LNCS 11541

Rosalie Iemhoff
Michael Moortgat
Ruy de Queiroz (Eds.)

Logic, Language, Information, and Computation

26th International Workshop, WoLLIC 2019
Utrecht, The Netherlands, July 2–5, 2019
Proceedings

Springer

More information about this series at http://www.springer.com/series/7407

Rosalie Iemhoff · Michael Moortgat ·
Ruy de Queiroz (Eds.)

Logic, Language, Information, and Computation

26th International Workshop, WoLLIC 2019
Utrecht, The Netherlands, July 2–5, 2019
Proceedings

Springer

Editors
Rosalie Iemhoff
Department of Philosophy and Religious Studies
Utrecht University
Utrecht, The Netherlands

Ruy de Queiroz
Centro de Informática
Universidade Federal de Pernambuco (UFPE)
Recife, Brazil

Michael Moortgat
Utrecht Institute of Linguistics OTS
Utrecht University
Utrecht, The Netherlands

ISSN 0302-9743 ISSN 1611-3349 (electronic)
Lecture Notes in Computer Science
ISBN 978-3-662-59532-9 ISBN 978-3-662-59533-6 (eBook)
https://doi.org/10.1007/978-3-662-59533-6

LNCS Sublibrary: SL1 – Theoretical Computer Science and General Issues

This Springer imprint is published by the registered company Springer-Verlag GmbH, DE
part of Springer Nature
The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

Preface

This volume contains the papers presented at the 26th Workshop on Logic, Language, Information, and Computation (WoLLIC 2019) held during July 2–5, 2019, at Utrecht University, The Netherlands. The WoLLIC series of workshops started in 1994 with the aim of fostering interdisciplinary research in pure and applied logic. The idea is to have a forum that is large enough in the number of possible interactions between logic and the sciences related to information and computation, and yet is small enough to allow for concrete and useful interaction among participants.

There were 60 submissions. Each submission was reviewed by at least two Program Committee members, with additional reviewers in the case of diverging scores. The committee decided to accept 41 papers; one paper was subsequently withdrawn. The program also included six invited lectures by Lev Beklemishev (Steklov Institute), Raffaella Bernardi (University of Trento), Marta Bílková (Czech Academy of Sciences), Johan Bos (University of Groningen), George Metcalfe (University of Bern), and Reinhard Muskens (University of Tilburg). The invited speakers were shared by two satellite workshops organized in conjunction with WoLLIC: Proof Theory in Logic (July 1–2, 2019, organizer R. Iemhoff) and Compositionality in Formal and Distributional Models of Natural Language Semantics (July 6 2019, organizer M. Moortgat).

In a special session during WoLLIC 2019, a short movie was screened in remembrance of the 85th anniversary of the award of a doctorate in mathematics to Paul Erdös (March 26, 1913 to September 20, 1996), a renowned Hungarian mathematician considered to be one of the most prolific mathematicians and producers of mathematical conjectures of the 20th century: *Erdös 100*, a 30-minute video prepared for the centennial celebration in 2013 of Paul Erdös's birth, directed by George Paul Csicsery (Zala Films).

We would very much like to thank all Program Committee members and external reviewers for the work they put into reviewing the submissions. The help provided by the EasyChair system created by Andrei Vorokonkov is gratefully acknowledged. Finally, we would like to acknowledge the support of the Netherlands Organisation for Scientific Research NWO (projects 639.073.807 and 360-89-070) and of the Faculty of Humanities at Utrecht University, and the scientific sponsorship of the following organizations: Interest Group in Pure and Applied Logics (IGPL), The Association for Logic, Language and Information (FoLLI), Association for Symbolic Logic (ASL), European Association for Theoretical Computer Science (EATCS), European Association for Computer Science Logic (EACSL), Sociedade Brasileira de Computação (SBC) and Sociedade Brasileira de Lógica (SBL).

July 2019

Rosalie Iemhoff
Michael Moortgat
Ruy de Queiroz

Organization

Program Committee

Raffaella Bernardi	University of Trento, Italy
Nick Bezhanishvili	ILLC, University of Amsterdam, The Netherlands
Ivano Ciardelli	Ludwig Maximilian University of Munich, Germany
Philippe de Groote	Inria, France
Valeria de Paiva	Samsung Research America and University of Birmingham, UK
Ruy De Queiroz	Universidade Federal de Pernambuco, Brazil
Giuseppe Greco	Utrecht University, The Netherlands
Rosalie Iemhoff (Chair)	Utrecht University, The Netherlands
Roberto Maieli	Roma Tre University, Italy
Michael Moortgat (Chair)	Utrecht University, The Netherlands
Richard Moot	CNRS (LIRMM) and University of Montpellier, France
Larry Moss	Indiana University Bloomington, USA
Sara Negri	University of Helsinki, Finland
Carlo Nicolai	University of Oxford, UK
Alessandra Palmigiano	Technical University of Delft, The Netherlands
Yde Venema	ILLC, University of Amsterdam, The Netherlands
Fan Yang	University of Helsinki, Finland

Local Organising Committee

Giuseppe Greco (Chair)	Utrecht University
Iris van der Giessen (Webmaster)	Utrecht University
Sam van Gool	Utrecht University
Rosalie Iemhoff	Utrecht University
Raheleh Jalali	Czech Academy of Sciences
Michael Moortgat	Utrecht University
Amir Tabatabai	Utrecht University

Additional Reviewers

Acclavio, Matteo
Aguzzoli, Stefano
Badia, Guillermo
Barenboim, Leonid
Beklemishev, Lev
Bezhanishvili, Guram
Bimbo, Katalin
Buszkowski, Wojciech
Chen, Jinsheng
Dorais, François
Farooque, Mahfuza
Fernández Gil, Oliver

Flaminio, Tommaso
Fujiwara, Makoto
Fussner, Wesley
Gabelaia, David
Galliani, Pietro
Gattinger, Malvin
Gore, Rajeev
Goubault-Larrecq, Jean
Grilletti, Gianluca
Grädel, Erich
Guerrieri, Giulio
Kanazawa, Makoto
Kopczynski, Eryk
Kulikov, Vadim
Kuznets, Roman
Lange, Martin
Leigh, Graham
Liang, Fei
Licata, Daniel R.
Lück, Martin
Ma, Minghui
Maffezioli, Paolo
Marcone, Alberto
Marcos, Joao
Mummert, Carl
Muskens, Reinhard
Nordvall Forsberg, Fredrik
Norrish, Michael
Oliva, Paulo
Orlandelli, Eugenio
Pagani, Michele
Pakhomov, Fedor
Pavlovic, Edi
Pedicini, Marco
Peltier, Nicolas
Pistone, Paolo
Poggiolesi, Francesca
Powell, Thomas
Protopopescu, Tudor
Puncochar, Vit
Ramos, Marcus
Rijke, Egbert
Rivieccio, Umberto
Salvati, Sylvain
Sano, Katsuhiko
Santocanale, Luigi
Standefer, Shawn
Sternagel, Christian
Straßburger, Lutz
Tedder, Andrew
Tzimoulis, Apostolos
Ugolini, Sara
Uustalu, Tarmo
Valentín, Oriol
Valota, Diego
van Ditmarsch, Hans
van Gool, Sam
Wang, Yanjing
Zohar, Yoni

Abstracts of Invited Talks

Reflection Algebras for Theories of Iterated Truth Definitions

Lev Beklemishev

Steklov Mathematical Institute of Russian Academy of Sciences
lbekl@yandex.ru

We consider extensions of the language of Peano arithmetic by iterated truth definitions satisfying uniform Tarskian biconditionals. Given a first-order language $\mathcal{L}$ containing that of arithmetic one can add a new symbol T for a truth predicate and postulate the equivalence of each formula $\phi(x)$ of $\mathcal{L}$ with $\mathsf{T}(\ulcorner\phi(\underline{x})\urcorner)$ where $\ulcorner\phi(\underline{x})\urcorner$ denotes the Gödel number of the result of substituting numeral for x into ϕ. This extension procedure can be repeated transfinitely many times.

Without further axioms, such theories are known to be weak conservative extensions of the original system of arithmetic. Much stronger systems, however, are obtained by adding either induction axioms or reflection axioms on top of them. Theories of this kind can interpret some well-known predicatively reducible fragments of second-order arithmetic such as iterated arithmetical comprehension. Feferman and Schütte studied related systems of ramified analysis. They used their systems to explicate the intuitive idea of a predicative proof and determined the ordinal Γ_0 as the bound to transfinite induction provable in predicative systems.

We obtain sharp results on the proof-theoretic strength of these systems using methods of provability logic, in particular, we calculate their proof-theoretic ordinals and conservativity spectra. We consider the semilattice of axiomatizable extensions of a basic theory of iterated truth definitions. We enrich the structure of this semilattice by suitable reflection operators and isolate the corresponding strictly positive modal logic (reflection calculus) axiomatizing the identities of this structure. The variable-free fragment of the logic provides a canonical ordinal notation system for the class of theories under investigation. This setup allows us to obtain in a technically smooth way conservativity relationships for iterated reflection principles of various strength which provide a sharp proof-theoretic analysis of our systems.

Joint work with F. Pakhomov and E. Kolmakov. This work is supported by a grant of the Russian Science Foundation (project No. 16-11-10252).

Jointly Learning to See, Ask, and Guess

Raffaella Bernardi

University of Trento
raffaella.bernardi@unitn.it

In our daily use of natural language, we constantly profit of our strong reasoning skills to interpret utterances we hear or read. At times we exploit implicit associations we have learned between words or between events, at others we explicitly think about a problem and follow the reasoning steps carefully and slowly. We could say that the latter are the realm of logical approaches based on symbolic representations, whereas the former are better modelled by statistical models, like Neural Networks (NNs), based on continuous representations.

My talk will focus on how NNs can learn to be engaged in a conversation on visual content. Specifically, I will present our work on Visual Dialogue (VD) taking as example two tasks oriented VDs, GuessWhat?! [2] and GuessWhich [1]. In these tasks, two NN agents interact to each other so that one of the two (the Questioner), by asking questions to the other (the Answerer), can guess which object the Answerer has in mind among all the entities in a given image (GuessWhat?!) or which image the Answerer sees among several ones seen by the Questioner at end of the dialogue (GuessWhich).

I will present our Questioner model: it encodes both visual and textual inputs, produces a multimodal representation, generates natural language questions, understands the Answerer's responses and guesses the object/image. I will show how training the NN agent's modules (Question generator and Guesser) jointly and cooperatively help the model performance and increase the quality of the dialogues. In particular, I will compare our model's dialogues with those of VD models which exploit much more complex learning paradigms, like Reinforcement Learning, showing that more complex machine learning methods do not necessarily correspond to better dialogue quality or even better quantitative performance.

The talk is based on [3] and other work available at https://vista-unitn-uva.github.io/.

References

1. Das, A., Kottur, S., Moura, J.M., Lee, S., Batra, D.: Learning cooperative visual dialog agents with deep reinforcement learning. In: International Conference on Computer Vision (ICCV) (2017)
2. de Vries, H., Strub, F., Chandar, S., Pietquin, O., Larochelle, H., Courville, A.C.: Guesswhat?! Visual object discovery through multi-modal dialogue. In: Conference on Computer Vision and Pattern Recognition (CVPR) (2017)
3. Shekhar, R., et al.: Beyond task success: a closer look at jointly learning to see, ask, and guesswhat. In: NAACL (2019)

On Infinitary Proof Theory of Logics of Information and Common Belief

Marta Bílková

Institute of Computer Science, the Czech Academy of Sciences
bilkova@cs.cas.cz

Abstract. Recently there has been a growing interest in applying non-classically based modal logics in the context of logics for agency and social behaviour. In particular, substructural or other information-based modal logics of knowledge and belief, or similar versions of PDL, have been designed. While basic modal extensions of substructural logics on one side, and classically based logics of common belief and other fixed point modalities, are relatively well understood when it comes to completeness and proof theory, with logics we have in mind it is not so.

In this talk, we will mainly concentrate on logics of common belief. We will consider two natural ways of axiomatizing the common belief over a basic modal logic (Belnap-Dunn logic or distributive substructural logics, extended by normal diamond and box modalities): one finitary, which is the standard Kozen's axiomatization, and the other infinitary, with an infinitary rule replacing the induction rule and using finite approximations of the fixed points. The finitary axiomatization is used to obtain, using an algebraic (and coalgebraic) insight, the soundness of the infinitary rule.

We will then concentrate on the infinitary part of the story and draw a general, duality based picture connecting the syntax and poset-based frame semantics of the infinitary axiomatizations, including a completeness argument based on a canonical model construction. Here, the infintary case differs from the usual finitary account of (non-classical) modal logics: in particular, one needs to use an appropriate version of Lindenbaum or Belnap Pair-Extension lemma.

Finally, we use the above insight to discuss proof theory of such logics within the framework of display calculi.

Interlingual Meaning Representations

Johan Bos

University of Groningen
johan.bos@rug.nl

What is *meaning*, and how can it be captured in a concrete representation? This is a challenging question, given the fact that meanings enjoy a large degree of abstraction. A question that directly follows from this is to what extent meanings (or if you prefer: *representations of meaning*), need to be language neutral. Most of the current large-scale meaning representations employed in natural language processing are tailored to specific object languages, often English [3–5] because it has been the dominating language of study in computational linguistics (but this is slowly changing, fortunately). This is completely understandable from a short-term, practical perspective. But from a theoretical point of view, this doesn't make sense at all (just think about how translations from one language into another preserve meaning). A natural question to ask, then, is how far we can stretch *interlingual* meaning representations. What is required to achieve this—what resources and (linguistic) knowledge do we need? What challenges are we facing? What role can and must logic play?

In most logical approaches to semantics, a part of the meaning representation is, by its very nature, independent of the object language: the logical symbols used to express negation, conjunction, disjunction, and quantification. The non-logical symbols are usually represented by strings resembling words of a specific language (again, usually English). This is a tradition started by Montague [11], and followed by many others [7, 8]. So logic only gives a partial guidance to our endeavour of making meaning representations more interlingual. Should we expect more from logic? What is a good balance between logical and non-logical ingredients in a meaning representation? Let us look at a concrete example.

The Parallel Meaning Bank, PMB [1], is a semantically annotated corpus for four languages (English, Dutch, German, and Italian). It comprises translations between these languages, and under the assumption that translations preserve meaning, the PMB is the perfect environment to investigate interlingual meaning representations. The meaning representations in the PMB combine the logical aspects of Discourse Representation Theory [9] with lexical resources including WordNet [6], VerbNet [10], and FrameNet [2]. The PMB data demonstrates that even closely related languages behave differently in (for instance) marking definiteness, realisation of verbal arguments, and multi-word expressions. Despite these new challenges, I will argue that providing interlingual meaning representations is a welcome direction not only in computational, but also in formal approaches to meaning.

References

1. Abzianidze, L., et al.: The parallel meaning bank: towards a multilingual corpus of translations annotated with compositional meaning representations. In: Proceedings of the 15th Conference of the European Chapter of the Association for Computational Linguistics, pp. 242–247, Valencia, Spain (2017)
2. Baker, C.F., Fillmore, C.J., Lowe, J.B.: The Berkeley FrameNet project. In: 36th Annual Meeting of the Association for Computational Linguistics and 17th International Conference on Computational Linguistics, Proceedings of the Conference, pp. 86–90. Université de Montréal, Montreal (1998)
3. Banarescu, L., et al.: Abstract meaning representation for sembanking. In: Proceedings of the 7th Linguistic Annotation Workshop and Interoperability with Discourse, pp. 178–186, Sofia, Bulgaria, August 2013. http://www.aclweb.org/anthology/W13-2322
4. Bos, J., Basile, V., Evang, K., Venhuizen, N., Bjerva, J.: The groningen meaning bank. In: Ide, N., Pustejovsky, J. (eds.) Handbook of Linguistic Annotation, vol. 2, pp. 463–496. Springer, Dordrecht (2017). https://doi.org/10.1007/978-94-024-0881-2_18
5. Butler, A.: The Semantics of Grammatical Dependencies, vol. 23. Emerald Group Publishing Limited (2010)
6. Fellbaum, C. (ed.): WordNet. An Electronic Lexical Database. The MIT Press (1998)
7. Heim, I.: The Semantics of Definite and Indefinite Noun Phrases. Ph.D. thesis, University of Massachusetts (1982)
8. Kamp, H.: A theory of truth and semantic representation. In: Groenendijk, J., Janssen, T.M., Stokhof, M. (eds.) Truth, Interpretation and Information, pp. 1–41. FORIS, Dordrecht, Holland/Cinnaminson, USA (1984)
9. Kamp, H., Reyle, U.: From Discourse to Logic; An Introduction to Modeltheoretic Semantics of Natural Language, Formal Logic and DRT. Kluwer, Dordrecht (1993)
10. Kipper, K., Korhonen, A., Ryant, N., Palmer, M.: A large-scale classification of English verbs. Lang. Resour. Eval. **42**(1), 21–40 (2008)
11. Montague, R.: The proper treatment of quantification in ordinary English. In: Hintikka, J., Moravcsik, J., Suppes, P. (eds.) Approaches to Natural Language, pp. 221–242. Reidel, Dordrecht (1973)

Proof Theory for Group-Like Structures

George Metcalfe

Mathematical Institute, University of Bern, Switzerland
george.metcalfe@math.unibe.ch

A central goal of structural proof theory is the development of analytic proof systems for logics and classes of structures that can be used to investigate their algorithmic and model-theoretic properties, notably, decidability and complexity bounds, (uniform) interpolation and amalgamation, and admissible rules and generation by subclasses. Although this endeavour has been successful for broad families of non-classical logics, it hits a roadblock when confronted with some of the most studied structures in mathematics, in particular, structures related in some way to groups. Not only is this an unfortunate limitation on the scope of proof-theoretic methods for tackling problems in algebra, these structures also serve as semantics for a wide range of substructural and many-valued logics.

In this talk, I will explore some recent attempts to address these limitations. First, I will explain how proof systems for classes of ordered groups introduced in [3, 5] relate to total orders on free groups [1, 2] and can be used to establish various decidability, complexity, and generation results. In the second part of the talk, I will consider how these results for ordered groups can be extended, via a Glivenko-style theorem, to classes of residuated lattices with close connections to BCI-algebras, Dubreil-Jacotin semigroups, and Casari's comparative logic [4]. Finally, I will describe some of the many open problems for this topic.

References

1. Colacito, A., Metcalfe, G.: Ordering groups and validity in lattice-ordered groups. J. Pure Appl. Algebra (to appear)
2. Colacito, A., Metcalfe, G.: Proof theory and ordered groups. In: Kennedy, J., de Queiroz, R. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 80–91. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_6
3. Galatos, N., Metcalfe, G.: Proof theory for lattice-ordered groups. Ann. Pure Appl. Log. **8**(167), 707–724 (2016)
4. Gil-Férez, J., Lauridsen, F.M., Metcalfe, G.: Self-cancellative residuated lattices (2019). http://arxiv.org/abs/1902.08144
5. Metcalfe, G., Olivetti, N., Gabbay, D.: Sequent and hypersequent calculi for abelian and Łukasiewicz logics. ACM Trans. Comput. Log. **6**(3), 578–613 (2005)

Logic, Lambdas, Vectors, and Concepts

Reinhard Muskens

Tilburg University
r.a.muskens@gmail.com
http://freevariable.nl

In this talk I will consider a range of approaches to modelling natural language meaning and explore possibilities to combine them. One approach I will consider is the one that I am most familiar with. It stems from Richard Montague's [4] observation that natural languages and logical languages can be treated on a par. Let's call this the *logical* approach. Another approach will be the *distributional* one, characterised by Firth's dictum that "you shall know a word by the company it keeps". A third category of approaches can be called *conceptual*. It includes forms of semantics that build on Peter Gärdenfors's [2] theory of "Conceptual Spaces", but also theories such as the one in Löbner [3], which is based on Barsalou's [1] frames.

Can these theories be combined? In particular, can work in the logical tradition be combined with any of the other theories? This would possibly be advantageous, as the virtues of the logical approach and any of the other approaches tend to be complementary. If a combined theory could be made to work, we could potentially profit from the best of two worlds.

One virtue that the logical tradition can bring to other approaches is *ease of composition*. In the talk I will emphasise that logical semantics in fact consists of two components. The first is the use of the (simply) typed lambda calculus as a composition engine. The second is logic in a more narrow sense, some theory of operators such as $\neg$. $\vee$, $\wedge$, $\forall$, $\exists$, $\Box$, $\Diamond$, and friends. It is entirely possible to have the first component without having the second and in fact in joint work with Mehrnoosh Sadrzadeh (e.g. [5]) we have used the lambda calculus to provide phrases with vector-based meanings on the basis of vector-based word meanings. The typed lambda calculus is a general theory of typed functions and in itself it is quite neutral with respect to the kind of functions it is applied to. It is also the case that many theories of syntax have a simple interface with semantics via lambdas. This means that once a semantic theory has been provided with a compositional mechanism via the lambda calculus, it will also connect with those syntactic formalisms.

I will explore to what extent this mechanism could also be put to use in the approaches to semantics I have dubbed "conceptual" and what may be good ways to combine the resulting compositional conceptual semantics with a logical semantics based on truth-conditions.

References

1. Barsalou, L.: Perceptual symbol systems. Behav. Brain Sci. **22**, 577–660 (1999)
2. Gärdenfors, P.: Conceptual Spaces: The Geometry of Thought. MIT Press (2004)
3. Löbner, S.: Functional concepts and frames. In: Gamerschlag, T., Gerland, D., Osswald, R., Petersen, W. (eds.) Meaning, Frames, and Conceptual Representation, pp. 15–42. Düsseldorf University Press, Düsseldorf (2015)
4. Montague, R.: The proper treatment of quantification in ordinary english. In: Hintikka, J., Moravcsik, J., Suppes, P. (eds.) Approaches to Natural Language, pp. 221–242. Reidel, Dordrecht (1973). Reprinted in [6]
5. Sadrzadeh, M., Muskens, R.: Static and dynamic vector semantics for lambda calculus models of natural language. J. Lang. Model. **6**(2), 319–351 (2019)
6. Thomason, R. (ed.): Formal Philosophy, Selected Papers of Richard Montague. Yale University Press (1974)

Contents

On Combinatorial Proofs for Logics of Relevance and Entailment

Matteo Acclavio[1(✉)] and Lutz Straßburger[2(✉)]

[1] Università Roma Tre, Rome, Italy
http://matteoacclavio.com/Math.html
[2] Inria Saclay, Palaiseau, France
http://www.lix.polytechnique.fr/Labo/Lutz.Strassburger/

Abstract. Hughes' combinatorial proofs give canonical representations for classical logic proofs. In this paper we characterize classical combinatorial proofs which also represent valid proofs for relevant logic with and without the mingle axiom. Moreover, we extend our syntax in order to represent combinatorial proofs for the more restrictive framework of entailment logic.

Keywords: Combinatorial proofs · Relevant logic · Entailment logic · Skew fibrations · Proof theory

1 Introduction

Combinatorial proofs have been conceived by Hughes [12] as a way to write *proofs* for classical propositional logic *without syntax*. Informally speaking, a combinatorial proof consists of two parts: first, a purely linear part, and second, a part that handles duplication and erasure. More formally, the first part is a variant of a proof net of multiplicative linear logic (MLL), and the second part is given by a skew fibration (or equivalently, a contraction-weakening map) from the cograph of the conclusion of the MLL proof net to the cograph of the conclusion of the whole proof. For the sake of a concise presentation, the MLL proof net is given as a cograph together with a perfect matching on the vertices of that graph. An important point is that in order to represent correct proofs, the proof nets have to obey a connectedness- and an acyclicity-condition.

To give an example, we show here the combinatorial proof of Pierce's law $((a \to b) \to a) \to a$, which can be written in negation normal form (NNF) as $(\bar{a} \vee b) \vee \bar{a}) \vee a$:

(1)

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 1–16, 2019.
https://doi.org/10.1007/978-3-662-59533-6_1

On the left above, we have written the conclusion of the proof as formula, and on the right as cograph, whose vertices are the atom occurrences of the formula, and whose edges are depicted as regular (red) lines. The linear part of the proof is given by the cograph determined by the four vertices and the regular (red) edge in the upper half of the diagram. The perfect matching is indicated by the bold (blue) edges. Finally, the downward arrows describe the skew fibration. In our example there is one atom in the conclusion (the a) that is the image of two vertices, indicating that it is the subject of a *contraction* in the proof. Then there is an atom (the b) that is not the image of any vertex above, indication that it is coming from a *weakening* in the proof.

Relevance logics have been studied by philosophers [2,3] to investigate when an implication is *relevant*, i.e., uses all its premises. In particular, in relevance logic, the implication $A \to (B \to A)$ is rejected because the B is not used to draw the conclusion A. In other words, we can no longer deduce A from $A \wedge B$. Put in proof theoretic terms, this corresponds to disallowing the *weakening* rule in a proof system. Carrying this observation to our combinatorial proofs mentioned above, this says that the skew fibration, that maps the linear part to the conclusion, must be surjective. The first contribution of this paper is to show that the converse also holds, provided that the surjection is with respect to the vertices *and* the edges. We will call such a skew fibration *relevant*. Then a classical combinatorial proof is a proof of relevance logic if and only if its skew fibration is relevant.

The mingle axiom is in its original form $A \to (A \to A)$ [2, p. 97]. In the implication-negation fragment of relevant logic, it can be derived from the more primitive form $A \to (B \to (\bar{B} \to A))$ (see also [2, p. 148]), which is equivalent to $(A \wedge B) \to (A \vee B)$, which is known as *mix* in the linear logic community. When mix is added to MLL, the connectedness-condition has to be dropped. This leads to the second result of this paper: adding mingle to relevance logic corresponds to dropping the connectedness condition from the combinatorial proofs.

Interestingly, Hughes' original version of combinatorial proofs included *mix* (i.e., there was no connectedness-condition). If *weakening* $\bot \to A$ is present, then mix is derivable, so that the presence or absence of mix does not have an effect on provability. However, when weakening is absent, as it is the case with many substructural logics, then mix/mingle has an impact on provability, and for this reason, we present combinatorial proofs in their basic form without mix, and follow the presentation in [18], using the notion of RB-cographs due to Retoré [16].

Entailment logic is a further refinement of relevance logic, insisting not only on the relevance of premises but also on their necessity (in the sense of the modal logic S4)[1]. More precisely, the logic rejects the implication $A \to ((A \to A) \to A)$. In terms of the sequent calculus, this means that the two sequents

[1] We do not discuss the philosophical considerations leading to this logic. For this, the reader is referred to the Book by Anderson and Belnap [2]. We take here the logic as given and discuss its proofs.

$$\Gamma \vdash A \to B \qquad \text{and} \qquad \Gamma, A \vdash B \tag{2}$$

are only equivalent if all formulas in the context Γ are of shape $C \to D$ for some C and D. If we write $A \to B$ as $\bar{A} \vee B$, then $\vee$ is not associative, as the rejected $A \to ((A \to A) \to A)$ would be written as $\bar{A} \vee ((A \wedge \bar{A}) \vee A)$, and the accepted $(A \to A) \to (A \to A)$ as $(A \wedge \bar{A}) \vee (\bar{A} \vee A)$. The consequence of this is that in combinatorial proofs we can no longer use simple cographs to encode formulas, as these identify formulas up to associativity and commutativity of $\wedge$ and $\vee$. We solve this problem by putting weights on the edges in the graphs. This leads us to our third contribution of this paper: combinatorial proofs for entailment logic.

Outline of the Paper. In this paper we study the *implication-negation-fragment* of relevance logic. For this, we recall in Sect. 2 the corresponding sequent calculi, following the presentation in [14] and [5]. Then, in Sect. 3 we introduce another set of sequent calculi, working with formulas in NNF, and we show the equivalence of the two presentations. The NNF presentation allows us to reuse standard results from linear logic. In Sects. 4 and 5, we introduce cographs and skew fibrations, so that in Sect. 6 we can finally define combinatorial proofs for relevance logic with and without mingle. Then, in Sects. 7 and 8 we extend our construction to the logic of entailment.

2 Sequent Calculus, Part I

In this section we recall the sequent calculi for the implication-negation-fragment of relevance logic (denoted by $\mathbf{R}_{\overset{\sim}{\to}}$), of relevance logic with mingle (denoted by $\mathbf{RM}_{\overset{\sim}{\to}}$), of entailment logic (denoted by $\mathbf{E}_{\overset{\sim}{\to}}$), and classical propositional logic (denoted by $\mathbf{CL}_{\overset{\sim}{\to}}$).

For this, we consider the class $\mathcal{I}$ of *formulas* (denoted by $A, B, \ldots$) generated by a countable set $\mathcal{A} = \{a, b, \ldots\}$ of *propositional variables* and the connectives $\to$ and $\bar{(\cdot)}$ by the following grammar:

$$A, B ::= a \mid \bar{A} \mid A \to B \tag{3}$$

A *sequent* Γ in $\mathcal{I}$ is a multiset of occurrence of formulas, written as list and separated by commas: $\Gamma = A_1, \ldots, A_n$. We denote by $\overline{\Gamma^{\to}}$ a sequent of formulas in $\mathcal{I}$ of the form $\overline{A_1 \to B_1}, \ldots, \overline{A_n \to B_n}$, and we write $\overline{\Gamma}$ for the sequent obtained from Γ by negating all its formulas, i.e., if $\Gamma = A_1, \ldots, A_n$ then $\overline{\Gamma} = \bar{A}_1, \ldots, \bar{A}_n$.

In Fig. 1 we give the standard sequent systems for the logics $\mathbf{E}_{\overset{\sim}{\to}}, \mathbf{R}_{\overset{\sim}{\to}}, \mathbf{RM}_{\overset{\sim}{\to}}$, and $\mathbf{CL}_{\overset{\sim}{\to}}$ as given in [5,14].

Theorem 2.1. *A formula is a theorem of the logic $\mathbf{E}_{\overset{\sim}{\to}}$ (resp. $\mathbf{R}_{\overset{\sim}{\to}}, \mathbf{RM}_{\overset{\sim}{\to}}$, $\mathbf{CL}_{\overset{\sim}{\to}}$) iff it is derivable in the sequent calculus $\mathsf{LE}_{\overset{\sim}{\to}}$, (resp. $\mathsf{LR}_{\overset{\sim}{\to}}, \mathsf{LRM}_{\overset{\sim}{\to}}$, $\mathsf{LK}_{\overset{\sim}{\to}}$) [14].*

Observe that the system $\mathsf{LE}_{\overset{\sim}{\to}}$ in Fig. 1 does contain the cut-rule, whereas the other systems are cut-free. The reason is that due to the form of the EAX, the cut cannot be eliminated.

In order to obtain cut-free systems for all four logics, we move to the negation normal form in the next section.

$\mathbf{E}_{\tilde{\rightarrow}}$	$\mathsf{LE}_{\tilde{\rightarrow}}$	$\mathsf{AX}, \mathsf{EAX}, \rightarrow^{\mathsf{E}}, \neg, \mathsf{C}, \mathsf{cut}$
$\mathbf{R}_{\tilde{\rightarrow}}$	$\mathsf{LR}_{\tilde{\rightarrow}}$	$\mathsf{AX}, \rightarrow, \overline{\rightarrow}, \neg, \mathsf{C}$
$\mathbf{RM}_{\tilde{\rightarrow}}$	$\mathsf{LRM}_{\tilde{\rightarrow}}$	$\mathsf{AX}, \rightarrow, \overline{\rightarrow}, \neg, \mathsf{C}, \mathsf{mAX}$
$\mathbf{CL}_{\tilde{\rightarrow}}$	$\mathsf{LK}_{\tilde{\rightarrow}}$	$\mathsf{AX}, \rightarrow, \overline{\rightarrow}, \neg, \mathsf{C}, \mathsf{W}$

$$\frac{}{A, \bar{A}}\,\mathsf{AX} \qquad \frac{}{\bar{A}, \overline{A \rightarrow B}, B}\,\mathsf{EAX} \qquad \frac{}{\Gamma, \bar{\Gamma}}\,\mathsf{mAX}$$

$$\frac{\Gamma, A}{\Gamma, \bar{\bar{A}}}\,\neg \qquad \frac{\Gamma, A \quad \bar{B}, \Delta}{\Gamma, \overline{A \rightarrow B}, \Delta}\,\overline{\rightarrow} \qquad \frac{\Gamma, \bar{A}, B}{\Gamma, A \rightarrow B}\,\rightarrow$$

$$\frac{\Gamma, A, A}{\Gamma, A}\,\mathsf{C} \qquad \frac{\overline{\Gamma^{\rightarrow}}, \bar{A}, B}{\overline{\Gamma^{\rightarrow}}, A \rightarrow B}\,\rightarrow^{\mathsf{E}} \qquad \frac{\Gamma}{\Gamma, A}\,\mathsf{W}$$

$$\frac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, \Delta}\,\mathsf{cut}$$

Fig. 1. Rules for the standard sequent systems for the logics $\mathbf{E}_{\tilde{\rightarrow}}$, $\mathbf{R}_{\tilde{\rightarrow}}$, $\mathbf{RM}_{\tilde{\rightarrow}}$, and $\mathbf{CL}_{\tilde{\rightarrow}}$.

3 Sequent Calculus, Part II

In this section we consider formulas in negation normal form (NNF), i.e., the class $\mathcal{L}$ of *formulas* (also denoted by $A, B, \ldots$) generated by the countable set $\mathcal{A} = \{a, b, \ldots\}$ of *propositional variables*, their duals $\bar{\mathcal{A}} = \{\bar{a}, \bar{b}, \ldots\}$, and the binary connectives $\wedge$ and $\vee$, via the following grammar:

$$A, B ::= a \mid \bar{a} \mid A \wedge B \mid A \vee B \tag{4}$$

An *atom* is a formula a or $\bar{a}$ with $a \in \mathcal{A}$. As before, a sequent Γ is a multiset of formulas separated by comma. We define *negation* as a function on all formulas in NNF via the De Morgan laws:

$$\bar{\bar{a}} = a \qquad \overline{A \wedge B} = \bar{A} \vee \bar{B} \qquad \overline{A \vee B} = \bar{A} \wedge \bar{B} \tag{5}$$

There is a correspondence between the class $\mathcal{I}$ defined in the previous section and the class $\mathcal{L}$ of formulas in NNF, defined via the two translations $[\cdot]_{\mathcal{L}} \colon \mathcal{I} \rightarrow \mathcal{L}$ and $[\cdot]_{\mathcal{I}} \colon \mathcal{L} \rightarrow \mathcal{I}$:

$$[a]_{\mathcal{L}} = a, \quad [\bar{A}]_{\mathcal{L}} = \overline{[A]_{\mathcal{L}}}, \quad [A \rightarrow B]_{\mathcal{L}} = \overline{[A]_{\mathcal{L}}} \vee [B]_{\mathcal{L}} \tag{6}$$

and

$$[a]_{\mathcal{I}} = a,\ [\bar{a}]_{\mathcal{I}} = \bar{a},\ [A \vee B]_{\mathcal{I}} = \overline{[A]_{\mathcal{I}}} \rightarrow [B]_{\mathcal{I}},\ [A \wedge B]_{\mathcal{I}} = \overline{[A]_{\mathcal{I}} \rightarrow \overline{[B]_{\mathcal{I}}}} \tag{7}$$

Proposition 3.1. *If A is a formula in NNF, then $[[A]_{\mathcal{I}}]_{\mathcal{L}} = A$.*

The proof is straightforward, but in general we do not have $[[B]_{\mathcal{L}}]_{\mathcal{I}} = B$ for formulas in $\mathcal{I}$, since we can have arbitrary nestings of negation and $[\bar{\bar{B}}]_{\mathcal{L}} = [B]_{\mathcal{L}}$. For this reason, we use here the NNF notation, as it is more concise and carries less redundancy.

We can use this correspondence to translate the sequent systems in Fig. 1 into the NNF notation. We go one step further and give cut-free systems LE', LR', LRM', and LK. They are given in Fig. 2, where we denote by $\Gamma^{\wedge}$ a sequent of the form $A_1 \wedge B_1, \ldots, A_n \wedge B_n$ (i.e., all formulas in $\Gamma^{\wedge}$ are conjunctions). That figure also defines the linear logic systems MLL, $\mathsf{MLL}_{\mathsf{mix}}$, and $\mathsf{MLL}_{\mathsf{E}}$ that we will need in the course of this paper.

MLL	$\mathsf{ax}, \vee, \wedge$
$\mathsf{MLL}_{\mathsf{mix}}$	$\mathsf{ax}, \vee, \wedge, \mathsf{mix}$
$\mathsf{MLL}_{\mathsf{E}}$	$\mathsf{ax}, \wedge, \vee_{\mathsf{E}}$
LE'	$\mathsf{ax}, \wedge, \vee_{\mathsf{E}}, \mathsf{C}$
LR'	$\mathsf{ax}, \vee, \wedge, \mathsf{C}$
LRM'	$\mathsf{ax}, \vee, \wedge, \mathsf{C}, \mathsf{mix}$
LK	$\mathsf{ax}, \vee, \wedge, \mathsf{C}, \mathsf{W}$

$$\frac{}{a, \bar{a}}\,\mathsf{ax} \qquad \frac{\Gamma^{\wedge}, A, B}{\Gamma^{\wedge}, A \vee B}\,\vee_{\mathsf{E}} \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\,\vee$$

$$\frac{\Gamma, A, A}{\Gamma, A}\,\mathsf{C} \qquad \frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \wedge B}\,\wedge$$

$$\frac{\Gamma}{\Gamma, A}\,\mathsf{W} \qquad \frac{\Gamma \quad \Delta}{\Gamma, \Delta}\,\mathsf{mix}$$

Fig. 2. The cut-free sequent systems for formulas in NNF

$$\frac{\Gamma\{(A \vee B) \wedge (C \vee D)\}}{\Gamma\{(A \vee C) \vee (B \vee D)\}}\,\mathsf{m}^{\downarrow} \qquad \frac{\Gamma\{A \vee A\}}{\Gamma\{A\}}\,\mathsf{C}^{\downarrow} \qquad \frac{\Gamma\{a \vee a\}}{\Gamma\{a\}}\,\mathsf{ac}^{\downarrow} \qquad \frac{\Gamma\{B\}}{\Gamma\{B \vee A\}}\,\mathsf{W}^{\downarrow}$$

Fig. 3. The *deep* rules for *medial*, *contraction*, *atomic contraction* and *weakening*.

We make also use of the *deep inference* rules in Fig. 3 (see also [6,10]), where a *context* $\Gamma\{\ \}$ is a sequent or a formula, where a hole $\{\ \}$ takes the place of an atom. We write $\Gamma\{A\}$ when we replace the hole with a formula A.

If S is a sequent system and Γ a sequent, we write $\overset{\mathsf{S}}{\vdash} \Gamma$ if Γ is derivable in S. Moreover, if S is a set of inference rules with exactly one premise, we write $\Gamma' \overset{\mathsf{S}}{\vdash} \Gamma$ whenever there is a derivation from Γ' to Γ using only rules in S.

Lemma 3.2. *If Γ is a sequent in $\mathcal{L}$, then*

$$\overset{\mathsf{LE}' \cup \{\mathsf{cut}\}}{\vdash} \Gamma \qquad \Longleftrightarrow \qquad \overset{\mathsf{LE}_{\to}}{\vdash} [\Gamma]_{\mathcal{I}}$$

Proof. The proof follows the definitions of the two translations $[\cdot]_{\mathcal{I}}$ and $[\cdot]_{\mathcal{L}}$. In fact, C- and cut-rules are the same in the two systems and $\to$-rule is equivalent to $\vee_{\mathsf{E}}$-rule. Moreover, it s trivial to prove by induction that $\overset{\mathsf{LE}'}{\vdash} A, \bar{A}$. Finally:

$$\frac{}{\bar{A}, \overline{A \to B}, B}\,\mathsf{EAX} \rightsquigarrow \frac{\begin{array}{c}\Vert\,{\scriptstyle \mathsf{LE}'}\\ [\bar{A}]_{\mathcal{L}}, [A]_{\mathcal{L}}\end{array} \quad \begin{array}{c}\Vert\,{\scriptstyle \mathsf{LE}'}\\ [\bar{B}]_{\mathcal{L}}, [B]_{\mathcal{L}}\end{array}}{[\bar{A}]_{\mathcal{L}}, ([A]_{\mathcal{L}} \wedge [\bar{B}]_{\mathcal{L}}), [B]_{\mathcal{L}}}\,\wedge_{\mathsf{E}}$$

and

$$\frac{\begin{array}{c}\Vert\,{\scriptstyle \mathsf{LE}'}\\ \Gamma, A\end{array} \quad \begin{array}{c}\Vert\,{\scriptstyle \mathsf{LE}'}\\ B, \Delta\end{array}}{\Gamma, A \wedge B, \Delta}\,\wedge_{\mathsf{E}} \rightsquigarrow \frac{\dfrac{\dfrac{}{\overline{[A]_{\mathcal{I}}}, \overline{[A]_{\mathcal{I}} \to \overline{\overline{[B]_{\mathcal{I}}}}}, \overline{[B]_{\mathcal{I}}}}\,\mathsf{EAX} \quad \begin{array}{c}\Vert\,{\scriptstyle \mathsf{LE}_{\to}}\\ [A]_{\mathcal{I}}, [\Gamma]_{\mathcal{I}}\end{array}}{[\Gamma]_{\mathcal{I}}, \overline{[A]_{\mathcal{I}} \to \overline{\overline{[B]_{\mathcal{I}}}}}, \overline{[B]_{\mathcal{I}}}}\,\mathsf{cut} \quad \begin{array}{c}\Vert\,{\scriptstyle \mathsf{LE}_{\to}}\\ [B]_{\mathcal{I}}, [\Delta]_{\mathcal{I}}\end{array}}{[\Gamma]_{\mathcal{I}}, \overline{[A]_{\mathcal{I}} \to \overline{\overline{[B]_{\mathcal{I}}}}}, [\Delta]_{\mathcal{I}}}\,\mathsf{cut}$$

$\square$

One important property of the systems in Fig. 2 is cut admissibility.

Theorem 3.3 (Cut admissibility). *Let Γ be a sequent in $\mathcal{L}$, and let* S *be one of the systems* $\mathsf{MLL}, \mathsf{MLL}_{\mathsf{mix}}, \mathsf{MLL}_{\mathsf{E}}, \mathsf{LE}', \mathsf{LR}', \mathsf{LRM}', \mathsf{LK}$. *Then*

$$\overset{\mathsf{S}\cup\{\mathsf{cut}\}}{\vdash} \Gamma \iff \overset{\mathsf{S}}{\vdash} \Gamma$$

Proof. The proof is a standard cut permutation argument. For LK it can already be found in [9] and for all other systems it is the same proof, observing that no reduction step introduces a rule that is not present in the system. $\square$

The following lemma relates the mix-rule from linear logic to the mingle axiom rule mAX:

Lemma 3.4. *Let* S *be a sequent system, if Γ is a sequent in $\mathcal{L}$ then*

$$\overset{\mathsf{S}\cup\{\mathsf{mAX}\}}{\vdash} F \iff \overset{\mathsf{S}\cup\{\mathsf{mix}\}}{\vdash} F$$

Proof. First, mAX can be derived using mix:

$$\dfrac{}{A_1, \ldots, A_n, \bar{A}_1, \ldots, \bar{A}_n}\,\mathsf{mAX} \rightsquigarrow \dfrac{\begin{array}{c}\dfrac{\overset{\mathsf{S}\,\|}{A_1, \bar{A}_1} \quad \overset{\mathsf{S}\,\|}{A_2, \bar{A}_2}}{A_1, \bar{A}_1, A_2, \bar{A}_2}\,\mathsf{mix} \\ \Big\| \mathsf{S}\cup\{\mathsf{mix}\} \\ A_1, \bar{A}_1, \ldots, A_{n-1}, \bar{A}_{n-1}\end{array} \quad \overset{\mathsf{S}\,\|}{A_n, \bar{A}_n}}{A_1, \bar{A}_1, \ldots, A_n, \bar{A}_n}\,\mathsf{mix}$$

Conversely, if Γ, Δ is the conclusion of a mix inference,

$$\dfrac{\begin{array}{c}\dfrac{}{a_1, \bar{a}_1}\,\mathsf{ax} \;\ldots\; \dfrac{}{a_n, \bar{a}_n}\,\mathsf{ax} \\ \bigtriangledown\,\mathsf{S} \\ \Gamma\end{array} \qquad \begin{array}{c}\dfrac{}{a_{n+1}, \bar{a}_{n+1}}\,\mathsf{ax} \;\ldots\; \dfrac{}{a_{n+m}, \bar{a}_{n+m}}\,\mathsf{ax} \\ \bigtriangledown\,\mathsf{S} \\ \Delta\end{array}}{\Gamma, \Delta}\,\mathsf{mix}$$

it suffices to replace one axiom of the derivation of Γ and one axiom of the derivation of Δ by a single mAX, that is

$$\begin{array}{c}\dfrac{}{a_2, \bar{a}_2}\,\mathsf{ax} \;\ldots\; \dfrac{}{a_{n+m-1}, \bar{a}_{n+m-1}}\,\mathsf{ax} \quad \dfrac{}{a_1, \bar{a}_1, a_{n+m}, \bar{a}_{n+m}}\,\mathsf{mAX} \\ \bigtriangledown\,\mathsf{S} \\ \Gamma, \Delta\end{array}$$

$\square$

This is enough to show the equivalence between the systems in Figs. 1 and 2.

Theorem 3.5. *If Γ is a sequent in $\mathcal{L}$, then*

$$\overset{\mathsf{LE}'}{\vdash} \Gamma \iff \overset{\mathsf{LE}_{\parr}}{\vdash} [\Gamma]_{\mathcal{I}} \qquad \overset{\mathsf{LR}'}{\vdash} \Gamma \iff \overset{\mathsf{LR}_{\parr}}{\vdash} [\Gamma]_{\mathcal{I}}$$
$$\overset{\mathsf{LRM}'}{\vdash} \Gamma \iff \overset{\mathsf{LRM}_{\parr}}{\vdash} [\Gamma]_{\mathcal{I}} \qquad \overset{\mathsf{LK}}{\vdash} \Gamma \iff \overset{\mathsf{LK}_{\parr}}{\vdash} [\Gamma]_{\mathcal{I}}$$

Proof. This follows from Lemma 3.2, Theorem 3.3, and Lemma 3.4, using the definitions of $[\cdot]_{\mathcal{I}}$ and $[\cdot]_{\mathcal{L}}$. $\square$

Finally, the most important reason to use the systems in Fig. 2 instead of the ones in Fig. 1 is the following decomposition theorem:

Theorem 3.6. *If Γ is a sequent in $\mathcal{L}$, then*

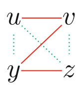
$$\overset{\mathsf{LE}'}{\vdash} \Gamma \iff \overset{\mathsf{MLL_E}}{\vdash} \Gamma' \overset{\mathsf{C}^{\downarrow}}{\vdash} \Gamma \qquad \overset{\mathsf{LR}'}{\vdash} \Gamma \iff \overset{\mathsf{MLL}}{\vdash} \Gamma' \overset{\mathsf{C}^{\downarrow}}{\vdash} \Gamma$$
$$\overset{\mathsf{LRM}'}{\vdash} \Gamma \iff \overset{\mathsf{MLL_{mix}}}{\vdash} \Gamma' \overset{\mathsf{C}^{\downarrow}}{\vdash} \Gamma \qquad \overset{\mathsf{LK}}{\vdash} \Gamma \iff \overset{\mathsf{MLL}}{\vdash} \Gamma' \overset{\mathsf{C}^{\downarrow},\mathsf{W}^{\downarrow}}{\vdash} \Gamma$$

Proof. The proof is given by rules permutation. It suffices to consider all W- and C-rules as their deep counterpart and move their instance as down as possible in the derivation. Conversely, it suffices to move up all occurrences of $\mathsf{C}^{\downarrow}$ and $\mathsf{W}^{\downarrow}$ until the context is shallow and then replace them by C and W instances. This permutation works because for each instance of $\mathsf{C}^{\downarrow}$ or $\mathsf{W}^{\downarrow}$ that occurs in the proof, the principle formula is a subformula of the conclusion of the proof. $\square$

4 Cographs

A *graph* $\mathcal{G} = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown} \rangle$ is a set $V_{\mathcal{G}}$ *vertices* and a set $\overset{\mathcal{G}}{\frown}$ of *edges*, which are two-element subsets of $V_{\mathcal{G}}$. We write $v \overset{\mathcal{G}}{\frown} w$ for $\{v, w\} \in \overset{\mathcal{G}}{\frown}$, and we write $v \overset{\mathcal{G}}{\not\frown} w$ if $\{v, w\} \notin \overset{\mathcal{G}}{\frown}$. We omit the index/superscript $\mathcal{G}$ when it is clear from the context. When drawing a graph we use v—w for $v \frown w$. If $v \not\frown w$ and $v \neq w$ we either draw no edge or use $v \cdots w$.

A *cograph* $\mathcal{G}$ is a P4-free graph, i.e. a graph $\mathcal{G}$ with no $u, v, y, z \in V$ such that their induced subgraph has the following shape:[2]

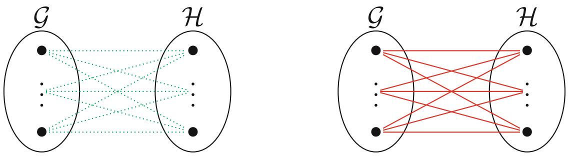


For two disjoint graphs $\mathcal{G}$ and $\mathcal{H}$, we define their *(disjoint) union* $\mathcal{G} \vee \mathcal{H}$ and their *join* $\mathcal{G} \wedge \mathcal{H}$ as follows:

$$\begin{aligned} \mathcal{G} \vee \mathcal{H} &= \langle V_{\mathcal{G}} \cup V_{\mathcal{H}} \ , \ \overset{\mathcal{G}}{\frown} \cup \overset{\mathcal{H}}{\frown} \rangle \\ \mathcal{G} \wedge \mathcal{H} &= \langle V_{\mathcal{G}} \cup V_{\mathcal{H}} \ , \ \overset{\mathcal{G}}{\frown} \cup \overset{\mathcal{H}}{\frown} \cup \{\{u, v\} \mid u \in V_{\mathcal{G}}, v \in V_{\mathcal{H}}\} \rangle \end{aligned} \tag{8}$$

[2] In the literature, this condition is also called Z-free or N-free.

which can be visualized as follows:



We say that a graph is $\mathcal{A}$*-labeled* if each vertex is marked with an atom in $\mathcal{A} \cup \bar{\mathcal{A}}$. We can associate to each formula F in $\mathcal{L}$ an $\mathcal{A}$-labeled cograph $[\![F]\!]$ inductively:

$$[\![a]\!] = \bullet_a, \quad [\![\bar{a}]\!] = \bullet_{\bar{a}}, \quad [\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!], \quad [\![A \wedge B]\!] = [\![A]\!] \wedge [\![B]\!]$$

If $\Gamma = A_1, \ldots, A_n$ is a sequent of formulas in $\mathcal{L}$, we define $[\![\Gamma]\!] = [\![A_1]\!] \vee \cdots \vee [\![A_n]\!]$.

The interest in cographs comes from the following two well-known theorems (see, e.g., [8,15]).

Theorem 4.1. *A $\mathcal{A}$-labeled graphs $\mathcal{G}$ is a cograph iff there is a formula $F \in \mathcal{L}$ such that $\mathcal{G} = [\![F]\!]$.*

Theorem 4.2. *$[\![F]\!] = [\![F']\!]$ iff F and F' are equivalent modulo associativity and commutativity of $\wedge$ and $\vee$.*

5 Skew Fibrations

Definition 5.1. Let $\mathcal{G}$ and $\mathcal{H}$ be graphs. A *skew fibration* $f \colon \mathcal{G} \to \mathcal{H}$ is a mapping from $V_{\mathcal{G}}$ to $V_{\mathcal{H}}$ that preserves $\frown$:

- if $u \overset{\mathcal{G}}{\frown} v$ then $f(u) \overset{\mathcal{H}}{\frown} f(v)$,

and that has the *skew lifting* property:

- if $w \overset{\mathcal{H}}{\frown} f(v)$ then there is $u \in V_{\mathcal{G}}$ such that $u \overset{\mathcal{G}}{\frown} v$ and $w \overset{\mathcal{H}}{\not\frown} f(u)$.

A skew fibration $f \colon \mathcal{G} \to \mathcal{H}$ is *relevant* if it is surjective on vertices and on $\frown$:

- for every $w \in V_{\mathcal{H}}$ there is a $u \in V_{\mathcal{G}}$ such that $f(u) = w$, and
- if $w \overset{\mathcal{H}}{\frown} t$ then there are $u, v \in V_{\mathcal{G}}$ such that $f(u) = w$ and $f(v) = t$ and $u \overset{\mathcal{G}}{\frown} v$.

The purpose of skew fibrations in this setting is to give a combinatorial characterization of derivations containing only contractions and weakenings.

Theorem 5.2. *If Γ, Γ' are sequents in $\mathcal{L}$ then*

1. *$\Gamma' \overset{\mathsf{c}^{\downarrow},\mathsf{w}^{\downarrow}}{\vdash} \Gamma$ iff there is a skew fibration $f \colon [\![\Gamma']\!] \to [\![\Gamma]\!]$.*
2. *$\Gamma' \overset{\mathsf{c}^{\downarrow}}{\vdash} \Gamma$ iff there is a relevant skew fibration $f \colon [\![\Gamma']\!] \to [\![\Gamma]\!]$.*

Proof. The first statement has been proved independently in [13] and in [17]. The proof of the second statement is similar, but the relevant condition rules out weakening. Let $\Gamma' = \Gamma_0, \Gamma_1, \ldots, \Gamma_n = \Gamma$ such that $\dfrac{\Gamma_i\{A_i \vee A_i\}}{\Gamma_{i+1} = \Gamma_i\{A_i\}}\,\mathsf{C}^{\downarrow}$. By induction over the size of A_i, there is a relevant skew fibration $f_i \colon \Gamma_i \to \Gamma_{i+1}$ for each $i \in \{0, \ldots n-1\}$ and the composition of such f_i is still a relevant skew fibration. Conversely, in case of f a relevant skew fibration, the lifting property becomes the following:

– if $w \overset{\mathcal{H}}{\frown} f(v)$ then there is $u \in V_{\mathcal{G}}$ such that $u \overset{\mathcal{G}}{\frown} v$ and $f(u) = w$.

which, by induction over Γ, allows to prove that:

– if $[\![\Gamma]\!] = \bullet_a$ then $[\![\Gamma']\!] = \bullet_a \vee \cdots \vee \bullet_a$;
– if $[\![\Gamma]\!] = \mathcal{G} \vee \mathcal{H}$ then $[\![\Gamma']\!] = \mathcal{G}' \vee \mathcal{H}'$ with $f(\mathcal{G}') = \mathcal{G}$ and $f(\mathcal{H}') = \mathcal{H}$;
– if $[\![\Gamma]\!] = \mathcal{G} \wedge \mathcal{H}$ then either $[\![\Gamma']\!] = \mathcal{G}' \wedge \mathcal{H}'$ with $f(\mathcal{G}') = \mathcal{G}$ and $f(\mathcal{H}') = \mathcal{H}$, or $[\![\Gamma']\!] = (\mathcal{G}'_1 \wedge \mathcal{H}'_1) \vee \cdots \vee (\mathcal{G}'_n \wedge \mathcal{H}'_n)$ with $f(\mathcal{G}'_i) = \mathcal{G}$ and $f(\mathcal{H}'_i) = \mathcal{H}$ for each $i \in \{1, \ldots, n\}$.

These decompositions guide the definition of a derivation $\Gamma' \overset{\mathsf{C}^{\downarrow}}{\longmapsto} \Gamma$. □

6 RB-cographs and Combinatorial Proofs

In this section we finally define combinatorial proofs. For this we use Retoré's RB-cographs [16]:

Definition 6.1 ([16]). An *RB-cograph* is a tuple $\mathcal{G} = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown}, \overset{\mathcal{G}}{\curlyvee} \rangle$ where $\mathcal{G}^R = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown} \rangle$ is a cograph and $\overset{\mathcal{G}}{\curlyvee}$ a irreflexive, symmetric binary relation such that for every $v \in V_{\mathcal{G}}$ there is a unique $w \in V_{\mathcal{G}}$ with $v \overset{\mathcal{G}}{\curlyvee} w$.

As done in (1) in the introduction, we use v—w for $v \frown w$, and v**—**w for $v \curlyvee w$ when drawing an RB-cograph.

Definition 6.2 ([16]). If u and v are two vertices of a RB-cograph, an *alternating elementary path (æ-path) from x_0 to x_n* is a sequence of pairwise disjoint vertices $x_0, \ldots, x_n \in V$ such that either $x_0 \frown x_1 \curlyvee x_2 \frown x_3 \curlyvee x_4 \cdots x_n$ or $x_0 \curlyvee x_1 \frown x_2 \curlyvee x_3 \frown x_4 \cdots x_n$. An *æ-cycle* is an æ-path of even length with $x_0 = x_n$. A *chord* of æ-path $x_0, \ldots, x_n$ is an edge $x_i \frown x_j$ with $i + 1 < j$. The æ-path is *chordless* if it has no chord. A RB-cograph is *æ-connected* if there is a chordless æ-path between each pair of vertices $\mathcal{G}$ and it is *æ-acyclic* if there are no chordless æ-cycle.

Theorem 6.3 ([16]). *If Γ is a sequent over $\mathcal{L}$ then*

1. $\overset{\mathsf{MLL}}{\longmapsto} \Gamma \iff$ *there is an æ-connected, æ-acyclic RB-cograph $\mathcal{G}$ with $\mathcal{G}^R = [\![\Gamma]\!]$*
2. $\overset{\mathsf{MLL}_{\mathsf{mix}}}{\longmapsto} \Gamma \iff$ *there is an æ-acyclic RB-cograph $\mathcal{G}$ with $\mathcal{G}^R = [\![\Gamma]\!]$*

We say that a map f from an RB-cograph $\mathcal{C}$ to a $\mathcal{A}$-labeled cograph is *axiom preserving* if for all u, v with $u \overset{\mathcal{C}}{\curlyvee} v$ we have that $f(u)$ and $f(v)$ are labeled by two dual atoms.

Definition 6.4. Let Γ be a sequent over $\mathcal{L}$.

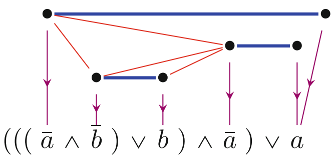
1. A *combinatorial* LK*-proof* of Γ is an axiom-preserving skew fibration $f : \mathcal{C} \to [\![\Gamma]\!]$ where $\mathcal{C}$ is an æ-connected, æ-acyclic RB-cograph.
2. A *combinatorial* LR'*-proof* of Γ is an is an axiom-preserving relevant skew fibration $f : \mathcal{C} \to [\![\Gamma]\!]$ where $\mathcal{C}$ is an æ-connected, æ-acyclic RB-cograph.
3. Finally, a *combinatorial* LRM'*-proof* of Γ is an is an axiom-preserving relevant skew fibration $f : \mathcal{C} \to [\![\Gamma]\!]$ where $\mathcal{C}$ is an æ-acyclic RB-cograph.

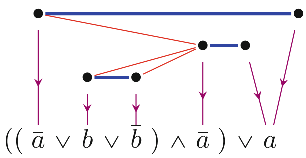
Theorem 6.5. *Let Γ be a sequent over $\mathcal{L}$, and let $\mathsf{S} \in \{\mathsf{LR}', \mathsf{LRM}', \mathsf{LK}\}$. Then*

$$\overset{\mathsf{S}}{\longmapsto} \Gamma \iff \text{there is a combinatorial } \mathsf{S}\text{-proof of } \Gamma.$$

Proof. This follows from Theorems 3.6, 5.2 and 6.3. For LK this has already been shown in [12, 13, 17, 18]. □

Below are a combinatorial LR'-proof (on the left) and a combinatorial LRM'-proof (on the right):





Theorem 6.6. *Let Γ be a sequent and $\mathcal{G}$ a graph together with a binary relation on its vertices, and let f be a map from $\mathcal{G}$ to $[\![\Gamma]\!]$. It can be decided in polynomial time in $|V_{\mathcal{G}}| + |\Gamma|$ whether $f : \mathcal{G} \to [\![\Gamma]\!]$ is a combinatorial LR'-proof (resp. a combinatorial LRM'-proof).*

Proof. All necessary properties can be checked in polynomial time. □

7 Sequent Calculus, Part III

In the remainder of the paper, we extend our results to the *entailment logic* $\mathbf{E}_{\bar{\circ}}$. The reason why we need a separate treatment is due to some intrinsic technical drawbacks occurring in the LE' sequent calculus. The first is that commas used to separate formulas in a sequent can not be interpreted as disjunction, as we usually do in classical logic. Using the *display calculi* [4] terminology, in LE' the comma is extensional while $\vee$ and $\wedge$ are intensional. Moreover, $\wedge$ and $\vee$ are not associative and this give birth to unusual behaviors. For example $(A \vee A) \vee (\bar{A} \wedge \bar{A})$ is provable in LE' while $A \vee (A \vee (\bar{A} \wedge \bar{A}))$ is not.

We first introduce the class of *entailed formulas* $\mathcal{E}$ which are generated by a countable set $\mathcal{A} = \{a, b, \dots\}$ of *propositional variables* and the following grammar:

$$A, B ::= a \mid \bar{a} \mid A \wedge B \mid A \vee B \mid A^n \tag{9}$$

where $n > 0$. Moreover, we consider the sequents $\Gamma\{A^{n+1}\}$ and $\Gamma\{A, A^n\}$ to be equal. In other words, A^n has to be thought of as an abbreviation for the sequent $A, \dots, A$ (n copies of A) that is allowed to occur as a subformula in a formula. We define the sequent systems $\mathsf{MLL}^{\bullet}_{\mathsf{E}}$ and $\mathsf{LE}^{\bullet}$ on entailed formulas given by the rules in Fig. 4.

$\mathsf{MLL}^{\bullet}_{\mathsf{E}}$	$\mathsf{ax}, \wedge^{\bullet}_{\mathsf{E}}, \vee^{\bullet}_{\mathsf{E}}$
$\mathsf{LE}^{\bullet}$	$\mathsf{ax}, \wedge^{\bullet}_{\mathsf{E}}, \vee^{\bullet}_{\mathsf{E}}, \mathsf{C}^{\bullet}_{\mathsf{E}}$

$$\frac{}{a, \bar{a}}\ \mathsf{ax} \qquad \frac{\Gamma, A^n \quad B^m, \Delta}{\Gamma, A^n \wedge B^m, \Delta}\ \wedge^{\bullet}_{\mathsf{E}} \qquad \frac{\Gamma^{\wedge}, A^n, B^m}{\Gamma^{\wedge}, A^n \vee B^m}\ \vee^{\bullet}_{\mathsf{E}} \qquad \frac{\Gamma, A^n}{\Gamma, A}\ \mathsf{C}^{\bullet}_{\mathsf{E}}$$

Fig. 4. The cut-free systems $\mathsf{MLL}^{\bullet}_{\mathsf{E}}$ and $\mathsf{LE}^{\bullet}$

Theorem 7.1. *If Γ is a sequent over $\mathcal{L}$ then*

$$\overset{\mathsf{LE}'}{\vdash} \Gamma \iff \overset{\mathsf{LE}^{\bullet}}{\vdash} \Gamma$$

Proof. It suffices to remark that $\mathsf{LE}^{\bullet}$ rules behave as LE' rules on standard NNF-formulas. ☐

Let $\mathsf{C}^{\downarrow}_{\mathsf{E}}$ be the deep inference rule $\dfrac{F\{A^n\}}{F\{A\}}\ \mathsf{C}^{\downarrow}_{\mathsf{E}}$. Then we have a result similar to Theorem 3.6.

Theorem 7.2. *If F is a formula in $\mathcal{E}$ then*

$$\overset{\mathsf{LE}^{\bullet}}{\vdash} F \qquad \iff \qquad \overset{\mathsf{MLL}^{\bullet}_{\mathsf{E}}}{\vdash} F' \overset{\mathsf{C}^{\downarrow}_{\mathsf{E}}}{\vdash} \Gamma$$

Proof. By rule permutations, similarly to the proof of Theorem 3.6. ☐

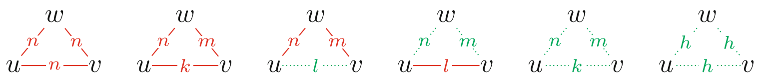
8 Weighted Cographs and Fibrations

Definition 8.1. A *weighted graph* $\mathcal{G} = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown}, \delta \rangle$ is a given by graph $\langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown} \rangle$ together with a *weight function* $\delta \colon V_{\mathcal{G}} \times V_{\mathcal{G}} \to \mathbb{N}$ such that if $u \overset{\mathcal{G}}{\frown} v$ then $\delta(u, v) > 0$ and $\delta(u, u) = 0$.

We use the following notations: we write $u \overset{\mathcal{G}}{\frown}_k v$ iff $u \overset{\mathcal{G}}{\frown} v$ and $\delta(u, v) = k$, and we write $u \overset{\mathcal{G}}{\smile}_k v$ iff $u \overset{\mathcal{G}}{\not\frown} v$ and $\delta(u, v) = k$. When drawing a graph we use $v \overset{k}{—} w$ for $v \frown_k w$ and we use $v \overset{k}{\cdots} w$ for $v \smile_k w$. If $v \smile_0 w$ we often draw no edges.

Definition 8.2. A *weighted cograph* $\mathcal{G} = \langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown}, \delta\rangle$ is a weighted graph such that:

1. the graphs $\langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\frown}_i\rangle$ and $\langle V_{\mathcal{G}}, \overset{\mathcal{G}}{\smile}_i\rangle$ are Z-free for all $i \neq 0$;
2. for all $u, v, w \in V_{\mathcal{G}}$, and any $n, m, k, l, h \in \mathbb{N}$, with n, m, k being pairwise distinct and $h > 0$, the following configurations are forbidden:



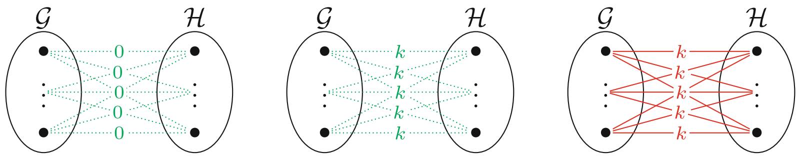
3. for all $u, v, w \in V_{\mathcal{G}}$ with

   either $n = 0$ or $m = 0$ or $n > m$.

We define the *juxtaposition*, *graded union* and *graded join* operations for weighted graphs:

$$\begin{aligned}
\mathcal{G} \star \mathcal{H} &= \langle V_{\mathcal{G}} \cup V_{\mathcal{H}} \, , \overset{\mathcal{G}}{\frown} \cup \overset{\mathcal{H}}{\frown} \, , \delta_{\mathcal{G}} \cup \delta_{\mathcal{H}} \cup \delta^0 \rangle \\
\mathcal{G} \vee \mathcal{H} &= \langle V_{\mathcal{G}} \cup V_{\mathcal{H}} \, , \overset{\mathcal{G}}{\frown} \cup \overset{\mathcal{H}}{\frown} \, , \delta_{\mathcal{G}} \cup \delta_{\mathcal{H}} \cup \delta^{\smile} \rangle \\
\mathcal{G} \wedge \mathcal{H} &= \langle V_{\mathcal{G}} \cup V_{\mathcal{H}} \, , \overset{\mathcal{G}}{\frown} \cup \overset{\mathcal{H}}{\frown} \cup \{\{u, v\} \mid u \in \mathcal{G}, v \in \mathcal{H}\} \, , \delta_{\mathcal{G}} \cup \delta_{\mathcal{H}} \cup \delta^{\frown} \rangle
\end{aligned}$$

where δ^0 is the weight function which assigns to each $(u, v) \in V_{\mathcal{G}} \times V_{\mathcal{H}} \cup V_{\mathcal{H}} \times V_{\mathcal{G}}$ the weight 0, while $\delta^{\frown}$ (resp. $\delta^{\smile}$) is the weight function which assigns to each $(u, v) \in V_{\mathcal{G}} \times V_{\mathcal{H}} \cup V_{\mathcal{H}} \times V_{\mathcal{G}}$ the weight $k = 1 + max\{\delta(w, z) \mid w \overset{\mathcal{G}}{\frown} z \text{ or } w \overset{\mathcal{H}}{\frown} z\}$ (respectively $k = 1 + max\{\delta(w, z) \mid w \overset{\mathcal{G}}{\not\frown} z \text{ or } w \overset{\mathcal{H}}{\not\frown} z\}$). We represent these operations as follows:



We associate to each entailed formula F (sequent Γ) a graded relation web:

$$[\![a]\!] = \bullet_a, \qquad [\![\bar{a}]\!] = \bullet_{\bar{a}}, \qquad [\![A \vee B]\!] = [\![A]\!] \vee [\![B]\!],$$

$$[\![A \wedge B]\!] = [\![A]\!] \wedge [\![B]\!], \qquad [\![A, B]\!] = [\![A]\!] \star [\![B]\!]$$

Two weighted graphs $\mathcal{G}$ and $\mathcal{H}$ are *isomorphic* (denoted $\mathcal{G} \simeq \mathcal{H}$) if there is a bijection ϕ between $V_{\mathcal{G}}$ and $V_{\mathcal{H}}$ which preserves edges and weights order, that is $u \overset{\mathcal{G}}{\frown} v$ iff $\phi(u) \overset{\mathcal{H}}{\frown} \phi(v)$, and $\delta(u, v) > \delta(u', v')$ iff $\delta(\phi(u), \phi(v)) > \delta(\phi(u'), \phi(v'))$. Then Theorem 4.1 can be extended to the following:

Theorem 8.3. *A $\mathcal{A}$-labeled weighted graph $\mathcal{G}$ is a weighted cograph iff there is a sequent Γ of entailed formulas such that $\mathcal{G} \simeq [\![\Gamma]\!]$.*

Proof. The proof is similar to the one of Theorem 4.1. However, the condition $\mathcal{G} \simeq [\![\Gamma]\!]$ (instead of $\mathcal{G} = [\![\Gamma]\!]$) is due to the existence of weighted cographs not of the form $[\![\Gamma]\!]$. By means of example take $a \overset{2}{—} b \simeq [\![a \wedge b]\!] = a \overset{1}{—} b$. $\square$

Definition 8.4. A *weighted skew fibration* $f : \mathcal{G} \to \mathcal{H}$ is a skew fibration between weighted graphs that preserves the weights.

Note that this means in particular that $f(u) = f(v)$ implies that $\delta(u, v) = 0$.

Theorem 8.5. *Let Γ and Γ' be sequents over $\mathcal{E}$. Then $\Gamma' \overset{\mathsf{C}^{\downarrow}_{\mathsf{E}}}{\vdash} \Gamma$ iff there is a weighted relevant skew fibration $f : [\![\Gamma']\!] \to [\![\Gamma]\!]$.*

Proof. The proof is similar to the one for (non-weighted) relevant skew fibrations. First, let $\Gamma' = \Gamma_0, \Gamma_1, \dots, \Gamma_n = \Gamma$ be a sequence of sequents such that

$$\frac{\Gamma_i\{A_i, A_i\}}{\Gamma_{i+1} = \Gamma_i\{A_i\}}\ \mathsf{C}^{\downarrow}_{\mathsf{E}}.$$

By definition of juxtaposition, join and union cograph operations we have that $f_i : [\![\Gamma_i]\!] \twoheadrightarrow [\![\Gamma_{i+1}]\!]$ is a relevant skew fibration and preserves $\frown$ and weights. Then also $f = f_{n-1} \circ \cdots \circ f_0$ is a weighted relevant skew fibration.

The converse follows by remarking that $f(u) = f(v)$ iff $u \smile_0 v$. $\square$

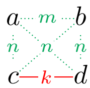
9 Weighted RB-cographs

Definition 9.1. A *weighted* RB-*cograph* is a tuple $\mathcal{G} = \langle V_\mathcal{G}, \overset{\mathcal{G}}{\frown}, \delta_\mathcal{G}, \overset{\mathcal{G}}{\curlyvee} \rangle$ where:

- $\mathcal{G}^{R\delta} = \langle V_\mathcal{G}, \overset{\mathcal{G}}{\frown}, \delta_\mathcal{G} \rangle$ is a weighted cograph;
- $\overset{\mathcal{G}}{\curlyvee}$ is a perfect matching on $V_\mathcal{G}$;

A weighted RB-cograph $\mathcal{G} = \langle V_\mathcal{G}, \overset{\mathcal{G}}{\frown}, \delta_\mathcal{G}, \overset{\mathcal{G}}{\curlyvee} \rangle$ is *æ-connected* (*æ-acyclic*) if the RB-cograph $\mathcal{G}^{RB} = \langle V_\mathcal{G}, \overset{\mathcal{G}}{\frown}, \overset{\mathcal{G}}{\curlyvee} \rangle$ is an æ-connected (æ-acyclic) RB-cograph. A weighted RB-cograph is *entailed* if it is æ-connected, æ-acyclic, and satisfies the following condition:

- if $a, b, c \in V$ such that $a \smile_m b$ for $m > 0$, and $c \smile_n a$ and $c \smile_n b$, with $n > m$ or $n = 0$, then there is $d \in V$ such that



Theorem 9.2. *If Γ is a sequent of entailed formulas then*

$\overset{\mathsf{MLL}^{\bullet}_{\mathsf{E}}}{\vdash} \Gamma \iff$ *there is an entailed weighted RB-cograph $\mathcal{G}$ with $\mathcal{G}^{R\delta} = [\![\Gamma]\!]$*

$$\dfrac{}{\langle [\![a]\!] \star [\![\bar{a}]\!] \mid \curlyvee = \{\{a,\bar{a}\}\}\rangle}\,[\![\mathsf{ax}]\!] \qquad \dfrac{\langle [\![\Gamma^{\frown}]\!] \star [\![A]\!] \star [\![B]\!] \mid \curlyvee\rangle}{\langle [\![\Gamma^{\frown}]\!] \star ([\![A]\!] \vee [\![B]\!]) \mid \curlyvee\rangle}\,[\![\vee^{\bullet}_{\mathsf{E}}]\!]$$

$$\dfrac{\langle [\![\Gamma]\!] \star [\![A]\!] \mid \overset{\Gamma,A}{\curlyvee}\rangle \quad \langle [\![B]\!] \star [\![\Delta]\!] \mid \overset{B,\Delta}{\curlyvee}\rangle}{\langle [\![\Gamma]\!] \star ([\![A]\!] \wedge [\![B]\!]) \star [\![\Delta]\!] \mid \overset{\Gamma,A}{\curlyvee} \cup \overset{B,\Delta}{\curlyvee}\rangle}\,[\![\wedge^{\bullet}_{\mathsf{E}}]\!]$$

Fig. 5. Construction rules for entailed weighted RB-cographs.

Proof. The proof piggybacks on Retoré's sequentialization proof [16]. Each proof in $\mathsf{MLL}_{\mathsf{E}}$ induces the construction of an entailed weighted cograph $\mathcal{G}$ by the operations shown in Fig. 5. In fact, each of these operations preserves æ-connectedness, æ-acyclicity and entailment conditions.

Conversely, let Γ be the sequent such that $[\![\Gamma]\!] = \mathcal{G}^{R\delta}$ and let F_Γ be the formula in $\mathcal{L}$ obtained by substituting each comma occurring in Γ by a $\vee$. By Theorem 6.3 we have derivation π_{MLL} of F_Γ in MLL. We construct a derivation π_{LE} of Γ in $\mathsf{MLL}^{\bullet}_{\mathsf{E}}$ by induction over the rules in π_{MLL}:

- If the last rule in $\pi_{\mathcal{L}}$ is an ax-rule, then the last rule in $\pi^{\bullet}_{\mathsf{LE}}$ is a ax-rule;
- If the last rule in $\pi_{\mathcal{L}}$ is an $\vee$-rule of the form $\dfrac{\Gamma, A, B}{\Gamma, A \vee B}\,\vee$, then $\delta(a,b) = \delta(a',b')$ for all $a,a' \in V_{[\![A]\!]}$ and $b,b' \in V_{[\![B]\!]}$. If $\delta(u,v) = 0$ we skip this rule inference in the construction of π_{LE} (the $\vee$ introduced by this rule in F_Γ is a comma in Γ). Otherwise, the last rule in $\pi^{\bullet}_{\mathsf{LE}}$ is a $\vee^{\bullet}_{\mathsf{E}}$-rule. In fact, for each $c \in V_{[\![\Gamma]\!]}$ by entailment condition there are $d \in V_{[\![\Gamma]\!]}$ such that $c \frown d$; that is $\Gamma = \Gamma^{\frown}$.
- If the last rule in $\pi_{\mathcal{L}}$ is an $\wedge$-rule, then the last rule in $\pi^{\bullet}_{\mathsf{LE}}$ is a $\wedge^{\bullet}_{\mathsf{E}}$-rule. ☐

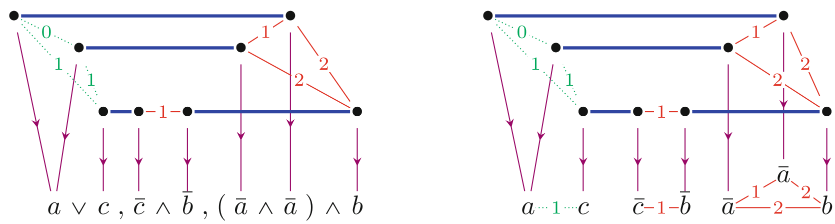
Definition 9.3. A combinatorial LE'-proof of a sequent Γ in $\mathcal{L}$ is given by an axiom-preserving weighted relevant skew fibration $f\colon \mathcal{C}^{R\delta} \to [\![\Gamma]\!]$ where $\mathcal{C}$ is an entailed weighted RB-cograph.

Theorem 9.4. *Let Γ be a sequent in $\mathcal{L}$ then*

$$\overset{\mathsf{LE}'}{\vdash} \Gamma \iff \textit{there is a combinatorial } \mathsf{LE}'\textit{-proof } f\colon \mathcal{C}^{R\delta} \to [\![\Gamma]\!]$$

Proof. This follows from Theorems 7.1, 7.2, 8.5 and 9.2. ☐

Below is an example of a combinatorial LE'-proof. On the left the conclusion is shown as sequent, and on the right as weighted cograph.



Theorem 9.5. *Let Γ be a sequent and $\mathcal{G}$ a graph together with a binary relation on it vertices and a weight function on its edges, and let f be a map from $\mathcal{G}$ to $[\![\Gamma]\!]$. It can be decided in polynomial time in $|V_{\mathcal{G}}| + |\Gamma|$ whether $f : \mathcal{G} \to [\![\Gamma]\!]$ is a combinatorial* LE'*-proof.*

Proof. All necessary properties (forbidden edges configurations for $\mathcal{G}$ being a weighted cograph, æ-connectedness and æ-acyclicity, and f being a weighted relevant skew fibration) can be checked in polynomial time. □

10 Conclusion

In this paper we presented combinatorial proofs for entailment logic $\mathbf{E}_{⅋}$, classical relevant logics $\mathbf{R}_{⅋}$ and classical relevant logic with mingle $\mathbf{RM}_{⅋}$. In some sense, combinatorial proof for entailment logic can be considered as a case study for logics with commutative but not associative connectives.

In fact, this paper can be seen as a small step in a larger research project showing that combinatorial proofs are a uniform, modular and bureaucratic-free way of representing proofs for a large class of logics. Apart from the logics studied in this paper, this goal has been achieved for multiplicative linear logic with and without mix in [16], for classical propositional logic in [12,13,18], and for intuitionistic propositional logic in [11]. For first-order logic, modal logics, and larger fragments of linear logic, this is work in progress.

A necessary condition for a logic to have combinatorial proofs seems to be the ability to separate the multiplicative (linear) fragment from the additive (contraction+weakening) fragment. This can happen inside some form of deep inference proof system [6,10], and is realized in this paper in Theorems 3.6 and 7.2.

A crucial condition that combinatorial proofs should obey, in order to be called *combinatorial proofs* for a chosen logic, is that all combinatorial properties needed for correctness of a given proof object can be checked in polynomial time with respect to its size. Then combinatorial proofs form a proof system (in the sense of Cook and Reckhow [7]) for the chosen logic. The combinatorial proofs we give in this paper have this property.

Thanks to their combinatorial (or bureaucracy-free) nature, combinatorial proofs allow us to capture a less coarser notion of proof identity with respect to the one given by syntactic formalisms like sequent calculus and analytic tableaux. Following the work in [1,13,19] we put forward the following notion of proof identity:

$$Two\ proofs\ are\ the\ same\ iff\ they\ have\ the\ same\ combinatorial\ proof.$$

References

1. Acclavio, M., Straßburger, L.: From syntactic proofs to combinatorial proofs. In: Galmiche, D., Schulz, S., Sebastiani, R. (eds.) IJCAR 2018. LNCS (LNAI), vol. 10900, pp. 481–497. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-94205-6_32

2. Anderson, A.R., Belnap Jr., N.D.: Entailment: The Logic of Relevance and Necessity, vol. 1. Princeton University Press, Princeton (1975)
3. Anderson, A.R., Belnap Jr., N.D., Dunn, J.M.: Entailment, Vol. II: The Logic of Relevance and Necessity, vol. 5009. Princeton University Press, Princeton (2017)
4. Belnap Jr., N.D.: Display logic. J. Philos. Log. **11**, 375–417 (1982)
5. Belnap Jr., N.D., Wallace, J.R.: A decision procedure for the system $e_{\bar{i}}$ of entailmengt with negation. Zeitschrift für Math. Log. Grundlagen der Math. **11**, 277–289 (1965)
6. Brünnler, K., Tiu, A.F.: A local system for classical logic. In: Nieuwenhuis, R., Voronkov, A. (eds.) LPAR 2001. LNCS (LNAI), vol. 2250, pp. 347–361. Springer, Heidelberg (2001). https://doi.org/10.1007/3-540-45653-8_24
7. Cook, S.A., Reckhow, R.A.: The relative efficiency of propositional proof systems. J. Symb. Log. **44**(1), 36–50 (1979)
8. Duffin, R.: Topology of series-parallel networks. J. Math. Anal. Appl. **10**(2), 303–318 (1965)
9. Gentzen, G.: Untersuchungen über das logische Schließen I. Math. Z. **39**, 176–210 (1935)
10. Guglielmi, A., Straßburger, L.: Non-commutativity and MELL in the calculus of structures. In: Fribourg, L. (ed.) CSL 2001. LNCS, vol. 2142, pp. 54–68. Springer, Heidelberg (2001). https://doi.org/10.1007/3-540-44802-0_5
11. Heijltjes, W., Hughes, D., Straßburger, L.: Intuitionistic proofs without syntax. In: LICS 2019 (2019)
12. Hughes, D.: Proofs without syntax. Ann. Math. **164**(3), 1065–1076 (2006)
13. Hughes, D.: Towards Hilbert's 24th problem: combinatorial proof invariants: (preliminary version). Electron. Notes Theor. Comput. Sci. **165**, 37–63 (2006)
14. McRobbie, M.A., Belnap, N.D.: Relevant analytic tableaux. Stud. Log. **38**(2), 187–200 (1979)
15. Möhring, R.H.: Computationally tractable classes of ordered sets. In: Rival, I. (ed.) Algorithms and Order, pp. 105–194. Kluwer Academic Publishers, Dordrecht (1989)
16. Retoré, C.: Handsome proof-nets: perfect matchings and cographs. Theor. Comput. Sci. **294**(3), 473–488 (2003)
17. Straßburger, L.: A characterization of medial as rewriting rule. In: Baader, F. (ed.) RTA 2007. LNCS, vol. 4533, pp. 344–358. Springer, Heidelberg (2007). https://doi.org/10.1007/978-3-540-73449-9_26
18. Straßburger, L.: Combinatorial flows and their normalisation. In: Miller, D. (ed.) 2nd International Conference on Formal Structures for Computation and Deduction, FSCD 2017. LIPIcs, Oxford, UK, 3–9 September 2017, vol. 84, pp. 31:1–31:17. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik (2017)
19. Straßburger, L.: The problem of proof identity, and why computer scientists should care about Hilbert's 24th problem. Philos. Trans. Roy. Soc. A **377** (2019)

An Infinitary Treatment of Full Mu-Calculus

Bahareh Afshari[1,2(✉)], Gerhard Jäger[2], and Graham E. Leigh[3]

[1] University of Amsterdam, Amsterdam, The Netherlands
b.afshari@uva.nl
[2] University of Bern, Bern, Switzerland
gerhard.jaeger@inf.unibe.ch
[3] University of Gothenburg, Gothenburg, Sweden
graham.leigh@gu.se

Abstract. We explore the proof theory of the modal μ-calculus with converse, aka the 'full μ-calculus'. Building on nested sequent calculi for tense logics and infinitary proof theory of fixed point logics, a cut-free sound and complete proof system for full μ-calculus is proposed. As a corollary of our framework, we also obtain a direct proof of the regular model property for the logic: every satisfiable formula has a tree model with finitely many distinct subtrees. To obtain the results we appeal to the basic theory of well-quasi-orderings in the spirit of Kozen's proof of the finite model property for μ-calculus without converse.

1 Introduction

Modal logic provides an effective language for expressing properties of state-based systems. When equipped with operators that can test for infinite behaviour like looping and reachability, the logic becomes a powerful tool for specifying correctness of nonterminating reactive processes such as communication protocols and control systems. An elegant example of such a logic is the *modal μ-calculus*, an extension of modal logic which captures the essence of inductive and co-inductive reasoning.

