$\mathbf{SIXTEEN}_3$ in Light of Routley Stars

Hitoshi Omori(✉) and Daniel Skurt(✉)

Department of Philosophy I, Ruhr-University Bochum, Bochum, Germany
{Hitoshi.Omori,Daniel.Skurt}@rub.de

Abstract. For one of the most well-known many-valued logics **FDE**, there are several semantics, including the star semantics by Richard Routley and Valerie Routley, the two-valued relational semantics by Michael Dunn and the four-valued semantics by Nuel Belnap. The last semantics inspired Yaroslav Shramko and Heinrich Wansing to introduce the trilattice $\mathbf{SIXTEEN}_3$. In this article, we offer two alternative semantical presentations for $\mathbf{SIXTEEN}_3$, by applying the Routleys' semantics and the Dunn semantics. Based on our new semantics, we discuss related systems with less truth values, as well as the relation to **FDE**-based modal logics.

Keywords: **FDE** · $\mathbf{SIXTEEN}_3$ · Routley star · Dunn semantics

1 Introduction

1.1 Background (I): From Belnap to Shramko-Wansing

Ever since Jan Łukasiewicz and Emil Post started to explore more than two truth values independently in the 1920s, infinitely many kinds of many-valued logics have been introduced. The one that plays the crucial role in this paper is the four-valued logic of Belnap and Dunn, also known as **FDE**.

The four-valued truth tables for **FDE** were known since the 1950s, when Timothy Smiley pointed this out to Nuel Belnap, but the four values did not have an intuitive reading. It was Dunn who explicitly connected these four values to the classical truth values, true and false (see [6]). This then inspired Belnap to write the two influential papers [2,3]. In particular, the four values are now seen as the power set $\mathcal{P}(\{1, 0\})$ of the set of the classical truth-values $\{1, 0\}$, and receive the following intuitive reading:

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$$\{0\} = \text{told false}, \quad \{\} \quad = \text{told neither true nor false},$$
$$\{1\} = \text{told true}, \ \{1, 0\} = \text{told both true and false}.$$

The above reading also inspired another perspective on the four-values, namely the bilattice of the power set of $\{1, 0\}$ (cf. [1,8]). In particular, two orders measure the degree of truth and the amount of information.

In [21–23] Shramko and Wansing then took this idea of Belnap even a step further. By arguing that the *computer* metaphor of Belnap can be transformed into considering a *computer network* communicating with each other about propositions, Shramko and Wansing developed the idea that such computers should be able to handle information that can be, for example, overcomplete and at the same time just true or false. In this way, they introduced **SIXTEEN**$_3$ which takes the power set of $\mathcal{P}(\{1, 0\})$ to generate a "useful sixteen-valued logic" which is meant to represent "how a computer network should think". This is thus a generalization of Belnap's "useful four-valued logic" which is meant to represent "how a computer should think". Moreover, **SIXTEEN**$_3$ is now a trilattice, rather than a bilattice, where an independent degree of falsity can be defined as an additional order.

Due to the interesting motivation, **SIXTEEN**$_3$ has now collected a lot of the attention it deserves. Just to mention some relevant work, Odintsov in [12], added some new algebraic insights and marked an important step on the problem of axiomatization. Heinrich Wansing considers sequent calculi related to **SIXTEEN**$_3$ in [25], and an analytic tableaux calculus is devised by Muskens and Wintein in [10]. Finally, the property of interpolation is studied again by Muskens and Wintein in [11].

1.2 Background (II): Routley and Dunn Semantics for FDE

As it is well-known, the four-valued interpretation of **FDE** is not the only semantics.[1] For the purpose of this paper, we focus on the following two: Routleys' star semantics and Dunn's relational semantics. Let us briefly highlight the key ideas of the two semantics which are both *two*-valued semantics.[2]

Routleys' star semantics, devised by Routley and Routley in [20], is a two-valued world semantics, as in the well-known Kripke semantics, but includes the so-called star operation which is an involutive operation on worlds. This star operation is used to interpret the negation. For conjunction and disjunction, it remains to be completely classical.

Dunn's relational semantics (or Dunn semantics in short) is yet another two-valued semantics which is also free of worlds. The crucial idea is to use a *relation* rather than a *function* in interpreting the language. In particular, formulas may be related to *both* true and false, or *neither* true nor false. As a consequence, truth and falsity conditions are both necessary, though in the case of **FDE**, those conditions remain completely classical.

[1] For a recent overview, see for example [17].

[2] The formal details will be given in the next section, so we are justified to be brief.

Both approaches have virtues of their own. On the one hand, Routleys' semantics is rather successful when applied to relevant logics. On the other hand, Dunn gives wonderful insights by giving an intuitive reading of truth values, as we already observed above through Belnap's semantics. In any case, the important thing here is that there are interesting two-valued semantics for **FDE**.

1.3 Aim

Based on these backgrounds, the motivation for this paper is rather simple: can we also devise two-valued semantics for logics related to $\mathbf{SIXTEEN}_3$? To the best of our knowledge, this seems to be not addressed yet in the literature. Therefore, we aim at marking the first step towards filling that gap.

On a broader scope, reducing the number of truth-values of a given system can be traced back to Suszko (cf. [24]), who believed that any multiplication of truth-values is a "mad idea". We do not wish to conflate our approach of reducing the number of truth-values with Suszko's critique about many-valued logics in general, but rather during the course of this article we will present an alternative strategy to obtain that goal.[3]

The paper is organized as follows. In Sects. 2 and 3 we will briefly recapitulate the basics of **FDE** and $\mathbf{SIXTEEN}_3$. These are followed by Sects. 4 and 5 in which we introduce the new two-valued semantics for $\mathbf{SIXTEEN}_3$. Based on the new semantics, we will reflect upon the implications in Sect. 6. Finally, we conclude the paper in Sect. 7 by summarizing our main observations and discuss some possible topics for further research.

2 Two-Valued Semantics for FDE

Our propositional languages consist of a finite set C of propositional connectives and a countable set Prop of propositional variables which we refer to as $\mathcal{L}_\mathsf{C}$. Furthermore, we denote by Form_C the set of formulas defined as usual in $\mathcal{L}_\mathsf{C}$. In this paper, we always assume that $\{\sim, \wedge, \vee\} \subseteq \mathsf{C}$ and just include the propositional connective(s) not from $\{\sim, \wedge, \vee\}$ in the subscript of $\mathcal{L}_\mathsf{C}$. Moreover, we denote a formula of $\mathcal{L}_\mathsf{C}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}_\mathsf{C}$ by Γ, Δ, Σ, etc.

First, we review Routleys' star semantics.

