E}}[\langle \eta B_t^s, B_t^s \rangle | \Omega_s] = 2G(\eta)t.$$

Exercise 3.10.4 Suppose that $X \in L_G^p(\Omega_T)$ for $p \geq 1$. Prove that there exists a sequence of bounded random variables $X_n \in Lip(\Omega_T)$, $n = 1, \cdots$, such that

$$\lim_{n\to\infty} \hat{\mathbb{E}}[|X - X_n|^p] = 0.$$

Exercise 3.10.5 Prove that for each $X \in Lip(\Omega_T)$, $\sup_{0\leq t\leq T} \hat{\mathbb{E}}_t[X] \in L_G^1(\Omega_T)$.

Exercise 3.10.6 Prove that $\varphi(B_t) \in L_G^1(\Omega_t)$ for each $\varphi \in C(\mathbb{R}^d)$ with a polynomial growth.

Exercise 3.10.7 Prove that, for a fixed $\eta \in M_G^2(0,T)$,

$$\underline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T \eta_t^2 dt\right] \leq \hat{\mathbb{E}}\left[(\int_0^T \eta_t dB_t)^2\right] \leq \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T \eta_t^2 dt\right],$$

where $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-B_1^2]$.

Exercise 3.10.8 Let $(B_t)_{t\geq 0}$ be a 1-dimensional G-Brownian motion and φ a bounded and Lipschitz function on $\mathbb{R}$. Show that

$$\lim_{N\to\infty} \hat{\mathbb{E}}\left[\left|\sum_{k=0}^{N-1} \varphi(B_{t_k^N})[(B_{t_{k+1}^N} - B_{t_k^N})^2 - (\langle B\rangle_{t_{k+1}^N} - \langle B\rangle_{t_k^N})]\right|\right] = 0,$$

where $t_k^N = kT/N, k = 0, \ldots, N-1$.

Exercise 3.10.9 Prove that, for a fixed $\eta \in M_G^1(0,T)$,

$$\underline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T |\eta_t| dt\right] \leq \hat{\mathbb{E}}\left[\int_0^T |\eta_t| d\langle B\rangle_t\right] \leq \overline{\sigma}^2\hat{\mathbb{E}}\left[\int_0^T |\eta_t| dt\right],$$

where $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-B_1^2]$.

Exercise 3.10.10 Complete the proof of Proposition 3.5.7.

Exercise 3.10.11 Let B be a 1-dimensional G-Brownian motion and $\widetilde{\mathbb{E}}$ a nonlinear expectation dominated by a G-expectation. Show that for any $\eta \in M_G^2(0,T)$:

(i) $\widetilde{\mathbb{E}}\left[\int_0^T \eta_s dB_s\right] = 0$;
(ii) $\widetilde{\mathbb{E}}\left[\left(\int_0^T \eta_s dB_s\right)^2\right] = \widetilde{\mathbb{E}}\left[\int_0^T |\eta_s|^2 d\langle B\rangle_s\right]$.

Notes and Comments

Bachelier [7] proposed to use the Brownian motion as a model of the fluctuations of stock markets. Independently, Einstein [56] used the Brownian motion to give experimental confirmation of the atomic theory, and Wiener [173] gave a mathematically rigorous construction of the Brownian motion. Here we follow Kolmogorov's idea [103] to construct G-Brownian motions by introducing finite dimensional cylinder function space and the corresponding family of infinite dimensional sublinear distributions, instead of (linear) probability distributions used in [103].

The notions of G-Brownian motions and the related stochastic calculus of Itô's type were firstly introduced by Peng [138] for the 1-dimensional case and then in

(2008) [141] for the multi-dimensional situation. It is very interesting that Denis and Martini [48] studied super-pricing of contingent claims under model uncertainty of volatility. They have introduced a norm in the space of continuous paths $\Omega = C([0, T])$ which corresponds to the L_G^2-norm and developed a stochastic integral. In that paper there are no notions such as nonlinear expectation and the related nonlinear distribution, G-expectation, conditional G-expectation, the related G-normal distribution and independence. On the other hand, by using powerful tools from capacity theory these authors obtained pathwise results for random variables and stochastic processes through the language of "quasi-surely" (see e.g. Dellacherie [42], Dellacherie and Meyer [43], Feyel and de La Pradelle [65]) in place of "almost surely" in classical probability theory.

One of the main motivations to introduce the notion of G-Brownian motions was the necessity to deal with pricing and risk measures under volatility uncertainty in financial markets (see Avellaneda, Lévy and Paras [6] and Lyons [114]). It was well-known that under volatility uncertainty the corresponding uncertain probability measures are singular with respect to each other. This causes a serious problem in the related path analysis to treat, e.g., when dealing with path-dependent derivatives, under a classical probability space. The notion of G-Brownian motions provides a powerful tool to study such a type of problems. Indeed, Biagini, Mancin and Meyer Brandis studied mean-variance hedging under the G-expectation framework in [18]. Fouque, Pun and Wong investigated the asset allocation problem among a risk-free asset and two risky assets with an ambiguous correlation through the theory of G-Brownian motions in [67]. We also remark that Beissner and Riedel [15] studied equilibria under Knightian price uncertainty through sublinear expectation theory, see also [14, 16].

The new Itô's calculus with respect to G-Brownian motion was inspired by Itô's groundbreaking work of [92] on stochastic integration, stochastic differential equations followed by a huge progress in stochastic calculus. We refer to interesting books cited in Chap. 4. Itô's formula given by Theorem 3.6.5 is from [138, 141]. Gao [72] proved a more general Itô's formula for G-Brownian motion. On this occasion an interesting problem appeared: can we establish an Itô's formula under conditions which correspond to the classical one? This problem will be solved in Chap. 8 with quasi surely analysis approach.

Using nonlinear Markovian semigroups known as Nisio's semigroups (see Nisio [119]), Peng [136] studied the processes with Markovian properties under a nonlinear expectation. Denk, Kupper and Nendel studied the relation between Lévy processes under nonlinear expectations, nonlinear semigroups and fully nonlinear PDEs, see [50].