In modal μ-calculus two quantifiers, μ and ν, binding propositional variables, are added to the syntax of modal logic. The formulæ $\mu x\phi$ and $\nu x\phi$ are interpreted over directed graphs as, respectively, the least and greatest fixed points of the monotone function $x \mapsto \phi(x)$. The calculus can thus be thought of as a logic that allows for restricted second-order quantification while still maintaining decidability. Indeed all standard computational problems, such as model-checking and satisfiability, are decidable for this logic (see e.g. [4,15]).

Despite its importance, many fundamental questions regarding μ-calculus, and in particular its intricate proof theory, remain open. There are two notable proof systems for modal μ-calculus. Kozen [19] proposed extending the axioms of basic modal logic K with the fixed point axioms

$$\phi(\mu x\phi) \to \mu x\phi(x) \qquad\qquad \phi(\psi) \to \psi \vdash \mu x\phi(x) \to \psi$$

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 17–34, 2019.
https://doi.org/10.1007/978-3-662-59533-6_2

$$\mu x\phi, \nu x\overline{\phi} \qquad \frac{\Gamma, \phi, \psi}{\Gamma, \phi \vee \psi}\vee \qquad \frac{\Gamma, \phi \quad \Gamma, \psi}{\Gamma, \phi \wedge \psi}\wedge \qquad \frac{\Gamma, \phi}{\langle\mathfrak{a}\rangle\Gamma, [\mathfrak{a}]\phi}\mathsf{mod}$$

$$\frac{\Gamma}{\Gamma, \phi}\mathsf{weak} \qquad \frac{\Gamma, \phi(x/\mu x\phi)}{\Gamma, \mu x\phi}\mu \qquad \frac{\Gamma, \phi(x/\overline{\Gamma})}{\Gamma, \nu x\phi}\mathsf{ind} \qquad \frac{\Gamma, \overline{\phi} \quad \Gamma, \phi}{\Gamma}\mathsf{cut}$$

$$\langle\mathfrak{a}\rangle\Gamma := \{\langle\mathfrak{a}\rangle\phi \mid \phi \in \Gamma\} \qquad \overline{\Gamma} := \bigwedge\{\overline{\phi} \mid \phi \in \Gamma\}$$

Fig. 1. Axioms and rules of Koz.

asserting that $\mu x\phi$ is a pre-fixed point of $\phi(x)$ and that it is the least such.

Completeness for the aconjunctive fragment of the language was established by Kozen [19], but full completeness of this axiomatisation was not proved until Walukiewicz' seminal work [31]. Walukiewicz' proof combines an analysis of tableaux, games and automata which, it is generally agreed, is highly complex [3,9]. A natural sequent representation of Kozen's axiomatisation, denoted Koz, is given in Fig. 1. The fixed point rule, μ, and the induction rule, ind, capture the two fixed point axioms above.

The second important axiomatisation for μ-calculus is a cut-free infinitary system due to Jäger, Kretz and Studer [16]. The system, denoted $K^+_\omega(\mu)$ in [16], is Koz with the cut and ind rules replaced by the single inference

$$\frac{\Gamma, \nu^0 x\phi \quad \Gamma, \nu^1 x\phi \quad \cdots}{\Gamma, \nu x\phi}\nu_\omega$$

The formula $\nu^n x\phi$ denotes the finite approximation to the greatest fixed point: $\nu^0 x\phi = \top$ and $\nu^{n+1} x\phi = \phi(\nu^n x\phi)$ for each $n < \omega$. The proof of completeness for the system $K^+_\omega(\mu)$ is established by adapting the method of canonical model construction for modal logics to the fixed point extension. To demonstrate soundness of the system, more specifically that of ν_ω-rule, the finite model property of μ-calculus [18,29] is invoked.

In this paper we are interested in the proof theory of μ-calculus extended by converse modalities. The extension, known as the *two-way μ-calculus* or *full μ-calculus*, assumes each action $\mathfrak{a}$ is associated a "converse" action $\overline{\mathfrak{a}}$ and that a transition system has an $\overline{\mathfrak{a}}$-edge from vertices u to v iff it has an $\mathfrak{a}$-edge from v to u. Axiomatically, one stipulates $\phi \to [\mathfrak{a}]\langle\overline{\mathfrak{a}}\rangle\phi$ for every formula and action.

Checking satisfiability for μ-calculus with converse was proven to be decidable by Vardi in [30] (see also [5]) where he introduces the *two-way automata* characterising this extension and shows the emptiness problem is decidable. In contrast to pure modal μ-calculus, the finite model property fails.

To the best of our knowledge, a sound and complete axiomatisation for full μ-calculus has not been given. This can seem somehow surprising if one were to speculate that the presence of converse can simplify completeness results – such as is the case for the computational tree logic CTL* with past [24,25]. One can add to Koz suitable converse axioms and ask whether the resulting system is complete. Walukiewicz' completeness proof for modal μ-calculus does not easily lend itself to this question because its machinery, particularly the parts based on tableaux, fall short of converse. Similarly, for the alternative proof of completeness via cyclic proofs given in [1] it is unclear how modalities operating in both directions can be incorporated.

There is another possibility for obtaining a sound and complete axiomatisation for full μ-calculus, namely an adaptation of the infinitary system $K_\omega^+(\mu)$ of [16], which is undertaken in this paper. There are two obstacles to this approach: accommodating converse in the canonical model construction and recovering any structural properties that remain in the absence of the finite model property that are needed to show soundness of an infinitary ν-rule.

We overcome the first issue by stepping into the framework of nested sequents in the style of Kashima's work for tense logics [17]. The failure of the finite model property shows the ν_ω-rule is unsound in the presence of converse. We establish soundness for the infinitary ν-rule with a premise for each approximant below ω^ω and prove that the ensuring nested sequent calculus is complete for the full μ-calculus. Moreover, we observe that this bound is optimal over trees: the greatest fixed point cannot be identified with its transfinite approximant for any ordinal below ω^ω.

Related Work. The history of modal logic with converse goes back to Prior and his introduction of tense logics.[1] Temporal logics with past have been widely studied. For example, completeness of converse PDL was first shown in [23], and a sound and complete axiomatisation of PCTL* (computation tree logic with past) is given in [24]. More recent work relevant to this paper include the treatment of tense logics in [14] and the completeness proof for the flat fragment of μ-calculus [8]. The literature on nested sequents is also rich: they have been used to establish algorithmic properties on a wide range of logics (e.g. [2,10,11,13,26]). The explicit use of ordinal approximations in the language of μ-calculus is a feature that has been used by other authors studying fixed-point logics. Of particular note is the work of [6] in which they are utilised for the correspondence between circular proofs and induction.

2 Full μ-calculus

Fix finite sets Act and Var of **actions** and **variables**, respectively. The μ-calculus **formulæ** are given by the following grammar, where $\mathfrak{a}$ ranges over actions and x over variables.

$$\phi := x \mid \phi \wedge \phi \mid \phi \vee \phi \mid [\mathfrak{a}]\phi \mid \langle \mathfrak{a} \rangle \phi \mid \mu x \phi \mid \nu x \phi$$

The two propositional quantifiers μ and ν are called the **least**, and **greatest**, **fixed point quantifier** respectively. The syntax above omits both negation and proposition constants. Constants for 'true' and 'false' can be defined via

[1] For a comprehensive account see, e.g., [12].

the quantifiers: $\top = \nu x.x$ and $\bot = \mu x.x$; other (unspecified) propositional constants can be represented via additional actions. Negation is representable via De Morgan duality in the usual way; see e.g., [7, Chap. 8]. We write $\phi(x/\psi)$ for the result of substituting the formula ψ for every free occurrence of x in ϕ subject to the proviso that no free variable of ψ becomes bound.

In the full μ-calculus every action $\mathfrak{a} \in \mathsf{Act}$ has an associated converse which we denote as $\overline{\mathfrak{a}}$. Thus we assume the presence of an involution, $\overline{\cdot}\colon \mathsf{Act} \to \mathsf{Act}$, on the set of actions: for every $\mathfrak{a} \in \mathsf{Act}$ we have $\mathfrak{a} \neq \overline{\mathfrak{a}}$, and $\overline{\overline{\mathfrak{a}}} = \mathfrak{a}$. If a formula ϕ contains at most one of $\mathfrak{a}$ or $\overline{\mathfrak{a}}$ for each $\mathfrak{a} \in \mathsf{Act}$ we may call ϕ **pure**.

We recall the standard Kripke semantics for the modal μ-calculus.

Definition 1. *A* ***labelled frame*** *is a pair (S, E) where S is a non-empty set of* ***vertices*** *and $E \subseteq \mathsf{Act} \times S \times S$ is a set of* ***(labelled) edges****. The* ***symmetric closure*** *of a frame (S, E) is the frame (S, E') where $E' = E \cup \{(\overline{\mathfrak{a}}, v, u) \mid (\mathfrak{a}, u, v) \in E\}$. A frame is* ***symmetric*** *if it is identical to its symmetric closure.*

If the labelled frame (S, E) is clear from the context, we write $u \xrightarrow{\mathfrak{a}} v$ (or simply $u \longrightarrow v$) if $(\mathfrak{a}, u, v) \in E$. Labelled frames provide a semantics for μ-calculus formulæ via the possible worlds interpretation of the modal connectives:

Definition 2. *Let ϕ be a formula, possibly with free variables, $\mathscr{S} = (S, E)$ a labelled frame, and $V \subseteq \mathsf{Var} \times S$. The* ***denotation*** *of ϕ in $\mathscr{S}$ relative to V, in symbols $\|\phi\|^{\mathscr{S}}_V$, is the subset of S defined with the standard semantics for boolean and modal operators (e.g. [7]) and the following equations for fixed point quantifiers, where*

$$V[x \mapsto T] = \{(y, u) \in V \mid y \neq x\} \cup (\{x\} \times T).$$
$$\|\mu x \phi\|^{\mathscr{S}}_V = \bigcap \{T \subseteq S \mid \|\phi\|^{\mathscr{S}}_{V[x \mapsto T]} \subseteq T\}$$
$$\|\nu x \phi\|^{\mathscr{S}}_V = \bigcup \{T \subseteq S \mid T \subseteq \|\phi\|^{\mathscr{S}}_{V[x \mapsto T]}\}$$

We write $(\mathscr{S}, u) \models \phi$ to express $u \in \|\phi\|^{\mathscr{S}}_{\emptyset}$. A formula ϕ is **satisfied** by $\mathscr{S}$ if $(\mathscr{S}, u) \models \phi$ for some vertex u and is **true** in $\mathscr{S}$ if $(\mathscr{S}, u) \models \phi$ for every vertex u. Given a class of frames $\mathcal{C}$, a formula ϕ has a **model** in C if ϕ is satisfied in some frame $\mathscr{S} \in C$, and is **valid** over C if ϕ is true in every frame in C. If mention of C is omitted we have in mind the class of all countable frames.

Definition 3. *Let ϱ be a well-formed formula. The* ***Fischer–Ladner closure*** *of ϱ, denoted $\mathbb{FL}(\varrho)$, is the smallest set of formulæ containing ϱ satisfying the following conditions.*

- *If $\phi \circ \psi \in \mathbb{FL}(\varrho)$ for $\circ \in \{\wedge, \vee\}$ then $\{\phi, \psi\} \subseteq \mathbb{FL}(\varrho)$.*
- *If $\triangle\phi \in \mathbb{FL}(\varrho)$ for $\triangle \in \{[\mathfrak{a}], \langle\mathfrak{a}\rangle \mid \mathfrak{a} \in \mathsf{Act}\}$ then $\phi \in \mathbb{FL}(\varrho)$.*
- *If $\sigma x \phi \in \mathbb{FL}(\varrho)$ for $\sigma \in \{\mu, \nu\}$ then $\phi(x/\sigma x \phi) \in \mathbb{FL}(\varrho)$.*

In what follows we utilise an extended language where the greatest fixed point quantifier ν spawns an infinite hierarchy of 'approximation' quantifiers indexed by ordinals. Fix an ordinal κ. The **κ-formulæ** are generated as follows

$$\phi := x \mid \phi \wedge \phi \mid \phi \vee \phi \mid [\mathfrak{a}]\phi \mid \langle\mathfrak{a}\rangle\phi \mid \mu x \phi \mid \nu x \phi \mid \nu^{\alpha} x \phi \quad (\alpha < \kappa)$$

The intended reading of the quantifier ν^α is of α-times unfolding the matrix, taking conjunctions at limits with $\nu^0 x\phi$ equivalent to $\top$. When κ is fixed it is convenient to identify the unannotated quantifier ν with ν^κ. From a κ-formula ϕ we derive a μ-calculus formula ϕ^-, called the **template** of ϕ, by removing the approximation of every quantifier.

The class of ω-formulæ corresponds to the language $\mathcal{L}_\mu^+$ of [16]. The following definition expands the generalisation of the Fischer–Ladner closure in [16] to a form appropriate for κ-formulæ.

Definition 4. *The* ***strong closure*** *of a κ-formula ϱ, denoted $\mathbb{SC}_\kappa(\varrho)$, is the smallest set containing ϱ closed under the formation rules for the Fischer–Ladner closure and the following clause.*

- *If $\nu^\alpha x\phi \in \mathbb{SC}_\kappa(\varrho)$ $(\alpha \leq \kappa)$ then $\{\phi(x/\nu^\beta x\phi) \mid \beta < \alpha\} \subseteq \mathbb{SC}_\kappa(\varrho)$.*

Semantics for κ-formulæ is obtained by extending the definition for μ-calculus formulæ to accommodate the approximating quantifiers subject to the equation $(\mathscr{S}, u) \models \nu^\alpha x\phi$ iff $(\mathscr{S}, u) \models \phi(x/\nu^\beta x\phi)$ for every $\beta < \alpha$. Standard arguments on the fixed point semantics show there exists κ s.t. $\nu x\phi \leftrightarrow \nu^\kappa x\phi$ is true in $\mathscr{S}$.

An important concept in μ-calculus is the relation of subsumption between variables occurring in a given formula, a syntactic constraint that mirrors the priority of quantifiers implicit in the semantics. In the present article, we take a pragmatic approach to subsumption, assuming a fixed strict partial order on Var, called the **subsumption order**, and constrain considerations to formulæ whose variables respect this relation, in the sense that if y occurs free in a sub-formula $\sigma x\phi$ then y subsumes x. We call such formulæ **well-formed**. The subsumption order must be irreflexive, asymmetric, transitive and for every $x \in \mathsf{Var}$ the set of variables subsuming x should be linearly ordered by subsumption.

If ϕ is well-formed then for a substitution $\phi(x/\psi)$ to be 'correct' it suffices that x does not subsume any free variable of ψ. Thus, every formula occurring in the Fischer–Ladner closure of a well-formed formula is well-formed. It is common to assume that each quantified formula uniquely determines a variable symbol that is bound; we call such a formula **well-named**. Note, however, that, unlike the notion of well-formed, the Fischer–Ladner closure conditions do not preserve well-namedness.

Definition 5. *L_μ denotes the set of closed formulæ that appear in the Fischer–Ladner closure of some well-named formula.*

We likewise need to isolate a class of κ-formulæ to assist the presentation of our results. The class of κ-formulæ with templates in L_μ is natural, but there is a strict sub-class of these formulæ that we should restrict attention to. This turns out to be the collection of κ-formulæ that arise when evaluating the denotation of an $\phi \in L_\mu$ subject to the identification of ν with ν^κ. These formulæ, which we call **well-annotated**, satisfy the following three conditions:

1. Their template is well-formed.
2. If $\nu^\alpha x$ and $\nu^\beta x$ are two quantifiers binding the same variable then $\alpha = \beta$.

3. The set of variable symbols bound by a quantifier ν^α with $\alpha < \kappa$ is linearly ordered by the subsumption relation.

The set of well-annotated κ-formulæ is denoted L_μ^κ. It is a simple exercise to check that every formula in the strong closure of a well-annotated formula is well-annotated.[2]

2.1 Nested Sequent Calculi

Nested sequents were utilised by Kashima to establish canonical completeness for tense logics [17]. In the following we adapt Kashima's approach to $L_\mu^{\omega_1}$. For the present section κ is an arbitrary ordinal $\leq \omega_1$.

Definition 6. *A* ***sequent*** *is a finite set of closed L_μ^κ formulæ. The* ***nested sequents*** *(ns) are defined inductively:*

1. *every plain sequent is a nested sequent,*
2. *if Γ is a nested sequent and $\mathfrak{a}$ is an action then $\mathfrak{a}\{\Gamma\}$ is a ns,*
3. *if Γ, Δ are ns then so is $\Gamma \cup \Delta$.*

As is usual, we use comma to abbreviate the union of two (nested) sequents and identify singleton sequents with their unique element. Hence, every nested sequent can be presented in the form

$$\Gamma = \phi_1, \ldots, \phi_m, \mathfrak{a}_1\{\Delta_1\}, \ldots, \mathfrak{a}_n\{\Delta_n\} \tag{1}$$

where $\phi_1, \ldots, \phi_n \in L_\mu^\kappa$, $\Delta_1, \ldots, \Delta_n$ are nested sequents and $\mathfrak{a}_1, \ldots, \mathfrak{a}_k \in \mathsf{Act}$. The intended interpretation of the nested sequent Γ in (1) is the formula

$$\iota(\Gamma) = \bigvee_{i=1}^{m} \phi_i \vee \bigvee_{i=1}^{n} [\mathfrak{a}_i]\iota(\Delta_i).$$

A **sequent with context** (simply **context**) is a nested sequent built from an additional unit [], called the context, which must have exactly one occurrence within the nested sequent. If Γ is a sequent with context and Δ is a nested sequent $\Gamma[\Delta]$ is the nested sequent given by substituting Δ for [] in Γ.

Definition 7. *Fix $\kappa \leq \omega_1$. $\mathsf{K}_{\mu+}^\kappa$ is the calculus deriving nested sequents given by the inferences in Fig. 2. K_μ^κ denotes the subsystem without the inference $\mathsf{con}_\mathfrak{a}$.*

A special case of the $\nu.\alpha$ inference is when $\alpha = 0$, whereby the sequent $\Gamma[\nu^0 x\phi]$ is derivable without premises. Hence, $\mathsf{K}_\mu^\kappa \vdash \Gamma[\top]$ for any sequent context $\Gamma[]$. Clearly, a smaller value of κ makes introducing greatest fixed points easier. The following properties can be established by induction on the length of derivations.

Lemma 1. *1. If $\mathsf{K}_{\mu+}^\alpha$ is complete so is $\mathsf{K}_{\mu+}^\beta$ for every $\beta \leq \alpha$; similarly for K_μ^κ.*
2. For all $\phi \in L_\mu$ and contexts $\Gamma[]$, $\mathsf{K}_\mu^\kappa \vdash \Gamma[\phi, \overline{\phi}]$ where $\overline{\phi}$ denotes the De Morgan dual of ϕ.
3. If $\mathsf{K}_{\mu+}^\kappa \vdash \Gamma[\nu^\alpha x\phi]$ then $\mathsf{K}_{\mu+}^\kappa \vdash \Gamma[\nu^\beta x\phi]$ for every $\beta < \alpha$; similarly for K_μ^κ.

[2] See Appendix A for precise definitions of the concepts of this section.

$$\frac{\Gamma[\phi,\psi]}{\Gamma[\phi\vee\psi]}\vee \qquad \frac{\Gamma[\phi] \quad \Gamma[\psi]}{\Gamma[\phi\wedge\psi]}\wedge \qquad \frac{\Gamma[\phi(x/\mu x\phi)]}{\Gamma[\mu x\phi]}\mu$$

$$\frac{\Gamma[\mathfrak{a}\{\phi\}]}{\Gamma[[\mathfrak{a}]\phi]}[\mathfrak{a}] \qquad \frac{\Gamma[\mathfrak{a}\{\Delta,\phi\}]}{\Gamma[\mathfrak{a}\{\Delta\},\langle\mathfrak{a}\rangle\phi]}\langle\mathfrak{a}\rangle \qquad \frac{\Gamma[\mathfrak{a}\{\Delta\},\phi]}{\Gamma[\mathfrak{a}\{\Delta,\langle\overline{\mathfrak{a}}\rangle\phi\}]}\mathsf{con}_{\mathfrak{a}}$$

$$\frac{\Gamma[\phi(x/\nu^{\beta}x\phi)] \quad \text{for all } \beta<\alpha\leq\kappa}{\Gamma[\nu^{\alpha}x\phi]}\nu.\alpha \qquad \frac{\Gamma[\phi(x/\nu^{\alpha}x\phi)] \quad \text{for all } \alpha<\kappa}{\Gamma[\nu x\phi]}\nu_{\kappa}$$

Fig. 2. System $\mathsf{K}^{\kappa}_{\mu+}$; $\mathsf{K}^{\kappa}_{\mu}$ is $\mathsf{K}^{\kappa}_{\mu+}$ without $\mathsf{con}_{\mathfrak{a}}$.

3 Completeness: Building Canonical Models

Definition 8. *A* ***κ-system*** *(in ϱ) is a tuple (S,E,λ) where (S,E) is a frame and $\lambda\colon S\to \mathrm{Pow}(\mathbb{SC}_{\kappa}(\varrho))$ assigns to each vertex of S a set of κ-formulæ from the strong closure of ϱ. A system (T,E,λ)* ***expands*** *a system (S,F,ρ) if $S\subseteq T$, $F\subseteq E$, and $\rho(u)\subseteq\lambda(u)$ for every $u\in S$.*

Explicit mention of κ and ϱ will be dropped if they can be inferred from context and, when there is no cause for confusion, vertices of a system will be identified with their labels: $\phi\in u$ in place of $\phi\in\lambda(u)$. Recall that edges of a labelled frame (and so of a system) are labelled by actions and that symmetry is not assumed. A nested sequent $\Gamma=\Delta_0,\mathfrak{a}_1\{\Delta_1\},\dots,\mathfrak{a}_l\{\Delta_l\}$ ($\Delta_0\subseteq L^{\kappa}_{\mu}$) has a natural representation as finite κ-system, $\mathrm{tree}(\Gamma)$, comprising a root with label Δ_0 and, for each $0<i\leq l$, an $\mathfrak{a}_i$-child with immediate subtree $\mathrm{tree}(\Delta_i)$.

For the proof of completeness, starting from an assumption that a sequent Γ is underivable we will construct a system expanding Γ by saturating the sequent through the $\mathsf{K}^{\kappa}_{\mu+}$ rules applied from conclusion to premise. Deconstructing a modality corresponds to creating, or saturating, other vertices in the system. This method combines saturation arguments for the (pure) modal μ-calculus [16] and the tableau-style constructions for tense logic [17]. If we obtain two different annotations of the same formula, say $\nu^{\alpha}x\phi$ and $\nu^{\beta}x\phi$, then clearly, from the perspective of non-derivability, the smaller approximation suffices. Thus, to maintain some control on the κ-formulæ enumerated via the process, we desire an ordering on L^{κ}_{μ} formulæ based on the ordinal approximations.

Recall a **quasi-order** is a reflexive, transitive relation. Let $\sqsubseteq$ be the quasi-order on L^{κ}_{μ} determined by $\phi\sqsubseteq\psi$ iff $\phi^-=\psi^-$ and for every maximal chain $x_1,\dots,x_n$ of the ν-quantified variables in ϕ such that x_i subsumes x_{i+1}, we have $(\alpha_1,\dots,\alpha_n)\leq(\beta_1,\dots,\beta_n)$, where α_i (β_i) is the ordinal assigned to x_i in ϕ (resp. ψ) and $\leq$ is the lexicographic ordering on sequences of ordinals.[3]

Definition 9. *A κ-system $\mathscr{S}$ is* ***saturated*** *if the following hold for every $u\in\mathscr{S}$, $\phi,\psi\in L^{\kappa}_{\mu}$, $\mathfrak{a}\in\mathsf{Act}$ and $\alpha\leq\kappa$.*

(a) $\phi\wedge\psi\in u$ implies $\phi\in u$ or $\psi\in u$,

[3] Cf. Appendix A.

(b) $\phi \vee \psi \in u$ *implies* $\phi \in u$ *and* $\psi \in u$*,*
(c) $\mu x\phi \in u$ *implies* $\phi(x/\mu x\phi) \in u$*,*
(d) $\nu^\alpha x\phi \in u$ *implies* $\phi(x/\nu^\beta x\phi) \in u$ *for some* $\beta < \alpha$*,*
(e) $[\mathfrak{a}]\phi \in u$ *implies for some* $u \xrightarrow{\mathfrak{a}} v$ *and* $\psi \in v$ *we have* $\psi \sqsubseteq \phi$*,*
(f) $\langle\mathfrak{a}\rangle\phi \in u$ *and either* $u \xrightarrow{\mathfrak{a}} v$ *or* $v \xrightarrow{\overline{\mathfrak{a}}} u$ *implies* $\psi \in v$ *for some* $\psi \sqsubseteq \phi$*.*

Our notion of saturation combines a number of features from other work. It is closely related to Kozen's well-annotations in [18] expanded to cover converse modalities in the style of Kashima [17]. Note, however, that our quasi-order differs from Kozen's. Dropping the two modal clauses e and f yields the definition of saturation in [16], for $\kappa = \omega$.

Lemma 2. *Let* $\mathscr{S}$ *be a saturated* κ*-system.*

1. *The symmetric closure of* $\mathscr{S}$ *is saturated.*
2. *For every* $u \in \mathscr{S}$ *and* $\phi \in u$*,* $(\mathscr{S}, u) \not\models \phi$ *(Truth Lemma).*

Proof. The first claim is immediate given the formulation of condition f. For 2, we refer the reader to [18, Lemma 4.2], noting that, like the quasi-order utilised in [18], denotation is monotone in $\sqsubseteq$: if $\phi \sqsubseteq \psi$ then $(\mathscr{S}, u) \models \psi$ implies $(\mathscr{S}, u) \models \phi$ for any u. A more detailed proof of the result, based on the assignment of a rank to each formula of L^κ_μ, is given in [16, Lemma 33].[4]

We establish weak completeness of the calculi $\mathsf{K}^\kappa_{\mu^+}$ and K^κ_μ, namely that every underivable sequent has a counter-model. In view of Lemma 2, it suffices to show that every underivable sequent expands to a saturated system. In contrast to the constructions in [16,17] (for pure μ-calculus and tense logic respectively), we cannot expect the result to be a finite system (i.e. a nested sequent); in general, an infinite tree will result.

Lemma 3 (Saturation Lemma). *Suppose* $\mathsf{K}^\kappa_{\mu^+} \nvdash \Gamma$*. There exists a saturated* κ*-system* $\mathscr{T}$ *expanding* Γ *such that every formula occurring in the label of a vertex of* $\mathscr{T}$ *is an element of* $\mathbb{SC}_\kappa(\varrho)$ *for some formula* ϱ*.*

Proof. We require an auxiliary notion of saturation. Let us call a κ-system 0-saturated if the saturation conditions hold with the possible exception of the clauses for modalities, e and f. Every underivable nested sequent can be expanded to a 0-saturated nested sequent that remains underivable. The proof of this fact follows the argument of Lemma 24 in [16].

Suppose $\mathsf{K}^\kappa_{\mu^+} \nvdash \Gamma$. We define a sequence of nested sequents $\Gamma = \Gamma_0, \Gamma_1, \Gamma_2, \dots$ such that Γ_{i+1} expands Γ_i and Γ_i is underivable. Given Γ_i, obtain Γ_{i+1} by

1. expanding Γ_i to a 0-saturated nested sequent Γ'_i;
2. expanding Γ'_i to a ns Γ_{i+1} by correcting any failure of conditions e or f:
   a. For any $v \in \Gamma'_i$ and formula $[\mathfrak{a}]\phi \in u$ for which there is no $\mathfrak{a}$-child of u in Γ'_i containing ϕ, create a $\mathfrak{a}$-child with label $\{\phi\}$;

[4] As already remarked, [16] deals only with the case $\kappa = \omega$. However, their notion of rank and the proof of the Truth Lemma readily generalises to arbitrary κ.

b. If v is a $\mathfrak{a}$-child of u in Γ_i', expand the label of u to include $\{\phi \mid \langle\overline{\mathfrak{a}}\rangle\phi \in v\}$ and the label of v to include $\{\phi \mid \langle\mathfrak{a}\rangle\phi \in u\}$.

The process of 0-saturation preserves underivability. Moreover, Γ_i' can be derived from Γ_{i+1} by a sequence of $\mathsf{con}_{\mathfrak{a}}$, $[\mathfrak{a}]$ and $\langle\mathfrak{a}\rangle$ inferences, hence Γ_{i+1} is underivable. Let $\mathscr{T}$ be the limit of trees $\mathrm{tree}(\Gamma_i)$ for $i < \omega$. By construction, $\mathscr{T}$ is a saturated κ-system fulfilling the requirements of the lemma. $\square$

As a consequence of the Saturation and Truth lemmas we deduce completeness for full μ-calculus. An analogous argument establishes completeness for the pure fragment.

Theorem 1. $\mathsf{K}^{\kappa}_{\mu^+}$ *is complete over symmetric frames.* $\mathsf{K}^{\kappa}_{\mu}$ *is complete for arbitrary frames.*

Proof. Suppose Γ is underivable in $\mathsf{K}^{\kappa}_{\mu^+}$ and let $\mathscr{S}$ be the symmetric closure of the κ-system expanding Γ provided by Lemma 3. As a consequence of Lemma 2, $(\mathscr{S}, r) \not\models \iota(\Gamma)$ where r is the root of $\mathscr{S}$. Hence, Γ is not valid. An analogous argument establishes completeness for $\mathsf{K}^{\kappa}_{\mu}$.

4 Soundness: Refining Canonical Models

We now turn to soundness theorems for the systems $\mathsf{K}^{\kappa}_{\mu^+}$ and $\mathsf{K}^{\kappa}_{\mu}$ for certain κ. It can be easily confirmed that for either system the only inference we need be concerned with is the introduction rule for the greatest fixed point, ν_κ.

Some cases of soundness can be inferred from known properties of the μ-calculus. For instance, the pure μ-calculus (without converse modalities) has the finite model property: every satisfiable formula has a finite model [18,29]. On the class of finite models the greatest fixed point coincides with the ω-th approximation, ν^ω. Thus soundness of $\mathsf{K}^{\omega}_{\mu}$ obtains.

Theorem 2. $\mathsf{K}^{\omega}_{\mu}$ *is sound and complete for arbitrary frames.*

The above theorem can also be deduced without directly appealing to the finite model property, by manipulating saturated systems. This argument was already made by Kozen [18] and will be extended below.

The full μ-calculus lacks the finite model property (there are satisfiable formulæ with no finite models) but every satisfiable formula has a model which is (the symmetric closure of) a finitely branching tree [30]. As a consequence we deduce $\mathsf{K}^{\kappa}_{\mu^+}$ is unsound for $\kappa \leq \omega$ but sound for $\kappa \geq \omega_1$.

Theorem 3. $\mathsf{K}^{\omega_1}_{\mu^+}$ *is sound and complete for arbitrary (symmetric) frames.*

In the sequel we prove a strengthening of Theorem 3: the calculus $\mathsf{K}^{\omega^\omega}_{\mu^+}$ is sound and complete for symmetric frames; and observe that, over trees, $\mathsf{K}^{\kappa}_{\mu^+}$ is is unsound for every $\kappa < \omega^\omega$. Our argument relies on a particular property of the quasi-order $\sqsubseteq$ we introduced earlier, which we now state. Given a set $X \subseteq \mathbb{SC}_\kappa(\varrho)$ let $\mathrm{Ker}\, X = \{\phi \in X \mid \forall \psi \in X\, \psi \not\sqsubset \phi\}$ be the set of $\sqsubseteq$-minimal elements of X. Recall, a quasi-order $\leq$ on a set Q is a **well-quasi-order** (wqo for short) if for every function $f\colon \omega \to Q$ there exists $i < j$ such that $f(i) \leq f(j)$.

Lemma 4. $(\mathbb{SC}_\kappa(\varrho), \sqsubseteq)$ *is a wqo. Moreover, there exists* k *such that for every* $X \subseteq \mathbb{SC}_\kappa(\varrho)$, $|\mathrm{Ker}\, X| < k$.

That $\sqsubseteq$ is a well-quasi-order follows from the observation that the ordering can be expressed as a sum of products of well-orders. Being a wqo we immediately deduce that $\mathrm{Ker}\, X$ is finite for every $X \subseteq \mathbb{SC}_\kappa(\varrho)$. The stronger result stated follows from the constraints we imposed in the definition of L^κ_μ, namely condition 3.[5] Specifically, it is this property that marks the essential difference between $\sqsubseteq$ and the wqo $\preccurlyeq$ in [18].

We require a lifting of $\sqsubseteq$ to sets of κ-formulæ. A natural candidate is the Smyth powerdomain introduced in [27] and given by $X \sqsubseteq Y$ iff for every $\psi \in Y$ there exists $\phi \in X$ such that $\phi \sqsubseteq \psi$. In general, this lifting does not preserve well-quasi-orders [20] but, rather, the stronger notion of better-quasi-order due to Nash-Williams [21,22]; $(\mathbb{SC}_\kappa(\varrho), \sqsubseteq)$ is readily seen to be a better-quasi-order.

For our strengthening of Theorem 3, however, we depend on a refinement of the Smyth powerdomain whereby X is bounded by Y if Y can be realised as the image of X under an endomorphism on $(\kappa, <)$. This choice is motivated by the observation that saturation is preserved under any change of annotating ordinals by a strictly monotone function on κ. The main technical result is to establish that this notion of boundedness is a well-quasi-order on $\mathrm{Pow}(\mathbb{SC}_\kappa(\varrho))$ for every $\varrho \in L_\mu$. We begin making the above definitions precise.

Let $\mathbb{I}(\kappa)$ be the set of strictly monotone functions on ordinals $\leq\kappa$. Note that such functions are increasing, so $\alpha \leq f(\alpha) \leq \kappa$ for every $\alpha \leq \kappa$. Each $f \in \mathbb{I}(\kappa)$ induces an operation on L^κ_μ mapping ϕ to the result of replacing each annotated quantifier ν^α by $\nu^{f(\alpha)}$, which we denote as ϕ^f. Similarly, for $X \subseteq L^\kappa_\mu$, define $X^f = \{\phi^f \mid \phi \in X\}$ and for a system $\mathscr{T} = (S, E, \lambda)$ we let $\mathscr{T}^f$ be the system (S, E, λ^f) where $\lambda^f \colon w \mapsto \lambda(w)^f$. The following is straightforward to verify.

Lemma 5. *Let* $f \in \mathbb{I}(\kappa)$. *If* $X \sqsubseteq Y$ *then* $X^f \sqsubseteq Y^f$. *Hence, if* $\mathscr{T}$ *is a saturated* κ*-system, so is* $\mathscr{T}^f$.

We are now in a position to define the quasi-order on $\mathrm{Pow}(\mathbb{SC}_\kappa(\varrho))$:

$$X \sqsubseteq^* Y := \exists f \in \mathbb{I}(\kappa) \text{ s.t. } \mathrm{Ker}\, Y = \mathrm{Ker}(X^f)$$

Since $(\mathrm{Ker}\, X)^f = \mathrm{Ker}\, X^f$ for every $X \subseteq \mathbb{SC}_\kappa(\varrho)$ and $f \in \mathbb{I}(\kappa)$, like the Smyth powerdomain, $\sqsubseteq^*$ is determined by its restriction to kernels: $X \sqsubseteq^* \mathrm{Ker}\, X \sqsubseteq^* X$. The fact that kernels are bounded (Lemma 4) is crucial for the following result.

We call κ **principal** if $\kappa = \omega^\alpha$ for some α.

Theorem 4. *If* κ *is principal then* $(\mathrm{Pow}(\mathbb{SC}_\kappa(\varrho)), \sqsubseteq^*)$ *is a wqo.*

Proof. Let $\mathsf{Var}_\nu(\phi)$ be the set of ν-quantified variables in ϕ. To each $\phi \in L^\kappa_\mu$ we may associate a function $o_\phi \colon \mathsf{Var}_\nu(\phi) \to \kappa + 1$ such that ϕ can be obtained from its template by replacing each quantifier νx in ϕ^- by $\nu^{o_\phi(x)} x$. We consider finite sequences in $\mathbb{FL}(\varrho) \times \mathsf{Var} \times (\kappa + 1)$, ordered pointwise by $(\phi_i, x_i, \alpha_i)_{i<m} \leq_{\mathrm{pw}}$

[5] See Appendix B for a proof of this fact.

$(\psi_i, y_i, \beta_i)_{i<n}$ iff $m = n$ and for all $i < m$, $\phi_i = \psi_i$, $x_i = y_i$ and $\alpha_i \leq \beta_i$. When restricted to a set of sequences of bounded length, $\leq_{\text{pw}}$ is a wqo. For $X \subseteq \mathbb{SC}_\kappa(\varrho)$, let $X^* = (\phi_i, x_i, \delta_i)_{i<k}$ be a sequence in $\mathbb{FL}(\varrho) \times \mathsf{Var} \times (\kappa + 1)$ such that

$$\operatorname{Ker} X = \{\phi \in \mathbb{SC}_\kappa(\varrho) \mid \forall x \in \mathsf{Var}_\nu(\phi) \exists i < k(\phi_i = \phi^- \wedge x_i = x \wedge o_\phi(x) = \sum_{j \leq i} \delta_j)\}.$$

Without loss of generality, we assume a total ordering $<$ of $\mathbb{FL}(\varrho) \times \mathsf{Var}$ and that $\delta_{i+1} = 0$ implies $(\phi_i, x_i) < (\phi_{i+1}, x_{i+1})$. By Lemma 4, k can be chosen independent of X. Hence it remains only to observe that for principal κ, $X \sqsubseteq^* Y$ iff $X^* \leq_{\text{pw}} Y^*$. □

Given systems $\mathscr{T}$ and $\mathscr{T}'$, write $\mathscr{T} \sqsubseteq^* \mathscr{T}'$ if $\mathscr{T}'$ is isomorphic to $\mathscr{T}^f$ for some $f \in \mathbb{I}(\kappa)$. If $\mathscr{T}$ is a tree, $\mathscr{T}_u$ denotes the sub-tree rooted at $u \in \mathscr{T}$. Suppose $\mathscr{T}$ is a system over a finite tree. We say $\mathscr{T}$ is **quasi-saturated** if:

1. $\mathscr{T}$ validates the saturation conditions for all vertices with the exception of a finite set L of leaves;
2. every $l \in L$ may fail the saturation requirements only in condition f;
3. for every $l \in L$ there exists a non-leaf vertex u in $\mathscr{T}$ such that $u \sqsubseteq^* l$.

Theorem 5. *Let Γ be a nested sequent. TFAE*

1. *There exists a saturated expansion of Γ.*
2. *There exists a finite quasi-saturated expansion of Γ.*
3. *There exists a saturated expansion of Γ, $\mathscr{T}$, which is a tree, and a finite set $U \subseteq \mathscr{T}$ such that for every $v \in \mathscr{T}$ there exists $u \in U$ satisfying $\mathscr{T}_u \sqsubseteq^* \mathscr{T}_v$.*
4. *There exists a saturated expansion of Γ with a regular underlying frame.*

Proof. The implications $3 \Rightarrow 4$ and $4 \Rightarrow 1$ follow from the definitions. Moreover, Theorem 4 yields $1 \Rightarrow 2$. We show $2 \Rightarrow 3$. Suppose $\mathscr{S} = (S, E, \lambda)$ is quasi-saturated and let $U = \mathscr{S} \backslash L$ be the vertices of $\mathscr{S}$ that fulfil all the saturation conditions. Fix a vertex $l \in L$. By assumption there exists $u \in U$ with $u \sqsubseteq^* l$. Let $\mathscr{S}_u = (S_u, E_u, \lambda | S_u)$ be the sub-tree of $\mathscr{S}$ rooted at u, and $f \in \mathbb{I}(\kappa)$ be such that $u^f = l$. Consider the system $\mathscr{S}' = (\mathscr{S}_u)^f = (S_u, E_u, \lambda')$. In particular, $\lambda'(u) = \lambda(l)$. Define $\mathscr{T}$ to be the system comprising the disjoint union of $\mathscr{S}$ and $\mathscr{S}'$ where the leaf l in $\mathscr{S}$ is identified with the root u of $\mathscr{S}'$. We claim $\mathscr{T}$ is quasi-saturated. Let l' be a leaf in $\mathscr{T} \backslash \mathscr{S}$ which fails the saturation conditions and let $u' \in U$ be such that $\lambda_{\mathscr{S}}(u') \sqsubseteq^* \lambda_{\mathscr{S}}(l')$. By construction $\lambda_{\mathscr{T}}(u') = \lambda_{\mathscr{S}}(u')$ and $\lambda_{\mathscr{T}}(l') = \lambda_{\mathscr{S}}(l')^f$, so $\lambda_{\mathscr{T}}(u') \sqsubseteq^* \lambda_{\mathscr{T}}(l')$ by transitivity. Repeating the method of unravelling the unsaturated leaves and considering the limit system yields a saturated system with the desired properties. □

The following, due to Vardi [30], is an immediate consequence of Theorem 5.

Corollary 1. *The full μ-calculus has the regular model property.*

We claim the above result enables us to lower the bound on $\mathsf{K}_{\mu^+}^{\omega_1}$. The idea is to find a refinement of Theorem 5 that controls the approximations appearing in a saturated system. This is the role of the next proposition.

Proposition 1. *Let $\mathscr{T}$ be a κ-system satisfying condition 3 in Theorem 5. Suppose for every $u \in U$ and $\phi \sqsubseteq \psi \in u$, if $(\mathscr{T}, u) \not\models \phi$ then $\phi \in u$. Then* $\mathrm{Ker}\, u \subseteq L_\mu^{\omega^\omega}$ *for every $u \in U$.*

Proof. Suppose $\mathscr{T} = (S, E, \lambda)$ is as stated. We may assume U is closed downwards in the accessibility relation on $\mathscr{T}$. Let $\mathscr{T}_0$ be the finite sub-system restricted to vertices in U and their immediate successors. By assumption, $\mathscr{T}_0$ is quasi-saturated. Let $L = \{l_0, \ldots, l_{n-1}\}$ be the vertices of $\mathscr{T}_0$ not in U. These are leaves and for each $i < n$, let $u_i \in U$ be such that $\lambda(u_i) \sqsubseteq^* \lambda(l_i)$. Consider the κ-system $\mathscr{S}_0 = (S_0, E_0, \lambda_0)$ where $S_0 = U \cup L$, $E_0 = E|S_0$ and $\lambda_0 = (\mathrm{Ker} \circ \lambda)|S_0$ with $|$ denoting restricting the domain of the function/relation.

We have that $\lambda(u) \sqsubseteq^* \lambda(v)$ implies $\lambda_0(u) \sqsubseteq^* \lambda_0(v)$, so $\mathscr{S}_0$ is quasi-saturated. Let O be the set of ordinals occurring in the sets $\lambda_0(u)$ for $u \in S_0$, which is finite, and $(\alpha_i)_{i<|O|}$ enumerate the elements in O in increasing order. Define $f \colon O \to \omega^\omega$ by $f(\alpha_i) = \min\{\alpha_i, \omega^i\}$. We claim $\mathscr{S}_0^f$ is quasi-saturated. Since f is strictly monotone it suffices to check, for each $i < n$, that

$$\lambda_0(u_i)^f \sqsubseteq^* \lambda_0(l_i)^f. \tag{2}$$

Fixing $i < n$, let $a_0 < \cdots < a_k$ be such that $\alpha_{a_0}, \ldots, \alpha_{a_k}$ enumerates the ordinals in $\lambda_0(u_i)$ and let $b_0 < \cdots < b_l < |O|$ be the analogous sequence for $\lambda_0(l_i)$. Given $\lambda_0(u_i) \sqsubseteq^* \lambda_0(l_i)$ we must have $k = l$ and $a_j \le b_j$ for each $j \le k$. By induction on $j \le k$ we may define $h \in \mathbb{I}(\omega^\omega)$ such that $h(f(\alpha_{a_j})) = f(\alpha_{b_j})$ for every $j \le k$. In other words, h witnesses (2). So $\mathscr{S}_0^f$ is quasi-saturated. Moreover, for every vertex u of $\mathscr{S}_0$, $\lambda_0(u)^f \sqsubseteq \lambda_0(u) \sqsubseteq \lambda(u)$ by the choice of f, hence $\lambda_0(u)^f \subseteq \lambda(u)$ by the Truth Lemma and the additional assumption on $\mathscr{T}$. But then $\mathrm{Ker}\, u \subseteq \lambda_0(u)^f \subseteq L_\mu^{\omega^\omega}$. $\square$

Thus we obtain the following theorem.

Theorem 6. $\mathsf{K}_{\mu^+}^{\omega^\omega}$ *is sound and complete system over symmetric frames.*

Proof. Suppose $\Gamma = \Delta[\nu x \phi]$ is not valid. Applying Theorem 5 we obtain a κ-system expanding Γ, which can be further expanded to a system $\mathscr{S}$ satisfying the assumptions of Proposition 1 with the additional property that the vertex u which contains the formula $\nu x \phi$ specified by the context is an element of the designated finite set U. As a consequence of the proposition, $\mathrm{Ker}\, u \subseteq L_\mu^{\omega^\omega}$. By saturation, $\nu^\alpha x \phi' \in u$ for some $\alpha < \omega^\omega$ and $\phi' \sqsubseteq \phi$, whence the Truth Lemma implies $\Gamma[\nu^\alpha x \phi]$ is not valid. Thus the rule ν_{ω^ω} is sound. Completeness is given by Theorem 1. $\square$

It is not difficult (though rather technical) to show that the ordinal ω^ω is optimal for obtaining soundness over trees by leveraging the failure of the finite model property. For instance, to observe that the inference ν_{ω^2} is unsound (over trees), consider the sequent $\Gamma = \varrho, \psi, \phi$ where ψ expresses the existence of a finite $\{\mathfrak{a}, \mathfrak{b}\}$-path, $\varrho = \langle \overline{\mathfrak{a}} \rangle \top \vee \langle \overline{\mathfrak{b}} \rangle \top$ and

$$\phi = \nu x([\overline{\mathfrak{b}}](x \wedge \varrho) \wedge \mu y \langle \mathfrak{b} \rangle (y \vee x) \wedge \mu y \langle \mathfrak{a} \rangle (y \vee x)).$$

This observation can be readily generalised to show ν_{ω^n} is unsound for each n. Combining with the previous theorem we conclude

Theorem 7. $\mathsf{K}^{\kappa}_{\mu^+}$ *is unsound over trees for every* $\kappa < \omega^\omega$.

5 Discussion

There is an interesting tradeoff between the difficulty in establishing soundness and completeness for different axiomatisations of μ-calculus. With Kozen's axiomatisation the difficulty lies in showing completeness (soundness being reasonably straightforward) whereas in the goal-oriented proof system of [28] or the circular axiomatisations proposed in [1] the proof of soundness is more involved. The infinitary proof system $\mathsf{K}^{\omega^\omega}_{\mu^+}$ belongs to this second category.

Finally, we wish to remark on one further result contained in Vardi's seminal article: the tree languages definable by μ-calculus formulæ with converse modalities are precisely those definable by formulæ *without* converse. Suppose $\phi \mapsto \phi^*$ is an effective translation of formulæ into pure formulæ such that $\phi \leftrightarrow \phi^*$ is true in the symmetric closure of every tree. To re-phrase Vardi's result, an arbitrary tree can be endowed with a saturated ω^ω-system containing ϕ in the root iff it can be given a saturated ω-system with root containing ϕ^*. Since we know that the ordinals ω^ω and ω are optimal for the respective languages (over trees), this leads us to wonder what features of the interpretation give rise to this necessary collapse (the 'only if' direction) and expansion ('if' direction) of ordinals. We cannot say at this stage, but believe questions in this vein demonstrate a clear gap in our understanding of the proof theory of fixed point logic.

Acknowledgements. The work was initiated during the authors' research visit to the Hausdorff Research Institute for Mathematics (HIM), University of Bonn, as part of the trimester program *Types, Sets and Constructions, May–Aug 2018*. Both the financial support and the hospitality of HIM, are gratefully acknowledged.

This research was supported by the Swedish Research Council (grants 2016-03502 and 2017-05111) and the Knut and Alice Wallenberg Foundation.

The authors also wish to thank Rajeev Goré for his unsparing advice to look at nested sequents, Steve Simpson for his interest and references to the theory of better-quasi-orders, and Valentin Goranko for his suggestions which have improved the final presentation of the results.

A Well-Annotated Formulæ

We begin by making more precise the definition of well-annotated κ-formulæ, and the properties that this class satisfy.

Fix $\kappa \leq \omega_1$ and let $\lhd$ denote the subsumption ordering on Var, where $x \lhd y$ reads as x **subsumes** y. We assume $\lhd$ is a strict partial order on Var which is downwards linear. Recall that we consider $\lhd$ fixed and that all formulæ respect $\lhd$. Hence, if $\mu y \phi$ is a formula with x free, then $x \lhd y$.

An κ-**assignment** is a partial function from Var into ordinals $<\kappa$ whose domain is linearly ordered by $\lhd$. $\mathbb{A}(\kappa)$ is the set of κ-assignments and we let $\operatorname{dom} o$ denote the domain of $o \in \mathbb{A}(\kappa)$. It proves convenient to occasionally treat assignments as total functions $o\colon \mathsf{Var} \to \kappa+1$, and set $\operatorname{dom} o = \{x \in \mathsf{Var} \mid o(x) < \kappa\}$. Given $o \in \mathbb{A}(\kappa)$ and $x \in \mathsf{Var}$, $o_{\lhd x}$ denotes the restriction of o to the variables subsuming x:

$$o_{\lhd x}(y) = \begin{cases} o(y), & \text{if } y \lhd x, \\ \kappa, & \text{otherwise.} \end{cases}$$

For $\phi \in L_\mu$ and $o \in \mathbb{A}(\kappa)$, ϕ^o is the κ-formula generated as follows.

$$\begin{array}{lllll} x^o = x & (\phi \wedge \psi)^o = \phi^o \wedge \psi^o & ([\mathfrak{a}]\phi)^o = [\mathfrak{a}]\phi^o & (\mu x\phi)^o = \mu x\phi^{o_{\lhd x}} \\ & (\phi \vee \psi)^o = \phi^o \vee \psi^o & (\langle\mathfrak{a}\rangle\phi)^o = \langle\mathfrak{a}\rangle\phi^o & (\nu x\phi)^o = \nu^{o(x)}x\phi^{o_{\lhd x}} \end{array}$$

That is, ν-quantifiers in ϕ^o are approximated by their value under o (which is no approximation if the variable is outside the domain) except for variables occurring within the scope of a variable lower in the subsumption ordering. The significance of constraining $\operatorname{dom} o$ to be linearly ordered will become apparent shortly when we consider a quasi-ordering of $\mathbb{A}(\kappa)$.

Example 1. Suppose $x \lhd y$ and $o(x) = \alpha$ and $o(y) = \beta$, with $\alpha, \beta < \kappa$. Let ϕ be a formula without quantifiers containing both x and y free. Then $(\nu y\nu x\phi)^o = \nu^\beta y\nu^\alpha\phi$, whereas $(\nu x\nu y\phi)^o = \nu^\alpha x\nu y\phi$. The requirement that $\operatorname{dom} o$ is linear means that if $((\nu x\phi) \vee (\nu z\psi))^o = (\nu^\alpha x\phi') \vee (\nu^\gamma z\chi')$ then either one of α and γ is κ, or x and z are comparable in $\lhd$.

Definition 10. *The image of a well-formed formula under a κ-assignment is **well-annotated**. We let L_μ^κ be the set of well-annotated κ-formulæ.*

Recall that substitution is well-defined for well-formed formulæ.

Lemma 6. *If ϱ is well-named then every formula in $\mathbb{SC}_\kappa(\varrho)$ is well-annotated.*

Proof. Suppose $\phi = (\nu x\,\psi)^o = \nu^\alpha x\,\psi^{o_{\lhd x}} \in \mathbb{SC}_\kappa(\varrho)$. Then for each $\beta < \alpha$, we have $\phi' = \psi^{o_{\lhd x}}(x/\nu^\beta x\,\psi^{o_{\lhd x}}) \in \mathbb{SC}_\kappa(\varrho)$ by the closure condition and we require to show that ϕ' is well-annotated. Assume $o(x) < \kappa$ (otherwise the result is immediate) and let o' be the assignment with domain $\{y \in \operatorname{dom} o \mid y \lhd x \vee y = x\}$ determined by $o'(y) = o(y)$ for $y \lhd x$ and $o'(x) = \beta$. Given the fact that ϱ is well-named, x does not appear bound in ψ, whence it is easy to check that $\phi' = \psi(x/\nu x\,\psi)^{o'}$.

The other closure conditions are straightforward.

As defined, κ-assignments do not uniquely determine the formulæ in L_μ^κ. Each $\psi \in L_\mu$ determines an obvious equivalence relation on $\mathbb{A}(\kappa)$, given by $o \sim_\psi o'$ iff $\psi^o = \psi^{o'}$. However, for each $\phi \in L_\mu^\kappa$ there exists a unique κ-assignment o with smallest domain such that $\phi = \psi^o$, where $\psi = \phi^-$ is the template of ϕ. We call this assignment the **ordinal assignment** of ϕ and denote it o_ϕ.

We can thus give the formal definition of the quasi-order $\sqsubseteq$ introduced immediately prior to Definition 9. This starts with a quasi-order $\leq$ on κ-assignments, defined by $o \leq \hat{o}$ iff $\operatorname{dom} o \subseteq \operatorname{dom} \hat{o}$ and for every maximal chain

$x_0 \lhd x_1 \lhd \cdots \lhd x_n \in \operatorname{dom} o$ the sequence $(o(x_0), \ldots, o(x_n))$ is lexicographically prior to $(\hat{o}(x_0), \ldots, \hat{o}(x_n))$.

Lemma 7. $(\mathbb{A}(\kappa), \leq)$ *is a well-quasi-order. Moreover, there exists* k *such that for every set* $X \subseteq \mathbb{A}(\kappa)$ *with* $|X| \geq k$ *there exists* $o, \hat{o} \in X$ *s.t.* $o < \hat{o}$.

Proof. Transitivity of $\leq$ is established by induction along $\lhd$ in Var. So, $\leq$ is a quasi-order. Moreover, this quasi-order is a well-order on sets of κ-assignments with the same domain since it reduces to the lexicographic ordering on κ^k for some k (as domains are linearly ordered by $\lhd$). Since Var is a finite set, both claims follow. □

Definition 11. *Fix* $\varrho \in L_\mu$ *and for* $\phi, \psi \in \mathbb{SC}_\kappa(\varrho)$ *define* $\phi \sqsubseteq \psi$ *iff* $\phi^- = \psi^-$ *and* $o_\phi \leq o_\psi$.

This relation is well-defined because of Lemma 6, which implies that every formula in the strong closure of an L_μ formula is well-annotated and, hence, has a defined ordinal assignment.

We consider it instructive to note that there is another natural quasi-order sitting strictly between Kozen's $\preccurlyeq$ and our $\sqsubseteq$, obtained by dropping the restriction of linearity of annotated quantifiers but otherwise applying the lexicographic ordering in $\sqsubseteq$. This too is a wqo, but does not satisfy the second part of Lemma 4.

B Omitted Proofs

We now present some missing arguments from the main text. We begin with Lemma 4 as this result follows directly from our work on ordinal assignments:

Lemma 4. $(\mathbb{SC}_\kappa(\varrho), \sqsubseteq)$ *is a wqo. Moreover, there exists* k *such that for every* $X \subseteq \mathbb{SC}_\kappa(\varrho)$, $|\operatorname{Ker} X| < k$.

Proof. We need only remark that the quasi-order $(\mathbb{SC}_\kappa(\varrho), \sqsubseteq)$ can be expressed as the disjoint union of finitely many copies of $(\mathbb{A}(\kappa), \leq)$, one for each formula in $\mathbb{FL}(\varrho)$, an operation that preserves wqo-ness. The second claim follows from this fact and Lemma 7.

Lemma 1(2): *For all* $\phi \in L_\mu$ *and contexts* $\Gamma[]$, $\mathsf{K}^\kappa_\mu \vdash \Gamma[\phi, \overline{\phi}]$.

Proof. Induction on $\phi = \phi(x_1, \ldots, x_k)$ shows the inference

$$\frac{\Gamma[\psi_1, \chi_1] \quad \cdots \quad \Gamma[\psi_k, \chi_k]}{\Gamma[\overline{\phi}(\psi_1, \ldots, \psi_k), \phi(\chi_1, \ldots, \chi_k)]}$$

is admissible in K^κ_μ and $\mathsf{K}^\kappa_{\mu+}$. For the case $\phi = \nu y \phi_0$, we have a derivation of $\Gamma[\overline{\phi}, \nu^0 y \phi_0]$ by $\nu.0$, and from $\Gamma[\overline{\phi}, \nu^\alpha y \phi_0]$ we derive $\Gamma[\overline{\phi}, \nu^{\alpha+1} y \phi_0]$ via the induction hypothesis and inferences μ and $\nu.(\alpha+1)$. Thus transfinite induction shows that $\Gamma[\overline{\phi}, \nu^\alpha y \phi_0]$ is derivable for every $\alpha < \kappa$, whence $\Gamma[\overline{\phi}, \phi]$ results.

Theorem 4. The proof of this theorem ends with a statement of the following equivalence:

$$\forall X, Y \subseteq \mathbb{SC}_\kappa(\varrho) : X \sqsubseteq^* Y \quad \text{iff} \quad X^* \leq_{\mathrm{pw}} Y^*$$

On first appearance this result appears non-trivial. However, it is an easy consequence of the following result relating finite sets of ordinals, the verification of which is straightforward.

Lemma 8. *Given a non-empty finite set of ordinals A, let $A^* = (\delta_i^A)_{i<|A|}$ denote the unique sequence such that $A = \{\sum_{j\leq i} \delta_j^A \mid i < |A|\}$. Fix a principal ordinal κ and let $A, B \subset \kappa$ be non-empty finite sets of the same cardinality. There exists $f \in \mathbb{I}(\kappa)$ such that $B = \{f(\alpha) \mid \alpha \in A\}$ iff $A^* \leq_{pw} B^*$.*

References

1. Afshari, B., Leigh, G.E.: Cut-free completeness for modal mu-calculus. In: 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, pp. 1–12. IEEE Computer Society (2017). https://doi.org/10.1109/LICS.2017.8005088
2. Arisaka, R., Das, A., Straßburger, L.: On nested sequents for constructive modal logics. Log. Methods Comput. Sci. **11**(3) (2015). https://doi.org/10.2168/LMCS-11(3:7)2015
3. Bradfield, J., Stirling, C.: 12 modal mu-calculi. In: Blackburn, P., Benthem, J.V., Wolter, F. (eds.) Handbook of Modal Logic. Studies in Logic and Practical Reasoning, vol. 3, pp. 721–756. Elsevier (2007). https://doi.org/10.1016/S1570-2464(07)80015-2
4. Bradfield, J., Walukiewicz, I.: The mu-calculus and model checking. In: Clarke, E., Henzinger, T., Veith, H., Bloem, R. (eds.) Handbook of Model Checking, pp. 871–919. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-10575-8_26
5. Cachat, T.: Two-way tree automata solving pushdown games. In: Grädel, E., Thomas, W., Wilke, T. (eds.) Automata Logics, and Infinite Games. LNCS, vol. 2500, pp. 303–317. Springer, Heidelberg (2002). https://doi.org/10.1007/3-540-36387-4_17
6. Dam, M., Gurov, D.: μ-calculus with explicit points and approximations. J. Log. Comput. **12**(2), 255–269 (2002). https://doi.org/10.1093/logcom/12.2.255
7. Demri, S., Goranko, V., Lange, M.: Temporal Logics in Computer Science: Finite-State Systems. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge (2016). https://doi.org/10.1017/CBO9781139236119
8. Enqvist, S.: Flat modal fixpoint logics with the converse modality. J. Log. Comput. **28**(6), 1065–1097 (2018). https://doi.org/10.1093/logcom/exy016
9. Enqvist, S., Seifan, F., Venema, Y.: Completeness for the modal μ-calculus: separating the combinatorics from the dynamics. Theor. Comput. Sci. **727**, 37–100 (2018). https://doi.org/10.1016/j.tcs.2018.03.001
10. Fitting, M.: Nested sequents for intuitionistic logics. Notre Dame J. Form. Log. **55**(1), 41–61 (2014). https://doi.org/10.1215/00294527-2377869
11. Fitting, M., Kuznets, R.: Modal interpolation via nested sequents. Ann. Pure Appl. Log. **166**(3), 274–305 (2015). https://doi.org/10.1016/j.apal.2014.11.002
12. Goranko, V., Galton, A.: Temporal logic. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy, winter 2015. Metaphysics Research Lab, Stanford University (2015)

13. Goré, R.: And-Or tableaux for fixpoint logics with converse: LTL, CTL, PDL and CPDL. In: Demri, S., Kapur, D., Weidenbach, C. (eds.) IJCAR 2014. LNCS (LNAI), vol. 8562, pp. 26–45. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-08587-6_3
14. Goré, R., Postniece, L., Tiu A.: On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics. Log. Methods Comput. Sci. **7**(2) (2011). https://doi.org/10.2168/LMCS-7(2:8)2011
15. Grädel, E., Thomas, W., Wilke, T. (eds.): Automata Logics, and Infinite Games: A Guide to Current Research. Springer, New York (2002). https://doi.org/10.1007/3-540-36387-4
16. Jäger, G., Kretz, M., Studer, T.: Canonical completeness of infinitary μ. J. Log. Algebr. Program. **76**(2), 270–292 (2008). https://doi.org/10.1016/j.jlap.2008.02.005
17. Kashima, R.: Cut-free sequent calculi for some tense logics. Stud. Log. **53**(1), 119–135 (1994). https://doi.org/10.1007/BF01053026
18. Kozen, D.: A finite model theorem for the propositional μ-calculus. Stud. Log. **47**(3), 233–241 (1988). https://doi.org/10.1007/BF00370554
19. Kozen, D.: Results on the propositional μ-calculus. Theor. Comput. Sci. **27**, 333–354 (1983). https://doi.org/10.1016/0304-3975(82)90125-6
20. Marcone, A.: Fine analysis of the quasi-orderings on the power set. Order **18**(4), 339–347 (2001). https://doi.org/10.1023/A:1013952225669
21. Nash-Williams, C.S.J.A.: On better-quasi-ordering transfinite sequences. Math. Proc. Camb. Philos. Soc. **64**(2), 273–290 (1968). https://doi.org/10.1017/S030500410004281X
22. Nash-Williams, C.S.J.A.: On well-quasi-ordering transfinite sequences. Math. Proc. Camb. Philos. Soc. **61**(1), 33–39 (1965). https://doi.org/10.1017/S0305004100038603
23. Parikh, R.: The completeness of propositional dynamic logic. In: Winkowski, J. (ed.) MFCS 1978. LNCS, vol. 64, pp. 403–415. Springer, Heidelberg (1978). https://doi.org/10.1007/3-540-08921-7_88
24. Reynolds, M.: An axiomatization of PCTL*. Inf. Comput. **201**(1), 72–119 (2005). https://doi.org/10.1016/j.ic.2005.03.005
25. Reynolds, M.: More past glories. In: 15th Annual IEEE Symposium on Logic in Computer Science, LICS 2000, pp. 229–240. IEEE Computer Society (2000). https://doi.org/10.1109/LICS.2000.855772
26. Shamkanov, D.S.: Nested sequents for provability logic GLP. Log. J. IGPL **23**(5), 789–815 (2015). https://doi.org/10.1093/jigpal/jzv029
27. Smyth, M.B.: Power domains. J. Comput. Syst. Sci. **16**, 23–36 (1978). https://doi.org/10.1016/0022-0000(78)90048-X
28. Stirling, C.: A tableau proof system with names for modal mu-calculus. In: Voronkov, A., Korovina, M.V. (eds.) HOWARD-60: a festschrift on the occasion of howard Barringer's 60th Birthday. EPiC Series in Computing, vol. 42, pp. 306–318. EasyChair (2014)
29. Streett, R.S., Emerson, E.A.: An automata theoretic decision procedure for the propositional mu-calculus. Inf. Comput. **81**, 249–264 (1989). https://doi.org/10.1016/0890-5401(89)90031-X

30. Vardi, M.Y.: Reasoning about the past with two-way automata. In: Larsen, K.G., Skyum, S., Winskel, G. (eds.) ICALP 1998. LNCS, vol. 1443, pp. 628–641. Springer, Heidelberg (1998). https://doi.org/10.1007/BFb0055090
31. Walukiewicz, I.: Completeness of Kozen's axiomatisation of the propositional mu-calculus. In: Proceedings of the 10th Annual IEEE Symposium on Logic in Computer Science, LICS 1995, pp. 14–24. IEEE Computer Society (1995). https://doi.org/10.1109/LICS.1995.523240

Algebraic and Topological Semantics for Inquisitive Logic via Choice-Free Duality

Nick Bezhanishvili[1], Gianluca Grilletti[1(✉)], and Wesley H. Holliday[2]

[1] ILLC, University of Amsterdam, Amsterdam, The Netherlands
N.Bezhanishvili@uva.nl, grilletti.gianluca@gmail.com
[2] University of California, Berkeley, USA
wesholliday@berkeley.edu

Abstract. We introduce new algebraic and topological semantics for inquisitive logic. The algebraic semantics is based on special Heyting algebras, which we call *inquisitive algebras*, with propositional valuations ranging over only the $\neg\neg$-fixpoints of the algebra. We show how inquisitive algebras arise from Boolean algebras: for a given Boolean algebra B, we define its *inquisitive extension* $H(B)$ and prove that $H(B)$ is the unique inquisitive algebra having B as its algebra of $\neg\neg$-fixpoints. We also show that inquisitive algebras determine Medvedev's logic of finite problems. In addition to the algebraic characterization of $H(B)$, we give a topological characterization of $H(B)$ in terms of the recently introduced choice-free duality for Boolean algebras using so-called upper Vietoris spaces (UV-spaces) [2]. In particular, while a Boolean algebra B is realized as the Boolean algebra of compact regular open elements of a UV-space dual to B, we show that $H(B)$ is realized as the algebra of compact open elements of this space. This connection yields a new topological semantics for inquisitive logic.

1 Introduction

The inquisitive logic `InqB` [7] is an extension of propositional logic that encompasses logical relations between *questions* in addition to statements. To define `InqB`, Ciardelli et al. [6] introduced a semantics based on states of partial information, called *support* semantics, which generalizes the standard truth-based semantics of propositional logic. In [4], connections between this semantics and several intermediate logics—including Medvedev's logic ML [10] and the Kreisel-Putnam logic KP [3, p. 148]—were studied: in particular, `InqB` can be characterized as the logic of general intuitionistic Kripke models based on Medvedev's frames for which the valuations of atomic propositions are principal upsets. Even though the algebraic structures arising from this characterization have been considered in the literature [8], a proper algebraic and topological semantics for inquisitive logic is still missing. The aim of this paper is to fill this gap.

The original version of this chapter was revised: An acknowledgement has been added. The correction to this chapter is available at https://doi.org/10.1007/978-3-662-59533-6_41

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 35–52, 2019.
https://doi.org/10.1007/978-3-662-59533-6_3

After reviewing inquisitive logic and some topological preliminaries in Sect. 2, we start in Sect. 3 with an algebraic semantics for inquisitive logic based on Heyting algebras with propositional valuations ranging over only the $\neg\neg$-fixpoints of the algebra. The Kripke semantics for inquistive logic can be seen as a particular instance of this algebraic semantics: for F a Medvedev frame, the algebra $\mathrm{Up}_p(F)$ of principal upsets of F is the algebra of $\neg\neg$-fixpoints of the Heyting algebra $\mathrm{Up}(F)$ of all upsets of F. For our algebraic semantics, we motivate restricting attention to only special Heyting algebras, which we call *inquisitive algebras*, of which $\mathrm{Up}(F)$ for a Medvedev frame F is an example.

We show how inquisitive algebras arise from Boolean algebras: for a given Boolean algebra B, we define in Sect. 4.1 its *inquisitive extension* $H(B)$ and prove in Sect. 4.2 that $H(B)$ is the unique inquisitive algebra having B as its algebra of $\neg\neg$-fixpoints. We also show that inquisitive algebras determine Medvedev's logic. In addition to the algebraic characterization of $H(B)$ in Sect. 4.2, we give a topological characterization of $H(B)$ in Sect. 4.3 in terms of the recently introduced choice-free duality for Boolean algebras using so-called upper Vietoris spaces (UV-spaces) [2], which we review in Sect. 2.2. In particular, while a Boolean algebra B is realized as the Boolean algebra of compact regular open elements of a UV-space dual to B, we show that $H(B)$ is realized as the algebra of compact open elements of this space.

The topological characterization of $H(B)$ leads in Sect. 5 to a new topological semantics for inquisitive logic based on UV-spaces. As an additional benefit, we obtain a new topological semantics for Medvedev's logic.

We conclude in Sect. 6 with some directions for future research. Several appendices contain proofs deferred in the main text.

2 Preliminaries

2.1 Inquisitive Logic

In this section, we introduce the syntax and the world-based semantics of inquisitive logic and present some basic results used throughout the paper. Further details can be found in [5,7].

Fix a set AP of atomic propositions.

Definition 2.1. *The set $\mathcal{L}$ of inquisitive formulas* (*over* AP) *is defined by the following grammar:*

$$\phi \;:=\; \bot \mid p \mid (\phi \wedge \phi) \mid (\phi \to \phi) \mid (\phi \vee\!\!\!\vee\, \phi)$$

where $p \in \mathrm{AP}$. We define $\neg\phi := \phi \to \bot$ and $(\phi \vee \psi) := \neg(\neg\phi \wedge \neg\psi)$.

The standard propositional language is the $\vee\!\!\!\vee$-free fragment of our language. We will refer to formulas in this fragment as *classical formulas*.

The intuitive interpretation of classical formulas is the same as in propositional logic. For example, the formula $p \vee \neg p$ is interpreted as the (tautological) statement "p holds or p does not hold." The role of the new connective $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, called *inquisitive disjunction*, is to introduce *questions* in the logic. For example, the intuitive reading of the formula $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$ is the question "Does p hold?" This intuition is formalized by the standard support semantics for this language [6].

Definition 2.2. *Let $\mathcal{W}$ be a set of valuations for* AP *(i.e., functions from* AP *to* $\{0, 1\}$*). We recursively define the* support relation $\vDash$ *for formulas in $\mathcal{L}$ by:*

$$\begin{array}{lll} \mathcal{W} \vDash \bot & \iff & \mathcal{W} = \emptyset \\ \mathcal{W} \vDash p & \iff & \forall w \in \mathcal{W}.\ w(p) = 1 \\ \mathcal{W} \vDash \phi \wedge \psi & \iff & \mathcal{W} \vDash \phi \textit{ and } \mathcal{W} \vDash \psi \\ \mathcal{W} \vDash \phi \to \psi & \iff & \forall \mathcal{V} \subseteq \mathcal{W}.\, [\ \textit{if } \mathcal{V} \vDash \phi \textit{ then } \mathcal{V} \vDash \psi\ ] \\ \mathcal{W} \vDash \phi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi & \iff & \mathcal{W} \vDash \phi \textit{ or } \mathcal{W} \vDash \psi. \end{array}$$

A set $\mathcal{W}$ of valuations is interpreted as a *state of partial information*: we know that the actual state of affairs is represented by one of the valuations in $\mathcal{W}$, but we do not know by which one. The information available is enough to assert that a statement holds if every $w \in \mathcal{W}$ agrees on the statement being true. Under this interpretation, every $\mathcal{W}$ supports a tautology such as $p \vee \neg p$ (cf. Lemma 2.3 below). When it comes to questions, the information available solves a question if every $w \in \mathcal{W}$ agrees on *the same* solution. For example "Does p hold?", represented by $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$, is solved in $\mathcal{W}$ if $w(p) = 1$ for every $w \in \mathcal{W}$ *or* $w(p) = 0$ for every $w \in \mathcal{W}$, that is, if $\mathcal{W} \vDash p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$.

The following lemma can be proven by a straightforward induction.

Lemma 2.3

1. *For every ϕ, $\emptyset \vDash \phi$;*
2. *If $\mathcal{W} \vDash \phi$ and $\mathcal{V} \subseteq \mathcal{W}$, then $\mathcal{V} \vDash \phi$;*
3. *If α is a classical formula, $\mathcal{W} \vDash \alpha$ iff $\forall w \in \mathcal{W}.\ w(\alpha) = 1$.*[1]

This lemma tells us that for a given $\mathcal{W}$ and ϕ, the set $[\![\phi]\!]^{\mathcal{W}} := \{\mathcal{V} \subseteq \mathcal{W} \mid \mathcal{V} \vDash \phi\}$ is a non-empty $\subseteq$-downset; and moreover, if ϕ is classical, it is a principal downset. These observations suggest the following connection with Medvedev's logic ML—recall that ML is the logic of *Medvedev frames*, which are Kripke frames of the form $(\mathcal{P}_0(W), \supseteq)$ for W a finite set, where $\mathcal{P}_0(W) = \{V \subseteq W \mid V \neq \emptyset\}$.

Lemma 2.4 **(**[4, Proposition 2.2.2]**).** *Let $\mathcal{W}$ be a set of valuations and consider the intuitionistic Kripke model $(\mathcal{P}_0(\mathcal{W}), \supseteq, V)$ where*

$$V(p) = \mathcal{P}_0(\{w \in \mathcal{W} \mid w(p) = 1\}).$$

Then for every formula $\phi \in \mathcal{L}$, we have[2]

$$\mathcal{W} \vDash \phi \quad \iff \quad (\mathcal{P}_0(\mathcal{W}), \supseteq, V) \Vdash \phi.$$

[1] Here we consider the standard extension of valuations over atomic propositions to arbitrary propositional formulas.

[2] Under the intuitionistic semantics, we interpret $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ as the intuitionistic disjunction.

If $\mathcal{W}$ is finite, then $(\mathcal{P}_0(\mathcal{W}), \supseteq)$ is a Medvedev frame; and $V(p)$ has to be a principal upset of this frame. Moreover, if we are interested in the validity of a fixed formula $\phi(p_1, \ldots, p_n)$, we can restrict our attention to sets of valuations over $p_1, \ldots, p_n$, which are always finite. Thus, we obtain the following.

Proposition 2.5. `InqB` *is the logic of the class of intuitionistic Kripke models*

$$\{(\mathcal{P}_0(X), \supseteq, V) \mid X \textit{ is finite and } V(p) \textit{is principal for all } p \in \mathrm{AP}\}.$$

In [5, Sect. 3.1] a sound and complete natural deduction system for `InqB` is presented, which is equivalent to the following Hilbert style system:

Axioms	*IPC:*	Axioms of IPC.	
	KP:	$(\neg\phi \to \psi \vee\!\!\!\vee \chi) \to (\neg\phi \to \psi) \vee\!\!\!\vee (\neg\phi \to \chi)$	for every $\phi, \psi, \chi \in \mathcal{L}$.
	DNE:	$\neg\neg p \to p$	for every $p \in \mathrm{AP}$.
Rules	*MP:*	$\phi, \phi \to \psi / \psi$.	

2.2 UV-Spaces

In this section, we recall the basic constructions of the choice-free duality for Boolean algebras recently developed in [2]. They will be used in Sects. 4.3 and 5, where we introduce a topological semantics for inquisitive logic.

Recall that for any poset $(X, \leq)$, we define

$$\mathtt{Cl}_{\leq}(U) = \{x \in X \mid \exists y \geq x.\ y \in U\}, \tag{1}$$

$$\mathtt{Int}_{\leq}(U) = X \setminus \mathtt{Cl}_{\leq}(X \setminus U) = \{x \in X \mid \forall y \geq x.\ y \in U\}. \tag{2}$$

We call a set U *$\leq$-regular open* if $U = \mathtt{Int}_{\leq}\mathtt{Cl}_{\leq}(U)$. Let X be a topological space and $\leq$ its specialization order. Let $\mathcal{RO}(X)$ be the collection of $\leq$-regular open subsets of X. Let $\mathsf{CO}(X)$ denote the collection of compact open subsets of X. Finally, let $\mathsf{CO}\mathcal{RO}(X) = \mathsf{CO}(X) \cap \mathcal{RO}(X)$.

Definition 2.6. *An* upper Vietoris space ($\textit{UV-space}$) *is a T_0 space X such that:*

1. $\mathsf{CO}\mathcal{RO}(X)$ *is closed under* $\cap$ *and* $\mathtt{Int}_{\leq}(X \setminus \cdot)$*;*
2. *if* $x \not\leq y$*, then there is a* $U \in \mathsf{CO}\mathcal{RO}(X)$ *such that* $x \in U$ *and* $y \notin U$*;*
3. *every proper filter in* $\mathsf{CO}\mathcal{RO}(X)$ *is* $\mathsf{CO}\mathcal{RO}(x) = \{U \in \mathsf{CO}\mathcal{RO}(X) \mid x \in U\}$ *for some* $x \in X$*.*

Given a UV-space X the set $\mathsf{CO}\mathcal{RO}(X)$ forms a Boolean algebra, where $\wedge$ is the intersection, $\vee$ is $\mathtt{Int}_{\leq}\mathtt{Cl}_{\leq}$ of the union, and $\neg$ is $\mathtt{Int}_{\leq}$ of the set-theoretic complement. It was observed in [2] that $\mathsf{CO}\mathcal{RO}(X)$ coincides with the set of compact regular open (in the topology of X) subsets of X. Conversely, for a Boolean algebra B we consider the set $UV(B)$ of all proper filters of B and define a topology generated by $\{\widehat{a} \mid a \in B\}$, where $\widehat{a} = \{x \in UV(B) \mid a \in x\}$. Then $UV(B)$ is a UV-space, where the specialization order is the inclusion order of filters, and B is isomorphic to the algebra $\mathsf{CO}\mathcal{RO}(UV(B))$. This correspondence can be extended to a full (choice-free) duality of the category of Boolean algebras and the category of UV-spaces [2]. The name "upper Vietoris" refers to the fact that, assuming the Axiom of Choice, the UV-dual of a Boolean algebra B is homeomorphic to the space of closed subsets of the Stone dual of B equipped with the upper Vietoris topology (for a choice-free version of this, see [2]).

3 Algebraic Semantics via Inquisitive Algebras

In this section, we define inquisitive algebras and a semantics for `InqB` via these algebras. We start with the following well-known result (see, e.g., [9, p. 51]).

Proposition 3.1. *For any Heyting algebra H, let $H_{\neg\neg} = \{\neg\neg x \mid x \in H\}$. Then:*

1. *$H_{\neg\neg}$ forms a bounded $\{\wedge, \to\}$-subalgebra of H;*
2. *$H_{\neg\neg}$ forms a Boolean algebra with join given by $a \vee_{H_{\neg\neg}} b = \neg\neg(a \vee_H b)$.*

Example 3.2. Let B be a *complete* Boolean algebra and consider the Heyting algebras $\mathrm{Dw}_0(B)$ and $\mathrm{Dw}_p(B)$ of its non-empty and principal downsets, respectively. The latter is isomorphic to B, with the join in $\mathrm{Dw}_p(B)$ given by $\{a\}^{\downarrow} \vee \{b\}^{\downarrow} = \neg\neg(\{a\}^{\downarrow} \cup \{b\}^{\downarrow}) = \{a \vee_B b\}^{\downarrow}$, where $U^{\downarrow}$ is the downset generated by U. Then as shown in Appendix A:

$$\mathrm{Dw}_p(B) = (\mathrm{Dw}_0(B))_{\neg\neg}. \tag{3}$$

Example 3.3. Let B be a Boolean algebra—not necessarily complete—and let $\mathrm{Dw}_{fg}(B)$ be the set of finitely generated downsets of B. Then as shown in Appendix A:

$$\mathrm{Dw}_p(B) = (\mathrm{Dw}_{fg}(B))_{\neg\neg}. \tag{4}$$

Elements of $\mathrm{Dw}_{fg}(B)$ can be represented in a special way that will be useful for later results. The proof of the next lemma is straightforward.

Lemma 3.4. *Every downset $D \in \mathrm{Dw}_{fg}(B)$ can be represented in a unique way as $D = \{a_1, \ldots, a_n\}^{\downarrow}$ with $a_i \not\leq a_j$ for $i \neq j$.*

We now define an algebraic semantics for inquisitive logic by restricting the interpretations of atoms to $H_{\neg\neg}$, as in the definition of *inquisitive validity* below. We will denote the meet, join, and implication in a Heyting algebra with the same symbols used for the connectives of our language, $\wedge, \vee\!\!\!\vee$, and $\to$.

Definition 3.5 (Algebraic semantics)

Let H be a Heyting algebra and $V : \mathrm{AP} \to H$. For each $\phi \in \mathcal{L}$, we define $[\![\phi]\!]^{H,V} \in H$ recursively as follows:

$$\begin{aligned} [\![\bot]\!]^{H,V} &= \bot & [\![\phi \wedge \psi]\!]^{H,V} &= [\![\phi]\!]^{H,V} \wedge [\![\psi]\!]^{H,V} \\ [\![p]\!]^{H,V} &= V(p) & [\![\phi \vee\!\!\!\vee \psi]\!]^{H,V} &= [\![\phi]\!]^{H,V} \vee\!\!\!\vee [\![\psi]\!]^{H,V} \\ & & [\![\phi \to \psi]\!]^{H,V} &= [\![\phi]\!]^{H,V} \to [\![\psi]\!]^{H,V}. \end{aligned}$$

Let $H, V \vDash \phi$ mean that $[\![\phi]\!]^{H,V} = \top$.

A formula ϕ is intuitionistically valid in H *if for every $V : \mathrm{AP} \to H$, we have $[\![\phi]\!]^{H,V} = \top$. Let* $\mathrm{IntLog}(H)$ *be the set of formulas intuitionistically valid in H. A formula is* intuitionistically valid *if it is intuitionistically valid in every Heyting algebra.*

A formula ϕ is inquisitively valid in H *if for every $V : \mathrm{AP} \to H_{\neg\neg}$, we have $[\![\phi]\!]^{H,V} = \top$. Let* $\mathrm{InqLog}(H)$ *be the set of formulas inquisitively valid in H. A formula is* inquisitively valid *if it is inquisitively valid in every Heyting algebra.*

From now on we write $[\![\phi]\!]$ instead of $[\![\phi]\!]^{H,V}$ if H and V are clear from context. Some properties of the semantics are straightforward to prove. For example:

Lemma 3.6. *If ϕ does not contain the symbol $\vee\!\!\!\vee$, then $[\![\phi]\!] \in H_{\neg\neg}$.*

It is immediate that every intuitionistic theorem is an inquisitve validity. And since the image of the valuations is restricted to $H_{\neg\neg}$, the formula $\neg\neg p \to p$ is also valid. But it is not the case that $\neg\neg\phi \to \phi$ is valid for every $\phi \in \mathcal{L}$, as Example 3.7 shows, so the set of validities is not closed under uniform substitution.

Example 3.7. Consider $H = \mathrm{Dw}_{fg}(\mathcal{P}(W))$ for a finite set W with at least two elements. Notice that $H = \mathrm{Dw}_0(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W))$. In this case the algebraic semantics boils down to the support semantics for inquisitive logic (cf. Lemma 2.4).

Given $A \subseteq W$, one can easily verify that $\neg\neg\{A\}^{\downarrow} = \{A\}^{\downarrow}$ and consequently $\neg\neg p \to p \in \mathrm{InqLog}(H)$. On the other hand, for $A, B \subseteq W$ we have $\neg\neg\{A, B\}^{\downarrow} = \{A \cup B\}^{\downarrow}$ and thus $\neg\neg(p \vee\!\!\!\vee q) \to (p \vee\!\!\!\vee q) \notin \mathrm{InqLog}(H)$.

A natural question to ask is for which Heyting algebra H we have $\mathtt{InqB} \subseteq \mathrm{InqLog}(H)$. The following obvious lemma gives a partial answer to this question. We call H a KP-*algebra* if H validates KP.

Lemma 3.8. *If H is a* KP*-algebra, then* $\mathtt{InqB} \subseteq \mathrm{InqLog}(H)$.

Combining Lemma 3.8 with the fact that the standard support semantics is a special case of our algebraic semantics (see Example 3.7), we obtain the following:

Proposition 3.9. *The set of formulas valid on* KP*-algebras is exactly the set of* $\mathtt{InqB}$ *validities.*

However, arbitrary KP-algebras are somewhat "too big" for our semantics. For example, if $H = \mathrm{Dw}_0(B)$ for a complete Boolean algebra B, then no matter what valuation we consider, the semantic value $[\![\phi]\!]$ of a formula ϕ must be an element of the subalgebra generated by $\mathrm{Dw}_p(B)$, that is, $\mathrm{Dw}_{fg}(B)$. This observation can be formalized as follows.

Lemma 3.10. *Let H be a Heyting algebra and H' the subalgebra of H generated by $H_{\neg\neg}$. Then:*

1. $(H')_{\neg\neg} = H_{\neg\neg}$;

2. *for every valuation* $V : \mathrm{AP} \to H_{\neg\neg}$ *and formula* ϕ *we have* $[\![\phi]\!]^{H,V} = [\![\phi]\!]^{H',V}$*;*
3. *if* H *is a* KP*-algebra, so is* H'*.*

Thus, without loss of generality, we can restrict attention to algebras in which $H_{\neg\neg}$ generates H.

Definition 3.11. *A Heyting algebra* H *is* regularly generated *if it is generated by* $H_{\neg\neg}$*.*

In fact, we can motivate one more restriction on the class of algebras we consider. As in Subsect. 2.1, formulas of `InqB` are interpreted as sentences (statements or questions) and the support semantics agrees with this interpretation. For example, a question $p \mathbin{\vee\!\!\!\vee} \neg p$ ("Does p hold?") is supported in an information model iff either p ("p holds") or $\neg p$ ("p does not hold") is supported in the model. However, this is not necessarily the case in the algebraic setting: for example, a Boolean algebra B is trivially a regularly generated KP-algebra, since $B_{\neg\neg} = B$; and $[\![p \mathbin{\vee\!\!\!\vee} \neg p]\!] = \top$ regardless of the value of $[\![p]\!]$ and $[\![\neg p]\!]$.