Definition 1. *A* Routley interpretation *for $\mathcal{L}$ is a structure $\langle W, *, v\rangle$ where $W \neq \emptyset$ is a set of worlds, $* : W \longrightarrow W$ is a function with $w^{**} = w$, and $v : W \times \mathsf{Prop} \longrightarrow \{0, 1\}$. The function v is extended to $I : W \times \mathsf{Form} \longrightarrow \{0, 1\}$ as follows:*

$$I(w, p) = v(w, p), \qquad I(w, A \wedge B) = 1 \textit{ iff } I(w, A) = 1 \textit{ and } I(w, B) = 1,$$
$$I(w, \sim A) = 1 \textit{ iff } I(w^*, A) \neq 1, \quad I(w, A \vee B) = 1 \textit{ iff } I(w, A) = 1 \textit{ or } I(w, B) = 1.$$

[3] For a mechanical procedure to reduce the number of truth values in **FDE** and its expansions, see [16].

Definition 2. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_* A$ *iff for all Routley interpretations* $\langle W, *, v\rangle$ *and for all* $w \in W$*, if* $I(w, B) = 1$ *for all* $B \in \Gamma$ *then* $I(w, A) = 1$*.*

Second, we review Dunn's relational semantics.

Definition 3. *A* Dunn-interpretation *for* $\mathcal{L}$ *is a relation,* r*, between propositional variables and the values 1 and 0, namely* $r \subseteq \mathsf{Prop} \times \{1, 0\}$*. Given an interpretation,* r*, this is extended to a relation between all formulas and truth values by the following clauses:*

$$\begin{array}{ll} {\sim}Ar1 \text{ iff } Ar0, & {\sim}Ar0 \text{ iff } Ar1, \\ A \wedge Br1 \text{ iff } Ar1 \text{ and } Br1, & A \wedge Br0 \text{ iff } Ar0 \text{ or } Br0, \\ A \vee Br1 \text{ iff } Ar1 \text{ or } Br1, & A \vee Br0 \text{ iff } Ar0 \text{ and } Br0. \end{array}$$

Definition 4. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_r A$ *iff for all Dunn-interpretations* r*, if* $Br1$ *for all* $B \in \Gamma$ *then* $Ar1$*.*

Then, the following result is rather well-known.

Fact 5. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_r A$ *iff* $\Gamma \models_* A$*.*

A proof can be found, e.g., in [18, 8.7.17, 8.7.18]. In fact, something stronger can be established by a careful examination of Graham Priest's proof. To this end, we introduce another semantic consequence relation.

Definition 6. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_{*,2} A$ *iff for all Routley interpretations* $\langle W, *, v\rangle$ *such that the number of worlds is 2 and for all* $w \in W$*, if* $I(w, B){=}1$ *for all* $B \in \Gamma$ *then* $I(w, A) = 1$*.*

Then, we obtain the following.

Lemma 1. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_r A$ *iff* $\Gamma \models_{*,2} A$*.*

Proof. For the proof of the left-to-right direction, Priest's construction works perfectly well with the two-world case. For the other direction, Priest's construction already establishes the desired result. □

As an immediate corollary, we obtain the following result, which can be regarded as logical folkore.

Theorem 1. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models_* A$ *iff* $\Gamma \models_{*,2} A$*. That is, two worlds suffice for the extensional fragment.*

Remark 1. In view of the above result, we may conclude that there is a clear understanding of the star in the context of the above language. The star world is simply *the other* world. Of course, this only works with the simple language, not in the language with the intensional conditional. In the latter case, the star operation is elegantly characterized by Restall (cf. [19]).

3 Basics of $\mathbf{SIXTEEN_3}$

3.1 Language

There are several languages discussed in relation to the trilattice $\mathbf{SIXTEEN}_3$. Following the convention specified in the previous section, we will mainly deal with $\mathcal{L}_{\sim_f}$ and $\mathcal{L}_{\sim_f,\wedge_f,\vee_f}$. The latter is referred to as $\mathcal{L}_{tf}$ in the literature, but for the sake of presentation, we will use the above notation with the hope of being more accessible to wider audience.

Note too that we are omitting the subscript t for connectives. We fully understand that this goes very much against the spirit of the trilattice in general, but for the sake of presentation, and ease of comparison between **FDE** and $\mathbf{SIXTEEN}_3$, we keep the basic connectives free of subscripts.

3.2 Semantics

Let **16** be the set of generalized truth values which consists of the following 16 values:

1. $\emptyset = \{\ \}$
2. $\mathbf{N} = \{\{\ \}\}$
3. $\mathbf{F} = \{\{0\}\}$
4. $\mathbf{T} = \{\{1\}\}$
5. $\mathbf{B} = \{\{0,1\}\}$
6. $\mathbf{NF} = \{\{\ \},\{0\}\}$
7. $\mathbf{NT} = \{\{\ \},\{1\}\}$
8. $\mathbf{NB} = \{\{\ \},\{0,1\}\}$
9. $\mathbf{FT} = \{\{0\},\{1\}\}$
10. $\mathbf{FB} = \{\{0\},\{0,1\}\}$
11. $\mathbf{TB} = \{\{1\},\{0,1\}\}$
12. $\mathbf{NFT} = \{\{\ \},\{0\},\{1\}\}$
13. $\mathbf{NFB} = \{\{\ \},\{0\},\{0,1\}\}$
14. $\mathbf{NTB} = \{\{\ \},\{1\},\{0,1\}\}$
15. $\mathbf{FTB} = \{\{0\},\{1\},\{0,1\}\}$
16. $\mathbf{A} = \{\{\ \},\{0\},\{1\},\{0,1\}\}$

Note here that we changed the notation slightly from the original presentation. More specifically, we replaced T and F by 1 and 0. Moreover, the naming strategy for the truth values is very simple. Recall the following representation:

1. **n**=$\{\ \}$, for **n**either true nor false
2. **f**=$\{0\}$, for **f**alse only
3. **t**=$\{1\}$, for **t**rue only
4. **b**=$\{0,1\}$, for **b**oth true and false

Then, except for the value **A**, the inclusion of capital letters **N**, **F**, **T** and **B** corresponds to the fact that **n**, **f**, **t** and **b** are members of the generalized truth value. And, for **A**, it stands for **a**ll values **n**, **f**, **t** and **b** are members of the set.

Now we can define three different orderings on **16**.

Definition 7. *For every* $x, y \in \mathbf{16}$*:*

1. $x \leq_i y$ *iff* $x \subseteq y$*;*
2. $x \leq_t y$ *iff* $x^1 \subseteq y^1$ *and* $y^{-1} \subseteq x^{-1}$*,*
 where $x^1 := \{z \in x : 1 \in z\}$ *and* $x^{-1} := \{z \in x : 1 \notin z\}$*;*
3. $x \leq_f y$ *iff* $x^0 \subseteq y^0$ *and* $y^{-0} \subseteq x^{-0}$*,*
 where $x^0 := \{z \in x : 0 \in z\}$ *and* $x^{-0} := \{z \in x : 0 \notin z\}$*.*

We can then easily see that meets and joins exist in **16** for all three partial orders. Therefore, we use $\sqcap$ and $\sqcup$ with the appropriate subscripts for these operations under the corresponding orders. Then, the algebraic structure of **16** comes out as the trilattice $\mathbf{SIXTEEN}_3 = \langle \mathbf{16}, \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f \rangle$.