This motivates us to recall the following standard definition [3, p. 455].

Definition 3.12. *A Heyting algebra* H *is* well connected *if for all* $a, b \in H$*, if* $a \mathbin{\vee\!\!\!\vee} b = 1$*, then* $a = 1$ *or* $b = 1$*.*

Thus, we finally arrive at our definition of the class of inquisitive algebras.

Definition 3.13 (Inquisitive algebra). *An* inquisitive algebra *is a regularly generated well-connected* KP*-algebra.*

In the next section, we show how to construct inquisitive algebras from Boolean algebras.

4 Inquisitive Extension of a Boolean Algebra

4.1 Construction of the Inquisitive Extension

We will show that for a given Boolean algebra B, there exists a *unique* inquisitive algebra H such that B is isomorphic to $H_{\neg\neg}$. We will construct this H as a quotient of the free Heyting algebra built using elements of B as constants. Consider the set

$$\mathcal{T} = \left\{ t(b_1, \ldots, b_n) \;\middle|\; t \text{ is a term in the signature } \left\{ \dot{\wedge}, \dot{\vee}, \dot{\to}, \dot{\bot}, \dot{\top} \right\} \right\}.$$

We also introduce the shorthand $\dot{\neg} t$ for $t \mathbin{\dot{\to}} \dot{\bot}$.

Define the binary relation $\approx$ on $\mathcal{T}$ as the smallest equivalence relation such that:

- $\approx$ respects all Heyting algebra equations (e.g., for commutativity of $\dot{\wedge}$ we require $t_1 \mathbin{\dot{\wedge}} t_2 \approx t_2 \mathbin{\dot{\wedge}} t_1$);
- $\approx$ respects KP: $\dot{\neg} t_1 \mathbin{\dot{\to}} (t_2 \mathbin{\dot{\vee}} t_3) \approx (t_1 \mathbin{\dot{\to}} t_2) \mathbin{\dot{\vee}} (t_1 \mathbin{\dot{\to}} t_3)$.

– $\approx$ agrees with the operations on B: for $a, b \in B$, $a \dot{\wedge} b \approx a \wedge b$; $a \dot{\to} b \approx a \to b$; $\dot{\bot} \approx \bot$; $\dot{\top} \approx \top$.

$\mathcal{T}/\approx$ has a natural structure of a KP-algebra, with operations defined as

$$[t_1] \wedge [t_2] = [t_1 \dot{\wedge} t_2] \qquad [t_1] \vee\!\!\!\vee [t_2] = [t_1 \dot{\vee} t_2] \qquad [t_1] \to [t_2] = [t_1 \dot{\to} t_2].$$

We call this algebra the *inquisitive extension of* B and denote it by $H(B)$. Notice that by construction it is a regularly generated KP-algebra. To simplify the notation, subsequently we will drop the square brackets. By construction, the following universal property holds.

Lemma 4.1. *Let B be a Boolean algebra and H a* KP*-algebra such that $B = H_{\neg\neg}$. Then there exists a unique homomorphism $h : H(B) \to H$ such that $h|_B = id_B$. Moreover, if H is regularly generated, then h is surjective.*

Proof. Consider the map $f : \mathcal{T} \to H$ defined by the clauses

$$\begin{array}{ll} f(b) = b, \text{ for } b \in B & f(t_1 \dot{\wedge} t_2) = f(t_1) \wedge f(t_2) \\ f(t_1 \dot{\vee} t_2) = f(t_1) \vee\!\!\!\vee f(t_2) & f(t_1 \dot{\to} t_2) = f(t_1) \to f(t_2). \end{array}$$

Since H is a KP-algebra and agrees with the operations on B, f factors through $H(B)$, and thus we obtain a quotient map $h : H(B) \to H$. Moreover, by construction, h is a Heyting algebra homomorphism.

The image of B is fixed and $H(B)$ is generated by B, so uniqueness follows. Moreover, if H is regularly generated, then h is surjective, since $B \subseteq h[H(B)]$ and B generates H.

The previous result allows us to understand the structure of the algebra $H(B)$. In particular, elements of $H(B)$ can be represented in a *disjunctive normal form*, corresponding to the normal form of `InqB` formulas (see [5, Prop. 2.4.4]).

Proposition 4.2

1. *Every $x \in H(B)$ can be represented in a unique way as $x = a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n$ with $a_1, \ldots, a_n \in B$ and $a_i \not\leq a_j$ for $i \neq j$.*
2. *$H(B) \cong \mathrm{Dw}_{fg}(B)$.*

We will call a representation of x as in item 1 *non-redundant*.

Proof. For the proof of item 1, see Appendix B.

For item 2, consider the map $h : H(B) \to \mathrm{Dw}_{fg}(B)$. Since

$$h(a_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_n) = h(a_1) \cup \cdots \cup h(a_n) = \{a_1, \ldots, a_n\}^{\downarrow},$$

h is injective. It is then easy to see that h is an isomorphism.

A direct consequence of Proposition 4.2 is that $H(B)$ is well connected and thus an inquisitive algebra. We can also prove the following interesting property of $H(B)$, which will be useful for later applications.

Lemma 4.3. *Let H' be a finitely generated subalgebra of $H(B)$. Then H' is a subalgebra of a finite subalgebra of $H(B)$ of the form $H(B')$, where B' a Boolean subalgebra of B.*

Proof. Let $a_1^1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_{k_1}^1, \ldots \; a_1^n \vee\!\!\!\vee \ldots \vee\!\!\!\vee a_{k_n}^n$ be the non-redundant representations of the generators of H', and let A be the set $A = \{a_j^i \mid i \leq n, j \leq k_i\}$. Let B' be the Boolean subalgebra of B generated by A. Notice that this is a finite algebra. Clearly $H' \subseteq H(B') \subseteq H(B)$.

Finally, the isomorphism of Proposition 4.2.2 maps $H(B')$ onto $\mathrm{Dw}_{fg}(B')$—which is finite, since $|\mathrm{Dw}_{fg}(B')|$ is equal to the number of antichains in B'. Therefore, $H(B')$ is finite.

The results of this section allow us to draw a strong connection between regularly generated KP-algebras and Medvedev's logic ML.

Theorem 4.4. *If H is a regularly generated* KP-*algebra, then H is an* ML-*algebra.*

Proof. Let H be a regularly generated KP-algebra. Then, by Lemma 4.1, H is a homomorphic image of some algebra of the form $H(B)$. Thus, it suffices to show that $H(B)$ is an ML-algebra.

It is well known that for every Heyting algebra A and intermediate logic L we have that A is an L-algebra iff every finitely generated subalgebra of A is an L-algebra. Therefore, by Lemma 4.3, we obtain that $H(B)$ is an ML-algebra iff $H(B')$ is an ML-algebra for every finite Boolean subalgebra B' of B.

Thus, we only need to prove the result for algebras of the form $H(B')$ where B' is finite. Then $B' \cong \mathcal{P}(W)$ for some finite set W. By Proposition 4.2,

$$H(B') \cong \mathrm{Dw}_{fg}(B') \cong \mathrm{Dw}_{fg}(\mathcal{P}(W)) \cong \mathrm{Dw}_0(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W)),$$

which is exactly the algebra corresponding to the Medvedev frame $(\mathcal{P}_0(W), \supseteq)$. We conclude that $H(B)$ is an ML-algebra and therefore H is also an ML-algebra.

Corollary 4.5

$$\begin{aligned} &\mathrm{IntLog}(\{H \mid H \textit{ is a regularly generated } \mathrm{KP}\textit{-algebra}\}) \\ =\ &\mathrm{IntLog}(\{H(B) \mid B \textit{ is a finite Boolean algebra}\}) \\ =\ &\mathrm{ML}. \end{aligned}$$

Proof. Let $\mathcal{C}_1$ be the class of regularly generated KP-algebras and $\mathcal{C}_2$ the class of $H(B)$'s for a finite Boolean algebra B. Firstly, notice that every $H(B)$ is a regularly generated KP-algebra, so $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Consequently $\mathrm{IntLog}(\mathcal{C}_1) \subseteq \mathrm{IntLog}(\mathcal{C}_2)$. Therefore, we just need to prove that $\mathrm{ML} \subseteq \mathrm{IntLog}(\mathcal{C}_1)$ and $\mathrm{IntLog}(\mathcal{C}_2) \subseteq \mathrm{ML}$.

The first inclusion follows directly from Theorem 4.4. For the second inclusion, consider an arbitrary Medvedev frame $(\mathcal{P}_0(W), \supseteq)$—recall that W is finite. As noticed in the proof of Theorem 4.4, the Heyting algebra corresponding to this frame is $\mathrm{Dw}(\mathcal{P}_0(W)) \cong H(\mathcal{P}(W))$. Hence it is isomorphic to an element of $\mathcal{C}_2$. It follows that $\mathrm{IntLog}(\mathcal{C}_2) \subseteq \mathrm{ML}$, as required.

4.2 Algebraic Characterization of the Inquisitive Extension

We are now ready to provide our first characterization of $H(B)$.

Theorem 4.6. *For a Boolean algebra B, its inquisitive extension $H(B)$ is the unique (up to isomorphism) inquisitive algebra such that $H(B)_{\neg\neg}$ is isomorphic to B.*

Proof. Let H be an inquisitive algebra where $H_{\neg\neg} \cong B$, and fix an isomorphism $g : H_{\neg\neg} \to B$. By Lemma 4.1, there exists a unique morphism $h : H(B) \to H$ such that $h|_{H(B)} = g$, which is surjective since H is regularly generated.

It only remains to show that h is also injective, thus proving that h is an isomorphism. For the proof of injectivity, see Appendix C.

Corollary 4.7. *A Heyting algebra A is an inquisitive algebra iff A is isomorphic to $H(A_{\neg\neg})$.*

Proof. The right-to-left implication is clear. For the left-to-right, consider an inquisitive algebra A. By Theorem 4.6, $H(A_{\neg\neg})$ is isomorphic to any inquisitive algebra with $A_{\neg\neg}$ as the set of $\neg\neg$-fixpoints. In particular, $A \cong H(A_{\neg\neg})$.

We conclude this section with a result analogous to Corollary 4.5 but now for inquisitive logic.

Corollary 4.8

$$\begin{aligned} &\mathrm{InqLog}(\{H \mid H \textit{ is a } \mathrm{KP}\textit{-algebra}\}) \\ =\, &\mathrm{InqLog}(\{H(B) \mid B \textit{ is a finite Boolean algebra}\}) \\ =\, &\mathtt{InqB}\,. \end{aligned}$$

Proof. By Lemma 3.8, $\mathtt{InqB}$ is included in the inquisitive logic of the two classes of algebras. For the other inclusion: by Proposition 4.2, given a finite set W we have $H(\mathcal{P}(W)) \cong \mathrm{Dw}(\mathcal{P}_0(W))$. So by Proposition 2.5, the inquisitive logic of the second class of algebras is indeed $\mathtt{InqB}$; and since the first class of algebras includes the second, we obtain both equalitites.

4.3 Topological Characterization of the Inquisitive Extension

Using the UV-spaces of Sect. 2.2, we can give a topological realization of $H(B)$, which in the next section will lead to a topological semantics of inquisitive logic. By item 2 of the following theorem, $H(B)$ may be characterized as (isomorphic to) the Heyting algebra of compact open sets of the UV-space dual to B.

Theorem 4.9. *Let B be a Boolean algebra and X its dual UV-space.*

1. $(\mathsf{O}(X), \subseteq) \cong \mathrm{Dw}_0(B)$.
2. $(\mathsf{CO}(X), \subseteq) \cong \mathrm{Dw}_{fg}(B) \cong H(B)$.

Proof. See Appendix D.

For those familiar with Esakia duality for Heyting algebras, we can further exploit Theorem 4.9 to obtain a connection between the choice-free duality for Boolean algebras and Esakia duality. This connection uses the following.

Proposition 4.10. *The following function defines an order isomorphism between the set* $\mathrm{Spec}(H(B))$ *of prime filters of* $H(B)$*, ordered by inclusion, and the set* $\mathrm{Filt}(B)$ *of filters of* B*, ordered by inclusion:*

$$\begin{array}{rccc} r: & (\mathrm{Spec}(H(B)), \subseteq) & \to & (\mathrm{Filt}(B), \subseteq) \\ & F & \mapsto & F \cap B \end{array}$$

Proof. See Appendix E.

Proposition 4.11. *Given* B *a Boolean algebra, the Esakia space* $\mathrm{Spec}(H(B))$ *dual to* $H(B)$ *is homeomorphic to the UV-space* $UV(B)$ *dual to* B*.*

Proof. The map r defined in Proposition 4.10 above is a homeomorphism; all the verifications are standard and left to the reader.

In particular, this gives us an alternative proof of Theorem 4.9.2.

The results of this section are summarized in Fig. 1.

algebras	**spaces**
$B \cong \mathsf{CO}\mathcal{RO}(UV(B))$	$UV(B)$
$H(B) \cong \mathsf{CO}(UV(B))$	$\wr\!\!\mid$
$H(B) \cong \mathsf{CO}(\mathrm{Spec}(H(B)))$	$\mathrm{Spec}(H(B))$

Fig. 1. Summary of results of Sect. 4.3.

5 Topological Semantics for Inquisitive Logic

Theorem 4.9 and Lemma 5.2 allow us to define a topological semantics for `InqB` using the duality based on UV-spaces.

Definition 5.1 (Topological semantics)

Let X *be a UV-space and* $V : \mathrm{AP} \to \mathsf{CO}\mathcal{RO}(X)$ *an atomic valuation. For each inquisitive formula* $\phi \in \mathcal{L}$*, we define its semantic valuation* $[\![\phi]\!]^{X,V} \in \mathsf{CO}(X)$ *by recursion as follows:*[3]

$$\begin{array}{ll} [\![\bot]\!]^{X,V} = \emptyset & [\![\phi \wedge \psi]\!]^{X,V} = [\![\phi]\!]^{X,V} \cap [\![\psi]\!]^{X,V} \\ [\![p]\!]^{X,V} = V(p) & [\![\phi \vee\!\!\!\vee \psi]\!]^{X,V} = [\![\phi]\!]^{X,V} \cup [\![\psi]\!]^{X,V} \\ & [\![\phi \to \psi]\!]^{X,V} = \mathtt{Int}\left((X \setminus [\![\phi]\!]^{X,V}) \cup [\![\psi]\!]^{X,V}\right). \end{array}$$

We adopt the same notational conventions for validity as in Definition 3.5.

[3] Notice that Theorem 4.9 ensures that $[\![\phi \to \psi]\!]^{X,V} \in \mathsf{CO}(X)$.

In the Boolean algebra $\mathsf{CO}\mathcal{RO}(X)$, implication is given by $U \to V = \neg U \vee V = \mathtt{Int}_{\leq}\mathtt{Cl}_{\leq}(\mathtt{Int}_{\leq}(X \setminus U) \cup V)$, and it is easy to check that the right-hand side is equal to $\mathtt{Int}_{\leq}((X \setminus U) \cup V)$. By the next result, we can also think in terms of the interior operator $\mathtt{Int}$ of the main topology, as in Definition 5.1, instead of the interior operator $\mathtt{Int}_{\leq}$ of the order topology.

Lemma 5.2. *Given $A, B \in \mathsf{CO}(X)$, $\mathtt{Int}((X \setminus A) \cup B) = \mathtt{Int}_{\leq}((X \setminus A) \cup B)$.*

Proof. See Appendix F.

Corollary 5.3. *The set of formulas valid on UV-spaces under this semantics is exactly the set of theorems of* $\mathtt{InqB}$.

Proof. Let X be a UV-space. By Theorem 4.9, $\mathsf{CO}(X) \cong H(\mathsf{CO}\mathcal{RO}(X))$. Moreover, by [2], every Boolean algebra is isomorphic to one of the form $\mathsf{CO}\mathcal{RO}(X)$. Combining this result with Corollary 4.8, we obtain:

$$\mathrm{InqLog}(\{X \mid X \text{ a UV-space}\}) = \mathrm{InqLog}(\{H(B) \mid B \text{ a Boolean algebra}\}) = \mathtt{InqB}\,.$$

We conclude this section by pointing out a connection with Medvedev's logic ML. UV-spaces can be used to give a new topological semantics for ML in a way analogous to inquisitive logic, namely by allowing valuations to range over CO-sets in Definition 5.1—and not only $\mathsf{CO}\mathcal{RO}$-sets.

Corollary 5.4. ML *is sound and complete with respect to the topological semantics presented above.*

Proof. This follows directly from Corollary 4.5 and Theorem 4.9.

6 Conclusion

In this paper, we introduced algebraic and topological semantics for inquisitive logic and connected them via choice-free duality for Boolean algebras [2]. This opens up new avenues for further research, three of which we will briefly mention.

The main results of this paper are concerned with KP-algebras, since the KP-axiom is essential for inquisitive logic. However, one could consider the more general case of arbitrary (regularly generated) Heyting algebras and study the corresponding generalized inquisitive logics.

Another generalization to consider is to replace the double negation nucleus $\neg\neg$ with an arbitrary (perhaps definable) nucleus on a Heyting algebra. Of course, the algebra of fixed points of such a nucleus will no longer be Boolean. This yields the nuclear semantics for "inquisitive intuitionistic logic" in [1]. How to characterize inquisitive extensions in that setting and what topological duality to use for their representation remain open problems.

Finally, just as in the case of intermediate and modal logics, where algebraic semantics and duality provide tools for studying lattices of these logics, we hope that this newly developed algebraic semantics and duality will open the door for investigations of lattices of inquisitive logics.

Acknowledgment. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 680220).

A Examples 1 and 2

Proof of (3)

Given a complete Boolean algebra B, we show that $\mathrm{Dw}_p(B) = (\mathrm{Dw}_0(B))_{\neg\neg}$. First, if we consider a principal downset, we have

$$\neg\{b\}^{\downarrow} = \{a \in B \mid a \wedge b = \bot\} = \{\neg b\}^{\downarrow} \quad \Longrightarrow \quad \neg\neg\{b\}^{\downarrow} = \{b\}^{\downarrow}.$$

So $\mathrm{Dw}_p(B) \subseteq (\mathrm{Dw}_0(B))_{\neg\neg}$. For the other inclusion, it suffices to show that $\neg D$ is principal for every downset D. We have

$$\neg D = \{a \in B \mid \forall d \in D.\ a \wedge d \leq \bot\} \subseteq \left\{\bigvee \neg D\right\}^{\downarrow}.$$

On the other hand, $\bigvee \neg D \in \neg D$, since for every $e \in D$, we have

$$e \wedge \bigvee \neg D = \bigvee\{e \wedge a \mid \forall d \in D.\ a \wedge d \leq \bot\} = \bigvee\{\bot\} = \bot.$$

It follows that $\neg D = \{\bigvee \neg D\}^{\downarrow}$. Thus, $\neg D$ is principal.

Proof of (4)

Given a Boolean algebra B, we show that $\mathrm{Dw}_p(B) = (\mathrm{Dw}_{fg}(B))_{\neg\neg}$. The inclusion $\mathrm{Dw}_p(B) \subseteq (\mathrm{Dw}_{fg}(B))_{\neg\neg}$ is proved as above. For the other inclusion it suffices to show that for any $b_1, \ldots, b_n \in B$, $\neg\{b_1, \ldots, b_n\}^{\downarrow}$ is principal. This follows from the equalities

$$\neg\{b_1, \ldots, b_n\}^{\downarrow} = \{a \in B \mid \forall i \leq n.\ a \wedge b_i = \bot\} = \{\neg b_1 \wedge \cdots \wedge \neg b_n\}^{\downarrow}.$$

B Proof of Proposition 4.2

We divide the proof in two steps: proving that every element $x \in H(B)$ can be written in the form $x = b_1 \vee\!\!\vee \ldots \vee\!\!\vee b_m$ with $b_1, \ldots, b_m \in B$; and proving that from this form we can obtain a non-redundant representation.

For the first part: since $H(B)$ is the quotient of the set $\mathcal{T}$ of terms, we can proceed by induction on $t \in \mathcal{T}$.

- If $x \in B$, then we are done.
- If $x = y \wedge z$, then consider two representations $y = c_1 \vee\!\!\vee \ldots \vee\!\!\vee c_k$ and $z = d_1 \vee\!\!\vee \ldots \vee\!\!\vee d_l$. Then

$$x = y \wedge z = (c_1 \vee\!\!\vee \ldots \vee\!\!\vee c_k) \wedge (d_1 \vee\!\!\vee \ldots \vee\!\!\vee d_l) = \bigvee\!\!\!\bigvee\{c_i \wedge d_j \mid i \leq k, j \leq l\}.$$

– If $x = y \vee\!\!\!\vee z$, then

$$x = y \vee\!\!\!\vee z = c_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee c_k \vee\!\!\!\vee d_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee d_l.$$

– If $x = y \to z$, then

$$\begin{aligned} x = y \to z &= (c_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee c_k) \to (d_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee d_l) \\ &= (c_1 \to d_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee d_l) \wedge \dots \wedge (c_k \to d_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee d_l) \\ &= \wedge_{i=1}^{l} ((c_i \to d_1) \vee\!\!\!\vee \dots \vee\!\!\!\vee (c_i \to d_l)) && \text{(by } KP\text{)} \\ &= \vee_{f:[n]\to[m]} \left(\wedge_{i=1}^{l} (c_i \to d_{f(i)})\right). \end{aligned}$$

For the second part: let $x = b_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee b_m$ be an arbitrary representation of x. If $\forall i, j.\ b_i \not\leq b_j$, then we are done. Otherwise, suppose (without loss of generality) that $b_1 \leq b_2$. Then

$$b_1 \vee\!\!\!\vee b_2 \vee\!\!\!\vee \dots \vee\!\!\!\vee b_n = b_2 \vee\!\!\!\vee \dots \vee\!\!\!\vee b_n.$$

Repeating this procedure, we obtain a non-redundant representation of x.

C Proof of Theorem 4.6

It only remained to prove that h is injective. Let $x, y \in H(B)$ and suppose that $h(x) = h(y)$. Let $x = a_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee a_n$ and $y = b_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee b_m$ be their non-redundant representations. Then where $\sqcap, \sqcup, \Rightarrow$ are the operations of H, we have

$$\begin{aligned} & a_1 \sqcup \dots \sqcup a_n = b_1 \sqcup \dots \sqcup b_m \\ \Longrightarrow\ & (a_1 \sqcup \dots \sqcup a_n) \Leftrightarrow (b_1 \sqcup \dots \sqcup b_m) = \top \\ \Longrightarrow\ & \begin{cases} \bigsqcup_{f:[n]\to[m]} \bigsqcap_{i\leq n} (a_i \Rightarrow b_{f(i)}) = \top \\ \bigsqcup_{g:[m]\to[n]} \bigsqcap_{j\leq m} (b_j \Rightarrow a_{g(j)}) = \top \end{cases} \\ \Longrightarrow\ & \begin{cases} \exists f : [n] \to [m].\ \bigsqcap_{i\leq n} (a_i \Rightarrow b_{f(i)}) = \top \\ \exists g : [m] \to [n].\ \bigsqcap_{j\leq m} (b_j \Rightarrow a_{g(j)}) = \top \end{cases} && \text{(since } H \text{ is inquisitive)} \\ \Longrightarrow\ & \begin{cases} \forall i \leq n.\ \exists j \leq m.\ (a_i \Rightarrow b_j) = \top \\ \forall j \leq m.\ \exists i \leq n.\ (b_j \Rightarrow a_i) = \top \end{cases} \\ \Longrightarrow\ & \begin{cases} \forall i \leq n.\ \exists j \leq m.\ a_i \leq b_j \\ \forall j \leq m.\ \exists i \leq n.\ b_j \leq a_i \end{cases} && \text{(since } h|_B = id_B \text{)} \\ \Longrightarrow\ & \begin{cases} x \leq y \\ y \leq x \end{cases} \\ \Longrightarrow\ & x = y. \end{aligned}$$

So h is injective and thus an isomorphism, as required.

D Proof of Theorem 4.9

To prove Theorem 4.9, we will use the following lemma.

Lemma D.1. *Let $A = \bigcup_{i\in I} U_i$ and $B = \bigcup_{j\in J} V_j$ be open sets of a UV-space X, where U_i, V_j are $\mathsf{C}\mathcal{O}\mathcal{R}\mathcal{O}$-sets. Then $A \subseteq B$ iff $\forall i \in I.\ \exists j \in J.\ U_i \subseteq V_j$.*

Proof. Firstly, we show that every $\mathsf{CO}\mathcal{RO}$-set U is the upset of a singleton: since $\{U\}^{\uparrow}$ is a filter in $\mathsf{CO}\mathcal{RO}(X)$, there exists a point x such that $\{U\}^{\uparrow} = \mathsf{CO}\mathcal{RO}(X)$. It follows that $U = \bigcap \mathsf{CO}\mathcal{RO}(x) = \{x\}^{\uparrow}$.

We can use this to prove the result. Call x_i the generator of U_i for each $i \in I$.

$$\begin{aligned} A \subseteq B \quad &\iff \textstyle\bigcup_{i\in I} U_i \subseteq \bigcup_{j\in J} V_j &&\iff \forall i \in I.\, U_i \subseteq \textstyle\bigcup_{j\in J} V_j \\ &\iff \forall i \in I.\, U_i \subseteq \textstyle\bigcup_{j\in J} V_j &&\iff \forall i \in I.\, x_i \in \textstyle\bigcup_{j\in J} V_j \\ &\iff \forall i \in I.\, \exists j \in J.\, x_i \in V_j &&\iff \forall i \in I.\, \exists j \in J.\, U_i \subseteq V_j. \end{aligned}$$

We are now ready to prove Theorem 4.9.

Proof of Theorem 4.9.

For the first part: consider the map $f : O(X) \to \mathrm{Dw}_0(B)$ defined by[4]

$$f\left(\bigcup_{i\in I} \widehat{a_i}\right) = \{a_i \mid i \in I\}^{\downarrow}.$$

To show that f is well defined and order preserving and reflecting, we observe the following equivalences, using Lemma D.1 for the first:

$$\begin{aligned} \bigcup_{i\in I} \widehat{a_i} \subseteq \bigcup_{j\in J} \widehat{b_j} &\iff \forall i \in I.\, \exists j \in J.\, \widehat{a_i} \subseteq \widehat{b_j} \\ &\iff \forall i \in I.\, \exists j \in J.\, a_i \leq b_j \\ &\iff \forall i \in I.\, \exists j \in J.\, \{a_i\}^{\downarrow} \subseteq \{b_j\}^{\downarrow} \\ &\iff \{a_i \mid i \in I\}^{\downarrow} \subseteq \{b_j \mid j \in J\}^{\downarrow}. \end{aligned}$$

Thus, f is also injective. Notice that surjectivity is trivially satisfied. Hence f is an isomorphism.

For the second part: since elements of $\mathsf{CO}(X)$ are exactly the sets of the form $\widehat{a_1} \cup \cdots \cup \widehat{a_n}$ for some $a_1, \ldots, a_n \in B$, we obtain that $f|_{\mathsf{CO}(X)}$ is an isomorphism with range $\mathrm{Dw}_{fg}(B)$, as required.

E Proof of Proposition 4.10

It is immediate that r is well defined and order preserving. For injectivity, notice that a prime filter $\mathfrak{p}$ of $H(B)$ is completely determined by the elements of B it contains, since for every non-redundant representation $a_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} a_n$, we have

$$a_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} a_n \in \mathfrak{p} \iff a_1 \in \mathfrak{p} \text{ or } \ldots \text{ or } a_n \in \mathfrak{p}. \tag{5}$$

Using this fact, we can also show surjectivity: let F be a filter of B and define $\mathfrak{p}_F$ as the smallest set including F and respecting (5). Then clearly $\mathfrak{p}_F$ is an

[4] Here we are adopting the convention $\{\}^{\downarrow} := \{\bot\}$, so that $f(\emptyset) = \{\bot\}$.

upset and respects the $\Vvee$-condition of prime filters. Moreover, it is closed under meets, since

$$\begin{aligned}
&a_1 \Vvee \ldots \Vvee a_n \in \mathfrak{p}_F \text{ and } b_1 \Vvee \ldots \Vvee b_m \in \mathfrak{p}_F \\
\iff &\exists i.\ a_i \in \mathfrak{p}_F \text{ and } \exists j.\ b_j \in \mathfrak{p}_F \\
\iff &\exists i.\ a_i \in F \text{ and } \exists j.\ b_j \in F \\
\iff &\exists i.\ \exists j.\ a_i \wedge b_j \in F \\
\iff &\exists i.\ \exists j.\ a_i \wedge b_j \in \mathfrak{p}_F \\
\iff &(a_1 \Vvee \ldots \Vvee a_n) \wedge (b_1 \Vvee \ldots \Vvee b_m) = \bigvee\!\!\!\bigvee \{a_i \wedge b_j \mid i \leq n, j \leq m\} \in \mathfrak{p}_F.
\end{aligned}$$

Since $r(\mathfrak{p}_F) = F$, we also have surjectivity.

F Proof of Lemma 5.2

To prove Lemma 5.2, we first need to establish some technical results. In the following we denote $X \setminus A$ by $\overline{A}$. For a UV space X and $x, y \in X$, let $x \sqcap y$ be the greatest lower bound of x and y in the specialization order of X [2, Corollary 5.4].

Lemma F.1. *Let $U \in \mathsf{CO}\mathcal{RO}(X)$ and $x_1, x_2 \in U$. Then $x_1 \sqcap x_2 \in U$.*

Proof. By Corollary 5.4 of [2], $U = U \vee U = U \cup \{x \sqcap y \mid x, y \in U\}$.

Lemma F.2. *Given $U, V \in \mathsf{CO}\mathcal{RO}(X)$, $\mathtt{Int}_{\leq}\left(\overline{U} \cup V\right) = \neg U \vee V$.*

Proof.

Left-to-right inclusion. Consider an element $x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$. If $x \in \neg U \cup V$, then there is nothing to prove; so suppose this is not the case. By Corollary 5.4 of [2], there is a decomposition $x = x_1 \sqcap x_2$ such that $x_1 \in \neg U$ and $x_2 \in U$.

Since $x_2 \notin \overline{U}$ and $x_2 \geq x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$, it follows that $x_2 \in V$. So $x \in \{y \sqcap z \mid y \in \neg U, z \in V\} \subseteq \neg U \vee V$, as desired.

Right-to-left inclusion. Consider $x \in \neg U \vee V$ and take an arbitrary $w \geq x$. We want to show that $w \in \overline{U} \cup V$.

If $w \in \neg U \cup V \subseteq \overline{U} \cup V$, then there is nothing to prove; so suppose this is not the case. By Corollary 5.4 of [2], we can write $w = w_1 \sqcap w_2$ with $w_1 \in \neg U$ and $w_2 \in V$. In particular, w_1 is a successor of w not in U, and since $\overline{U}$ is a $\leq$-downset, it follows that $w \in \overline{U} \subseteq \overline{U} \cup V$.

Since w was an arbitrary successor of x, it follows $x \in \mathtt{Int}_{\leq}\left(\overline{U} \cup V\right)$.

Lemma F.3. *Given $U_i, V_j \in \mathsf{CO}\mathcal{RO}(X)$, the following identity holds:*

$$\mathtt{Int}_{\leq}\left(\left(\bigcap_{i=1}^{m} \overline{U_i}\right) \cup \left(\bigcup_{j=1}^{n} V_j\right)\right) = \bigcup_{f:[m]\to[n]} \bigcap_{i=1}^{m} \left(\neg U_i \vee V_{f(i)}\right).$$

Proof. By Lemma F.2, the identity is equivalent to

$$\mathtt{Int}_{\leq}\left(\left(\bigcap_{i=1}^{m}\overline{U_i}\right)\cup\left(\bigcup_{j=1}^{n}V_j\right)\right)=\bigcup_{f:[m]\to[n]}\mathtt{Int}_{\leq}\left(\bigcap_{i=1}^{m}\left(\overline{U}_i\cup V_{f(i)}\right)\right).$$

Let L and R be the left-hand side and right-hand side, respectively.

Right-to-left inclusion. Consider $x\in R$. This means that:

$$\exists f:[m]\to[n].\ \forall y\geq x.\ y\in\bigcap_{i=1}^{m}\left(\overline{U}_i\cup V_{f(i)}\right).$$

So with fixed f as above, given $y\geq x$, we have:

$$y\in\bigcap_{i=1}^{m}\left(\overline{U}_i\cup V_{f(i)}\right)\subseteq\bigcap_{i=1}^{m}\left(\overline{U}_i\cup\left(\bigcup_{j=1}^{n}V_j\right)\right)=\left(\bigcap_{i=1}^{m}\overline{U_i}\right)\cup\left(\bigcup_{j=1}^{n}V_j\right).$$

As y was an arbitrary successor of x, it follows that $x\in L$.

Left-to-right inclusion. We will show this step by contradiction. Suppose that $x\notin R$. This means that:

$$\forall f:[m]\to[n].\ \exists y\geq x.\ \exists i\in[m].\ y\notin\overline{U}_i\cup V_{f(i)},$$

or equivalently

$$\exists i\in[m].\ \forall j\in[n].\ \{x\}^{\uparrow}\cap U_i\cap\overline{V}_j\neq\emptyset.$$

Fix an index k instantiating the first quantifier, and consider for each $j\in[n]$ an element $y_j\in\{x\}^{\uparrow}\cap U_k\cap\overline{V}_j$. Define $y=y_1\sqcap\cdots\sqcap y_n$. We have:

- For every $j\in[n]$, $y_j\geq x$, and thus $y\geq x$.
- Since $y_j\in\overline{V}_j$ and V_j is open, it follows that $\mathtt{Cl}(\{y_j\})\subseteq\overline{V}_j$; and consequently $y\in\overline{V}_j$, since $y\leq y_j$.
- Since $y_1,\ldots,y_n\in U_k$, we have $y\in U_k$ (see Lemma F.1).

So it follows that $y\geq x$ and $y\in U_k\cap\overline{V_1}\cap\cdots\cap\overline{V_n}$. Thus in particular $y\notin\left(\bigcap_{i=1}^{m}\overline{U}_i\right)\cup\left(\bigcup_{j=1}^{n}V_j\right)$, from which we obtain $x\notin L$, as desired.

We are now able to prove Lemma 5.2.

Proof (Proof of Lemma 5.2). By Lemma F.3, $\mathtt{Int}_{\leq}(\overline{A}\cup B)\in\mathsf{CO}(X)$. Since the order topology is finer than the main topology, we have

$$\mathtt{Int}(\overline{A}\cup B)=\mathtt{Int}\left(\mathtt{Int}_{\leq}(\overline{A}\cup B)\right)=\mathtt{Int}_{\leq}(\overline{A}\cup B).$$

References

1. Bezhanishvili, G., Holliday, W.H.: Inquisitive intuitionistic logic. Manuscript (2019)
2. Bezhanishvili, N., Holliday, W.H.: Choice-free Stone duality. J. Symb. Log. (Forthcoming)
3. Chagrov, A., Zakharyaschev, M.: Modal logic. In: Oxford Logic Guides, vol. 35. The Clarendon Press, New York (1997)
4. Ciardelli, I.: Inquisitive semantics and intermediate logics. MSc thesis, University of Amsterdam (2009)
5. Ciardelli, I.: Questions in logic. Ph.D. thesis, Institute for Logic, Language and Computation, University of Amsterdam (2016)
6. Ciardelli, I., Groenendijk, J., Roelofsen, F.: Inquisitive semantics: a new notion of meaning. Lang. Linguist. Compass **7**(9), 459–476 (2013)
7. Ciardelli, I., Groenendijk, J., Roelofsen, F.: Inquisitive Semantics. Oxford University Press, Oxford (2018)
8. Frittella, S., Greco, G., Palmigiano, A., Yang, F.: A multi-type calculus for inquisitive logic. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 215–233. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_14
9. Johnstone, P.T.: Stone spaces. In: Cambridge Studies in Advanced Mathematics, vol. 3. Cambridge University Press, Cambridge (1982)
10. Medvedev, Y.T.: Interpretation of logical formulas by means of finite problems. Sov. Math. Dokl. **7**(4), 857–860 (1966)

Rigid First-Order Hybrid Logic

Patrick Blackburn[1(✉)], Manuel Martins[2], María Manzano[3], and Antonia Huertas[4]

[1] IKH, Roskilde University, Roskilde, Denmark
patrick.rowan.blackburn@gmail.com
[2] CIDMA, University of Aveiro, Aveiro, Portugal
martins@ua.pt
[3] University of Salamanca, Salamanca, Spain
mara@usal.es
[4] Universitat Oberta de Catalunya, Barcelona, Spain
mhuertass@uoc.edu

Abstract. Hybrid logic is usually viewed as a variant of modal logic in which it is possible to refer to worlds. But when one moves beyond propositional hybrid logic to first or higher-order hybrid logic, it becomes useful to view it as a systematic *modal language of rigidification*. The key point is this: @ can be used to rigidify not merely formulas, but other types of symbol as well. This idea was first explored in first-order hybrid logic (without function symbols) where @ was used to rigidify the first-order constants. It has since been used in hybrid type-theory: here one only has function symbols, but they are of every finite type, and @ can rigidify any of them. This paper fills the remaining gap: it introduces a first-order hybrid language which handles function symbols, and allows predicate symbols to be rigidified. The basic idea is straightforward, but there is a slight complication: transferring information about rigidity between the level of terms and formulas. We develop a syntax to deal with this, provide an axiomatization, and prove a strong completeness result for a varying domain (actualist) semantics.

Keywords: Hybrid logic · First-order modal logic · Rigidity · Rigid predicate symbols · Function symbols · Varying domains · Actualist semantics · Henkin models

1 Introduction

Hybrid logic is usually viewed as a variant of modal logic in which it is possible to refer to worlds. But when one moves beyond propositional hybrid logic to first- or even higher-order hybrid logic, it becomes more useful to view it as a systematic *modal language of rigidification*. Rigidity has long been an important concept in first-order modal logic: a first-order constant is said to be rigid if it denotes the same individual in all worlds, and free first-order variables in most first-order modal logics are interpreted rigidly. But rigidity is also central to hybrid logic, and it is central at all levels, from the propositional to the type-theoretic.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 53–69, 2019.
https://doi.org/10.1007/978-3-662-59533-6_4

Recall that propositional hybrid logic makes use of special propositional symbols called *nominals* (typically written as i, j and k) that are true at exactly one world in any model: in effect, nominals act as names for the unique world they are true at. Basic hybrid logic also allows us to write expressions of the form $@_i p$, which means "evaluate p at the unique world called i". Read this way, $@_i$ is a *modality* whose task is to inspect what is going on at the i-world, and see whether p is true there or not. But $@_i$ can also be read as a *rigidifier*: "form the *rigid* proposition $@_i p$ out of the proposition p". Note that $@_i p$ is indeed rigid: it is either true at all worlds, or false at all worlds, depending on whether p is true or false at the i-world.

As was soon realized, we can use @ to rigidify more than propositions. The paper [5], which explores first-order hybrid logic, took the first step in this direction by introducing expressions of the form $(@_i c)$. Here c is a first-order constant, which may denote different individuals in different worlds, and $(@_i c)$ is the *rigid* constant that denotes (at all worlds) whatever it is that c denotes at world i.[1] For example, if we think of c as a (non-rigid) constant that names Donald-Trump in some world w, and Bernie-Sanders in some other world w', then, if j is the nominal that names world w', $(@_j c)$ is a *rigid* constant that names Bernie-Sanders at all worlds. On the technical side, it was observed that Henkin-stye model building techniques could be used to build first-order Kripke models whose frames were defined using equivalence classes of nominals, and whose domains of quantification were defined using equivalence classes of rigidified constants; this led to a number of general completeness and interpolation results (see [2,6]).

But the idea of using @ as a general-purpose rigidifier has been most widely applied in the setting of *higher-order* hybrid logic. A range of higher-order hybrid logics, most based on Church's theory of simple types, have been defined (see [1,4,9,10]) and general Henkin-style completeness results for them proved. Although these systems differ in various ways, they have two points in common. First, they all allow expressions of the form $(@_i f)$ to be formed where f is a function symbol of *any* finite type. That is: in the higher-order setting, @ is totally overloaded—it can rigidify *all* the types of information that these languages can describe. Second, the higher-order Kripke models needed to prove completeness are constructed out the equivalence classes[2] of nominals (to define the frame) and equivalence classes of rigidified function symbols are used to define the needed function hierarchies.

One gap in this picture remains, and the purpose of this paper is to start filling it: to strengthen the first-order hybrid language defined in [5] to handle first-order function symbols and rigidified predicate symbols. There are several

[1] Note that in this paper expressions of the form $(@_i c)$ were introduced *in addition* to expressions of the form $@_i p$. As the authors of this paper put it: they deliberately *overloaded* the @ symbol. In this paper, we are going to overload @ even more. Our basic convention will be to omit the enclosing out brackets when propositional information is rigidified (as in $@_i p$), and to use enclosing brackets when other types of information are rigidified (as in $(@_i c)$). More on this later.

[2] Or sets of rigidified function symbols in the partial type theory explored in [9].

reasons for exploring such languages. For a start, many of the classical conceptual problems surrounding modal logic arise in first-order modal logic. One could explore them in a higher-order logic, but these are more complex, and bring new philosophical problems along with them. An expressive first-order hybrid language—one that makes it possible to rigidify constants, function symbols, predicate symbols and formulas—offers useful resources for addressing classical conceptual issues. Rigidifying predicate symbols, for example, allows us to express precise distinctions: we can talk about the relation of Love with reference to the pairs of individuals in love in the i-world, that is, $(@_i\mathsf{Love})$; or we can specify that we are interested in its denotation in the j-world using $(@_j\mathsf{Love})$.

There is also a more down-to-earth reason for our interest in this language: it is rare to see function symbols treated in any detail in discussions of first-order modal logic. Most authors skate lightly over the omission, and the reader is left with the impression that extending first-order modal logic to cope with function is a routine extension of what is already known. This seems misguided. First-order modal logic raises a wide range of technical and conceptual challenges, especially if one wants to work with varying domains (that is: with an actualist semantics). This paper provides a general approach to handling function symbols in a first-order modal logic with an actualist semantics.

The basic ideas explored in this paper are straightforward and build on previous work in the hybrid literature (probably [6] and [3] are the two most directly relevant references). However there is one complication: transferring information about rigidity between the first-order terms and formulas (this is an issue that does not arise in the type-theoretic case, where one only has to deal with function symbols). We cope with this by defining a recursive notion of rigidification, which keeps the term syntax relatively simple.

We proceed as follows. In Sect. 2 we define the syntax of our first-order language of rigidification, and what it means to rigidify a term. In Sect. 3 we define a varying domain semantics for our language, and note a basic lemma about rigid terms. In Sect. 4 we provide an axiomatisation, and in Sect. 5 and the Appendix we prove that it is complete. Finally, in Sect. 6 we sketch the ways we are developing the work reported here.

2 Rigid First-Order Hybrid Logic

We start with a first-order signatures, consisting of n-ary function and relation symbols:

Definition 1 (Signature). *A first-order* signature Σ *is a pair* $((\mathrm{Func}_n)_{n\in\mathbb{N}}, (\mathrm{Rel}_n)_{n\in\mathbb{N}})$*, where* Func_n *and* Rel_n *are sets of functional and relational symbols of arity* n*, respectively. The indexed elements in either family may be empty, and if they are all empty, we have the* empty signature*. The elements of* Func_0 *(if any) are called* constants*, and the elements of* Rel_0 *(if any) are called* propositional symbols.

We intend to use such signatures in a first-order hybrid language, thus we next to add first-order variables and nominals, and then "rigidify the signature" by allowing any function or relation symbol (including any constants or propositional symbols) to be preceded by rigidifying operators of the form $@_i$.

Definition 2. *A* first-order hybrid similarity type τ *is a tuple* $\langle \Sigma, X, \mathrm{NOM}\rangle$ *where* Σ *is first-order signature,* X *is a countably infinite set of variables and* NOM *is a set of symbols, called* nominals*. The* NOM*-rigidification of* Σ *(with respect to* τ*) is the signature:* $@\Sigma = ((@\mathrm{Func}_n)_{n\in\mathbb{N}}, (@\mathrm{Rel}_n)_{n\in\mathbb{N}})$*, where* $@\mathrm{Func}_n = \{(@_i f) : i \in \mathrm{NOM}, f \in \mathrm{Func}_n\}$ *and* $@\mathrm{Rel}_n = \{(@_i P) : i \in \mathrm{NOM}, P \in \mathrm{Rel}_n\}$*.*

Given a similarity type τ, we define the set of rigid terms, and the set of terms, as follows:

Definition 3 (Terms). *Let* τ *be a first-order hybrid similarity type. The set of rigid* Σ-terms *over* τ*,* $@\mathrm{Term}(\tau)$*, is recursively defined by:*

- *for any* $x \in X$*,* $x \in @\mathrm{Term}(\tau)$*;*
- *for any* $f^{@} \in @\mathrm{Func}_n$*, and all terms* $t_i \in @\mathrm{Term}(\tau)$*,* $i = 1, \ldots, n$*,* $f^{@}(t_1, \ldots, t_n) \in @\mathrm{Term}(\tau)$*.*

The set of Σ-terms *over* τ*,* $\mathrm{Term}(\tau)$*, is recursively defined by:*

- *for any* $x \in X$*,* $x \in \mathrm{Term}(\tau)$*;*
- *for any* $f \in \mathrm{Func}_n \cup @\mathrm{Func}_n$*, and all terms* $t_i \in \mathrm{Term}(\tau)$*,* $i = 1, \ldots, n$*,* $f(t_1, \ldots, t_n) \in \mathrm{Term}(\tau)$*.*

Clearly every rigid term is a term, that is, $@\mathrm{Term}(\tau) \subseteq \mathrm{Term}(\tau)$*. We call a term* ground *if it contains no variables.*

The elements of Func_0 (that is, constants) will play an important role in the completeness proof, as we will then expand our language by adding denumerably many new constants ("Henkin witness constants") to prove our Lindenbaum lemma. So it is worth noting that (by the previous definition) elements of the form $(@_i c)$, where c is a constant symbol, are indeed rigid terms (that is, elements of $@\mathrm{Term}(\tau)$) as all such expressions belong to $@\mathrm{Func}_0$.

Now for an important definition. Given a term t and a nominal i, we can (recursively) rigidify t at i, as follows:

Definition 4 (Rigidification of a term). *Let* $t \in \mathrm{Term}(\tau)$ *and* $i \in \mathrm{NOM}$*. The rigidification of* t *at* i *is the term* $@_i t \in @\mathrm{Term}(\tau)$ *recursively defined by:*

- *if* $t \in X$*,* $@_i t := t$
- *if* $t = f(t_1, \ldots, t_n)$ *with* $f \in @\mathrm{Func}_n$*, then* $@_i t := f(@_i t_1, \ldots, @_i t_n)$
- *if* $t = f(t_1, \ldots, t_n)$ *with* $f \in \mathrm{Func}_n$*, then* $@_i t := (@_i f)(@_i t_1, \ldots, @_i t_n)$

To spell this out: first, the rigidification process ignores variables, as they will always be interpreted rigidly. Second, if the functor prefixing a term is of the form $(@_j f)$, which means that we have a syntactic guarantee that it is rigid, then we ignore it and go on to recursively rigidify its arguments. Third, if the functor prefixing a term is of the form f (that is, we have no syntactic guarantee of its rigidity) we replace the functor f by the rigid form $(@_i f)$ and go on to recursively rigidify its arguments. Note that for the special case of constants (functions of arity 0) we have: given a constant c, and nominals i and j, the rigidification of c with respect to i is $(@_i c)$, and the rigidification of c with respect to j is $(@_j c)$. So the base case of the recursion is simply the rigidification-of-first-order-constants used in [5]. Also note that when a term $t \in @\mathrm{Term}(\tau)$ is rigidified, the result is simply t itself. That is, rigidification is the identity map on $@\mathrm{Term}(\tau)$.

Definition 5. *The set of* $\mathrm{Fm}(\tau)$ *of first-order hybrid formulas is the smallest set such that:*

1. $\mathrm{NOM} \subseteq \mathrm{Fm}(\tau)$*;*
2. $t_1 \approx t_2 \in \mathrm{Fm}(\tau)$*, for any* $t_1, t_2 \in \mathrm{Term}(\tau)$
3. $P(t_1, \ldots, t_n) \in \mathrm{Fm}(\tau)$*, for any* $P \in \mathrm{Rel}_n \cup @\mathrm{Rel}_n$ *and* $t_1, \ldots, t_n \in \mathrm{Term}(\tau)$*;*
4. *if* $\varphi \in \mathrm{Fm}(\tau)$ *and* i *is a nominal, then* $@_i\varphi \in \mathrm{Fm}(\tau)$*;*
5. *if* $\varphi \in \mathrm{Fm}(\tau)$*, then* $\neg\varphi, \Box\varphi \in \mathrm{Fm}(\tau)$*;*
6. *if* $\varphi \in \mathrm{Fm}(\tau)$ *and* $\psi \in \mathrm{Fm}(\tau)$ *then* $\varphi \wedge \psi \in \mathrm{Fm}(\tau)$ *and* $\varphi \vee \psi \in \mathrm{Fm}(\tau)$*.*
7. *if* $x \in X$ *and* $\varphi \in \mathrm{Fm}(\tau)$*, then* $\forall x\varphi \in \mathrm{Fm}(\tau)$*.*

We use familiar abbreviations: $\Diamond\varphi$ is $\neg\Box\neg\varphi$, $\exists x\varphi$ is $\neg\forall x\neg\varphi$, $\varphi \to \psi$ is $\neg(\varphi \wedge \neg\psi)$, and so on. We define $\mathsf{EXISTS}(t)$ to be $\exists x\,(x \approx t)$, provided that x does not occur in t, as is standard in varying domain approaches to first-order modal logic.

It is worth explicitly noting some of the syntactic distinctions that can be drawn in this language. Let i and j be nominals, let c and d be constant symbols, and let P be a two-place predicate symbol. Then $P(c, d)$ is a formula, one that displays no syntactic indications concerning rigidity. $P((@_i c), (@_j d))$ is also a formula, though this time the two constants it contains have been rigidified. Furthermore, $(@_i P)(c, d)$ is also a formula, though here it is the initial predicate has been rigidified. Indeed, $(@_i P)((@_i c), (@_j d))$ is a formula too, though this time the predicate and both constants have been rigidified. But there are other possibilities. In particular, note that $@_i P(c, d)$ is also a formula: it is the formula $P(c, d)$ preceded by $@_i$. Note that this is *not* the same formula as $(@_i P)(c, d)$. Indeed, under the semantics we shall shortly define, the two formulas have importantly different properties: $@_i P(c, d)$ is guaranteed to be a rigid proposition (it will either be true at all worlds or false at all worlds) while $(@_i P)(c, d)$ may vary in truth value from world to world.

Hopefully these examples help make our basic bracketing convention clear: when we combine $@_i$ with any formula φ (that is: propositional information) then we write the resulting formula as $@_i\varphi$ (that is: with no enclosing brackets). On the other hand, when we combine $@_i$ with either a function symbol f, a constant symbol c, or a predicate symbol P of arity $\geqslant 1$, then we write the resulting rigidifications as $(@_i f)$, $(@_i c)$ and $(@_i P)$ respectively (that is: with enclosing brackets). In the case of a predicate symbol p of arity 0 (that is: the propositional symbol p) we write $@_i p$, since propositional symbols are formulas.

However one other point should be emphasized: in statements of lemmas and axiom schemas we sometimes write expressions of the form $@_i t$ (for i a nominal and t a term). Here it is important to recall that such expressions are *not* members of the *object* language, rather they are *metalinguistic* abbreviation for the rigidification of t at i as defined by Definition 4.

3 Semantics

We now define a varying domain (actualist) semantics for our language. There are several choices available; here we simply remark that we have aimed for a general semantics, and typically follow the decisions made in [8]. We will say more about this in the paper's conclusion.

Definition 6 (Skeleton). *A skeleton over τ is a tuple $M = (W, \mathrm{Dom}, D, R)$, where $W \neq \varnothing$, Dom is a nonempty set, $D : W \to P(\mathrm{Dom})$ such that $D(w) \neq \varnothing$ and $R \subseteq W^2$. We will usually write D_w for $D(w)$.*

That is: we have a non-empty set of worlds W, a binary accessibility relation R between these worlds, a global domain of objects Dom, and a function D which tells us which elements of these domain elements actually exist at any world w. We call D_w (for any $w \in W$) a local domain. Local domains can be distinct, which is why this is a "varying domain" semantics.

Definition 7. *A* model *for a rigid first-order hybrid similarity type τ is a pair $\mathcal{M} = (M, I)$, where M is a skeleton and I is the interpretation function such that:*

- *For any $i \in \mathrm{NOM}$, $I(i) \in W$,*
- *For any $P \in \mathrm{Rel}_n$ and any $w \in W$, $I_w(P) \subseteq (\mathrm{Dom})^n$, and*
- *For any $f \in \mathrm{Func}_n$ and any $w \in W$, $I_w(f) : (\mathrm{Dom})^n \to \mathrm{Dom}$.*

Note that (following [8]) we allow the interpretation of a predicate P to involve individuals that do not exist in the local domain. Analogously, we interpret function symbols in a way that lets them take as input entities that do not exist at the local domain, and to output non-local entities as well. This seems the simplest and most general starting point, but we'll say more about this decision in the paper's conclusion.

Definition 8. *Let $\mathcal{M} = (M, I)$ be a model and $g : X \to \mathrm{Dom}$ be a variable assignment. The interpretation of terms is recursively defined as follows:*

- *if $t \in X$, $[t]^{\mathcal{M},w,g} = g(t)$.*
- *if $t = f(t_1, \ldots, t_n), f \in \mathrm{Func}_n$ with $n \geqslant 0$,*
  $[t]^{\mathcal{M},w,g} = I_w(f)([t_1]^{\mathcal{M},w,g}, \ldots, [t_n]^{\mathcal{M},w,g})$
- *if $t = (@_i f)(t_1, \ldots, t_n), f \in \mathrm{Func}_n$ with $n \geqslant 0$*
  $[t]^{\mathcal{M},w,g} = I_{I(i)}(f)([t_1]^{\mathcal{M},w,g}, \ldots, [t_n]^{\mathcal{M},w,g})$

We can now give the satisfaction definition.

Definition 9. *Let $\mathcal{M} = (M, I)$ be a model, $g : X \to \mathrm{Dom}$ an assignment and $w \in W$. Then:*

$\mathcal{M}, w, g \vDash i$ *iff* $I(i) = w$

$\mathcal{M}, w, g \vDash t_1 \approx t_2$ *iff* $[t_1]^{\mathcal{M},w,g} = [t_2]^{\mathcal{M},w,g}$

$\mathcal{M}, w, g \vDash P(t_1, \ldots, t_n)$ *iff* $([t_1]^{\mathcal{M},w,g}, \ldots, [t_n]^{\mathcal{M},w,g}) \in I_w(P)$, *for* $P \in \mathrm{Rel}_n$ *and* $t_1, \ldots, t_n \in \mathrm{Term}(\tau)$

$\mathcal{M}, w, g \vDash (@_i P)(t_1, \ldots, t_n)$ *iff* $([t_1]^{\mathcal{M},w,g}, \ldots, [t_n]^{\mathcal{M},w,g}) \in I_{I(i)}(P)$, *for* $P \in \mathrm{Rel}_n$ *and* $t_1, \ldots, t_n \in \mathrm{Term}(\tau)$

$\mathcal{M}, w, g \vDash \neg\varphi$ *iff* $\mathcal{M}, w, g \nvDash \varphi$

$\mathcal{M}, w, g \vDash \varphi \wedge \psi$ *iff* $\mathcal{M}, w, g \vDash \varphi$ *and* $\mathcal{M}, w, g \vDash \psi$

$\mathcal{M}, w, g \vDash @_i\varphi$ *iff* $\mathcal{M}, I(i), g \vDash \varphi$

$\mathcal{M}, w, g \vDash \Box\varphi$ *iff for all* $w' \in W$ *such that* $wRw', \mathcal{M}, w', g \vDash \varphi$

$\mathcal{M}, w, g \vDash \forall x\varphi$ *iff for all* $d \in D_w, \mathcal{M}, w, g[x \mapsto d] \vDash \varphi$

A formula φ is said to be true at a world w *under the assignment g if and only if $\mathcal{M}, w, g \vDash \varphi$. It is* valid in a model $\mathcal{M}$, *denoted by $\mathcal{M} \vDash \varphi$, if and only if, for every world w and every assignment g we have that $\mathcal{M}, w, g \vDash \varphi$.*

Lemma 1. *For every $t \in \mathrm{Term}(\tau)$, every assignment g on $\mathcal{M}$, every world w, and every nominal i we have that:*

$$[t]^{M,I(i),g} = [@_i t]^{M,w,g}$$

Proof. By induction on the structure of t. Recall from Definition 4 that $@_i t$ is the (recursively defined) *rigidification* of term t.

4 Axiomatisation

This section gives an axiomatisation K_τ for first-order hybrid logic, given a first-order hybrid similarity type τ. We will take all propositional tautologies as axioms, and in addition:

Distributivity axioms

$(K_\Box)$ $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

$(K_@)$ $@_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi)$.

Quantifier axioms

(*Q1*) $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, where x does not occur free in φ.

(*Q2*) $\forall x\varphi \to (\mathsf{EXISTS}(\tau) \to \varphi(\frac{\tau}{x}))$, where τ is rigid.

(*Q3*) $\exists y \mathsf{EXISTS}(y)$

Basic hybrid axioms

$(Ref_@)$ $@_i i$.

(*Agree*) $@_i @_j\varphi \leftrightarrow @_j\varphi$.

$(Selfdual_@)$ $@_i\varphi \leftrightarrow \neg @_i \neg\varphi$.

(*Intro*) $i \to (\varphi \leftrightarrow @_i\varphi)$.

(*Back*) $\Diamond @_i\varphi \to @_i\varphi$.

Axioms for $\approx$

($Ref_{\approx}$) $t_1 \approx t_1$, for all $t_1 \in \mathrm{Term}(\tau)$.

($Sym_{\approx}$) $(t_1 \approx t_2) \to (t_2 \approx t_1)$, for all $t_1, t_2 \in \mathrm{Term}(\tau)$.

($Trans_{\approx}$) $\big((t_1 \approx t_2) \wedge (t_2 \approx t_3)\big) \to (t_1 \approx t_3)$, for all $t_1, t_2, t_3 \in \mathrm{Term}(\tau)$.

($Func$) $(t_1 \approx t_1' \wedge ... \wedge t_n \approx t_n') \to f(t_1, \dots, t_n) \approx f(t_1', \dots, t_n')$, where $f \in \mathrm{Func} \cup @\mathrm{Func}$, and $t_i, t_i' \in \mathrm{Term}(\tau)$, for $i = 1, \dots, n$, $n \geqslant 0$.

($Pred$) $(t_1 \approx t_1' \wedge ... \wedge t_n \approx t_n') \to P(t_1, \dots, t_n) \leftrightarrow P(t_1' \dots, t_n')$, where $P \in \mathrm{Rel} \cup @\mathrm{Rel}$, and $t_i, t_i' \in \mathrm{Term}(\tau)$, for $i = 1, \dots, n$, $n \geqslant 0$.

Interactions between @ and $\approx$

($Rigidify$) $@_i(c \approx (@_i c))$, for any constant c.

($K_{@\approx}$) $@_i(t_1 \approx t_2) \leftrightarrow (@_i t_1 \approx @_i t_2)$, for all $t_1, t_2 \in \mathrm{Term}(\tau)$.

($Nom_{\approx}$) $@_i j \to (@_i t \approx @_j t)$, $t \in \mathrm{Term}(\tau)$.

($Agree_{\approx}$) $@_i(t_1 \approx t_2) \leftrightarrow (t_1 \approx t_2)$, for all $t_1, t_2 \in @\mathrm{Term}(\tau)$.

Linking formula rigidity with predicate-and-term rigidity

($Shuffle$-1) $@_i P(t_1, \dots, t_n) \leftrightarrow (@_i P)(@_i t_1, \dots, @_i t_n)$.

($Shuffle$-2) $@_i(@_j P)(t_1, \dots, t_n) \leftrightarrow (@_j P)(@_i t_1, \dots, @_i t_n)$.

As rules of proof we take the following (these proof rules are discussed in detail in [6], and we shall note some results from this paper in what follows). For any formulas φ and ψ, and any nominals i and j we have:

(MP) $\dfrac{\varphi \to \psi \qquad \varphi}{\psi}$

($Gen_{@}$) $\dfrac{\varphi}{@_i \varphi}$

($Gen_{\Box}$) $\dfrac{\varphi}{\Box \varphi}$

($Gen_{\forall}$) $\dfrac{\varphi}{\forall x \varphi}$

($Name$) $\dfrac{@_i \varphi}{\varphi}$, where i does not occur in φ.

(BG) $\dfrac{@_i \Diamond j \to @_j \varphi}{@_i \Box \varphi}$, if $j \neq i$ and j does not occur in φ.

($Subs$) $\dfrac{\varphi}{\varphi'}$, where φ' is any formula obtained from φ by replacing nominals by nominals and variables by rigidified terms.

As usual, we say that a proof of a formula φ is a finite sequence of formulas such that every formula in the sequence is either an axiom, or is obtained from previous formula(s) in the sequence using the rules of proof. We write $\vdash \varphi$ whenever we have such a sequence and say that φ is a K_τ-theorem. If $\Gamma \cup \{\varphi\}$ is a set of formulas, a proof of φ from Γ is a proof of $\vdash_{K\tau} (\gamma_1 \wedge \ldots \wedge \gamma_n) \to \varphi$ where $\{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$. A formula φ is provable from a set of formulas Γ (officially written as $\Gamma \vdash_{K_\tau} \varphi$, though we will usually just write $\Gamma \vdash \varphi$ instead) if and only if there is a proof of φ from Γ. The Deduction Theorem holds: $\Gamma \cup \{\varphi\} \vdash \psi$ iff $\Gamma \vdash \varphi \to \psi$.

Proposition 1. *The following are all K_τ-theorems:*
($K^{-1}{}_@$) $\vdash (@_i\varphi \to @_i\psi) \to @_i(\varphi \to \psi)$
(Nom) $\vdash @_ij \to (@_i\varphi \to @_j\varphi)$
(Sym) $\vdash @_ij \to @_ji$
(Bridge) $\vdash @_i\Diamond j \wedge @_j\varphi \to @_i\Diamond\varphi$
(Conj) $\vdash @_i(\varphi \wedge \psi) \leftrightarrow (@_i\varphi \wedge @_i\psi)$
(Elim) $\vdash (i \wedge @_i\varphi) \to \varphi$

Proof. See [6].

Proposition 2. *The following rules are admissible in K_τ:*
(Name′) $\dfrac{i \to \varphi}{\varphi}$, *where i does not occur in φ.*

(Paste$_\Diamond$) $\dfrac{(@_i\Diamond j \wedge @_j\varphi) \to \psi}{@_i\Diamond\varphi \to \psi}$, *if $j \neq i$ does not occur in φ or ψ.*

(Paste$_\forall$) $\dfrac{(@_i\mathsf{EXISTS}(t) \wedge @_i\varphi(\frac{@_it}{x})) \to \psi}{@_i\exists x\varphi \to \psi}$, *$t$ is ground and does not occur in ψ.*

Proof. See [6].

Corollary 1. *Let $\Gamma \cup \{\varphi, \psi\}$ be a set of formulas and i, j nominals. Then:*

1. *if i does not occur in $\Gamma \cup \{\varphi\}$, then*
$$\Gamma \vdash i \to \varphi \;\Rightarrow\; \Gamma \vdash \varphi.$$
2. *if $j \neq i$ does not occur in $\Gamma \cup \{\varphi, \psi\}$, then*
$$\Gamma \vdash (@_i\Diamond j \wedge @_j\varphi) \to \psi \;\Rightarrow\; \Gamma \vdash @_i\Diamond\varphi \to \psi.$$
3. *if t is ground and does not occur in $\Gamma \cup \{\varphi, \psi\}$, then*
$$\Gamma \vdash (@_i\mathsf{EXISTS}(t) \wedge @_i\varphi(\frac{@_it}{x})) \to \psi \;\Rightarrow\; \Gamma \vdash @_i\exists x\varphi \to \psi.$$

Proof. Immediate from the previous proposition.

5 Soundness and Completeness

Theorem 1 (Soundness). *Every theorem of K_τ is valid. That is, for any formula $\varphi \in \mathrm{Fm}(\tau)$, we have that $\vdash \varphi \;\Rightarrow\; \models \varphi$.*

Proof. Fairly straightforward. The Distributivity, Quantifier, Basic Hybrid Axioms and the Axioms for $\approx$ are all familiar from modal, hybrid, first-order or equational logic. The soundness of $K_{@\approx}$, $Nom_\approx$, and $Agree_\approx$ rests on Definitions 4 and 8. Note that *Shuffle-2* also holds in the special case $i = j$. If you are unfamiliar with hybrid logic, the soundness of the ($Name$) and (BG) rules may not be obvious: they are best thought of as analogous to natural deduction rules (the conclusion of each rule "discharges" a nominal in the premiss) and the side conditions are important. For detailed discussion of both rules (and some variants) see [6].

Definition 10. *Let $\Gamma \subseteq \mathrm{Fm}(\tau)$.*

- Γ *is said to be* K_τ-inconsistent *if* $\Gamma \vdash_{K_\tau} \varphi$ *for any* $\varphi \in \mathrm{Fm}(\tau)$*. Otherwise we say that* Γ *is* K_τ-consistent*.*
- Γ *is* maximal K_τ-consistent *if* Γ *is consistent and any set of formulas that properly extends* Γ *is* K_τ*-inconsistent.*
- Γ *is* named *if it contains at least one nominal.*
- Γ *is* $\Diamond$-saturated *if for all* $@_i\Diamond\varphi \in \Gamma$*, there is a nominal* j *such that* $@_i\Diamond j$ *and* $@_j\varphi$ *belong to* Γ*.*
- Γ is $\exists$-saturated *if for all formula* $@_i\exists x\varphi \in \Gamma$ *there is a constant* c *such that* $@_i(\mathsf{EXISTS}(c)) \in \Gamma$ *and* $@_i\varphi\frac{(@_i c)}{x} \in \Gamma$*.*

Lemma 2. *Let $\Gamma \subseteq \mathrm{Fm}(\tau)$. Then*

1. *Γ is inconsistent iff there is a formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.*
2. *$\varphi \in \Gamma$ then $\Gamma \vdash \varphi$.*
3. *$\Gamma \cup \{\varphi\}$ is inconsistent iff $\Gamma \vdash \neg\varphi$.*
4. *If Γ is maximal consistent then, $\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$.*

Proof. Standard.

We are ready to prove the Lindenbaum lemma we require: every K_τ-consistent set of formulas can be extended to a named, $\Diamond$-saturated, $\exists$-saturated, maximal K_τ-consistent set.

Lemma 3 (Lindenbaum). *Let $(i_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be countably infinite sets of new nominals and new constants, respectively. Let $\overline{\tau}$ be the new signature obtained by extending Σ and* NOM *with these symbols, and $K_{\overline{\tau}}$ the first-order hybrid logic over the extended signature. (Note that by the substitution rule, $K_{\overline{\tau}}$ is a conservative extension of K_τ.) Every K_τ-consistent set of formulas Γ can be extended to a named, $\Diamond$-saturated, $\exists$-saturated and maximal $K_{\overline{\tau}}$-consistent set.*

Proof. Let Γ be a K_τ-consistent set of formulas. We also have $(i_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$, countably infinite sets of new nominals and constants respectively, at our disposal. We define the set Γ^* to be $\bigcup_{n\in\mathbb{N}} \Gamma^n$, where:

$\Gamma^0 = \Gamma \cup \{i_0\}$;

$$\Gamma^{n+1} = \begin{cases} \Gamma^n & \text{, if } \Gamma^n \cup \{\varphi_n\} \text{ is inconsistent} \\ \Gamma^n \cup \{\varphi_n, @_i\Diamond i_m, @_{i_m}\psi\} & \text{, if } \varphi_n = @_i\Diamond\psi \text{ and } \Gamma^n \cup \{\varphi_n\} \\ & \text{ is consistent} \\ \Gamma^n \cup \{\varphi_n, @_i(\mathsf{EXISTS}(c_m)), @_i\psi\frac{@_i c_m}{x}\} & \text{, if } \varphi_n = @_i\exists x\psi \text{ and } \Gamma^n \cup \{\varphi_n\} \\ & \text{is consistent,} \\ \Gamma^n \cup \{\varphi_n\} & \text{, otherwise} \end{cases}$$

In these clauses, i_m is the first new nominal not occurring in Γ^n or in φ_n and c_m is the first new constant not in Γ^n or in φ_n. We now prove by induction that Γ^* is $K_{\overline{\tau}}$-consistent.

Suppose that Γ^0 is not consistent. Then $\Gamma \cup \{i_0\} \vdash \bot$. Hence, by the Deduction Theorem, $\Gamma \vdash i_0 \to \bot$. Since i_0 does not occur in $\Gamma \cup \{\varphi\}$, by Corollary 1 clause 1 $\Gamma \vdash \bot$, which is absurd since Γ is consistent.

Next, assume that Γ^n is $K_{\overline{\tau}}$-consistent and consider φ_n of the form $@_i\Diamond\psi$. Suppose for the sake of a contradiction that $\Gamma^n \cup \{\varphi_n\}$ is consistent, but that Γ^{n+1} is not. Then $\Gamma^n \cup \{@_i\Diamond\psi, @_i\Diamond i_m,\ @_{i_m}\psi\} \vdash \bot$. Hence, by the Deduction Theorem, $\Gamma^n \cup \{@_i\Diamond\psi\} \vdash (@_i\Diamond i_m \wedge @_{i_m}\psi) \to \varphi$. By Corollary 1 clause 2, $\Gamma^n \cup \{@_i\Diamond\psi\} \vdash @_i\Diamond\psi \to \bot$. Applying modus ponens yields $\Gamma^n \cup \{@_i\Diamond\psi\} \vdash \bot$, which contradicts our assumption that $\Gamma^n \cup \{\varphi_n\}$ is consistent.

Next, assume that Γ^n is $K_{\overline{\tau}}$-consistent and consider φ_n of the form $@_i\exists x\psi$. Suppose for the sake of a contradiction that $\Gamma^n \cup \{\varphi_n\}$ is consistent, but that Γ^{n+1} is not. This means that

$$\Gamma^n \cup \{\varphi_n, @_i(\mathsf{EXISTS}(c_m)), @_i\psi\frac{(@_ic_m)}{x}\} \vdash \bot.$$

Then, using the Deduction Theorem, we have that

$$\Gamma^n \cup \{\varphi_n\} \vdash (@_i(\mathsf{EXISTS}(c_m)) \wedge @_i\psi\frac{(@_ic_m)}{x}) \to \bot.$$

Then, using Corollary 1 clause 3, we have that

$$\Gamma^n \cup \{\varphi_n\} \vdash @_i\exists x\psi \to \bot.$$

Thus, $\Gamma^n \cup \{\varphi_n\} \vdash \bot$, contradicting its consistency.

Since Γ^n is $K_{\overline{\tau}}$-consistent for $n \in \mathbb{N}$, it follows that $\Gamma^* := \bigcup_{n\in\mathbb{N}} \Gamma^n$ is also $K_{\overline{\tau}}$-consistent. Moreover, Γ^* is also maximal. For suppose for the sake of a contradiction that it is not: that is, suppose that there exists a formula $\varphi \notin \Gamma^*$ such that $\Gamma^* \cup \{\varphi\}$ is $K_{\overline{\tau}}$-consistent. Then $\varphi = \varphi_n$, for some $n \in \mathbb{N}$, and $\Gamma^n \cup \{\varphi_n\}$ is consistent. Consequently, $\varphi_n \in \Gamma^{n+1}$ which is an absurd since we assumed that $\varphi \notin \Gamma^*$. So Γ^* is maximal, and we have proved our Lindenbaum lemma.

In the sequel, given a K_τ-consistent set of formulas Γ, Γ^* will denote the named, $\Diamond$-saturated, $\exists$-saturated, maximal consistent extension of Γ, defined in the proof of Lemma 3.

Definition 11. *Let Γ be a named, maximal K_τ-consistent set of formulas. Binary relations $\sim_n$ and $\sim_r$, over* NOM *and* $@\mathrm{Term}(\tau)$*, respectively, are defined as follows:*

- $i \sim_n j \ \Leftrightarrow\ @_ij \in \Gamma,\ i, j \in \mathrm{NOM}$
- $t \sim_r t' \ \Leftrightarrow\ t \approx t' \in \Gamma,\ t, t' \in @\mathrm{Term}(\tau)$

Lemma 4. *The relations $\sim_n$ and $\sim_r$ are equivalence relations. Moreover, if $t_k \sim_r t'_k$ for $k = 1, \ldots, n$, then $(@_if)(t_1, \ldots, t_n) \sim_r (@_if)(t'_1, \ldots, t'_n)$.*

Proof. The proofs that $\sim_n$ and $\sim_r$ are equivalence relations are straightforwrd (and standard). The proof of the last statement uses the axiom:

$$(t_1 \approx t'_1 \wedge ... \wedge t_n \approx t'_n) \to f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n),$$

where $f \in \mathrm{Func}_n \cup @\mathrm{Func}_n$, $n \geqslant 0$.

Definition 12. *Suppose Γ is a named, $\Diamond$-saturated, $\exists$-saturated and maximal K_τ-consistent set of formulas. Then the* Henkin structure

$$\mathcal{M}^\Gamma = ((W^\Gamma, \mathrm{Dom}^\Gamma, D^\Gamma, R^\Gamma), I^\Gamma)$$

is defined by:

– $W^\Gamma = \{|i| : i$ *is a nominal*$\}$
– $\mathrm{Dom}^\Gamma = \{|t| \in @\mathrm{Term}(\tau) : t$ *is ground*$\}$
– $D^\Gamma_{|i|} = \{|t| \in \mathrm{Dom} : @_i\mathsf{EXISTS}(t) \in \Gamma\}$
– $|i|R^\Gamma|j|$ *iff* $@_i\Diamond j \in \Gamma$
– $I^\Gamma_{|i|}(i) = |i|$, *for each nominal* i
– *for each* $f \in \mathrm{Func}_n$ *and* $|t_1|, \ldots, |t_n| \in \mathrm{Dom}^\Gamma$,
  $I^\Gamma_{|i|}(f)(|t_1|, \ldots, |t_n|) = |(@_i f)(t_1, \ldots, t_n)|$
– *for each* $P \in \mathrm{Rel}_n$,
  $I^\Gamma_{|i|}(P) = \{(|t_1|, \ldots, |t_n|) \in \mathrm{Dom}^\Gamma : (@_i P)(t_1, \ldots, t_n) \in \Gamma)\}$

Let us briefly check this definition. Note that R^Γ is well defined. For suppose $i' \in |i|$, then $@_i i' \in \Gamma$ so, if $@_i\Diamond j \in \Gamma$, by (*Nom*), $@_{i'}\Diamond j \in \Gamma$. Now suppose $j' \in |j|$, then $@_j j' \in \Gamma$ so, if $@_i\Diamond j \in \Gamma$, by (*Bridge*), $@_i\Diamond j' \in \Gamma$. We leave the reader to check that the functions and predicate interpretations are well-defined as well.

With this done, we are ready to state the Truth Lemma which establishes that the Henkin structure $\mathcal{M}^\Gamma$ is the model we are looking for; the Truth Lemma is stated and proved in the appendix. This leads to:

Theorem 2 (Completeness). *Let τ be a first-order hybrid similarity type φ be a sentence and Γ a set of sentences. Then*

$$\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi.$$

6 Conclusions and Future Work

We want to view hybrid logic as a general language of rigidification, and use it to explore conceptual and technical issues in first-order modal logic; this paper is our first step in this direction. The completeness result just proved takes us closer to this goal, because it covers not merely the basic logic, but also completeness with respect to any extension obtained by adding pure axioms or existential saturation rules (for a definition and detailed discussion of these concepts, see [6]). Adding pure axioms automatically yields completeness for many different frame conditions (for example, transitivity, reflexivity, and irreflexivity), and for additional modalities (such as the Priorean tense operators and the universal modality). More importantly for present purposes, such tools also immediately yield completeness for conditions of particular relevance to *first-order* modal logic. To give three such examples from [6], adding the pure axiom

$$@_i\mathsf{EXISTS}(@_k c) \to @_j\mathsf{EXISTS}(@_k c)$$

gives us a complete axiomatisation for constant domain (possibilist) semantics, adding

$$@_i\mathsf{EXISTS}(@_k c) \wedge @_j\mathsf{EXISTS}(@_k c) \rightarrow @_i j$$

gives us completeness with respect to the condition that all local domains be disjoint, and adding the existential saturation rule

$$\textbf{if} \vdash @_i\mathsf{EXISTS}(@_j c) \rightarrow \varphi \textbf{ then} \vdash \varphi,$$

where i is a nominal distinct from j not occurring in φ, gives us completeness with respect to the class of models in which every object in the domain (that is: every element of Dom) is also an element of some local domain. Thus the system defined in this paper already achieves a reasonable degree of generality.

But there are a number of issues that should be explored further. In this paper we have taken a minimalist approach to rigidification syntax. In particular, we did not have expressions of the form $@_i t$ in the *object* language, we instead used such expressions as metalinguistic abbreviations for the rigidification of t at i (as defined in Definition 4). However, having explored this minimal choice, we are now experimenting with extended versions of the language in which all such expressions are part of the object language. This seems useful for at least two reasons.

First, we want to develop and axiomatize richer forms of rigid first-order hybrid logic which incorporate the $\downarrow$ binder. This binder is a standard tool in hybrid logic: it binds nominals to the world of evaluation. For example, $\downarrow i.\Diamond\neg i$ is a formula that is true at any world w in any model if and only if w is an irreflexive world. Now, when we add the $\downarrow$ binder to the language explored in this paper, it will let us bind the nominals in rigidified function and predicate symbols. This is an extension worth exploring, but it seems more interesting to add $\downarrow$ to an extended version of the language in which *all* expressions of the form $@_i t$ are available. Why? Because once we add $\downarrow$, it seems both natural and desirable to be able to form arbitrary terms of the form $\downarrow i.@_i t$, and this of course requires that we have all terms of the form $@_i t$ available (and open for binding) in the object language.

Similar remarks apply to the other extension we are exploring: a general treatment of *partial* functions in a varying domain setting. We have explored this combination of ideas in the setting of higher-order hybrid logic [9], and are currently transferring the key ideas down to the first-order setting defined in this paper. Because the standard hybrid logical results concerning pure axioms and existential saturation rules still hold in our approach to partiality (which draws on ideas due to William Farmer [7]), we are confident that this can be done smoothly, and that the result will be a general first-order modal framework for working with partiality in an actualist semantics. But, once again, it seems that this extension may be more usefully carried out in a language in which all expressions of the form $@_i t$ are available at the object-level. In this paper we have interpreted function symbols in a way that lets them take as input entities that do not exist at the local domain, and to output non-local entities as well. But this hard-wires a lot into the semantics. We hope to find a flexible language in which a wide variety of choices about the semantics of functions (and predicates) can simply be axiomatised using such standard hybrid tools such as pure axioms and existential saturation rules. Partialising the semantics Farmer-style, adding object-level expressions of the form $@_i t$, and exploring the impact of $\downarrow$, seems a promising route to such a system.

Acknowledgements. The authors are grateful to the Spanish Ministerio de Economía y Competitividad for funding the project *Intensionality as a unifier: Logic, Language and Philosophy*, FFI2017-82554, hosted by the Universidad de Salamanca. Patrick Blackburn would also like to thank the Danish Council for Independent Research (FKK) for funding as part of the project: *The Primacy of Tense: A. N. Prior Now and Then*. Manuel Martins was also supported by ERDF, through the COMPETE 2020 Programme, and by FCT, within the projects `POCI-01-0145-FEDER-016692` and `UID/MAT/04106/2019`.

Appendix

This appendix sketches the definitions and lemmas that lead to the Truth Lemma, and thus to the Completeness Theorem stated in the main text. As a first step, given an assignment function g on the Henkin structure $\mathcal{M}^\Gamma$ defined in Definition 12, we need an inductive definition of how to substitute a suitable rigid term for a variables inside terms and formulas; the substitution syntactically mirrors the assignment function.

We do so as follows. Given a variable assignment g into $\mathcal{M}^\Gamma$ (that is, $g : X \to \mathrm{Dom}^\Gamma$) we first define a substitution function $\hat{g} : X \to @\mathrm{Term}(\tau)$ in the following way: for any variable x, we define $x^{\hat{g}} := t_k$, where t_k is the first rigid ground term in $@\mathrm{Term}(\tau)$ with lowest k such that $g(x) = |t_k|$. Here we assume that $@\mathrm{Term}(\tau)$ is ordered. We extend $\hat{g}$ to arbitrary terms t by defining: if $t = f(t_1, \ldots, t_n)$ then $t^{\hat{g}} = f(t_1^{\hat{g}}, \ldots, t_n^{\hat{g}})$.

We extend $\hat{g}$ to formulas in the following way:

- $i^{\hat{g}} := i$, $i \in \mathrm{NOM}$
- $(t_1 \approx t_2)^{\hat{g}} := (t_1^{\hat{g}} \approx t_2^{\hat{g}})$, $t_1, t_2 \in \mathrm{Term}(\tau)$
- $(P(t_1, \ldots, t_n))^{\hat{g}} := P(t_1^{\hat{g}}, \ldots, t_n^{\hat{g}})$, $P \in \mathrm{Rel}_n \cup @\mathrm{Rel}_n$ and $t_1, \ldots, t_n \in \mathrm{Term}(\tau)$
- $(@_i\varphi)^{\hat{g}} := @_i(\varphi^{\hat{g}})$, $\varphi \in \mathrm{Fm}(\tau)$ and $i \in \mathrm{NOM}$
- $(\neg\varphi)^{\hat{g}} := \neg(\varphi^{\hat{g}})$ and $(\Diamond\varphi)^{\hat{g}} := \Diamond(\varphi^{\hat{g}})$, $\varphi \in \mathrm{Fm}(\tau)$
- $(\varphi \wedge \psi)^{\hat{g}} := \varphi^{\hat{g}} \wedge \psi^{\hat{g}}$ and $(\varphi \vee \psi)^{\hat{g}} := \varphi^{\hat{g}} \vee \psi^{\hat{g}}$, for $\varphi \in \mathrm{Fm}(\tau)$ and $\psi \in \mathrm{Fm}(\tau)$
- $(\exists x\varphi)^{\hat{g}} := \exists x(\varphi^{\hat{g}_x^x})$, $x \in X$ and $\varphi \in \mathrm{Fm}(\tau)$, where $\hat{g}_x^x = \hat{g} \setminus \{(x, \hat{g}(x))\}) \cup \{(x, x)\})$

For any $t \in \mathrm{Term}(\tau)$ and any assignment g on $\mathcal{M}^\Gamma$, in what follows we will simply write t^g for $t^{\hat{g}}$. A similar simplification will be adopted for formulas.

Lemma 5. *For any $t \in \mathrm{Term}(\tau)$ and any assignment g on $\mathcal{M}^\Gamma$ we have*

$$[t]^{\mathcal{M}^\Gamma, |i|, g} = |@_i t^g|$$

Proof. By induction on term structure.

$(t \in X)$

$$\begin{aligned}
[x]^{\mathcal{M}^{\Gamma},|i|,g} &= g(x) \\
&= |t_k|,\ \text{where } t_k \text{ is the first ground (and rigid) term in} \\
&\quad @\mathrm{Term}(\tau) \text{ with lowest } k \text{ such that } g(x) = |t_k|. \\
&= |@_i t_k|,\ \text{since } t_k \in @\mathrm{Term}(\tau), \text{ by definition } @_i t_k = t_k \\
&= |@_i x^g|
\end{aligned}$$

$(t = f(t_1, \ldots, t_n), f \in \mathrm{Func}_n,\ n \geqslant 0)$

$$\begin{aligned}
[f(t_1, \ldots, t_n)]^{\mathcal{M}^{\Gamma},|i|,g} &= I_{|i|}(f)([t_1]^{\mathcal{M},|i|,g}, \ldots, [t_n]^{\mathcal{M},|i|,g}) \\
&= I_{|i|}(f)(|@_i t_1^g|, \ldots, |@_i t_n^g|) && \text{(Ind. Hyp.)} \\
&= |(@_i f)(@_i t_1^g, \ldots, @_i t_n^g)| \\
&= |@_i(f(t_1^g, \ldots, t_n^g))| \\
&= |@_i t^g|
\end{aligned}$$

$(t = (@_j f)(t_1, \ldots, t_n), f \in \mathrm{Func}_n,\ n \geqslant 0)$

$$\begin{aligned}
[(@_j f)(t_1, \ldots, t_n)]^{\mathcal{M}^{\Gamma},|i|,g} &= I_{|j|}(f)([t_1]^{\mathcal{M},|i|,g}, \ldots, [t_n]^{\mathcal{M},|i|,g}) \\
&= I_{|j|}(f)(|@_i t_1^g|, \ldots, |@_i t_n^g|) && \text{(Ind. Hyp.)} \\
&= |(@_j f)(@_i t_1^g, \ldots, @_i t_n^g)| \\
&= |@_i((@_j f)(t_1^g, \ldots, t_n^g))| \\
&= |@_i t^g|
\end{aligned}$$

Lemma 6 (Truth Lemma). *For every nominal i, any assignment g on $\mathcal{M}^{\Gamma}$ and every formula φ*

$$\mathcal{M}^{\Gamma}, |i|, g \vDash \varphi \quad \Leftrightarrow \quad @_i\varphi^g \in \Gamma$$

Proof. The proof proceeds by induction on the complexity of φ.