We can associate with each of the lattice orders of **SIXTEEN**$_3$ a unary operation which is an involution of order two with respect to this ordering and preserves the other orders. The unary operations $-_t$, $-_f$, and $-_i$ corresponding to the orders $\leq_t$, $\leq_f$ and $\leq_i$, respectively, are defined as follows.

x	$-_t x$	$-_f x$	$-_i x$	x	$-_t x$	$-_f x$	$-_i x$	x	$-_t x$	$-_f x$	$-_i x$	x	$-_t x$	$-_f x$	$-_i x$
$\emptyset$	$\emptyset$	$\emptyset$	**A**	**B**	**F**	**T**	**FTB**	**NB**	**FT**	**FT**	**FT**	**NFB**	**FTB**	**NFT**	**F**
N	**T**	**F**	**NFT**	**NF**	**TB**	**NF**	**NF**	**FB**	**FB**	**NT**	**FB**	**NTB**	**NFT**	**FTB**	**T**
F	**B**	**N**	**NFB**	**NT**	**NT**	**FB**	**NT**	**TB**	**NF**	**TB**	**TB**	**FTB**	**NFB**	**NTB**	**B**
T	**N**	**B**	**NTB**	**FT**	**NB**	**NB**	**NB**	**NFT**	**NTB**	**NFB**	**N**	**A**	**A**	**A**	$\emptyset$

We are now ready to assign generalized truth values of **16** to our language. More specifically, given a **16**-valuation $v : \mathsf{Prop} \to \mathbf{16}$, we extend the valuation to $\mathsf{Form}_{\sim_f, \wedge_f, \vee_f}$ as follows.

Definition 8. *For every* $A, B \in \mathsf{Form}_{\sim_f, \wedge_f, \vee_f}$*:*

1. $v(A \wedge B) = v(A) \sqcap_t v(B)$
2. $v(A \vee B) = v(A) \sqcup_t v(B)$
3. $v(\sim A) = -_t v(A)$
4. $v(A \wedge_f B) = v(A) \sqcap_f v(B)$
5. $v(A \vee_f B) = v(A) \sqcup_f v(B)$
6. $v(\sim_f A) = -_f v(A)$

Based on this, we can finally define the semantic consequence relations.

Definition 9. *For every* $A, B \in \mathsf{Form}_{\sim_f, \wedge_f, \vee_f}$*:*

- $A \models_t B$ *iff for all* **16***-valuations* v*:* $v(A) \leq_t v(B)$*;*
- $A \models_f B$ *iff for all* **16***-valuations* v*:* $v(A) \leq_f v(B)$*.*

Remark 2. We are *not* using the information order at all to interpret our language, but we introduced them above to emphasize that **16** is a trilattice. We will come back to the unary connective interpreted via $-_i$ towards the end of this paper, but only briefly, in the conclusion section. For discussions on the language including informational connectives, see e.g. [14].

3.3 Proof Systems

We now turn to the proof system. Note that we will only offer the proof system for the language $\mathcal{L}_{\sim_f}$, and just remark on the case of full language, namely the language $\mathcal{L}_{\sim_f, \wedge_f, \vee_f}$.

Definition 10. $\vdash$ *is a binary consequence relation on the language* $\mathcal{L}_{\sim_f}$ *satisfying the following axioms and rules.*

$A \wedge B \vdash A$	(a_t1)		
$A \wedge B \vdash B$	(a_t2)	$\dfrac{A \vdash B \quad B \vdash C}{A \vdash C}$	(r_t1)
$A \vdash A \vee B$	(a_t3)		
$B \vdash A \vee B$	(a_t4)	$\dfrac{A \vdash B \quad A \vdash C}{A \vdash B \wedge C}$	(r_t2)
$A \wedge (B \vee C) \vdash (A \wedge B) \vee C$	(a_t5)		
$A \vdash {\sim}{\sim}A$	(a_t6)	$\dfrac{A \vdash C \quad B \vdash C}{A \vee B \vdash C}$	(r_t3)
${\sim}{\sim}A \vdash A$	(a_t7)		
$A \vdash {\sim}_f{\sim}_f A$	(a_t8)	$\dfrac{A \vdash B}{{\sim}B \vdash {\sim}A}$	(r_t4)
${\sim}_f{\sim}_f A \vdash A$	(a_t9)	$\dfrac{A \vdash B}{{\sim}_f A \vdash {\sim}_f B}$	(r_t5)
${\sim}_f{\sim}A \vdash {\sim}{\sim}_f A$	(a_t10)		

Remark 3. Note that the binary consequence relation characterized in terms of the axioms from (a_t1) to (a_t7), as well as the rules from (r_1) to (r_t4) is sound and complete with respect to **FDE** for the language $\mathcal{L}$.

Finally, the following result was established by Shramko and Wansing in [22, Theorems 4.10, 4.13].

Theorem 2 (Shramko & Wansing). *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \vdash B$ iff $A \models_t B$.*

Remark 4. The problem of axiomatizing $\models_t$ for the language $\mathcal{L}_{\sim_f,\wedge_f,\vee_f}$ was left open in [22], but Odintsov in [12] marked the first step by showing that $\models_t$ is axiomatizable and that the consequence relation can be characterized by the intersection of two related consequence relations. Odintsov also introduced an expansion of $\mathcal{L}_{\sim_f,\wedge_f,\vee_f}$ by adding an implication, and presented an axiomatization of $\models_t$ in the expanded language. A definite solution to the original problem was given in [14] by Odintsov and Wansing by making use of algebraic results related to $\mathbf{SIXTEEN}_3$.

4 Alternative Semantics for $\mathbf{SIXTEEN_3}$ (I)

The first alternative semantics will have two star operations. More specifically, we take the star semantics for **FDE**, and add one more star to capture the additional connective ${\sim}_f$. Our strategy here is to prove the soundness and completeness with respect to the proof system given by Shramko and Wansing to establish the equivalence between the original semantics and the two-star semantics.