- $\varphi = j$
  We have that
  $$\mathcal{M}^{\Gamma^*}, |i|, g \vDash j \text{ iff } |i| = |j| \text{ iff } @_i j \in \Gamma \text{ iff } @_i j^g \in \Gamma.$$
- $\varphi = t_1 \approx t_2$,
  $$\begin{aligned}
  \mathcal{M}^{\Gamma}, |i|, g \vDash t_1 \approx t_2 &\text{ iff } [t_1]^{\mathcal{M},|i|,g} = [t_2]^{\mathcal{M},|i|,g} \\
  &\text{ iff } |@_i t_1^g| = |@_i t_2^g|, \text{ by Lemma 5} \\
  &\text{ iff } @_i t_1^g \sim_r @_i t_2^g \\
  &\text{ iff } @_i t_1^g \approx @_i t_2^g \in \Gamma \\
  &\text{ iff } @_i(t_1^g \approx t_2^g) \in \Gamma, \text{ by axiom } K_{@\approx} \\
  &\text{ iff } @_i(t_1 \approx t_2)^g \in \Gamma
  \end{aligned}$$
- $\varphi = P(t_1, \ldots, t_n)$, with $P \in \mathrm{Rel}_n \cup @\mathrm{Rel}_n$ and $t_1, \ldots, t_n \in \mathrm{Term}(\tau)$;
  If $P \in \mathrm{Rel}_n$:
  $$\begin{aligned}
  \mathcal{M}^{\Gamma}, |i|, g \vDash P(t_1, \ldots, t_n) &\text{ iff } ([t_1]^{\mathcal{M},|i|,g}, \ldots, [t_n]^{\mathcal{M},|i|,g}) \in I_{|i|}(P) \\
  &\text{ iff } (|@_i t_1^g|, \ldots, |@_i t_n^g|) \in I_{|i|}(P), \text{ by Lemma 5} \\
  &\text{ iff } (@_i P)(@_i t_1^g, \ldots, @_i t_n^g) \in \Gamma \\
  &\text{ iff } @_i(P(t_1^g, \ldots, t_n^g)) \in \Gamma, \\
  &\qquad \text{by the Shuffle-1 Axiom} \\
  &\qquad (@_i P)(@_i t_1, \ldots, @_i t_n) \leftrightarrow @_i(P(t_1, \ldots, t_n)) \\
  &\text{ iff } @_i((P(t_1, \ldots, t_n)^g) \in \Gamma
  \end{aligned}$$

If $P = (@_j S)$, with $S \in \mathrm{Rel}_n$:

$$\begin{aligned}\mathcal{M}^\Gamma, |i|, g \models (@_j S)(t_1, \ldots, t_n) \text{ iff } & ([t_1]^{\mathcal{M},|i|,g}, \ldots, [t_n]^{\mathcal{M},|i|,g}) \in I_{|j|}(S)\\ \text{iff } & (|@_i t_1^g|, \ldots, |@_i t_n^g|) \in I_{|j|}(S), \text{ by Lemma 5}\\ \text{iff } & (@_j S)(@_i t_1^g, \ldots, @_i t_n^g) \in \Gamma\\ \text{iff } & @_i((@_j S)(t_1^g, \ldots, t_n^g)) \in \Gamma,\\ & \quad \text{by the Shuffle-2 axiom}\\ & \quad (@_j S)(@_i t_1, \ldots, @_i t_n) \leftrightarrow @_i((@_j S)(t_1, \ldots, t_n))\\ \text{iff } & @_i(((@_j S)(t_1, \ldots, t_n)^g) \in \Gamma\end{aligned}$$

– $\varphi = @_j \psi$.

$$\begin{aligned}\mathcal{M}^\Gamma, |i|, g \models @_j\psi \text{ iff } & \mathcal{M}^\Gamma, |j|, g \models \psi\\ \text{iff } & @_j(\psi)^g \in \Gamma, \text{ IH}\\ \text{iff } & (@_j\psi)^g \in \Gamma\\ \text{iff } & @_i(@_j\psi)^g \in \Gamma, \text{ by } \mathit{Agree}\end{aligned}$$

– $\varphi = \neg\psi$.

$$\begin{aligned}\mathcal{M}^\Gamma, |i|, g \models \neg\psi \text{ iff } & \mathcal{M}^\Gamma, |i|, g \not\models \psi\\ \text{iff } & @_i(\psi)^g \notin \Gamma, \text{ IH}\\ \text{iff } & \neg @_i(\psi)^g \in \Gamma, \text{ as } \Gamma \text{ is maximal consistent}\\ \text{iff } & @_i\neg(\psi)^g \in \Gamma, \text{ by } \mathit{Selfdual}_@\\ \text{iff } & @_i(\neg\psi)^g \in \Gamma\end{aligned}$$

– $\varphi = \Diamond\psi$.

$$\begin{aligned}\mathcal{M}^\Gamma, |i|, g \models \Diamond\psi \text{ iff } & \text{there is } j \text{ such that } |i| R^\Gamma |j| \text{ and } \mathcal{M}^\Gamma, |j|, g \models \psi\\ \text{iff } & \text{there is } j \text{ such that } |i| R^\Gamma |j| \text{ and } @_i\psi^g \in \Gamma, \text{ by IH}\\ \text{iff } & @_i\Diamond\psi^g \in \Gamma,\\ & \quad \text{by } \mathit{Bridge} \text{ (since } @_i\Diamond j \in \Gamma) \text{ and } \Diamond\text{-saturation}\\ \text{iff } & @_i(\Diamond\psi)^g \in \Gamma\end{aligned}$$

– $\varphi = \psi_1 \wedge \psi_2$

$$\begin{aligned}\mathcal{M}^\Gamma, |i|, g \models \psi_1 \wedge \psi_2 \text{ iff } & \mathcal{M}^\Gamma, |i|, g \models \psi_1 \text{ and } \mathcal{M}^\Gamma, |i|, g \models \psi_2\\ \text{iff } & @_i(\psi_1)^g \in \Gamma \text{ and } @_i(\psi_2)^g \in \Gamma \text{ , IH}\\ \text{iff } & @_i(\psi_1)^g \wedge @_i(\psi_2)^g \in \Gamma, \text{ as } \Gamma \text{ is maximal consistent}\\ \text{iff } & @_i((\psi_1)^g \wedge (\psi_2)^g) \in \Gamma\\ \text{iff } & @_i(\psi_1 \wedge \psi_2)^g) \in \Gamma\end{aligned}$$

– $\varphi = \exists x\psi$.

$$\begin{aligned}\mathcal{M}^\Gamma, |i|, g \models \exists x\psi \quad \text{iff} \quad & \text{exists } \theta \in D_{|i|} \text{ s.t } \mathcal{M}, w, g[x \mapsto \theta] \models \varphi\\ \text{iff} \quad & \text{exists } \theta \in D_{|i|} \text{ s.t } @_i\varphi^{g[x\mapsto\theta]} \in \Gamma, \text{ induction hypothesis}\\ \text{iff}^{(*)} \quad & @_i(\exists x\varphi)^g \in \Gamma\end{aligned}$$

Proof of $(*)$

The implication "$\Rightarrow$" holds by the Corollary 1 clause 3.

The implication "$\Leftarrow$" holds by $\exists$- saturation. $@_i(\exists x\varphi)^g \in \Gamma$ implies that there exists a constant c such that $@_i\mathsf{EXISTS}(c) \in \Gamma$ and $(\varphi)^{g^x_x}(x \mapsto @_i c) \in \Gamma$. So there is $\theta := @_i c \in D_{|i|}$(because $@_i\mathsf{EXISTS}(c) \in \Gamma$) s.t $@_i\varphi^{g[x\mapsto\theta]} \in \Gamma$.

Lemma 7. *Let Γ be a consistent set of sentences. Then, there is a nominal k such that for every $\varphi \in \Gamma$,*

$$\mathcal{M}^\Gamma, |k| \models \varphi$$

Theorem 3 (Completeness). *Let τ be a first-order hybrid similarity type φ be a sentence and Γ a set of sentences. Then*

$$\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi.$$

References

1. Areces, C., Blackburn, P., Huertas, A., Manzano, M.: Completeness in hybrid type theory. J. Philos. Log. **43**, 209–238 (2014)
2. Areces, C., Blackburn, P., Marx, M.: Repairing the interpolation theorem in quantified modal logic. Ann. Pure Appl. Log. **124**(1–3), 287–299 (2003)
3. Barbosa, L.S., Martins, M.A., Carreteiro, M.: A Hilbert-style axiomatisation for equational hybrid logic. J. Log. Lang. Inf. **23**(1), 31–52 (2014)
4. Blackburn, P., Huertas, A., Manzano, M., Jørgensen, K.F.: Henkin and hybrid logic. In: Manzano, M., Sain, I., Alonso, E. (eds.) The Life and Work of Leon Henkin. SUL, pp. 279–306. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-09719-0_19
5. Blackburn, P., Marx, M.: Tableaux for quantified hybrid logic. In: Egly, U., Fermüller, C.G. (eds.) TABLEAUX 2002. LNCS (LNAI), vol. 2381, pp. 38–52. Springer, Heidelberg (2002). https://doi.org/10.1007/3-540-45616-3_4
6. Blackburn, P., ten Cate, B.: Pure extensions, proof rules, and hybrid axiomatics. Studia Logica **84**, 277–322 (2006)
7. Farmer, W.M.: A partial functions version of Church's simple theory of types. J. Symb. Log. **55**(3), 1269–1291 (1990)
8. Fitting, M., Mendelsohn, R.: First-Order Modal Logic. Springer, Heidelberg (1998). https://doi.org/10.1007/978-94-011-5292-1
9. Manzano, M., Huertas, A., Blackburn, P., Martins, M.: Hybrid partial type theory (2019, Submitted)
10. Manzano, M., Martins, M., Huertas, A.: Completeness in equational hybrid propositional type theory. Studia Logica (2018). https://doi.org/10.1007/s11225-018-9833-5

The One-Variable Fragment of Corsi Logic

Xavier Caicedo[1], George Metcalfe[2(✉)], Ricardo Rodríguez[3,4], and Olim Tuyt[2]

[1] Departamento de Matemáticas, Universidad de los Andes, Bogotá, Colombia
xcaicedo@uniandes.edu.co
[2] Mathematical Institute, University of Bern, Bern, Switzerland
{george.metcalfe,olim.tuyt}@math.unibe.ch
[3] UBA. FCEyN. Departamento de Computación, Buenos Aires, Argentina
ricardo@dc.uba.ar
[4] CONICET-UBA. Inst. de Invest. en Cs. de la Computación, Buenos Aires, Argentina

Abstract. The one-variable fragment of the first-order logic of linear intuitionistic Kripke models, referred to here as Corsi logic, is shown to have as its modal counterpart the many-valued modal logic $\mathsf{S5}(\mathbf{G})$. It is also shown that $\mathsf{S5}(\mathbf{G})$ can be interpreted in the crisp many-valued modal logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, the modal counterpart of the one-variable fragment of first-order Gödel logic. Finally, an algebraic finite model property is proved for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ and used to establish co-NP-completeness for validity in the aforementioned modal logics and one-variable fragments.

1 Introduction

One-variable fragments of first-order logics are often studied as propositional modal logics, where each unary predicate $P(x)$ is replaced with a propositional variable p and quantifiers $(\forall x)$ and $(\exists x)$ are replaced with modalities $\Box$ and $\Diamond$, respectively. This shift in perspective can be useful in obtaining axiomatization, finite model property, and complexity results both for the fragments and for corresponding classes of algebraic models. In particular, the modal logic $\mathsf{S5}$ and intuitionistic modal logic MIPC (corresponding to monadic Boolean algebras and monadic Heyting algebras) are modal counterparts of the one-variable fragments of first-order classical logic and intuitionistic logic, respectively. Both these modal logics have the finite model property and are decidable. The correspondence between one-variable fragments of first-order intermediate logics and varieties of monadic Heyting algebras has been considered in some depth in [2,14,15]. Decidability and complexity results have also been obtained for intermediate modal logics viewed as fragments of classical bimodal logics (see [8] for details).

In this paper, we investigate the one-variable fragment of the first-order logic of linear intuitionistic Kripke models, axiomatized by Corsi in [7] as the extension of first-order intuitionistic logic with the prelinearity axiom schema $(\alpha \to \beta) \vee (\beta \to \alpha)$, and referred to here as *Corsi logic*. In particular, we prove that the

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 70–83, 2019.
https://doi.org/10.1007/978-3-662-59533-6_5

modal counterpart of this one-variable fragment is the many-valued modal logic $\mathsf{S5}(\mathbf{G})$, with propositional connectives interpreted using the standard semantics of Gödel logic and $\Box$ and $\Diamond$ interpreted as infima and suprema relative to $[0,1]$-valued accessibility relations. It has been shown in [6] that an axiomatization of $\mathsf{S5}(\mathbf{G})$ is obtained by extending MIPC with the prelinearity axiom schema, and that adding also the axiom schema $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box p)$ yields an axiomatization of the crisp version $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ of $\mathsf{S5}(\mathbf{G})$, obtained by restricting to $\{0,1\}$-valued accessibility relations. The logic $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is the modal counterpart of the one-variable fragment of first-order Gödel logic or, equivalently (see [1,16]), the first-order logic of linear intuitionistic Kripke models with constant domains.

The logic $\mathsf{S5}(\mathbf{G})$ lacks the finite model property with respect to its standard Kripke semantics, but is complete with respect to a variety of monadic Heyting algebras that has this property (see [2]) and is hence decidable. We provide here an alternative decidability proof that also establishes co-NP-completeness. First, we give an interpretation of the one-variable fragment of Corsi logic in the one-variable fragment of first-order Gödel logic, yielding an interpretation of $\mathsf{S5}(\mathbf{G})$ in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. Although $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ also lacks the finite model property, decidability (indeed co-NP-completeness) has been established in [5] using an alternative Kripke semantics that does have the property. We show here that this rather ad hoc alternative semantics emerges naturally from a well-known representation of monadic Heyting algebras (see [2]). Finally, an algebraic finite model property is established for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, and used to prove co-NP-completeness for the two many-valued modal logics and their associated one-variable fragments.

2 The One-Variable Fragments

In this section, we present the one-variable fragments of first-order intermediate logics defined over all linear Kripke models and the linear Kripke models that have constant domains. For convenience, we restrict our definitions here to the set Fm_1 of one-variable first-order formulas $\alpha, \beta, \ldots$, built inductively as usual from a countably infinite set of unary predicates $\{P_i\}_{i\in\mathbb{N}}$, propositional connectives $\wedge, \vee, \to, \bot, \top$, a fixed variable x, and quantifiers $\forall, \exists$.

A *monadic intuitionistic Kripke model* (for short, IK_1*-model*) is a 4-tuple $\mathfrak{M} = \langle W, \preceq, \{D_w\}_{w\in W}, \{I_w\}_{w\in W}\rangle$ consisting of a non-empty poset $\langle W, \preceq\rangle$, a non-empty set D_w for each $w \in W$ called the *domain* of w, and functions $\{I_w\}_{w\in W}$ mapping each P_i to some $I_w(P_i) \subseteq D_w$, satisfying for all $w, v \in W$ and $i \in \mathbb{N}$,

$$w \preceq v \implies D_w \subseteq D_v \text{ and } I_w(P_i) \subseteq I_v(P_i).$$

Satisfaction in $\mathfrak{M}$ is then defined inductively as follows for $w \in W$ and $a \in D_w$:

$$\begin{array}{lll}
\mathfrak{M}, w \models^a \bot & \iff & \text{never} \\
\mathfrak{M}, w \models^a \top & \iff & \text{always} \\
\mathfrak{M}, w \models^a P_i(x) & \iff & a \in I_w(P_i) \\
\mathfrak{M}, w \models^a \alpha \land \beta & \iff & \mathfrak{M}, w \models^a \alpha \text{ and } \mathfrak{M}, w \models^a \beta \\
\mathfrak{M}, w \models^a \alpha \lor \beta & \iff & \mathfrak{M}, w \models^a \alpha \text{ or } \mathfrak{M}, w \models^a \beta \\
\mathfrak{M}, w \models^a \alpha \to \beta & \iff & \mathfrak{M}, v \models^a \alpha \text{ implies } \mathfrak{M}, v \models^a \beta \text{ for all } v \succeq w \\
\mathfrak{M}, w \models^a (\forall x)\alpha & \iff & \mathfrak{M}, v \models^b \alpha \text{ for all } v \succeq w \text{ and } b \in D_v \\
\mathfrak{M}, w \models^a (\exists x)\alpha & \iff & \mathfrak{M}, w \models^b \alpha \text{ for some } b \in D_w.
\end{array}$$

Let us call $\mathfrak{M}$ an IKL_1-*model* if $\preceq$ is linear, a CDIK_1-*model* if it has constant domains (i.e., $D_w = D_v$ for all $v, w \in W$), and a CDIKL_1-*model* if both these conditions are satisfied. We say that $\alpha \in \mathrm{Fm}_1$ is *valid* in $\mathfrak{M}$ if $\mathfrak{M}, w \models^a \alpha$ for all $w \in W$ and $a \in D_w$. Given $\mathsf{L} \in \{\mathsf{IK}_1, \mathsf{IKL}_1, \mathsf{CDIK}_1, \mathsf{CDIKL}_1\}$, we say that $\alpha \in \mathrm{Fm}_1$ is L-*valid*, denoted by $\models_{\mathsf{L}} \alpha$, if it is valid in all L-models.

Let $\mathcal{IQC}$ be an axiomatization for first-order intuitionistic logic and consider the following axiom schema for all (i.e., not just one-variable) first-order formulas α and β, where x is not free in β for (cd):

$$\text{(prl) } (\alpha \to \beta) \lor (\beta \to \alpha) \quad \text{and} \quad \text{(cd) } (\forall x)(\alpha \lor \beta) \to ((\forall x)\alpha \lor \beta).$$

By known completeness results for first-order logics, we obtain for any $\alpha \in \mathrm{Fm}_1$:

$$\begin{array}{llll}
\models_{\mathsf{IK}_1} \alpha \iff \vdash_{\mathcal{IQC}} \alpha & [12]; & \models_{\mathsf{CDIK}_1} \alpha \iff \vdash_{\mathcal{IQC}+\text{(cd)}} \alpha & [9]; \\
\models_{\mathsf{IKL}_1} \alpha \iff \vdash_{\mathcal{IQC}+\text{(prl)}} \alpha & [7]; & \models_{\mathsf{CDIKL}_1} \alpha \iff \vdash_{\mathcal{IQC}+\text{(cd)}+\text{(prl)}} \alpha & [16].
\end{array}$$

Now let $\mathrm{Fm}_{\Box\Diamond}$ be the set of modal formulas $\varphi, \psi, \ldots$, built inductively over a set of propositional variables $\{p_i\}_{i \in \mathbb{N}}$, propositional connectives $\land, \lor, \to, \bot, \top$, and modal connectives $\Box, \Diamond$. Recall also the standard translations $(-)^*$ and $(-)^\circ$ between $\mathrm{Fm}_{\Box\Diamond}$ and Fm_1, where $\star \in \{\land, \lor, \to\}$, $c \in \{\bot, \top\}$:

$$\begin{array}{rlrl}
(P_i(x))^* &= p_i & p_i^\circ &= P_i(x) \\
c^* &= c & c^\circ &= c \\
(\alpha \star \beta)^* &= \alpha^* \star \beta^* & (\varphi \star \psi)^\circ &= \varphi^\circ \star \psi^\circ \\
((\forall x)\alpha)^* &= \Box\alpha^* & (\Box\varphi)^\circ &= (\forall x)\varphi^\circ \\
((\exists x)\alpha)^* &= \Diamond\alpha^* & (\Diamond\varphi)^\circ &= (\exists x)\varphi^\circ.
\end{array}$$

Clearly $(\alpha^*)^\circ = \alpha$ for any $\alpha \in \mathrm{Fm}_1$ and $(\varphi^\circ)^* = \varphi$ for any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$.

Let $\mathcal{MIPC}$ be an axiomatization of intuitionistic propositional logic extended with the necessitation rule $\varphi/\Box\varphi$ and the axiom schema

$$\begin{array}{ll}
\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) & \Diamond(\varphi \lor \psi) \to (\Diamond\varphi \lor \Diamond\psi) \\
\Box\varphi \to \varphi & \varphi \to \Diamond\varphi \\
\Diamond\varphi \to \Box\Diamond\varphi & \Diamond\Box\varphi \to \Box\varphi \\
\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi), &
\end{array}$$

and consider the additional axiom schema

$$\text{(prl)}\ (\varphi \to \psi) \vee (\psi \to \varphi) \quad \text{and} \quad (\text{cd})_\Box\ \Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi).$$

The following completeness results are known:

$$\begin{array}{llll} \models_{\mathsf{IK}_1} \alpha & \iff & \vdash_{\mathcal{MIPC}} \alpha^* & [4]; \\ \models_{\mathsf{CDIK}_1} \alpha & \iff & \vdash_{\mathcal{MIPC}+(\text{cd})_\Box} \alpha^* & [13]; \\ \models_{\mathsf{CDIKL}_1} \alpha & \iff & \vdash_{\mathcal{MIPC}+(\text{cd})_\Box+(\text{prl})} \alpha^* & [6]. \end{array}$$

In Sect. 3 of this paper, we establish the missing result for Corsi logic.

Theorem 1. *For any* $\alpha \in \mathrm{Fm}_1$, $\models_{\mathsf{IKL}_1} \alpha \iff \vdash_{\mathcal{MIPC}+(\text{prl})} \alpha^*$.

The one-variable fragment of first-order intuitionistic logic IK_1 has the finite model property and is decidable [13], but the precise complexity is not known, whereas CDIKL_1, the one-variable fragment of first-order Gödel logic [1,16], lacks the finite model property but is co-NP-complete [5]. The one-variable fragment IKL_1 of Corsi logic also lacks the finite model property. For example, the formula $(\forall x)\neg\neg P_0(x) \to \neg\neg(\forall x)P_0(x)$ (where $\neg\alpha$ is defined as $\alpha \to \bot$) is valid in all finite IKL_1-models, but not in the infinite IKL_1-model $\mathfrak{M} = \langle \mathbb{N}, \leq, \{D_n\}_{n\in\mathbb{N}}, \{I_n\}_{n\in\mathbb{N}}\rangle$ with $D_n = \{a_0, \dots, a_n\}$ and $I_n(P_0) = \{a_0, \dots, a_{n-1}\}$ for each $n \in \mathbb{N}$. On the other hand, it is known (see [2]) that the variety of monadic Heyting algebras corresponding to the axiom system $\mathcal{MIPC} + \text{(prl)}$ has the finite model property, implying, by Theorem 1, that validity in IKL_1 is decidable. We prove a stronger result here, giving first an interpretation of IKL_1 in CDIKL_1 (Sect. 4) and then establishing an algebraic finite model property for CDIKL_1 (Sect. 5), to obtain the following complexity bound.

Theorem 2. *Validity in* IKL_1 *is co-NP-complete.*

3 The Many-Valued Modal Logics

In this section, we prove that the one-variable fragment of Corsi logic has as its modal counterpart the many-valued modal logic $\mathsf{S5}(\mathbf{G})$. Since the latter was axiomatized in [6] as an extension of $\mathcal{MIPC}$ with the prelinearity axiom schema (prl), this result yields a proof of Theorem 1.

Consider first the *standard Gödel algebra* $\mathbf{G} = \langle [0,1], \wedge, \vee, \to, 0, 1\rangle$ where $x \to y$ is y if $x > y$, and 1 otherwise. An $\mathsf{S5}(\mathbf{G})$-*model* is a triple $\mathfrak{M} = \langle W, R, V\rangle$ consisting of a non-empty set W, a map $R\colon W \times W \to [0,1]$ satisfying $Rww = 1$ (reflexivity); $Rwv = Rvw$ (symmetry); and $Rwv \wedge Rvu \leq Rwu$ (transitivity), and a map $V\colon \{p_i\}_{i\in\mathbb{N}} \times W \to [0,1]$. The map V is extended to $V\colon \mathrm{Fm}_{\Box\Diamond} \times W \to [0,1]$ inductively by the clauses $V(\bot, w) = 0$, $V(\top, w) = 1$, $V(\varphi \star \psi, w) = V(\varphi, w) \star V(\psi, w)$ for $\star \in \{\wedge, \vee, \to\}$, and

$$V(\Box\varphi, w) = \bigwedge\{Rwv \to V(\varphi, v) \mid v \in W\}$$

$$V(\Diamond\varphi, w) = \bigvee\{Rwv \wedge V(\varphi, v) \mid v \in W\}.$$

If $Rvw \in \{0,1\}$ for all $v, w \in W$, then $\mathfrak{M}$ is called an $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-*model.* A formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is said to be *valid* in $\mathfrak{M}$ if $V(\varphi, w) = 1$ for all $w \in W$, and L-*valid* for $\mathsf{L} \in \{\mathsf{S5}(\mathbf{G}), \mathsf{S5}(\mathbf{G})^{\mathsf{C}}\}$, written $\models_{\mathsf{L}} \varphi$, if φ is valid in all L-models.

An $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, V\rangle$ is called *universal* if $Rwv = 1$ for all $w, v \in W$; we then write $\mathfrak{M} = \langle W, V\rangle$, since the conditions for $\Box, \Diamond$ simplify to

$$V(\Box\varphi, w) = \bigwedge\{V(\varphi, v) \mid v \in W\} \quad \text{and} \quad V(\Diamond\varphi, w) = \bigvee\{V(\varphi, v) \mid v \in W\}.$$

It is easily proved that $\models_{\mathsf{S5}(\mathbf{G})^{\mathsf{C}}} \varphi$ if and only if φ is valid in all universal $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-models, and that this holds if and only if φ° is valid in first-order Gödel logic. The equivalence between first-order Gödel logic and the logic of linear Kripke models with constant domains (see [1,16]) therefore yields the following correspondence.

Theorem 3. *For any* $\alpha \in \mathrm{Fm}_1$, $\models_{\mathsf{CDIKL_1}} \alpha \iff \models_{\mathsf{S5}(\mathbf{G})^{\mathsf{C}}} \alpha^*$.

The rest of this section is devoted to proving the analogous result for $\mathsf{IKL_1}$.

Theorem 4. *For any* $\alpha \in \mathrm{Fm}_1$, $\models_{\mathsf{IKL_1}} \alpha \iff \models_{\mathsf{S5}(\mathbf{G})} \alpha^*$.

We consider first the right-to-left direction. Proceeding contrapositively, let (without loss of generality) $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ and suppose that $\not\models_{\mathsf{IKL_1}} \varphi^\circ$. Then there exists a countable $\mathsf{IKL_1}$-model $\mathfrak{M} = \langle W, \preceq, \{D_w\}_{w\in W}, \{I_w\}_{w\in W}\rangle$, $w_0 \in W$, and $a_0 \in D_{w_0}$ such that $\mathfrak{M}, w_0 \not\models^{a_0} \varphi^\circ$. Let $\mathrm{Up}(\langle W, \preceq\rangle)$ be the complete linearly ordered set of upsets of $\langle W, \preceq\rangle$ ordered by inclusion with W and $\emptyset$ as greatest and least elements, respectively. Since W is countable, there exists a complete (i.e., preserving all suprema and infima) order-embedding of $\mathrm{Up}(\langle W, \preceq\rangle)$ into $\langle [0,1], \leq\rangle$ (see [1]) and we may therefore implicitly identify $\mathrm{Up}(\langle W, \preceq\rangle)$ with a subset of $[0,1]$.

Let $W^* = \bigcup_{v\in W} D_v$ and for each $a \in W^*$, let $U(a) = \{v \in W \mid a \in D_v\}$, i.e., $U(a)$ is the largest (with respect to $\subseteq$) $U \in \mathrm{Up}(\langle W, \preceq\rangle)$ such that $a \in \bigcap_{v\in U} D_v$. We define an $\mathsf{S5}(\mathbf{G})$-model $\mathfrak{M}^* = \langle W^*, R, V\rangle$ where for all $a, b \in W^*$,

$$Rab = \begin{cases} W & a = b \\ U(a) \cap U(b) & a \neq b \end{cases} \quad \text{and} \quad V(p_i, a) = \{v \in W \mid a \in I_v(P_i)\}.$$

Note that each $V(p_i, a)$ is an upset of $\langle W, \preceq\rangle$ since $u \preceq v$ implies $I_u(P_i) \subseteq I_v(P_i)$, and that $Raa = W$, $Rab = Rba$, and $Rab \wedge Rbc \leq Rac$ for all $a, b, c \in W^*$.

The following lemma yields $V(\varphi, a_0) \neq W$ and hence $\not\models_{\mathsf{S5}(\mathbf{G})} \varphi$ as required.

Lemma 1. *For any* $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $w \in W$, *and* $a \in D_w$,

$$\mathfrak{M}, w \models^a \varphi^\circ \iff w \in V(\varphi, a).$$

Proof. The following observation will be useful. If $a \in D_w$ then for any $b \in W^*$, $b \in D_w$ if and only if $w \in Rab$. Indeed, if $b = a$, this is trivial. If $b \neq a$, then as $a \in D_w$, $w \in U(a) \cap U(b)$ if and only if $w \in U(b)$, i.e., $b \in D_w$.

We prove the claim by induction on the length of φ. The base cases for $\bot$, $\top$, and p_i are immediate from the definitions. The cases for $\wedge$ and $\vee$ are also straightforward, so let us just consider the cases when φ is of the form $\psi_1 \to \psi_2$, $\Box\psi$, or $\Diamond\psi$. Let $w \in W$ and $a \in D_w$, and set $[w) = \{v \in W \mid v \succeq w\}$.

- Suppose that $\varphi = \psi_1 \to \psi_2$.

$$\begin{aligned}\mathfrak{M}, w \models^a (\psi_1 \to \psi_2)^\circ &\iff \mathfrak{M}, v \models^a \psi_1^\circ \text{ implies } \mathfrak{M}, v \models^a \psi_2^\circ \text{ for all } v \succeq w\\ &\iff v \in V(\psi_1, a) \text{ implies } v \in V(\psi_2, a) \text{ for all } v \succeq w\\ &\iff [w) \cap V(\psi_1, a) \subseteq V(\psi_2, a)\\ &\iff [w) \subseteq (V(\psi_1, a) \to V(\psi_2, a))\\ &\iff w \in V(\psi_1 \to \psi_2, a).\end{aligned}$$

- Suppose that $\varphi = \Box\psi$.

$$\begin{aligned}\mathfrak{M}, w \models^a (\Box\psi)^\circ &\iff \mathfrak{M}, v \models^b \psi^\circ \text{ for all } v \succeq w \text{ and } b \in D_v\\ &\iff v \in V(\psi, b) \text{ for all } v \succeq w \text{ such that } v \in Rab\\ &\iff [w) \cap Rab \subseteq V(\psi, b) \text{ for all } b \in W^*\\ &\iff w \in (Rab \to V(\psi, b)) \text{ for all } b \in W^*\\ &\iff w \in V(\Box\psi).\end{aligned}$$

- Suppose that $\varphi = \Diamond\psi$.

$$\begin{aligned}\mathfrak{M}, w \models^a (\Diamond\psi)^\circ &\iff \mathfrak{M}, w \models^b \psi^\circ \text{ for some } b \in D_w\\ &\iff w \in Rab \text{ and } w \in V(\psi, b) \text{ for some } b \in D_w\\ &\iff w \in \bigvee\{Rab \cap V(\psi, b) \mid b \in W^*\}\\ &\iff w \in V(\Diamond\psi, b).\end{aligned}$$

The second-to-last equivalence follows from the fact that in $\mathrm{Up}(\langle W, \preceq\rangle)$ suprema are interpreted as unions. □

For the left-to-right direction, we also proceed contrapositively. For technical reasons, however, we show first that we can restrict our attention to a restricted class of $\mathsf{S5}(\mathbf{G})$-models. We say that an $\mathsf{S5}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R, V\rangle$ is *irrational* if $V(\varphi, w)$ is irrational, 0, or 1 for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ and $w \in W$.

Lemma 2. *For any countable* $\mathsf{S5}(\mathbf{G})$*-model* $\mathfrak{M} = \langle W, R, V\rangle$*, there exists an irrational* $\mathsf{S5}(\mathbf{G})$*-model* $\mathfrak{M}' = \langle W, R', V'\rangle$ *such that for all* $\varphi, \psi \in \mathrm{Fm}_{\Box\Diamond}$, $w \in W$*:*

$$V(\varphi, w) < V(\psi, w) \iff V'(\varphi, w) < V'(\psi, w).$$

Proof. By [10, Lemma 3.7], there exists a complete order-embedding f from the countable set $S = \{V(\varphi, w) \mid w \in W,\ \varphi \in \mathrm{Fm}_{\Box\Diamond}\} \cup R[W \times W]$ into $\mathbb{Q} \cap [0, 1]$. Now for each $q \in \mathbb{Q} \cap [0, 1]$, let

$$g(q) = \begin{cases} \frac{\pi}{3}q & q \leq \frac{1}{2}\\ \frac{\pi}{6} + (2 - \frac{\pi}{3})(q - \frac{1}{2}) & q > \frac{1}{2}.\end{cases}$$

Then g is a complete order-embedding from $\mathbb{Q} \cap [0, 1]$ into $([0, 1]\backslash\mathbb{Q}) \cup \{0, 1\}$ with $g(0) = 0$, $g(1) = 1$. So $h = g \circ f$ is a complete order-embedding from S

into $([0,1]\backslash\mathbb{Q}) \cup \{0,1\}$ with $h(0) = 0$, $h(1) = 1$. Finally, let $\mathfrak{M}' = \langle W, R', V' \rangle$ where $R'wv = h(Rwv)$ and $V'(p_i, w) = h(V(p_i, w))$ for $w, v \in W$ and $i \in \mathbb{N}$. A straightforward induction on formula length yields $V'(\varphi, w) = h(V(\varphi, w))$ for any $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ and $w \in W$ and the claim follows immediately. $\Box$

Now let $\mathfrak{M} = \langle W, R, V \rangle$ be any irrational $\mathsf{S5}(\mathbf{G})$-model and fix $w_0 \in W$. Let $(0,1)_{\mathbb{Q}}$ denote $(0,1) \cap \mathbb{Q}$. We define the IKL_1-model

$$\mathfrak{M}^\circ = \langle (0,1)_{\mathbb{Q}}, \geq, \{D_q\}_{q\in(0,1)_{\mathbb{Q}}}, \{I_q\}_{q\in(0,1)_{\mathbb{Q}}} \rangle$$

such that for each $q \in (0,1)_{\mathbb{Q}}$ and unary predicate P_i,

$$D_q = \{v \in W \mid Rw_0v \geq q\} \quad \text{and} \quad I_q(P_i) = \{v \in W \mid V(p_i, v) \geq q\} \cap D_q.$$

Lemma 3. *For any* $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $q \in (0,1)_{\mathbb{Q}}$, *and* $w \in D_q$,

$$\mathfrak{M}^\circ, q \models^w \varphi^\circ \iff V(\varphi, w) \geq q.$$

Proof. We prove the claim by induction on the length of φ. The base cases follow by definition and the cases of the propositional connectives are straightforward. We consider the modal cases.

- For $\varphi = \Box\psi$, observe first that

$$\begin{aligned} \mathfrak{M}^\circ, q \models^w (\forall x)\psi^\circ &\iff \mathfrak{M}^\circ, r \models^v \psi^\circ \text{ for all } r \leq q \text{ and } v \in D_r \\ &\iff V(\psi, v) \geq r \text{ for all } r \leq q \text{ and } v \in D_r; \\ V(\Box\psi, w) \geq q &\iff \bigwedge\{Rwv \to V(\psi, v) \mid v \in W\} \geq q \\ &\iff Rwv \to V(\psi, v) \geq q \text{ for all } v \in W \\ &\iff V(\psi, v) \geq q \wedge Rwv \text{ for all } v \in W. \end{aligned}$$

  For the left-to-right direction suppose that $V(\psi, v) \geq r$ for all $r \leq q$ and $v \in D_r$. By assumption, $w \in D_q$, so $Rw_0w \geq q$. Let $v \in W$. If $q \leq Rwv$, then, by symmetry and transitivity, $Rw_0v \geq q$, i.e., $v \in D_q$, and hence $V(\psi, v) \geq q = q \wedge Rwv$. If $q > Rwv$, then $Rw_0w \geq q > Rwv$. By symmetry and transitivity, $Rw_0v = Rwv$. For all $r \in (0,1)_{\mathbb{Q}}$ such that $r \leq Rw_0v$, it holds that $v \in D_r$, so $V(\psi, v) \geq r$. Since $(0,1)_{\mathbb{Q}}$ is dense in $(0,1)\backslash\mathbb{Q}$, we have $\sup\{r \in (0,1)_{\mathbb{Q}} \mid Rw_0v \geq r\} = Rw_0v$ and hence $V(\psi, v) \geq Rw_0v = Rwv$.
  For the right-to-left direction, suppose that $V(\psi, v) \geq q \wedge Rwv$ for all $v \in W$. Let $r \leq q$ and $v \in D_r$. Then $Rw_0v \geq r$. Since $w \in D_q$, also $Rw_0w \geq q \geq r$, and by symmetry and transitivity, $Rwv \geq r$. Hence $V(\psi, v) \geq q \wedge Rwv \geq r$.
- For $\varphi = \Diamond\psi$, observe first that since $\mathfrak{M}$ is irrational and $q \in (0,1)_{\mathbb{Q}}$, $V(\varphi, w) \geq q$ if and only if $V(\varphi, w) > q$. Now observe that

$$\begin{aligned} \mathfrak{M}^\circ, q \models^w (\exists x)\psi^\circ &\iff \mathfrak{M}^\circ, q \models^v \psi^\circ \text{ for some } v \in D_q \\ &\iff V(\psi, v) \geq q \text{ for some } v \in D_q; \\ V(\Diamond\psi, w) \geq q &\iff \bigvee\{Rwv \wedge V(\psi, v) \mid v \in W\} \geq q \\ &\iff \bigvee\{Rwv \wedge V(\psi, v) \mid v \in W\} > q \\ &\iff Rwv \wedge V(\psi, v) \geq q \text{ for some } v \in W. \end{aligned}$$

For the left-to-right direction, suppose that $V(\psi, v) \geq q$ for some $v \in D_q$. Since $w, v \in D_q$, by transitivity, $Rwv \geq q$ and hence $Rwv \wedge V(\psi, v) \geq q$. For the right-to-left direction, suppose that there exists $v \in W$ such that $Rwv \wedge V(\psi, v) \geq q$, i.e., $Rwv \geq q$ and $V(\psi, v) \geq q$. Since $w \in D_q$, also $Rwv \geq q$, so $v \in D_q$ and $V(\psi, v) \geq q$. □

To conclude the proof of Theorem 4 suppose that $\not\models_{\mathsf{S5}(\mathbf{G})} \varphi$. It follows that there exist an $\mathsf{S5}(\mathbf{G})$-model $\langle W, R, V \rangle$ and $w \in W$ such that $V(\varphi, w) < 1$. By Lemma 2, there exist an irrational $\mathsf{S5}(\mathbf{G})$-model $\mathfrak{M} = \langle W, R', V' \rangle$ and $r \in (0,1)_{\mathbb{Q}}$ such that $V'(\varphi, w) < r < 1$. But then, by Lemma 3, for the $\mathsf{IKL_1}$-model $\mathfrak{M}^\circ$ defined above, $\mathfrak{M}^\circ, r \not\models^w \varphi^\circ$. That is, $\not\models_{\mathsf{IKL_1}} \varphi^\circ$.

4 An Interpretation of $\mathsf{S5}$(G) in $\mathsf{S5}(\mathrm{G})^{\mathsf{C}}$

In this section, we provide an interpretation of the one-variable fragment of Corsi logic in the one-variable fragment of first-order Gödel logic, thereby obtaining also an interpretation of $\mathsf{S5}(\mathbf{G})$ in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. The key idea of this interpretation is to use a distinguished unary predicate P_0 for a $\mathsf{CDIKL_1}$-model to describe the domains of a corresponding $\mathsf{IKL_1}$-model. To this end, we let $\mathrm{Fm}_1^r \subseteq \mathrm{Fm}_1$ denote the set of one-variable first-order formulas not containing P_0, and define an $\mathsf{IKL}_1^{\mathsf{r}}$-model to be an $\mathsf{IKL_1}$-model $\mathfrak{M} = \langle W, \preceq, \{D_w\}_{w\in W}, \{I_w\}_{w\in W} \rangle$ such that the functions $\{I_w\}_{w\in W}$ are restricted to $\{P_i\}_{i\in\mathbb{N}^+}$.

With every $\mathsf{CDIKL_1}$-model $\mathfrak{M} = \langle W, \preceq, \{D\}, \{I_w\}_{w\in W} \rangle$, we associate an $\mathsf{IKL}_1^{\mathsf{r}}$-model $\mathfrak{M}^r = \langle W, \preceq, \{D_w\}_{w\in W}, \{I_w^r\}_{w\in W} \rangle$, where for each $w \in W$ and $i \in \mathbb{N}^+$,

$$D_w = I_w(P_0) \quad \text{and} \quad I_w^r(P_i) = I_w(P_i) \cap D_w.$$

Notice that $\mathfrak{M} \mapsto \mathfrak{M}^r$ is a surjective map from $\mathsf{CDIKL_1}$-models to $\mathsf{IKL}_1^{\mathsf{r}}$-models.

Now for each $\alpha \in \mathrm{Fm}_1^r$, we define $\alpha^c \in \mathrm{Fm}_1$ by relativizing quantifiers to the unary predicate P_0. Inductively, we let $(P_i(x))^c = P_i(x)$ for each $i \in \mathbb{N}^+$, $\bot^c = \bot$, $\top^c = \top$, $(\alpha \star \beta)^c = \alpha^c \star \beta^c$ for $\star \in \{\wedge, \vee, \to\}$,

$$((\forall x)\alpha)^c = (\forall x)(P_0(x) \to \alpha^c), \text{ and } ((\exists x)\alpha)^c = (\exists x)(P_0(x) \wedge \alpha^c).$$

Lemma 4. *Given any $\alpha \in \mathrm{Fm}_1^r$, $\mathsf{CDIKL_1}$-model $\mathfrak{M} = \langle W, \preceq, \{D\}, \{I_w\}_{w\in W} \rangle$, $w \in W$, and $a \in I_w(P_0)$,*

$$\mathfrak{M}^r, w \models^a \alpha \iff \mathfrak{M}, w \models^a \alpha^c.$$

Proof. We prove the claim by induction on the length of α. For the base case, for each $i \in \mathbb{N}^+$, using the assumption that $a \in D_w$,

$$\mathfrak{M}^r, w \models^a P_i(x) \iff a \in I_w^r(P_i) \iff a \in I_w(P_i) \iff \mathfrak{M}, w \models^a P_i(x).$$

The cases for the propositional connectives follow easily using the induction hypothesis and the definition of α^c, so we just check the cases for the quantifiers:

$$\begin{aligned}
\mathfrak{M}^r, w \models^a (\forall x)\beta &\iff \mathfrak{M}^r, v \models^b \beta \text{ for all } v \succeq w \text{ and } b \in D_v \\
&\iff \mathfrak{M}, v \models^b \beta^c \text{ for all } v \succeq w \text{ and } b \in I_v(P_0) \\
&\iff (\mathfrak{M}, v \models^b P_0(x) \Rightarrow \mathfrak{M}, v \models^b \beta^c) \text{ for all } v \succeq w \text{ and } b \in D \\
&\iff \mathfrak{M}, v \models^b P_0(x) \to \beta^c \text{ for all } v \succeq w \text{ and } b \in D \\
&\iff \mathfrak{M}, w \models^a (\forall x)(P_0(x) \to \beta^c) \\
&\iff \mathfrak{M}, w \models^a ((\forall x)\beta)^c. \\
\mathfrak{M}^r, w \models^a (\exists x)\beta &\iff \mathfrak{M}^r, w \models^b \beta \text{ for some } b \in D_w \\
&\iff \mathfrak{M}, w \models^b \beta^c \text{ for some } b \in I_{w(P_0)} \\
&\iff (\mathfrak{M}, w \models^b P_0(x) \text{ and } \mathfrak{M}, w \models^b \beta^c) \text{ for some } b \in D \\
&\iff \mathfrak{M}, w \models^b P_0(x) \wedge \beta^c \text{ for some } b \in D \\
&\iff \mathfrak{M}, w \models^a (\exists x)(P_0(x) \wedge \beta^c) \\
&\iff \mathfrak{M}, w \models^a ((\exists x)\beta)^c.
\end{aligned}$$

□

Corollary 1. *For any sentence $\alpha \in \mathrm{Fm}_1^r$, $\models_{\mathsf{IKL}_1} \alpha \iff \models_{\mathsf{CDIKL}_1} \alpha^c$.*

Proof. Consider a CDIKL_1-model $\mathfrak{M} = \langle W, \preceq, \{D\}, \{I_w\}_{w \in W}\rangle$ and any $a \in D$. Since $\alpha \in \mathrm{Fm}_1^r$ is a sentence, $\mathfrak{M} \models \alpha^c$ if and only if $\mathfrak{M}, w \models^a \alpha^c$ for all $w \in W$. So, by the previous lemma, $\mathfrak{M} \models \alpha^c$ if and only if $\mathfrak{M}^r, w \models^a \alpha$ for all $w \in W$, which holds, since $\alpha \in \mathrm{Fm}_1^r$ is a sentence, if and only if $\mathfrak{M}^r \models \alpha$. The result now follows immediately using the fact that the map $\mathfrak{M} \mapsto \mathfrak{M}^r$ is surjective. □

Now let $\mathrm{Fm}_{\Box\Diamond}^r \subseteq \mathrm{Fm}_{\Box\Diamond}$ denote the set of modal formulas not containing p_0. For each $\varphi \in \mathrm{Fm}_{\Box\Diamond}^r$, we define $\varphi^c \in \mathrm{Fm}_1$ by relativizing modalities to p_0. Inductively, we let $(p_i)^c = p_i$ for each $i \in \mathbb{N}^+$, $\bot^c = \bot$, $\top^c = \top$, $(\varphi \star \psi)^c = \varphi^c \star \psi^c$ for $\star \in \{\wedge, \vee, \to\}$,

$$(\Box\varphi)^c = \Box(p_0 \to \varphi^c), \quad \text{and} \quad (\Diamond\varphi)^c = \Diamond(p_0 \wedge \varphi^c).$$

The main result of this section then follows directly using Theorems 3 and 4 and Corollary 1.

Theorem 5. *For all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$, $\models_{\mathsf{S5}(\mathbf{G})} \varphi \iff \models_{\mathsf{S5}(\mathbf{G})^{\mathsf{C}}} (\Box\varphi)^c$.*

Let us remark that the above proof generalizes in a straightforward way to provide an interpretation of the full first-order Corsi logic in the first-order logic of linear Kripke models with constant domains, or, equivalently, first-order Gödel logic. Moreover, the predicate used in this interpretation corresponds exactly to the existence predicate considered in the context of Scott logics by Iemhoff in [11] and is closely related also to the normalized probability distribution used for the possibilistic logic studied in [3]. We intend to investigate these connections in more detail in future work.

5 A Complexity Result

As has been mentioned already, neither $\mathsf{S5}(\mathbf{G})$ nor $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ admits the finite model property with respect to their standard Kripke semantics. It is known, however, that $\mathsf{S5}(\mathbf{G})$ does admit the finite model property with respect to its algebraic semantics (see [2]), and we prove here that the same result holds also for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. We then use this finite model property to give a new proof that validity in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is co-NP-complete (first proved in [5]), and hence also, by Theorem 5, the same result for $\mathsf{S5}(\mathbf{G})$.

An algebra $\langle H, \wedge, \vee, \to, \bot, \top, \Box, \Diamond\rangle$ (also shortened to $\langle \mathbf{H}, \Box, \Diamond\rangle$) is called a *monadic Heyting algebra* if $\mathbf{H} = \langle H, \wedge, \vee, \to, \bot, \top\rangle$ is a Heyting algebra and $\Box, \Diamond$ are unary operators on H satisfying for all $a, b \in H$,

(1a) $\Box a \leq a$	(1b) $a \leq \Diamond a$
(2a) $\Box(a \wedge b) = \Box a \wedge \Box b$	(2b) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
(3a) $\Box\top = \top$	(3b) $\bot = \Diamond\bot$
(4a) $\Box\Diamond a = \Diamond a$	(4b) $\Diamond\Box a = \Box a$
(5a) $\Diamond(\Diamond a \wedge b) = \Diamond a \wedge \Diamond b.$	

If a monadic Heyting algebra satisfies the prelinearity law $(x \to y) \vee (y \to x) \approx \top$, then we call it a *monadic linear Heyting algebra*, and if it satisfies also the constant domain law $\Box(\Box x \vee y) \approx \Box x \vee \Box y$, we call it a *monadic Gödel algebra*. It is straightforward to prove that the varieties (equivalently, equational classes) of monadic linear Heyting algebras and monadic Gödel algebras provide equivalent algebraic semantics for $\mathsf{S5}(\mathbf{G})$ and $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, respectively. Indeed, the lattices of axiomatic extensions of $\mathcal{MIPC}$ and varieties of monadic Heyting algebras are dual (see [2]).

These algebras also admit a useful alternative representation. For any monadic Heyting algebra $\langle \mathbf{H}, \Box, \Diamond\rangle$, the set $H_0 = \{\Box a \mid a \in H\} = \{\Diamond a \mid a \in H\}$ forms a subuniverse of $\mathbf{H}$ satisfying for all $a \in H$,

$$\Box a = \bigvee\{b \in H_0 \mid b \leq a\} \quad \text{and} \quad \Diamond a = \bigwedge\{b \in H_0 \mid b \geq a\}.$$

Conversely, call any subuniverse H_0 of a Heyting algebra $\mathbf{H}$ where all such suprema and infima exist in H_0 *relatively complete*. Defining $\Box$ and $\Diamond$ as above for any relatively complete subuniverse H_0 of a Heyting algebra $\mathbf{H}$ yields a monadic Heyting algebra $\langle \mathbf{H}, \Box, \Diamond\rangle$.

Theorem 6 (cf. [2]). *There exists a one-to-one correspondence between monadic Heyting algebras $\langle \mathbf{H}, \Box, \Diamond\rangle$ and pairs $\langle \mathbf{H}, H_0\rangle$ of Heyting algebras where H_0 is a relatively complete subuniverse of $\mathbf{H}$.*

We use this alternative representation to establish the finite model property for the variety of monadic Gödel algebras. Let us call a monadic Gödel algebra *standard* if it is of the form $\langle \mathbf{G}^W, \Box, \Diamond\rangle$, where W is any non-empty set, $\mathbf{G}^W$ is the Heyting algebra with universe $[0,1]^W$ and operations defined pointwise, and for each $f \in [0,1]^W$ and $w \in W$,

$$\Box(f)(w) = \bigwedge\{f(v) \mid v \in W\} \quad \text{and} \quad \Diamond(f)(w) = \bigvee\{f(v) \mid v \in W\}.$$

Using the completeness results of [6], a formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$-valid if and only if $\varphi \approx \top$ is valid in all standard monadic Gödel algebras. However, this equivalence fails when restricted to standard monadic Gödel algebras $\langle \mathbf{G}^W, \Box, \Diamond\rangle$ where W is finite.

Observe now that for any standard monadic Gödel algebra $\langle \mathbf{G}^W, \Box, \Diamond\rangle$, the subuniverse $\{\Box f \mid f \in [0,1]^W\}$ consists of all constant functions for $r \in [0,1]$,

$$f_r \colon W \to [0,1]; \quad w \mapsto r.$$

We broaden the class of standard monadic Gödel algebras by considering also subuniverses consisting of only some of these constant functions.

Lemma 5. *For any complete sublattice T of $[0,1]$ containing $\{0,1\}$, the set $\{f_r \mid r \in T\}$ is a relatively complete subuniverse of $\mathbf{G}^W$ and yields a monadic Gödel algebra with modal operators*

$$\Box f(w) = \bigvee\{r \in T \mid r \le \bigwedge\{f(v) \mid v \in W\}\}$$
$$\Diamond f(w) = \bigwedge\{r \in T \mid r \ge \bigvee\{f(v) \mid v \in W\}\}.$$

Proof. It is easy to check that $\{f_r \mid r \in T\}$ is a subuniverse of $\mathbf{G}^W$. To show that it is relatively complete, consider $\bigvee\{f_r \mid f_r \le g, r \in T\}$ for some $g \in \mathbf{G}^W$. Then $f_r \le g$ for $r \in T$ amounts to $r \le g(v)$ for all $v \in W$, i.e., $r \le \bigwedge\{g(v) \mid v \in W\}$. So $\bigvee\{f_r \mid f_r \le g, r \in T\} = \bigvee\{f_r \mid r \le \bigwedge\{g(v) \mid v \in W\}, r \in T\}$, which exists in T since T is complete. Similarly, $\bigwedge\{f_r \mid f_r \ge g\}$ exists in T, so $\{f_r \mid r \in T\}$ is relatively complete. Hence $\langle \mathbf{G}^W, \{f_r \mid r \in T\}\rangle$ corresponds to a monadic Heyting algebra. Clearly, this algebra also satisfies the prelinearity law and it is easy to check that the constant domain law is satisfied using properties of T and relative completeness. $\Box$

Note that the algebras described in the previous lemma correspond exactly to the alternative semantics used in [5] to prove decidability and complexity results for $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$. Here we obtain simpler proofs of these results (avoiding a rather complicated "squeezing" of truth values argument) by establishing a finite model property with respect to this class of monadic Gödel algebras.

Lemma 6. *Suppose that the equation $\varphi \approx \top$ is not valid in a standard monadic Gödel algebra $\langle \mathbf{G}^W, \Box, \Diamond\rangle$ for some $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ of length $n-2$. Then there exist a non-empty set $W' \subseteq W$, a set $T \subseteq [0,1]$ with $\{0,1\} \subseteq T$, and a subalgebra $\mathbf{A}$ of $\mathbf{G}$ with $T \subseteq A$ satisfying $|W'| \le n$, $|T| \le n$, and $|A| \le n^2$, such that $\varphi \approx \top$ is not valid in the finite monadic Gödel algebra corresponding to $\langle \mathbf{A}^{W'}, \{f_r \mid r \in T\}\rangle$.*

Proof. Suppose that $\varphi \approx \top$ is not valid in some standard monadic Gödel algebra $\langle \mathbf{G}^W, \Box, \Diamond \rangle$ for some $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ of length $n-2$. Then there exists an evaluation e from $\mathrm{Fm}_{\Box\Diamond}$ to G^W satisfying $e(\varphi)(w) < 1$ for some $w \in W$. Let $\Sigma \subseteq \mathrm{Fm}_{\Box\Diamond}$ be the set of subformulas of φ, noting that $|\Sigma| \leq n-2$. We define

$$T = \{0,1\} \cup \{e(\Box\psi)(w) \mid \Box\psi \in \Sigma\} \cup \{e(\Diamond\psi)(w) \mid \Diamond\psi \in \Sigma\}.$$

Clearly, $|T| \leq n$. For each $\Box\psi \in \Sigma$ and $\Diamond\psi \in \Sigma$, we pick a witness $v_{\Box\psi} \in W$ or $v_{\Diamond\psi} \in W$, respectively, such that

$$\begin{aligned} e(\Box\psi)(w) &= \bigvee\{r \in T \mid r \leq e(\psi)(v_{\Box\psi})\} \\ e(\Diamond\psi)(w) &= \bigwedge\{r \in T \mid r \geq e(\psi)(v_{\Diamond\psi})\}. \end{aligned}$$

We define

$$W' = \{w\} \cup \{v_{\Box\psi} \mid \Box\psi \in \Sigma\} \cup \{v_{\Diamond\psi} \mid \Diamond\psi \in \Sigma\} \quad \text{and} \quad e' = e{\restriction}_{G^{W'}}.$$

Clearly, $|W'| \leq n$. Moreover, $e'(\varphi)(w) = e(\varphi)(w) < 1$ and hence $\varphi \approx \top$ is not valid in the monadic Gödel algebra corresponding to $\langle \mathbf{G}^{W'}, \{f_r \mid r \in T\}\rangle$. Finally, we define

$$A = \{0,1\} \cup \bigcup_{\psi \in \Sigma} e(\psi)[W'].$$

Clearly, $T \subseteq A$ and $|A| \leq n^2$. Moreover, $A^{W'}$ is a finite subuniverse of $\mathbf{G}^{W'}$, and $\varphi \approx \top$ is not valid in the finite monadic Gödel algebra corresponding to $\langle A^{W'}, \{f_r \mid r \in T\}\rangle$. $\Box$

An analysis of the number of steps needed to find a finite countermodel yields an upper complexity bound for checking validity in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$.

Theorem 7

(a) *The variety of monadic Gödel algebras has the finite model property.*
(b) *Checking validity in* $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ *is co-NP-complete.*

Proof. (a) The variety of monadic Gödel algebras is generated by its standard members, and hence the finite algebras described in the previous lemma also generate this variety.

(b) To check the non-validity of an equation $\varphi \approx \top$, we fix sets A and W' that we may identify with $K = \{1, 2, \ldots, n^2\}$, where $n-2$ is the length of φ, letting $\mathbf{A}$ denote the unique Gödel algebra induced by the standard order. It suffices to find a relative subuniverse $T \subseteq A$ and an evaluation $e\colon \mathrm{Pr}(\Sigma) \to A^{W'}$ (where $\mathrm{Pr}(\Sigma)$ is the set of propositional variables occurring in Σ that we may also identify with K) and check $e(\varphi) < \top$ when evaluated in the algebra $\langle A^{W'}, \{f_r \mid r \in T\}\rangle$. Finding such T and e is equivalent to finding a characteristic function $\widetilde{T}\colon A \to \{0,1\}$ and a function $\tilde{e}\colon \mathrm{Pr}(\Sigma) \times W' \to A$; that is, finding a pair of sequences of length n^2 and n^4 respectively with entries in K. The tasks of guessing non-deterministically these sequences and checking $e(\varphi) < \top$ in the resulting algebra can be performed in polynomial time. Hence checking non-validity is in NP. But also non-modal formulas are valid in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ if and only if they are valid in Gödel logic, which is known to be co-NP-complete. Hence checking validity in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$ is co-NP-complete. $\square$

Using the interpretation of $\mathsf{S5}(\mathbf{G})$ in $\mathsf{S5}(\mathbf{G})^{\mathsf{C}}$, provided by Theorem 5, that is linear in the length of the input formula, it follows that also checking validity in $\mathsf{S5}(\mathbf{G})$ is co-NP-complete. The correspondence between validity in $\mathsf{S5}(\mathbf{G})$ and the one-variable fragment of Corsi logic provided by Theorem 1, again linear in the length of the input formula, then completes the proof of Theorem 2.

Acknowledgements. The second and fourth authors were supported by the Swiss National Science Foundation grant 200021_165850, the first author by the Universidad de los Andes Science Faculty Research Fund, and the third author by the research projects PIP 112-20150100412CO, CONICET, UBA-CyT 20020150100002BA and PICT/O 2016-0215. The authors have also received funding from the EU Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 689176.

References

1. Beckmann, A., Preining, N.: Linear Kripke frames and Gödel logics. J. Symb. Log. **72**, 26–44 (2007)
2. Bezhanishvili, G.: Varieties of monadic Heyting algebras - part I. Studia Logica **61**(3), 367–402 (1998)
3. Bou, F., Esteva, F., Godo, L., Rodriguez, R.O.: Possibilistic semantics for a modal *KD*45 extension of Gödel fuzzy logic. In: Carvalho, J.P., Lesot, M.-J., Kaymak, U., Vieira, S., Bouchon-Meunier, B., Yager, R.R. (eds.) IPMU 2016. CCIS, vol. 611, pp. 123–135. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-40581-0_11
4. Bull, R.A.: MIPC as formalisation of an intuitionist concept of modality. J. Symb. Log. **31**, 609–616 (1966)
5. Caicedo, X., Metcalfe, G., Rodríguez, R., Rogger, J.: Decidability in order-based modal logics. J. Comput. Syst. Sci. **88**, 53–74 (2017)
6. Caicedo, X., Rodríguez, R.: Bi-modal Gödel logic over [0,1]-valued Kripke frames. J. Log. Comput. **25**(1), 37–55 (2015)
7. Corsi, G.: Completeness theorem for Dummett's LC quantified. Studia Logica **51**, 317–335 (1992)
8. Gabbay, D.M., Kurucz, A., Wolter, F., Zakharyaschev, M.: Many-Dimensional Modal Logics. Elsevier, Amsterdam (2003)
9. Görnemann, S.: A logic stronger than intuitionism. J. Symb. Log. **36**(2), 249–261 (1971)
10. Horn, A.: Logic with truth values in a linearly ordered Heyting algebra. J. Symb. Log. **34**(3), 395–409 (1969)
11. Iemhoff, R.: A note on linear Kripke models. J. Log. Comput. **15**(4), 489–506 (2005)
12. Kripke, S.A.: Semantical analysis of intuitionistic logic I. In: Crossley, J.N., Dummett, M.A.E. (eds.) Formal Systems and Recursive Functions, Studies in Logic and the Foundations of Mathematics, vol. 40, pp. 92–130. Elsevier (1965)

13. Ono, H.: On some intuitionistic modal logics. Publ. RIMS, Kyoto Univ. **13**, 687–722 (1977)
14. Ono, H., Suzuki, N.-Y.: Relations between intuitionistic modal logics and intermediate predicate logics. Rep. Math. Log. **22**, 65–87 (1988)
15. Suzuki, N.-Y.: Kripke bundles for intermediate predicate logics and Kripke frames for intuitionistic modal logics. Studia Logica **49**(3), 289–306 (1990)
16. Takano, M.: Ordered sets R and Q as bases of Kripke models. Studia Logica **46**, 137–148 (1987)

Analytic Calculi for Monadic PNmatrices

Carlos Caleiro and Sérgio Marcelino(✉)

SQIG - Instituto de Telecomunicações, Dep. Matemática - Instituto Superior Técnico, Universidade de Lisboa, Lisbon, Portugal
{ccal,smarcel}@math.tecnico.ulisboa.pt

Abstract. Analytic calculi are a valuable tool for a logic, as they allow for effective proof-search and decidability results. We study the axiomatization of generalized consequence relations determined by monadic partial non-deterministic matrices (PNmatrices). We show that simple axiomatizations can always be obtained, using inference rules which can have more than one conclusion. Further, we prove that these axiomatizations are always analytic, which seems to raise a contrast with recent non-analyticity results for sequent-calculi with PNmatrix semantics.

1 Introduction

PNmatrices were introduced in [5], as a generalization of non-deterministic matrices (Nmatrices) [1,2]. Adding non-determinism and also partiality to the traditional notion of logical matrix (see [17]) has proven quite relevant in a myriad of recent compositional results in logic [3,5,6,11,13], namely as semantical counterparts of certain families of sequent-calculi. However, while Nmatrices still inherit from logical matrices a local semantical form of analyticity (a well formed valuation on a set of formulas closed for subformulas can always be extended to a full valuation), the partiality allowed by PNmatrices spoils this property. It turns out that for the sequent-calculi using these semantical tools, partiality seems to devoid them of a usable (even if generalized) subformula property capable of guaranteeing analyticity (and elimination of non-analytic cuts) [5,11].

Concerning other types of calculi, traditional Hilbert-style calculi are clearly not an option if any form of analyticity is expected. However, a very simple and powerful (and too often neglected) generalization of Hilbert-calculi has been proposed in [16]. Along with their proposal, Shoesmith and Smiley already showed that every logical matrix can be given a *multiple conclusion* axiomatization with rules of the form $\frac{\Gamma}{\Delta}$ where both Γ (read conjunctively, as usual) and Δ (read disjunctively) are sets of formulas. This axiomatization is finite for a given finite

This research was done under the scope of Project UID/EEA/50008/2019 of Instituto de Telecomunicações (IT, financed by the applicable framework (FCT/MEC through national funds and cofunded by FEDER-PT2020), and is part of the MoLC project of SQIG at IT. Thanks are due to two anonymous referees for their valuable feedback, which helped improving an earlier version of this paper.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 84–98, 2019.
https://doi.org/10.1007/978-3-662-59533-6_6

matrix, in contrast with the existence of finite matrices whose logic cannot be finitely axiomatized in Hilbert-style [18].

In a recent paper [14], we have extended the result of [16] and shown that multiple conclusion axiomatizations can be easily obtained for every Nmatrix with the additional expressiveness requirement of being *monadic*. Further, we showed that the calculi obtained enjoy a suitably generalized subformula property which allowed us to prove the analyticity of the axiomatizations obtained. Herein, we go a step further and analyze the case of PNmatrices. It turns out that the same exact methods can be used to obtain sound and complete axiomatizations of the (generalized) consequence relations of any monadic PNmatrix. However, perhaps expectedly, the calculi obtained are not analytic, again due to partiality. What is really interesting, though, is that in the framework of multiple conclusion calculi it is extremely easy to remedy this situation: with the addition of a few sound rules, the calculus not only remains complete, of course, but it also becomes analytic.

The paper is organized as follows. Section 2 introduces and illustrates multiple conclusion logics and calculi (after [16] and [14]), as well as analyticity; and then recalls the fundamental aspects of (P)Nmatrices (after [1] and [5]), and the key property of being monadic (see [14,16]). Section 3 defines the calculi to be associated with each monadic PNmatrix and proves our main results, i.e., their completeness and analyticity. Along these sections we will illustrate our methods using the implication-free fragment of Kleene's strong three-valued logic (see [12]). In Sect. 4 we present a detailed example, one of the problematic paraconsistent logics of [11]. We close, in Sect. 5, discussing the results obtained, their import and limitations, and possible extensions of this work.

2 Preliminaries

In any context, given a function $h : X \to Y$ and $Z \subseteq X$ we use $h(Z)$ to denote the set $\{h(z) : z \in Z\}$.

A propositional *signature* Σ is an $\mathbb{N}$-indexed set $\Sigma = \{\Sigma^{(k)} : k \in \mathbb{N}\}$, where each $\Sigma^{(k)}$ contains the *k-ary connectives* of Σ. As usual, we may write $© \in \Sigma$ when $© \in \Sigma^{(k)}$ for some $k \in \mathbb{N}$. The language $L_\Sigma(P)$ is the carrier of the absolutely free Σ-algebra generated over a given set of propositional variables P. Elements of $L_\Sigma(P)$ are called *formulas*. Notationwise, we use $A, B, C, \ldots$ to denote formulas, and $\Gamma, \Delta, \Omega, \ldots$ to denote sets of formulas. For convenience, we often use commas and write Γ, Δ instead of $\Gamma \cup \Delta$, or Γ, A instead of $\Gamma \cup \{A\}$, or A, B instead of $\{A, B\}$.

Given a formula $A \in L_\Sigma(P)$, we denote by $\mathsf{var}(A)$ (resp. $\mathsf{sub}(A)$) the set of propositional variables (resp. subformulas) of A. A *substitution* is a mapping $\sigma : P \to L_\Sigma(P)$, uniquely extendable into an endomorphism $\cdot^\sigma : L_\Sigma(P) \to L_\Sigma(P)$. We also use $\overline{\Gamma}$ to denote $L_\Sigma(P) \setminus \Gamma$. If $A, B_1, \ldots, B_n \in L_\Sigma(P)$ is such that $\mathsf{var}(A) \subseteq \{p_1, \ldots, p_n\}$ then we use $A(B_1, \ldots, B_n)$ to denote the formula A^σ where σ is any substitution such that $\sigma(p_1) = B_1, \ldots, \sigma(p_n) = B_n$.

For added self-containment, as well as to fix notation, we recall the main notions regarding multiple conclusion logics and calculi, as well Nmatrices, PNmatrices and monadicity.

2.1 Multiple Conclusions

Fixed a signature Σ, a *(schematic) (multiple conclusion) inference rule* is a pair $\langle \Gamma, \Delta \rangle \in \wp(L_\Sigma(P)) \times \wp(L_\Sigma(P))$, usually simply written as $\frac{\Gamma}{\Delta}$, where Γ is the set of *premises* and Δ the set of *conclusions*. A *(multiple conclusion) calculus* is a set of inferences rules. A calculus is *finitary* if each of its rules has finitely many premises and conclusions.

Example 1. Consider a signature Σ containing a unary connective $\neg$ and a binary connective $\wedge$. The following four rules define a calculus R_1.

$$\frac{q\,,\,\neg q\,,\,\neg(p\wedge q)}{p\,,\,\neg p\,,\,p\wedge q}\,r_{\wedge ab0} \qquad \frac{q\,,\,\neg q\,,\,p\wedge q}{p\,,\,\neg p\,,\,\neg(p\wedge q)}\,r_{\wedge ab1}$$

$$\frac{q\,,\,\neg q}{p\,,\,\neg p\,,\,p\wedge q\,,\,\neg(p\wedge q)}\,r_{\wedge aba} \qquad \frac{q\,,\,\neg q\,,\,p\wedge q\,,\,\neg(p\wedge q)}{p\,,\,\neg p}\,r_{\wedge abb}$$

The next three rules define another calculus, R_2.

$$\frac{p\,,\,q\,,\,\neg q}{\neg p\,,\,p\wedge q}\,r_{\wedge 120} \qquad \frac{p\,,\,q\,,\,\neg q\,,\,p\wedge q}{\neg p\,,\,\neg(p\wedge q)}\,r_{\wedge 121} \qquad \frac{p\,,\,q\,,\,\neg q\,,\,p\wedge q\,,\,\neg(p\wedge q)}{\neg p}\,r_{\wedge 122}$$

$\triangle$

Inference rules can be used in derivations of conclusions from premises. However, contrarily to the case of Hilbert-style rules where derivations correspond to sequences of formulas resulting from premises by application of instances of rules, in this generalized setting derivations must now have a tree structure [16]. In order to show that Δ follows from Γ using the rules in R one must be able to build a tree starting from formulas in Γ and branching out whenever applying an instance of a rule, in such a way that all branches of the tree finally include some formula of Δ.

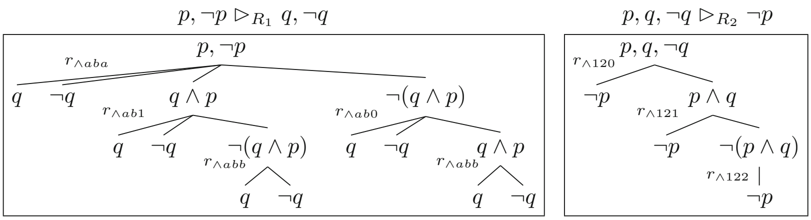
Given a rooted tree t let $<^{\mathsf{t}}$ be the (partial) order induced by t on its nodes by the relation of descendency. We denote: the set of nodes of t as $\mathsf{nodes}^{\mathsf{t}}$; the root of t as $\mathsf{root}^{\mathsf{t}}$ (the minimal element in $<^{\mathsf{t}}$); the set of leaf nodes of t as $\mathsf{leaves}^{\mathsf{t}}$ (the maximal elements in $<^{\mathsf{t}}$); the set of immediate children of node s as $\mathsf{childn}^{\mathsf{t}}(s)$ (the minimal descendants of node s); and the ancestors of node s as $\mathsf{ancest}^{\mathsf{t}}(s) = \{s' : s' <^{\mathsf{t}} s\}$. A tree t is said to be *bounded* if $\mathsf{nodes}^{\mathsf{t}} \setminus \mathsf{leaves}^{\mathsf{t}} = \bigcup_{s \in \mathsf{leaves}^{\mathsf{t}}} \mathsf{ancest}^{\mathsf{t}}(s)$, which means that every branch of the tree has a maximal element (leaf).

We say that a bounded rooted tree t labelled by $\ell : \mathsf{nodes}^{\mathsf{t}} \to \wp(L_\Sigma(P)) \cup \{*\}$ is an *R-derivation* provided that for each node $s \notin \mathsf{leaves}^{\mathsf{t}}$ we have that $\ell(s) \subseteq L_\Sigma(P)$ and there is a rule $\frac{\Gamma}{\Delta} \in R$ and a substitution $\sigma : P \to L_\Sigma(P)$ such that $\Gamma^\sigma \subseteq \ell(s)$ and:

– if $\Delta = \emptyset$ then $\mathsf{childn}^{\mathsf{t}}(s) = \{s^*\}$ and $\ell(s^*) = *$,
– if $\Delta \neq \emptyset$ then $\mathsf{childn}^{\mathsf{t}}(s) = \{s^A : A \in \Delta^\sigma\}$ and each $\ell(s^A) = \ell(s) \cup \{A\}$.

Given $\Gamma, \Delta \subseteq L_\Sigma(P)$, we say that an R-derivation t is a *R-proof* of Δ from Γ whenever $\ell(\mathsf{root}^{\mathsf{t}}) \subseteq \Gamma$ and $\ell(s) \cap \Delta \neq \emptyset$ for every $s \in \mathsf{leaves}^{\mathsf{t}}$ with $\ell(s) \neq *$. Note that leaves labelled by $*$ signal *discontinued* branches of a derivation. It should be noted that whenever R is finitary it is sufficient to consider finite proof trees. We write $\Gamma \rhd_R \Delta$ whenever there exists an R-proof of Δ from Γ.

Example 2. Below, we depict examples of derivations, namely of $p, \neg p \rhd_{R_1} q, \neg q$, and of $p, q, \neg q \rhd_{R_2} \neg p$, using the calculi defined in Example 1. Note that we label each child node with only the new formula, instead of the whole set, which can be collected from the labels of its ancestors.



△

A *generalized consequence relation* on $L_\Sigma(P)$, or *Scottian*, or *multiple conclusion consequence relation*, after [15,16] is a relation $\rhd \subseteq \wp(L_\Sigma(P)) \times \wp(L_\Sigma(P))$ satisfying the properties below for every $\Gamma, \Delta, \Gamma', \Delta' \subseteq L_\Sigma(P)$:

(O) if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \rhd \Delta$,
(D) if $\Gamma \rhd \Delta$ then $\Gamma, \Gamma' \rhd \Delta, \Delta'$,
(C) if $\Gamma, \Omega \rhd \overline{\Omega}, \Delta$ for each $\Omega \subseteq L_\Sigma(P)$, then $\Gamma \rhd \Delta$,
(S) if $\Gamma \rhd \Delta$ then $\Gamma^\sigma \rhd \Delta^\sigma$ for each substitution $\sigma : P \to L_\Sigma(P)$.

Furthermore, if R is finitary then $\rhd_R$ further satisfies the following property for every $\Gamma \subseteq L_\Sigma(P)$:

(F) if $\Gamma \rhd \Delta$ then there exist finite sets $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$ such that $\Gamma_0 \rhd \Delta_0$.

Property (C) is best known as *cut for sets* or *transitivity*, though we prefer to call it *case exhaustion*. The other properties are usually known as *overlap* (O) or *reflexivity*, *dilution* (D) or *monotonicity*, *substitution invariance* (S) or *structurality*, and *finitariness* (F) (see [15–17]).

Proposition 1. *For a calculus R over signature Σ, $\rhd_R$ is the smallest generalized consequence relation on $L_\Sigma(P)$ which contains R.*

Proof. The proof is a straightforward generalization of [16, Theorem 3.5], where the finitary case is dealt with using a property known as *cut for formulas* ($\mathrm{C^F}$), which is equivalent to (C) for finitary consequence relations[1]. □

[1] Cut for formulas demands, for every $\Gamma, \Delta, \{A\} \subseteq L_\Sigma(P)$:
($\mathrm{C^F}$) if $\Gamma, A \rhd \Delta$ and $\Gamma \rhd A, \Delta$ then $\Gamma \rhd \Delta$.

In the context of a given generalized consequence relation $\rhd$, we denote by $\rhd^{\mathrm{T}} = \rhd \cap (\wp(L_\Sigma(P)) \times L_\Sigma(P))$ the Tarskian *companion* of $\rhd$. Recall that, in general, there may be many different generalized consequence relations with exactly the same companion [16].

We will say that a calculus R defines an *axiomatization* of a generalized consequence relation $\rhd$ when $\rhd_R = \rhd$. Of course, in such a case, R can also be used as a calculus for $\rhd^{\mathrm{T}}$.

Fix a signature Σ and a calculus R. Given $\Lambda \subseteq L_\Sigma(P)$ we write[2] $\Gamma \rhd_R^\Lambda \Delta$ when there exists an R-proof of Δ from Γ where all occurring formulas are in Λ. Controlling the possible formulas appearing in a derivation is key to defining a suitable notion of *analyticity* for multiple conclusion axiomatizations.

Let $\Phi \subseteq L_\Sigma(P)$. We say that R is *Φ-analytic* if when $\Gamma \rhd_R \Delta$ then $\Gamma \rhd_R^{\Upsilon_\Phi} \Delta$ with $\Upsilon = \mathsf{sub}(\Gamma \cup \Delta)$ and $\Upsilon_\Phi = \Upsilon \cup \{A^\sigma : A \in \Phi, \sigma : P \to \Upsilon\}$. Intuitively, this means that an R-proof of Δ from Γ needs only to use formulas which are subformulas of $\Gamma \cup \Delta$, or instances of Φ with such subformulas. Hence, formulas in Υ_Φ can be seen as a certain notion of *generalized subformula*. Clearly, a Φ-analytic calculus R is *consistent* (i.e., $\emptyset \not\rhd_R \emptyset$) if and only if the rule $\frac{\emptyset}{\emptyset} \notin R$. Analyticity is even more interesting for finite sets Φ, as in these cases we know that deciding the logic is in coNP, and there is an algorithm for proof-search in $\mathsf{EXPTIME}$ (see the discussion in the conclusion of [14]).

2.2 Logical Matrices, Non-determinism, Partiality, and Monadicity

A *partial non-deterministic matrix* $\mathbb{M}$ over a signature Σ, or *Σ-PNmatrix*, is a tuple $\langle V, D, \cdot_{\mathbb{M}} \rangle$ where V is a set (of *truth-values*), $D \subseteq V$ is the set of *designated* values and, for each $k \in \mathbb{N}$ and $\copyright \in \Sigma^{(k)}$, $\cdot_{\mathbb{M}}$ gives the *interpretation function* $\copyright_{\mathbb{M}} : V^k \to \wp(V)$ of $\copyright$ in $\mathbb{M}$. Given $X \subseteq V$ we will use $\overline{X}$ to denote $V \setminus X$. In particular, the values in $\overline{D}$ shall be referred to as *undesignated*. Whenever the interpretation function is always different from the empty set we say the PNmatrix is *total*, or *proper*, or simply say it is an *Nmatrix*. The common deterministic notion of a *logical matrix* is recovered by considering (P)Nmatrices for which the interpretation function always yields a singleton.

Given a PNmatrix $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$, each set $X \subseteq V$ defines a *sub-PNmatrix (or simple refinement)* of $\mathbb{M}$ defined by $\mathbb{M}_X = \langle X, D_X, \cdot_X \rangle$ with $D_X = D \cap X$ and $\copyright_X(x_1, \ldots, x_k) = \copyright_{\mathbb{M}}(x_1, \ldots, x_k) \cap X$ for each $\copyright \in \Sigma^{(k)}$ and $x_1, \ldots, x_k \in X$. We denote by $\mathcal{T}_{\mathbb{M}}$ the set of all subsets of the values of each non-empty total sub-PNmatrix of $\mathbb{M}$, that is,

$$\mathcal{T}_{\mathbb{M}} = \bigcup_{\substack{\emptyset \neq X \subseteq V \\ \mathbb{M}_X \text{ total}}} \wp(X).$$

Example 3. The Tarskian consequence relation of the implication-free fragment of Kleene's strong three-valued logic can be defined over a signature with one

[2] Note that in general $\rhd_R^\Lambda$ is not a generalized consequence relation. It still satisfies properties (D) and (C), but only weaker versions of (O) and (S).

unary connective $\neg$ and two binary connectives $\wedge, \vee$ by means of two three-valued matrices: those arising from the three-valued chain with only the top element designated, or both non-bottom elements designated [12]. Equivalently, the logic is given by the P(N)matrix $\mathbb{K} = \langle\{0, a, b, 1\}, \{b, 1\}, \cdot_{\mathbb{K}}\rangle$ defined by the following truth-tables, where we omit brackets for non-empty (in this case, singleton) sets.

$\wedge_{\mathbb{K}}$	0	a	b	1
0	0	0	0	0
a	0	a	$\emptyset$	a
b	0	$\emptyset$	b	b
1	0	a	b	1

$\vee_{\mathbb{K}}$	0	a	b	1
0	0	a	b	1
a	a	a	$\emptyset$	1
b	b	$\emptyset$	b	1
1	1	1	1	1

	$\neg_{\mathbb{K}}$
0	1
a	a
b	b
1	0

Note that $\mathcal{T}_{\mathbb{K}} = \{X \subseteq \{0, a, b, 1\} : \{a, b\} \nsubseteq X\}$. The three-valued matrices mentioned above clearly correspond to $\mathbb{K}_{\{0,a,1\}}$ and $\mathbb{K}_{\{0,b,1\}}$, respectively. $\triangle$

A $\mathbb{M}$-*valuation* is a function $v : L_\Sigma(P) \to V$ such that for each $© \in \Sigma^{(k)}$ and $A_1, \ldots, A_k \in L_\Sigma(P)$ we have $v(©(A_1, \ldots, A_k)) \in ©_{\mathbb{M}}(v(A_1), \ldots, v(A_k))$. Note that this implies that $v(L_\Sigma(P)) \in \mathcal{T}_{\mathbb{M}}$. We extend the interpretation in a PNmatrix $\mathbb{M}$ to any formula $A \in L_\Sigma(P)$ with $\mathsf{var}(A) \subseteq \{p_1 \ldots, p_n\}$ by letting $A_{\mathbb{M}}(x_1, \ldots, x_n) = \{v(A) : v \text{ is an } \mathbb{M}\text{-valuation}, v(p_1) = x_1, \ldots, v(p_n) = x_n\}$.

As is well known, if $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}}\rangle$ is a matrix then every function $f : Q \to V$ with $Q \subseteq P$ can be extended to a $\mathbb{M}$-valuation (in an essentially unique way for all formulas A with $\mathsf{var}(A) \subseteq Q$). When $\mathbb{M}$ is a Nmatrix, however, we know from [2] that a function $f : \Gamma \to V$ with $\Gamma \subseteq L_\Sigma(P)$ can be extended to a $\mathbb{M}$-valuation provided that $\mathsf{sub}(\Gamma) \subseteq \Gamma$ and that $f(©(A_1, \ldots, A_k)) \in ©_{\mathbb{M}}(f(A_1), \ldots, f(A_n))$ whenever $©(A_1, \ldots, A_k) \in \Gamma$. In case $\mathbb{M}$ is a PNmatrix, in general, one does not even have such a guarantee, unless $f(\Gamma) \in \mathcal{T}_{\mathbb{M}}$ [5] (take, for instance, $\Gamma = \{p, q\}$ and $f(p) = a, f(q) = b$ in the PNmatrix $\mathbb{K}$ of Example 3).

Every $\mathbb{M}$-valuation v defines a set $\Omega_v \subseteq L_\Sigma(P)$ with $\Omega_v = \{A : v(A) \in D\}$. Of course, it follows that $\overline{\Omega_v} = \{A : v(A) \notin D\}$. Let $\Gamma, \Delta \subseteq L_\Sigma(P)$ be arbitrary sets of formulas. We write $\Gamma \rhd_{\mathbb{M}} \Delta$ if every $\mathbb{M}$-valuation v is such that $\Gamma \cap \overline{\Omega_v} \neq \emptyset$ or $\Delta \cap \Omega_v \neq \emptyset$. It is well known that $\rhd_{\mathbb{M}}$ is a generalized consequence relation, and $\rhd^{\mathrm{T}}_{\mathbb{M}}$ the usual Tarskian consequence relation defined from a (partial) (non-deterministic) matrix. If $\mathbb{M}$ is finite (i.e., its underlying set of truth-values is finite) then $\rhd_{\mathbb{M}}$ and $\rhd^{\mathrm{T}}_{\mathbb{M}}$ are known to be finitary. Every Tarskian, or Scottian consequence relation is known to be characterized by a set of logical matrices [16, 17] (as usual, as the intersection of the consequence relations characterized by each of the matrices). Still, only logics satisfying *cancellation* can be given by a single logical matrix [17]. Easily, every logic can be given by a single PNmatrix, as one can use partiality to merge a set of matrices (or Nmatrices) into a single PNmatrix, as in Example 3 above. This ability of PNmatrices adds to the power of non-determinism already present in Nmatrices. In [7], we have completely characterized those Tarskian logics definable by finitely many finite matrices. However, there are logics which cannot be defined by finitely many finite matrices but can still be defined by one finite Nmatrix [1, 13].

When axiomatizing the consequence relation determined by a PNmatrix $\mathbb{M}$, we say that a set of rules R is *sound (with respect to* $\mathbb{M}$*)* if $\rhd_R \subseteq \rhd_{\mathbb{M}}$. This means that every $\mathbb{M}$-valuation v *respects* the rules of R, in the sense that for every rule $\frac{\Gamma}{\Delta} \in R$ we have that $\Gamma \cap \overline{\Omega_v} \neq \emptyset$ or $\Delta \cap \Omega_v \neq \emptyset$. Conversely, we say that R is *complete (with respect to* $\mathbb{M}$*)* if $\rhd_{\mathbb{M}} \subseteq \rhd_R$. This means that if $\Gamma \not\rhd_R \Delta$ then there exists a $\mathbb{M}$-valuation v such that $\Gamma \subseteq \Omega_v$ and $\Delta \subseteq \overline{\Omega_v}$. Soundess and completeness jointly imply $\rhd_R = \rhd_{\mathbb{M}}$.