4.1 Semantics

Definition 11. *A* two-star interpretation *for $\mathcal{L}_{\sim_f}$ is at tuple $\mathcal{M} = \langle W, g, *_1, *_2, v\rangle$ where $W \neq \emptyset$ is a set of worlds, $g \in W$; $*_i : W \longrightarrow W$ is a function with $w^{*_i *_i} = w$ and $w^{*_i *_j} = w^{*_j *_i}$; $v : W \times \mathsf{Prop} \to \{0, 1\}$. The function v is extended to $I : W \times \mathsf{Form} \to \{0, 1\}$ by the following condition:*

$I(w, p) = v(w, p)$,
$I(w, {\sim}A){=}1$ *iff* $I(w^{*_1}, A){\neq}1$, $I(w, A \wedge B){=}1$ *iff* $I(w, A){=}1$ *and* $I(w, B){=}1$,
$I(w, {\sim}_f A){=}1$ *iff* $I(w^{*_2}, A){=}1$, $I(w, A \vee B){=}1$ *iff* $I(w, A){=}1$ *or* $I(w, B){=}1$.

Remark 5. It should be clear, from the definition, that the fragment with only the "truth connectives" will coincide with **FDE**. Note also that the truth condition for ${\sim}_f$ does *not* look like a truth condition for negation. We will reflect upon this connective in Sect. 6.

We then define two kinds of semantic consequence relation.

Definition 12. *Let $\Gamma \cup \{A\}$ be set of sentences in $\mathcal{L}_{\sim_f}$. Then,*

- *$\Gamma \models_{*,\forall} A$ iff for all two-star interpretations $\langle W, g, *_1, *_2, v\rangle$ and for all $w \in W$, $I(w, A) = 1$ if $I(w, B) = 1$ for all $B \in \Gamma$.*
- *$\Gamma \models_{*,g} A$ iff for all two-star interpretations $\langle W, g, *_1, *_2, v\rangle$, $I(g, A) = 1$ if $I(g, B) = 1$ for all $B \in \Gamma$.*

Remark 6. As we will establish below, these two consequence relations are equivalent as in some (not all!) modal logics (recall Kripke's seminal paper and the more recent text books). However, it will be useful to have both for our purposes.

4.2 Equivalence of Three Semantic Consequence Relations

We will now establish the equivalence of $\models_t$, $\models_{*,\forall}$ and $\models_{*,g}$ via the proof system. More specifically, in view of Theorem 2 of Shramko and Wansing, we prove the following three statements: for all $A, B \in \mathsf{Form}_{\sim_f}$,

$$\text{if } A \vdash B \text{ then } A \models_{*,\forall} B, \text{ if } A \models_{*,\forall} B \text{ then } A \models_{*,g} B, \text{ if } A \models_{*,g} B \text{ then } A \vdash B.$$

Note here that the second item is obvious. Therefore, we prove the first and the third item. The first item, which is soundness, is quite straightforward.

Proposition 1. *For all $A, B \in \mathsf{Form}_{\sim_f}$, if $A \vdash B$ then $A \models_{*,\forall} B$.*

Proof. We only note that we need $\models_{*,\forall}$, instead of $\models_{*,g}$, to establish the soundness, especially for the rules (r_t4) and (r_t5). □

For the purpose of establishing the third item, we construct a suitable canonical model. To this end, we introduce some standard notions.

Definition 13. *Let Γ be a set of sentences. Then, Γ is*

- *a* theory *iff Γ is closed under $\vdash$ and $\wedge$, i.e., for all A, B, if $A \in \Gamma$ and $A \vdash B$ then $B \in \Gamma$, and if $A \in \Gamma$ and $B \in \Gamma$, then $A \wedge B \in \Gamma$;*
- prime *iff for all A, B, if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$.*

The following fact is well known, due to Lindenbaum.

Lemma 2 (Lindenbaum). *For all A, B, if $A \nvdash B$ then there is a prime theory Γ such that $A \in \Gamma$ and $B \notin \Gamma$.*

We will also make use of the following lemma which is already established by Shramko and Wansing in [22, Lemma 4.11].

Lemma 3 (Shramko & Wansing). *Let Γ be a theory, and let Γ^* be defined as follows:*

$$\Gamma^* := \{A : \sim_f A \in \Gamma\}$$

Then Γ^* is a theory, $\sim_f A \in \Gamma^*$ iff $A \in \Gamma$, and Γ^* is prime iff Γ is prime.

We can then prove completeness as well.

Theorem 3. *For all $A, B \in \mathsf{Form}_{\sim_f}$, if $A \models_{*,g} B$ then $A \vdash B$.*

Proof. The details can be found in Appendix A. □

As a corollary, we obtain the following desired result:

Corollary 1. *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_t B$ iff $A \models_{*,g} B$ iff $A \models_{*,\forall} B$.*

We will now turn to two observations related to this result.

4.3 Two Basic Observations

First, we observe that we only need four worlds for two-star interpretations to characterize the syntactic consequence relation $\vdash$. To this end, we introduce one more semantic consequence relation.

Definition 14. *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_{*,g,4} B$ iff for all two-star interpretations $\langle W, g, *_1, *_2, v\rangle$ such that the number of worlds is 4, $I(g, B) = 1$ if $I(g, A) = 1$.*

Then, we obtain in analogy to Theorem 1 the following result:

Proposition 2. *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_{*,g} B$ iff $A \models_{*,g,4} B$.*

Proof. The left-to-right direction is obvious. For the other direction, it suffices to prove that $A \vdash B$ if $A \models_{*,g,4} B$ in view of Proposition 1. But this is already established by the proof for Theorem 3. □

Remark 7. We have a relatively clear formal understanding of star operations. However, as in the case for **FDE**, we do not know what they *mean*. Only that each star corresponds to a different "mate" relation, cf. [18, p. 151].

The second observation, which relies on the first observation, is that $\models_t$ is equivalent to yet another semantic consequence relation defined in terms of preservation of designated values. More precisely, we introduce the following consequence relation.

Definition 15. *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_{16} B$ iff for all* **16***-valuations v: $v(B) \in \mathcal{D}$ if $v(A) \in \mathcal{D}$, where $\mathcal{D} := \{x \in \mathbf{16} : \mathbf{T} \in x\}$.*

Then, by unpacking the definition of $\models_{*,g,4}$, we obtain the following result:

Proposition 3. *For all* $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_t B$ *iff* $A \models_{16} B$.

Remark 8. The reason of introducing $\models_{*,g}$ is to establish this connection to the 16-valued semantic consequence relation defined via designated values.

Note also that the result of the proposition above was already discussed in Lemma 4.3 in [22], for the language $\mathcal{L}$. In Lemma 4.9 of the same paper an additional restriction for the consequence relation is discussed for the language $\mathcal{L}_{\sim_f,\wedge_f,\vee_f}$ In the language $\mathcal{L}_{\sim_f}$, however, we do not need such additional restriction.

5 Alternative Semantics for SIXTEEN$_3$ (II)

The second alternative semantics will have only one star operation, but will be based on four-valued worlds, in analogy to the relational semantics of **FDE**. Therefore, the new semantics presented in this section can be seen as a hybrid of Routleys' semantics and Dunn semantics. The equivalence of the semantics will be established through the semantics given in the previous section.

5.1 Semantics

Definition 16. *A* one-star interpretation *for* $\mathcal{L}_{\sim_f}$ *is a tuple* $\mathcal{M} = \langle W, g, *, r\rangle$ *where* W *is a non-empty set of worlds,* $g \in W$*;* $* : W \longrightarrow W$ *is a function with* $w^{**} = w$*; and* $r_w \subseteq \mathsf{Prop} \times \{0, 1\}$ *for all* $w \in W$*. Given an interpretation,* r_w*, this is extended to a relation between all formulas and truth values by the following clauses:*

$$
\begin{array}{ll}
{\sim}Ar_w1 \textit{ iff } Ar_{w^*}0, & {\sim}Ar_w0 \textit{ iff } Ar_{w^*}1,\\
A \wedge Br_w1 \textit{ iff } Ar_w1 \textit{ and } Br_w1, & A \wedge Br_w0 \textit{ iff } Ar_w0 \textit{ or } Br_w0,\\
A \vee Br_w1 \textit{ iff } Ar_w1 \textit{ or } Br_w1, & A \vee Br_w0 \textit{ iff } Ar_w0 \textit{ and } Br_w0,\\
{\sim_f}Ar_w1 \textit{ iff } Ar_{w^*}1, & {\sim_f}Ar_w0 \textit{ iff } Ar_{w^*}0.
\end{array}
$$

Remark 9. As one can see from the above definition, the one-star interpretation is a hybrid of Routleys' semantics, for the use of the star operation, and Dunn semantics, for the use of the relation instead of the function.

Definition 17. *For all* $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_r B$ *iff for all one-star interpretations* $\mathcal{M}$, Br_g1 *if* Ar_g1.

5.2 Equivalence of Two Semantics

Proposition 4. *For all* $A, B \in \mathsf{Form}_{\sim_f}$*, if* $A \models_{*,g} B$ *then* $A \models_r B$.

Proof. The details are spelled out in Appendix B. □

Proposition 5. *For all* $A, B \in \mathsf{Form}_{\sim_f}$*, if* $A \models_r B$ *then* $A \models_{*,g} B$.

Proof. The details are spelled out in Appendix C. □

Remark 10. As in the case for **FDE** it is possible that the number of worlds for $\models_{*,g}$ can be reduce to 2. This can be seen by careful examination of the proofs of Lemma 1 and Proposition 5.

6 Reflections on $\sim_f$

The operator $\sim_f$ can be regarded as the negation with respect to the falsity order of the trilattice **SIXTEEN**$_3$. However, in the context of this article, in which we focus solely on truth-order, it can be observed that $\sim_f$ is more than just a simple negation.

6.1 $\sim_f$ in Special Cases

The introduction of **SIXTEEN**$_3$ inspired Dmitri Zaitsev to consider some variants with less truth values in [26]. In brief, Zaitsev suggests to apply the power set of a three-element set, rather than the four-element set used by Shramko and Wansing. Due to the limitation of space, we cannot discuss the details of how our two-valued semantics will capture one of Zaitsev's systems.

However, since it is rather natural to consider some variants with less truth values, we briefly consider three special cases of two-star interpretations, and connect the resulting system to those known in the literature.

First, as expected, if we require $w^{*_2} = w$ for all $w \in W$, then we simply obtain an expansion of **FDE** with $\sim_f A \vdash A$ and $A \vdash \sim_f A$. Second, if we require $w^{*_1} = w^{*_2}$ for all $w \in W$, then we obtain an expansion of **FDE** with $\sim_f$ as conflation.[4] Since classical negation is definable in terms of de Morgan negation and conflation, and conflation is definable in terms of de Morgan negation and classical negation, the resulting system is equivalent to the expansion of **FDE** by classical negation, called **BD+** in [5]. Finally, if we require $w^{*_1} = w$ for all $w \in W$, then $\sim$ is a classical negation, and $\sim_f$ is again conflation. Since de Morgan negation is definable in terms of classical negation and conflation, the resulting system is again equivalent to **BD+**.

6.2 $\sim_f$ as a Modal Operator

In **SIXTEEN**$_3$, the operator $\sim_f$ serves as a negation over the falsity ordering. In what follows, we will, however, show that truth condition for $\sim_f$, understood as in Sect. 4, suffice to interpret $\sim_f$ as a modal operator satisfying the **K**-axiom, as well as the rule of necessitation. Since our language is rather weak, we add $\rightarrow$ which satisfies the following truth condition in a two-star interpretation.

$$I(w, A \rightarrow B) = 1 \text{ iff } I(w, A) \neq 1 \text{ or } I(w, B) = 1.$$

In fact, this connective is the implication introduced by Odintsov in [12] as $\rightarrow_t$.

It is now possible to prove the following proposition.

Proposition 6. *For all* $A, B \in \mathsf{Form}_{\sim_f,\rightarrow}$,

1. $\models_{*,\forall} \sim_f(A \rightarrow B) \rightarrow (\sim_f A \rightarrow \sim_f B)$,

[4] Given a Dunn interpretation, conflation, written as $-$, is characterized by the following truth and falsity conditions: $-Ar1$ iff not $Ar0$, and $-Ar0$ iff not $Ar1$.

2. $\dfrac{\models_{*,\forall} A}{\models_{*,\forall} \sim_f A}$ *and* $\dfrac{\models_{*,\forall} \sim_f A}{\models_{*,\forall} A}$.

Remark 11. The **T** and **S4** axiom are not valid in this semantics. Furthermore, the equivalence $\sim\sim_f\sim A \leftrightarrow \sim_f A$ shows that $\sim_f$ is self-dual and hence also contains properties of a possibility operator. The negative modality $\sim$ behaves in a similar way.[5]

Given that $\sim_f$ is not defined via an accessibility relation over worlds, but rather a function that maps worlds to worlds, one may doubt that $\sim_f$ counts as modal operator at all. However, as described by van Benthem in [4], it is possible to model propositional modal logic with a family of functions $\mathcal{F}$, rather than accessibility relations. A model $\mathcal{M} = \langle W, \mathcal{F}, V\rangle$ is then a tuple in the usual manner, with the following clause for the necessity operator: $I(w, \Box A) = 1$ iff $I(f(w), A) = 1$ for all $f \in \mathcal{F}$. For example, the modal logic **T** is complete with respect "for all frames whose function set $\mathcal{F}$ contains the identity function" [4].