Fix a signature Σ and a Σ-Nmatrix $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$. We want to use the resources of the logic to distinguish between the different truth-values. Namely, we require that the syntax of the logic, granted the shadow of bivalence present in the contrast between designated and undesignated values, is enough to distinguish among the truth-values [8,14,16]. A pair of non-empty sets of elements $\emptyset \neq X, Y \subseteq V$ are *separated*, $X \# Y$, if $X \subseteq D$ and $Y \subseteq \overline{D}$, or vice versa. A formula S with $\mathsf{var}(S) \subseteq \{p\}$ such that $S_{\mathbb{M}}(x) \# S_{\mathbb{M}}(y)$ is said to *separate* x and y, and called a *monadic separator* for $\mathbb{M}$. The PNmatrix $\mathbb{M}$ is said to be *monadic* if there is a monadic separator for every pair of distinct elements of V.

3 Axiomatizing Monadic PNmatrices

We extend to PNmatrices the results obtained in [14] about the construction of analytic calculi for monadic Nmatrices. Granted a monadic PNmatrix $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$ and some set $\Theta = \{S^{xy} : x, y \in V, x \neq y\}$ of monadic separators for $\mathbb{M}$ such that each S^{xy} separates x and y, a *discriminator* for $\mathbb{M}$ is the V-indexed family $\widetilde{\Theta} = \{\widetilde{\Theta}_x\}_{x \in V}$, with each $\widetilde{\Theta}_x = \{S^{xy} : y \in V \setminus \{x\}\}$. Each $\widetilde{\Theta}_x$ is naturally partitioned into $\Omega_x = \{S \in \widetilde{\Theta}_x : S_{\mathbb{M}}(x) \subseteq D\}$ and $\mho_x = \{S \in \widetilde{\Theta}_x : S_{\mathbb{M}}(x) \subseteq \overline{D}\}$. This partition is easily seen to characterize precisely the truth-values of $\mathbb{M}$.

Lemma 1. *Let* $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$ *be a monadic* Σ*-PNmatrix with discriminator* $\widetilde{\Theta}$*. For every* $\mathbb{M}$*-valuation* v*,* $x \in V$ *and* $A \in L_\Sigma(P)$*, we have*

$$v(A) = x \text{ if and only if } v(\Omega_x(A)) \subseteq D \text{ and } v(\mho_x(A)) \subseteq \overline{D}.$$

Proof. Let $v(A) = x$. For each $S \in \widetilde{\Theta}_x$, clearly, we have that $v(S(A)) \in S_{\mathbb{M}}(v(A)) = S_{\mathbb{M}}(x) \subseteq D$ if and only if $S \in \Omega_x$.

Now, let $v(A) = y \neq x$ and consider $S^{xy} \in \widetilde{\Theta}_x$. Since $S^{xy}_{\mathbb{M}}(x) \# S^{xy}_{\mathbb{M}}(y)$, it follows that $v(S^{xy}(A)) \in S^{xy}_{\mathbb{M}}(v(A)) = S^{xy}_{\mathbb{M}}(y) \subseteq D$ if and only if $S^{xy}_{\mathbb{M}}(x) \subseteq \overline{D}$ if and only if $S^{xy} \in \mho_x$. $\square$

Given $X \subseteq V$, let Ω^*_X denote any set built by choosing one element from each Ω_x for $x \in X$, and $\mho^*_X$ denote any set built by choosing one element from each $\mho_x$ for $x \in X$. In particular, if $X = \emptyset$ then $\Omega^*_X = \mho^*_X = \emptyset$ are the only possibilities. On the other hand, if for some $x \in X$ one has $\Omega_x = \emptyset$ then there is no possible choice for Ω^*_X, and similarly there is no possible choice for $\mho^*_X$ whenever $\mho_x = \emptyset$ for some $x \in X$.

Example 4. The PNmatrix $\mathbb{K}$ introduced in Example 3 is monadic. Indeed we have that $\Theta = \{p, \neg p\}$ is a set separators for $\mathbb{K}$, and setting $\widetilde{\Theta}_0 = \widetilde{\Theta}_a = \widetilde{\Theta}_b = \widetilde{\Theta}_1 = \Theta$ defines a discriminator for $\mathbb{K}$. In this case we have that

x	$\mho_x$	$\mho_x$

x	Ω_x	$\mho_x$
0	$\{\neg p\}$	$\{p\}$
a	$\emptyset$	$\{p, \neg p\}$
b	$\{p, \neg p\}$	$\emptyset$
1	$\{p\}$	$\{\neg p\}$

We also have that $\Omega^*_{\{0\}} = \mho^*_{\{1\}} = \{\neg p\}$ and $\Omega^*_{\{1\}} = \mho^*_{\{0\}} = \{p\}$. Furthermore, $\Omega^*_{\{b\}}$ has two possible values, either $\Omega^*_{\{b\}} = \{p\}$ or $\Omega^*_{\{b\}} = \{\neg p\}$. Similarly, $\mho^*_{\{a\}}$ also has the same two possible values. On the contrary, there is no possible choice for $\Omega^*_{\{a\}}$ (nor for Ω^*_X if $a \in X$) or $\mho^*_{\{b\}}$ (nor for $\mho^*_X$ if $b \in X$). △

We now define a set of inference rules respected by any monadic PNmatrix.

Definition 1. *Let* $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$ *be a monadic PNmatrix,* $\widetilde{\Theta}$ *a discriminator for* $\mathbb{M}$*. We define the set of rules* $R^{\widetilde{\Theta}}_{\mathbb{M}} = R_\exists \cup R_{\mathsf{D}} \cup R_\Sigma \cup R_{\mathcal{T}}$ *as follows:*

- $R_\exists$ *contains, for each* $X \subseteq V$ *and each possible* $\mho^*_X$ *and* $\Omega^*_{\overline{X}}$*, the rule*

$$\frac{\mho^*_X(p)}{\Omega^*_{\overline{X}}(p)}$$

- R_{D} *contains, for each* $x \in V$*, the rule*

$$\frac{\Omega_x(p)}{p, \mho_x(p)} \;\textit{if } x \in D \qquad \frac{\Omega_x(p), p}{\mho_x(p)} \;\textit{if } x \notin D$$

- $R_\Sigma = \bigcup_{© \in \Sigma} R_©$ *where, for* $© \in \Sigma^{(k)}$*,* $R_©$ *contains, for each* $x_1, \ldots, x_k \in V$ *and* $y \notin ©_{\mathbb{M}}(x_1, \ldots, x_k)$*, the rule*

$$\frac{\bigcup\limits_{1 \leq i \leq k} \Omega_{x_i}(p_i)\,,\ \Omega_y(©(p_1 \ldots, p_k))}{\bigcup\limits_{1 \leq i \leq k} \mho_{x_i}(p_i)\,,\ \mho_y(©(p_1 \ldots, p_k))}$$

- $R_{\mathcal{T}}$ *contains, for each* $X \subseteq V$ *with* $X \notin \mathcal{T}_{\mathbb{M}}$*, the rule*

$$\frac{\bigcup\limits_{x_i \in X} \Omega_{x_i}(p_i)}{\bigcup\limits_{x_i \notin X} \mho_{x_i}(p_i)}$$

Note that the rules above form a finite collection of finite rules whenever Θ is finite, which is always possible for finite $\mathbb{M}$ over finite Σ. The number of propositional variables used in the inference rules $R^{\widetilde{\Theta}}_{\mathbb{M}} \setminus R_{\mathcal{T}}$ is $k+1$ where k is the maximum arity of a connective in Σ, when it exists. Further, the number of variables in $R_{\mathcal{T}}$ is bounded by the number of values of $\mathbb{M}$.

Note also that, often, many of the rules obtained by this general process are useless (e.g., in the sense that they are instances of overlap), or can be substantially simplified, or are simply derivable from other rules.

Example 5. Recall the PNmatrix $\mathbb{K}$ introduced in Example 3 and its descriminator $\widetilde{\Theta}$ from Example 4. A simplified version of the axiomatization $R_{\mathbb{K}}^{\widetilde{\Theta}}$ consists of the following rules.

$$\frac{p\,,\,q}{p\wedge q}\,r_1 \qquad \frac{p\wedge q}{p}\,r_2 \qquad \frac{p\wedge q}{q}\,r_3 \qquad \frac{\neg p}{\neg(p\wedge q)}\,r_4 \qquad \frac{\neg q}{\neg(p\wedge q)}\,r_5 \qquad \frac{\neg(p\wedge q)}{\neg p\,,\,\neg q}\,r_6$$

$$\frac{p}{p\vee q}\,r_7 \qquad \frac{q}{p\vee q}\,r_8 \qquad \frac{\neg(p\vee q)}{\neg p}\,r_9 \qquad \frac{\neg(p\vee q)}{\neg q}\,r_{10} \qquad \frac{\neg p\,,\,\neg q}{\neg(p\vee q)}\,r_{11} \qquad \frac{p\vee q}{p\,,\,q}\,r_{12}$$

$$\frac{p}{\neg\neg p}\,r_{13} \qquad \frac{\neg\neg p}{p}\,r_{14} \qquad \frac{p\,,\,\neg p}{q\,,\,\neg q}\,r_{15}$$

Note that every rule resulting from $R_\exists$ and R_D is a case of overlap and was omitted. After simplification, the rules r_1–r_6 correspond to $R_\wedge$, r_7–r_{12} to $R_\vee$, r_{13}–r_{14} to $R_\neg$, and r_{15} results from $R_\mathcal{T}$ (with $X = \{a, b\}$).

Notice that the four rules of the calculus R_1 introduced in Example 1 (where each $r_{\wedge aby}$ corresponds to $R_\wedge$ for $y \notin (a \wedge_\mathbb{K} b) = \emptyset$) have been omitted, as they are easily derivable from r_{15}. Several other innocuous simplifications have been applied. △

It is not hard to understand in general that the rules proposed in Definition 1 capture the behaviour of the given PNmatrix $\mathbb{M}$. Namely, $R_\exists$ allows one to exclude combinations of separators that do not correspond to truth-values, R_D distinguishes those combinations of separators that characterize designated values from those that characterize undesignated values, R_Σ completely determines the interpretation of connectives in $\mathbb{M}$. The novelty with respect to [14] consists in the rules $R_\mathcal{T}$, which do not apply to Nmatrices, as they guarantee that values are taken within a total sub-PNmatrix of $\mathbb{M}$. The following results rigorously capture these intuitions.

Proposition 2. *Given a monadic PNmatrix $\mathbb{M} = \langle V, D, \cdot_\mathbb{M}\rangle$ with discriminator $\widetilde{\Theta}$, $R_{\mathbb{M}}^{\widetilde{\Theta}}$ is a calculus sound with respect to $\mathbb{M}$.*

Proof. We show that every $\mathbb{M}$-valuation v respects the rules of $R_{\mathbb{M}}^{\widetilde{\Theta}} = R_\exists \cup R_\mathsf{D} \cup R_\Sigma \cup R_\mathcal{T}$. For rules of each type, we show that if v fails to respect a rule then a contradiction can be obtained. Lemma 1 is instrumental, in all cases.

$R_\exists$**:** If (i) $v(\mho_X^*(p)) \subseteq D$ and (ii) $v(\Omega_X^*(p)) \subseteq \overline{D}$, then it easily follows that (i) for each $x \in X$ there is $y \neq x$ with $S^{xy} \in \mho_x$ and $v(S^{xy}(p)) \in D$ and (ii) for each $x \in \overline{X}$ there is $y \neq x$ with $S^{xy} \in \Omega_x$ and $v(S^{xy}(p)) \in \overline{D}$, and thus Lemma 1 guarantees that (i) $v(p) \notin X$ and (ii) $v(p) \notin \overline{X}$, which contradicts the fact that $v(p) \in V = X \cup \overline{X}$.

R_D**:** If we have $v(\Omega_x(p)) \subseteq D$ and $v(\mho_x(p)) \subseteq \overline{D}$ then Lemma 1 guarantees that $v(p) = x$, therefore $v(p) \in \overline{D}$ if and only if $x \in D$ is a contradiction.

R_Σ**:** If $v(\Omega_{x_i}(p_i)) \subseteq D$ and $v(\mho_{x_i}(p_i)) \subseteq \overline{D}$ then Lemma 1 guarantees that $v(p_i) = x_i$ for each $1 \leq i \leq k$, further, if $v(\Omega_y(©(p_1 \ldots, p_k))) \subseteq D$ and $v(\mho_y(©(p_1 \ldots, p_k))) \subseteq \overline{D}$ then Lemma 1 guarantees that $v(©(p_1 \ldots, p_k)) = y$, and thus $y = v(©(p_1 \ldots, p_k)) \in ©_{\mathbb{M}}(v(p_1), \ldots, v(p_k)) = ©_{\mathbb{M}}(x_1, \ldots, x_k)$ which contradicts the fact that $y \notin ©_{\mathbb{M}}(x_1, \ldots, x_k)$.

$R_\mathcal{T}$**:** If $v(\Omega_{x_i}(p_i)) \subseteq D$ and $v(\mho_{x_i}(p_i)) \subseteq \overline{D}$ then Lemma 1 guarantees that $v(p_i) = x_i$ for each $x_i \in X$, therefore $X \subseteq v(L_\Sigma(P)) \in \mathcal{T}_{\mathbb{M}}$ which contradicts the fact that $X \notin \mathcal{T}_{\mathbb{M}}$. □

Having established the soundness of the calculi $R^{\widetilde{\Theta}}_{\mathbb{M}}$, we now proceed to prove their completeness and analyticity. We first need another auxiliary result.

Lemma 2. *Let $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}} \rangle$ be a monadic PNmatrix, $\widetilde{\Theta}$ a discriminator for $\mathbb{M}$, and $R^{\widetilde{\Theta}}_{\mathbb{M}} = R_\exists \cup R_\mathsf{D} \cup R_\Sigma \cup R_\mathcal{T}$. For every $\Omega, \Upsilon \subseteq L_\Sigma(P)$ with $\mathsf{sub}(\Upsilon) \subseteq \Upsilon$, we have:*

(a) *if $\Omega \not\triangleright^{\Upsilon_\Theta}_{R_\exists} \overline{\Omega}$ then for every $A \in \Upsilon$ there is $x \in V$ such that $\Omega_x(A) \subseteq \Omega$ and $\mho_x(A) \subseteq \overline{\Omega}$,*

(b) *if $\Omega \not\triangleright^{\Upsilon_\Theta}_{R_\mathsf{D}} \overline{\Omega}$ then for every $A \in \Upsilon$ and $x \in V$ with $\Omega_x(A) \subseteq \Omega$ and $\mho_x(A) \subseteq \overline{\Omega}$, we have that $x \in D$ iff $A \in \Omega$,*

(c) *if $\Omega \not\triangleright^{\Upsilon_\Theta}_{R_\Sigma} \overline{\Omega}$ then for every $© \in \Sigma^{(k)}$, $A = ©(A_1, \ldots, A_k) \in \Upsilon$ and $x_1, \ldots, x_k \in V$ with $\Omega_{x_i}(A_i) \subseteq \Omega$ and $\mho_{x_i}(A_i) \subseteq \overline{\Omega}$ for each $1 \leq i \leq k$, we have that $\Omega_y(A) \subseteq \Omega$ and $\mho_y(A) \subseteq \overline{\Omega}$ implies $y \in ©_{\mathbb{M}}(x_1, \ldots, x_k)$,*

(d) *if $\Omega \not\triangleright^{\Upsilon_\Theta}_{R_\mathcal{T}} \overline{\Omega}$ then $\{x \in V : \Omega_x(A) \subseteq \Omega, \mho_x(A) \subseteq \overline{\Omega}$ for $A \in \Upsilon\} \in \mathcal{T}_{\mathbb{M}}$.*

Proof. We prove each of the items.

(a) Assume that for some $A \in \Upsilon$ there is no $x \in V$ with $\Omega_x(A) \subseteq \Omega$ and $\mho_x(A) \subseteq \overline{\Omega}$. Then, we can consider $X = \{x \in V : \mho_x(A) \cap \Omega \neq \emptyset\}$, and $\overline{X} = V \setminus X$. Define $\mho^*_X$ by choosing some $S \in \mho_x$ such that $S(A) \in \mho_x(A) \cap \Omega$ for each $x \in X$, and $\Omega^*_{\overline{X}}$ by choosing some $S \in \Omega_x$ such that $S(A) \in \Omega_x(A) \cap \overline{\Omega}$ for each $x \in \overline{X}$. We have that $\mho^*_X(A) \subseteq \Omega \cap \Upsilon_\Theta$ and $\Omega^*_{\overline{X}}(A) \subseteq \overline{\Omega} \cap \Upsilon_\Theta$. Hence, $\Omega \triangleright^{\Upsilon_\Theta}_{R_\exists} \overline{\Omega}$.

(b) Assume that there is $A \in \Upsilon$ such that $\Omega_x(A) \subseteq \Omega$, $\mho_x(A) \subseteq \overline{\Omega}$ and $x \in D$. Then $\Omega_x(A) \subseteq \Omega \cap \Upsilon_\Theta$ and $\mho_x(A) \subseteq \overline{\Omega} \cap \Upsilon_\Theta$. Hence, $\Omega \triangleright^{\Upsilon_\Theta}_{R_\mathsf{D}} \overline{\Omega}$. The case where $x \notin D$ is analogous.

(c) Assume that there is $A = ©(A_1, \ldots, A_k) \in \Upsilon$, $\Omega_{x_i}(A_i) \subseteq \Omega$ and $\mho_{x_i}(A_i) \subseteq \overline{\Omega}$ for $1 \leq i \leq k$, and for some $y \notin ©_{\mathbb{M}}(x_1 \ldots, x_n)$ we have $\Omega_y(A) \subseteq \Omega$ and $\mho_y(A) \subseteq \overline{\Omega}$. Then $\bigcup\limits_{1 \leq i \leq k} \Omega_{x_i}(p_i) \cup \Omega_y(©(p_1 \ldots, p_k)) \subseteq \Omega \cap \Upsilon_\Theta$ and $\bigcup\limits_{1 \leq i \leq k} \mho_{x_i}(p_i) \cup \mho_y(©(p_1 \ldots, p_k)) \subseteq \overline{\Omega} \cap \Upsilon_\Theta$. Hence, $\Omega \triangleright^{\Upsilon_\Theta}_{R_\Sigma} \overline{\Omega}$.

(d) Let $X = \{x \in V : \Omega_x(A) \subseteq \Omega, \mho_x(A) \subseteq \overline{\Omega}$ for $A \in \Upsilon\}$. For each $x_i \in X$ pick $A_i \in \Upsilon$ such that $\Omega_{x_i}(A_i) \subseteq \Omega$ and $\mho_{x_i}(A_i) \subseteq \overline{\Omega}$. Easily, then, $\bigcup\limits_{x_i \in X} \Omega_{x_i}(A_i) \subseteq \Omega \cap \Upsilon_\Theta$ and $\bigcup\limits_{x_i \in X} \mho_{x_i}(A_i) \subseteq \overline{\Omega} \cap \Upsilon_\Theta$, and $\Omega \triangleright^{\Upsilon_\Theta}_{R_\mathcal{T}} \overline{\Omega}$ if $X \notin \mathcal{T}_{\mathbb{M}}$.

□

With Proposition 2 and Lemma 2 in hand, it is relatively straightforward to show that $R_{\mathbb{M}}^{\widetilde{\Theta}}$ is a Θ-analytic calculus that provides an axiomatization of the generalized consequence relation determined by $\mathbb{M}$.

Theorem 1. *Given a monadic PNmatrix $\mathbb{M} = \langle V, D, \cdot_{\mathbb{M}}\rangle$ with discriminator $\widetilde{\Theta}$, $R_{\mathbb{M}}^{\widetilde{\Theta}}$ is a Θ-analytic axiomatization of $\rhd_{\mathbb{M}}$.*

Proof. Let $R = R_{\mathbb{M}}^{\widetilde{\Theta}} = R_{\exists} \cup R_{\mathsf{D}} \cup R_{\Sigma} \cup R_{\mathcal{T}}$. Soundness follows from Proposition 2, that is, $\rhd_R \subseteq \rhd_{\mathbb{M}}$. Let us detail the analytic completeness part.

Given $\Gamma, \Delta \subseteq L_{\Sigma}(P)$, it is clear that $\rhd_R^{\Upsilon_\Theta} \subseteq \rhd_R \subseteq \rhd_{\mathbb{M}}$ for $\Upsilon = \mathsf{sub}(\Gamma \cup \Delta)$. We show that if $\Gamma \not\rhd_R^{\Upsilon_\Theta} \Delta$ then $\Gamma \not\rhd_{\mathbb{M}} \Delta$. Knowing that $\Gamma \not\rhd_R^{\Upsilon_\Theta} \Delta$, by property (C), we get that there is $\Omega \subseteq L_{\Sigma}(P)$ such that $\Gamma, \Omega \not\rhd_R^{\Upsilon_\Theta} \Delta, \overline{\Omega}$. Now, using Lemma 2 **(a)**, **(b)** and **(c)**, one can build a function $f : \Upsilon \to V$ with $f(A) \in D$ iff $A \in \Omega$, and such that $f(\copyright(A_1, \ldots, A_k)) \in \copyright_{\mathbb{M}}(f(A_1), \ldots, f(A_k))$ whenever $\copyright(A_1, \ldots, A_k) \in \Upsilon$. At last, Lemma 2 **(d)** guarantees that $f(\Upsilon) \in \mathcal{T}_{\mathbb{M}}$, and we conclude that f can be extended to a full $\mathbb{M}$-valuation and thus $\Gamma \not\rhd_{\mathbb{M}} \Delta$. $\square$

We must emphasize here that the $R_{\mathcal{T}}$ rules play no role when we are interested in proving just the completeness of $R_{\mathbb{M}}^{\widetilde{\Theta}}$. Indeed, taking $\Upsilon = L_{\Sigma}(P)$, we can use Lemma 2 **(a)**, **(b)** and **(c)**, which only depend on $R_{\exists} \cup R_{\mathsf{D}} \cup R_{\Sigma}$, and directly obtain a valuation over $\mathbb{M}$. The precise role of the $R_{\mathcal{T}}$ rules can be made clearer. In [14], we showed that the local demands necessary in order to guarantee the extension of valuation functions to full valuations over Nmatrices, were sufficient to show that the axiomatization $R = R_{\mathbb{M}}^{\widetilde{\Theta}} \setminus R_{\mathcal{T}}$ would grant an Θ-*analytic* calculus for $\rhd_{\mathbb{M}}$. Expectedly, this property may not be true, in general, if $\mathbb{M}$ is a PNmatrix. At this point, we could simply have adopted the strategy delineated in [11], decomposing the given PNmatrix into its total sub-Nmatrices, then providing analytic calculi for each of them, and finally using these calculi together in order to deal with $\rhd_{\mathbb{M}}$. But we could do better. The rules in $R_{\mathcal{T}}$ are sound, and therefore they must be derivable in $R_{\mathbb{M}}^{\widetilde{\Theta}} \setminus R_{\mathcal{T}}$. Example 2, on the left, examplifies precisely this fact, given the explanation in Example 5 above. In general, however, the derivation we manage to obtain is not Θ-analytical. In the example, we see that in order to obtain the derivation we need to use the rules of the connectives that lead to the relevant partial entry of the PNmatrix (just conjunction, in this case), thus loosing the generalized subformula property. However, notably, adding $R_{\mathcal{T}}$ to the axiomatization restored analyticity.

4 A Detailed Example

In this section we consider as a full fledge example a *logic of formal inconsistency* [9], over a signature containing two single unary connective $\neg$ and $\circ$ and three binary connectives $\wedge$, $\vee$ and $\to$. Namely, we take the logic resulting from adding the axioms $\circ p \to \circ(p \wedge q)$ and $(\neg p \vee \neg q) \to \neg(p \wedge q)$ to the basic logic of formal inconsistency $\mathcal{KB}$ [4,10].

As shown in [11], the logic is characterized by the three-valued PNmatrix $\mathbb{P} = \langle \{0,1,2\}, \{1,2\}, \cdot_{\mathbb{P}} \rangle$ defined by the following truth-tables (again we omit brackets for non-empty sets.).

$\wedge_{\mathbb{P}}$	0	1	2
0	0	0	0
1	0	1	$\emptyset$
2	0	2	2

$\vee_{\mathbb{P}}$	0	1	2
0	0	1, 2	1, 2
1	1, 2	1, 2	1, 2
2	1, 2	1, 2	1, 2

$\to_{\mathbb{P}}$	0	1	2
0	1, 2	1, 2	1, 2
1	0	1, 2	1, 2
2	0	1, 2	1, 2

	$\neg_{\mathbb{P}}$	$\circ_{\mathbb{P}}$
0	1, 2	1, 2
1	0	1, 2
2	1, 2	0

It is clear that $\mathcal{T}_{\mathbb{P}} = \{X \subseteq \{0,1,2\} : \{1,2\} \not\subseteq X\}$. This happens because there is an empty entry in the truth-tables of $\mathbb{P}$, namely $1 \wedge_{\mathbb{P}} 2 = \emptyset$, which implies that no $\mathbb{P}$-valuation v can have $v(A) = 1$ and $v(B) = 2$ for $A, B \in L_{\Sigma}(P)$. Thus, all the truth-table entries corresponding to applications of any of the binary connectives $\wedge, \vee, \to$ to a pair of values formed with 1 and 2 are irrelevant and could be empty as well, namely $2 \wedge_{\mathbb{P}} 1, 1 \vee_{\mathbb{P}} 2, 2 \vee_{\mathbb{P}} 1, 1 \to_{\mathbb{P}} 2, 2 \to_{\mathbb{P}} 1$. This would not change the logic, but would potentially introduce subtle differences in the rules obtained directly by our method.

It is easy to see that $\Theta = \{p, \neg p\}$ is a set of separators for $\mathbb{P}$, and therefore it is monadic. Furthermore, setting $\widetilde{\Theta}_0 = \{p\}$ and $\widetilde{\Theta}_1 = \widetilde{\Theta}_2 = \{p, \neg p\}$ defines a discriminator for $\mathbb{P}$, which yields the following partitions:

x	Ω_x	$\mho_x$
0	$\emptyset$	$\{p\}$
1	$\{p\}$	$\{\neg p\}$
2	$\{p, \neg p\}$	$\emptyset$

Note that there is no possible choice for Ω^*_X if $0 \in X$, and also no possible choice for $\mho^*_X$ if $2 \in X$. Applying Definition 1 we obtain, after simplification of the axiomatization $R^{\widetilde{\Theta}}_{\mathbb{P}}$, the following inference rules.

$$\frac{p\,,\,q}{p \wedge q}\,r_1 \qquad \frac{p \wedge q}{p}\,r_2 \qquad \frac{p \wedge q}{q}\,r_3 \qquad \frac{\neg p}{\neg(p \wedge q)}\,r_4$$

$$\frac{p}{p \vee q}\,r_5 \qquad \frac{q}{p \vee q}\,r_6 \qquad \frac{p \vee q}{p\,,\,q}\,r_7 \qquad \frac{p\,,\,p \to q}{q}\,r_8 \qquad \frac{q}{p \to q}\,r_9 \qquad \frac{}{p\,,\,p \to q}\,r_{10}$$

$$\frac{}{p\,,\,\circ p}\,r_{11} \qquad \frac{p}{\neg p\,,\,\circ p}\,r_{12} \qquad \frac{p\,,\,\neg p\,,\,\circ p}{}\,r_{13} \qquad \frac{}{p, \neg p}\,r_{14} \qquad \frac{p\,,\,q\,,\,\neg q}{\neg p}\,r_{15}$$

Note that every rule resulting from $R_{\exists}$ and R_{D} is a case of overlap and was omitted. After simplification, the rules r_1–r_4 correspond to $R_{\wedge}$, r_5–r_7 to $R_{\vee}$, r_8–r_{10} to $R_{\to}$, r_{11}–r_{13} to $R_{\circ}$, and r_{14} to $R_{\neg}$, whereas r_{15} results from $R_{\mathcal{T}}$ (with $X = \{1,2\}$).

Of course, several innocuous (Θ-analytical) simplifications have been applied. For instance, the three rules of the calculus R_2 introduced in Example 1 (where each $r_{\wedge 12y}$ corresponds to $R_{\wedge}$ for $y \notin (1 \wedge_{\mathbb{P}} 2) = \emptyset$) are all subsumed by r_{15}. More interestingly, as already explained in regard to the previous example, rule

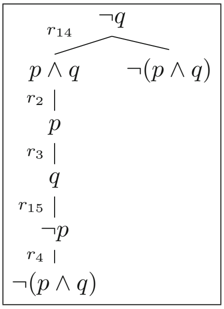
r_{15} is derivable from the other three rules too, as shown in Example 2, on the right. That derivation is, of course, not Θ-analytical.

It is interesting to see that the following two additional, useful, rules

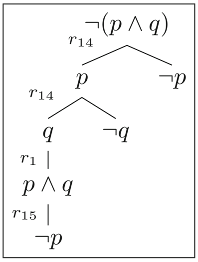
$$\frac{\neg q}{\neg(p \land q)}\, r_{16} \qquad\qquad \frac{\neg(p \land q)}{\neg p, \neg q}\, r_{17}$$

can be analytically derived from the others.

$\neg q \rhd_{\mathbb{P}} \neg(p \land q)$

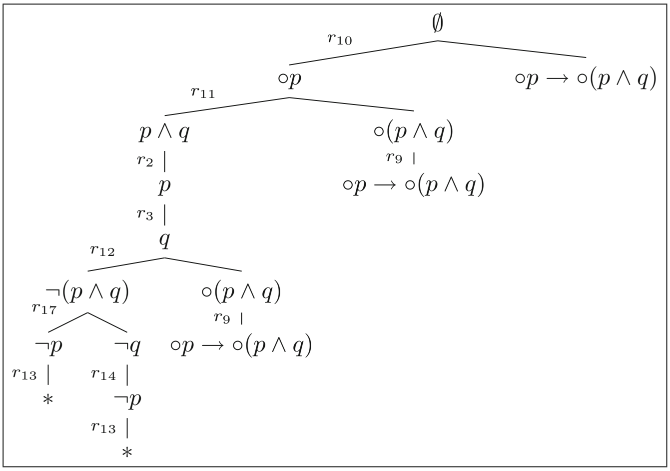


$\neg(p \land q) \rhd_{\mathbb{P}} \neg p, \neg q$

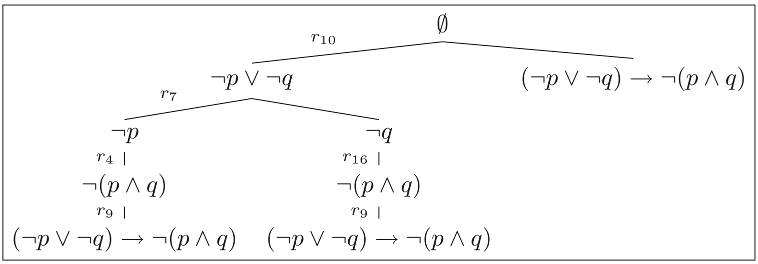


Finally, we present derivations of the two intended axioms.

$\rhd_{\mathbb{P}} \circ p \to \circ(p \land q)$



$\rhd_{\mathbb{P}} (\neg p \lor \neg q) \to \neg(p \land q)$



5 Conclusion and Future Work

In this paper we have shown the usefulness of multiple conclusion calculi, namely by proving that one can mechanically obtain analytic calculi for any given monadic PNmatrix. The monadicity requirement is fundamental, here, and corresponds to the *sufficient expressiveness* used in [3,8,11]. However, there is possibly still some room for improvement. Shoesmith and Smiley in [16] also used monadicity for logical matrices, but then showed that a more general notion of separability, using parameters, (readily available for reduced matrices) would suffice. We believe that a deeper understanding of what it means to reduce (P)Nmatrices, as well as how to deal with parameters, may help to generalize the present results.

Continued exploration of further compositional results that may be covered by these techniques is important. Still, we can identify two relatively obvious topics of further work: a detailed implementation of our methods and related proof search and decidability algorithms; and a deeper study of the relationship between multiple conclusion calculi and sequent-calculi, that may render our methods useful in designing analytic sequent-calculi, even when their semantics is not given by proper PNmatrices.

References

1. Avron, A., Lev, I.: Non-deterministic multiple-valued structures. J. Log. Comput. **15**(3), 241–261 (2005)
2. Avron, A., Zamansky, A.: Non-deterministic semantics for logical systems. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic. HALO, vol. 16, pp. 227–304. Springer, Heidelberg (2011). https://doi.org/10.1007/978-94-007-0479-4_4
3. Avron, A., Ben-Naim, J., Konikowska, B.: Cut-free ordinary sequent calculi for logics having generalized finite-valued semantics. Logica Universalis **1**(1), 41–70 (2007)
4. Avron, A., Konikowska, B., Zamansky, A.: Cut-free sequent calculi for C-systems with generalized finite-valued semantics. J. Log. Comput. **23**(3), 517–540 (2013)
5. Baaz, M., Lahav, O., Zamansky, A.: Finite-valued semantics for canonical labelled calculi. J. Autom. Reason. **51**(4), 401–430 (2013)
6. Caleiro, C., Marcelino, S., Marcos, J.: Combining fragments of classical logic: when are interaction principles needed? Soft Comput. **23**(7), 2213–2231 (2019)
7. Caleiro, C., Marcelino, S., Rivieccio, U.: Characterizing finite-valuedness. Fuzzy Sets Syst. **345**, 113–125 (2018)
8. Caleiro, C., Marcos, J., Volpe, M.: Bivalent semantics, generalized compositionality and analytic classic-like tableaux for finite-valued logics. Theor. Comput. Sci. **603**, 84–110 (2015)
9. Carnielli, W., Coniglio, M., Marcos, J.: Logics of formal inconsistency. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, vol. 14. Kluwer (2007)
10. Carnielli, W., Marcos, J.: A taxonomy of C-systems. In: Carnielli, W., Coniglio, M., D'Ottaviano, I. (eds.) Paraconsistency: The Logical Way to the Inconsistent. Lecture Notes in Pure and Applied Mathematics, vol. 228, pp. 1–94. Marcel Dekker (2002)

11. Ciabattoni, A., Lahav, O., Spendier, L., Zamansky, A.: Taming paraconsistent (and other) logics: an algorithmic approach. ACM Trans. Comput. Log. **16**(1), 5:1–5:23 (2014)
12. Font, J.: Belnap's Four-valued logic and De Morgan lattices. Log. J. IGPL **5**(3), 1–29 (1997)
13. Marcelino, S., Caleiro, C.: Disjoint fibring of non-deterministic matrices. In: Kennedy, J., de Queiroz, R.J.G.B. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 242–255. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_17
14. Marcelino, S., Caleiro, C.: Axiomatizing non-deterministic many-valued generalized consequence relations. Synthese (to appear). https://doi.org/10.1007/s11229-019-02142-8
15. Scott, D.: Completeness and axiomatizability in many-valued logic. In: Henkin, L., Addison, J., Chang, C., Craig, W., Scott, D., Vaught, R. (eds.) Proceedings of the Tarski Symposium, volume XXV of Proceedings of Symposia in Pure Mathematics, pp. 411–435. American Mathematical Society (1974)
16. Shoesmith, D., Smiley, T.: Multiple-Conclusion Logic. Cambridge University Press, Cambridge (1978)
17. Wójcicki, R.: Theory of Logical Calculi, Synthese Library, vol. 199. Kluwer (1998)
18. Wroński, A.: A three element matrix whose consequence operation is not finitely based. Bull. Sect. Log. **2**(8), 68–70 (1979)

Non Normal Logics: Semantic Analysis and Proof Theory

Jinsheng Chen[1(✉)], Giuseppe Greco[2], Alessandra Palmigiano[1,3], and Apostolos Tzimoulis[4]

[1] Delft University of Technology, Delft, The Netherlands
jinsheng.chen@foxmail.com
[2] University of Utrecht, Utrecht, The Netherlands
[3] University of Johannesburg, Johannesburg, South Africa
[4] Vrije Universiteit Amsterdam, Amsterdam, The Netherlands

Abstract. We introduce proper display calculi for basic monotonic modal logic, the conditional logic CK and a number of their axiomatic extensions. These calculi are sound, complete, conservative and enjoy cut elimination and subformula property. Our proposal applies the multi-type methodology in the design of display calculi, starting from a semantic analysis based on the translation from monotonic modal logic to normal bi-modal logic.

Keywords: Monotonic modal logic · Conditional logic · Proper display calculi

1 Introduction

By *non normal logics* we understand in this paper those propositional logics algebraically captured by varieties of *Boolean algebra expansions*, i.e. algebras $\mathbb{A} = (\mathbb{B}, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ such that $\mathbb{B}$ is a Boolean algebra, and $\mathcal{F}^{\mathbb{A}}$ and $\mathcal{G}^{\mathbb{A}}$ are finite, possibly empty families of operations on $\mathbb{B}$ in which the requirement is dropped that each operation in $\mathcal{F}^{\mathbb{A}}$ be finitely join-preserving or meet-reversing in each coordinate and each operation in $\mathcal{G}^{\mathbb{A}}$ be finitely meet-preserving or join-reversing in each coordinate. Very well known examples of non normal logics are *monotonic modal logic* [4] and *conditional logic* [3,29], which have been intensely investigated, since they capture key aspects of agents' reasoning, such as the epistemic [34], strategic [31,32], and hypothetical [13,26].

Non normal logics have been extensively investigated both with model-theoretic tools [23] and with proof-theoretic tools [28,30]. Specific to proof theory, the main challenge is to endow non normal logics with analytic calculi which can be modularly expanded with additional rules so as to uniformly capture wide classes of axiomatic extensions of the basic frameworks, while preserving

This research is supported by the NWO Vidi grant 016.138.314, the NWO Aspasia grant 015.008.054, and a Delft Technology Fellowship awarded to the third author.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 99–118, 2019.
https://doi.org/10.1007/978-3-662-59533-6_7

key properties such as cut elimination. In this paper, we propose a method to achieve this goal. We will illustrate this method for the two specific signatures of monotonic modal logic and conditional logic.

Our starting point is the very well known observation that, under the interpretation of the modal connective of monotonic modal logic in neighbourhood frames $\mathbb{F} = (W, \nu)$, the monotonic 'box' operation can be understood as the composition of a *normal* (i.e. finitely join-preserving) semantic diamond $\langle\nu\rangle$ and a *normal* (i.e. finitely meet-preserving) semantic box $[\ni]$. The binary relations R_ν and $R_\ni$ corresponding to these two normal operators are not defined on one and the same domain, but span over two domains, namely $R_\nu \subseteq W \times \mathcal{P}(W)$ is s.t. $wR_\nu X$ iff $X \in \nu(w)$ and $R_\ni \subseteq \mathcal{P}(W) \times W$ is s.t. $XR_\ni w$ iff $w \in X$ (cf. [23, Definition 5.7], see also [14,25]). We refine and expand these observations so as to: (a) introduce a semantic environment of two-sorted Kripke frames (cf. Definition 1) and their heterogeneous algebras (cf. Definition 2); (b) outline a network of discrete dualities and adjunctions among these semantic structures and the algebras and frames for monotone modal logic and conditional logic (cf. Propositions 1, 2, 3 and 4); (c) based on these semantic relationships, introduce multi-type *normal* logics into which the original non normal logics can embed via suitable translations (cf. Sect. 4) following a methodology which was successful in several other cases [7,9–11,16,17,19,22,27,33]; (d) retrieve well known dual characterization results for axiomatic extensions of monotone modal logic and conditional logics as instances of general algorithmic correspondence theory for normal (multi-type) LE-logics applied to the translated axioms (cf. Sect. B); (e) extract analytic structural rules from the computations of the first order correspondents of the translated axioms, so that, again by general results on proper display calculi [20] (which, as discussed in [1], can be applied also to multi-type logical frameworks) the resulting calculi are sound, complete, conservative and enjoy cut elimination and subformula property.

2 Preliminaries

Notation. Throughout the paper, the superscript $(\cdot)^c$ denotes the relative complement of the subset of a given set. When the given set is a singleton $\{x\}$, we will write x^c instead of $\{x\}^c$. For any binary relation $R \subseteq S \times T$, and any $S' \subseteq S$ and $T' \subseteq T$, we let $R[S'] := \{t \in T \mid (s,t) \in R \text{ for some } s \in S'\}$ and $R^{-1}[T'] := \{s \in S \mid (s,t) \in R \text{ for some } t \in T'\}$. As usual, we write $R[s]$ and $R^{-1}[t]$ instead of $R[\{s\}]$ and $R^{-1}[\{t\}]$, respectively. For any ternary relation $R \subseteq S \times T \times U$ and subsets $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$, we also let

- $R^{(0)}[T', U'] = \{s \in S \mid \exists t \exists u (R(s,t,u) \;\&\; t \in T' \;\&\; u \in U')\}$,
- $R^{(1)}[S', U'] = \{t \in T \mid \exists s \exists u (R(s,t,u) \;\&\; s \in S' \;\&\; u \in U')\}$,
- $R^{(2)}[S', T'] = \{u \in U \mid \exists s \exists t (R(s,t,u) \;\&\; s \in S' \;\&\; t \in T')\}$.

Any binary relation $R \subseteq S \times T$ gives rise to the *modal operators* $\langle R\rangle, [R], [R\rangle, \langle R] : \mathcal{P}(T) \to \mathcal{P}(S)$ s.t. for any $T' \subseteq T$

- $\langle R\rangle T' := R^{-1}[T'] = \{s \in S \mid \exists t (sRt \;\&\; t \in T')\}$;

– $[R]T' := (R^{-1}[T'^c])^c = \{s \in S \mid \forall t(sRt \ \rightarrow \ t \in T')\}$;
– $[R\rangle T' := (R^{-1}[T'])^c = \{s \in S \mid \forall t(sRt \ \rightarrow \ t \notin T')\}$;
– $\langle R]T' := R^{-1}[T'^c] = \{s \in S \mid \exists t(sRt \ \& \ t \notin T')\}$.

By construction, these modal operators are normal. In particular, $\langle R\rangle$ is completely join-preserving, $[R]$ is completely meet-preserving, $[R\rangle$ is completely join-reversing and $\langle R]$ is completely meet-reversing. Hence, their adjoint maps exist and coincide with $[R^{-1}]\langle R^{-1}\rangle, [R^{-1}\rangle, \langle R^{-1}] : \mathcal{P}(S) \rightarrow \mathcal{P}(T)$, respectively. Any ternary relation $R \subseteq S \times T \times U$ gives rise to the *modal operators* $\rhd_R$: $\mathcal{P}(T) \times \mathcal{P}(U) \rightarrow \mathcal{P}(S)$ and $\blacktriangle_R : \mathcal{P}(T) \times \mathcal{P}(S) \rightarrow \mathcal{P}(U)$ and $\blacktriangleright_R$: $\mathcal{P}(S) \times \mathcal{P}(U) \rightarrow \mathcal{P}(T)$ s.t. for any $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$,

– $T' \rhd_R U' := (R^{(0)}[T', U'^c])^c = \{s \in S \mid \ \forall t \forall u(R(s,t,u) \ \& \ t \in T' \Rightarrow u \in U')\}$;
– $T' \blacktriangle_R S' := R^{(2)}[T', S'] = \{u \in U \mid \ \exists t \exists s(R(s,t,u) \ \& \ t \in T' \ \& \ s \in S')\}$;
– $S' \blacktriangleright_R U' := (R^{(1)}[S', U'^c])^c = \{t \in T \mid \ \forall s \forall u(R(s,t,u) \ \& \ s \in S' \Rightarrow u \in U')\}$.

The stipulations above guarantee that these modal operators are normal. In particular, $\rhd_R$ and $\blacktriangleright_R$ are completely join-reversing in their first coordinate and completely meet-preserving in their second coordinate, and $\blacktriangle_R$ is completely join-preserving in both coordinates. These three maps are residual to each other, i.e. $S' \subseteq T' \rhd_R U'$ iff $T' \blacktriangle_R S' \subseteq U'$ iff $T' \subseteq S' \blacktriangleright_R U'$ for any $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$.

2.1 Basic Monotonic Modal Logic and Conditional Logic

Syntax. For a countable set of propositional variables Prop, the languages $\mathcal{L}_\nabla$ and $\mathcal{L}_>$ of monotonic modal logic and conditional logic over Prop are defined as follows:

$$\mathcal{L}_\nabla \ni \phi ::= \ p \mid \neg\phi \mid \phi \wedge \phi \mid \nabla\phi \qquad\qquad \mathcal{L}_> \ni \phi ::= \ p \mid \neg\phi \mid \phi \wedge \phi \mid \phi > \phi.$$

The connectives $\top, \wedge, \vee, \rightarrow$ and $\leftrightarrow$ are defined as usual. The *basic monotone modal logic* $\mathbf{L}_\nabla$ (resp. *basic conditional logic* $\mathbf{L}_>$) is a set of $\mathcal{L}_\nabla$-formulas (resp. $\mathcal{L}_>$-formulas) containing the axioms of classical propositional logic and closed under modus ponens, uniform substitution and M (resp. RCEA and RCK$_n$ for all $n \geq 0$):

$$\text{M}\ \frac{\varphi \rightarrow \psi}{\nabla\varphi \rightarrow \nabla\psi} \qquad \text{RCEA}\ \frac{\varphi \leftrightarrow \psi}{(\varphi > \chi) \leftrightarrow (\psi > \chi)}$$

$$\text{RCK}_n\ \frac{\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \psi}{(\chi > \varphi_1) \wedge \ldots \wedge (\chi > \varphi_n) \rightarrow (\chi > \psi)}$$

Algebraic Semantics. A *monotone Boolean algebra expansion*, abbreviated as *m-algebra* (resp. *conditional algebra*, abbreviated as *c-algebra*) is a pair $\mathbb{A} = (\mathbb{B}, \nabla^{\mathbb{A}})$ (resp. $\mathbb{A} = (\mathbb{B}, >^{\mathbb{A}})$) s.t. $\mathbb{B}$ is a Boolean algebra and $\nabla^{\mathbb{A}}$ is a unary monotone operation on $\mathbb{B}$ (resp. $>^{\mathbb{A}}$ is a binary operation on $\mathbb{B}$ which is finitely meet-preserving in its second coordinate). Interpretation of formulas in algebras under assignments $h : \mathcal{L}_\nabla \rightarrow \mathbb{A}$ (resp. $h : \mathcal{L}_> \rightarrow \mathbb{A}$) and validity of formulas in algebras (in symbols: $\mathbb{A} \models \phi$) are defined as usual. By a routine Lindenbaum-Tarski construction one can show that $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) is sound and complete w.r.t. the class of m-algebras (resp. c-algebras).

Canonical Extensions. The *canonical extension* of an m-algebra (resp. c-algebra) $\mathbb{A}$ is $\mathbb{A}^\delta := (\mathbb{B}^\delta, \nabla^{\mathbb{A}^\delta})$ (resp. $\mathbb{A}^\delta := (\mathbb{B}^\delta, >^{\mathbb{A}^\delta})$), where $\mathbb{B}^\delta$ is the canonical extension of $\mathbb{B}$ [24], and $\nabla^{\mathbb{A}^\delta}$ (resp. $>^{\mathbb{A}^\delta}$) is the σ-extension of $\nabla^{\mathbb{A}}$ (resp. the π-extension of $>^{\mathbb{A}}$). By general results of π-extensions of maps (cf. [15]), the canonical extension of an m-algebra (resp. c-algebra) is a *perfect* m-algebra (resp. c-algebra), i.e. the Boolean algebra $\mathbb{B}$ on which it is based can be identified with a powerset algebra $\mathcal{P}(W)$ up to isomorphism.

Frames and Models. A *neighbourhood frame*, abbreviated as *n-frame* (resp. *conditional frame*, abbreviated as *c-frame*) is a pair $\mathbb{F} = (W, \nu)$ (resp. $\mathbb{F} = (W, f)$) s.t. W is a non-empty set and $\nu : W \to \mathcal{P}(\mathcal{P}(W))$ is a *neighbourhood function* ($f : W \times \mathcal{P}(W) \to \mathcal{P}(W)$ is a *selection function*). In the remainder of the paper, even if it is not explicitly indicated, we will assume that n-frames are *monotone*, i.e. s.t. for every $w \in W$, if $X \in \nu(w)$ and $X \subseteq Y$, then $Y \in \nu(w)$. For any n-frame (resp. c-frame) $\mathbb{F}$, the *complex algebra* of $\mathbb{F}$ is $\mathbb{F}^* := (\mathcal{P}(W), \nabla^{\mathbb{F}^*})$ (resp. $\mathbb{F}^* := (\mathcal{P}(W), >^{\mathbb{F}^*})$) s.t. for all $X, Y \in \mathcal{P}(W)$,

$$\nabla^{\mathbb{F}^*} X := \{w \mid X \in \nu(w)\} \qquad\qquad X >^{\mathbb{F}^*} Y := \{w \mid f(w, X) \subseteq Y\}.$$

The complex algebra of an n-frame (resp. c-frame) is an m-algebra (resp. a c-algebra). *Models* are pairs $\mathbb{M} = (\mathbb{F}, V)$ such that $\mathbb{F}$ is a frame and $V : \mathcal{L} \to \mathbb{F}^*$ is a homomorphism of the appropriate type. Hence, truth of formulas at states in models is defined as $\mathbb{M}, w \Vdash \varphi$ iff $w \in V(\varphi)$, and unravelling this stipulation for ∇- and $>$-formulas, we get:

$$\mathbb{M}, w \Vdash \nabla\varphi \quad \text{iff} \quad V(\varphi) \in \nu(w) \qquad\qquad \mathbb{M}, w \Vdash \varphi > \psi \quad \text{iff} \quad f(w, V(\varphi)) \subseteq V(\psi).$$

Global satisfaction (notation: $\mathbb{M} \Vdash \phi$) and frame validity (notation: $\mathbb{F} \Vdash \phi$) are defined in the usual way. Thus, by definition, $\mathbb{F} \Vdash \phi$ iff $\mathbb{F}^* \models \phi$, from which the soundness of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) w.r.t. the corresponding class of frames immediately follows from the algebraic soundness. Completeness follows from algebraic completeness, by observing that (a) the canonical extension of any algebra refuting ϕ will also refute ϕ; (b) canonical extensions are perfect algebras; (c) perfect algebras can be associated with frames as follows: for any $\mathbb{A} = (\mathcal{P}(W), \nabla^{\mathbb{A}})$ (resp. $\mathbb{A} = (\mathcal{P}(W), >^{\mathbb{A}})$) let $\mathbb{A}_* := (W, \nu_\nabla)$ (resp. $\mathbb{A}_* := (W, f_>)$) s.t. for all $w \in W$ and $X \subseteq W$,

$$\nu_\nabla(w) := \{X \subseteq W \mid w \in \nabla X\} \qquad\qquad f_>(w, X) := \bigcap\{Y \subseteq W \mid w \in X > Y\}.$$

If $X \in \nu_\nabla(w)$ and $X \subseteq Y$, then the monotonicity of ∇ implies that $\nabla X \subseteq \nabla Y$ and hence $Y \in \nu_\nabla(w)$, as required. By construction, $\mathbb{A} \models \phi$ iff $\mathbb{A}_* \Vdash \phi$. This is enough to derive the frame completeness of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) from its algebraic completeness.

Proposition 1. *If $\mathbb{A}$ is a perfect m-algebra (resp. c-algebra) and $\mathbb{F}$ is an n-frame (resp. c-frame), then $(\mathbb{F}^*)_* \cong \mathbb{F}$ and $(\mathbb{A}_*)^* \cong \mathbb{A}$.*

Axiomatic Extensions. A *monotone modal logic* (resp. a *conditional logic*) is any extension of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) with $\mathcal{L}_\nabla$-axioms (resp. $\mathcal{L}_>$-axioms). Below we collect correspondence results for axioms that have cropped up in the literature [23, Theorem 5.1], [30].

Theorem 1. *For every n-frame (resp. c-frame)* $\mathbb{F}$,

N	$\mathbb{F} \Vdash \nabla\top$	*iff*	$\mathbb{F} \models \forall w[W \in \nu(w)]$
P	$\mathbb{F} \Vdash \neg\nabla\bot$	*iff*	$\mathbb{F} \models \forall w[\varnothing \notin \nu(w)]$
C	$\mathbb{F} \Vdash \nabla p \wedge \nabla q \rightarrow \nabla(p \wedge q)$	*iff*	$\mathbb{F} \models \forall w \forall X \forall Y[(X \in \nu(w) \ \&\ Y \in \nu(w)) \Rightarrow X \cap Y \in \nu(w)]$
T	$\mathbb{F} \Vdash \nabla p \rightarrow p$	*iff*	$\mathbb{F} \models \forall w \forall X[X \in \nu(w) \Rightarrow w \in X]$
4	$\mathbb{F} \Vdash \nabla\nabla p \rightarrow \nabla p$	*iff*	$\mathbb{F} \models \forall w \forall Y X[(X \in \nu(w) \ \&\ \forall x(x \in X \Rightarrow Y \in \nu(x))) \Rightarrow Y \in \nu(w)]$
4'	$\mathbb{F} \Vdash \nabla p \rightarrow \nabla\nabla p$	*iff*	$\mathbb{F} \models \forall w \forall X[X \in \nu(w) \Rightarrow \{y \mid X \in \nu(y)\} \in \nu(w)]$
5	$\mathbb{F} \Vdash \neg\nabla\neg p \rightarrow \nabla\neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w \forall X[X \notin \nu(w) \Rightarrow \{y \mid X \in \nu(y)\}^c \in \nu(w)]$
B	$\mathbb{F} \Vdash p \rightarrow \nabla\neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w \forall X[w \in X \Rightarrow \{y \mid X^c \in \nu(y)\}^c \in \nu(w)]$
D	$\mathbb{F} \Vdash \nabla p \rightarrow \neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w \forall X[X \in \nu(w) \Rightarrow X^c \notin \nu(w)]$
CS	$\mathbb{F} \Vdash (p \wedge q) \rightarrow (p > q)$	*iff*	$\mathbb{F} \models \forall x \forall Z[f(x, Z) \subseteq \{x\}]$
CEM	$\mathbb{F} \Vdash (p > q) \vee (p > \neg q)$	*iff*	$\mathbb{F} \models \forall X \forall y[\lvert f(y, X)\rvert \leq 1]$
ID	$\mathbb{F} \Vdash p > p$	*iff*	$\mathbb{F} \models \forall x \forall Z[f(x, Z) \subseteq Z]$.

3 Semantic Analysis

3.1 Two-Sorted Kripke Frames and Their Discrete Duality

Structures similar to those below are considered implicitly in [23], and explicitly in [12].

Definition 1. *A* two-sorted n-frame *(resp.* c-frame*) is a structure* $\mathbb{K} := (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ *(resp.* $\mathbb{K} := (X, Y, R_\ni, R_{\not\ni}, T_f)$*) such that* X *and* Y *are nonempty sets,* $R_\ni, R_{\not\ni} \subseteq Y \times X$ *and* $R_\nu, R_{\nu^c} \subseteq X \times Y$ *and* $T_f \subseteq X \times Y \times X$*. Such an n-frame is* supported *if for every* $D \subseteq X$,

$$R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c. \tag{1}$$

For any two-sorted n-frame (resp. c-frame) $\mathbb{K}$*, the* complex algebra *of* $\mathbb{K}$ *is*

$$\mathbb{K}^+ := (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{K}^+}, \langle\not\ni\rangle^{\mathbb{K}^+}, \langle\nu\rangle^{\mathbb{K}^+}, [\nu^c]^{\mathbb{K}^+})$$
$$\textit{(resp. } \mathbb{K}^+ := (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{K}^+}, [\not\ni\rangle^{\mathbb{K}^+}, \rhd^{\mathbb{K}^+})\textit{), s.t.}$$

$$\begin{array}{rrr}
\langle\nu\rangle^{\mathbb{K}^+} : \mathcal{P}(Y) \to \mathcal{P}(X) & [\ni]^{\mathbb{K}^+} : \mathcal{P}(X) \to \mathcal{P}(Y) & \langle\not\ni\rangle^{\mathbb{K}^+} : \mathcal{P}(X) \to \mathcal{P}(Y) \\
U \mapsto R_\nu^{-1}[U] & D \mapsto (R_\ni^{-1}[D^c])^c & D \mapsto R_{\not\ni}^{-1}[D] \\
 & & \\
[\nu^c]^{\mathbb{K}^+} : \mathcal{P}(Y) \to \mathcal{P}(X) & [\not\ni\rangle^{\mathbb{K}^+} : \mathcal{P}(X) \to \mathcal{P}(Y) & \rhd^{\mathbb{K}^+} : \mathcal{P}(Y) \times \mathcal{P}(X) \to \mathcal{P}(X) \\
U \mapsto (R_{\nu^c}^{-1}[U^c])^c & D \mapsto (R_{\not\ni}^{-1}[D])^c & (U, D) \mapsto (T_f^{(0)}[U, D^c])^c
\end{array}$$

The adjoints and residuals of the maps above (cf. Sect. 2) are defined as follows:

$$\begin{array}{rrr}
[\reflectbox{ν}]^{\mathbb{K}^+} : \mathcal{P}(X) \to \mathcal{P}(Y) & \langle\in\rangle^{\mathbb{K}^+} : \mathcal{P}(Y) \to \mathcal{P}(X) & [\notin]^{\mathbb{K}^+} : \mathcal{P}(Y) \to \mathcal{P}(X) \\
D \mapsto (R_\nu[D^c])^c & U \mapsto R_\ni[U] & U \mapsto (R_{\not\ni}[U^c])^c \\
 & & \\
\langle\reflectbox{ν}^c\rangle^{\mathbb{K}^+} : \mathcal{P}(X) \to \mathcal{P}(Y) & [\notin\rangle^{\mathbb{K}^+} : \mathcal{P}(Y) \to \mathcal{P}(X) & \blacktriangleright^{\mathbb{K}^+} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(Y) \\
D \mapsto R_{\nu^c}[D] & U \mapsto (R_{\not\ni}[U])^c & (C, D) \mapsto (T_f^{(1)}[C, D^c])^c \\
 & \blacktriangle^{\mathbb{K}^+} : \mathcal{P}(Y) \times \mathcal{P}(X) \to \mathcal{P}(X) & \\
 & (U, D) \mapsto T_f^{(2)}[U, D] &
\end{array}$$

Complex algebras of two-sorted frames can be recognized as heterogeneous algebras (cf. [2]) of the following kind:

Definition 2. *A* heterogeneous m-algebra *(resp.* c-algebra*) is a structure*

$$\mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}}) \qquad \textit{(resp. } \mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}})\textit{)}$$

such that $\mathbb{A}$ and $\mathbb{B}$ are Boolean algebras, $\langle\nu\rangle^{\mathbb{H}}, [\nu^c] : \mathbb{B} \to \mathbb{A}$ are finitely join-preserving and finitely meet-preserving respectively, $[\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}} : \mathbb{A} \to \mathbb{B}$ are finitely meet-preserving, finitely join-reversing, and finitely join-preserving respectively, and $\rhd^{\mathbb{H}}: \mathbb{B} \times \mathbb{A} \to \mathbb{A}$ is finitely join-reversing in its first coordinate and finitely meet-preserving in its second coordinate. Such an $\mathbb{H}$ is complete *if $\mathbb{A}$ and $\mathbb{B}$ are complete Boolean algebras and the operations above enjoy the complete versions of the finite preservation properties indicated above, and is* perfect *if it is complete and $\mathbb{A}$ and $\mathbb{B}$ are perfect. The* canonical extension *of a heterogeneous m-algebra (resp. c-algebra) $\mathbb{H}$ is $\mathbb{H}^\delta := (\mathbb{A}^\delta, \mathbb{B}^\delta, [\ni]^{\mathbb{H}^\delta}, \langle\not\ni\rangle^{\mathbb{H}^\delta}, \langle\nu\rangle^{\mathbb{H}^\delta}, [\nu^c]^{\mathbb{H}^\delta})$ (resp. $\mathbb{H}^\delta := (\mathbb{A}^\delta, \mathbb{B}^\delta, [\ni]^{\mathbb{H}^\delta}, [\not\ni\rangle^{\mathbb{H}^\delta}, \rhd^{\mathbb{H}^\delta})$), where $\mathbb{A}^\delta$ and $\mathbb{B}^\delta$ are the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ respectively [24], moreover $[\ni]^{\mathbb{H}^\delta}$, $[\not\ni\rangle^{\mathbb{H}^\delta}$, $[\nu^c]^{\mathbb{H}^\delta}, \rhd^{\mathbb{H}^\delta}$ are the π-extensions of $[\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}}, \rhd^{\mathbb{H}}$ respectively, and $\langle\nu\rangle^{\mathbb{H}^\delta}$, $\langle\not\ni\rangle^{\mathbb{H}^\delta}$ are the σ-extensions of $\langle\nu\rangle^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}$ respectively.*

Definition 3. *A heterogeneous m-algebra $\mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ is* supported *if $\langle\nu\rangle^{\mathbb{H}}[\ni]^{\mathbb{H}}a = [\nu^c]^{\mathbb{H}}\langle\not\ni\rangle^{\mathbb{H}}a$ for every $a \in \mathbb{A}$.*

It immediately follows from the definitions that

Lemma 1. *The complex algebra of a supported two-sorted n-frame is a heterogeneous supported m-algebra.*

Definition 4. *If $\mathbb{H} = (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ is a perfect heterogeneous m-algebra (resp. $\mathbb{H} = (\mathcal{P}(X), \mathcal{P}(Y), [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}})$ is a perfect heterogeneous c-algebra), its associated two-sorted n-frame (resp. c-frame) is*

$$\mathbb{H}_+ := (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c}) \qquad \textit{(resp. } \mathbb{H}_+ := (X, Y, R_\ni, R_{\not\ni}, T_f)\textit{), s.t.}$$

- *$R_\ni \subseteq Y \times X$ is defined by $yR_\ni x$ iff $y \notin [\ni]^{\mathbb{H}}x^c$,*
- *$R_{\not\ni} \subseteq Y \times X$ is defined by $xR_{\not\ni}y$ iff $y \in \langle\not\ni\rangle^{\mathbb{H}}\{x\}$ (resp. $y \notin [\not\ni\rangle^{\mathbb{H}}\{x\}$),*
- *$R_\nu \subseteq X \times Y$ is defined by $xR_\nu y$ iff $x \in \langle\nu\rangle^{\mathbb{H}}\{y\}$,*
- *$R_{\nu^c} \subseteq X \times Y$ is defined by $xR_{\nu^c}y$ iff $x \notin [\nu^c]^{\mathbb{H}}y^c$,*
- *$T_f \subseteq X \times Y \times X$ is defined by $(x', y, x) \in T_f$ iff $x' \notin \{y\} \rhd^{\mathbb{H}} x^c$.*

From the definition above it readily follows that:

Lemma 2. *If $\mathbb{H}$ is a perfect supported heterogeneous m-algebra, then $\mathbb{H}_+$ is a supported two-sorted n-frame.*

The theory of canonical extensions (of maps) and the duality between perfect BAOs and Kripke frames can be readily extended to the present two-sorted case. The following proposition collects these well known facts, the proofs of which are analogous to those of the single-sort case, hence are omitted.

Proposition 2. *For every heterogeneous m-algebra (resp. c-algebra)* $\mathbb{H}$ *and every two-sorted n-frame (resp. c-frame)* $\mathbb{K}$,

1. $\mathbb{H}^\delta$ *is a perfect heterogeneous m-algebra (resp. c-algebra);*
2. $\mathbb{K}^+$ *is a perfect heterogeneous m-algebra (resp. c-algebra);*
3. $(\mathbb{K}^+)_+ \cong \mathbb{K}$, *and if* $\mathbb{H}$ *is perfect, then* $(\mathbb{H}_+)^+ \cong \mathbb{H}$.

3.2 Equivalent Representation of m-Algebras and c-Algebras

Every supported heterogeneous m-algebra (resp. c-algebra) can be associated with an m-algebra (resp. a c-algebra) as follows:

Definition 5. *For every supported heterogeneous m-algebra* $\mathbb{H} = (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ *(resp. c-algebra* $\mathbb{H} = (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}})$*), let* $\mathbb{H}_\bullet := (\mathbb{A}, \nabla^{\mathbb{H}_\bullet})$ *(resp.* $\mathbb{H}_\bullet := (\mathbb{A}, >^{\mathbb{H}_\bullet})$*), where for every* $a \in \mathbb{A}$ *(resp.* $a, b \in \mathbb{A}$*),*

$$\nabla^{\mathbb{H}_\bullet} a = \langle\nu\rangle^{\mathbb{H}}[\ni]^{\mathbb{H}} a = [\nu^c]^{\mathbb{H}}\langle\not\ni\rangle^{\mathbb{H}} a \qquad (\text{resp. } a >^{\mathbb{H}_\bullet} b := ([\ni]^{\mathbb{H}} a \wedge [\not\ni\rangle^{\mathbb{H}} a) \rhd^{\mathbb{H}} b).$$

It immediately follows from the stipulations above that $\nabla^{\mathbb{H}_\bullet}$ is a monotone map (resp. $>^{\mathbb{H}_\bullet}$ is finitely meet-preserving in its second coordinate), and hence $\mathbb{H}_\bullet$ is an m-algebra (resp. a c-algebra). Conversely, every complete m-algebra (resp. c-algebra) can be associated with a supported heterogeneous m-algebra (resp. a c-algebra) as follows:

Definition 6. *For every complete m-algebra* $\mathbb{C} = (\mathbb{A}, \nabla^{\mathbb{C}})$ *(resp. complete c-algebra* $\mathbb{C} = (\mathbb{A}, >^{\mathbb{C}})$*), let* $\mathbb{C}^\bullet := (\mathbb{A}, \mathcal{P}(\mathbb{A}), [\ni]^{\mathbb{C}^\bullet}, \langle\not\ni\rangle^{\mathbb{C}^\bullet}, \langle\nu\rangle^{\mathbb{C}^\bullet}, [\nu^c]^{\mathbb{C}^\bullet})$ *(resp.* $\mathbb{C}^\bullet := (\mathbb{A}, \mathcal{P}(\mathbb{A}), [\ni]^{\mathbb{C}^\bullet}, [\not\ni\rangle^{\mathbb{C}^\bullet}, \rhd^{\mathbb{C}^\bullet})$*), where for every* $a \in \mathbb{A}$ *and* $B \in \mathcal{P}(\mathbb{A})$,

$$[\ni]^{\mathbb{C}^\bullet} a := \{b \in \mathbb{A} \mid b \leq a\} \qquad \langle\nu\rangle^{\mathbb{C}^\bullet} B := \bigvee\{\nabla^{\mathbb{C}} b \mid b \in B\} \qquad [\not\ni\rangle^{\mathbb{C}^\bullet} a := \{b \in \mathbb{A} \mid a \leq b\}$$

$$[\nu^c]^{\mathbb{C}^\bullet} B := \bigwedge\{\nabla^{\mathbb{C}} b \mid b \notin B\} \quad B \rhd^{\mathbb{C}^\bullet} a := \bigwedge\{b >^{\mathbb{C}} a \mid b \in B\} \quad \langle\not\ni\rangle^{\mathbb{C}^\bullet} a := \{b \in \mathbb{A} \mid a \not\leq b\}.$$

One can readily see that the operations defined above are all normal by construction, and that they enjoy the complete versions of the preservation properties indicated in Definition 2. Moreover, $\langle\nu\rangle^{\mathbb{C}^\bullet}[\ni]^{\mathbb{C}^\bullet} a = \nabla^{\mathbb{C}} a = [\nu^c]^{\mathbb{C}^\bullet}\langle\not\ni\rangle^{\mathbb{C}^\bullet} a$ for every $a \in \mathbb{A}$. Hence,

Lemma 3. *If* $\mathbb{C}$ *is a complete m-algebra (resp. complete c-algebra), then* $\mathbb{C}^\bullet$ *is a complete supported heterogeneous m-algebra (resp. c-algebra).*

The assignments $(\cdot)^\bullet$ and $(\cdot)_\bullet$ can be extended to functors between the appropriate categories of single-type and heterogeneous algebras and their homomorphisms. These functors are adjoint to each other and form a section-retraction pair. Hence:

Proposition 3. *If* $\mathbb{C}$ *is a complete m-algebra (resp. c-algebra), then* $\mathbb{C} \cong (\mathbb{C}^\bullet)_\bullet$*. Moreover, if* $\mathbb{H}$ *is a complete supported heterogeneous m-algebra (resp. c-algebra), then* $\mathbb{H} \cong \mathbb{C}^\bullet$ *for some complete m-algebra (resp. c-algebra)* $\mathbb{C}$ *iff* $\mathbb{H} \cong (\mathbb{H}_\bullet)^\bullet$*.*

The proposition above characterizes up to isomorphism the supported heterogeneous m-algebras (resp. c-algebras) which arise from single-type m-algebras (resp. c-algebras). Thanks to the discrete dualities discussed in Sects. 2.1 and 3.1, we can transfer this algebraic characterization to the side of frames, as detailed in the next subsection.

3.3 Representing n-Frames and c-Frames as Two-Sorted Kripke Frames

Definition 7. *For any n-frame (resp. c-frame)* $\mathbb{F}$*, we let* $\mathbb{F}^\star := ((\mathbb{F}^*)^\bullet)_+$*, and for every supported two-sorted n-frame (resp. c-frame)* $\mathbb{K}$*, we let* $\mathbb{K}_\star := ((\mathbb{K}^+)_\bullet)_*$*.*

Spelling out the definition above, if $\mathbb{F} = (W, \nu)$ (resp. $\mathbb{F} = (W, f)$) then $\mathbb{F}^\star = (W, \mathcal{P}(W), R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ (resp. $\mathbb{F}^\star = (W, \mathcal{P}(W), R_{\not\ni}, R_\ni, T_f)$) where:

- $R_\nu \subseteq W \times \mathcal{P}(W)$ is defined as $xR_\nu Z$ iff $Y \in \nu(x)$;
- $R_{\nu^c} \subseteq W \times \mathcal{P}(W)$ is defined as $xR_{\nu^c}Z$ iff $Z \notin \nu(x)$;
- $R_\ni \subseteq \mathcal{P}(W) \times W$ is defined as $ZR_\ni x$ iff $x \in Z$;
- $R_{\not\ni} \subseteq \mathcal{P}(W) \times W$ is defined as $ZR_{\not\ni}x$ iff $x \notin Z$;
- $T_f \subseteq W \times \mathcal{P}(W) \times W$ is defined as $T_f(x, Z, x')$ iff $x' \in f(x, Z)$.

Moreover, if $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ (resp. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, T_f)$), then $\mathbb{K}_\star = (X, \nu_\star)$ (resp. $\mathbb{K}_\star = (X, f_\star)$) where:

- $\nu_\star(x) = \{D \subseteq X \mid x \in R_\nu^{-1}[(R_\ni^{-1}[D^c])^c]\} = \{D \subseteq X \mid x \in (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c\}$;
- $f_\star(x, D) = \bigcap\{C \subseteq X \mid x \in T_f^{(0)}[\{C\}, D^c]\}$.

Lemma 4. *If* $\mathbb{F} = (W, \nu)$ *is an n-frame, then* $\mathbb{F}^\star$ *is a supported two-sorted n-frame.*

Proof. By definition, $\mathbb{F}^\star$ is a two-sorted n-frame. Moreover, for any $D \subseteq W$,

$$\begin{aligned}
(R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c &= \{w \mid \forall X(X \notin \nu(w) \Rightarrow \exists u(X \not\ni u \ \&\ u \in D))\} \\
&= \{w \mid \forall X(X \notin \nu(w) \Rightarrow D \not\subseteq X)\} \\
&= \{w \mid \forall X(D \subseteq X \Rightarrow X \in \nu(w))\} \\
&= \{w \mid \exists X(X \in \nu(w) \ \&\ X \subseteq D)\} \qquad (*) \\
&= R_\nu^{-1}[(R_\ni^{-1}[D^c])^c].
\end{aligned}$$

To show the identity marked with $(*)$, from top to bottom, take $X := D$; conversely, if $D \subseteq Z$ then $X \subseteq Z$, and since by assumption $X \in \nu(w)$ and $\nu(w)$ is upward closed, we conclude that $Z \in \nu(w)$, as required.

The next proposition is the frame-theoretic counterpart of Proposition 3.

Proposition 4. *If* $\mathbb{F}$ *is an n-frame (resp. c-frame), then* $\mathbb{F} \cong (\mathbb{F}^\star)_\star$*. Moreover, if* $\mathbb{K}$ *is a supported two-sorted n-frame (resp. c-frame), then* $\mathbb{K} \cong \mathbb{F}^\star$ *for some n-frame (resp. c-frame)* $\mathbb{F}$ *iff* $\mathbb{K} \cong (\mathbb{K}_\star)^\star$*.*

4 Embedding Non-Normal Logics into Two-Sorted Normal Logics

The two-sorted frames and heterogeneous algebras discussed in the previous section serve as semantic environment for the multi-type languages defined below.

Multi-type Languages. For a denumerable set Prop of atomic propositions, the languages $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ in types S (sets) and N (neighbourhoods) over Prop are defined as follows:

$$\begin{array}{ll} \mathsf{S} \ni A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \langle\nu\rangle\alpha \mid [\nu^c]\alpha & \mathsf{S} \ni A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \alpha \rhd A \\ \mathsf{N} \ni \alpha ::= 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid [\ni]A \mid \langle\not\ni\rangle\alpha & \mathsf{N} \ni \alpha ::= 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid [\ni]A \mid [\not\ni\rangle A. \end{array}$$

Algebraic Semantics. Interpretation of $\mathcal{L}_{MT\nabla}$-formulas (resp. $\mathcal{L}_{MT>}$formulas) in heterogeneous m-algebras (resp. c-algebras) under homomorphic assignments $h : \mathcal{L}_{MT\nabla} \to \mathbb{H}$ (resp. $h : \mathcal{L}_{MT>} \to \mathbb{H}$) and validity of formulas in heterogeneous algebras ($\mathbb{H} \models \Theta$) are defined as usual.

Frames and Models. $\mathcal{L}_{MT\nabla}$*-models* (resp. $\mathcal{L}_{MT>}$*-models*) are pairs $\mathbb{N} = (\mathbb{K}, V)$ s.t. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ is a supported two-sorted n-frame (resp. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, T_f)$ is a two-sorted c-frame) and $V : \mathcal{L}_{MT} \to \mathbb{K}^+$ is a heterogeneous algebra homomorphism of the appropriate signature. Hence, truth of formulas at states in models is defined as $\mathbb{N}, z \Vdash \Theta$ iff $z \in V(\Theta)$ for every $z \in X \cup Y$ and $\Theta \in \mathsf{S} \cup \mathsf{N}$, and unravelling this stipulation for formulas with a modal operator as main connective, we get:

- $\mathbb{N}, x \Vdash \langle\nu\rangle\alpha$ iff $\mathbb{N}, y \Vdash \alpha$ for some y s.t. $xR_\nu y$;
- $\mathbb{N}, x \Vdash [\nu^c]\alpha$ iff $\mathbb{N}, y \Vdash \alpha$ for all y s.t. $xR_{\nu^c} y$;
- $\mathbb{N}, y \Vdash [\ni]A$ iff $\mathbb{N}, x \Vdash A$ for all x s.t. $yR_\ni x$;
- $\mathbb{N}, y \Vdash \langle\not\ni\rangle A$ iff $\mathbb{N}, x \Vdash A$ for some x s.t. $yR_{\not\ni} x$;
- $\mathbb{N}, y \Vdash [\not\ni\rangle A$ iff $\mathbb{N}, x \nVdash A$ for all x s.t. $yR_{\not\ni} x$;
- $\mathbb{N}, x \Vdash \alpha \rhd A$ iff for all y and all x', if $T_f(x, y, x')$ and $\mathbb{N}, y \Vdash \alpha$ then $\mathbb{N}, x' \Vdash A$.

Global satisfaction (notation: $\mathbb{N} \Vdash \Theta$) is defined relative to the domain of the appropriate type, and frame validity (notation: $\mathbb{K} \Vdash \Theta$) is defined as usual. Thus, by definition, $\mathbb{K} \Vdash \Theta$ iff $\mathbb{K}^+ \models \Theta$, and if $\mathbb{H}$ is a perfect heterogeneous algebra, then $\mathbb{H} \models \Theta$ iff $\mathbb{H}_+ \Vdash \Theta$.

Sahlqvist Theory for Multi-type Normal Logics. This semantic environment supports a straightforward extension of Sahlqvist theory for multi-type normal logics, which includes the definition of inductive and analytic inductive formulas and inequalities in $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ (cf. Sect. A), and a corresponding version of the algorithm ALBA [6] for computing their first-order correspondents and analytic structural rules.

Translation. Sahlqvist theory and analytic calculi for the non-normal logics $\mathbf{L}_\nabla$ and $\mathbf{L}_>$ and their analytic extensions can be then obtained 'via translation', i.e. by recursively defining translations $\tau_1, \tau_2 : \mathcal{L}_\nabla \to \mathcal{L}_{MT\nabla}$ and $(\cdot)^\tau : \mathcal{L}_> \to \mathcal{L}_{MT>}$ as follows:

$$\begin{array}{rlrlrl} \tau_1(p) &= p & \tau_2(p) &= p & p^\tau &= p \\ \tau_1(\phi \wedge \psi) &= \tau_1(\phi) \wedge \tau_1(\psi) & \tau_2(\phi \wedge \psi) &= \tau_2(\phi) \wedge \tau_2(\psi) & (\phi \wedge \psi)^\tau &= \phi^\tau \wedge \psi^\tau \\ \tau_1(\neg\phi) &= \neg\tau_2(\phi) & \tau_2(\neg\phi) &= \neg\tau_1(\phi) & (\neg\phi)^\tau &= \neg\phi^\tau \\ \tau_1(\nabla\phi) &= \langle\nu\rangle[\ni]\tau_1(\phi) & \tau_2(\nabla\phi) &= [\nu^c]\langle\not\ni\rangle\tau_2(\phi) & (\phi > \psi)^\tau &= ([\ni]\phi^\tau \wedge [\not\ni\rangle\phi^\tau) \rhd \psi^\tau \end{array}$$

The following proposition is shown by a routine induction.

Proposition 5. *If $\mathbb{F}$ is an n-frame (resp. c-frame) and $\phi \vdash \psi$ is an $\mathcal{L}_\nabla$-sequent (resp. ϕ is an $\mathcal{L}_>$-formula), then $\mathbb{F} \Vdash \phi \vdash \psi$ iff $\mathbb{F}^\star \Vdash \tau_1(\phi) \vdash \tau_2(\psi)$ (resp. $\mathbb{F} \Vdash \phi$ iff $\mathbb{F}^\star \Vdash \phi^\tau$).*

With this framework in place, we are in a position to (a) retrieve correspondence results in the setting of *non normal* logics, such as those collected in Theorem 1, as instances of the general Sahlqvist theory for multi-type *normal* logics, and (b) recognize whether the translation of a non normal axiom is analytic inductive, and compute its corresponding analytic structural rules (cf. Sect. B).

	Axiom	Translation	Inductive	Analytic
N	$\nabla\top$	$\top \leq [\nu^c]\langle\not\ni\rangle\top$	✓	✓
P	$\neg\nabla\bot$	$\top \leq \neg\langle\nu\rangle[\ni]\bot$	✓	✓
C	$\nabla p \wedge \nabla q \to \nabla(p \wedge q)$	$\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \leq [\nu^c]\langle\not\ni\rangle(p \wedge q)$	✓	✓
T	$\nabla p \to p$	$\langle\nu\rangle[\ni]p \leq p$	✓	✓
4	$\nabla\nabla p \to \nabla p$	$\langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \leq [\nu^c]\langle\not\ni\rangle p$	✓	×
4’	$\nabla p \to \nabla\nabla p$	$\langle\nu\rangle[\ni]p \leq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$	✓	×
5	$\neg\nabla\neg p \to \nabla\neg\nabla\neg p$	$\neg[\nu^c]\langle\not\ni\rangle\neg p \leq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	✓	×
B	$p \to \nabla\neg\nabla\neg p$	$p \leq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	✓	×
D	$\nabla p \to \neg\nabla\neg p$	$\langle\nu\rangle[\ni]p \leq \neg\langle\nu\rangle[\ni]\neg p$	✓	✓
CS	$(p \wedge q) \to (p > q)$	$(p \wedge q) \leq (([\ni]p \wedge [\not\ni\rangle p) \rhd q)$	✓	✓
CEM	$(p > q) \vee (p > \neg q)$	$\top \leq (([\ni]p \wedge [\not\ni\rangle p) \rhd q) \vee (([\ni]p \wedge [\not\ni\rangle p) \rhd \neg q)$	✓	✓
ID	$p > p$	$\top \leq ([\ni]p \wedge [\not\ni\rangle p) \rhd p$	✓	✓

5 Proper Display Calculi

In this section we introduce proper multi-type display calculi for $\mathbf{L}_\nabla$ and $\mathbf{L}_>$ and their axiomatic extensions generated by the analytic axioms in the table above.

Languages. The language $\mathcal{L}_{DMT\nabla}$ of the calculus D.MT∇ for $\mathbf{L}_\nabla$ is defined as follows:

$$\mathsf{S}\begin{cases} A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \langle\nu\rangle\alpha \mid [\nu^c]\alpha \\ X ::= A \mid \hat{\top} \mid \check{\bot} \mid \tilde{\neg} X \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid \langle\hat{\nu}\rangle\Gamma \mid [\check{\nu^c}]\Gamma \mid \langle\hat{\in}\rangle\Gamma \mid [\check{\not\in}]\Gamma \end{cases}$$

$$\mathsf{N}\begin{cases} \alpha ::= [\ni]A \mid \langle\not\ni\rangle A \\ \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\sim}\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid [\check{\ni}]X \mid \langle\hat{\not\ni}\rangle X \mid [\check{\nu}]X \mid \langle\hat{\nu}^c\rangle X \end{cases}$$

The language $\mathcal{L}_{DMT>}$ of the calculus D.MT> for $\mathbf{L}_>$ is defined as follows:

$$\mathsf{S}\begin{cases} A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \alpha \rhd A \\ X ::= A \mid \hat{\top} \mid \check{\bot} \mid \tilde{\neg} X \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid \langle\hat{\in}\rangle\Gamma \mid \Gamma \mathbin{\check{\rhd}} X \mid \Gamma \blacktriangle X \mid [\check{\not\in}\rangle\Gamma \end{cases}$$

$$\mathsf{N}\begin{cases} \alpha ::= [\ni]A \mid [\not\ni\rangle A \mid \alpha \cap \alpha \\ \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\sim}\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid [\check{\ni}]X \mid [\check{\not\ni}\rangle X \mid X \blacktriangleright X \end{cases}$$

Multi-type Display Calculi. In what follows, we use X, Y, W, Z as structural S-variables, and $\Gamma, \Delta, \Sigma, \Pi$ as structural N-variables.

Propositional Base. The calculi D.MT∇ and D.MT> share the rules listed below.

– Identity and Cut:

$$Id_{\mathsf{S}}\,\frac{}{p \vdash p} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\,Cut_{\mathsf{S}} \qquad \frac{\Gamma \vdash \alpha \qquad \alpha \vdash \Delta}{\Gamma \vdash \Delta}\,Cut_{\mathsf{N}}$$

– Pure S-type display rules:

$$\bot \, \frac{}{\bot \vdash \check{\bot}} \qquad \frac{}{\hat{\top} \vdash \top} \, \top \qquad gal_{\mathsf{S}} \, \frac{\tilde{\neg} X \vdash Y}{\tilde{\neg} Y \vdash X} \qquad \frac{X \vdash \tilde{\neg} Y}{Y \vdash \tilde{\neg} X} \, gal_{\mathsf{S}}$$

$$res_{\mathsf{S}} \, \frac{X \mathbin{\hat{\wedge}} Y \vdash Z}{Y \vdash \tilde{\neg} X \mathbin{\check{\vee}} Z} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Z}{\tilde{\neg} Y \mathbin{\hat{\wedge}} X \vdash Z} \, res_{\mathsf{S}}$$

– Pure N-type display rules:

$$res_{\mathsf{N}} \, \frac{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \Sigma}{\Delta \vdash \tilde{\sim} \Gamma \mathbin{\check{\cup}} \Sigma} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Sigma}{\tilde{\sim} \Delta \mathbin{\hat{\cap}} \Gamma \vdash \Sigma} \, res_{\mathsf{N}} \qquad gal_{\mathsf{N}} \, \frac{\tilde{\sim} \Gamma \vdash \Delta}{\tilde{\sim} \Delta \vdash \Gamma} \qquad \frac{\Gamma \vdash \tilde{\sim} \Delta}{\Delta \vdash \tilde{\sim} \Gamma} \, gal_{\mathsf{N}}$$

– Pure-type structural rules (these include standard Weakening (W), Contraction (C), Commutativity (E) and Associativity (A) in each type which we omit to save space):

$$cont_{\mathsf{S}} \, \frac{X \vdash Y}{\tilde{\neg} Y \vdash \tilde{\neg} X} \qquad \hat{\top} \, \frac{X \vdash Y}{X \mathbin{\hat{\wedge}} \hat{\top} \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \mathbin{\check{\vee}} \check{\bot}} \, \check{\bot}$$

$$cont_{\mathsf{N}} \, \frac{\Gamma \vdash \Delta}{\tilde{\sim} \Delta \vdash \tilde{\sim} \Gamma} \qquad \hat{1} \, \frac{\Gamma \vdash \Delta}{\Gamma \mathbin{\hat{\cap}} \hat{1} \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \mathbin{\check{\cup}} \check{0}} \, \check{0}$$

– Pure S-type logical rules:

$$\wedge \, \frac{A \mathbin{\hat{\wedge}} B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \mathbin{\hat{\wedge}} Y \vdash A \wedge B} \, \wedge \qquad \neg \, \frac{\tilde{\neg} A \vdash X}{\neg A \vdash X} \qquad \frac{X \vdash \tilde{\neg} A}{X \vdash \neg A} \, \neg$$

Monotonic Modal Logic. D.MT∇ also includes the rules listed below.