In analogy to van Benthem's approach, we may regard our two-star interpretation as a model $\mathcal{M} = \langle W, g, *_1, \mathcal{F}, V\rangle$ where $\mathcal{F} = \{*_2\}$ (recall Definition 11). We would then have $I(w, \sim_f A) = 1$ iff $I(f(w), A) = 1$ for all $f \in \mathcal{F}$. Therefore, if van Benthem's approach is seen as an approach to modality, then $\sim_f$ will be also counted as a modality at least in that sense. Hence, the language $\mathcal{L}_{\sim_f}$ can be interpreted as an **FDE**-based modal language, where **FDE** is captured in terms of the star semantics (recall Definition 1), as, for example, in [7,9].[6]

7 Concluding Remarks

What we hope to have established in this paper is that it is possible to provide two-valued semantics for a logic based on **SIXTEEN**$_3$. In particular, we made essential use of Routleys' star operation for both two-valued semantics. However, our result here is just a first step, and there seem to be a number of problems to be explored in more details. We will mention two of them.

The first problem is related to the language. In this paper, we focused on the most simple language associated to **SIXTEEN**$_3$, namely $\mathcal{L}_{\sim_f}$. However, this is only one of the many possible choices. In particular, it seems more than natural to deal with $\wedge_f$ and $\vee_f$, but these connectives seem to be resistant. For example, if we consider the truth condition for $\wedge_f$ in a two-star interpretation, then a straightforward application of our method suggests to split truth condition depending on the number of stars applied at the state. We do not know, at the time of writing, if we can capture $\wedge_f$ in a two-star interpretation by a single truth condition. We should also note that some connectives discussed in the literature can be captured. For example, $\neg$ and $\sim_i$, in a two-star interpretation, will have the following truth conditions respectively:

[5] We thank Sergei Odintsov for pointing this out.

[6] For a different approach to **FDE**-based modal logic, where **FDE** is captured in terms of the Dunn semantics (recall Definition 3), see, for example [13,15]. Comparing the two approaches will be future work.

- $I(w, \neg A) = 1$ iff $I(w, A) \neq 1$
- $I(w, \sim_i A) = 1$ iff $I(w^{*_1 *_2}, A) = 1$

The second problem is to explore the relation between the two-valued semantics and the trilattice. Note that in our two-valued semantics, we are making essential use of the star operation, but this seems to give rise to some difficulties. Here is a reason: In the context of **FDE**, informational join and meet of the bilattice naturally inspire to introduce binary connectives, and these connectives can be captured easily in terms of Dunn semantics by giving truth and falsity conditions. However, it is far from obvious if we can capture the same connectives based on the star semantics by equally simple conditions. And a similar issue may carry over to the case with $\mathbf{SIXTEEN}_3$. In fact, this might also be related to the first problem related to $\wedge_f$ and $\vee_f$.

A Details of the Proof of Theorem 3

We prove the contrapositive. Assume $A \not\vdash B$. Then, by Lindenbaum's lemma, there is a prime theory Γ such that $A \in \Gamma$ and $B \notin \Gamma$. We then define a two-star interpretation $\langle W, g, *_1, *_2, v\rangle$ as follows:

- $W = \{a, b, c, d\}$, $g = a$;
- $a^{*_1} = b$, $b^{*_1} = a$, $c^{*_1} = d$, $d^{*_1} = c$, $a^{*_2} = c$, $b^{*_2} = d$, $c^{*_2} = a$, $d^{*_2} = b$;
- $v : W \times \mathsf{Prop} \to \{0, 1\}$ is defined as follows:

$$\begin{array}{ll} v(a,p)=1 \text{ iff } p \in \Gamma; & v(c,p)=1 \text{ iff } p \in \Gamma^*; \\ v(b,p)=1 \text{ iff } {\sim}p \notin \Gamma; & v(d,p)=1 \text{ iff } {\sim}p \notin \Gamma^*. \end{array}$$

If we can show that the above condition holds for all formulas, then the result follows since at $a \in W$, $I(a, A) = 1$ but $I(a, B) \neq 1$, i.e. $A \not\models_* B$. We prove this by induction on the complexity of A. We only prove the cases for $\sim$ and $\sim_f$, since the cases for $\wedge$ and $\vee$ are straightforward.

Case 1. If A is an element of Prop, the result holds by definition.

Case 2. If $A = {\sim}B$, then

$$\begin{array}{llll} v(a,{\sim}B){=}1 & \text{iff } v(a^{*_1},B){\neq}1 & & \\ & \text{iff } v(b,B){\neq}1 & \text{Def. } *_1 \\ & \text{iff } {\sim}B \in \Gamma & \text{IH} \end{array} \qquad \begin{array}{llll} v(c,{\sim}B){=}1 & \text{iff } v(c^{*_1},B){\neq}1 & \\ & \text{iff } v(d,B){\neq}1 & \text{Def. } *_1 \\ & \text{iff } {\sim}B \in \Gamma^* & \text{IH} \end{array}$$

$$\begin{array}{lll} v(b,{\sim}B){=}1 & \text{iff } v(b^{*_1},B){\neq}1 & \\ & \text{iff } v(a,B){\neq}1 & \text{Def. } *_1 \\ & \text{iff } B \notin \Gamma & \text{IH} \\ & \text{iff } {\sim}{\sim}B \notin \Gamma & (\mathrm{a}_t6), (\mathrm{a}_t7) \end{array} \qquad \begin{array}{lll} v(d,{\sim}B){=}1 & \text{iff } v(d^{*_1},B){\neq}1 & \\ & \text{iff } v(c,B){\neq}1 & \text{Def. } *_1 \\ & \text{iff } B \notin \Gamma^* & \text{IH} \\ & \text{iff } {\sim}{\sim}B \notin \Gamma^* & (\mathrm{a}_t6), (\mathrm{a}_t7) \end{array}$$

Case 3. If $A = \sim_f B$, then

$$\begin{array}{lll} v(a, \sim_f B)=1 & \text{iff } v(a^{*_2}, B)=1 & \\ & \text{iff } v(c, B)=1 & \text{Def. } *_2 \\ & \text{iff } B \in \Gamma^* & \text{IH} \\ & \text{iff } \sim_f\sim_f B \in \Gamma^* & (\mathrm{a}_t 8), (\mathrm{a}_t 9) \\ & \text{iff } \sim_f B \in \Gamma & \text{Lem. 3} \end{array} \qquad \begin{array}{lll} v(c, \sim_f B)=1 & \text{iff } v(c^{*_2}, B)=1 & \\ & \text{iff } v(a, B)=1 & \text{Def. } *_2 \\ & \text{iff } B \in \Gamma & \text{IH} \\ & \text{iff } \sim_f B \in \Gamma^* & \text{Lem. 3} \end{array}$$

$$\begin{array}{lll} v(b, \sim_f B)=1 & \text{iff } v(b^{*_2}, B)=1 & \\ & \text{iff } v(d, B)=1 & \text{Def. } *_2 \\ & \text{iff } \sim B \notin \Gamma^* & \text{IH} \\ & \text{iff } \sim_f\sim B \notin \Gamma & \text{Lem. 3} \\ & \text{iff } \sim\sim_f B \notin \Gamma & (\mathrm{a}_t 10) \end{array} \qquad \begin{array}{lll} v(d, \sim_f B)=1 & \text{iff } v(d^{*_2}, B)=1 & \\ & \text{iff } v(b, B)=1 & \text{Def. } *_2 \\ & \text{iff } \sim B \notin \Gamma & \text{IH} \\ & \text{iff } \sim_f\sim B \notin \Gamma^* & \text{Lem. 3} \\ & \text{iff } \sim\sim_f B \notin \Gamma^* & (\mathrm{a}_t 10) \end{array}$$

This completes the proof. □

B Details of the Proof of Proposition 4

We prove the contrapositive. Assume $A \not\models_r B$. Then, there is a one-star interpretation $\langle W, g, *, r\rangle$ such that $Ar_g 1$, but not $Br_g 1$. We then define a two-star interpretation $\langle W, g *_1, *_2, v\rangle$ as follows:

- $W = \{a, b, c, d\}$, $g = a$;
- $a^{*_1} = b$, $b^{*_1} = a$, $c^{*_1} = d$, $d^{*_1} = c$, $a^{*_2} = c$, $b^{*_2} = d$, $c^{*_2} = a$, $d^{*_2} = b$;
- $v : W \times \mathsf{Prop} \to \{0, 1\}$ is defined as follows:

$$\begin{array}{ll} v(a, p) = 1 \text{ iff } pr_g 1; & v(c, p) = 1 \text{ iff } pr_{g^*} 1; \\ v(b, p) = 1 \text{ iff not } pr_{g^*} 0 & v(d, p) = 1 \text{ iff not } pr_g 0. \end{array}$$

If we can show that the above condition holds for all formulas, then the result follows since at $a \in W$, $v(a, A) = 1$ but $v(a, B) \neq 1$, i.e. $A \not\models_{*,g} B$. We prove this by induction. We only prove the cases for $\sim$ and $\sim_f$, since the cases for $\wedge$ and $\vee$ are straightforward.

Case 1. If A is an element of Prop, the result holds by definition.

Case 2. If $A = \sim B$, then

$$\begin{array}{lll} v(a, \sim B) = 1 & \text{iff } v(a^{*_1}, B) \neq 1 & \\ & \text{iff } v(b, B) \neq 1 & \text{Def. } *_1 \\ & \text{iff } Br_{g^*} 0 & \text{IH} \\ & \text{iff } \sim Br_g 1 & \end{array} \qquad \begin{array}{lll} v(c, \sim B) = 1 & \text{iff } v(c^{*_1}, B) \neq 1 & \\ & \text{iff } v(d, B) \neq 1 & \text{Def. } *_1 \\ & \text{iff } Br_g 0 & \text{IH} \\ & \text{iff } \sim Br_{g^*} 1 & \end{array}$$

$$\begin{array}{lll} v(b, \sim B) = 1 & \text{iff } v(b^{*_1}, B) \neq 1 & \\ & \text{iff } v(a, B) \neq 1 & \text{Def. } *_1 \\ & \text{iff not } Br_g 1 & \text{IH} \\ & \text{iff not } \sim Br_{g^*} 0 & \end{array} \qquad \begin{array}{lll} v(d, \sim B) = 1 & \text{iff } v(d^{*_1}, B) \neq 1 & \\ & \text{iff } v(c, B) \neq 1 & \text{Def. } *_1 \\ & \text{iff not } Br_{g^*} 1 & \text{IH} \\ & \text{iff not } \sim Br_g 0 & \end{array}$$

Case 3. If $A = {\sim_f} B$, then

$$\begin{array}{llll} v(a, {\sim_f} B) = 1 & \text{iff } v(a^{*_2}, B) = 1 & & \\ & \text{iff } v(c, B) = 1 & \text{Def. } *_2 \\ & \text{iff } Br_{g^*}1 & \text{IH} \\ & \text{iff } {\sim_f} Br_g 1 & \end{array} \qquad \begin{array}{lll} v(c, {\sim_f} B) = 1 & \text{iff } v(c^{*_2}, B) = 1 & \\ & \text{iff } v(a, B) = 1 & \text{Def. } *_2 \\ & \text{iff } Br_g 1 & \text{IH} \\ & \text{iff } {\sim_f} Br_{g^*}1 & \text{Lem. 3} \end{array}$$

$$\begin{array}{lll} v(b, {\sim_f} B) = 1 & \text{iff } v(b^{*_2}, B) = 1 & \\ & \text{iff } v(d, B) = 1 & \text{Def. } *_2 \\ & \text{iff not } Br_g 0 & \text{IH} \\ & \text{iff not } {\sim_f} Br_{g^*}0 & \end{array} \qquad \begin{array}{lll} v(d, {\sim_f} B) = 1 & \text{iff } v(d^{*_2}, B) = 1 & \\ & \text{iff } v(b, B) = 1 & \text{Def. } *_2 \\ & \text{iff not } Br_{g^*}0 & \text{IH} \\ & \text{iff not } {\sim_f} Br_g 0 & \end{array}$$

This completes the proof. □

C Details of the Proof for Proposition 5

We prove the contrapositive. Assume $A \not\models_{*,g} B$. Then, there is a two-star interpretation $\langle W, g, *_1, *_2, v\rangle$ such that $I(g, A) = 1$ but $I(g, B) \neq 1$. We then define a one-star interpretation $\langle W, g, *, r\rangle$ as follows:

- $W = \{a, b\}$, $g = a$;
- $a^* = b$, $b^* = a$;
- $r_w \subseteq \mathsf{Prop} \times \{0, 1\}$ is defined as follows:

$$\begin{array}{ll} pr_a 1 \text{ iff } I(g, p) = 1; & pr_b 1 \text{ iff } I(g^{*_2}, p) = 1; \\ pr_a 0 \text{ iff } I(g^{*_1 *_2}, p) \neq 1; & pr_b 0 \text{ iff } I(g^{*_1}, p) \neq 1. \end{array}$$

If we can show that the above condition holds for all formulas, then the result follows since at $a \in W$, $Ar_a 1$ but not $Br_a 1$, i.e. $A \not\models_r B$. We prove this by induction. We only prove the cases for $\sim$ and $\sim_f$, since the cases for $\wedge$ and $\vee$ are straightforward.

Case 1. If A is an element of Prop, the result holds by definition.