– Multi-type display rules:

$$\langle \hat{\nu} \rangle [\check{\ni}] \, \frac{\langle \hat{\nu} \rangle \Gamma \vdash X}{\Gamma \vdash [\check{\ni}] X} \qquad \langle \hat{\ni}^c \rangle [\check{\nu}^c] \, \frac{\langle \hat{\ni}^c \rangle X \vdash \Gamma}{X \vdash [\check{\nu}^c] \Gamma} \qquad \langle \hat{\in} \rangle [\check{\ni}] \, \frac{\langle \hat{\in} \rangle \Gamma \vdash X}{\Gamma \vdash [\check{\ni}] X}$$

$$\langle \hat{\in} \rangle [\check{\ni}] \, \frac{\langle \hat{\in} \rangle \Gamma \vdash X}{\Gamma \vdash [\check{\ni}] X} \qquad \langle \hat{\not\ni} \rangle [\check{\not\in}] \, \frac{\langle \hat{\not\ni} \rangle X \vdash \Gamma}{X \vdash [\check{\not\in}] \Gamma}$$

– Logical rules for multi-type connectives:

$$\langle \nu \rangle \, \frac{\langle \hat{\nu} \rangle \alpha \vdash X}{\langle \nu \rangle \alpha \vdash X} \qquad \frac{\Gamma \vdash \alpha}{\langle \hat{\nu} \rangle \Gamma \vdash \langle \nu \rangle \alpha} \, \langle \nu \rangle \qquad [\nu^c] \, \frac{\alpha \vdash \Gamma}{[\nu^c] \alpha \vdash [\check{\nu}^c] \Gamma} \qquad \frac{X \vdash [\check{\nu}^c] \alpha}{X \vdash [\nu^c] \alpha} \, [\nu^c]$$

$$\langle \not\ni \rangle \, \frac{\langle \hat{\not\ni} \rangle A \vdash \Gamma}{\langle \not\ni \rangle A \vdash \Gamma} \qquad \frac{X \vdash A}{\langle \hat{\not\ni} \rangle X \vdash \langle \not\ni \rangle A} \, \langle \not\ni \rangle \qquad [\ni] \, \frac{A \vdash X}{[\ni] A \vdash [\check{\ni}] X} \qquad \frac{\Gamma \vdash [\check{\ni}] A}{\Gamma \vdash [\ni] A} \, [\ni]$$

Conditional Logic. D.MT> includes left and right logical rules for $[\ni]$, the display postulates $\langle \hat{\in} \rangle [\check{\ni}]$ and the rules listed below.

– Multi-type display rules:

$$\hat{\blacktriangle}\ \check{\vartriangleright}\ \frac{X \vdash \Gamma \,\check{\vartriangleright}\, Y}{\Gamma \,\hat{\blacktriangle}\, X \vdash Y} \qquad \frac{\Gamma \vdash X \,\check{\blacktriangleright}\, Y}{X \vdash \Gamma \,\check{\vartriangleright}\, Y}\ \check{\blacktriangleright}\ \check{\vartriangleright} \qquad \frac{X \vdash [\check{\not\in}\rangle \Gamma}{\Gamma \vdash [\check{\not\ni}\rangle X}\ [\check{\not\in}\rangle[\check{\not\ni}\rangle$$

– Logical rules for multi-type connectives and pure G-type logical rules:

$$\vartriangleright\ \frac{\Gamma \vdash \alpha \qquad A \vdash X}{\alpha \vartriangleright A \vdash \Gamma \,\check{\vartriangleright}\, X} \qquad \frac{X \vdash \alpha \,\check{\vartriangleright}\, A}{X \vdash \alpha \vartriangleright A}\ \vartriangleright \qquad [\not\ni\rangle\ \frac{X \vdash A}{[\not\ni\rangle A \vdash [\check{\not\ni}\rangle X} \qquad \frac{\Gamma \vdash [\check{\not\ni}\rangle A}{\Gamma \vdash [\not\ni\rangle A}\ [\not\ni\rangle$$

$$\cap\ \frac{\alpha \,\hat{\cap}\, \beta \vdash \Gamma}{\alpha \cap \beta \vdash \Gamma} \qquad \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma \,\hat{\cap}\, \Delta \vdash \alpha \cap \beta}\ \cap$$

Axiomatic Extensions. Each rule is labelled with the name of its corresponding axiom.

$$\text{N}\ \frac{\langle \hat{\not\ni} \rangle \hat{\top} \vdash \Gamma}{\hat{\top} \vdash [\check{\nu^c}] \Gamma} \qquad \text{ID}\ \frac{\Delta \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Gamma \qquad \langle \hat{\in} \rangle \Gamma \vdash X}{\hat{\top} \vdash (\Gamma \,\hat{\cap}\, \Delta) \,\check{\vartriangleright}\, X} \qquad \text{C}\ \frac{\langle \hat{\not\ni} \rangle (\langle \hat{\in} \rangle \Gamma \,\hat{\wedge}\, \langle \hat{\in} \rangle \Delta) \vdash \Theta}{\langle \hat{\nu} \rangle \Gamma \,\hat{\wedge}\, \langle \hat{\nu} \rangle \Delta \vdash [\check{\nu^c}] \Theta}$$

$$\text{D}\ \frac{\Gamma \vdash [\check{\ni}]\ \tilde{\neg} \langle \hat{\in} \rangle \Delta}{\langle \hat{\nu} \rangle \Delta \vdash\ \tilde{\neg} \langle \hat{\nu} \rangle \Gamma} \qquad \text{P}\ \frac{\Gamma \vdash [\check{\ni}] \check{\bot}}{\hat{\top} \vdash\ \tilde{\neg} \langle \hat{\nu} \rangle \Gamma} \qquad \text{CS}\ \frac{\Gamma \vdash [\check{\ni}][\check{\not\in}\rangle \Delta \qquad X \vdash [\check{\not\in}\rangle \Delta \qquad Y \vdash Z}{X \,\hat{\wedge}\, Y \vdash (\Gamma \,\hat{\cap}\, \Delta) \,\check{\vartriangleright}\, Z}$$

$$\text{CEM}\ \frac{\Pi \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Gamma \quad \Pi \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Theta \quad \Delta \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Gamma \quad \Delta \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Theta \quad Y \vdash X}{\hat{\top} \vdash ((\Gamma \,\hat{\cap}\, \Delta) \,\check{\vartriangleright}\, X)\ \check{\vee}\ ((\Theta \,\hat{\cap}\, \Pi) \,\check{\vartriangleright}\ \tilde{\neg} Y)} \qquad \text{T}\ \frac{\Gamma \vdash [\check{\ni}] X}{\langle \hat{\nu} \rangle \Gamma \vdash X}$$

Properties. The calculi introduced above are proper (cf. [20,35]), and hence the general theory of proper multi-type display calculi guarantees that they enjoy *cut elimination* and *subformula property* [8], and are *sound* w.r.t. their corresponding class of perfect heterogeneous algebras (or equivalently, two-sorted frames) [20]). In particular, key to the soundness argument for the axiomatic extensions is the observation that (multi-type) analytic inductive inequalities are canonical (i.e. preserved under taking canonical extensions of heterogeneous algebras [6]). Canonicity is also key to the proof of *conservativity* of the calculi w.r.t. the original logics (this is a standard argument which is analogous to those in e.g. [18,21]). *Completeness* is argued by showing that the translations of each rule and axiom is derivable in the corresponding calculus, and is sketched below.

N. $\nabla \top \ \rightsquigarrow\ [\nu^c]\langle \not\ni \rangle \top$ P. $\neg \nabla \bot \ \rightsquigarrow\ \neg \langle \nu \rangle [\ni] \bot$ T. $\nabla A \to A \ \rightsquigarrow\ \langle \nu \rangle [\ni] A \vdash A$

$$\text{N}\ \frac{\dfrac{\hat{\top} \vdash \top}{\langle \hat{\ni} \rangle \hat{\top} \vdash \langle \ni \rangle \top}}{\hat{\top} \vdash [\check{\nu^c}] \langle \ni \rangle \top} \qquad \text{P}\ \frac{\dfrac{\bot \vdash \check{\bot}}{[\ni] \bot \vdash [\check{\ni}] \check{\bot}}}{\hat{\top} \vdash\ \tilde{\neg} [\ni] \bot} \qquad \text{T}\ \frac{\dfrac{A \vdash A}{[\ni] A \vdash [\check{\ni}] A}}{\langle \hat{\nu} \rangle [\ni] A \vdash A}$$

ID. $A > A \ \rightsquigarrow\ ([\ni] A \wedge [\not\ni\rangle A) \vartriangleright A$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle A$}
\UnaryInfC{$A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}][\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\LeftLabel{ID}
\BinaryInfC{$\hat{\top} \vdash ([\check{\ni}]A \,\hat{\cap}\, [\check{\not\ni}\rangle A) \,\check{\rhd}\, A$}
\end{prooftree}
$$

CS. $(A \wedge B) \rightarrow (A > B) \;\rightsquigarrow\; (A \wedge B) \vdash ([\ni]A \cap [\not\ni\rangle A) \rhd B$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle A$}
\UnaryInfC{$A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}][\check{\notin}\rangle[\not\ni\rangle A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle A$}
\UnaryInfC{$A \vdash [\check{\notin}\rangle[\not\ni\rangle A$}
\AxiomC{$B \vdash B$}
\LeftLabel{CS}
\TrinaryInfC{$A \,\hat{\wedge}\, B \vdash ([\check{\ni}]A \,\hat{\cap}\, [\check{\not\ni}\rangle A) \,\check{\rhd}\, B$}
\end{prooftree}
$$

CEM. $(A > B) \vee (A > \neg B) \;\rightsquigarrow\; ([\ni]A \cap [\not\ni\rangle A) \rhd B \vee ([\ni]A \cap [\not\ni\rangle A) \rhd \neg B$

$$
\begin{prooftree}
\AxiomC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\AxiomC{$[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A$}
\LeftLabel{CEM}
\QuaternaryInfC{$\hat{\top} \vdash ([\ni]A \,\hat{\cap}\, [\not\ni\rangle A) \,\check{\rhd}\, B \,\check{\vee}\, ([\ni]A \,\hat{\cap}\, [\not\ni\rangle A) \,\check{\rhd}\, \tilde{\neg} B$}
\end{prooftree}
$$

C. $\nabla A \wedge \nabla B \rightarrow \nabla(A \wedge B) \rightsquigarrow \langle\nu\rangle[\ni]A \wedge \langle\nu\rangle[\ni]B \vdash [\nu^c]\langle\not\ni\rangle(A \wedge B)$
D. $\nabla A \rightarrow \neg\nabla\neg A \rightsquigarrow \langle\nu\rangle[\ni]A \vdash \neg\langle\nu\rangle[\ni]\neg A$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\ni]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\AxiomC{$B \vdash B$}
\UnaryInfC{$[\ni]B \vdash [\ni]B$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]B \vdash B$}
\BinaryInfC{$\langle\hat{\in}\rangle[\ni]A \,\hat{\wedge}\, \langle\hat{\in}\rangle[\ni]B \vdash A \wedge B$}
\UnaryInfC{$\langle\hat{\not\ni}\rangle(\langle\hat{\in}\rangle[\ni]A \,\hat{\wedge}\, \langle\hat{\in}\rangle[\ni]B) \vdash \langle\not\ni\rangle(A \wedge B)$}
\LeftLabel{C}
\UnaryInfC{$\langle\hat{\nu}\rangle[\ni]A \,\hat{\wedge}\, \langle\hat{\nu}\rangle[\ni]B \vdash [\check{\nu^c}]\langle\not\ni\rangle(A \wedge B)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$[\ni]A \vdash [\check{\ni}]A$}
\UnaryInfC{$\langle\hat{\in}\rangle[\ni]A \vdash A$}
\UnaryInfC{$\neg A \vdash \tilde{\neg}\langle\hat{\in}\rangle[\ni]A$}
\UnaryInfC{$[\ni]\neg A \vdash [\check{\ni}]\,\tilde{\neg}\langle\hat{\in}\rangle[\ni]A$}
\LeftLabel{D}
\UnaryInfC{$\langle\hat{\nu}\rangle[\ni]A \vdash \tilde{\neg}\langle\hat{\nu}\rangle[\ni]\neg A$}
\end{prooftree}
$$

The (translations of the) rules M, RCEA and RCK_n are derivable via the logical rules for the corresponding multi-type connectives, adjunction/residuation, weakening, contraction, the usual definition of $\leftrightarrow$ and the fact that if $(A \rightarrow B) \wedge (B \rightarrow A)$ is derivable, then each conjoint is derivable too.

A Analytic Inductive Inequalities

In the present section, we specialize the definition of *analytic inductive inequalities* (cf. [20]) to the multi-type languages $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ reported below.

$$
\begin{array}{ll}
\mathsf{S} \ni A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \langle\nu\rangle\alpha \mid [\nu^c]\alpha & \quad \mathsf{S} \ni A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \alpha \rhd A \\
\mathsf{N} \ni \alpha ::= 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid [\ni]A \mid \langle\not\ni\rangle A & \quad \mathsf{N} \ni \alpha ::= 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid [\ni]A \mid [\not\ni\rangle A.
\end{array}
$$

An *order-type* over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. If ϵ is an order type, ϵ^∂ is its *opposite* order type; i.e. $\epsilon^\partial(i) = 1$ iff $\epsilon(i) = \partial$ for every $1 \leq i \leq n$. The connectives of the language above are grouped together into the families $\mathcal{F} := \mathcal{F}_{\mathsf{S}} \cup \mathcal{F}_{\mathsf{N}} \cup \mathcal{F}_{\mathrm{MT}}$ and $\mathcal{G} := \mathcal{G}_{\mathsf{S}} \cup \mathcal{G}_{\mathsf{N}} \cup \mathcal{G}_{\mathrm{MT}}$, defined as follows:

$$\begin{array}{ll}\mathcal{F}_{\mathsf{S}} := \{\neg\} & \mathcal{G}_{\mathsf{S}} = \{\neg\} \\ \mathcal{F}_{\mathsf{N}} := \{\sim\} & \mathcal{G}_{\mathsf{N}} := \{\sim\} \\ \mathcal{F}_{\mathrm{MT}} := \{\langle\nu\rangle, \langle\not\ni\rangle\} & \mathcal{G}_{\mathrm{MT}} := \{[\ni], [\nu^c], \rhd, [\not\ni\rangle\}\end{array}$$

For any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$), we let $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) denote the arity of f (resp. g), and the order-type ϵ_f (resp. ϵ_g) on n_f (resp. n_g) indicate whether the ith coordinate of f (resp. g) is positive ($\epsilon_f(i) = 1$, $\epsilon_g(i) = 1$) or negative ($\epsilon_f(i) = \partial$, $\epsilon_g(i) = \partial$).

Definition 8 (Signed Generation Tree). *The* positive *(resp.* negative*)* generation tree *of any $\mathcal{L}_{MT}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows: For any node labelled with $\ell \in \mathcal{F} \cup \mathcal{G}$ of arity n_ℓ, and for any $1 \leq i \leq n_\ell$, assign the same (resp. the opposite) sign to its ith child node if $\epsilon_\ell(i) = 1$ (resp. if $\epsilon_\ell(i) = \partial$). Nodes in signed generation trees are* positive *(resp.* negative*) if are signed $+$ (resp. $-$).*

For any term $s(p_1, \ldots p_n)$, any order type ϵ over n, and any $1 \leq i \leq n$, an ϵ-*critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\epsilon(i) = 1$ or $-p_i$ with $\epsilon(i) = \partial$. An ϵ-*critical branch* in the tree is a branch ending in an ϵ-critical node. For any term $s(p_1, \ldots p_n)$ and any order type ϵ over n, we say that $+s$ (resp. $-s$) *agrees with* ϵ, and write $\epsilon(+s)$ (resp. $\epsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ϵ-critical. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\epsilon(s') \prec *s$ (resp. $\epsilon^\partial(s') \prec *s$) to indicate that the signed subtree s', with the sign inherited from $*s$, agrees with ϵ (resp. with ϵ^∂).

Definition 9 (Good branch). *Nodes in signed generation trees will be called* Δ-adjoints*,* syntactically left residual (SLR)*,* syntactically right residual (SRR)*, and* syntactically right adjoint (SRA)*, according to the specification given in Table 1. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a* good branch *if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes and P_2 consists (apart from variable nodes) only of Skeleton-nodes.*

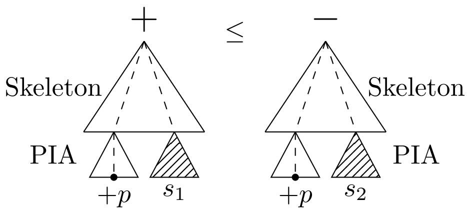


Table 1. Skeleton and PIA nodes.

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee \quad \cup$	$+ \quad \wedge \quad \cap \quad [\ni] \quad [\nu^c] \quad \rhd \quad [\not\ni\rangle \quad \neg \quad \sim$
$- \quad \wedge \quad \cap$	$- \quad \vee \quad \cup \quad \langle\nu\rangle \quad \langle\not\ni\rangle \quad \neg \quad \sim$
SLR	SRR
$+ \quad \wedge \quad \cap \quad \langle\nu\rangle \quad \langle\not\ni\rangle \quad \neg \quad \sim$	$+ \quad \vee \quad \cup$
$- \quad \vee \quad \cup \quad [\ni] \quad [\nu^c] \quad \rhd \quad [\not\ni\rangle \quad \neg \quad \sim$	$- \quad \wedge \quad \cap$

Definition 10 (Analytic inductive inequalities). *For any order type ϵ and any irreflexive and transitive relation $<_\Omega$ on $p_1, \ldots p_n$, the signed generation tree $*s$ $(* \in \{-,+\})$ of an $\mathcal{L}_{MT}$ term $s(p_1, \ldots p_n)$ is* analytic (Ω, ϵ)-inductive *if*

1. *every branch of $*s$ is good (cf. Definition 9);*
2. *for all $1 \leq i \leq n$, every SRR-node occurring in any ϵ-critical branch with leaf p_i is of the form $\circledast(s, \beta)$ or $\circledast(\beta, s)$, where the critical branch goes through β and*
   (a) $\epsilon^\partial(s) \prec *s$ (cf. discussion before Definition 9), and
   (b) $p_k <_\Omega p_i$ for every p_k occurring in s and for every $1 \leq k \leq n$.

An inequality $s \leq t$ is analytic (Ω, ϵ)-inductive *if the signed generation trees $+s$ and $-t$ are analytic (Ω, ϵ)-inductive. An inequality $s \leq t$ is* analytic inductive *if is analytic (Ω, ϵ)-inductive for some Ω and ϵ.*

B Algorithmic Proof of Theorem 1

In what follows, we show that the correspondence results collected in Theorem 1 can be retrieved as instances of a suitable multi-type version of algorithmic correspondence for normal logics (cf. [5,6]), hinging on the usual order-theoretic properties of the algebraic interpretations of the logical connectives, while admitting nominal variables of two sorts. For the sake of enabling a swift translation into the language of m-frames and c-frames, we write nominals directly as singletons, and, abusing notation, we quantify over the elements defining these singletons. These computations also serve to prove that each analytic structural rule is sound on the heterogeneous perfect algebras validating its correspondent axiom. In the computations relative to each analytic axiom, the line marked with $(\star)$ marks the quasi-inequality that interprets the corresponding analytic rule. This computation does *not* prove the equivalence between the axiom and the rule, since the variables occurring in each starred quasi-inequality are restricted rather than arbitrary. However, the proof of soundness is completed by observing that all ALBA rules in the steps above the marked inequalities are (inverse) Ackermann and adjunction rules, and hence are sound also when arbitrary variables replace (co-)nominal variables.

N. $\mathbb{F} \Vdash \nabla\top \;\rightsquigarrow\; \top \subseteq [\nu^c]\langle\not\ni\rangle\top$

$\top \subseteq [\nu^c]\langle\not\ni\rangle\top$

iff $\forall X \forall w[\langle\not\ni\rangle\top \subseteq \{X\}^c \Rightarrow \{w\} \subseteq [\nu^c]\{X\}^c]$ $(\star)$ first. app.

iff $\forall X \forall w[X = W \Rightarrow \{w\} \subseteq [\nu^c]\{X\}^c)$ $(\langle\ni\rangle\top = \{W\}^c)$

iff $\forall w[\{w\} \subseteq [\nu^c]\{W\}^c]$

iff $\forall w[\{w\} \subseteq (R_{\nu^c}^{-1}[W])^c]$

iff $\forall w[\{w\} \subseteq R_{\nu}^{-1}[W]]$

iff $\forall w[W \in \nu(w)]$

P. $\mathbb{F} \models \neg\nabla\bot \;\rightsquigarrow\; \top \subseteq \neg\langle\nu\rangle[\ni]\bot$

$\top \subseteq \neg\langle\nu\rangle[\ni]\bot$

iff $\forall X[X \subseteq [\ni]\bot \Rightarrow T \subseteq \neg\langle\nu\rangle X]$ $(\star)$ first. app.

iff $W \subseteq \neg\langle\nu\rangle[\ni]\emptyset$

iff $W \subseteq \neg\langle\nu\rangle\{\emptyset\}$ $[\ni]\emptyset = \{Z \subseteq W \mid Z \subseteq \emptyset\}$

iff $W \subseteq \{w \in W \mid wR_\nu\emptyset\}^c$

iff $\forall w[\emptyset \notin \nu(w)]$.

C. $\mathbb{F} \models \nabla p \wedge \nabla q \rightarrow \nabla(p \wedge q) \;\rightsquigarrow\; \langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q)$

$\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q)$

iff $\forall Z_1 Z_2 Z_3 \forall pq[\{Z_1\} \subseteq [\ni]p \;\&\; \{Z_2\} \subseteq [\ni]q \;\&\; \langle\not\ni\rangle(p \wedge q) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$
first approx.

iff $\forall Z_1 Z_2 Z_3 \forall pq[\langle\in\rangle\{Z_1\} \subseteq p \;\&\; \langle\in\rangle\{Z_2\} \subseteq q \;\&\; \langle\not\ni\rangle(p \wedge q) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$
Residuation

iff $\forall Z_1 \forall Z_2 \forall Z_3[\langle\not\ni\rangle(\langle\in\rangle\{Z_1\} \wedge \langle\in\rangle\{Z_2\}) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ $(\star)$ Ackermann

iff $\forall Z_1 \forall Z_2 \forall Z_3[(\langle\in\rangle\{Z_1\} \wedge \langle\in\rangle\{Z_2\}) \subseteq [\notin]\{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ Residuation

iff $\forall Z_1 \forall Z_2 \forall Z_3[\forall x(xR_\in Z_1 \;\&\; xR_\in Z_2 \Rightarrow \neg xR_\notin Z_3) \Rightarrow \forall x(xR_\nu Z_1 \;\&\; xR_\nu Z_2 \Rightarrow \neg xR_{\nu^c} Z_3)]$
Standard translation

iff $\forall Z_1 \forall Z_2 \forall Z_3[\forall x(x \in Z_1 \;\&\; x \in Z_2 \Rightarrow x \in Z_3) \Rightarrow \forall x(Z_1 \in \nu(x) \;\&\; Z_2 \in \nu(x) \Rightarrow Z_3 \in \nu(x))]$
Relations interpretation

iff $\forall Z_1 \forall Z_2 \forall Z_3[Z_1 \cap Z_2 \subseteq Z_3 \Rightarrow \forall x(Z_1 \in \nu(x) \;\&\; Z_2 \in \nu(x) \Rightarrow Z_3 \in \nu(x))]$

iff $\forall Z_1 \forall Z_2 \forall x(Z_1 \in \nu(x) \;\&\; Z_2 \in \nu(x) \Rightarrow Z_1 \cap Z_2 \in \nu(x))]$. Monotonicity

4'. $\mathbb{F} \models \nabla p \rightarrow \nabla\nabla p \;\rightsquigarrow\; \langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$

$\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$

iff $\forall Z_1 \forall x' \forall p[\{Z_1\} \subseteq [\ni]p \;\&\; [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p \subseteq \{x'\}^c) \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ first approx.

iff $\forall Z_1 \forall x' \forall p[\langle\in\rangle\{Z_1\} \subseteq p \;\&\; [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p \subseteq \{x'\}^c) \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ Residuation

iff $\forall Z_1 \forall x'[[\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle\langle\in\rangle\{Z_1\} \subseteq \{x'\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ Ackermann

iff $\forall Z_1[\langle\nu\rangle\{Z_1\} \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle\langle\in\rangle\{Z_1\}]$

iff $\forall Z_1 \forall x[xR_\nu Z_1 \Rightarrow \forall Z_2(xR_{\nu^c} Z_2 \Rightarrow \exists y(Z_2 R_{\not\ni} y \;\&\; \forall Z_3(yR_{\nu^c} Z_3 \Rightarrow \exists w(Z_3 R_{\not\ni} w \;\&\; wR_\in Z_1))))]$
Standard translation

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow \forall Z_2(Z_2 \notin \nu(x) \Rightarrow \exists y(y \notin Z_2 \;\&\; \forall Z_3(Z_2 \notin \nu(y) \Rightarrow \exists w(w \notin Z_3 \;\&\; w \in Z_1))))]$
Relations translation

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow \forall Z_2(Z_2 \notin \nu(x) \Rightarrow \exists y(y \notin Z_2 \;\&\; \forall Z_3(Z_2 \notin \nu(y) \Rightarrow Z_1 \not\subseteq Z_3)))]$
Relations translation

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow (\forall Z_2(\forall y(\forall Z_3(Z_1 \subseteq Z_3 \Rightarrow Z_3 \in \nu(y)) \Rightarrow y \in Z_2) \Rightarrow Z_2 \in \nu(x)))]$
Contraposition

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow (\forall Z_2(\forall y(Z_1 \in \nu(y)) \Rightarrow y \in Z_2) \Rightarrow Z_2 \in \nu(x)))]$ Monotonicity

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow \{y \mid Z_1 \in \nu(y)\} \in \nu(x)]$. Monotonicity

4. $\mathbb{F} \models \nabla\nabla p \rightarrow \nabla p \;\rightsquigarrow\; \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle p$

$\langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle p$

iff $\forall x \forall Z_1 \forall p[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \;\&\; \langle\not\ni\rangle p \subseteq \{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$ first approx.

iff $\forall x \forall Z_1 \forall p[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \;\&\; p \subseteq [\notin]\{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$ Adjunction

iff $\forall x \forall Z_1[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni][\notin]\{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$ Ackermann

iff $\forall x \forall Z_1[(\exists Z_2(xR_\nu Z_2 \;\&\; \forall y(Z_2 R_\ni y \Rightarrow \exists Z_3(yR_\nu Z_3 \;\&\; \forall w(Z_3 R_\ni w \Rightarrow \neg wR_\notin Z_1))))) \Rightarrow \neg xR_{\nu^c} Z_1]$
Standard translation

iff $\forall x \forall Z_1[((\exists Z_2 \in \nu(x))(\forall y \in Z_2)(\exists Z_3 \in \nu(y))(\forall w \in Z_3)(w \in Z_1)) \Rightarrow Z_1 \in \nu(x)]$
Relation translation

iff $\forall x \forall Z_1[((\exists Z_2 \in \nu(x))(\forall y \in Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_1 \in \nu(x)]$

iff $\forall x \forall Z_1 \forall Z_2[(Z_2 \in \nu(x) \;\&\; (\forall y \in Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_1 \in \nu(x)]$

iff $\forall x \forall Z_1 \forall Z_2[(Z_2 \in \nu(x) \;\&\; (\forall y \in Z_2)(Z_1 \in \nu(y))) \Rightarrow Z_1 \in \nu(x)]$ Monotonicity

5. $\mathbb{F} \models \neg\nabla\neg p \to \nabla\neg\nabla\neg p \ \rightsquigarrow\ \neg[\nu^c]\langle\not\ni\rangle\neg p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$

$\neg[\nu^c]\langle\not\ni\rangle\neg p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$
iff $\forall x\forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p \subseteq \{x\}^c \ \&\ \langle\not\ni\rangle\neg p \subseteq \{Z_1\}^c \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$ first approx.
iff $\forall x\forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p \subseteq \{x\}^c \ \&\ \neg[\not\in]\{Z_1\}^c \subseteq p \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$ Residuation
iff $\forall x\forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\neg[\not\in]\{Z_1\}^c \subseteq \{x\}^c \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$ Ackermann
iff $\forall Z_1[\neg[\nu^c]\{Z_1\}^c \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\neg[\not\in]\{Z_1\}^c]$
iff $\forall Z_1\forall x[xR_{\nu^c}Z_1 \Rightarrow \forall Z_2(xR_{\nu^c}Z_2 \Rightarrow \exists y(Z_2R_{\not\ni}y \ \&\ \forall Z_3(yR_\nu Z_3 \Rightarrow \exists w(Z_3R_\ni w \ \&\ wR_{\not\in}Z_1)))))]$
Standard translation
iff $\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow (\forall Z_2 \notin \nu(x))(\exists y \notin Z_2)(\forall Z_3 \in \nu(y))(\exists w \in Z_3)(w \notin Z_1)]$
Relation translation
iff $\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow (\forall Z_2 \notin \nu(x))(\exists y \notin Z_2)(\forall Z_3 \in \nu(y))(Z_3 \nsubseteq Z_1)]$
iff $\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow \forall Z_2(((\forall y \notin Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_2 \in \nu(x))]$
Contraposition
iff $\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow \forall Z_2((\forall y \notin Z_2)(Z_1 \in \nu(y)) \Rightarrow Z_2 \in \nu(x))]$ Monotonicity
iff $\forall Z_1\forall x[Z_1 \notin \nu(x) \Rightarrow \{y \mid Z_1 \in \nu(y)\}^c \in \nu(x))]$ Monotonicity

B. $\mathbb{F} \models p \to \nabla\neg\nabla\neg p \ \rightsquigarrow\ p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$

$p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$
iff $\forall x\forall p[\{x\} \subseteq p \Rightarrow \{x\} \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p]$ first approx.
iff $\forall x[\{x\} \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\{x\}]$ Ackermann
iff $\forall x[\{x\} \subseteq [\nu^c]\langle\not\ni\rangle[\nu]\langle\ni\rangle\{x\}]$
iff $\forall x[\forall Z_1(xR_{\nu^c}Y \Rightarrow \exists y(YR_{\not\ni}x \ \&\ \forall Z_2(yR_\nu Z_2 \Rightarrow Z_2R_\ni x)))]$ Standard translation
iff $\forall x[\forall Z_1(Z_1 \notin \nu(x) \Rightarrow \exists y(x \notin Z_1 \ \&\ \forall Z_2(Z_2 \in \nu(y) \Rightarrow x \in Z_2)))]$ Relations translation
iff $\forall x[\forall Z_1(\forall y(\forall Z_2(x \notin Z_2 \Rightarrow Z_2 \notin \nu(y)) \Rightarrow y \in Z_1) \Rightarrow Z_1 \in \nu(x))]$ Contrapositive
iff $\forall x[\forall Z_1(\forall y(\{x\}^c \notin \nu(y_1)) \Rightarrow y \in Z_1) \Rightarrow Z_1 \in \nu(x))]$ Monotonicity
iff $\forall x[\{y \mid \{x\}^c \notin \nu(y)\} \in \nu(x))]$ Monotonicity
iff $\forall x\forall X[x \in X \Rightarrow \{y \mid X^c \notin \nu(y)\} \in \nu(x)]$ Monotonicity

D. $\mathbb{F} \models \nabla p \to \neg\nabla\neg p \ \rightsquigarrow\ \langle\nu\rangle[\ni]p \subseteq \neg\langle\nu\rangle[\ni]\neg p$

$\langle\nu\rangle[\ni]p \subseteq \neg\langle\nu\rangle[\ni]\neg p$
iff $\forall Z\forall Z'[\{Z\} \subseteq [\ni]p \ \&\ Z' \subseteq [\ni]\neg p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle Z']$ first approx.
iff $\forall Z\forall Z'[\langle\in\rangle\{Z\} \subseteq p \ \&\ \{Z'\} \subseteq [\ni]\neg p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle\{Z'\}]$ Residuation
iff $\forall Z\forall Z'[\{Z'\} \subseteq [\ni]\neg\langle\in\rangle\{Z\} \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle\{Z'\}]$ ($\star$) Ackermann
iff $\forall Z[\langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle[\ni]\neg\langle\in\rangle\{Z\}]$
iff $\forall Z[\langle\nu\rangle\{Z\} \subseteq [\nu]\langle\ni\rangle\langle\in\rangle\{Z\}]$
iff $\forall Z\forall x[xR_\nu Z \Rightarrow \forall Y(xR_\nu Y \Rightarrow \exists w(YR_\ni w \ \&\ wR_\in Z))]$ Standard Translation
iff $\forall Z\forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \in \nu(x) \Rightarrow \exists w(w \in Y \ \&\ w \in Z))]$ Relation translation
iff $\forall Z\forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \in \nu(x) \Rightarrow Y \nsubseteq Z^c)]$
iff $\forall Z\forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \subseteq Z^c \Rightarrow Y \notin \nu(x))]$ Contrapositive
iff $\forall Z\forall x\forall Y[Z \in \nu(x) \Rightarrow Z^c \notin \nu(x)]$ Monotonicity

CS. $\mathbb{F} \models (p \wedge q) \to (p \succ q) \ \rightsquigarrow\ (p \wedge q) \subseteq ([\ni]p \wedge [\not\ni\rangle p)\rhd q$

$(p \wedge q) \subseteq ([\ni]p \cap [\not\ni\rangle p)\rhd q$
iff $\forall x\forall Z\forall x'\forall pq[\{x\} \subseteq p \wedge q \ \&\ \{Z\} \subseteq [\ni]p \cap [\not\ni\rangle p \ \&\ q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$
first. approx.
iff $\forall x\forall Z\forall x\forall p\forall q[\{x\} \subseteq p \ \&\ \{x\} \subseteq q \ \&\ \{Z\} \subseteq [\ni]p \ \&\ \{Z\} \subseteq [\not\ni\rangle p \ \&\ q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$
Splitting rule
iff $\forall x\forall Z\forall x'\forall p\forall q[\{x\} \subseteq p \ \&\ \{x\} \subseteq q \ \&\ \{Z\} \subseteq [\ni]p \ \&\ p \subseteq [\not\in\rangle\{Z\} \ \&\ q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$
Residuation
iff $\forall x\forall Z\forall x'\forall q[\{x\} \subseteq [\not\in\rangle\{Z\} \ \&\ \{x\} \subseteq q \ \&\ \{Z\} \subseteq [\ni][\not\in\rangle\{Z\} \ \&\ q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$
Ackermann
iff $\forall x\forall Z\forall x'[\{x\} \subseteq [\not\in\rangle\{Z\} \ \&\ \{Z\} \subseteq [\ni][\not\in\rangle\{Z\} \ \&\ \{x\} \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$
($\star$) Ackermann
iff $\forall x\forall Z[\{x\} \subseteq [\not\in\rangle\{Z\} \ \&\ \{Z\} \subseteq [\ni][\not\in\rangle\{Z\} \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x\}]$
iff $\forall x\forall Z[\neg xR_{\not\in}Z \ \&\ \forall y(ZR_\ni y \Rightarrow \neg yR_{\not\in}Z) \Rightarrow \forall y(T_f(x,Z,y) \Rightarrow y = x)]$ Standard translation
iff $\forall x\forall Z[x \in Z \ \&\ \forall y(y \in Z \Rightarrow Z \in y) \Rightarrow \forall y(y \in f(x,Z) \Rightarrow y = x)]$ Relation interpretation
iff $\forall x\forall Z[x \in Z \ \Rightarrow\ \forall y(y \in f(x,Z) \Rightarrow y = x)]$
iff $\forall x\forall Z[x \in Z \Rightarrow f(x,Z) \subseteq \{x\}]$

ID. $\mathbb{F} \models p \succ p \rightsquigarrow ([\ni]p \cap [\not\ni\rangle p) \rhd p$

$$\top \subseteq ([\ni]p \cap [\not\ni\rangle p) \rhd p$$
$$\text{iff } \forall Z Z' \forall x' p[(\{Z\} \subseteq [\ni]p \ \&\ \{Z'\} \subseteq [\not\ni\rangle p \ \&\ p \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \{x'\}^c] \quad \text{first approx.}$$
$$\text{iff } \forall Z Z' \forall x' p[(\langle\in\rangle\{Z\} \subseteq p \ \&\ \{Z'\} \subseteq [\not\ni\rangle p \ \&\ p \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \{x'\}^c] \quad \text{Adjunction}$$
$$\text{iff } \forall Z \forall Z' \forall x'[(\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \ \&\ \langle\in\rangle\{Z\} \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \{x'\}^c \quad \text{Ackermann}$$
$$\text{iff } \forall Z \forall Z'[\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \ \Rightarrow \forall x'[\langle\in\rangle\{Z\} \subseteq \{x'\}^c \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \{x'\}^c]] \quad \text{Currying}$$
$$\text{iff } \forall Z \forall Z'[\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \ \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\}) \rhd \langle\in\rangle\{Z\}] \quad (\star)\ \text{Ackermann}$$
$$\text{iff } \forall x \forall Z \forall Z'[\forall w(Z' R_{\not\ni} w \Rightarrow \neg w R_{\in} Z) \Rightarrow \forall y(T_f(x, Z, y) \ \&\ Z = Z' \Rightarrow y \in Z)] \quad \text{Standard Translation}$$
$$\text{iff } \forall x \forall Z \forall Z' \forall y[\forall w(Z' R_{\not\ni} w \Rightarrow \neg w R_{\in} Z) \ \&\ (T_f(x, Z, y) \ \&\ Z = Z' \Rightarrow y \in Z)]$$
$$\text{iff } \forall x \forall Z \forall Z' \forall y[\forall w(w \notin Z' \Rightarrow w \notin Z) \ \&\ (y \in f(x, Z) \ \&\ Z = Z' \Rightarrow y \in Z)] \quad \text{Relation interpretation}$$
$$\text{iff } \forall x \forall Z \forall Z' \forall y[Z \subseteq Z' \ \&\ (y \in f(x, Z) \ \&\ Z = Z' \Rightarrow y \in Z)]$$
$$\text{iff } \forall x \forall Z \forall y[(y \in f(x, Z) \Rightarrow y \in Z)]$$
$$\text{iff } \forall x \forall Z[f(x, Z) \subseteq Z]$$

T. $\mathbb{F} \models \nabla p \rightarrow p \rightsquigarrow \langle\nu\rangle[\ni]p \subseteq p$

$$\langle\nu\rangle[\ni]p \subseteq p$$
$$\text{iff } \forall x \forall Z \forall p[p \subseteq \{x\}^c \ \&\ \{Z\} \subseteq [\ni]p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \{x\}^c] \quad \text{first approx.}$$
$$\text{iff } \forall x \forall Z \forall p[p \subseteq \{x\}^c \ \&\ \langle\in\rangle\{Z\} \subseteq p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \{x\}^c] \quad \text{Adjunction}$$
$$\text{iff } \forall x \forall Z[\langle\in\rangle\{Z\} \subseteq \{x\}^c \Rightarrow \langle\nu\rangle\{Z\} \subseteq \{x\}^c] \quad (\star)\ \text{Ackermann}$$
$$\text{iff } \forall Z[\langle\nu\rangle\{Z\} \subseteq \langle\ni\rangle\{Z\}] \quad \text{inverse approx.}$$
$$\text{iff } \forall x \forall Z[x R_\nu Z \Rightarrow x R_\ni Z] \quad \text{Standard translation}$$
$$\text{iff } \forall x \forall Z[Z \in \nu(x) \Rightarrow x \in Z]. \quad \text{Relation translation}$$

CEM. $\mathbb{F} \models (p \succ q) \vee (p \succ \neg q) \rightsquigarrow (([\ni]p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni]p \cap [\not\ni\rangle p) \rhd \neg q)$

$$\top \subseteq (([\ni]p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni]p \cap [\not\ni\rangle p) \rhd \neg q)$$
$$\text{iff } \forall p \forall q \forall X \forall Y \forall x \forall y(\{X\} \subseteq [\ni]p \cap [\not\ni\rangle p \ \&\ \{Y\} \subseteq [\ni]p \cap [\not\ni\rangle p \ \&\ q \subseteq \{x\}^c \ \&\ \{y\} \subseteq q$$
$$\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\}) \quad \text{first approx.}$$
$$\text{iff } \forall p \forall q \forall X \forall Y \forall x \forall y(\{X\} \subseteq [\ni]p \ \&\ \{X\} \subseteq [\not\ni\rangle p \ \&\ \{Y\} \subseteq [\ni]p \ \&\ \{Y\} \subseteq [\not\ni\rangle p \ \&\ q \subseteq \{x\}^c \ \&\ \{y\} \subseteq q$$
$$\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\}) \ (\star) \quad \text{Splitting}$$
$$\text{iff } \forall p \forall q \forall X \forall Y \forall x \forall y(\{X\} \subseteq [\ni]p \ \&\ p \subseteq [\notin\rangle\{X\} \ \&\ \{Y\} \subseteq [\ni]p \ \&\ p \subseteq [\notin\rangle\{Y\} \ \&\ q \subseteq \{x\}^c \ \&\ \{y\} \subseteq q$$
$$\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\}) \quad \text{Residuation}$$
$$\text{iff } \forall X \forall Y \forall x \forall y(\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle\{X\} \wedge [\notin\rangle\{Y\}) \ \&\ \{y\} \subseteq \{x\}^c$$
$$\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\}) \quad \text{Ackermann}$$
$$\text{iff } \forall X \forall Y \forall x(\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle\{X\} \wedge [\notin\rangle\{Y\}) \Rightarrow \forall y(\{y\} \subseteq \{x\}^c \Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\})) \quad \text{Currying}$$
$$\text{iff } \forall X \forall Y \forall x(\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle\{X\} \wedge [\notin\rangle\{Y\}) \Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{x\}^c))$$
$$\text{iff } \forall X \forall Y \forall x[(\forall y(X R_\ni y \ \text{ or } \ Y R_\ni y) \Rightarrow \neg y R_{\notin} X \ \&\ \neg y R_{\notin} Y)$$
$$\Rightarrow \forall y(\neg T_f(y, X, x) \ \text{ or } \ (\forall z(T_f(y, Y, z) \Rightarrow z = x)))] \quad \text{Standard translation}$$
$$\text{iff } \forall X \forall Y \forall x[(\forall y(y \in X \text{ or } \ y \in Y) \Rightarrow y \in X \ \&\ y \in Y)$$
$$\Rightarrow \forall y(x \notin f(y, X) \ \text{ or } \ (\forall z(z \in f(y, Y) \Rightarrow z = x)))] \quad \text{Relation interpretation}$$
$$\text{iff } \forall X \forall Y \forall x[(X \cup Y \subseteq X \cap Y) \ \Rightarrow \forall y(x \notin f(y, X) \ \text{ or } \ (\forall z(z \in f(y, Y) \Rightarrow z = x)))]$$
$$\text{iff } \forall X \forall Y \forall x[X = Y \Rightarrow \forall y(x \notin f(y, X) \ \text{ or } \ (\forall z(z \in f(y, Y) \Rightarrow z = x)))]$$
$$\text{iff } \forall X \forall x \forall y[(x \notin f(y, X) \ \text{ or } \ (\forall z(z \in f(y, X) \Rightarrow z = x)))]$$
$$\text{iff } \forall X \forall x \forall y[(x \in f(y, X) \ \Rightarrow \ f(y, X) = \{x\})]$$
$$\text{iff } \forall X \forall y[|f(y, X)| \leq 1]$$

References

1. Bílková, M., Greco, G., Palmigiano, A., Tzimoulis, A., Wijnberg, N.: The logic of resources and capabilities. Rev. Symb. Log. **11**(2), 371–410 (2018)
2. Birkhoff, G., Lipson, J.: Heterogeneous algebras. J. Comb. Theory **8**(1), 115–133 (1970)
3. Chellas, B.F.: Basic conditional logic. J. Philos. Log. **4**(2), 133–153 (1975)
4. Chellas, B.F.: Modal Logic: An Introduction. Cambridge University Press, Cambridge (1980)
5. Conradie, W., Ghilardi, S., Palmigiano, A.: Unified correspondence. In: Baltag, A., Smets, S. (eds.) Johan van Benthem on Logic and Information Dynamics. OCL, vol. 5, pp. 933–975. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-06025-5_36

6. Conradie, W., Palmigiano, A.: Algorithmic correspondence and canonicity for non-distributive logics. Ann. Pure Appl. Log. (2019, in press). ArXiv preprint arXiv:1603.08515
7. Frittella, S., Greco, G., Kurz, A., Palmigiano, A.: Multi-type display calculus for propositional dynamic logic. J. Log. Comput. **26**(6), 2067–2104 (2016)
8. Frittella, S., Greco, G., Kurz, A., Palmigiano, A., Sikimić, V.: Multi-type sequent calculi. In: Indrzejczak, A., et al. (eds.) Proceedings of Trends in Logic XIII, pp. 81–93 (2014)
9. Frittella, S., Greco, G., Kurz, A., Palmigiano, A., Sikimic, V.: Multi-type display calculus for dynamic epistemic logic. J. Log. Comput. **26**(6), 2017–2065 (2016)
10. Frittella, S., Greco, G., Kurz, A., Palmigiano, A., Sikimic, V.: A proof-theoretic semantic analysis of dynamic epistemic logic. J. Log. Comput. **26**(6), 1961–2015 (2016)
11. Frittella, S., Greco, G., Palmigiano, A., Yang, F.: A multi-type calculus for inquisitive logic. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 215–233. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_14
12. Frittella, S., Palmigiano, A., Santocanale, L.: Dual characterizations for finite lattices via correspondence theory for monotone modal logic. JLC **27**(3), 639–678 (2017)
13. Gabbay, D., Giordano, L., Martelli, A., Olivetti, N., Sapino, M.L.: Conditional reasoning in logic programming. J. Log. Program. **44**(1–3), 37–74 (2000)
14. Gasquet, O., Herzig, A.: From classical to normal modal logics. In: Wansing, H. (ed.) Proof Theory of Modal Logic. APLS, vol. 2, pp. 293–311. Springer, Dordrecht (1996). https://doi.org/10.1007/978-94-017-2798-3_15
15. Gehrke, M., Jónsson, B.: Bounded distributive lattice expansions. Mathematica Scandinavica, 13–45 (2004)
16. Greco, G., Jipsen, P., Manoorkar, K., Palmigiano, A., Tzimoulis, A.: Logics for rough concept analysis. In: Khan, M.A., Manuel, A. (eds.) ICLA 2019. LNCS, vol. 11600, pp. 144–159. Springer, Heidelberg (2019). https://doi.org/10.1007/978-3-662-58771-3_14
17. Greco, G., Liang, F., Manoorkar, K., Palmigiano, A.: Proper multi-type display calculi for rough algebras. ArXiv preprint arXiv:1808.07278 (2018)
18. Greco, G., Liang, F., Moshier, M.A., Palmigiano, A.: Multi-type display calculus for semi De Morgan logic. In: Kennedy, J., de Queiroz, R.J.G.B. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 199–215. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_14
19. Greco, G., Liang, F., Palmigiano, A., Rivieccio, U.: Bilattice logic properly displayed. Fuzzy Sets Syst. **363**, 138–155 (2018)
20. Greco, G., Ma, M., Palmigiano, A., Tzimoulis, A., Zhao, Z.: Unified correspondence as a proof-theoretic tool. J. Log. Comput. **28**(7), 1367–1442 (2018)
21. Greco, G., Palmigiano, A.: Linear logic properly displayed. arXiv preprint arXiv:1611.04184 (2016)
22. Greco, G., Palmigiano, A.: Lattice logic properly displayed. In: Kennedy, J., de Queiroz, R.J.G.B. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 153–169. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_11
23. Hansen, H.H.: Monotonic modal logics. Institute for Logic, Language and Computation (ILLC), University of Amsterdam (2003)
24. Jónsson, B., Tarski, A.: Boolean algebras with operators. Part I. Am. J. Math. **73**(4), 891–939 (1951)

25. Kracht, M., Wolter, F.: Normal monomodal logics can simulate all others. J. Symb. Log. **64**(1), 99–138 (1999)
26. Lewis, D.: Counterfactuals. Wiley, Hoboken (2013)
27. Manoorkar, K., Nazari, S., Palmigiano, A., Tzimoulis, A., Wijnberg, N.M.: Rough concepts (2018, Submitted)
28. Negri, S.: Proof theory for non-normal modal logics: the neighbourhood formalism and basic results. IFCoLog J. Log. Appl. **4**, 1241–1286 (2017)
29. Nute, D.: Topics in Conditional Logic, vol. 20. Springer, Heidelberg (2012)
30. Olivetti, N., Pozzato, G., Schwind, C.: A sequent calculus and a theorem prover for standard conditional logics. ACM Trans. Comput. Log. **8**, 40–87 (2007)
31. Pauly, M.: A modal logic for coalitional power in games. JLC **12**(1), 149–166 (2002)
32. Pauly, M., Parikh, R.: Game logic - an overview. Studia Logica **75**(2), 165–182 (2003)
33. Tzimoulis, A.: Algebraic and proof-theoretic foundations of the logics for social behaviour. Ph.D. thesis. TU Delft (2018)
34. van Benthem, J., Pacuit, E.: Dynamic logics of evidence-based beliefs. Studia Logica **99**(1–3), 61 (2011)
35. Wansing, H.: Displaying Modal Logic, vol. 3. Springer, Heidelberg (2013)

Modeling the Interaction of Computer Errors by Four-Valued Contaminating Logics

Roberto Ciuni[1], Thomas Macaulay Ferguson[2,3(✉)], and Damian Szmuc[4]

[1] Department FISPPA, Section of Philosophy, University of Padova, Padua, Italy
roberto.ciuni@unipd.it
[2] Cycorp, Austin, USA
[3] Saul Kripke Center, CUNY Graduate Center, New York, USA
tferguson@gradcenter.cuny.edu
[4] IIF-SADAF, CONICET and Department of Philosophy, University of Buenos Aires, Buenos Aires, Argentina
szmucdamian@gmail.com

Abstract. Logics based on weak Kleene algebra (**WKA**) and related structures have been recently proposed as a tool for reasoning about *flaws* in computer programs. The key element of this proposal is the presence, in **WKA** and related structures, of a non-classical truth-value that is "contaminating" in the sense that whenever the value is assigned to a formula ϕ, any complex formula in which ϕ appears is assigned that value as well. Under such interpretations, the contaminating states represent occurrences of a flaw. However, since different programs and machines can interact with (or be nested into) one another, we need to account for different kind of errors, and this calls for an evaluation of systems with *multiple* contaminating values. In this paper, we make steps toward these evaluation systems by considering two logics, HYB_1 and HYB_2, whose semantic interpretations account for two contaminating values beside classical values 0 and 1. In particular, we provide two main formal contributions. First, we give a characterization of their relations of (multiple-conclusion) logical consequence—that is, necessary and sufficient conditions for a set Δ of formulas to logically follow from a set Γ of formulas in HYB_1 or HYB_2. Second, we provide sound and complete sequent calculi for the two logics.

1 Introduction

Applications of logic to the topic of reasoning about computer errors date back at least to McCarthy's [16]. There, *critical errors* affecting *sequential* (*i.e.*, 'lazy' or 'short-circuit') evaluation are considered, and the main intuition concerning such errors is that the exact step of the computation in which the error occurs determine which string of information fails to be computed. For instance, if one is computing the value of $\phi \vee \psi$ and an error occurs while computing the value of ϕ, then the computation will stop without establishing a value. By contrast,

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 119–139, 2019.
https://doi.org/10.1007/978-3-662-59533-6_8

if the error occurs when computing the value of ψ *and after* computing the value of ϕ—assigning, say, 1—then the computation will prove successful and assign value 1 to $\phi \vee \psi$. This kind of error, in a nutshell, suggests the need for a non-commutative disjunction (and conjunction). Avron and Konikowska's [1] made progress with respect to [16] by providing tools for reasoning about errors in *parallel* computing, and by proposing a four-valued logic for reasoning about the interaction of the two kinds of errors. In particular, the logic for reasoning on errors in parallel computing is the strong Kleene logic K_3, which is in turn interpreted on the so-called (strong) Kleene algebra from [15].

More recently, [11] has discussed another kind of error, such as those that emerge from the failure to declare a variable in programs that are based on C++. An interesting consequence by the discussion in [11] is that these kinds of error are not suitably represented by structures such as strong Kleene algebra, or by the matrix proposed in [16]. By contrast, such errors would be better represented by weak Kleene algebra (**WKA**). This algebra comprises a non-classical value n beside 0 and 1; such a value, in turn, *contaminates* the other, in the sense that, if it is assigned to any input of a truth function, it will be assigned to the output of the truth function, independently from the value of the other inputs.

Two non-trivial logics are based on **WKA**, namely systems $\mathsf{K}_3^{\mathsf{w}}$ from [4] and PWK from [14]. [11] considers the former in his computational interpretation of the third value from **WKA**.

The kinds of error pointed at in [11] are novel with respect to [1,16], but contrary to [1], [11] does not consider the interaction of distinct types of error. This is limiting, since a host of different errors may display the contaminating behavior of value n, or a very close behavior—see Sect. 4 below. This seems to call for sublogics of $\mathsf{K}_3^{\mathsf{w}}$, where each contaminating value represent a different kind of error.

In this paper, we consider two families of computer errors that, in our view, can be represented by a contaminating value. These are *code errors* involving errors in software, and *physical errors* involving the physical machine on which the software runs—see Sect. 4 below. In order to provide reasoning tools that can, in principle, capture the interaction of these two kinds of errors, we introduce a four-valued algebra that is somehow inspired to WKA; we call this further structure 'hybrid algebra', since it hybridizes two different contaminating values. Also, we introduce two logics that are interpreted on the hybrid algebra, namely the four-valued systems HYB_1 and HYB_2. These in turn provides reasoning tools that can account for the interaction of the two kinds of errors above.

Given the role played by contaminating values in the systems we present here, we call $\mathsf{K}_3^{\mathsf{w}}$, PWK, HYB_1 and HYB_2 'contaminating logics'. We provide some semantic and proof-theoretical results for the two four-valued contaminating logics HYB_1 and HYB_2. In particular, we provide sound and complete sequent calculi, and a characterization of the two logics—that is, a way to establish HYB_1- and HYB_2-consequence relations on the ground of $\mathsf{K}_3^{\mathsf{w}}$- or PWK-consequence relations, or, alternatively, on the ground of classical consequence. In view of the results concerning sequent calculi, we focus on so-called multiple-conclusion consequence relations. We believe this secures a nice uniformity between the semantic and the proof-theoretic parts of the paper.

The paper proceeds as follows. After introducing some preliminaries in Sect. 2, in Sect. 3 we present the basic contaminating logics $\mathsf{K}_3^{\mathsf{w}}$ and PWK from [4] and [14], respectively. This will allow the reader to familiarize with contaminating logics and their envisaged applications to computer errors. In Sect. 4, we discuss the interaction of different sources of computer errors, with each of the sources discussed operating at different levels. In order to capture such an interaction, we introduce the *hybrid algebra* **HYB** and two four-valued contaminating logics interpreted on it, namely systems HYB_1 and HYB_2. In Sect. 5, we provide sound and complete annotated sequent calculi for the two logics. Section 6 provides conclusions and discuss some research directions.

2 Preliminaries

Given a similarity type ν and a countably infinitely set $X = \{p, q, r, \dots\}$ of generators (the *propositional variables*), we define the *formula algebra* **Fml** over X of type ν as the absolutely free algebra defined on X, with Fml denoting the universe of **Fml**, and the members of Fml being *formulas*, which we denote by $\phi, \psi, \theta, \dots$. $\Gamma, \Delta, \Psi \dots$ denote *sets of formulas*. In this paper, **Fml** will be a formula algebra of type $(1, 2, 2)$, namely, of the type containing the connectives $\neg, \vee, \wedge$. Given this, we feel free to omit reference to the type ν in what follows.[1]

We define a *logic* (of type ν) as a pair $\mathsf{S} = \langle \mathbf{Fml}, \vdash_{\mathsf{S}} \rangle$, with **Fml** a formula algebra (of type ν) and $\vdash_{\mathsf{S}} \subseteq \mathcal{P}(Fml) \times \mathcal{P}(Fml)$ a substitution invariant *multiple-conclusion consequence relation*.

We define a *matrix* as a pair $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$ with $\mathbf{A}$ an algebra (of some given type ν) with universe A and $\mathcal{D} \subset A$. $\mathcal{D}$ is called the *filter* of $\mathcal{M}$. Informally, we think of the members of A as *truth-values*, and of members of $\mathcal{D}$ as *designated truth values*.[2]

The following notion of a submatrix is relevant for our purposes:

Definition 1. *A matrix $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$ is a submatrix of a matrix $\mathcal{M}' = \langle \mathbf{A}', \mathcal{D}' \rangle$ ($\mathcal{M} \sqsubseteq \mathcal{M}'$) if and only if $\mathbf{A}$ is a subalgebra of $\mathbf{A}'$ and $\mathcal{D} = \mathcal{D}' \cap A$.*

In this paper, we focus on matrices that have the $\mathcal{M}_{\mathsf{CL}}$ of classical logic as a submatrix. In particular, classical logic CL can be defined as $\langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{CL}}} \rangle$, and

[1] Throughout the paper, we adopt the standard notation and basic definitions from *Abstract Algebraic Logic*, as presented *e.g.* in [13]. One important exception with regard to [13], however, concerns our definition of multiple-conclusion matrix consequence (see below).

[2] Notice that, in using these notions, we do not assume or even try to stress that we do not allow the presence of matrices whose algebraic reduct is the trivial algebra. However, as will become clear shortly, in this paper our interest is in investigating logics induced by matrices having contaminating values which, in turn, extend the two-valued matrix inducing classical logic—*i.e.* the matrix whose algebraic reduct is the two-element Boolean algebra. We would like to thank an anonymous reviewer for urging us to clarify this issue.

$\mathcal{M}_{\mathsf{CL}}$ is defined as $\langle \mathbf{B}_2, \{1\}\rangle$, where $\mathbf{B}_2 = \langle \{0,1\}, \neg, \vee, \wedge\rangle$ is the well-known two-element Boolean algebra of type $(1,2,2)$. The elements 0 and 1 of its universe are informally interpreted as 'false' and 'true', respectively, with 1 being the only designated value in $\mathcal{M}_{\mathsf{CL}}$.

A further relevant notion is that of a *valuation*:

Definition 2. *A valuation is a* homomorphism $v : \mathbf{Fml} \longrightarrow \mathbf{A}$ *from a formula algebra* $\mathbf{Fml}$ *into an algebra* $\mathbf{A}$ *of the same type.*

We denote by $\mathrm{Hom}_{\mathbf{Fml},\mathbf{A}}$ the set of valuations for $\mathbf{Fml}$ defined on $\mathbf{A}$. When $\mathbf{Fml}$ is clear by the context and we wish to focus on the matrix rather than on the algebra, we write $\mathrm{Hom}_{\mathcal{M}}$. For every $\mathcal{M} = \langle \mathbf{A}, \mathcal{D}\rangle$, we let $\mathrm{Hom}_{\mathcal{M}}(\Gamma)$ be the set $\{v \in \mathrm{Hom}_{\mathcal{M}} \mid v[\Gamma] \subseteq \mathcal{D}\}$ of the *models* of Γ based on $\mathcal{M}$.

Logical matrices, in turn, can be seen to give raise to substitution invariant multiple-conclusion consequence relations—the so-called *matrix consequence relation*—as the next definition illustrates:

Definition 3. *Given a matrix* $\mathcal{M} = \langle \mathbf{A}, \mathcal{D}\rangle$*, the relation* $\models_{\mathcal{M}} \subseteq \mathcal{P}(Fml) \times \mathcal{P}(Fml)$ *defined as follows:*

$$\Gamma \models_{\mathcal{M}} \Delta \Leftrightarrow \text{for every } v \in \mathrm{Hom}_{\mathcal{M}}, \nu[\Gamma] \subseteq \mathcal{D} \text{ implies } \nu(\psi) \in \mathcal{D} \text{ for some } \psi \in \Delta$$

is a multiple-conclusion matrix consequence *relation.*

We follow standard terminology and say that Δ is a *tautology* if and only if $\varnothing \models_{\mathcal{M}} \Delta$, and we say that Γ is *unsatisfiable* if and only if $\Gamma \models_{\mathcal{M}} \varnothing$—*i.e.*, if Γ has no models. We write $\phi \models_{\mathcal{M}} \psi$ instead of $\{\phi\} \models_{\mathcal{M}} \{\psi\}$, and $\phi, \psi \models_{\mathcal{M}} \gamma, \delta$ instead of $\{\phi, \psi\} \models_{\mathcal{M}} \{\gamma, \delta\}$. We also use other notation, writing *e.g.* Γ, Δ for $\Gamma \cup \Delta$, or Γ, ϕ for $\Gamma \cup \{\phi\}$.[3] Finally, when $\models_{\mathcal{M}_{\mathsf{S}}}$ is the matrix consequence relation of a logic S, we refer to $\models_{\mathcal{M}_{\mathsf{S}}}$ as to S-consequence.

Notice that the notion of multiple-conclusion consequence from Definition 3 differs from the one given in [13] in that the present interpretation comes with a *disjunctive* reading of the right side of $\models_{\mathcal{M}}$, while [13, Definition 1.7] comes with a *conjunctive* reading of it—implying that *all* the formulas in the conclusion-set have to be satisfied. In [13], the author himself notices that his definition is not standard.

Since the disjunctive reading of the right side of $\models_{\mathcal{M}}$ fits the interpretation of two-sided sequents in sequent calculi, we believe that in the present paper Definition 3 proves more suitable than the one from [13]. In particular, a uniform reading seems more appropriate in view of the results on sequent calculi from Sect. 5.

3 Basic Contaminating Logics

Here, we introduce two logics that are based on the so-called weak Kleene algebra (**WKA**) from [15]. These are relevant for our purposes, since **WKA** is a submatrix of the structures on which HYB_1 and HYB_2 are based.

[3] For this notation, see also [13, Chap. 1].

Definition 4. *[Weak Kleene Algebra] The weak Kleene algebra* ***WKA*** *is the algebra* $\mathbf{WKA}$ *of type* $(1, 2, 2)$ *such that (1)* $\mathbf{WKA} = \langle\{0, n, 1\}, \neg, \vee, \wedge\rangle$ *and (2) has operations* $\neg, \vee, \wedge$ *behaving as per Table 1.*

Table 1. Matrices for **WKA**.

	$\neg$
1	0
n	n
0	1

$\vee$	1	n	0
1	1	n	1
n	n	n	n
0	1	n	0

$\wedge$	1	n	0
1	1	n	0
n	n	n	n
0	0	n	0

Given its behavior w.r.t. the connectives, value n from Table 1 is usually said to be *contaminating* [6,9] or *infectious* [11,18]. Here, we prefer the first label. The following gives a straightforward and intuitive expression to this intuitive notion:

Observation 1 (Contamination). For all formulas ϕ in Fml and valuation $v \in \mathrm{Hom}_{\mathbf{Fml},\mathbf{WKA}}$:

$$v(\phi) = n \;\text{ iff }\; v(p) = n \,\text{for some}\, p \in var(\phi)$$

The LTR (left-to-right) direction is shared by all the most widespread three-valued logics; the RTL (right-to-left) direction is clear from Table 1, and it implies that ϕ takes value n if some $p \in var(\phi)$ has the value, and *no matter* what the value of q is for any $q \in var(\phi) \setminus \{p\}$.

WKA provides the *simplest* case of contamination, where a value n contaminates *all* the values in the universe A of the algebra in question. Another example of this is the four-valued matrix used to interpret the system $\mathsf{S_{fde}}$ from [10].

Two distinct non-trivial systems can be defined on **WKA**:[4] The logic $\mathsf{K}^{\mathsf{w}}_{3}$ has been introduced in [4] in order to reason about Russell's paradox and related set-theoretic antinomies. [11] has later proposed $\mathsf{K}^{\mathsf{w}}_{3}$ and cognate formalisms as a tool to reason about the way C++ processes information (see below). The logic PWK has been first introduced in [14] in order to reason about meaningless expressions and is investigated by [5,6,8]. We discuss some background and motivations for these logics at the end of the present section. $\mathsf{K}^{\mathsf{w}}_{3}$ and PWK are defined as follows:

Definition 5. $\mathsf{K}^{\mathsf{w}}_{3} = \langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{K}^{\mathsf{w}}_{3}}} \rangle$ *and* $\mathsf{PWK} = \langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{PWK}}} \rangle$*, where:*

$$\mathcal{M}_{\mathsf{K}^{\mathsf{w}}_{3}} = \langle \mathbf{WK}, \{1\} \rangle \qquad \mathcal{M}_{\mathsf{PWK}} = \langle \mathbf{WK}, \{n, 1\} \rangle$$

The following observation details some validities and the most notable failures of $\mathsf{K}^{\mathsf{w}}_{3}$ and PWK:

[4] We do not consider here the trivial systems resulting from $\mathcal{D} = A$ and $\mathcal{D} = \varnothing$.

Observation 2. *The following holds for* $\mathsf{K}_3^{\mathsf{w}}$*-consequence and* PWK*-consequence:*

$1a$ $\varnothing \not\models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \phi$ for every $\phi \in Fml$	$1b$ $\varnothing \models_{\mathcal{M}_{\mathsf{PWK}}} \phi$ for ϕ a classical tautology
$2a$ $\beta \not\models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \alpha$ for α a classical tautology	$2b$ $\beta \models_{\mathcal{M}_{\mathsf{PWK}}} \alpha$ for α a classical tautology
$3a$ $\alpha, \neg\alpha \models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \beta$	$3b$ $\alpha, \neg\alpha \not\models_{\mathcal{M}_{\mathsf{PWK}}} \beta$
$4a$ $\alpha \supset (\beta \wedge \neg\beta) \models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \neg\alpha$	$4b$ $\alpha \supset (\beta \wedge \neg\beta) \not\models_{\mathcal{M}_{\mathsf{PWK}}} \neg\alpha$
$5a$ $\alpha, \alpha \supset \beta \models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \beta$	$5b$ $\alpha, \alpha \supset \beta \not\models_{\mathcal{M}_{\mathsf{PWK}}} \beta$

We refer the reader to [6,11] for these failures and validities. Given the standard definitions of *paraconsistency* and *paracompleteness*,[5] an immediate consequence of Observation 2 is that $\mathsf{K}_3^{\mathsf{w}}$ is *paracomplete* and PWK is *paraconsistent*.

$\mathsf{K}_3^{\mathsf{w}}$ shares the above failures and validities with the related strong Kleene logic K_3 from [15], while PWK shares the above failures and validities with the related LP by [17].

The contaminating behavior of n contributes to some further failure, which are distinctive of $\mathsf{K}_3^{\mathsf{w}}$, PWK, and their sublogics:

$$\phi \not\models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \phi \vee \psi \quad \textit{Failure of Addition}$$
$$\phi \wedge \psi \not\models_{\mathcal{M}_{\mathsf{PWK}}} \phi \quad \textit{Failure of Simplification}$$

In particular, we have $v(\phi \vee \psi) = n$ in any valuation v such that $v(\phi) = 1$ and $v(\psi) = n$. Since $n \notin \mathcal{D}_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}}$, this implies failure of disjunctive addition in $\mathsf{K}_3^{\mathsf{w}}$. Also, $v(\phi \wedge \psi) = n$ in any valuation v such that $v(\phi) = 0$ and $v(\psi) = n$. Since $\mathcal{D}_{\mathcal{M}_{\mathsf{PWK}}} = \{n, 1\}$, this implies failure of conjunctive simplification.

By contrast, the following *local* versions of these properties hold:

$$\phi \vee \psi \models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \phi \vee \neg\phi \quad \textit{Local Excluded Middle}$$
$$\phi \wedge \neg\phi \models_{\mathcal{M}_{\mathsf{PWK}}} \phi \wedge \psi \quad \textit{Local Explosion}$$

[8] provides sound and complete sequent calculi for $\mathsf{K}_3^{\mathsf{w}}$ and PWK while [5] provides a sound and complete Hilbert-style axiomatization of PWK. [7] proves that $\mathsf{K}_3^{\mathsf{w}}$-consequence and PWK-consequence can be characterized in terms of classical consequence via two different variable-inclusion requirements. In particular:

Proposition 1 ([7], Theorems 3.4 and 4.3). $\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}$*- and* $\mathcal{M}_{\mathsf{PWK}}$*-consequence can be characterized as follows:*

$$\Gamma \models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \Delta \ \Leftrightarrow Var(\Delta') \subseteq Var(\Gamma) \textit{ for some } \Delta' \subseteq \Delta \textit{ s.t. } \Gamma \models_{\mathcal{M}_{\mathsf{CL}}} \Delta'$$

$$\Gamma \models_{\mathcal{M}_{\mathsf{PWK}}} \Delta \Leftrightarrow Var(\Gamma') \subseteq Var(\Delta) \textit{ for some } \Gamma' \subseteq \Gamma \textit{ s.t. } \Gamma' \models_{\mathcal{M}_{\mathsf{CL}}} \Delta$$

[5] A system is *paraconsistent* if it falsifies Ex Falso Quodlibet $\phi, \neg\phi \models \beta$, and is *paracomplete* if it falsifies Excluded Middle $\varnothing \models \phi \vee \neg\phi$.

This result provides a *characterization* of $\mathsf{K}_3^{\mathsf{w}}$ and PWK—that is, they specify necessary and sufficient conditions for a set Δ to follow from a set Γ of formulas in $\mathsf{K}_3^{\mathsf{w}}$ and PWK, respectively.

Discussion. PWK has been introduced by [14] in order to capture the impact of meaningless expressions on reasoning. One point worth noting is that [14] supports that some formulas should be valid even if there are occasions in which they are meaningless—in our matrix-based setting, this equates with designating n alongside 1. [14] defends this by arguing that the *validity* of a formula should be judged solely on the basis of its *meaningful* instances.

The idea motivating introduction of $\mathsf{K}_3^{\mathsf{w}}$ in [4] is that statements such as Russell's paradox would be *meaningless*. Under this interpretation, the third value n in **WKA** represents this further semantical category, and any $v(p) = n$ is a valuation where p is deemed meaningless.

Notice that this 'meaninglessness' interpretation fares well with the fact that n is a contaminating element in **WKA**: if p is meaningless, then it is not possible to process any information involving it—that is, it is not possible to process any formula ϕ in which p occurs.

[11] provides a computational application of this conceptual and formal apparatus, with particular attention paid to programming language C++. In this language, if some syntactical object p is to be used as a Boolean variable, the interpreter must be informed that p is to be used in this way. When the program is executed, an instruction is made to allocate sufficient memory for p to take a value. To *declare* the Boolean variable p is to allocate the necessary resources, *e.g.*, by reserving a physical address for its value. If a variable is *undeclared*, then it is *meaningless*: even if a formula is well-formed, if its atomic variables have not yet been declared, it is no more serviceable than an ill-formed string of symbols. In particular, the algorithm from Fig. 1 exemplifies how a C++-based program would react when fed with an undeclared variable.

procedure DECLARATION(y)
    boolean $p \leftarrow 1$
    $x \leftarrow (p$ **or** $q)$
end procedure

Fig. 1. Algorithm with undeclared variables

The fact that undeclared variables in a formula prevents the entire formula from being evaluated matches the contaminating behavior of the third value n from **WKA**; also, a formula that cannot be processed cannot be assigned values 0 or 1 as well, and this fits with $n \notin \mathcal{D}$. The algorithm in Fig. 1 shows that addition is bound to fail exactly as it does in $\mathsf{K}_3^{\mathsf{w}}$ [11, p. 352]. This example suggests that $\mathsf{K}_3^{\mathsf{w}}$ enjoys some legitimacy as a tool to reason about the way C++ processes information.

Figure 1 also suggests that undeclared variables also bring a kind of computer error: if they are involved, a C++-based program becomes unable to process relevant information along the lines of classical logic (or the logic of choice on which the program is based).

Application of logic to computer errors is not new. [16] proposes non-commutative disjunction and conjunction in order to reason about errors in sequential computing. Crucial to this proposal is a matrix-based semantics involving a third truth value beside 1 and 0. [1] applies the system by [16] to *critical errors* from sequential computing—that is, errors that make the computation stop—and the strong Kleene logic from [15] to *non-critical errors* from parallel computing—that is, errors that can be somehow remedied. Also, [1] proposes a four-valued sublogic of both McCarthy's and Kleene's systems, which allows for reasoning on both kinds of errors.

Errors due to undeclared variables differ from those considered by [1] and [16], insofar as they represent computation stops that are due to a syntactic failure. This, and the logical features of errors due to undeclared variables, justify the application of a different system such as $\mathsf{K}_3^{\mathsf{w}}$ or some relevant subsystem.

4 Four-Valued Contaminating Logics

Failure to assign a value to an *undeclared* variable in C++ is an error in code, and hence on the level of *software*. Errors of this kind may cause a process to halt. Beside these kinds of errors, we have errors on the level of *hardware*. An instance of these are the physical errors that are caused when an environment attempts to retrieve a value from a physical address that is corrupt.

Catastrophic, physical errors that cause some failure at the level of hardware clearly affect the software running on this hardware. By contrast, errors causing failure at the level of a virtual machine need not propagate to all the environments in the system. Both errors may happen, but they will affect the entire system in different ways. This calls for an *interaction* of different types of errors, and this interaction seems to be hierarchical in a sense, since the level at which an error takes place contributes to determine how much of the environment is affected by that error. We believe that this interaction is suitably captured by applying *two* contaminating values similar to the value n from Table 1, and we briefly discuss why.

Errors in Code and Physical Errors. Consider errors in code. The triggering of the syntactic error at the local level—that is, within the virtual machine—may cause the *environment within which the executable was run* to halt prematurely. This calls for some truth value that displays a contaminating behavior in the style of n from Table 1, since the situation we have described represents the capacity of an error to affect any string of information or environment *in which* the error takes place. At the same time, however, this error happens within the scope of a virtual machine, which in turn insulates the operating system from such local errors. This is better represented by a value that is just *partially* contaminating, that is a value that contaminates *some*, but not *all* other values. Value n from Table 1 cannot capture this, since it contaminates all other values in the universe of **WKA**. Thus, we need to adjust n to fit our current purpose.

Going to physical errors at the level of the hardware, if the *operating system* attempts to retrieve a value *on behalf of* a virtual machine from a bad address, the error that causes the operating system to fail will bring down the virtual machine alongside it. This calls for a contaminating value that affects *all* other values, including possibly partially contaminating values like the one discussed above.

In a nutshell, the logical representation of the interaction of the two kinds of errors above requires *two* values that are contaminating in some sense—in particular, we need one value to be contaminating in a weaker sense than n, and the other to be contaminating in exactly the same sense of n.

Reasoning About the Interaction of the Two Errors. In this section we propose the two logics HYB_1 and HYB_2 as tools that can be used for reasoning about the interaction of the kind of errors that we have been discussed above. Both HYB_1 and HYB_2 display the two different types of contaminating values that we see fit in capturing the two different kinds of errors that we have discussed above. Also, HYB_1 and HYB_2 include both a designated and an undesignated contaminating value.

Whether one or more contaminating values should be designated or not is, in our view, a *pragmatic* issue, determined by an end user's interest. For instance, a developer may be concerned with the stability of the code itself and *not* with the stability of the physical memory. Thus, one might be justified in modeling this global error via a designated value. Take the concrete case of a large ontology with an integrated theorem prover, for example. Here, one might wish for certain theorems to be derivable, in spite of the potential for hardware errors. In this case, practical concerns make lead the ontology's developers to *discount* this type of situation from consideration when judging validity. Also, when one is testing code, some tiers of errors are important to acknowledge while others are not. Simply put, whether one's code leads to a software error is part of a developer's concern; the fact that a particular piece of hardware upon which the software runs crashes due to faulty RAM is not.

We acknowledge that the examples above do not bring conclusive evidence for designating a contaminating values, but we also believe that they provide reasons for it, which would deserve further discussion and testing. We postpone a detailed discussion of this issue to a further paper. In this paper, we take this provisional reasons as strong enough to support the elaboration of four-valued logics with both designated and undesignated values.

Thus, in the next sections we extend our previous considerations to build appropriate semantic tools to model such settings. We do this by appealing to the idea of a linear order of contaminating values, such that the greater contaminating values contaminate the smaller ones and, of course, the non-contaminating values.

4.1 An Algebra for the Interaction of Different Computer Errors

First, we introduce a structure that can represent the interaction between computer errors that we have envisaged above:

Definition 6. *[Hybrid Algebra] The hybrid algebra is the algebra* **HYB** *of type* $(1, 2, 2)$ *such that (1)* $\mathbf{HYB} = \langle\{0, n_1, n_2, 1\}, \neg, \vee, \wedge\rangle$ *and (2) has operations* $\neg, \vee, \wedge$ *behaving as per Table 2.*

Table 2. Matrices for **HYB**.

	$\neg$
1	0
n_1	n_1
n_2	n_2
0	1

$\vee$	1	n_1	n_2	0
1	1	n_1	n_2	1
n_1	n_1	n_1	n_2	n_1
n_2	n_2	n_2	n_2	n_2
0	1	n_1	n_2	0

$\wedge$	1	n_1	n_2	0
1	1	n_1	n_2	0
n_1	n_1	n_1	n_2	n_1
n_2	n_2	n_2	n_2	n_2
0	0	n_1	n_2	0

Values n_1 and n_2 from Table 2 enjoys a sort of contaminating behavior in the style of n, but notice that the behavior of n_1 does not satisfy the conditions sorted out by Observation 1. In order to qualify their different behaviors, we adjust the notion of contamination from Observation 1 and we define a full-fledged, general notion of contamination:

Definition 7. *An algebra* $\mathbf{A}$ *of type* ν *has a contaminating element* k *if and only if there is a non-empty* $A' \subseteq A$*, with* $A' \neq \{k\}$*, such that for every* m*-ary* $g \in \nu$ *and every* $\{a_1, \ldots, a_m\} \subseteq A'$*:*

$$\textit{if } k \in \{a_1, \ldots, a_m\} \textit{ then } g^{\mathbf{A}}(a_1, \ldots, a_m) = k$$

Both n_1 and n_2 satisfy Definition 7. Since n_2 contaminates every other value, we will say that n_2 is *absolutely* contaminating. By contrast, we will say that n_1 is just *partially* contaminating, since it contaminates all values *but* n_2.

Discussion of the Two Contaminating Values. Given its *partially contaminating* behavior, value n_1 fits our description of how errors in code work. Indeed, n_1 does not trump any other value, and this seems to fit the fact that errors in code do not necessarily affect *any* environment, while they do prevent computation to proceed in the virtual machine where they occur. By contrast, given its *absolutely contaminating* behavior, value n_2 fits our description of how physical errors work at the level of physical hardware. Again, the former trumps any other value, and this seems fit the fact that a physical errors occurring in the operating system affects any environment.

4.2 Logics Based on HYB

We now discuss two non-trivial logics induced by logical matrices built using the **HYB** algebra, the systems HYB_1 and HYB_2. We give sound and complete sequent calculi in Sect. 5. In what follows, we familiarize with the two systems, and provide characterizations in the style of Theorem 3.4 and Theorem 4.3 from [7]. The two logics are defined as follows:

Definition 8. $\mathsf{HYB}_1 = \langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{HYB}_1}} \rangle$ *and* $\mathsf{HYB}_2 = \langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{HYB}_2}} \rangle$*, where:*

$$\mathcal{M}_{\mathsf{HYB}_1} = \langle \mathbf{HYB}, \{n_1, 1\} \rangle \qquad \mathcal{M}_{\mathsf{HYB}_2} = \langle \mathbf{HYB}, \{n_2, 1\} \rangle$$

Each of HYB_1 and HYB_2 shares all the failures of $\mathsf{K}^{\mathsf{w}}_3$ and PWK, since the former are subsystems of the latter. Additionally, the following distinguish the two logics HYB_1 and HYB_2 from $\mathsf{K}^{\mathsf{w}}_3$ and PWK:

$$\phi \vee \psi \models_{\mathcal{M}_{\mathsf{HYB}_1}} \phi \vee \neg\phi \qquad \phi \wedge \neg\phi \not\models_{\mathcal{M}_{\mathsf{HYB}_1}} \phi \wedge \psi$$

$$\phi \vee \psi \not\models_{\mathcal{M}_{\mathsf{HYB}_2}} \phi \vee \neg\phi \qquad \phi \wedge \neg\phi \models_{\mathcal{M}_{\mathsf{HYB}_2}} \phi \wedge \psi$$

As for Local Excluded Middle, any valuation v such that $v(\psi) = v(\phi \vee \psi) = n_2$ and $v(\phi) = n_1$ is such that $v(\phi \vee \psi) \in \mathcal{D}_{\mathcal{M}_{\mathsf{HYB}_2}}$ and $v(\phi \vee \neg\phi) \notin \mathcal{D}_{\mathcal{M}_{\mathsf{HYB}_2}}$. Also, for every valuation v such that $v(\phi \vee \psi) \in \{n_1, 1\}$, we have $v(\phi \vee \neg\phi) \in \{n_1, 1\}$. Since $\mathcal{D}_{\mathcal{M}_{\mathsf{HYB}_1}} = \{n_1, 1\}$, the inference has no countermodel in $\mathcal{M}_{\mathsf{HYB}_1}$. As for Local Explosion, any valuation v where $v(\phi \wedge \neg\phi) = n_1$ and $v(\psi) = n_2$ provides a countermodel to the inference in HYB_1; for every valuation v where $v(\phi) = v(\phi \wedge \neg\phi) = n_2$, we have $v(\phi \wedge \psi) = n_2$ by contamination. Since $\mathcal{D}_{\mathcal{M}_{\mathsf{HYB}_2}} = \{n_2, 1\}$, the rule has no countermodel in $\mathcal{M}_{\mathsf{HYB}_2}$.

The following lemma plays a crucial role in proving Theorem 2 from this section and Theorem 6 from Sect. 5:

Lemma 1. *The consequence relations* $\models_{\mathcal{M}_{\mathsf{HYB}_1}}$ *and* $\models_{\mathcal{M}_{\mathsf{HYB}_2}}$ *are* dual*, that is:*

$$\Gamma \models_{\mathcal{M}_{\mathsf{HYB}_1}} \Delta \Leftrightarrow \Delta^{\neg} \models_{\mathcal{M}_{\mathsf{HYB}_2}} \Gamma^{\neg}$$

where, for every $\Gamma \subseteq Fml$*,* $\Gamma^{\neg} = \{\neg\phi \mid \phi \in \Gamma\}$*.*

Discussion of the Interaction of n_1 and n_2 in HYB_1 and HYB_2. Values n_1 and n_2 may represent, as we have discussed above, code errors in a virtual machine, and physical errors in the operating system. Given what we have proposed about the pragmatic nature of designation of a contaminating value, two combinations are possible: code errors are taken as non-catastrophic and physical errors as fatal, or vice versa. The two options correspond to taking HYB_1 and HYB_2 as one's logic of choice, respectively. Under this informal reading, $\phi \vee \psi \models_{\mathcal{M}_{\mathsf{HYB}_1}} \phi \vee \neg\phi$ can be seen as a way of expressing that, if both ϕ and ψ are free from fatal errors at the software level, then any of the involved formulas can be assigned a value—which implies that either ϕ or its negation will receive a designated value, given the behavior of $\neg$. By contrast,

$\phi \vee \psi \not\models_{\mathcal{M}_{\mathsf{HYB}_2}} \phi \vee \neg\phi$ implies that, if some supposedly non-catastrophic error occurs in processing either ϕ or ψ, nothing excludes that the other piece of information is not involved in some error in code, which is less contaminating but fatal, under this specific interpretation.

Although we have a preference for the option that supports HYB_1 over HYB_2—we feel that physical errors at the level of the operating system can be hardly seen as unthreatening—we believe that it is worth exploring both options.

4.3 Characterizating Logical Consequence in HYB_1

The following is a characterization result for HYB_1:

Theorem 1

$$\Gamma \models_{\mathsf{HYB}_1} \Delta \text{ iff } \Gamma \models_{\mathsf{PWK}} \Delta' \text{ for at least a non-empty } \Delta' \subseteq \Delta \text{ s.t. } \mathrm{var}(\Delta') \subseteq \mathrm{var}(\Gamma).$$

This fits with the way HYB_1 conceives of the hierarchy of errors represented by n_1 and n_2: since the fatal errors are those represented by the most contaminating value, any information that is included in the premises will be safe from fatal errors, since the premises itself must be, if they are to be designated.

Theorem 1 in turn explains why the PWK-valid inference from $\phi\wedge\neg\phi$ to $\phi\wedge\psi$ fails in HYB_1, while the PWK-valid inference from $\phi \vee \psi$ to $\phi \vee \neg\phi$ is valid in HYB_1. The former violates the variable-inclusion requirement from Theorem 1, while the latter complies with it.

4.4 Characterizing Logical Consequence in HYB_2

With the above notions and facts at hand, we are ready to provide the characterization result for HYB_2:

Theorem 2

$$\Gamma \models_{\mathsf{HYB}_2} \Delta \text{ iff } \Gamma' \models_{\mathsf{K}_3^{\mathsf{w}}} \Delta \text{ for at least a non-empty } \Gamma' \subseteq \Gamma \text{ s.t. } \mathrm{var}(\Gamma') \subseteq \mathrm{var}(\Delta).$$

This fits with the way HYB_2 conceives of the hierarchy of errors represented by n_1 and n_2: since the fatal errors are those represented by the least contaminating value, if information from part of the premise is included in the conclusion, then the latter will be safe from fatal errors, since otherwise the premises would not be.

Theorem 2 explains $\phi \vee \psi \not\models_{\mathsf{HYB}_2} \phi \vee \neg\phi$. Indeed, although the inference is $\mathsf{K}_3^{\mathsf{w}}$-valid, there is no guarantee that the variables of $\phi \vee \psi$ are all contained in those of ϕ—notice that $\phi \vee \psi$ is, in turn, the only non-empty subset of $\phi \vee \psi$.

The following corollary will be helpful in proving Lemma 2 from Sect. 5. It is a straightforward consequence of Proposition 1, Theorems 1 and 2:

Corollary 1. *$\mathcal{M}_{\mathsf{HYB}_1}$-consequence and $\mathcal{M}_{\mathsf{HYB}_2}$-consequence can be characterized as follows:*

$$\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_1}} \Delta \Leftrightarrow Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma) \text{ for some } \Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta \text{ s.t. } \Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$$

$$\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_2}} \Delta \Leftrightarrow Var(\Delta') \subseteq Var(\Gamma') \subseteq Var(\Delta) \text{ for some } \Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta \text{ s.t. } \Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$$

4.5 Discussion of Theorems 1 and 2

Sublogics like HYB_1 and HYB_2 are attracting increasing attention [3,18], and they are natural way to generalize the three-valued contaminating settings based on **WKA** to four-valued settings involving more than one contaminating value. However, these logics have not yet received detailed investigations. The two theorems from the present section make a significant contribution to our knowledge of such logics, and we believe that this explains their relevance.

Additionally, we believe that our results make a significant progress with respect to [12, Observation 1], that also provides a clear direction for a general characterization methods for logics endowed with many contaminating values. First, [12, Observation 1] is concerned with single-conclusion consequence relations, while our results suggest a method that would apply to the more general multiple-conclusion case. Second, and more important, [12, Observation 1] concerns logics where contaminating values are not designated, while Theorem 2 provides an insight that is relevant also for logics that comprise one (or more) designated contaminating values. Although the insight from [12, Observation 1] easily extends to HYB_1, it is not clear if it extends naturally to HYB_2. Thus, the present results offer an insight that is more general than the insight offered by [12, Observation 1].

5 Sequent Calculi

We go now to the sequent calculi for HYB_1 and HYB_2. More precisely, we provide sound and complete calculi of *annotated* sequents for the two four-valued logics. An *annotated* sequent is an object of the form $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ where $\Gamma, \Gamma', \Delta, \Delta'$ are sets of formulas of the language. In annotated sequent calculi, additional rules are provided in order to capture the interaction among formulas within squared brackets, outside square brackets, and the interaction of formulas within square brackets and formulas outside the brackets.

Our results extend the ones from [8] for $\mathsf{K}_3^{\mathsf{w}}$ and PWK. As in [8], each of our calculi places *restrictions* on several rules—more precisely, the rules need some variable inclusion condition to be satisfied in order to be applicable. We will specify the relevant restrictions when needed.

Below, we present the rules for the two annotated calculi and proceed to demonstrate the soundness and completeness of the two calculi.

5.1 Rules

Both systems include the following three rules, where for every $\mathit{\Gamma} \subseteq Fml$, $\mathit{\Gamma}^*$ is any modification of $\mathit{\Gamma}$ by permuting elements, absorbing redundancies, or duplicating formulas:

$$\frac{}{\varnothing, [\![p]\!] \Rightarrow \varnothing, [\![p]\!]}\ [\text{Axiom}]$$

$$\frac{\mathit{\Gamma}, [\![\mathit{\Xi}]\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Theta}]\!]}{\mathit{\Gamma}^*, [\![\mathit{\Xi}^*]\!] \Rightarrow \mathit{\Delta}^*, [\![\mathit{\Theta}^*]\!]}\ [\text{Structural}]$$

$$\frac{\mathit{\Gamma}, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}']\!]}{\mathit{\Gamma}, \mathit{\Xi}, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, \mathit{\Theta}, [\![\mathit{\Delta}']\!]}\ [\text{Weak}]$$

[Axiom] secures the validity of those classical axioms in which a propositional variable is within the scope of a square bracket in each sequent. [Structural] grants standard structural rules, but Weakening, *within any of the four slots*. [Weak] differs from the Weakening for non-annotated calculi as we can only allow Weakening *outside* the scope of the bracket. The following "push" rules below meet the need to shift formulas from outside the scope of a square bracket into the brackets. It is with these rules that variable-inclusion restrictions come into play:

$$\frac{\mathit{\Gamma}, \phi, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}']\!]}{\mathit{\Gamma}, [\![\mathit{\Gamma}', \phi]\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}']\!]}\ [\text{PushL}] \qquad \frac{\mathit{\Gamma}, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, \psi, [\![\mathit{\Delta}']\!]}{\mathit{\Gamma}, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}', \psi]\!]}\ [\text{PushR}]$$

***Restrictions for* PushL *and* PushR.** In the HYB_1 calculus, [PushL] requires the restriction $Var(\phi) \subseteq Var(\mathit{\Delta}')$ and [PushR] requires $Var(\psi) \subseteq Var(\mathit{\Gamma} \cup \mathit{\Gamma}')$. In the HYB_2 calculus, the [PushL] and [PushR] rules require $Var(\phi) \subseteq Var(\mathit{\Delta} \cup \mathit{\Delta}')$ and $Var(\psi) \subseteq Var(\mathit{\Gamma}')$, respectively.

Negation rules come with a pair of right rules and a pair of left rules, since we need to distinguish the case where we are introducing the sign within the scope of a square bracket from the case in which we are introducing the sign out of such a scope. The right rules:

$$\frac{\mathit{\Gamma}, [\![\mathit{\Gamma}', \phi]\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}']\!]}{\mathit{\Gamma}, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}', \neg\phi]\!]}\ [\neg\text{R}_1] \qquad \frac{\mathit{\Gamma}, \phi, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}']\!]}{\mathit{\Gamma}, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, \neg\phi, [\![\mathit{\Delta}']\!]}\ [\neg\text{R}_2]$$

The left rules:

$$\frac{\mathit{\Gamma}, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}', \psi]\!]}{\mathit{\Gamma}, [\![\mathit{\Gamma}', \neg\psi]\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}']\!]}\ [\neg\text{L}_1] \qquad \frac{\mathit{\Gamma}, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, \psi, [\![\mathit{\Delta}']\!]}{\mathit{\Gamma}, \neg\psi, [\![\mathit{\Gamma}']\!] \Rightarrow \mathit{\Delta}, [\![\mathit{\Delta}']\!]}\ [\neg\text{L}_2]$$

Restrictions for* $\neg\mathbf{R}_1$*,* $\neg\mathbf{R}_2$*, and* $\neg\mathbf{L}_1$*,* $\neg\mathbf{R}_2$*. In the HYB_1 calculus, $[\neg\text{R}_1]$ and $[\neg\text{R}_2]$ require $Var(\phi) \subseteq Var(\mathit{\Gamma} \cup \mathit{\Gamma}')$; in the HYB_2 calculus, $[\neg R_1]$ requires that $Var(\phi) \subseteq Var(\mathit{\Gamma}')$, and $[\neg R_1]$ has no proviso. In both calculi, $[\neg\text{L}_1]$ requires that $Var(\psi) \subseteq Var(\mathit{\Delta}')$ and $[\neg\text{L}_2]$ has no proviso.

Conjunction rules also come in pairs:

$$\frac{\Gamma, [\![\Gamma', \phi, \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma', \phi \wedge \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]} [\wedge \mathrm{L}_1] \qquad \frac{\Gamma, \phi, \psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, \phi \wedge \psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]} [\wedge \mathrm{L}_2]$$

Rules $[\wedge \mathrm{L}_1]$ and $[\wedge \mathrm{L}_2]$ require no provisos in either HYB_1 or HYB_2. However, the following mixed rule requires a variable-inclusion restriction:

$$\frac{\Gamma, \phi, [\![\Gamma', \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma', \phi \wedge \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]} [\wedge \mathrm{L}^*]$$

In HYB_1, the rule is definable provided that $Var(\phi) \subseteq Var(\Delta')$, while in HYB_2, $Var(\phi) \subseteq Var(\Delta \cup \Delta')$ is required. For the right rules, we consider the case in which both conjuncts are outside of the scope of $[\![-]\!]$ and the case in which both are within its scope. Note, again, that we can appeal to [PushR] in order to cover mixed cases.

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \phi]\!] \qquad \Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \psi]\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \phi \wedge \psi]\!]} [\wedge \mathrm{R}_1]$$

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \phi, [\![\Delta']\!] \qquad \Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \psi, [\![\Delta']\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \phi \wedge \psi, [\![\Delta']\!]} [\wedge \mathrm{R}_2]$$

Again, neither $[\wedge \mathrm{R}_1]$ nor $[\wedge \mathrm{R}_2]$ requires a proviso in the two logics, but one could introduce a definable rule that requires that $Var(\phi) \subseteq Var(\Gamma \cup \Gamma')$ in HYB_1 and $Var(\phi) \subseteq Var(\Gamma')$ in HYB_2:

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \phi, [\![\Delta']\!] \qquad \Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \psi]\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \phi \wedge \psi]\!]} [\wedge \mathrm{R}^*]$$

Disjunction rules are as follows:

$$\frac{\Gamma, [\![\Gamma', \phi]\!] \Rightarrow \Delta, [\![\Delta']\!] \qquad \Gamma, [\![\Gamma', \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma', \phi \vee \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]} [\vee \mathrm{L}_1]$$

$$\frac{\Gamma, \phi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!] \qquad \Gamma, \psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, \phi \vee \psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]} [\vee \mathrm{L}_2]$$

Neither $[\vee \mathrm{L}_1]$ nor $[\vee \mathrm{L}_2]$ require provisos. Again, for the right rules, we consider the case in which both disjuncts are outside of the scope of $[\![-]\!]$ and the case in which both are within its scope. Note, again, that we can appeal to [PushR] in order to cover mixed cases.

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \phi, \psi]\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \phi \vee \psi]\!]} [\vee \mathrm{R}_1] \qquad \frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \phi, \psi, [\![\Delta']\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \phi \vee \psi, [\![\Delta']\!]} [\vee \mathrm{R}_2]$$

5.2 Soundness and Completeness

Now we state *soundness* and *completeness* of HYB_1 is *sound* and *complete* with respect to $\mathcal{M}_{\mathsf{HYB}_1}$ (Theorems 3 and 4), and HYB_2 is *sound* and *complete* with respect to $\mathcal{M}_{\mathsf{HYB}_2}$ (Theorems 5 and 6).

Theorem 3 (Soundness of HYB_1). *If $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ is provable in HYB_1, then $\Gamma \cup \Gamma' \vDash_{\mathcal{M}_{\mathsf{HYB}_2}} \Delta \cup \Delta'$.*

In what follows, when we talk about 'the two-sided sequent calculi for PWK and $\mathsf{K}_3^{\mathsf{w}}$', we will be referring to the calculi from [8], which are presented there as fragments of Gentzen's sequent calculus for classical logic (indeed, as fragments where some of the operational rules were restricted with *variable inclusion* requirements). This is important for understanding the following lemma, which helps prove the completeness of HYB_1 with respect to $\mathcal{M}_{\mathsf{HYB}_2}$.

Lemma 2. *If $\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_2}} \Delta$ such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma)$ and $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is provable in the calculus for PWK.*

Definition 9. *In the HYB_1 calculus, a PWK rule that applies only to formulas within brackets is a "bracketed rule".*

Theorem 4 (Completeness of HYB_1). *If $\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_2}} \Delta$ such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma)$ and $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$, then $\Gamma', [\![\Gamma'']\!] \Rightarrow \Delta', [\![\Delta'']\!]$ is provable in HYB_1, where $\Gamma = \Gamma' \cup \Gamma''$ and $\Delta = \Delta' \cup \Delta''$.*

The duality of HYB_1 and HYB_2 allows us to leverage Lemma 1 to establish the corresponding results for HYB_2.

Theorem 5 (Soundness of HYB_2). *If $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ is provable in HYB_2, then $\Gamma \cup \Gamma' \vDash_{\mathcal{M}_{\mathsf{HYB}_1}} \Delta \cup \Delta'$.*

Theorem 6 (Completeness of HYB_2). *If $\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_1}} \Delta$ such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, $Var(\Delta') \subseteq Var(\Gamma') \subseteq Var(\Delta)$ and $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$, then $\Gamma', [\![\Gamma'']\!] \Rightarrow \Delta', [\![\Delta'']\!]$ is provable in HYB_2, where $\Gamma = \Gamma' \cup \Gamma''$ and $\Delta = \Delta' \cup \Delta''$.*

It is clear from the above rules that the *Subformula Property* holds of HYB_1 and HYB_2.

We will finish by considering how to approach the admissibility of the Cut rule in the calculi HYB_1 and HYB_2. Our calculi for HYB_1 and HYB_2 are *decorated*, since we use a bracketing device in each of the antecedent and succedent to track variable-inclusion properties. Although we feel that there are conceptual differences between the bracketing device employed in our calculi and the labeling employed by many-sided sequent calculi (like those described in [2]), similar issues arise in formulating the Cut rule. Given a set $A = \{a_1, a_2, \ldots, a_n\}$ whose members are interpreted as truth values and where $a_1 = 0$ and $a_2 = 1$, many-side sequent calculi allow for sequents of the form $\Gamma_1 \mid \cdots \mid \Gamma_n$. The standard informal meaning of such a sequent is: 'for some i between 1 and n, and some ϕ in Γ_i, ϕ has value a_i'. In a nutshell, each "side" of a sequent plays the role of a distinct truth-value.[6]

[6] This illustrates the difference between our calculi and many-sided sequent calculi. Contrary to the latter, the bracketing in our calculi for HYB_1 and HYB_2 does not *a priori* correspond to truth-values.

This leads Baaz *et al.* to define not one Cut, but *many*, depending on the two sides in which the cut formula is found. Treating our calculi as many-sided calculi would lead us to *six* distinct structural rules that look like Cut. Clearly, not all of these are plausibly admissible. While

$$\frac{\Gamma, [\![\Gamma', \phi]\!] \Rightarrow \Delta, [\![\Delta']\!] \qquad \Theta, [\![\Theta']\!] \Rightarrow \Xi, [\![\Xi', \phi]\!]}{\Gamma, \Theta, [\![\Gamma', \Theta']\!] \Rightarrow \Delta, \Xi, [\![\Delta', \Xi']\!]}\ [\text{Cut } 2, 4]$$

seems plausible,

$$\frac{\Gamma, \phi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!] \qquad \Theta, [\![\Theta', \phi]\!] \Rightarrow \Xi, [\![\Xi']\!]}{\Gamma, \Theta, [\![\Gamma', \Theta']\!] \Rightarrow \Delta, \Xi, [\![\Delta', \Xi']\!]}\ [\text{Cut } 1, 2]$$

seems wildly implausible. The questions of which candidate versions of Cut are admissible in these calculi and how the case differs from the many-sided case are intriguing but left for future research.

6 Conclusions

In this paper, we have discussed the interaction of computer errors that come from different sources and, especially, takes place at different levels in the system. Some of these errors are suitably represented by values that are contaminating in a sense closely resembling the third value from **WKA**. The paper discusses this structure together with the two non-trivial logics that can be interpreted on it. These are the systems PWK from [14] and $\mathsf{K}_3^{\mathsf{w}}$ from [4]. In particular, $\mathsf{K}_3^{\mathsf{w}}$ and cognate formalisms have been given a computational interpretation in [11], where the logic is used in order to reason about those failures in C++-based programs that are due to the presence of undeclared variables. Since computer errors may have a variety of different sources, and differ in their effects on the environment, we discuss the interaction of two different kinds of computer errors, namely those which occur at the level of software, and those which occur at the level of hardware. In order to capture the interaction of these two kinds of errors, we introduce the four-valued algebra **HYB**, and two logics based on that: the systems HYB_1 and HYB_2. We provide characterization results for the two logics—that is, we provide necessary and sufficient conditions for two sets Γ and Δ of formulas to be in the relation of HYB_1- or HYB_2-consequence. Before closing, we discuss some directions for future research.

First, we plan to devote future work to an investigation into the matter of designation (or not) of contaminating truth-values (see Sect. 4 for the issue). This is a very important point. Indeed, the increasing use of virtualization and cloud computing entails that one frequently encounters programs running in a cascade of virtual machines nested in one another. Interest of the user and specific application may lead to *discount* some errors and consider them uninteresting and unthreatening. In this case, one might want to designate the relevant contaminating truth value, since this represent the ability of the computation to go on, the error notwithstanding. We wish to cast this general framework against the background of concrete scenarios of nested computer errors.

Another interesting issue raised by a referee concerns the role of error detection and correction. Our model of computation in this paper presupposes that any error is fatal to the system in which it occurs but there are numerous techniques employed allowing a process to recover in the face of otherwise catastrophic errors; in the present day, any important transmission of data is accompanied by a host of safeguards to preserve its integrity, through the use of *e.g.* checksums. This suggests that a more accurate model allows not only for error-free states and catastrophic states, but also states intermediate between these, in which a process has encountered—and recovered from—an otherwise fatal error. Whether this type of case can be accurately modeled by many-valued matrices and, if so, whether the inclusion of such states influences the consequence relations is clearly worth investigating.

Appendix

Proof of Lemma 1: We start with the LTR direction. Suppose that $\varGamma \vDash_{\mathcal{M}_{\mathsf{HYB}_1}} \varDelta$. This means that, if $v(\psi) \in \{0, n_2\}$ for every $\psi \in \varDelta$, then $v(\phi) \in \{0, n_2\}$ for some $\phi \in \varGamma$ and every $v \in \mathrm{Hom}_{\mathbf{Fml},\mathbf{HYB}}$. Given the behavior of n_2 w.r.t. negation, this implies that, if $v(\theta) = \{1, n_2\}$ for every $\theta \in \varDelta^{\neg}$, then $v(\zeta) = \{1, n_2\}$ for some $\zeta \in \varGamma^{\neg}$ and every $v \in \mathrm{Hom}_{\mathbf{Fml},\mathbf{HYB}}$. Since $\mathcal{D}_{\mathsf{HYB}_2} = \{1, n_2\}$, this implies $\varDelta^{\neg} \models_{\mathcal{M}_{\mathsf{HYB}_2}} \varGamma^{\neg}$. The RTL direction is proved along the very same lines. ■

Proof of Theorem 1: We start with the LTR direction. We first prove that *if* $\varGamma \models_{\mathsf{HYB}_1} \varDelta$, *then* $\varGamma' \models_{\mathsf{HYB}_1} \varDelta$ for at least a non-empty $\varDelta' \subseteq \varDelta$ such that $var(\varDelta') \subseteq var(\varGamma)$. Assume the antecedent as the initial hypothesis, and suppose that $\varGamma \not\models_{\mathsf{HYB}_1} \varDelta'$ for every $\varDelta \subseteq \varDelta'$ such that $var(\varDelta') \subseteq var(\varGamma)$. This implies that there is valuation $v \in \mathrm{Hom}_{\mathbf{Fml},\mathbf{HYB}}$ such that $v(\psi) \in \{n_2, 0\}$ for every $\psi \in \varDelta'$ and yet $v(\phi) \in \{1, n_1\}$ for every $\phi \in \varGamma$. By the contaminating behavior of n_2 from Table 2 and $var(\varDelta') \subseteq var(\varGamma)$, this implies $v(\psi) = 0$ for every $\psi \in \varDelta'$. More in general, we have $v(p) \neq n_2$ for every $p \in var(\varGamma)$, and by this, we have $v(q) \neq n_2$ for every $q \in var(\bigcup_{\varDelta' \in \mathbf{G}_{\varDelta,\varGamma}})$. This implies that $v(\phi) = \{n_1, 1\}$ for every $\phi \in \varGamma$. v can be extended to a valuation $v' \in \mathrm{Hom}_{\mathbf{Fml},\mathbf{HYB}}$ such that $v'(p) = v(p)$ if $p \in var(\varGamma)$, and $v'(p) = n_2$ otherwise. This implies that $v'(\phi) \in \{1, n_1\}$ for every $\phi \in \varGamma$, $v'(\theta) = n_2$ for every $\theta \in \varDelta \setminus \bigcup_{\varDelta' \in \mathbf{G}_{\varDelta,\varGamma}}$, and $v(\psi) = 0$ for every $\psi \in \varDelta$. But this in turn contradicts the initial hypothesis, given the definition of HYB_1-consequence. Thus, we have that, *if* $\varGamma \models_{\mathsf{HYB}_1} \varDelta$, *then* $\varGamma \models_{\mathsf{HYB}_2} \varDelta'$ for at least a non-empty $\varDelta' \subseteq \varDelta$ such that $var(\varDelta') \subseteq var(\varGamma)$. Since HYB_2 is a sublogic of PWK, we conclude that $\varGamma \models_{\mathsf{HYB}_1} \varDelta$ implies $\varGamma \models_{\mathsf{PWK}} \varDelta'$ for at least a non-empty $\varDelta' \subseteq \varDelta$ such that $var(\varDelta') \subseteq var(\varGamma)$.

As for the RTL direction, assume as the initial hypothesis that $\varGamma \models_{\mathsf{PWK}} \varDelta'$ for at least a non-empty $\varDelta' \subseteq \varDelta$ such that $var(\varDelta') \subseteq var(\varGamma)$. To establish $\varGamma \models_{\mathsf{HYB}_1} \varDelta'$, fix any valuation $U \in \mathrm{Hom}$ such that $v(\phi) \in \{1, n_1\}$ for every $\phi \in \varGamma'$. Our goal is to show that $v(\psi) \in \{1, n_1\}$ for some $\psi \in \varDelta$. We consider two cases:

Case (1): $v(\phi) = n_1$ for some $\phi \in \Gamma$. Fix some formula $\theta \in \Gamma$ such that $v(\theta) = n_1$. By the contaminating behavior of n_1 from Table 2, there is a $q \in var(\theta)$ such that $v(q) = n_1$. Remember that $var(\Delta') \subseteq var(\Gamma)$, and suppose that $q \in var(\Gamma) \cap var(\Delta')$. Since $var(\Delta') \subseteq var(\Gamma)$ and $v(p) \neq n_2$ for every $p \in var(\Gamma)$, we have $v(q) \neq n_2$ for every $q \in var(\Delta')$. Suppose now that $v(\phi) \in \{1, n_1\}$ for every $\phi \in \Gamma$ and $v(\psi) = 0$ for every $\psi \in \Delta'$. This implies that there is a valuation $V \in \text{Hom}$ such that $v(\phi) \in \{1, n_\}$ for every $\phi \in \Gamma$ and $v(\psi) = 0$ for every $\psi \in \Delta'$. But this contradicts the initial hypothesis that $\Gamma \models_{\mathsf{PWK}} \Delta'$.
Case (2): $v(\phi) \neq n_1$ for every $\phi \in \Gamma'$. This implies that $v(\phi) = 1$ for every $\phi \in \Gamma$, and, by the contaminating behavior of n_1, n_2 from Table 2, $v(p) = 1$ for every $p \in var(\Gamma)$. From this and $\Gamma \models_{\mathsf{CL}} \Delta'$ (which follows from the initial hypothesis $\Gamma \models_{\mathsf{PWK}} \Delta'$), we have that $v(\psi) = 1$ for some $\psi \in \Delta$, as desired.

Since these two cases are jointly exhaustive, we conclude $\Gamma \models_{\mathsf{HYB}_1} \Delta'$. From this and the Definition of $\models_{\mathsf{HYB}_1}$, it follows that $\Gamma \models_{\mathsf{HYB}_1} \Delta$. ■

Proof of Theorem 2: By Theorem 1 and Lemma 1. ■

Proof of Theorem 3: Any initial sequent $\varnothing, [\![p]\!] \Rightarrow \varnothing, [\![p]\!]$ has the form $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ in which Γ and Δ are empty and $\Gamma' = \Delta' = \{p\}$. In this case, the sequent enjoys the property that:[7]

1. $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma \cup \Gamma')$
2. $\Gamma' \subseteq \Gamma \cup \Gamma'$ and $\Delta' \subseteq \Delta \cup \Delta'$
3. The sequent $\Gamma' \Rightarrow \Delta'$ is derivable in **LK**

It can be easily checked that this property is preserved under each of the foregoing rules. The case of the Exchange and Contraction rules, and Weakening (outside the scope of the square brackets) can be noted to preserve this property, since they correspond to properties that are valid in every Tarskian logic and HYB_1 is a Tarskian logic, as every matrix logic is—see [19]. We notice that this property is preserved by the other rules as follows. Moreover, this can also be checked to apply straightforwardly to the "push" rules and the operational rules (in- and outside the square brackets). Hence, any derivable sequent enjoys the above tripartite property.

Now, we know that $\Xi \vDash_{\mathcal{M}_{\mathsf{HYB}_2}} \Theta$ if and only if there exists a $\Xi' \subseteq \Xi$ and a $\Theta' \subseteq \Theta$ such that $Var(\Xi') \subseteq Var(\Theta') \subseteq Var(\Xi)$ and $\Xi' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Theta'$. Because of soundness of **LK** (a presentation of which is described in [8]), the above tripartite property entails validity in $\mathcal{M}_{\mathsf{HYB}_2}$. Soundness of HYB_2 with respect to $\mathcal{M}_{\mathsf{HYB}_1}$ is proved by similar reasoning. ■

Proof of Lemma 2: Assume $\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_2}} \Delta$. Then by Corollary 1 for $\mathcal{M}_{\mathsf{HYB}_2}$, we know that there are $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, with $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma)$ and $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$. By completeness of **LK**, this implies that $\Gamma' \Rightarrow \Delta'$ is provable in **LK**. We also know that $Var(\Gamma') \subseteq Var(\Delta')$. Hence, by [8, Lemma 21], these two observations jointly imply that $\Gamma' \Rightarrow \Delta'$ is provable in the sequent calculus for PWK. ■

Proof of Theorem 4: Assume that $\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_2}} \Delta$. Then, by Lemma 2, there is a PWK proof of $\Gamma' \Rightarrow \Delta'$. Call this proof, *i.e.* a rooted binary tree, Π. We can

[7] As usual, this label denotes the standard sequent calculus for classical logic CL.

design an algorithm to transform a PWK proof of this sequent into an HYB_1 proof of $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$.

First, replace every node $\Xi \Rightarrow \Theta$ of Π by a node $\varnothing, [\![\Xi]\!] \Rightarrow \varnothing, [\![\Theta]\!]$. Then, place below each leaf, or axiom node, one instance of [Weak], such that from an axiom $\varnothing, [\![p]\!] \Rightarrow \varnothing, [\![p]\!]$ we infer in one step the sequent $\Gamma, [\![p]\!] \Rightarrow \Delta, [\![p]\!]$. After that, for each non-axiom node place Γ to the left of the square brackets in the antecedent and Δ to the left of the square brackets in the succedent. In the resulting proof, each PWK rule is applied within the scope of the square brackets. Moreover, we can check that every application of a PWK rule corresponds to a "bracketed rule" in HYB_1 that respects the corresponding provisos.

Actually, since Weakening is not fully admissible within the scope of square brackets, something must be said about this case. Suppose in an H proof of $\Gamma' \Rightarrow \Gamma'$ there is an *ineliminable* application of Weakening that allows to go from a node $\Xi \Rightarrow \Theta$ to a node $\Xi, \Xi' \Rightarrow \Theta, \Theta'$—whence we can legitimately call Ξ' and Θ' the active (sets of) formulas in this step. Then the current algorithm can be further specified by saying that if Π is a proof which has no ineliminable application of Weakening, then we proceed as previously stated. However, if Π *has* an ineliminable application of Weakening, then we enlarge every node (outside the square brackets) with Γ and Ξ', and Δ and Θ', in their respective sides. Finally, when the Π requires the corresponding application of Weakening, we mimic this in HYB_1 applying the [PushL] and [PushR] rules to Ξ' and Θ', as needed.

This renders a rooted binary tree Π^* with $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ as its terminal sequent. We then proceed to apply the rules [PushL], [PushR] followed by elimination of duplicate formulas in Γ' and Δ'. We end up with a HYB_1 proof ending with $\Gamma'', [\![\Gamma']\!] \Rightarrow \Delta'', [\![\Delta']\!]$, for which $\Gamma'' \cup \Gamma' = \Gamma$ and $\Delta'' \cup \Delta' = \Delta$ and $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma'' \cup \Gamma') = \Gamma$. ■

Proof of Theorem 5: By Theorem 3 and Lemma 1. ■

Proof of Theorem 6: By Theorem 4 and Lemma 1. ■

References

1. Avron, A., Konikowska, B.: Proof systems for reasoning about computation errors. Studia Logica **91**(2), 273–293 (2009)
2. Baaz, M., Fermüller, C., Zach, R.: Elimination of cuts in first-order many-valued logic. J. Inf. Process. Cybern. **29**, 333–355 (1994)
3. Barrio, E., Pailos, F., Szmuc, D.: A cartography of logics of formal inconsistency and truth (2016, manuscript)
4. Bochvar, D.: On a three-valued calculus and its application in the analysis of the paradoxes of the extended functional calculus. Matematicheskii Sbornik **4**, 287–308 (1938)
5. Bonzio, S., Gil-Ferez, J., Paoli, F., Peruzzi, L.: On paraconsistent weak Kleene logic: axiomatization and algebraic analysis. Studia Logica **105**(2), 253–297 (2017)
6. Ciuni, R., Carrara, M.: Characterizing logical consequence in paraconsistent weak Kleene. In: Felline, L., Ledda, A., Paoli, F., Rossanese, E. (eds.) New Developments in Logic and the Philosophy of Science, pp. 165–176. College Publications, London (2016)

7. Ciuni, R., Carrara, M.: Semantical analysis of weak Kleene logic (Under submission, ms)
8. Coniglio, M.E., Corbalan, M.I.: Sequent calculi for the classical fragment of Bochvar and Halldén's nonsense logic. In: Kesner, D., Petrucio, V., (eds.) Proceedings of the 7th LSFA Workshop, Electronic Proceedings in Computer Science, pp. 125–136 (2012)
9. Correia, F.: Weak necessity on weak Kleene matrices. In: Advances in Modal Logic, vol. 4 (2004)
10. Deutsch, H.: Relevant analytic entailment. Relevance Log. Newsl. **2**, 26–44 (1977)
11. Ferguson, T.M.: A computational interpretation of conceptivism. J. Appl. Non-Class. Log. **24**(4), 333–367 (2014)
12. Ferguson, T.M.: Logics of nonsense and Parry systems. J. Philos. Log. **44**(1), 65–80 (2015)
13. Font, J.M.: Abstract Algebraic Logic. College Publications, London (2016)
14. Halldén, S.: The Logic of Nonsense. Lundequista Bokhandeln, Uppsala (1949)
15. Kleene, S.: Introduction to Metamathematics. North Holland, Amsterdam (1952)
16. McCarthy, J.: A basis for a mathematical theory of computation. In: Braffort, P., Hirschberg, D. (eds.) Computer Programming and Formal Systems, pp. 33–70. North-Holland Publishing Company, Amsterdam (1963)
17. Priest, G.: In Contradiction, 2nd edn. Oxford University Press, Oxford (2006)
18. Szmuc, D.: Defining LFIs and LFUs in extensions of infectious logics. J. Appl. Non-Class. Log. **26**(4), 286–314 (2016)
19. Wójcicki, R.: Logical matrices strongly adequate for structural sentential calculi. Bulletin de l'Academie Polonaise des Sciences Série des Sciences Mathématiques Astronomiques et Physiques **17**, 333–335 (1969)

Modelling Informational Entropy

Willem Conradie[1], Andrew Craig[2(✉)], Alessandra Palmigiano[2,3], and Nachoem M. Wijnberg[2,4]

[1] University of the Witwatersrand, Johannesburg, South Africa
[2] University of Johannesburg, Johannesburg, South Africa
acraig@uj.ac.za
[3] Delft University of Technology, Delft, The Netherlands
[4] University of Amsterdam, Amsterdam, The Netherlands

Abstract. By 'informational entropy', we understand an inherent boundary to knowability, due e.g. to perceptual, theoretical, evidential or linguistic limits. In this paper, we discuss a logical framework in which this boundary is incorporated into the semantic and deductive machinery, and outline how this framework can be used to model various situations in which informational entropy arises.

Keywords: Lattice-based modal logic · Epistemic logic · Concept lattice · Graph-based semantics · Polarity-based semantics

1 Introduction

This paper contributes to a line of research stemming from the theory of canonicity and correspondence of lattice expansions [4,8,9,18], which aims at defining and studying relational semantic frameworks for lattice-based logics. The present contribution specifically builds on the *graph-based* semantics introduced in [2], on the basis of a 'modal expansion' of Ploščica's representation [23], its relationship with canonical extensions of bounded lattices [11,13], and the ensuing algebraic canonicity and correspondence results [2,9]. The resulting relational structures introduced in this paper, called *graph-based frames* (cf. Definition 2), are more general than those in [2], as the 'TiRS' conditions have been removed. Hence, rather than being characterized as discrete duals of perfect modal lattices, the graph-based structures considered here are in a discrete adjunction with complete modal lattices, much in the same way in which the class of the relational structures interpreting the same logic in [6], which are based on polarities rather than on graphs, was generalized in [7] so as to remove the 'RS' conditions. However, the notions of satisfaction and refutation of formulas at states of graph-based frames can be extracted from their interpretation on the complex algebras of graph-based frames by an analogous 'dual characterization'

The research of the third author is supported by the NWO Vidi grant 016.138.314, the NWO Aspasia grant 015.008.054, and a Delft Technology Fellowship awarded in 2013.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 140–160, 2019.
https://doi.org/10.1007/978-3-662-59533-6_9

process which the frames-to-algebras direction of the adjunction is enough to convey.

Besides this technical contribution, there is also a conceptual contribution which consists of making sense of this semantic framework in a more fundamental way. Our proposal in this respect is to use graph-based frames to provide a purely qualitative representation of the notion of *relative entropy* in information theory [24], which is a stochastic measure of *noise* in communication systems. As is argued by Weaver [24], the significance of the key notions and insights developed in information theory goes very much beyond the original "engineering aspects of communication", and invests also such aspects as meaning and knowledge. If the notion of relative entropy is construed more broadly in this way, so as to capture *conceptual noise*, then it can be understood as the inherent boundary to knowability due e.g. to perceptual, theoretical, evidential or linguistic limits. In this paper, as specific examples, we model phenomena of informational entropy (under this broader understanding) arising in natural language and visual perception. The interpretation proposed in the present paper is further pursued in [3], where informational entropy arises from the scientific theories on which empirical studies are grounded, and in [10], where it arises from socio-political theories.

Of course, the interpretation and use of graph-based structures proposed in the present paper does not exclude the possibility of other interpretations and uses, as is suggested by the fact that the 'companion' polarity-based semantics for lattice-based modal logic has been used to provide different interpretations of the lattice-based modal logic, including one in which lattice-based modal logic is viewed as an *epistemic logic of categories* [6,7] and one [5,19] in which the same logic is viewed as the *logic of rough concepts*, where polarity-based semantics is used as an encompassing framework for the integration of rough set theory [22] and formal concept analysis [17], and as a basis for further developments such as a Dempster–Shafer theory of concepts [16].

2 Preliminaries

Notation. We let Δ_U denote the identity relation on a set U, and we will drop the subscript when it causes no ambiguity. The superscript $(\cdot)^c$ denotes the relative complement of the subset of a given set. Hence, for any binary relation $R \subseteq S \times T$, we let $R^c \subseteq S \times T$ be defined by $(s,t) \in R^c$ iff $(s,t) \notin R$. For any such R and any $S' \subseteq S$ and $T' \subseteq T$, we let $R[S'] := \{t \in T \mid (s,t) \in R \text{ for some } s \in S'\}$ and $R^{-1}[T'] := \{s \in S \mid (s,t) \in R \text{ for some } t \in T'\}$, and write $R[s]$ and $R^{-1}[t]$ for $R[\{s\}]$ and $R^{-1}[\{t\}]$, respectively. Any such R gives rise to the *semantic modal operators* $\langle R\rangle, [R] : \mathcal{P}(T) \to \mathcal{P}(S)$ s.t. $\langle R\rangle W := R^{-1}[W]$ and $[R]W := (R^{-1}[W^c])^c$ for any $W \subseteq T$. For any $T \subseteq U \times V$, and any $U' \subseteq U$ and $V' \subseteq V$, let

$$T^{(1)}[U'] := \{v \mid \forall u(u \in U' \Rightarrow uTv)\} \qquad T^{(0)}[V'] := \{u \mid \forall v(v \in V' \Rightarrow uTv)\}. \tag{1}$$

Known properties of this construction (cf. [14, Sects. 7.22–7.29]) are collected below.

Lemma 1. *1. $X_1 \subseteq X_2 \subseteq U$ implies $T^{(1)}[X_2] \subseteq T^{(1)}[X_1]$, and $Y_1 \subseteq Y_2 \subseteq V$ implies $T^{(0)}[Y_2] \subseteq T^{(0)}[Y_1]$.*
2. $U' \subseteq T^{(0)}[V']$ iff $V' \subseteq T^{(1)}[U']$.
3. $U' \subseteq T^{(0)}[T^{(1)}[U']]$ and $V' \subseteq T^{(1)}[T^{(0)}[V']]$.
4. $T^{(1)}[U'] = T^{(1)}[T^{(0)}[T^{(1)}[U']]]$ and $T^{(0)}[V'] = T^{(0)}[T^{(1)}[T^{(0)}[V']]]$.
5. $T^{(0)}[\bigcup \mathcal{V}] = \bigcap_{V' \in \mathcal{V}} T^{(0)}[V']$ and $T^{(1)}[\bigcup \mathcal{U}] = \bigcap_{U' \in \mathcal{U}} T^{(1)}[U']$.

For any relation $T \subseteq U \times V$, and any $U' \subseteq U$ and $V' \subseteq V$, let

$$T^{[1]}[U'] := \{v \mid \forall u(u \in U' \Rightarrow uT^c v)\} \qquad T^{[0]}[V'] := \{u \mid \forall v(v \in V' \Rightarrow uT^c v)\}. \tag{2}$$

Hence, $T^{[1]}[U'] = (T^c)^{(1)}[U']$ and $T^{[0]}[V'] = (T^c)^{(0)}[V']$, therefore, the following lemma is an immediate consequence of Lemma 1 instantiated to $T := T^c$.

Lemma 2. *1. $X_1 \subseteq X_2 \subseteq U$ implies $T^{[1]}[X_2] \subseteq T^{[1]}[X_1]$, and $Y_1 \subseteq Y_2 \subseteq V$ implies $T^{[0]}[Y_2] \subseteq T^{[0]}[Y_1]$.*
2. $U' \subseteq T^{[0]}[V']$ iff $V' \subseteq T^{[1]}[U']$.
3. $U' \subseteq T^{[0]}[T^{[1]}[U']]$ and $V' \subseteq T^{[1]}[T^{[0]}[V']]$.
4. $T^{[1]}[U'] = T^{[1]}[T^{[0]}[T^{[1]}[U']]]$ and $T^{[0]}[V'] = T^{[0]}[T^{[1]}[T^{[0]}[V']]]$.
5. $T^{[0]}[\bigcup \mathcal{V}] = \bigcap_{V' \in \mathcal{V}} T^{[0]}[V']$ and $T^{[1]}[\bigcup \mathcal{U}] = \bigcap_{U' \in \mathcal{U}} T^{[1]}[U']$.

2.1 Basic Normal Non-distributive Modal Logic

The logic discussed below was considered in [6] as an instance of a logic to which a general methodology applies for endowing lattice-based logics with relational semantics (cf. [9, Sect. 2]). The semantics of this logic was based on a restricted class of formal contexts. These restrictions were lifted in [7].

Basic Logic. Let Prop be a (countable or finite) set of atomic propositions. The language $\mathcal{L}$ of the *basic normal non-distributive modal logic* is defined as follows:

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in \mathsf{Prop}$. The *basic*, or *minimal normal $\mathcal{L}$-logic* is a set $\mathbf{L}$ of sequents $\phi \vdash \psi$ with $\phi, \psi \in \mathcal{L}$, containing the following axioms:

$$\begin{array}{llll}
p \vdash p, & \bot \vdash p, & p \vdash \top, & \\
p \vdash p \vee q, & q \vdash p \vee q, & p \wedge q \vdash p, & p \wedge q \vdash q, \\
\top \vdash \Box\top, & \Box p \wedge \Box q \vdash \Box(p \wedge q), & \Diamond\bot \vdash \bot, & \Diamond p \vee \Diamond q \vdash \Diamond(p \vee q)
\end{array}$$

and closed under the following inference rules:

$$\frac{\phi \vdash \chi \quad \chi \vdash \psi}{\phi \vdash \psi} \quad \frac{\phi \vdash \psi}{\phi\,(\chi/p) \vdash \psi\,(\chi/p)} \quad \frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi} \quad \frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi} \quad \frac{\phi \vdash \psi}{\Box\phi \vdash \Box\psi} \quad \frac{\phi \vdash \psi}{\Diamond\phi \vdash \Diamond\psi}$$

By an *$\mathcal{L}$-logic* we understand any extension of $\mathbf{L}$ with $\mathcal{L}$-axioms $\phi \vdash \psi$.

Algebraic Semantics. The logic above is sound and complete w.r.t. the class $\mathbb{LE}$ of normal lattice expansions $\mathbb{A} = (\mathbb{L}, \Box, \Diamond)$, where $\mathbb{L} = (L, \wedge, \vee, \top, \bot)$ is a general lattice, and $\Box$ and $\Diamond$ are unary operations on $\mathbb{L}$ satisfying the following identities:

$$\Box\top = \top \quad \Box(a \wedge b) = \Box a \wedge \Box b \quad \Diamond\bot = \bot \quad \Diamond(a \vee b) = \Diamond a \vee \Diamond b.$$

In what follows, we will sometimes refer to elements of $\mathbb{LE}$ as $\mathcal{L}$-algebras. Since **L** is selfextensional (i.e. the interderivability relation is a congruence of the formula algebra), a standard Lindenbaum–Tarski construction is sufficient to show its completeness w.r.t. $\mathbb{LE}$, i.e. that an $\mathcal{L}$-sequent $\phi \vdash \psi$ is in **L** iff $\mathbb{LE} \models \phi \vdash \psi$.

3 Graph-Based Semantics for the Basic Non-distributive Modal Logic

Graph-based models for non-distributive logics arise in close connection with the topological structures dual to general lattices in Ploščica's representation [23], see also [11,13]. However, an important difference in the current paper is that we do not require the TiRS conditions [11, Sect. 2].

A *reflexive graph* is a structure $\mathbb{X} = (Z, E)$ such that Z is a nonempty set, and $E \subseteq Z \times Z$ is a reflexive relation. From now on, we will assume that all graphs we consider are reflexive even when we drop the adjective. Any graph $\mathbb{X} = (Z, E)$ defines the polarity[1] $\mathbb{P}_{\mathbb{X}} = (Z_A, Z_X, I_{E^c})$ where $Z_A = Z = Z_X$ and $I_{E^c} \subseteq Z_A \times Z_X$ is defined as $aI_{E^c}x$ iff aE^cx. More generally, any relation $R \subseteq Z \times Z$ 'lifts' to relations $I_{R^c} \subseteq Z_A \times Z_X$ and $J_{R^c} \subseteq Z_X \times Z_A$ defined as $aI_{R^c}x$ iff aR^cx and $xJ_{R^c}a$ iff xR^ca. The next lemma follows directly from these definitions:

Lemma 3. *For any relation $R \subseteq Z \times Z$ and any $Y, B \subseteq Z$,*

$$I^{(0)}_{R^c}[Y] = R^{[0]}[Y] \quad I^{(1)}_{R^c}[B] = R^{[1]}[B] \quad J^{(0)}_{R^c}[B] = R^{[0]}[B] \quad J^{(1)}_{R^c}[Y] = R^{[1]}[Y].$$

The complete lattice $\mathbb{X}^+$ associated with a graph $\mathbb{X}$ is defined as the concept lattice of $\mathbb{P}_{\mathbb{X}}$. For any lattice $\mathbb{L}$, let $\mathsf{Flt}(\mathbb{L})$ and $\mathsf{Idl}(\mathbb{L})$ denote the set of filters and

[1] A *formal context* [17], or *polarity*, is a structure $\mathbb{P} = (A, X, I)$ such that A and X are sets, and $I \subseteq A \times X$ is a binary relation. Every such $\mathbb{P}$ induces maps $(\cdot)^{\uparrow} : \mathcal{P}(A) \to \mathcal{P}(X)$ and $(\cdot)^{\downarrow} : \mathcal{P}(X) \to \mathcal{P}(A)$, respectively defined by the assignments $B^{\uparrow} := I^{(1)}[B]$ and $Y^{\downarrow} := I^{(0)}[Y]$. A *formal concept* of $\mathbb{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^{\uparrow} = Y$ and $Y^{\downarrow} = B$. Given a formal concept $c = (B, Y)$ we will often write $[\![c]\!]$ for B and $(\![c]\!)$ for Y and, consequently, $c = ([\![c]\!], (\![c]\!))$. The set $L(\mathbb{P})$ of the formal concepts of $\mathbb{P}$ can be partially ordered as follows: for any $c = (B_1, Y_1), d = (B_2, Y_2) \in L(\mathbb{P})$,

$$c \leq d \quad \text{iff} \quad B_1 \subseteq B_2 \quad \text{iff} \quad Y_2 \subseteq Y_1.$$

With this order, $L(\mathbb{P})$ is a complete lattice, the *concept lattice* $\mathbb{P}^+$ of $\mathbb{P}$. Any complete lattice $\mathbb{L}$ is isomorphic to the concept lattice $\mathbb{P}^+$ of some polarity $\mathbb{P}$.

ideals of $\mathbb{L}$, respectively. The graph associated with $\mathbb{L}$ is $\mathbb{X}_{\mathbb{L}} := (Z, E)$ where Z is the set of tuples $(F, J) \in \mathsf{Flt}(\mathbb{L}) \times \mathsf{Idl}(\mathbb{L})$ such that $F \cap J = \varnothing$. For $z \in Z$, we denote by F_z the filter part of z and by J_z the ideal part of z. Clearly, filter parts and ideal parts of states of $\mathbb{X}_{\mathbb{L}}$ must be proper. The (reflexive) E relation is defined by zEz' if and only if $F_z \cap J_{z'} = \emptyset$.

Definition 1. *[18, Sect. 2] Let $\mathbb{L}$ be a (bounded) sublattice of a complete lattice $\mathbb{L}'$.*

1. *$\mathbb{L}$ is* dense *in $\mathbb{L}'$ if every element of $\mathbb{L}'$ can be expressed both as a join of meets and as a meet of joins of elements from $\mathbb{L}$.*
2. *$\mathbb{L}$ is* compact *in $\mathbb{L}'$ if, for all $S, T \subseteq L$, if $\bigvee S \leq \bigwedge T$ then $\bigvee S' \leq \bigwedge T'$ for some finite $S' \subseteq S$ and $T' \subseteq T$.*
3. *The* canonical extension *of a lattice $\mathbb{L}$ is a complete lattice $\mathbb{L}^\delta$ containing $\mathbb{L}$ as a dense and compact sublattice.*

The canonical extension of any bounded lattice exists [18, Proposition 2.6] and is unique up to isomorphism [18, Proposition 2.7].

Proposition 1. *[12, Proposition 4.2] For any lattice $\mathbb{L}$, the complete lattice ${\mathbb{X}_{\mathbb{L}}}^+$ is the canonical extension of $\mathbb{L}$.*

Furthermore, from results in [18, Sects. 5 and 6], we know that if $\mathbb{A} = (\mathbb{L}, \Box, \Diamond)$ is an $\mathcal{L}$-algebra, then the additional operations can be extended to ${\mathbb{X}_{\mathbb{L}}}^+$ in order to get a complete $\mathcal{L}$-algebra.

Definition 2. *A* graph-based $\mathcal{L}$-frame *is a structure $\mathbb{F} = (\mathbb{X}, R_\Diamond, R_\Box)$ where $\mathbb{X} = (Z, E)$ is a reflexive graph,*[2] *and $R_\Diamond$ and $R_\Box$ are binary relations on Z satisfying the following* E-compatibility *conditions (notation defined in (2)): for all $b, y \in Z$,*

$$(R_\Box^{[0]}[y])^{[10]} \subseteq R_\Box^{[0]}[y] \qquad\qquad (R_\Box^{[1]}[b])^{[01]} \subseteq R_\Box^{[1]}[b]$$
$$(R_\Diamond^{[0]}[b])^{[10]} \subseteq R_\Diamond^{[0]}[b] \qquad\qquad (R_\Diamond^{[1]}[y])^{[01]} \subseteq R_\Diamond^{[1]}[y].$$

The complex algebra *of a graph-based $\mathcal{L}$-frame $\mathbb{F} = (\mathbb{X}, R_\Diamond, R_\Box)$ is the complete $\mathcal{L}$-algebra $\mathbb{F}^+ = (\mathbb{X}^+, [R_\Box], \langle R_\Diamond\rangle)$, where $\mathbb{X}^+$ is the concept lattice of $\mathbb{P}_\mathbb{X}$, and $[R_\Box]$ and $\langle R_\Diamond\rangle$ are unary operations on $\mathbb{P}^+_\mathbb{X}$ defined as follows: for every $c = ([\![c]\!], (\![c]\!)) \in \mathbb{P}^+_\mathbb{X}$,*

$$[R_\Box]c := (R_\Box^{[0]}[(\![c]\!)], (R_\Box^{[0]}[(\![c]\!)])^{[1]}) \quad \textit{and} \quad \langle R_\Diamond\rangle c := ((R_\Diamond^{[0]}[[\![c]\!]])^{[0]}, R_\Diamond^{[0]}[[\![c]\!]]).$$

[2] Applying the notation (2) to a graph-based $\mathcal{L}$-frame $\mathbb{F}$, we will sometimes abbreviate $E^{[0]}[Y]$ and $E^{[1]}[B]$ as $Y^{[0]}$ and $B^{[1]}$, respectively, for each $Y, B \subseteq Z$. If $Y = \{y\}$ and $B = \{b\}$, we write $y^{[0]}$ and $b^{[1]}$ for $\{y\}^{[0]}$ and $\{b\}^{[1]}$, and write $Y^{[01]}$ and $B^{[10]}$ for $(Y^{[0]})^{[1]}$ and $(B^{[1]})^{[0]}$, respectively. Notice that, by Lemma 3, $Y^{[0]} = I_{E^c}^{(0)}[Y] = Y^{\downarrow}$ and $B^{[1]} = I_{E^c}^{(1)}[B] = B^{\uparrow}$, where the maps $(\cdot)^{\downarrow}$ and $(\cdot)^{\uparrow}$ are those associated with the polarity $\mathbb{P}_\mathbb{X}$.

The following lemma is an immediate consequence of Lemma 9 in the appendix, using Lemma 3 and the observation in Footnote 2.

Lemma 4. *1. The following are equivalent for every graph (Z, E) and every relation $R \subseteq Z \times Z$:*
(i) $(R^{[0]}[y])^{[10]} \subseteq R^{[0]}[y]$ for every $y \in Z$;
(ii) $(R^{[0]}[Y])^{[10]} \subseteq R^{[0]}[Y]$ for every $Y \subseteq Z$;
(iii) $R^{[1]}[B] = R^{[1]}[B^{[10]}]$ for every $B \subseteq Z$.
2. The following are equivalent for every graph (Z, E) and every relation $R \subseteq Z \times Z$:
(i) $(R^{[1]}[b])^{[01]} \subseteq R^{[1]}[b]$ for every $b \in Z$;
(ii) $(R^{[1]}[B])^{[01]} \subseteq R^{[1]}[B]$ for every $B \subseteq Z$;
(iii) $R^{[0]}[Y] = R^{[0]}[Y^{[01]}]$ for every $Y \subseteq Z$.

For any graph-based $\mathcal{L}$-frame $\mathbb{F}$, let us define $R_\blacklozenge \subseteq Z \times Z$ by $xR_\blacklozenge a$ iff $aR_\Box x$, and $R_\blacksquare \subseteq Z \times Z$ by $aR_\blacksquare x$ iff $xR_\Diamond a$. Hence, for every $B, Y \subseteq Z$,

$$R_\blacklozenge^{[0]}[B] = R_\Box^{[1]}[B] \quad R_\blacklozenge^{[1]}[Y] = R_\Box^{[0]}[Y] \quad R_\blacksquare^{[0]}[Y] = R_\Diamond^{[1]}[Y] \quad R_\blacksquare^{[1]}[B] = R_\Diamond^{[0]}[B]. \tag{3}$$

By Lemma 4, the E-compatibility of $R_\Box$ and $R_\Diamond$ guarantees that the operations $[R_\Box], \langle R_\Diamond \rangle$ (as well as $[R_\blacksquare]$, $\langle R_\blacklozenge \rangle$) are well defined on $\mathbb{X}^+$.

Lemma 5. *Let $\mathbb{F} = (\mathbb{X}, R_\Box, R_\Diamond)$ be a graph-based $\mathcal{L}$-frame. Then the algebra $\mathbb{F}^+ = (\mathbb{X}^+, [R_\Box], \langle R_\Diamond \rangle)$ is a complete lattice expansion such that $[R_\Box]$ is completely meet-preserving and $\langle R_\Diamond \rangle$ is completely join-preserving.*

Proof. As mentioned above, the E-compatibility of $R_\Box$ and $R_\Diamond$ guarantees that the maps $[R_\Box], \langle R_\Diamond \rangle, [R_\blacksquare]$, $\langle R_\blacklozenge \rangle$: $\mathbb{X}^+ \to \mathbb{X}^+$ are well defined. Since $\mathbb{X}^+$ is a complete lattice, by [14, Proposition 7.31], to show that $[R_\Box]$ is completely meet-preserving and $\langle R_\Diamond \rangle$ is completely join-preserving, it is enough to show that $\langle R_\blacklozenge \rangle$ is the left adjoint of $[R_\Box]$ and $[R_\blacksquare]$ is the right adjoint of $\langle R_\Diamond \rangle$. For any $c = ([\![c]\!], (\![c]\!)), d = ([\![d]\!], (\![d]\!)) \in \mathbb{X}^+$,

$$\begin{array}{lll} \langle R_\blacklozenge \rangle c \leq d & \text{iff } (\![d]\!) \subseteq R_\blacklozenge^{[0]}[[\![c]\!]] & \text{ordering of concepts} \\ & \text{iff } (\![d]\!) \subseteq R_\Box^{[1]}[[\![c]\!]] & (3) \\ & \text{iff } [\![c]\!] \subseteq R_\Box^{[0]}[(\![d]\!)] & \text{Lemma 2.2} \\ & \text{iff } c \leq [R_\Box]d. & \text{ordering of concepts} \end{array}$$

Likewise, one shows that $[R_\blacksquare]$ is the right adjoint of $\langle R_\Diamond \rangle$.

For an $\mathcal{L}$-algebra $\mathbb{L}$ and $K \subseteq L$, we let

$$\Box K = \{\Box u \mid u \in K\} \qquad \text{and} \qquad \Diamond K = \{\Diamond v \mid v \in K\}.$$

Further, for $K \subseteq L$, we denote by $\lceil K \rceil$ ($\lfloor K \rfloor$) the ideal (filter) generated by K.

Lemma 6. *Let $\mathbb{L}$ be an $\mathcal{L}$-algebra with $F \in \mathsf{Flt}(\mathbb{L})$ and $J \in \mathsf{Idl}(\mathbb{L})$. Then*

1. $F \cap \Box J \neq \emptyset$ if and only if $F \cap \lceil \Box J \rceil \neq \emptyset$;
2. $\Diamond F \cap J \neq \emptyset$ if and only if $\lfloor \Diamond F \rfloor \cap J \neq \emptyset$.

Proof. Let us prove item 1. The left-to-right direction is immediate since $\Box J \subseteq \lceil \Box J \rceil$. Conversely, assume that there are elements $u_1, \ldots, u_n \in J$ such that $\Box u_1 \vee \cdot \vee \Box u_n \in F$. Because $\Box$ is monotone and F is upward closed, then $\Box(u_1 \vee \ldots \vee u_n) \in F$. Because $u_1, \ldots, u_n \in J$ and J is an ideal, then $u_1 \vee \cdot \vee u_n \in J$, which completes the proof that $F \cap \Box J \neq \emptyset$. The proof of item 2 is similar and omitted.

Definition 3. *Given a complete $\mathcal{L}$-algebra $\mathbb{A} = (\mathbb{L}, \Box, \Diamond)$ we define its associated $\mathcal{L}$-frame to be the structure $\mathbb{F}_{\mathbb{A}} = (\mathbb{X}_{\mathbb{L}}, R_\Box, R_\Diamond)$ where $R_\Box, R_\Diamond \subseteq Z \times Z$ are given by $xR_\Box y$ iff $F_x \cap \Box J_y = \emptyset$ and $xR_\Diamond y$ iff $J_x \cap \Diamond F_y = \emptyset$.*

Proposition 2. *For any $\mathcal{L}$-algebra $\mathbb{A}$, the associated $\mathcal{L}$-frame $\mathbb{F}_{\mathbb{A}}$ is a graph-based $\mathcal{L}$-frame.*

Proof. We show that $(R_\Box^{[0]}[y])^{[10]} \subseteq R_\Box^{[0]}[y]$. The other three properties will follow by similar arguments. With the help of Lemma 6(1), we observe that

$$R_\Box^{[0]}[y] = \{u \in Z \mid (u, y) \notin R_\Box\} = \{u \in Z \mid F_u \cap \Box J_y \neq \emptyset\} = \{u \in Z \mid F_u \cap \lceil \Box J_y \rceil \neq \emptyset\}.$$

We have

$$\begin{aligned}(R_\Box^{[0]}[y])^{[1]} &= \{z \in Z \mid \forall u(u \in R_\Box^{[0]}[y] \Rightarrow (u, z) \notin E)\}\\ &= \{z \in Z \mid \forall u(F_u \cap \Box J_y \neq \emptyset \Rightarrow F_u \cap J_z \neq \emptyset)\}\\ &= \{z \in Z \mid \Box J_y \subseteq J_z\}.\end{aligned}$$

Hence

$$\begin{aligned} a \in (R_\Box^{[0]}[y])^{[10]} &\text{ iff } \forall z \in Z(\Box J_y \subseteq J_z \Rightarrow (a, z) \notin E)\\ &\text{ iff } \forall z \in Z(\Box J_y \subseteq J_z \Rightarrow F_a \cap J_z \neq \emptyset)\\ &\text{ iff } F_a \cap \lceil \Box J_y \rceil \neq \emptyset\\ &\text{ iff } F_a \cap \Box J_y \neq \emptyset && \text{Lemma 6}\\ &\text{ iff } a \in R_\Box^{[0]}[y].\end{aligned}$$

Definition 4. *A* graph-based $\mathcal{L}$-model *is a tuple $\mathbb{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a graph-based $\mathcal{L}$-frame and $V : \mathsf{Prop} \to \mathbb{F}^+$. Since $V(p)$ is therefore a formal concept, we will write $V(p) = ([\![p]\!], (\![p]\!))$.*

For every graph-based $\mathcal{L}$-model $\mathbb{M} = (\mathbb{F}, V)$, the valuation V can be extended compositionally to all $\mathcal{L}$-formulas as follows:

$$\begin{array}{rlrl} V(p) &= ([\![p]\!], (\![p]\!)) & & \\ V(\top) &= (Z, \emptyset) & V(\bot) &= (\emptyset, Z)\\ V(\phi \wedge \psi) &= ([\![\phi]\!] \cap [\![\psi]\!], ([\![\phi]\!] \cap [\![\psi]\!])^{[1]}) & V(\phi \vee \psi) &= (((\![\phi]\!) \cap (\![\psi]\!))^{[0]}, (\![\phi]\!) \cap (\![\psi]\!))\\ V(\Box\phi) &= (R_\Box^{[0]}[(\![\phi]\!)], (R_\Box^{[0]}[(\![\phi]\!)])^{[1]}) & V(\Diamond\phi) &= ((R_\Diamond^{[0]}[[\![\phi]\!]])^{[0]}, R_\Diamond^{[0]}[[\![\phi]\!]]) \end{array}$$

and moreover, the existence of the adjoints of $[R_\Box]$ and $\langle R_\Diamond \rangle$ supports the interpretation of the following expansion:

$$V(\blacksquare\phi) = (R_\blacksquare^{[0]}[(\![\phi]\!)], (R_\blacksquare^{[0]}[(\![\phi]\!)])^{[1]}) \qquad V(\blacklozenge\phi) = ((R_\blacklozenge^{[0]}[[\![\phi]\!]])^{[0]}, R_\blacklozenge^{[0]}[[\![\phi]\!]])$$

Spelling out the definition above (cf. [2]), we can define the satisfaction and co-satisfaction relations $\mathbb{M}, z \Vdash \phi$ and $\mathbb{M}, z \succ \phi$ for every graph-based $\mathcal{L}$-model $\mathbb{M} = (\mathbb{F}, V)$, $z \in Z$, and any $\mathcal{L}$-formula ϕ, by the following simultaneous recursion:

$\mathbb{M}, z \Vdash \bot$	never	$\mathbb{M}, z \succ \bot$	always
$\mathbb{M}, z \Vdash \top$	always	$\mathbb{M}, z \succ \top$	never
$\mathbb{M}, z \Vdash p$	iff $z \in [\![p]\!]$	$\mathbb{M}, z \succ p$	iff $\forall z'[z'Ez \Rightarrow z' \not\Vdash p]$
$\mathbb{M}, z \succ \phi \vee \psi$	iff $\mathbb{M}, z \succ \phi$ and $\mathbb{M}, z \succ \psi$	$\mathbb{M}, z \Vdash \phi \vee \psi$	iff $\forall z'[zEz' \Rightarrow \mathbb{M}, z' \not\succ \phi \vee \psi]$
$\mathbb{M}, z \Vdash \phi \wedge \psi$	iff $\mathbb{M}, z \Vdash \phi$ and $\mathbb{M}, z \Vdash \psi$	$\mathbb{M}, z \succ \phi \wedge \psi$	iff $\forall z'[z'Ez \Rightarrow \mathbb{M}, z' \not\Vdash \phi \wedge \psi]$
$\mathbb{M}, z \succ \Diamond\phi$	iff $\forall z'[zR_\Diamond z' \Rightarrow \mathbb{M}, z' \not\Vdash \phi]$	$\mathbb{M}, z \Vdash \Diamond\phi$	iff $\forall z'[zEz' \Rightarrow \mathbb{M}, z' \not\succ \Diamond\phi]$
$\mathbb{M}, z \Vdash \Box\psi$	iff $\forall z'[zR_\Box z' \Rightarrow \mathbb{M}, z' \not\succ \psi]$	$\mathbb{M}, z \succ \Box\psi$	iff $\forall z'[z'Ez \Rightarrow \mathbb{M}, z' \not\Vdash \Box\psi]$

An $\mathcal{L}$-sequent $\phi \vdash \psi$ is *true* in $\mathbb{M}$, denoted $\mathbb{M} \models \phi \vdash \psi$, if for all $z, z' \in Z$, if $\mathbb{M}, z \Vdash \phi$ and $\mathbb{M}, z' \succ \psi$ then $zE^c z'$. An $\mathcal{L}$-sequent $\phi \vdash \psi$ is *valid* in $\mathbb{F}$, denoted $\mathbb{F} \models \phi \vdash \psi$, if it is true in every model based on $\mathbb{F}$.

The next lemma follows immediately from the definition of an $\mathcal{L}$-sequent being true in a graph-based $\mathcal{L}$-model.

Lemma 7. *Let $\mathbb{F}$ be a graph-based $\mathcal{L}$-frame and $\phi \vdash \psi$ an $\mathcal{L}$-sequent. Then $\mathbb{F} \models \phi \vdash \psi$ iff $\mathbb{F}^+ \models \phi \vdash \psi$.*

The next proposition follows from the fact that **L** is sound and complete with respect to the class of $\mathcal{L}$-algebras and Lemma 7.

Proposition 3. *The basic non-distributive modal logic* **L** *is sound w.r.t. the class of graph-based $\mathcal{L}$-frames. I.e., if an $\mathcal{L}$-sequent $\phi \vdash \psi$ is provable in* **L**, *then $\mathbb{F} \models \phi \vdash \psi$ for every graph-based frame $\mathbb{F}$.*

Let $\mathbb{A}_{\mathbf{L}}$ be the Lindenbaum–Tarski algebra of **L**. We will abuse notation and write ϕ instead $[\phi]$ (i.e. formulas instead of their equivalence classes) for the elements of the Lindenbaum–Tarski algebra $\mathbb{A}_{\mathbf{L}}$. Define the *canonical graph-based model* to be $\mathbb{M}_{\mathbf{L}} = (\mathbb{F}_{\mathbb{A}_{\mathbf{L}}}, V)$ where $V(p) = (\{z \in Z \mid p \in F_z\}, \{z \in Z \mid p \in J_z\})$. By Proposition 2, $\mathbb{F}_{\mathbb{A}_{\mathbf{L}}}$ is a graph-based $\mathcal{L}$-frame. That V is well defined can be shown as follows:

$$\begin{aligned}(\{z \in Z \mid p \in F_z\})^{[1]} &= \{z \in Z \mid \forall z'(p \in F_z \Rightarrow (z, z') \notin E)\} \\ &= \{z \in Z \mid \forall z'(p \in F_z \Rightarrow F_z \cap J_{z'} \neq \emptyset)\} \\ &= \{z \in Z \mid p \in J_z\}\end{aligned}$$

Lemma 8. *Let $\phi \in \mathcal{L}$. Then*

1. $\mathbb{M}_{\mathbf{L}}, z \Vdash \phi$ *iff* $\phi \in F_z$
2. $\mathbb{M}_{\mathbf{L}}, z \succ \phi$ *iff* $\phi \in J_z$.

Proof. Let us show item 1 under the additional assumption that ϕ is a theorem of **L** (i.e. **L** derives $\top \vdash \phi$). Then ϕ belongs to every filter, hence to show the required equivalence, we need to show that $[\![\phi]\!]_{\mathbb{M}_{\mathbf{L}}} = Z$. If **L** derives $\top \vdash \phi$, then, by soundness, $\mathbb{M}_{\mathbf{L}} \models \top \vdash \phi$. Then for every state z in $\mathbb{M}_{\mathbf{L}}$, we have $\mathbb{M}_{\mathbf{L}}, z \not\succ \phi$. Indeed, suppose for contradiction that $\mathbb{M}_{\mathbf{L}}, z \succ \phi$ for some state z. Since $\mathbb{M}_{\mathbf{L}}, z \Vdash \top$, then by spelling out the definition of satisfaction of a sequent in a model in the instance $\mathbb{M}_{\mathbf{L}} \models \top \vdash \phi$, we would conclude that $(z, z) \notin E$, i.e. E is not reflexive, which contradicts the fact that E is reflexive by construction. This finishes the proof that if $\mathbf{L}$ derives $\top \vdash \phi$, then $(\![\phi]\!)_{\mathbb{M}_{\mathbf{L}}} = \emptyset$. Hence, $[\![\phi]\!]_{\mathbb{M}_{\mathbf{L}}} = ((\![\phi]\!)_{\mathbb{M}_{\mathbf{L}}})^{[1]} = \emptyset^{[1]} = Z$, as required.

Likewise, one can show item 2 of the lemma under the additional assumption that $\mathbf{L}$ derives $\phi \vdash \bot$.

Now, assuming that $\mathbf{L}$ derives neither $\top \vdash \phi$ nor $\phi \vdash \bot$, we proceed by induction on ϕ. The base cases are straightforward. Consider $\phi = \alpha \vee \beta$. Now

$$\begin{array}{lll} \mathbb{M}_{\mathbf{L}}, z \Vdash \alpha \vee \beta & \text{iff } \forall z' \in Z[zEz' \Rightarrow \mathbb{M}_{\mathbf{L}}, z' \not\succ \alpha \vee \beta] & \\ & \text{iff } \forall z' \in Z[zEz' \Rightarrow (\mathbb{M}_{\mathbf{L}}, z' \not\succ \alpha \text{ or } \mathbb{M}_{\mathbf{L}}, z' \not\succ \beta)] & \\ & \text{iff } \forall z' \in Z[F_z \cap J_{z'} = \emptyset \Rightarrow (\alpha \notin J_{z'} \text{ or } \beta \notin J_{z'})] & \text{inductive hypothesis} \\ & \text{iff } \forall z' \in Z[F_z \cap J_{z'} \neq \emptyset \text{ or } (\alpha \notin J_{z'} \text{ or } \beta \notin J_{z'})]. & \end{array}$$

Consider $z' \in Z$ defined by $z' = (\lfloor \top \rfloor, \lceil (\alpha \vee \beta) \rceil)$, where $\lfloor \top \rfloor$ and $\lceil (\alpha \vee \beta) \rceil$ denote, respectively, the filter generated by $\top$ and the ideal generated by $\alpha \vee \beta$. The state z' is indeed well-defined since by assumption $(\alpha \vee \beta) \notin \lfloor \top \rfloor$. Moreover, since $\top \not\vdash \alpha \vee \beta$, this filter and ideal are disjoint. Clearly $\alpha \in J_{z'}$ and $\beta \in J_{z'}$ so we must have $F_z \cap \lceil (\alpha \vee \beta) \rceil \neq \emptyset$ so $\alpha \vee \beta \in F_z$. Conversely, suppose $\alpha \vee \beta \in F_z$ and consider $z' \in Z$ with zEz'. Then $F_z \cap J_{z'} = \emptyset$ so $\alpha \vee \beta \notin J_{z'}$ and since this is a down-set we have $\alpha \notin J_{z'}$ and $\beta \notin J_{z'}$ and by the inductive hypothesis we have $\mathbb{M}_{\mathbf{L}}, z' \not\succ \alpha$ and $\mathbb{M}_{\mathbf{L}}, z' \not\succ \beta$.

The proof that $\mathbb{M}_{\mathbf{L}}, z \succ \alpha \vee \beta$ iff $\alpha \vee \beta \in J_z$ follows easily from the fact that J_z is an ideal. The proof of $\phi = \alpha \wedge \beta$ is similar to $\phi = \alpha \vee \beta$ but with the role of $\Vdash$ and $\succ$ interchanged.

Now consider $\phi = \Box\psi$ and assume that $\mathbb{M}_{\mathbf{L}}, z \Vdash \Box\psi$. We have

$$\begin{array}{lll} \mathbb{M}_{\mathbf{L}}, z \Vdash \Box\psi & \text{iff } \forall z' \in Z[zR_{\Box}z' \Rightarrow \mathbb{M}_{\mathbf{L}}, z' \not\succ \psi] & \\ & \text{iff } \forall z' \in Z[zR_{\Box}z' \Rightarrow \psi \notin J_{z'}] & \text{inductive hypothesis} \\ & \text{iff } \forall z' \in Z[F_z \cap \Box J_{z'} = \emptyset \Rightarrow \psi \notin J_{z'}] & \\ & \text{iff } \forall z' \in Z[\psi \in J_{z'} \Rightarrow F_z \cap \Box J_{z'} \neq \emptyset] & \end{array}$$

Consider $z' = (\lfloor \top \rfloor, \lceil \psi \rceil)$. Clearly $\psi \in J_{z'}$ so there exists $\alpha \in F_z \cap \Box J_{z'}$. Now $\alpha = \Box\beta$ for some $\beta \leq \psi$ (in the lattice order of $\mathbb{A}_{\mathbf{L}}$), i.e. $\beta \vdash \psi$ and therefore $\Box\beta \vdash \Box\psi$, whence $\Box\psi \in F_z$. For the converse, if $\Box\psi \in F_z$ then clearly the statement $\forall z' \in Z[F_z \cap \Box J_{z'} = \emptyset \Rightarrow \psi \notin J_{z'}]$ is true and so $\mathbb{M}_{\mathbf{L}}, z \Vdash \Box\psi$. Now

$$\begin{array}{lll} \mathbb{M}_{\mathbf{L}}, z \succ \Box\psi & \text{iff } \forall z' \in Z[z'Ez \Rightarrow \mathbb{M}_{\mathbf{L}}, z' \not\Vdash \Box\psi] & \\ & \text{iff } \forall z' \in Z[z'Ez \Rightarrow \Box\psi \notin F_{z'}] & \text{from above} \\ & \text{iff } \forall z' \in Z[F_{z'} \cap J_z = \emptyset \Rightarrow \Box\psi \notin F_{z'}] & \\ & \text{iff } \forall z' \in Z[\Box\psi \in F_{z'} \Rightarrow F_{z'} \cap J_z \neq \emptyset] & \\ & \text{iff } \Box\psi \in J_z. & \end{array}$$

The forward implication of the last equivalence follows by taking $z' = (\lfloor \Box\psi \rfloor, \lceil \bot \rceil)$.

The case of $\phi = \Diamond\psi$ follows using a similar proof to that of $\phi = \Box\psi$ except starting by first showing $\mathbb{M}_{\mathbf{L}}, z \succ \Diamond\psi$ iff $\Diamond\psi \in J_z$.

Theorem 1. *The basic non-distributive modal logic* $\mathbf{L}$ *is complete w.r.t. the class of graph-based* $\mathcal{L}$*-frames.*

Proof. Consider an $\mathcal{L}$-sequent $\phi \vdash \psi$ that is not derivable in $\mathbf{L}$. Then $\lfloor\phi\rfloor \cap \lceil\psi\rceil = \emptyset$ in the Lindenbaum-Tarski algebra. Let $z := (\lfloor\phi\rfloor, \lceil\psi\rceil)$ be the corresponding state in $\mathbb{M}_{\mathbf{L}}$ By Lemma 8 we have $\mathbb{M}, z \Vdash \phi$ and $\mathbb{M}, z \succ \psi$, but zEz. Hence $\mathbb{M} \not\models \phi \vdash \psi$.

4 Sahlqvist Correspondence on Graph-Based Frames

Parametric Notions. We find it useful to phrase the correspondence results of the present section in terms of a number of notions, parametric in E, which generalize familiar notions about sets and relations which are staples of correspondence theory in Kripke frames. The following definition will make it possible to concisely express relevant first order conditions. Properties of this definition are collected in Sect. B.

Definition 5. *For any graph $\mathbb{X} = (Z, E)$ and relations $R, S \subseteq Z \times Z$, the* E-compositions *of R and S are the relations $R \circ_E S \subseteq Z \times Z$ and $R \bullet_E S \subseteq Z \times Z$ defined as follows: for any $a, x \in Z$,*

$$x(R \circ_E S)a \quad \textit{iff} \quad \exists b(xRb \ \&\ E^{(1)}[b] \subseteq S^{(0)}[a]).$$

$$a(R \bullet_E S)x \quad \textit{iff} \quad \exists y(aRy \ \&\ E^{(0)}[y] \subseteq S^{(0)}[x]).$$

If $E = \Delta$, then $E^{(1)}[b] = E^{(0)}[b] = \{b\}$ for every $b \in Z$, and hence $(R \circ_E S)$ and $(R \bullet_E S)$ reduce both to the usual relational composition of R and S. The interpretation of E-compositions will be discussed in Sect. 5, while a number of their key properties are proven in Appendix B.

Definition 6. *For any graph $\mathbb{X} = (Z, E)$, the relation $R \subseteq Z \times Z$ is:*

E-reflexive *iff $E \subseteq R$;* sub-E *iff $R \subseteq E$;* $E_\circ$-transitive *iff $R \circ_E R \subseteq R$;* $E_\bullet$-transitive *iff $R \bullet_E R \subseteq R$.*

When $E := \Delta$, we obtain the usual reflexivity, transitivity etc.

Proposition 4. *For any graph-based $\mathcal{L}$-frame $\mathbb{F} = (\mathbb{X}, R_\Box, R_\Diamond)$,*

1. $\mathbb{F} \models \Box\phi \vdash \phi$ *iff* $E \subseteq R_\Box$ *($R_\Box$ is E-reflexive).*
2. $\mathbb{F} \models \phi \vdash \Diamond\phi$ *iff* $E \subseteq R_\blacksquare$ *($R_\Diamond$ is E-reflexive).*
3. $\mathbb{F} \models \Box\phi \vdash \Box\Box\phi$ *iff* $R_\Box \bullet_E R_\Box \subseteq R_\Box$ *($R_\Box$ is $E_\bullet$-transitive).*
4. $\mathbb{F} \models \Diamond\Diamond\phi \vdash \Diamond\phi$ *iff* $R_\Diamond \circ_E R_\Diamond \subseteq R_\Diamond$ *($R_\Diamond$ is $E_\circ$-transitive).*
5. $\mathbb{F} \models \phi \vdash \Box\phi$ *iff* $R_\Box \subseteq E$ *($R_\Box$ is sub-E).*
6. $\mathbb{F} \models \Diamond\phi \vdash \phi$ *iff* $R_\blacksquare \subseteq E$ *($R_\Diamond$ is sub-E).*

Proof. The modal principles above are all Sahlqvist (cf. [9, Definition 3.5]). Hence, they all have first-order correspondents, *both* on Kripke frames *and* on graph-based $\mathcal{L}$-frames, which can be computed e.g. via the algorithm ALBA (cf. [9, Sect. 4]). Below, we do so for the modal axiom in item 1 (for the remaining items, see Appendix C). In what follows, the variables j are interpreted as elements of the set $J := \{(a^{[10]}, a^{[1]}) \mid a \in Z\}$ which completely join-generates $\mathbb{F}^+$, and the variables m as elements of $M := \{(x^{[0]}, x^{[01]}) \mid x \in Z\}$ which completely meet-generates $\mathbb{F}^+$.

$$\begin{array}{lll}
& \forall p\ [\Box p \leq p] & \\
\text{iff} & \forall p \forall j \forall m[(j \leq \Box p\ \&\ p \leq m) \Rightarrow j \leq m] & \text{first approximation} \\
\text{iff} & \forall j \forall m[j \leq \Box m \Rightarrow j \leq m] & \text{Ackermann's Lemma} \\
\text{iff} & \forall m[\Box m \leq m] & J \text{ completely join-generates } \mathbb{F}^+
\end{array}$$

Translating the universally quantified algebraic inequality above into its concrete representation in $\mathbb{F}^+$ requires using the interpretation of m as ranging in M and the definition of $[R_\Box]$ and $[R_\blacksquare]$, as follows:

$$\begin{array}{llll}
& \forall x \in Z & R_\Box^{[0]}[x^{[01]}] \subseteq E^{[0]}[x] & \text{translation} \\
\text{iff} & \forall x \in Z & R_\Box^{[0]}[x] \subseteq E^{[0]}[x] & \text{Lemma 4 since } R_\Box \text{ is } E\text{-compatible} \\
\text{iff} & R_\Box^c \subseteq E^c & & (2) \\
\text{iff} & E \subseteq R_\Box. & &
\end{array}$$

5 Graph-Based Frames as Models of Informational Entropy

As shown in the previous sections, graph-based frames – such as those defined for the language $\mathcal{L}$ – provide a mathematically grounded semantic environment for lattice-based logics such as **L**. However, in order for this environment to 'make sense' in a more fundamental way, we need to: (a) specify how it generalizes the Kripke semantics of classical normal modal logic; (b) couple it with an extra-mathematical interpretation which simultaneously accounts for the meaning of *all* connectives, and coherently extends to the meaning of axioms and of their first order correspondents. Below, we propose a way to address these issues.

By assumption, the graphs $\mathbb{X} = (Z, E)$ on which the semantics of **L** is based are reflexive, i.e. $\Delta \subseteq E$. Hence, a good starting point to address (a) is to understand this semantics when $E = \Delta$. In this case, the polarity arising from $\mathbb{X}$ is $\mathbb{P}_\mathbb{X} = (Z_A, Z_X, I_{\Delta^c})$, and, as is well known and easy to see (cf. [5, Proposition 1]), the complete lattice $\mathbb{X}^+$ arising from $\mathbb{X}$ is (isomorphic to) the powerset algebra $\mathcal{P}(Z)$, and can be represented as a concept lattice the join-generators of which are $(a^{[10]}, a^{[1]}) = (\{a\}, \{a\}^c)$ for every $a \in Z$, and the meet generators of which are $(x^{[0]}, x^{[01]}) = (\{x\}^c, \{x\})$ for every $x \in Z$. Notice also that if $E := \Delta$, then $B^\uparrow = B^c$ and $Y^\downarrow = Y^c$ for all $B, Y \subseteq Z$. Hence, the interpretation of $\mathcal{L}$-formulas on frames based on $\mathbb{X} = (Z, \Delta)$ reduces as shown below. These computations show that indeed, when $E := \Delta$, we recover the usual Kripke-style interpretation of the logical connectives, both propositional and modal.

$$\begin{array}{rll}
V(p) = ([\![p]\!], (\![p]\!)) & = ([\![p]\!], [\![p]\!]^c) & \\
V(\top) = (Z, Z^{[1]}) & = (Z, Z^c) & \\
V(\bot) = (Z^{[0]}, Z) & = (Z^c, Z) & \\
V(\phi \wedge \psi) = ([\![\phi]\!] \cap [\![\psi]\!], ([\![\phi]\!] \cap [\![\psi]\!])^{[1]}) & = ([\![\phi]\!] \cap [\![\psi]\!], ([\![\phi]\!] \cap [\![\psi]\!])^c) & \\
V(\phi \vee \psi) = (((\![\phi]\!) \cap (\![\psi]\!))^{[0]}, (\![\phi]\!) \cap (\![\psi]\!)) & = ([\![\phi]\!] \cup [\![\psi]\!], ([\![\phi]\!] \cup [\![\psi]\!])^c) & \\
V(\Box\phi) = (R_\Box^{[0]}[(\![\phi]\!)], (R_\Box^{[0]}[(\![\phi]\!)])^{[1]}) & = ((R_\Box^{-1}[[\![\phi]\!]^c])^c, R_\Box^{-1}[[\![\phi]\!]^c]) & (*) \\
V(\Diamond\phi) = ((R_\Diamond^{[0]}[[\![\phi]\!]])^{[0]}, R_\Diamond^{[0]}[[\![\phi]\!]]) & = (R_\Diamond^{-1}[[\![\phi]\!]], (R_\Diamond^{-1}[[\![\phi]\!]])^c) & (**)
\end{array}$$

To justify the lines marked with $(*)$ and $(**)$,

$$\begin{aligned} R_\Box^{[0]}[(\![\phi]\!)] &= (R_\Box^c)^{(0)}[[\![\phi]\!]^c] \\ &= \{z \mid \forall y(y \notin [\![\phi]\!] \Rightarrow zR_\Box^c y)\} \\ &= \{z \mid \forall y(zR_\Box y \Rightarrow y \in [\![\phi]\!])\} \\ &= (\{z \mid \exists y(zR_\Box y \ \&\ y \in [\![\phi]\!]^c)\})^c \\ &= (R_\Box^{-1}[[\![\phi]\!]^c])^c \end{aligned} \qquad \begin{aligned} R_\Diamond^{[0]}[[\![\phi]\!]] &= (R_\Diamond^c)^{(0)}[[\![\phi]\!]] \\ &= \{z \mid \forall y(y \in [\![\phi]\!] \Rightarrow zR_\Diamond^c y)\} \\ &= \{z \mid \forall y(zR_\Diamond y \Rightarrow y \in [\![\phi]\!]^c)\} \\ &= (\{z \mid \exists y(zR_\Diamond y \ \&\ y \in [\![\phi]\!])\})^c \\ &= (R_\Diamond^{-1}[[\![\phi]\!]])^c \end{aligned}$$

Earlier on, we observed that the E-composition of relations reduces to the usual relational composition when $E := \Delta$, and so do the 'E-versions' of relational properties such as reflexivity and transitivity (cf. Definition 6). So, in a slogan, the graph-based interpretation of the modal operators is *classical modulo a shift* from Δ to E. In what follows we focus on this shift.

Drawing from the literature in information science and modal logic, we can regard the vertices of $\mathbb{X} = (Z, E)$ as states, and interpret zEy as 'z is *indiscernible* from y'. The reflexivity of E is the minimal property we assume of such a relation, i.e. that every state is indiscernible from itself.[3] The closure $a^{[10]}$ of any $a \in Z$ arises by first considering the set $a^{[1]}$ of all the states from which a is not indiscernible, and then the set of all the states that can be told apart from every state in $a^{[1]}$. Then clearly, a is an element of $a^{[10]}$, but this is as far as we can go: $a^{[10]}$ represents a *horizon* to the possibility of completely 'knowing' a. This horizon could be epistemic, cognitive, technological, or evidential. Hence, $E := \Delta$ represents the limit case in which $a^{[10]} = \{a\}$ for each state, i.e. there are no bounds to the 'knowability' of each state of Z.

As we saw in Definition 2, the elements of the complex algebra of a graph-based frame are tuples (B, Y) such that $Y = B^{[1]}$ and $B = Y^{[0]}$. This two-sided representation yields a corresponding interpretation of $\mathcal{L}$-formulas φ as tuples $([\![\varphi]\!], (\![\varphi]\!))$ which, as discussed above, reduce to $([\![\varphi]\!], [\![\varphi]\!]^c)$ when $E := \Delta$. Hence, formulas φ are assigned both a *satisfaction set* $[\![\varphi]\!]$ and a *refutation set* $(\![\varphi]\!)$ which, as is the case when $E := \Delta$, determine each other, i.e. $(\![\varphi]\!) = [\![\varphi]\!]^{[1]}$ and $[\![\varphi]\!] = (\![\varphi]\!)^{[0]}$. The latter identities imply that $[\![\varphi]\!]^{[10]} = [\![\varphi]\!]$ and $(\![\varphi]\!)^{[01]} = (\![\varphi]\!)$, i.e. both the satisfaction and the refutation set of any formula are *stable*. The stability requirement, which is mathematically justified by the need of defining a compositional semantics for $\mathcal{L}$, can also be understood at a more fundamental level: if E encodes an *inherent boundary* to perfect knowability (i.e. the *informational entropy* of the title), this boundary should be incorporated in the meaning of formulas which are both satisfied and refuted 'up to E', i.e. not by arbitrary subsets of the domain of the graph, but only by subsets which are preserved (i.e. faithfully translated) in the shift from Δ to E.