Case 2. If $A = {\sim} B$, then

$$\begin{array}{lll} {\sim} Br_a 1 & \text{iff } Br_{a^*}0 & \\ & \text{iff } Br_b 0 & \text{Def. } * \\ & \text{iff } I(g^{*_1}, B) \neq 1 & \text{IH} \\ & \text{iff } I(g, {\sim} B) = 1 & \end{array} \qquad \begin{array}{lll} {\sim} Br_b 1 & \text{iff } Br_{b^*}0 & \\ & \text{iff } Br_a 0 & \text{Def. } * \\ & \text{iff } I(g^{*_1 *_2}, B) \neq 1 & \text{IH} \\ & \text{iff } I(g^{*_2}, {\sim} B) = 1 & \end{array}$$

$$\begin{array}{lll} {\sim} Br_a 0 & \text{iff } Br_{a^*}1 & \\ & \text{iff } Br_b 1 & \text{Def. } * \\ & \text{iff } I(g^{*_2}, B) = 1 & \text{IH} \\ & \text{iff } I(g^{*_2 *_1}, {\sim} B) \neq 1 & \end{array} \qquad \begin{array}{lll} {\sim} Br_b 0 & \text{iff } Br_{b^*}1 & \\ & \text{iff } Br_a 1 & \text{Def. } * \\ & \text{iff } I(g, B) = 1 & \text{IH} \\ & \text{iff } I(g^{*_1}, {\sim} B) \neq 1 & \end{array}$$

Case 3. If $A = {\sim_f} B$, then

$$
\begin{array}{llllll}
{\sim_f} B r_a 1 & \text{iff } B r_{a^*} 1 & & {\sim_f} B r_b 1 & \text{iff } B r_{b^*} 1 & \\
 & \text{iff } B r_b 1 & \text{Def. } * & & \text{iff } B r_a 1 & \text{Def. } * \\
 & \text{iff } I(g^{*_2}, B) = 1 & \text{IH} & & \text{iff } I(g, B) = 1 & \text{IH} \\
 & \text{iff } I(g, {\sim_f} B) = 1 & & & \text{iff } I(g^{*_2}, {\sim_f} B) = 1 &
\end{array}
$$

$$
\begin{array}{llllll}
{\sim_f} B r_a 0 & \text{iff } B r_{a^*} 0 & & {\sim_f} B r_b 0 & \text{iff } B r_{b^*} 0 & \\
 & \text{iff } B r_b 0 & \text{Def. } * & & \text{iff } B r_a 0 & \text{Def. } * \\
 & \text{iff } I(g^{*_1}, B) \neq 1 & \text{IH} & & \text{iff } I(g^{*_1 *_2}, B) \neq 1 & \text{IH} \\
 & \text{iff } I(g^{*_1 *_2}, {\sim_f} B) \neq 1 & & & \text{iff } I(g^{*_1}, {\sim_f} B) \neq 1 &
\end{array}
$$

This completes the proof. □

References

1. Arieli, O., Avron, A.: Reasoning with logical bilattices. J. Log. Lang. Inf. **5**(1), 25–63 (1996)
2. Belnap, N.: How a computer should think. In: Ryle, G. (ed.) Contemporary Aspects of Philosophy, pp. 30–55. Oriel Press (1976)
3. Belnap, N.: A useful four-valued logic. In: Dunn, J., Epstein, G. (eds.) Modern Uses of Multiple-Valued Logic, pp. 8–37. D. Reidel Publishing Co. (1977)
4. van Benthem, J.: Beyond accessibility. In: de Rijke, M. (ed.) Diamonds and Defaults: Studies in Pure and Applied Intensional Logic. SYLI, vol. 229, pp. 1–18. Springer, Dordrecht (1993). https://doi.org/10.1007/978-94-015-8242-1_1
5. De, M., Omori, H.: Classical negation and expansions of Belnap-Dunn logic. Stud. Log. **103**(4), 825–851 (2015)
6. Dunn, J.M.: Intuitive semantics for first-degree entailment and 'coupled trees'. Philos. Stud. **29**, 149–168 (1976)
7. Fuhrmann, A.: Models for relevant modal logics. Stud. Log. **49**(4), 501–514 (1990)
8. Ginsberg, M.: Multi-valued logics: a uniform approach to AI. Comput. Intell. **4**, 243–247 (1988)
9. Mares, E.D., Meyer, R.K.: The semantics of R4. J. Philos. Log. **22**(1), 95–110 (1993)
10. Muskens, R., Wintein, S.: Analytic tableaux for all of SIXTEEN$_3$. J. Philos. Log. **44**(5), 473–487 (2015)
11. Muskens, R., Wintein, S.: Interpolation in 16-valued trilattice logics. Stud. Log. **106**(2), 345–370 (2018)
12. Odintsov, S.P.: On axiomatizing Shramko-Wansing's logic. Stud. Log. **91**(3), 407–428 (2009)
13. Odintsov, S.P., Wansing, H.: Modal logics with Belnapian truth values. J. Appl. Non-Class. Log. **20**, 279–301 (2010)
14. Odintsov, S.P., Wansing, H.: The logic of generalized truth values and the logic of bilattices. Stud. Log. **103**(1), 91–112 (2015)
15. Odintsov, S.P., Wansing, H.: Disentangling FDE-based paraconsistent modal logics. Stud. Log. **105**(6), 1221–1254 (2017)
16. Omori, H., Sano, K.: Generalizing functional completeness in Belnap-Dunn logic. Stud. Log. **103**(5), 883–917 (2015)
17. Omori, H., Wansing, H.: 40 years of FDE: an introductory overview. Stud. Log. **105**(6), 1021–1049 (2017)

18. Priest, G.: An Introduction to Non-Classical Logic: From If to Is, 2nd edn. Cambridge University Press, Cambridge (2008)
19. Restall, G.: Negation in relevant logics (how i stopped worrying and learned to love the routley star). In: Gabbay, D.M., Wansing, H. (eds.) What is Negation?, pp. 53–76. Kluwer Academic Publishers (1999)
20. Routley, R., Routley, V.: Semantics for first degree entailment. Noûs **6**, 335–359 (1972)
21. Shramko, Y., Wansing, H.: Truth and Falsehood - An Inquiry into Generalized Logical Values, 1st edn. Springer, Dordrecht (2012). https://doi.org/10.1007/978-94-007-0907-2
22. Shramko, Y., Wansing, H.: Some useful 16-valued logics: how a computer network should think. J. Philos. Log. **34**(2), 121–153 (2005)
23. Shramko, Y., Wansing, H.: Hyper-contradictions, generalized truth values and logics of truth and falsehood. J. Log. Lang. Inf. **15**(4), 403–424 (2006)
24. Suszko, R.: Remarks on Łukasiewicz's three-valued logic. Bull. Sect. Log. **4**, 87–90 (1975)
25. Wansing, H.: The power of Belnap: sequent systems for $\mathrm{SIXTEEN}_3$. J. Philos. Log. **39**, 369–393 (2010)
26. Zaitsev, D.: A few more useful 8-valued logics for reasoning with tetralattice EIGHT_4. Stud. Log. **92**(2), 265–280 (2009)