This is similar to the *persistency* restriction in the interpretation of formulas of intuitionistic (modal) logic. Just like the interpretation of implication changes

[3] In well-known settings (e.g. [15,22]), indiscernibility is modelled as an equivalence relation. However, transitivity will fail, for example, when zEy iff $d(z, y) < \alpha$ for some distance function d. It has been argued in the psychological literature (cf. [21, 25]) that symmetry will fail in situations where indiscernibility is understood as similarity, defined e.g. as z is similar to y iff z has all the features y has.

in the shift from classical to intuitionistic semantics, the interpretation of *disjunction* changes from classical to graph-based semantics and becomes *weaker*: the stipulation $[\![\phi \vee \psi]\!] = ((\![\phi]\!) \cap (\![\psi]\!))^{[0]}$ requires a state z to satisfy $\phi \vee \psi$ exactly when z can be told apart from any state that refutes both ϕ and ψ. All states in $[\![\phi]\!] \cup [\![\psi]\!]$ will satisfy this requirement, but more states might as well which neither satisfy ϕ nor ψ, provided that E detects their being different from every state that refutes both ϕ and ψ.

Additional relations on graphs-based frames can be regarded as encoding *subjective indiscernibility*, i.e. $zR_\Box y$ iff z is indiscernible from y according to a given agent. Under this interpretation, the stipulation $[\![\Box\phi]\!] = R_\Box^{[0]}[(\![\phi]\!)]$ requires $\Box\phi$ to be satisfied at exactly those states that the agent can tell apart from each state refuting ϕ, and the stipulation $(\![\Diamond\phi]\!) = R_\Diamond^{[0]}[[\![\phi]\!]]$ requires $\Diamond\phi$ to be refuted at exactly those states that the agent can tell apart from each state satisfying ϕ, and be satisfied at the states that can be told apart from every state in $(\![\Diamond\phi]\!)$. Hence, under the interpretation indicated above, these semantic clauses support the usual reading of $\Box\phi$ as 'the agent knows/believes ϕ' and $\Diamond\phi$ as 'the agent considers ϕ plausible'.

Finally, we illustrate, by way of examples, how this interpretation coherently extends to axioms. In Proposition 4, we show that, also on graph-based frames, well known modal axioms from classical modal logic have first-order correspondents, which are the parametrized 'E-counterparts' of the first order correspondents on Kripke frames. Interestingly, this surface similarity goes deeper, and in fact guarantees that the intended meaning of a given axiom under a given interpretation is preserved in the translation from Δ to E. As a first illustration of this phenomenon, consider the axiom $\Box\phi \vdash \phi$, which, under the epistemic reading, in classical modal logic captures the characterizing property of the *factivity* of knowledge (if the agent knows ϕ, then ϕ is true). This axiom corresponds to $E \subseteq R_\Box$ on graph-based frames (cf. Proposition 4). This condition requires that if the agent tells apart z from y, then indeed z is not indistinguishable from y. That is, the agent's assessments are correct, which *mutatis mutandis*, is exactly what factivity is about.

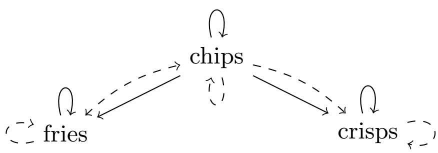
Likewise, as is well known, under the epistemic reading, axiom $\Box\phi \vdash \Box\Box\phi$ captures the so called *positive introspection* condition: knowledge of ϕ implies knowledge of knowing ϕ. This axiom corresponds to $R_\Box \bullet_E R_\Box \subseteq R_\Box$ on graph-based frames (cf. Proposition 4). This condition requires that if the agent cannot distinguish a state y from a and nothing from which y is (in principle) indistinguishable she can distinguish from x, then she cannot distinguish x from a. Equivalently, if she can distinguish x from a, then every state which she cannot distinguish from a cannot be distinguished (in principle) from some state from which she can distinguish x. This is exactly what positive introspection is about. As a third example, consider the axiom $\phi \vdash \Box\phi$, which in the epistemic logic literature is referred to as the *omniscience* principle (if ϕ is true, then the agent knows ϕ). This axiom corresponds to $R_\Box \subseteq E$ on graph-based frames (cf. Proposition 4). This condition requires the agent to tell apart z from every state y from which z is not indistinguishable, which is indeed what an omniscient agent should be able to do.

6 Sources of Informational Entropy

In this section we discuss two examples of the use graph-based models to capture situations where informational entropy arises. The first considers synonymy in natural a language while the second deals with colour perception an the limits of the human visual apparatus.

Synonymy in Natural Language. The exact nature of synonymy is debated, but there is evidence to suggest that this relation, although reflexive, can fail to be an equivalence, both on symmetry and transitivity. For example, one study [1] looks at English synonyms in an online thesaurus and finds high degree of asymmetry. For example, http://thesaurus.com lists *cushion* in the entry for *pillow*, but does not list *pillow* in the entry for *cushion*, suggesting that cushion is a synonym for pillow but not vice versa. To take another example, in a South African context, the term *chips* covers both what Americans would call *fries* and what the British would call *crisps*. A South African English speaker would thus regards *chips* as a synonym for both *fries* and *crisps*, but would regard neither *fries* nor *crisps* as synonyms for *chips*. *Chips* is by far the most commonly used word, with *fries* and *crisps* only used when disambiguation is required. This can be modelled with the graph-based frame in the figure below, where the solid arrows represent the E-relation, taken as the South African synonymity relation. As the reader can easily verify, the closed sets of this graph are exactly $\varnothing$, $\{\mathit{fries}\}$, $\{\mathit{crisps}\}$, $\{\mathit{fries}, \mathit{crisps}\}$ and $\{\mathit{crisps}, \mathit{chips}, \mathit{fries}\}$[4]. For any given word, the smallest of these sets containing it can be thought of as its 'semantic scope'. In particular, this accurately represents the fact that the words *fries* and *crisps* have unambiguous meanings while, without the benefit of context, *chips* could mean either of the others.

Now consider an American tourist trying to make sense of local usage. Having some experience with British usage, she assumes *chips* and *fries* as interchangeable terms, and say she also knows that South Africans use *chips* as a synonym for *crisps*. This epistemic situation is modelled by the dashed arrows in the figure below which define the E-compatible relation R.



We could evaluate a proposition letter p, with intended interpretation 'specific terms for fried potatoes', to $(\llbracket p \rrbracket, (\!| p |\!)) = (\{\mathit{fries}, \mathit{crisps}\}, \{\mathit{chips}\})$, which would yield $\llbracket \Box_R p \rrbracket = \{\mathit{crisps}\}$ capturing the fact that *crisps* is the only term the tourist can be sure denotes a specific kind of fried potato.

4 Notice that since the E-relation in this example is only 'one step', it is automatically transitive and therefore a pre-order. Hence, unsurprisingly, the associated concept lattice is distributive.

Perceptual Limits. The wavelength of visible light lies roughly in the rage from 380 to 780 nm. The smallest difference between wavelengths in this range which is detectable by the human eye is known as the *differentiation minimum.* The differentiation minimum varies with wavelengths and is best in the green-blue (around 490 nm) and orange (around 590 nm) spectra, where it is as low as 1 nm. It goes as high as 7 nm in the low 400 and middle 600 ranges, but averages round 4 nm over the spectrum of visible light. Deficient colour vision is characterized by significantly higher individual differentiation minima in certain ranges [20].

We model this situation using a graph-based frame. Firstly, write $[380, 780]$ for $\{x \in \mathbb{N} \mid 380 \leq x \leq 780\}$ and represent the differentiation minimum by the function $\delta : [380, 780] \to \mathbb{N}$ mapping every integer valued wavelength between 380 nm and 780 nm to the associated differentiation minimum. Represent the (possibly deficient) colour vision of an agent A by $\delta_A : [380, 780] \to \mathbb{N}$ such that $\delta_A(x) \geq \delta(x)$ for all $x \in [380, 780]$. We will make the assumption that δ has no sudden "jumps", specifically, that for all $x \in [380, 779]$, $|\delta(x) - \delta(x+1)| \leq 1$. We will assume that for all $x \in [380, 780]$, if $(x - \delta_A(x)) \geq 380$, there exists $x_\ell \in [x - \delta_A(x) + 1, x]$ such that $\delta(x_\ell) = x_\ell - (x - \delta_A(x))$ and, symmetrically, that if $(x+\delta_A(x)) \leq 780$, there exists $x_r \in [x, x+\delta_A(x)-1]$ such that $\delta(x_r) = (x+\delta_A(x))-x_r$. This assumption is needed for technical reasons. However, is justified in the case of x_ℓ (and symmetrically in the case of x_r) by the consideration that, since $x - \delta_A(x)$ is the first point to the left of x in the spectrum which agent can discern from x, there should be a point in between $x - \delta_A(x) + 1$ and x which is minimally discernible from $x - \delta_A(x)$ according to differentiation minimum (and could be x itself, if the agent's perception at this point coincides with the differentiation minimum).

Let $\mathbb{F} = (\mathbb{X}, R_\Diamond, R_\Box)$ where $\mathbb{X} = ([380, 780], E)$ such that xEy iff $|x-y| < \delta(x)$ and $xR_\Diamond y$ iff $xR_\Box y$ iff $|x - y| < \delta_A(x)$. Note that E is reflexive, but need be neither symmetric nor transitive. Using the assumptions above, one can prove that $R_\Box$ is E-compatible.

Suitable proposition letters to interpret on $\mathbb{F}$ would be colour terms like *green*, *yellow*, *orange* etc. For example, according to the standard division of the spectrum into colours, one would evaluate $[\![\text{green}]\!] = [520, 560]$, $[\![\text{yellow}]\!] = [560, 590]$ and $[\![\text{orange}]\!] = [590, 635]$. As a simplified and stylized example (but one nevertheless not too unrealistic for the range we focus on subsequently), let us take δ and δ_A to be defined as in the following table:

Interval	δ	δ_A
370–519	3	7
520–550	4	8
551–570	3	7
571–780	2	6

In this model we get $[\![\Box\text{green}]\!] = R^{[0]}[(\![\text{green}]\!)] = R^{[0]}[[370, 516] \cup [563, 780]] = [524, 556]$ which represent the range of wavelengths that the agent definitely perceives as green. On the other hand $(\!\![\Diamond\text{green}]\!\!) = R^{[1]}[[\![\text{green}]\!]] = R^{[1]}[[520, 560]] = [370, 512] \cup [567, 780]$ which is the set of wavelengths which the agent definitely perceives as *not* green. This leaves the intervals $[513, 523]$ and $[557, 568]$ where the agent cannot tell whether the corresponding colour is green or not.

7 Conclusions

The present contributions lay the ground for a number of further developments, some of which are listed below.

Parametric Sahlqvist Theory. In Proposition 4 we were able to formulate our correspondence results as parametric versions (where E is the parameter) of well known relational properties such as reflexivity and transitivity (cf. Definition 6). This phenomenon was also observed in [5, Proposition 5]. A natural question is whether these instances can be subsumed by a more general and systematic parametric Sahlqvist theory, where the generalized frame correspondent of any Sahlqvist formula would be obtainable directly as a parametrization of its classical frame correspondents.

Gödel-McKinsey-Tarski Translation. As mentioned in Sect. 5, one way of making sense of the present framework is by comparing it with the relational semantics of intuitionistic logic. In the later, the relation E is reflexive and transitive, and rather than being used to generate the semantics of modal operators on powerset algebras, it is used to generate an algebra of stable sets, namely the persistent (i.e. upward closed or downward closed) sets. Hence a natural direction is to build a non-distributive version of the transfer results induced by a suitable counterpart of Gödel-McKinsey-Tarski translation. We are presently pursuing this direction.

Many-Valued Graph-Based Semantics. In this paper, we only treat examples of informational entropy due to linguistic and perceptual limits. However, a very interesting area of application for this framework is the formal analysis of informational entropy induced by theoretical frameworks adopted to conduct scientific experiments. These situations are also amenable to be studied using a many-valued version of the present framework, which we have started to outline in [3].

A Equivalent Compatibility Conditions in Formal Contexts

Lemma 9. *1. The following are equivalent for every formal context $\mathbb{P} = (A, X, I)$ and every relation $R \subseteq A \times X$:*

(i) $R^{(0)}[x]$ is Galois-stable for every $x \in X$;
(ii) $R^{(0)}[Y]$ is Galois-stable for every $Y \subseteq X$;
(iii) $R^{(1)}[B] = R^{(1)}[B^{\uparrow\downarrow}]$ for every $B \subseteq A$.

2. *The following are equivalent for every formal context* $\mathbb{P} = (A, X, I)$ *and every relation* $R \subseteq A \times X$*:*
   (i) $R^{(1)}[a]$ *is Galois-stable for every* $a \in A$*;*
   (ii) $R^{(1)}[B]$ *is Galois-stable, for every* $B \subseteq A$*;*
   (iii) $R^{(0)}[Y] = R^{(0)}[Y^{\downarrow\uparrow}]$ *for every* $Y \subseteq X$*.*

Proof. We only prove item 1, the proof of item 2 being similar. For $(i) \Rightarrow (ii)$, see [7, Lemma 4]. The converse direction is immediate.

$(i) \Rightarrow (iii)$. Since $(\cdot)^{\uparrow\downarrow}$ is a closure operator, $B \subseteq B^{\uparrow\downarrow}$. Hence, Lemma 1.1 implies that $R_\Box^{(1)}[B^{\uparrow\downarrow}] \subseteq R_\Box^{(1)}[B]$. For the converse inclusion, let $x \in R_\Box^{(1)}[B]$. By Lemma 1.2, this is equivalent to $B \subseteq R_\Box^{(0)}[x]$. Since $R_\Box^{(0)}[x]$ is Galois-stable by assumption, this implies that $B^{\uparrow\downarrow} \subseteq R_\Box^{(0)}[x]$, i.e., again by Lemma 1.2, $x \in R_\Box^{(1)}[B^{\uparrow\downarrow}]$. This shows that $R_\Box^{(1)}[B] \subseteq R_\Box^{(1)}[B^{\uparrow\downarrow}]$, as required.

$(iii) \Rightarrow (i)$. Let $x \in X$. It is enough to show that $(R_\Box^{(0)}[x])^{\uparrow\downarrow} \subseteq R_\Box^{(0)}[x]$. By Lemma 1.2, $R_\Box^{(0)}[x] \subseteq R_\Box^{(0)}[x]$ is equivalent to $x \in R_\Box^{(1)}[R_\Box^{(0)}[x]]$. By assumption, $R_\Box^{(1)}[R_\Box^{(0)}[x]] = R_\Box^{(1)}[(R_\Box^{(0)}[x])^{\uparrow\downarrow}]$, hence $x \in R_\Box^{(1)}[(R_\Box^{(0)}[x])^{\uparrow\downarrow}]$. Again by Lemma 1.2, this is equivalent to $(R_\Box^{(0)}[x])^{\uparrow\downarrow} \subseteq R_\Box^{(0)}[x]$, as required.

B Composing Relations on Graph-Based Structures

The present section collects properties of the E-compositions (cf. Definition 5).

Lemma 10. *For any graph* $\mathbb{X} = (Z, E)$*, relations* $R, S \subseteq Z \times Z$ *and* $a, x \in Z$*,*

$$(R \circ_E S)^{[0]}[a] = R^{[0]}[E^{[0]}[S^{[0]}[a]]], \qquad (R \circ_E S)^{[1]}[x] = R^{[1]}[E^{[1]}[S^{[1]}[x]]],$$
$$(R \bullet_E S)^{[0]}[x] = R^{[0]}[E^{[1]}[S^{[0]}[x]]] \quad \text{and} \quad (R \bullet_E S)^{[1]}[a] = R^{[1]}[E^{[0]}[S^{[1]}[a]]].$$

Proof. We only prove the identities in the left column.

$$\begin{aligned}
R^{[0]}[E^{[0]}[S^{[0]}[a]]] &= R^{[0]}[E^{[0]}[\{x \mid xS^c a\}]] && \text{definition of } S^{[0]}[a]\\
&= R^{[0]}[\{b \mid \forall x(xS^c a \Rightarrow bE^c x)\}] && \text{definition of } E^{[0]}[-]\\
&= R^{[0]}[\{b \mid S^{[0]}[a] \subseteq E^{[1]}[b]\}] &&\\
&= R^{[0]}[\{b \mid E^{(1)}[b] \subseteq S^{(0)}[a]\}] && \text{Lemma 3}\\
&= \{x \mid \forall b(E^{(1)}[b] \subseteq S^{(0)}[a] \Rightarrow xR^c b)\} && \text{definition of } R^{[0]}[-]\\
&= (\{x \mid \exists b(xRb \;\&\; E^{(1)}[b] \subseteq S^{(0)}[a])\})^c &&\\
&= (\{x \mid x(R \circ_E S)a\})^c && \text{Definition 5}\\
&= \{x \mid x(R \circ_E S)^c a\} &&\\
&= (R \circ_E S)^{[0]}[a]. &&
\end{aligned}$$

$$\begin{aligned}
R^{[0]}[E^{[1]}[S^{[0]}[x]]] &= R^{[0]}[E^{[1]}[\{a \mid aS^c x\}]] && \text{definition of } S^{[0]}[x]\\
&= R^{[0]}[\{y \mid \forall a(aS^c x \Rightarrow aE^c y)\}] && \text{definition of } E^{[1]}[-]\\
&= R^{[0]}[\{y \mid S^{[0]}[x] \subseteq E^{[0]}[y]\}] &&\\
&= R^{[0]}[\{y \mid E^{(0)}[y] \subseteq S^{(0)}[x]\}] && \text{Lemma 3}\\
&= \{b \mid \forall y(E^{(0)}[y] \subseteq S^{(0)}[x] \Rightarrow bR^c y)\} && \text{definition of } R^{[0]}[-]\\
&= (\{b \mid \exists y(bRy \;\&\; E^{(0)}[y] \subseteq S^{(0)}[x])\})^c &&\\
&= (\{b \mid b(R \bullet_E S)x\})^c && \text{Definition 5}\\
&= \{b \mid b(R \bullet_E S)^c x\} &&\\
&= (R \bullet_E S)^{[0]}[x]. &&
\end{aligned}$$

Lemma 11. *If $R, T \subseteq Z \times Z$ and R is E-compatible, then so are $R \circ_E T$ and $R \bullet_E T$.*

Proof. Let $a \in Z$. By Lemma 10, $(R;T)^{[0]}[a] = R^{[0]}[I^{[0]}[T^{[0]}[a]]]$, hence the following chain of identities holds:

$$((R\,;T)^{[0]}[a])^{[01]} = (R^{[0]}[I^{[0]}[T^{[0]}[a]]])^{[01]} = R^{[0]}[I^{[0]}[T^{[0]}[a]]] = (R;T)^{[0]}[a],$$

the second identity in the chain above following from the E-compatibility of R and Lemma 4.1. The remaining conditions for the E-compatibility of $R \circ_E T$ and $R \bullet_E T$ are shown similarly.

The following lemma is the counterpart of [5, Lemma 6] in graph-based semantics.

Lemma 12. *If $R, T \subseteq Z \times Z$ are E-compatible, then for any $B, Y \subseteq Z$,*

$$(R \circ_E T)^{[1]}[Y] = R^{[1]}[E^{[1]}[T^{[1]}[Y]]] \qquad (R \circ_E T)^{[0]}[B] = R^{[0]}[E^{[0]}[T^{[0]}[B]]].$$

$$(R \bullet_E T)^{[1]}[B] = R^{[1]}[E^{[0]}[T^{[1]}[B]]] \qquad (R \bullet_E T)^{[0]}[Y] = R^{[0]}[E^{[1]}[T^{[0]}[Y]]].$$

Proof. We only prove the first identity, the remaining ones being proved similarly.

$$\begin{aligned}
R^{[1]}[E^{[1]}[T^{[1]}[Y]]] &= R^{[1]}[E^{[1]}[T^{[1]}[\textstyle\bigcup_{x\in Y}\{x\}]]] & \\
&= R^{[1]}[E^{[1]}[\textstyle\bigcap_{x\in Y} T^{[1]}[x]]] & \text{Lemma 2.5}\\
&= R^{[1]}[E^{[1]}[\textstyle\bigcap_{x\in Y} E^{[0]}[E^{[1]}[T^{[1]}[x]]]]] & T \text{ is } E\text{-compatible}\\
&= R^{[1]}[E^{[1]}[E^{[0]}[\textstyle\bigcup_{x\in Y} E^{[1]}[T^{[1]}[x]]]]] & \text{Lemma 2.5}\\
&= R^{[1]}[\textstyle\bigcup_{x\in Y} E^{[1]}[T^{[1]}[x]]] & \text{Lemma 4}\\
&= \textstyle\bigcap_{x\in Y} R^{[1]}[E^{[1]}[T^{[1]}[x]]] & \text{Lemma 2.5}\\
&= \textstyle\bigcap_{x\in Y} (R \circ_E T)^{[1]}[x] & \text{Lemma 10}\\
&= (R \circ_E T)^{[1]}[\textstyle\bigcup_{x\in Y}\{x\}] & \text{Lemma 2.5}\\
&= (R \circ_E T)^{[1]}[Y]. &
\end{aligned}$$

Lemma 13. *If $R, T, U \subseteq Z \times Z$ are E-compatible, then $(R \circ_E T) \circ_E U = R \circ_E (T \circ_E U)$ and $(R \bullet_E T) \bullet_E U = R \bullet_E (T \bullet_E U)$.*

Proof. For every $x \in Z$, repeatedly applying Lemma 12 we get:

$$\begin{aligned}
(R \circ_E (T \circ_E U))^{[1]}[x] &= R^{[1]}[E^{[1]}[(T \circ_E U)^{[1]}[x]]]\\
&= R^{[1]}[E^{[1]}[T^{[1]}[E^{[1]}[U^{[1]}[x]]]]]\\
&= (R \circ_E T)^{[1]}[E^{[1]}[U^{[1]}[x]]]\\
&= ((R \circ_E T) \circ_E U)^{[1]}[x],
\end{aligned}$$

which shows that $x(R \circ_E (T \circ_E U))^c a$ iff $x((R \circ_E T) \circ_E U)^c a$ for any $x, a \in Z$, and hence $x(R \circ_E (T \circ_E U))a$ iff $x((R \circ_E T) \circ_E U)a$ for any $x, a \in Z$, as required. The remaining statements are proven similarly.

C Proof of Proposition 4

2.

$$\begin{array}{lll}
& \forall p\ [p \leq \Diamond p] & \\
\text{iff} & \forall p \forall j \forall m[(j \leq p\ \&\ \Diamond p \leq m) \Rightarrow j \leq m] & \text{first approximation} \\
\text{iff} & \forall p \forall j \forall m[(j \leq p\ \&\ p \leq \blacksquare m) \Rightarrow j \leq m] & \text{adjunction} \\
\text{iff} & \forall j \forall m[j \leq \blacksquare m \Rightarrow j \leq m] & \text{Ackermann's Lemma} \\
\text{iff} & \forall m[\blacksquare m \leq m] & J \text{ completely join-generates } \mathbb{F}^+ \\
\text{i.e.} & \forall x \in Z \quad R_{\blacksquare}^{[0]}[x^{[01]}] \subseteq E^{[0]}[x] & \text{translation} \\
\text{iff} & \forall x \in Z \quad R_{\blacksquare}^{[0]}[x] \subseteq E^{[0]}[x] & \text{Lemma 4 since } R_{\blacksquare} \text{ is } E\text{-compatible} \\
\text{iff} & R_{\blacksquare}^c \subseteq E^c & (2) \\
\text{iff} & E \subseteq R_{\blacksquare}. &
\end{array}$$

3.

$$\begin{array}{lll}
& \forall p\ [\Box p \leq \Box\Box p] & \\
\text{iff} & \forall p \forall j \forall m[(j \leq \Box p\ \&\ p \leq m) \Rightarrow j \leq \Box\Box m] & \text{first approximation} \\
\text{iff} & \forall j \forall m[j \leq \Box m \Rightarrow j \leq \Box\Box m] & \text{Ackermann's Lemma} \\
\text{iff} & \forall m[\Box m \leq \Box\Box m] & J \text{ completely join-generates } \mathbb{F}^+ \\
\text{i.e.} & \forall x \in Z \quad R_{\Box}^{[0]}[x^{[01]}] \subseteq R_{\Box}^{[0]}[E^{[1]}[R_{\Box}^{[0]}[x^{[01]}]]] & \text{translation} \\
\text{iff} & \forall x \in Z \quad R_{\Box}^{[0]}[x] \subseteq R_{\Box}^{[0]}[E^{[1]}[R_{\Box}^{[0]}[x]]] & \text{Lemma 4 since } R_{\Box} \text{ is } E\text{-compatible} \\
\text{iff} & \forall x \in Z \quad R_{\Box}^{[0]}[x] \subseteq (R_{\Box} \bullet_E R_{\Box})^{[0]}[x] & \text{Lemma 12} \\
\text{iff} & R_{\Box}^c \subseteq (R_{\Box} \bullet_E R_{\Box})^c & (2) \\
\text{iff} & R_{\Box} \bullet_E R_{\Box} \subseteq R_{\Box}. &
\end{array}$$

4.

$$\begin{array}{lll}
& \forall p\ [\Diamond\Diamond p \leq \Diamond p] & \\
\text{iff} & \forall p \forall j \forall m[(j \leq p\ \&\ \Diamond p \leq m) \Rightarrow \Diamond\Diamond j \leq m] & \text{first approximation} \\
\text{iff} & \forall j \forall m[\Diamond j \leq m \Rightarrow \Diamond\Diamond j \leq m] & \text{Ackermann's Lemma} \\
\text{iff} & \forall j[\Diamond\Diamond j \leq \Diamond j] & M \text{ completely meet-generates } \mathbb{F}^+ \\
\text{i.e.} & \forall a \in Z \quad R_{\Diamond}^{[0]}[a^{[10]}] \subseteq R_{\Diamond}^{[0]}[(R_{\Diamond}^{[0]}[a^{[10]}])^{[0]}] & \text{translation} \\
\text{iff} & \forall a \in Z \quad R_{\Diamond}^{[0]}[a] \subseteq R_{\Diamond}^{[0]}[(R_{\Diamond}^{[0]}[a])^{[0]}] & \text{Lemma 4 since } R_{\Diamond} \text{ is } E\text{-compatible} \\
\text{iff} & \forall a \in Z \quad R_{\Diamond}^{[0]}[a] \subseteq (R_{\Diamond} \circ R_{\Diamond})^{[0]}[a] & \text{Lemma 12} \\
\text{iff} & R_{\Diamond}^c \subseteq (R_{\Diamond} \circ_E R_{\Diamond})^c & (2) \\
\text{iff} & R_{\Diamond} \circ_E R_{\Diamond} \subseteq R_{\Diamond}. &
\end{array}$$

5.

$$\begin{array}{lll}
& \forall p[p \leq \Box p] & \\
\text{iff} & \forall p \forall j \forall m[(j \leq p\ \&\ p \leq m) \Rightarrow j \leq \Box m] & \text{first approximation} \\
\text{iff} & \forall j \forall m[j \leq m \Rightarrow j \leq \Box m] & \text{Ackermann's Lemma} \\
\text{iff} & \forall m[m \leq \Box m] & J \text{ completely join-generates } \mathbb{F}^+ \\
\text{i.e.} & \forall x \in Z \quad E^{[0]}[x] \subseteq R_{\Box}^{[0]}[x^{[01]}] & \text{translation} \\
\text{iff} & \forall x \in Z \quad E^{[0]}[x] \subseteq R_{\Box}^{[0]}[x] & \text{Lemma 4 since } R_{\Box} \text{ is } E\text{-compatible} \\
\text{iff} & E^c \subseteq R_{\Box}^c & (2) \\
\text{iff} & R_{\Box} \subseteq E. &
\end{array}$$

6.

$$
\begin{array}{lll}
& \forall p\ [\Diamond p \leq p] & \\
\text{iff} & \forall p \forall j \forall m[(j \leq p\ \&\ p \leq m) \Rightarrow \Diamond j \leq m] & \text{first approximation} \\
\text{iff} & \forall j \forall m[j \leq m \Rightarrow \Diamond j \leq m] & \text{Ackermann's Lemma} \\
\text{iff} & \forall j \forall m[j \leq m \Rightarrow j \leq \blacksquare m] & \text{adjunction} \\
\text{iff} & \forall m[m \leq \blacksquare m] & J \text{ completely join-generates } \mathbb{F}^+ \\
\text{i.e.} & \forall x \in Z \quad E^{[0]}[x] \subseteq R_{\blacksquare}^{[0]}[x^{[01]}] & \text{translation} \\
\text{iff} & \forall x \in Z \quad E^{[0]}[x] \subseteq R_{\blacksquare}^{[0]}[x] & \text{Lemma 4 since } R_{\blacksquare} \text{ is } E\text{-compatible} \\
\text{iff} & E^c \subseteq R_{\blacksquare}^c & (2) \\
\text{iff} & R_{\blacksquare} \subseteq E. &
\end{array}
$$

References

1. Chodorow, M.S., Ravin, Y., Sachar, H.E.: A tool for investigating the synonymy relation in a sense disambiguated thesaurus. In: Second Conference on Applied Natural Language Processing, pp. 144–151 (1988)
2. Conradie, W., Craig, A.: Relational semantics via TiRS graphs. In: TACL 2015 Extended Abstract (2015)
3. Conradie, W., Craig, A., Palmigiano, A., Wijnberg, N.: Modelling competing theories. In: Proceedings of the EUSFLAT 2019, Atlantis Studies in Uncertainty Modelling (2019, accepted)
4. Conradie, W., Craig, A., Palmigiano, A., Zhao, Z.: Constructive canonicity for lattice-based fixed point logics. In: Kennedy, J., de Queiroz, R.J.G.B. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 92–109. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_7. ArXiv preprint arXiv:1603.06547
5. Conradie, W., et al.: Rough concepts (2019, submitted)
6. Conradie, W., Frittella, S., Palmigiano, A., Piazzai, M., Tzimoulis, A., Wijnberg, N.M.: Categories: how I learned to stop worrying and love two sorts. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 145–164. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_10
7. Conradie, W., Frittella, S., Palmigiano, A., Piazzai, M., Tzimoulis, A., Wijnberg, N.M.: Toward an epistemic-logical theory of categorization. In: Proceedings of the TARK 2017. EPTCS, vol. 251, pp. 167–186 (2017)
8. Conradie, W., Palmigiano, A.: Constructive canonicity of inductive inequalities. arXiv preprint arXiv:1603.08341 (2016)
9. Conradie, W., Palmigiano, A.: Algorithmic correspondence and canonicity for non-distributive logics. Ann. Pure Appl. Log. (2019). https://doi.org/10.1016/j.apal.2019.04.003
10. Conradie, W., Palmigiano, A., Robinson, C., Tzimoulis, A., Wijnberg, N.M.: Modelling socio-political competition (2019, submitted)
11. Craig, A., Gouveia, M., Haviar, M.: TiRS graphs and TiRS frames: a new setting for duals of canonical extensions. Algebra Universalis **74**(1–2), 123–138 (2015)
12. Craig, A., Haviar, M.: Reconciliation of approaches to the construction of canonical extensions of bounded lattices. Mathematica Slovaca **64**, 1–22 (2014)
13. Craig, A., Haviar, M., Priestley, H.A.: A fresh perspective on canonical extensions for bounded lattices. Appl. Categor. Struct. **21**(6), 725–749 (2013)
14. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order. Cambridge University Press, Cambridge (2002)

15. Fagin, R., Halpern, J., Moses, Y., Vardi, M.: Reasoning About Knowledge. The MIT Press, Cambridge (1995)
16. Frittella, S., Manoorkar, K., Palmigiano, A., Tzimoulis, A., Wijnberg, N.M.: Towards a Dempster-Shafer theory of concepts (2019, submitted)
17. Ganter, B., Wille, R.: Formal Concept Analysis: Mathematical Foundations. Springer, Heidelberg (2012)
18. Gehrke, M., Harding, J.: Bounded lattice expansions. J. Algebra **238**(1), 345–371 (2001)
19. Greco, G., Jipsen, P., Manoorkar, K., Palmigiano, A., Tzimoulis, A.: Logics for rough concept analysis. In: Khan, M.A., Manuel, A. (eds.) ICLA 2019. LNCS, vol. 11600, pp. 144–159. Springer, Heidelberg (2019). https://doi.org/10.1007/978-3-662-58771-3_14
20. Krúdy, Á., Ladunga, K.: Measuring wavelength discrimination threshold along the entire visible spectrum. Periodica Polytechnica Mechanical Engineering **45**(1), 41–48
21. Nosofsky, R.M.: Stimulus bias, asymmetric similarity, and classification. Cogn. Psychol. **23**(1), 94–140 (1991)
22. Pawlak, Z.: Rough set theory and its applications to data analysis. Cybern. Syst. **29**(7), 661–688 (1998)
23. Ploščica, M.: A natural representation of bounded lattices. Tatra Mt. Math. Publ. **5**, 75–88 (1995)
24. Shannon, C.E., Weaver, W.: The Mathematical Theory of Communication. University of Illinois Press, Champaign (1949)
25. Tversky, A.: Features of similarity. Psychol. Rev. **84**(4), 327 (1977)

Hennessy-Milner Properties for (Modal) Bi-intuitionistic Logic

Jim de Groot(✉) and Dirk Pattinson(✉)

The Australian National University, Canberra, Australia
{jim.degroot,dirk.pattinson}@anu.edu.au

Abstract. Bi-intuitionistic logic is an extension of intuitionistic propositional logic with a binary operator that is residuated with respect to disjunction. Our main result is a Hennessy-Milner property for bi-intuitionistic logic interpreted over certain classes of Kripke models. We generalise this to obtain a corresponding result for modal bi-intuitionistic logic. Our main technical tools are a categorical duality between (modal) descriptive Kripke frames and (modal) bi-Heyting algebras, and the use of behavioural equivalence.

Keywords: Bi-intuitionistic logic · Modal logic · Bisimulation · Hennessy-Milner property

1 Introduction

Bi-intuitionistic logic, also known as subtractive logic [4] and Heyting-Brouwer logic [12], is the extension of intuitionistic logic with a subtraction arrow $\prec$ which is dual to Heyting implication. It was introduced by Rauszer with Kripke semantics and a Hilbert calculus [14].

Bi-intuitionistic logic has been studied from various perspectives. From the point of view of computer science, the subtraction arrow can be used to describe control mechanisms such as co-routines [4]. In philosophy the subtraction arrow can be used to reason about refutation [15,16]. Within logic, bi-intuitionistic logic is interesting because it is a non-classical logic which is more expressive than intuitionistic logic. Its proof theory has been studied in, amongst other papers, [7,10,16].

In this paper we study bisimulations for frame semantics of bi-intuitionistic logic. We show that the class of (Kripke) models with finite connected components enjoys the Hennessy-Milner property. In fact, this follows from a stronger theorem which states that logical equivalence, bisimilarity, and behavioural equivalence coincide for the collection of so-called bi-descriptive Kripke frames.

A key ingredient in the proof of the latter is the duality between the category of bi-Heyting algebras and the category of bi-descriptive Kripke frames. This allows us to view bi-descriptive Kripke frames from a different perspective. In particular, the existence of an initial object in the category of bi-Heyting models

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 161–176, 2019.
https://doi.org/10.1007/978-3-662-59533-6_10

entails a final bi-descriptive Kripke model which is used to show that logical equivalence implies behavioural equivalence.

We then generalise these results to the extension of bi-intuitionistic logic with two unary modal operators, $\Box$ and $\Diamond$. We introduce these modal operators in a way similar to [17]. We define modal bi-Heyting algebras, modal Kripke frames and modal bisimulations, and show that logical equivalence, modal bisimilarity, and behavioural equivalence coincide on bi-descriptive modal Kripke frames.

Structure of the Paper. Section 2 reviews bi-intuitionistic logic and bi-Heyting algebras; in Sect. 3 we define bi-descriptive Kripke frames and exhibit its duality with bi-Heyting algebras as a restriction of known results from intuitionistic logic; in Sect. 4 we compare bisimilarity to logical equivalence and behavioural equivalence, and arrive at a Hennessy-Milner type theorem; and finally, in Sect. 5, we outline a similar procedure for modal bi-intuitionistic logic.

Related Work. Bisimulations for bi-intuitionistic logic were studied by Badia [1] in the context of characterising bi-intuitionistic logic as the bisimulation invariant fragment of the first order language with a distinguished binary relation symbol and a unary relation symbol for every proposition letter. The equivalence of bisimilarity and logical equivalence is not addressed. Our semantic models for modal bi-intuitionistic logic are a generalisation of the modal intuitionistic frames of [17] where again Hennessy-Milner theorems are not discussed. Results of this type for intuitionistic (not bi-intuitionistic) propositional (not modal) logic and finite frames are presented in [9].

2 Preliminaries: Bi-intuitionistic Logic

We recall bi-intuitionistic logic, bi-Heyting algebras and bi-Esakia spaces and their relationships, crucially the dual equivalence of bi-Heyting algebras and bi-Esakia spaces. Throughout, Prop is a set of proposition letters.

Definition 2.1. The language $\mathbb{L}$ of bi-intuitionistic logic is given by

$$\phi ::= p \mid \bot \mid \top \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \prec \phi$$

where $p \in \text{Prop}$. We abbreviate $\neg\phi = (\phi \to \bot)$ and $\sim\phi = (\top \prec \phi)$.

This language is equivalent to the one given by Rauszer in [13] who gave a Hilbert-style axiomatisation in [13, Sect. 9]. The algebraic structures corresponding to bi-intuitionistic logic are bi-Heyting algebras, also introduced by Rauszer, as semi-Boolean algebras [13, Sect. 1.1].

Definition 2.2. A bi-Heyting algebra is a bounded distributive lattice $(H, \bot, \top, \vee, \wedge)$ with two binary operations $\to$ and $\prec$ that satisfy

$$a \to b \leq c \quad \text{iff} \quad c \wedge a \leq b, \qquad\qquad a \prec b \leq c \quad \text{iff} \quad a \leq b \vee c.$$

A *bi-Heyting morphism* is a bounded distributive lattice morphism which preserves $\to$ and $\prec$. We write biHA for the category of bi-Heyting algebras and bi-Heyting morphisms, and usually identify an algebra with its carrier.

Example 2.3. 1. Every Boolean algebra is a bi-Heyting algebra, with $a \to b = \neg a \vee b$ and $a \prec b = a \wedge \neg b$.

2. Every complete distributive lattice that satisfies the infinite distributive laws $a \wedge \bigvee_{i\in I} b_i = \bigvee_{i \in I}(a \wedge b_i)$ and $a \vee \bigwedge_{i \in I} b_i = \bigwedge_{i\in I}(a \vee b_i)$ is a bi-Heyting algebra, with $a \to b = \bigvee\{c \mid c \wedge a \leq b\}$ and $a \prec b = \bigwedge\{c \mid a \leq b \vee c\}$.
3. An example of complete distributive lattices that satisfy the infinite distributive laws above is given by Alexandrov spaces, i.e. topological spaces where an arbitrary intersection of opens is again open. If (X, τ) is an Alexandrov space, then τ is a bi-Heyting algebra with

$$a \to b = \bigcup\{c \in \tau \mid c \cap a \subseteq b\} \quad \text{and} \quad a \prec b = \bigcap\{c \in \tau \mid a \subseteq b \cup c\}$$

where $\vee$ and $\wedge$ are given by union and intersection, respectively.

Formulae are interpreted in bi-Heyting algebras via valuations.

Definition 2.4. A *valuation* for a bi-Heyting algebra H is a map $V : \mathrm{Prop} \to H$ and a pair $\mathfrak{H} = (H, V)$ of a bi-Heyting algebra and a valuation is called a *bi-Heyting model.* A *bi-Heyting model morphism* from $\mathfrak{H} = (H, V)$ to $\mathfrak{H}' = (H', V')$ is a bi-Heyting morphism $f : H \to H'$ satisfying $V' = f \circ V$. The category of bi-Heyting models and their morphisms is denoted by biHM.

The *interpretation* of an $\mathbb{L}$-formula in $\mathfrak{H} = (H, V)$ is defined recursively by $[\![p]\!]^{\mathfrak{H}} = V(p)$ and $[\![\phi \star \psi]\!]^{\mathfrak{H}} = [\![\phi]\!]^{\mathfrak{H}} \star [\![\psi]\!]^{\mathfrak{H}}$ for $\star \in \{\vee, \wedge, \to, \prec\}$. We say that a formula ϕ is *valid* on a bi-Heyting model $\mathfrak{H}$ if $[\![\phi]\!]^{\mathfrak{H}} = \top$, notation: $\mathfrak{H} \Vdash \phi$, and ϕ is *valid* on a bi-Heyting algebra H if it is valid on (H, V) for every valuation V, notation: $H \Vdash \phi$.

Bi-Heyting morphisms preserve truth in a standard way:

Proposition 2.5. *If $f : \mathfrak{H} \to \mathfrak{H}'$ is a bi-Heyting model morphism and ϕ a formula, then $f([\![\phi]\!]^{\mathfrak{H}}) = [\![\phi]\!]^{\mathfrak{H}'}$.*

As usual, the Lindenbaum-Tarski algebra (L, V_L), consisting of formulae modulo provable equivalence in the Hilbert-axiomatisation [13, Sect. 9] is the initial model.

Proposition 2.6. *The bi-Heyting model $\mathfrak{L} = (L, V_L)$ is initial in* biHM.

Proof. For a bi-Heyting model $\mathfrak{H} = (H, V_H)$, the assignment $[p] \mapsto [\![p]\!]^{\mathfrak{H}}$ extends to a bi-Heyting model morphism $f_{\mathfrak{H}}$ which is unique by Proposition 2.5. $\square$

The equivalence between bi-Heyting algebras and bi-Esakia spaces needs the following notions about sets with binary relations.

Definition 2.7. Let (X, R) be a set with a binary relation. Then

$$\uparrow_R a = \{x \in X \mid \exists y \in a \text{ s.t. } yRx\}, \qquad \downarrow_R a = \{x \in X \mid \exists y \in a \text{ s.t. } xRy\}$$

are the upward (resp. downward) closure of a subset $a \subseteq X$. We omit the subscript R if no confusion is likely to arise.

Definition 2.8. Let (X, R) and (X', R') be two sets with arbitrary binary relations, and $f : X \to X'$ a function. Then f is called *bounded* if

(i) If xRy then $f(x)R'f(y)$;
(ii) If $f(x)R'y'$ then there exists $y \in X$ such that xRy and $f(y) = y'$.

The map f is called *bi-bounded* if moreover it satisfies

(iii) If $y'R'f(x)$ then there exists $y \in X$ such that yRx and $f(y) = y'$.

$$\begin{tikzcd}
y \arrow[r] & f(y) \\
x \arrow[u, dash, "\leq"] \arrow[r, "f"'] & f(x) \arrow[u, dash, dashed, "\leq'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[r, dashed] & y' \\
x \arrow[u, dash, dashed, "\leq"] \arrow[r, "f"'] & f(x) \arrow[u, dash, "\leq'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r] & f(x) \\
y \arrow[u, dash, dashed, "\leq"] \arrow[r, dashed, "f"'] & y' \arrow[u, dash, "\leq'"']
\end{tikzcd}$$

Fig. 1. Conditions (i), (ii) and (iii) of bi-bounded morphisms.

This now puts us into a position to define bi-Esakia spaces. As (bi-)Esakia spaces are special cases of Priestly and Esakia spaces, we recall their definitions. A *Priestly Space* is an ordered topological space (X, τ, R) where (X, τ) is a Stone space and $R \subseteq X \times X$ is a partial order that satisfies the *Priestley separation axiom*: whenever $(x, y) \notin R$ are not related, there exists a clopen upset a such that $x \in a$ and $y \notin a$.

An *Esakia Space* is a Priestley space where the relation R is additionally clopen, that is, $\downarrow_R (a)$ is clopen for every clopen subset $a \subseteq X$.

Definition 2.9 (Bi-Esakia Spaces and Morphisms). A *bi-Esakia space* is an Esakia space (X, τ, R) where R is *forward clopen*, that is, $\uparrow_R (a)$ is clopen whenever a is clopen. A *bi-Esakia morphism* is a continuous function that is bi-bounded with respect to the underlying Kripke frames.

It turns out that the Priestley separation axiom is automatic once we have a relation that is both clopen, and forward clopen.

Lemma 2.10. *Let $\boldsymbol{\mathcal{X}} = (X, \tau)$ be a Stone space and R a relation on X. Then $(\boldsymbol{\mathcal{X}}, R)$ is a bi-Esakia space if and only if R clopen and forward clopen.*

Proof. If R is forward clopen, then it is point closed, that is, $\uparrow_R (\{x\})$ is closed for all $x \in X$. This is because $R(x) = \bigcap\{R[a] \mid x \in a, a \text{ clopen}\}$, which is the intersection of clopen sets, hence closed. The result then follows from [2, Proposition 2.3.21]. $\square$

The following theorem was proven by Esakia in [6]. The duality is a restriction of Esakia duality [5], which is in turn a restriction of Priestley duality [11].

Theorem 2.11. *We have a dual equivalence of categories* $\mathsf{biHA} \equiv^{\mathrm{op}} \mathsf{biES}$.

3 Frame Semantics and Duality for Bi-intuitionistic Logic

We construct a categorical duality between (certain) frames and bi-Heyting algebras which is used to establish the connection between logical and behavioural equivalence in Sect. 4. We recall the Kripke semantics for bi-intuitionistic logic of [14, Sect. 2.8] and introduce bi-general and bi-descriptive Kripke frames (and models). The latter congregate in a category dually equivalent to biHA.

Definition 3.1. A *Kripke frame* is a poset $(X, \leq)$. A *valuation* for $(X, \leq)$ is a map $V : \mathrm{Prop} \to \mathrm{Up}(X, \leq) = \{a \subseteq X \mid a = {\uparrow}a\}$ and a tuple $(X, \leq, V)$ of a Kripke frame and a valuation is called a *Kripke model.* A *bi-bounded model morphism* from $(X, \leq, V)$ to $(X', \leq', V')$ is a bi-bounded morphism f between the underlying frames satisfying $V = f^{-1} \circ V'$. We write biKF for the category of Kripke frames and bi-bounded morphisms and biKM for the category of Kripke models and bi-bounded model morphisms.

Note that the objects of biKF are the usual intuitionistic Kripke frames [3, Sect. 2.2], but we require morphisms to be bi-bounded rather than bounded.

Definition 3.2. Define the operators $\to$ and $\prec$ on the collection $\mathrm{Up}(X, \leq)$ of upsets of a Kripke frame by

$$a \to b = \{x \in X \mid \forall y \geq x \,(y \in a \to y \in b)\} = X \setminus {\downarrow}(a \setminus b),$$
$$a \prec b = \{x \in X \mid \exists y \leq x \text{ s.t. } y \in a \text{ and } y \notin b\} = {\uparrow}(a \setminus b).$$

The *semantics* of a formula in $\mathbb{L}$ in a Kripke model $(X, \leq, V)$ is given by $[\![p]\!] = V(p)$ for propositions, $\cup$ and $\cap$ for disjunction and conjunction, and $\to$ and $\prec$ via the above operations. We call two states x and x' in two Kripke models *logically equivalent* if they satisfy precisely the same formulas, notation: $x \equiv_{\mathbb{L}} x'$.

Proposition 4.5 below states that bi-bounded morphisms preserve logical truth. The Hennessy-Milner results of the next section rely on the following notion of connected $\leq$-components. Recall that a connected component of an undirected graph G is a subgraph in which every two points in the subgraph are connected by a (finite) path, and which is not connected to any other vertices in G. For our purposes, we require that components are moreover non-empty.

Definition 3.3. A *connected* $\leq$*-component* of a Kripke frame $(X, \leq)$ is a non-empty connected component of the underlying undirected graph.

We give examples of a Kripke frame and model with infinite components. As we will see later this gives rise to an example where logical equivalence and bisimilarity disagree.

Example 3.4. Let $W = \{(n,k) \in (\mathbb{N} \cup \{\infty\}) \times \mathbb{N} \mid k < n\} \cup \{x\}$ and define an order $\preccurlyeq$ by: $(n,k) \preccurlyeq x$ for all $(n,k) \in W$ and $(n,k) \preccurlyeq (n',k')$ iff $n = n'$ and $k \leq k'$. It is easy to see that the only connected $\preccurlyeq$-component of $(W, \preccurlyeq)$ is W itself.

For Prop $= \{p_i \mid i \in \mathbb{N}\} \cup \{q\}$ define the valuation V by $V(q) = \{x\}$ and $V(p_i) = \{(n,k) \in W \mid i \leq k\} \cup \{x\}$. Then the triple $(W, \preccurlyeq, V)$ is a Kripke model with an infinite connected $\preccurlyeq$-component. See Fig. 3 below for a for a pictorial presentation.

The following lemma is a consequence of the definition of bi-bounded morphisms:

Lemma 3.5. *Let $f : (X, \leq) \to (X', \leq')$ be a bi-bounded morphism and a' a connected $\leq'$-component. Then $(\operatorname{im} f) \cap a' \neq \emptyset$ iff $a' \subseteq (\operatorname{im} f)$.*

Let $\mathfrak{M} = (X, \leq, V)$ be a Kripke model and a a connected $\leq$-component of $\mathfrak{M}$. Setting $\leq_a = \leq_{\restriction a \times a}$ and $V_a(p) = V(p) \cap a$, we obtain a sub-Kripke model $\mathfrak{M}_a = (a, \leq_a, V_a)$ generated by a. We have:

Lemma 3.6. *The restriction of the identity morphism on $\mathfrak{M}$ to a is a bi-bounded model morphism $\mathfrak{M}_a \to \mathfrak{M}$.*

We generalise Kripke frames to bi-general Kripke frames. These are Kripke frames together with a set of cones, which is closed under certain operations. They are general intuitionistic Kripke frames [3, Sect. 8.1] with an extra closure operator on the set of cones.

Definition 3.7. A *bi-general Kripke frame* is a tuple $(X, \leq, P)$ of a Kripke frame $(X, \leq)$ and a set of cones $P \subseteq \mathrm{Up}(X, \leq)$ containing $\emptyset$ and X, closed under union, intersection, and the operations $\to$ and $\prec$ from Definition 3.2. A *bi-general morphism* $f : (X, \leq, P) \to (X', \leq', P')$ is a bi-bounded morphism between the underlying Kripke frames such that $f^{-1}(a') \in P$ for all $a' \in P'$. The category of bi-general Kripke frames and morphisms is called biGKF.

An *admissible valuation* for a bi-general Kripke frame is a map $V : \mathrm{Prop} \to P$ and a pair of a bi-general Kripke frame with an admissible valuation is called a *bi-general model*. The interpretation of formulas is the same as for Kripke frames. A *bi-general model morphism* is a bi-general morphism which is also a bi-bounded model morphism.

Every Kripke frame $(X, \leq)$ can be viewed as a bi-general Kripke frame by putting $P = \mathrm{Up}(X, \leq)$. As this makes every bi-bounded morphism automatically bi-general, we have an embedding $\mathsf{biKF} \to \mathsf{biGKF}$.

To construct algebras from frames, note that the complex algebra $\mathfrak{X}^* = (P, \emptyset, X, \cup, \cap, \to, \prec)$ of a bi-general frame $\mathfrak{X} = (X, \leq, P)$ is a bi-Heyting algebra. If $f : \mathfrak{X} \to \mathfrak{X}'$ is a bi-general frame morphism, $f^* = f^{-1}$ is a bi-Heyting algebra morphism as f^{-1} preserves all unions and all intersections. There is a bijective correspondence between valuations for $\mathfrak{X}$ and valuations for $\mathfrak{X}^*$:

Lemma 3.8. *Let $\mathfrak{X} = (X, \leq, P)$ be a bi-general frame. A map $V : \mathrm{Prop} \to P$ is an admissible valuation iff it is a valuation for $\mathfrak{X}^*$.*

Conversely, if H is a bi-Heyting algebra then $H_* = (\mathrm{pf}H, \subseteq, \widetilde{H})$ is a bi-general Kripke frame where pf$H$ is the collection of prime filters of H (e.g. [2, Definition 2.2.20]), $\widetilde{H} = \{\widetilde{a} \mid a \in H\}$ and $\widetilde{a} = \{u \in \mathrm{pf}H \mid a \in u\}$. In general we have $(H_*)^* \simeq H$ for every bi-Heyting algebra H, but not $(\mathfrak{X}^*)_* \simeq \mathfrak{X}$, for every bi-general Kripke frame $\mathfrak{X}$.

To obtain a dual equivalence, we need to restrict bi-general Kripke frames.

Definition 3.9. A bi-general Kripke frame $\mathfrak{X} = (X, \leq, P)$ is called *bi-descriptive* if it is *refined*, i.e. for every $x, y \in X$, $x \not\leq y$ implies that there is $a \in P$ such that $x \in a$ and $y \notin a$; and *compact*, i.e. for every $A \subseteq P$ and $B \subseteq \{X \setminus a \mid a \in P\}$, if $A \cup B$ has the finite intersection property, then $\bigcap(A \cup B) \neq \emptyset$.

The category of bi-descriptive Kripke frames and bi-general morphisms is denoted by biDKF. A bi-descriptive model is a bi-general model whose underlying bi-general frame is bi-descriptive. The category of bi-descriptive Kripke models and bi-general model morphisms is denoted by biDKM.

In other words: a bi-descriptive Kripke frame is a descriptive frame in the sense of [3, Sect. 8.4] which is also a bi-general Kripke frame. It is proven in [3, Proposition 8.50] that a Kripke frame is compact if and only if it is finite.An easy example of an infinite bi-descriptive Kripke frame is the set $\mathbb{N} \cup \{\infty\}$ ordered by $\leq$ in the standard way, with cofinite upsets and the empty set as admissible cones.

A well-known result in intuitionistic logic is the isomorphism between the category of descriptive intuitionistic Kripke frames and the category of Esakia spaces [5] (see [2] for an English reference). It is routine to see that this isomorphism restricts to the bi-intuitionistic counterparts defined in Definitions 2.9 and 3.9.

Theorem 3.10. *We have an isomorphism and equivalences of categories:*

$$\mathsf{biES} \cong \mathsf{biDKF}, \qquad \mathsf{biHA}^{\mathrm{op}} \equiv \mathsf{biDKF}, \qquad \mathsf{biHM}^{\mathrm{op}} \equiv \mathsf{biDKM}.$$

Proof (Sketch). The middle equivalence follows from combining the left isomorphism and Theorem 2.11. The right equivalence follows from the middle one and Lemma 3.8. So we prove the left isomorphism.

For a bi-descriptive Kripke frame $\mathfrak{X} = (X, \leq, P)$ let $-P = \{X \setminus a \mid a \in P\}$ and let $\boldsymbol{\mathcal{X}}$ be the set X with the topology generated by the clopen subbase $P \cup -P$. Then $(\boldsymbol{\mathcal{X}}, \leq)$ is a bi-Esakia space. Conversely, for a bi-Esakia space $(\boldsymbol{\mathcal{X}}, \leq)$ with underlying set X let $\mathrm{ClpUp}(\boldsymbol{\mathcal{X}}, \leq)$ be the collection of clopen upsets of $\boldsymbol{\mathcal{X}}$ and define $\mathfrak{X} = (X, \leq, \mathrm{ClpUp}\boldsymbol{\mathcal{X}})$. It is routine to see that a function $f : (X, \leq, P) \to (X', \leq', P')$ is a bi-general morphism if and only if it is an Esakia morphism. $\square$

4 Bisimulation and the Hennessy-Milner Property

We define bisimulations and prove a Hennessy-Milner theorem for Kripke models whose connected components are finite. This follows as logical equivalence, bisimilarity and behavioural equivalence coincide on bi-descriptive Kripke frames.

Definition 4.1. Let $\mathfrak{M} = (X, \leq, V)$ and $\mathfrak{M}' = (X', \leq', V')$ be two Kripke models. A *bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ is a relation $B \subseteq X \times X'$ such that for all (x, x') with xBx', we have

(B_1) For all $p \in \text{Prop}$, $x \in V(p)$ iff $x' \in V'(p)$;
(B_2) If $x' \leq y'$ then there exists $y \in X$ such that $x \leq y$ and yBy';
(B_3) If $x \leq y$ then there exists $y' \in X'$ such that $x' \leq y'$ and yBy';
(B_4) If $y' \leq x'$ then there exists $y \in X$ such that $y \leq x$ and yBy';
(B_5) If $y \leq x$ then there exists $y' \in X'$ such that $y' \leq x'$ and yBy'.

If there is a bisimulation linking two state x and x' we say that they are *bisimilar*, notation: $x \leftrightarrows x'$. A bisimulation between bi-descriptive Kripke models is simply a bisimulation between the underlying Kripke models.

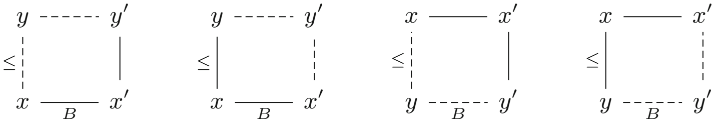


Fig. 2. Conditions (B_2) to (B_5) of a bisimulation.

If B is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$, then we call $\{x \in X \mid \exists x' \in X' \text{ s.t. } (x, x') \in B\}$ the *domain* of B and $\{x' \in X' \mid \exists x \in X \text{ s.t. } (x, x') \in B\}$ the *codomain* of B. If B' is a bisimulation between $\mathfrak{M}'$ and $\mathfrak{M}''$ whose domain equals the codomain of B, then an easy verification shows that $B \circ B'$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}''$.

Remark 4.2. Directed bisimulations [1, Definition 4] between Kripke models are pairs (Z_1, Z_2) of simulations, i.e. pairs (Z_1, Z_2) of two relations $Z_1 \subseteq X \times X'$ and $Z_2 \subseteq X' \times X$ satisfying certain conditions. This is closely related to bisimulation as just introduced: if B is a bisimulation then (B, B^{-1}) is a directed bisimulation, and if (Z_1, Z_2) is a directed bisimulation, then $Z_1 \cap Z_2^{-1}$ is a bisimulation.

Although not carried out in *op.cit.*, one could define x and x' to be directed bisimilar if there is a directed bisimulation (Z_1, Z_2) with $(x, x') \in Z_1$ and $(x', x) \in Z_2$. Directed bisimilarity and bisimilarity as defined in Definition 4.1 above are then easily seen to coincide.

As usual, morphisms between Kripke models give rise to bisimulations:

Example 4.3. Let $f : (X, \leq, V) \to (X', \leq', V')$ be bi-bounded model morphism and $\text{Gr}\, f = \{(x, f(x)) \mid x \in X\}$ the graph of f. Let $a' \subseteq X'$ be a nonempty union of connected $\leq'$-components, then $(\text{Gr}\, f) \cap (X \times a')$ is a bisimulation. (This follows easily from the definition of bi-bounded morphisms.) If moreover a' is a connected $\leq'$-component and $(\text{im}\, f) \cap a' \neq \emptyset$, then by Lemma 3.5 the codomain of this bisimulation is a'.

The following proposition follows from [1, Lemma 4], but can also be proven by a straightforward induction on the complexity of the formula.

Proposition 4.4. *Let B be a bisimulation between Kripke models $\mathfrak{M}$ and $\mathfrak{M}'$ and ϕ a formula in $\mathbb{L}$. If xBx' then $\mathfrak{M}, x \Vdash \phi$ iff $\mathfrak{M}', x' \Vdash \phi$.*

As a consequence of Example 4.3 and Proposition 4.4 we obtain:

Proposition 4.5. *Let $f : \mathfrak{M} \to \mathfrak{M}'$ be a bi-bounded model morphism between Kripke models, $x \in \mathfrak{M}$ and $\phi \in \mathbb{L}$. Then $\mathfrak{M}, x \Vdash \phi$ iff $\mathfrak{M}', f(x) \Vdash \phi$.*

A competing notion, behavioural equivalence, is given via co-spans.

Definition 4.6. Let $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$ be two states in two Kripke models. We call x and x' *behaviourally equivalent* in biKM if there is a cospan

$$\mathfrak{M} \xrightarrow{f} \mathfrak{N} \xleftarrow{f'} \mathfrak{M}' \tag{1}$$

in biKM such that $f(x) = f'(x')$, notation: $x \simeq_{\mathsf{biKM}} x'$. Behavioural equivalence in biDKM is defined similarly, and we require the cospan (1) to be in biDKM.

Note that behavioural equivalence relies on the category we are working in; caution is commendable. It is easy to see that behavioural equivalence in biDKM implies behavioural equivalence in biKM: simply forget about the descriptive structure of the models in use. We will now show that behavioural equivalence implies bisimilarity (hence logical equivalence).

Proposition 4.7. *Any two behaviourally equivalent states are bisimilar.*

Proof. If $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$ are behaviourally equivalent, there is a Kripke model $\mathfrak{N} = (Y, \leq_Y, V_Y)$ and model morphisms $f : \mathfrak{M} \to \mathfrak{N}$ and $f' : \mathfrak{M}' \to \mathfrak{N}$ satisfying $f(x) = f'(x')$. Let $a \subseteq Y$ be the connected $\leq_Y$-component containing $f(x)$ (hence also $f'(x')$). Then $B = (\mathrm{Gr}\, f) \cap (X \times a)$ and $B' = (\mathrm{Gr}\, f') \cap (X' \times a)$ are bisimulations with codomain a by Example 4.3. Hence the composition $B \circ (B')^{-1}$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ linking x and x'. □

Restricting to bi-descriptive Kripke models yields an equivalence between logical and behavioural equivalence, and bisimilarity. This is similar to corresponding results in modal logic, but additionally considers behavioural equivalence.

Theorem 4.8. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two bi-descriptive Kripke models, $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$. Then*

$$x \leftrightarrows x' \quad \textit{iff} \quad x \equiv_{\mathbb{L}} x' \quad \textit{iff} \quad x \simeq_{\mathsf{biKM}} x'.$$

Proof. Behavioural equivalence implies bisimilarity by Proposition 4.7 and bisimilarity implies logical equivalence by Proposition 4.4. We just show that logical equivalence implies behavioural equivalence.

Recall from Proposition 2.6 that $\mathfrak{L} = (L, V_L)$, the Lindenbaum-Tarski algebra with canonical valuation, is initial in biHM. Let $Z = \mathrm{pf} L$ and $V_Z(p) = \phi([p]) =$

$\{u \in \mathrm{pf}L \mid p \in u\}$. Then $\mathfrak{L}_* = \mathfrak{Z} = (Z, V_Z)$ and by Theorem 3.10 $\mathfrak{Z}$ is final in biDKM. For an arbitrary bi-descriptive Kripke model $\mathfrak{M} = (X, \leq, P, V)$ define

$$\mathrm{th}_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{Z}, x \mapsto \{[\phi] \in L \mid \mathfrak{M}, x \Vdash \phi\}.$$

Then $\mathrm{th}_{\mathfrak{M}}^{-1} : \mathfrak{L} \to \mathrm{pf}\mathfrak{M}$ sends an element $[\phi] \in \mathfrak{L}$ to $\mathrm{th}_{\mathfrak{M}}^{-1}([\phi]) = \{x \in \mathfrak{M} \mid \mathfrak{M}, x \Vdash \phi\} = [\![\phi]\!]^{\mathfrak{M}} = [\![\phi]\!]^{\mathrm{pf}\mathfrak{M}}$. Therefore $\mathrm{th}_{\mathfrak{M}}^{-1}$ is precisely one of the maps described in Proposition 2.6, hence a bi-Heyting model morphism. Consequently $\mathrm{th}_{\mathfrak{M}}$ is a bi-general model morphism. If x and x' are logically equivalent then $\mathrm{th}_{\mathfrak{M}}(x) = \mathrm{th}_{\mathfrak{M}'}(x')$ and hence $x \simeq_{\mathsf{biDKM}} x'$. □

Since finite Kripke models are finite bi-descriptive Kripke models, the following theorem is a consequence of Theorem 4.8.

Corollary 4.9. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be finite Kripke models, $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$. Then $x \equiv_{\mathbb{L}} x'$ iff $x \leftrightarroweq x'$.*

We can obtain the same result for Kripke models with finite connected components. (These are precisely the filtered colimits [8] of the finite Kripke models.)

Corollary 4.10. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two Kripke models whose connected components are finite, $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$. Then*

$$x \equiv_{\mathbb{L}} x' \quad \textit{iff} \quad x \leftrightarroweq x'.$$

Proof. 'If' is Proposition 4.4. For 'only if' suppose $x \equiv_{\mathbb{L}} x'$. Let a and a' be the connected components containing x and x' respectively. Let $\mathfrak{M}_a$ be the submodel of $\mathfrak{M}$ with underlying set a and similar for $\mathfrak{M}'_{a'}$. It follows from combining Lemma 3.6 and Proposition 4.5 that $\mathfrak{M}_a, x \Vdash \phi$ iff $\mathfrak{M}'_{a'}, x' \Vdash \phi$ for all formulas, so by Theorem 4.9 there is a bisimulation $B \subseteq a \times a'$ linking x and x'. It is easy to check that this B also defines a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. □

The following example shows that we cannot drop the assumption that the connected components be finite.

Example 4.11. Recall the Kripke model $\mathfrak{W} = (W, \preccurlyeq, V)$ from Example 3.4 and let $\mathfrak{W}' = (W', \preccurlyeq', V')$ be the submodel of $\mathfrak{W}$ with underlying set $W' = \{(n, k) \in \mathbb{N} \times \mathbb{N} \mid k < n\} \cup \{x\}$. (Note that $\mathfrak{W}'$ does not have an infinite branch.) We shall write x' for the point x in W'. See Fig. 3 for pictorial presentations of the two models. We claim that x and x' are logically equivalent but not bisimilar.

Suppose towards a contradiction that there exists a bisimulation B linking x and x'. Since $(\infty, 0) \preccurlyeq x$ in W there must be some $y' \in W'$ such that $(\infty, 0)By$ and $y' \preccurlyeq' x'$. Then y' cannot be x', because $W, (\infty, 0) \not\Vdash q$, hence $W', y' \not\Vdash q$, whereas $W', x' \Vdash q$. So y is of the form (n', k') for some $n', k' \in \mathbb{N}$ with $k' < n'$. But then $W', (n', k') \Vdash p_{n'+1} \to q$, while $W, (\infty, 0) \not\Vdash p_{n'+1} \to q$. Therefore $(\infty, 0)$ and (n', k') are not logically equivalent, hence by Proposition 4.4 they cannot be bisimilar. This contradicts the assumption that there exists a bisimulation B linking x and x', thus x and x' are not bisimilar.

Next we show that $x \in W$ and $x' \in W'$ are logically equivalent.Recall from Example 3.4 that we take $\text{Prop} = \{p_i \mid i \in \mathbb{N}\} \cup \{q\}$ and let $\text{Prop}_m = \{p_i \mid i \in \mathbb{N}, i \leq m\} \cup \{q\}$. Then $\mathbb{L}(\text{Prop}) = \bigcup_{m \in \mathbb{N}} \mathbb{L}(\text{Prop}_m)$. Define $R_m \subseteq W \times W'$ by

$$R_m = \{(x, x')\} \cup \big\{\big((n,k),(n',k')\big) \mid \text{either } [n = n' \text{ and } k = k'] \\ \text{or } [k, k' \geq m] \\ \text{or } [n, n' > m \text{ and } k = k' < m]\big\}.$$

It can be shown by induction that whenever $(z, z') \in R_m$, we have $W, z \Vdash \phi$ iff $W', z' \Vdash \phi$ for all $\phi \in \mathbb{L}(\text{Prop}_m)$. It follows that x and x' are logically equivalent because $(x, x') \in R_m$ for all $m \in \mathbb{N}$.

Thus we have found two logically equivalent states that are not bisimilar.

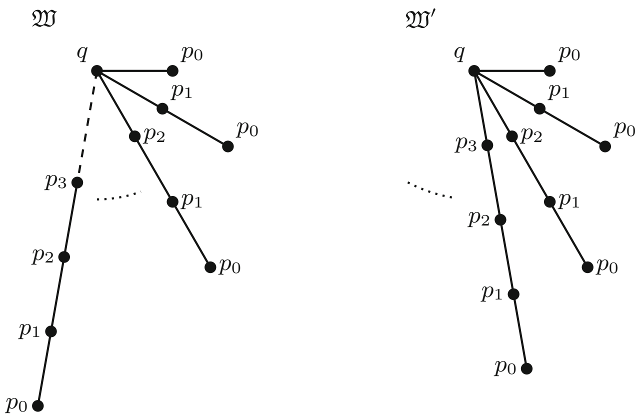


Fig. 3. The figures depicts the models $\mathfrak{W}$ and $\mathfrak{W}'$ from Example 4.11. The p_i denote the lowest occurrence of a proposition letter in each branch of the models. That is, if p_i is true in some state, then it is also true in all states above.

5 Generalisation to Modal Bi-intuitionistic Logic

We extend the language of bi-intuitionistic logic with two modal operators, $\Box$ and $\Diamond$. Our approach resembles the one in [17], where the authors extend intuitionistic logic with two modal operators using similar structures to interpret formulae.

The frames we use for interpreting this language are the ones used in [17]. We extend this viewpoint by also considering morphism which allows us to study behavioural equivalence. Our descriptive versions of these frames require additional closure properties compared to the descriptive frames in *op.cit*, to account for the subtraction connective.

After introducing the language and its algebraic semantics, we proceed similar to Sects. 3 and 4 and define modal (bi-descriptive) Kripke frames, obtain a categorical duality with the category of modal bi-Heyting algebras and modal bi-Heyting morphisms, and investigate the relations between the notions of logical equivalence, bisimilarity, and behavioural equivalence.

Definition 5.1. Let $\mathbb{L}_{\Box\Diamond}$ be the language given by the grammar

$$\phi ::= p \mid \bot \mid \top \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \prec \phi \mid \Box\phi \mid \Diamond\phi,$$

for $p \in \mathrm{Prop}$. We add the following axioms to those of bi-intuitionistic logic [13]:

$(\Box_1)$ $\Box\top \leftrightarrow \top$ $\quad$ $(\Diamond_1)$ $\Diamond\bot \leftrightarrow \bot$
$(\Box_2)$ $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$ $\quad$ $(\Diamond_2)$ $\Diamond(\phi \vee \psi) \leftrightarrow \Diamond\phi \vee \Diamond\psi$.

Modal bi-intuitionistic logic corresponds to modal bi-Heyting algebras.

Definition 5.2. A *modal bi-Heyting algebra* is a tuple $(H, \Box, \Diamond)$ of a bi-Heyting algebra H and two unary operators $\Box, \Diamond : H \to H$ such that $\Box$ satisfies $(\Box_1)$ and $(\Box_2)$, and $\Diamond$ satisfies $(\Diamond_1)$ and $(\Diamond_2)$.

A *valuation* on a modal bi-Heyting algebra is an assignment $V : \mathrm{Prop} \to H$ and a modal bi-Heyting algebra together with a valuation is called a modal bi-Heyting model. The interpretation of formulas from $\mathbb{L}_{\Box\Diamond}$ in a modal bi-Heyting model is defined recursively, with the bi-intuitionistic connectives as in Definition 2.4, together with $[\![\Box\phi]\!] = \Box([\![\phi]\!])$ and $[\![\Diamond\phi]\!] = \Diamond([\![\phi]\!])$.

A *modal bi-Heyting morphism* is a bi-Heyting morphism $f : H \to H'$ such that $f(\Box a) = \Box f(a)$ and $f(\Diamond a) = \Diamond f(a)$ and a *modal bi-Heyting model morphism* $f : (H, V) \to (H', V')$ is a modal bi-Heyting morphism $f : H \to H'$ satisfying $V' = f \circ V$. The category of modal bi-Heyting algebras and morphisms is denoted mbiHA and the category of modal bi-Heyting models and the corresponding morphisms is denoted by mbiHM.

As in the non-modal case, the Lindenbaum-Tarski algebra L of $\mathbb{L}_{\Box\Diamond}$ (for the axioms of Definition 5.1) equipped with the canonical valuation V_L is initial in mbiHM. Formulae of $\mathbb{L}_{\Box\Diamond}$ can also be interpreted over modal Kripke frames.

Definition 5.3. A *modal Kripke frame* is a tuple $(X, \leq, R_\Box, R_\Diamond)$ where $(X, \leq)$ is a Kripke frame and $R_\Box$, $R_\Diamond$ are binary relations on X satisfying

(R_1) $\leq \circ R_\Box \circ \leq = R_\Box$; $\quad$ (R_2) $\leq^{-1} \circ R_\Diamond \circ \leq^{-1} = R_\Diamond$.

A *valuation* for a modal Kripke frame $(X, \leq, R_\Box, R_\Diamond)$ is a function $V : \mathrm{Prop} \to \mathrm{Up}(X, \leq)$ and a *modal Kripke model* is a modal Kripke frame together with a valuation. The *semantics* for modal bi-intuitionistic formulas in a modal Kripke model $\mathfrak{M} = (X, \leq, R_\Box, R_\Diamond, V)$ is as in Definition 3.2, with additionally

$$\mathfrak{M}, x \Vdash \Box\phi \quad \text{iff} \quad \forall y\,(xR_\Box y \text{ implies } y \Vdash \phi),$$
$$\mathfrak{M}, x \Vdash \Diamond\phi \quad \text{iff} \quad \exists y \text{ s.t. } xR_\Diamond y \text{ and } y \Vdash \phi.$$

It is easy to verify that the truth set of every formula is an upset in $(X, \leq)$.

A *modal Kripke morphism* from $(X, \leq, R_\Box, R_\Diamond)$ to $(X, \leq, R_\Box, R_\Diamond)$ is a function $f : X \to X'$ such that:

1. $f : (X, \leq) \to (X', \leq')$ is bi-bounded;
2. $f : (X, R_\Box) \to (X', R'_\Box)$ and $f : (X, R_\Diamond) \to (X', R'_\Diamond)$ are bounded.

A *modal Kripke model morphism* is a Kripke model morphism between the underlying Kripke models which is also a modal Kripke morphism. Denote the category of modal Kripke frames (models) and modal Kripke (model) morphisms by mbiKF (mbiKM).

Trivially, our results also apply for models that postulate additional coherence conditions between the various relations, because both logical equivalence and bisimilarity are oblivious to the class of models under consideration.

Definition 5.4. A *bi-general modal Kripke frame* is a tuple $(X, \leq, R_\Box, R_\Diamond, P)$ such that $(X, \leq, R_\Box, R_\Diamond)$ is a modal Kripke frame, $(X, \leq, P)$ is a bi-general Kripke frame, and moreover P is closed under

$$\Box a = \{x \in X \mid \forall y \in X \,(xR_\Box y \text{ implies } y \in a)\},$$
$$\Diamond a = \{x \in X \mid \exists y \in a \text{ s.t. } xR_{\Diamond y}\}.$$

A morphism between bi-general modal Kripke frames $(X, \leq, R_\Box, R_\Diamond, P)$ and $(X', \leq, R'_\Box, R'_\Diamond, P')$ is a modal Kripke morphism $f : (X, \leq, R_\Box, R_\Diamond) \to (X', \leq, R'_\Box, R'_\Diamond)$ such that $f^{-1}(a') \in P$ for all $a' \in P'$.

A bi-general modal Kripke frame $\mathfrak{X} = (X, \leq, R_\Box, R_\Diamond, P)$ gives rise to a modal bi-Heyting algebra $\mathfrak{X}^* = (P, \emptyset, X, \cup, \cap, \to, \prec, \Box, \Diamond)$. Conversely, from a modal bi-Heyting algebra $(H, \Box, \Diamond)$ we can construct a bi-general modal Kripke frame by taking $(X, \leq, P)$ to be the bi-descriptive Kripke frame dual to H and defining

$$uR_\Box v \quad \text{iff} \quad \forall a \in H \,(\Box a \in u \to a \in v)$$
$$uR_\Diamond v \quad \text{iff} \quad \forall a \in H \,(v \in a \to u \in \Diamond a).$$

In general, we have $(\mathfrak{H}_*)^* \simeq \mathfrak{H}$, but not $(\mathfrak{X}^*)_* \simeq \mathfrak{X}$. For this to be true, we need to restrict to the class of bi-descriptive modal Kripke frames as before.

Definition 5.5. A bi-general modal Kripke frame $(X, \leq, R_\Box, R_\Diamond, P)$ is called *bi-descriptive* if the underlying bi-general frame $(X, \leq, P)$ is bi-descriptive and the relations $R_\Box$ and $R_\Diamond$ satisfy

(R_3) $xR_\Box y$ iff $\forall a \in P \,(x \in \Box a \to y \in a)$;
(R_4) $xR_\Diamond y$ iff $\forall a \in P \,(y \in a \to x \in \Diamond a)$.

The category of bi-descriptive modal Kripke frames (models) and bi-general modal frame (model) morphisms is denoted by mbiDKF (mbiDKM).

Setting $f^* = f^{-1}$ for morphisms f in mbiDKF and $g_* = g^{-1}$ for morphisms g in mbiHA, we obtain two functors $(-)^* : \mathsf{mbiDKF} \to \mathsf{mbiHA}$ and $(-)_* : \mathsf{mbiHA} \to \mathsf{mbiDKF}$. These yield the following theorem:

Theorem 5.6. *We have a dual equivalences of categories* $\mathsf{mbiDKF} \equiv^{\mathrm{op}} \mathsf{mbiHA}$ *and* $\mathsf{mbiDKM} \equiv^{\mathrm{op}} \mathsf{mbiHM}$.

The object part of Theorem 5.6 is essentially a restriction of results in [17].

Proposition 5.7. *Let* $\mathfrak{H} = (H, \Box, \Diamond)$ *be a modal bi-Heyting algebra. Then* $\mathfrak{H} \Vdash \phi$ *if and only if* $\mathfrak{H}_* \Vdash \phi$.

Proof. First, observe that a valuation for $\mathfrak{H}$ is simply a valuation for the underlying bi-Heyting algebra, and an admissible valuation for $\mathfrak{H}_*$ is an admissible valuation for the underlying bi-descriptive Kripke frame. Therefore Lemma 3.8 states that valuations for $\mathfrak{H}$ and $\mathfrak{H}_*$ correspond one-to-one.

Let V be any valuation for $\mathfrak{H}$ and V_* the corresponding valuation on $\mathfrak{H}_*$. Then V_* is defined by $V_*(p) = \phi(V(p)) = \{u \mid V(p) \in u\}$. It is routine to show that $[\![\phi]\!]^{(\mathfrak{H}, V)} = \top$ iff $[\![\phi]\!]^{(\mathfrak{H}_*, V_*)} = \mathrm{pf}\mathfrak{H}$, i.e. the entire set of prime filters of $\mathfrak{H}$. $\Box$

Now that we have our duality in place, we turn our attention to bisimulations.

Definition 5.8. Let $\mathfrak{M} = (X, \leq, R_\Box, R_\Diamond, V)$ and $\mathfrak{M}' = (X', \leq', R'_\Box, R'_\Diamond, V')$ be two modal Kripke models. A *bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ is a relation $B \subseteq X \times X'$ such that B is a bisimulation between $(X, \leq, V)$ and $(X', \leq', V')$ (in the sense of Definition 4.1) and for all $(x, x') \in B$ we have:

(B_6) If $x' R'_\Box y'$ then there exists $y \in X$ such that $x R_\Box y$ and $y B y'$;
(B_7) If $x R_\Box y$ then there exists $y' \in X'$ such that $x' R'_\Box y'$ and $y B y'$;
(B_8) If $x' R'_\Diamond y'$ then there exists $y \in X$ such that $x R_\Diamond y$ and $y B y'$;
(B_9) If $x R_\Diamond y$ then there exists $y' \in X'$ such that $x' R'_\Diamond y'$ and $y B y'$.

Note that these are two pairs of back and forth conditions (Fig. 4).

$$\begin{tikzcd}
y \arrow[r, dash, dashed] \arrow[d, dash, dashed, "R_\Box"'] & y' \arrow[d, dash] & & y \arrow[r, dash, dashed] \arrow[d, dash, "R_\Box"'] & y' \arrow[d, dash, dashed] & & y \arrow[r, dash, dashed] \arrow[d, dash, dashed, "R_\Diamond"'] & y' \arrow[d, dash] & & y \arrow[r, dash, dashed] \arrow[d, dash, "R_\Diamond"'] & y' \arrow[d, dash, dashed] \\
x \arrow[r, dash, "B"'] & x' & & x \arrow[r, dash, "B"'] & x' & & x \arrow[r, dash, "B"'] & x' & & x \arrow[r, dash, "B"'] & x'
\end{tikzcd}$$

Fig. 4. Conditions (B_6) to (B_9) of bisimulations between modal Kripke models.

Proposition 5.9. *Let* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *be two models,* B *a bisimulation between* $\mathfrak{M}$ *and* $\mathfrak{M}'$*. Then for all* ϕ *and* $(x, x') \in B$ *we have* $\mathfrak{M}, x \Vdash \phi$ *iff* $\mathfrak{M}', x' \Vdash \phi$.

Behavioural equivalence is defined similar to Definition 4.6: Two states $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$ in two modal Kripke models are said to be *behaviourally equivalent* in mbiKM if there is a cospan $\mathfrak{M} \xrightarrow{f} \mathfrak{N} \xleftarrow{f'} \mathfrak{M}'$ in biKM such that $f(x) = f'(x')$, notation: $x \simeq_{\mathsf{mbiKM}} x'$. Behavioural equivalence in mbiDKM is defined analogously, except that we require the cospan to be in mbiDKM.

Proposition 5.10. *Let* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *be (bi-descriptive) modal Kripke models. If* $x \in \mathfrak{M}$ *and* $x' \in \mathfrak{M}'$ *are behaviourally equivalent, then they are bisimilar.*

Proof. We give a proof for the case where $\mathfrak{M}$ and $\mathfrak{M}'$ are modal Kripke models. The proof for bi-descriptive modal Kripke models is analogous.

If x and x' are behaviourally equivalent, then there must exist a bi-descriptive modal Kripke model $\mathfrak{N}$ and model morphisms $f : \mathfrak{M} \to \mathfrak{N}$ and $f' : \mathfrak{M}' \to \mathfrak{N}$ satisfying $f(x) = f'(x')$. Let $B = \{(y, y') \in X \times X' \mid f(y) = f'(y')\}$. It is routine to check that this is a bisimulation. □

Restricting to descriptive models yields the following analog of Theorem 4.8. The proof is similar to that of Theorem 4.8 and uses Propositions 5.9 and 5.10.

Theorem 5.11. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two descriptive modal Kripke models and $x \in \mathfrak{M}$, $x' \in \mathfrak{M}'$ two states. Then $x \leftrightarroweq x'$ iff $x \equiv_{\mathbb{L}} x'$ iff $x \simeq_{\mathsf{mbiKM}} x'$.*

Since finite descriptive modal Kripke models are precisely the finite modal Kripke models it follows that:

Corollary 5.12. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be finite modal Kripke models, $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$. Then $x \equiv_{\mathbb{L}} x'$ iff $x \leftrightarroweq x'$.*

Using a suitable notion of component, we can derive an analog of Theorem 4.10. A *component* in a modal Kripke frame $(X, \leq, R_\Box, R_\Diamond)$ is a nonempty subset $a \subseteq X$ satisfying $a = \uparrow_{\leq} a = \downarrow_{\leq} a = \uparrow_{R_\Box} a = \uparrow_{R_\Diamond} a$. Since these components are closed under intersection, we can define the *minimal component* of $x \in X$ as the intersection of all components containing x.

Corollary 5.13. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two modal Kripke models such that the minimal component for each element in both models is finite, $x \in \mathfrak{M}$ and $x' \in \mathfrak{M}'$. Then $x \equiv_{\mathbb{L}} x'$ iff $x \leftrightarroweq x'$.*

6 Conclusion

We have established previously unknown Hennessy-Milner theorems for both bi-intuitionistic logic and a version of modal bi-intuitionistic logic: logical equivalence and bisimilarity agree on models with finite connected components. As our models are based on posets (to interpret intuitionistic implication and its dual), this implies that the order relation and its inverse are image finite. Our main technical tool in the proofs are various categorical equivalences for bi-intuitionistic logic. An intriguing open question is to isolate a more general semantic property that implies Hennessy-Milner type results uniformly for a large class of logics.

References

1. Badia, G.: Bi-simulating in bi-intuitionistic logic. Studia Logica **104**, 1037–1050 (2016)
2. Bezhanishvili, N.: Lattices of intermediate and cylindric modal logics. Ph.D. thesis. University of Amsterdam (2006)

3. Chagrov, A., Zakharyaschev, M.: Modal Logic. Oxford University Press, Oxford (1997)
4. Crolard, T.: A formulae-as-types interpretation of subtractive logic. J. Log. Comput. **14**(4), 529–570 (2004)
5. Esakia, L.: Topological Kripke models. Soviet Mathematics Doklady **15**, 147–151 (1974)
6. Esakia, L.: The problem of dualism in the intuitionistic logic and Browerian lattices. In: V International Congress of Logic, Methodology and Philosophy of Science, Canada, pp. 7–8 (1975)
7. Goré, R.: Dual intuitionistic logic revisited. In: Dyckhoff, R. (ed.) TABLEAUX 2000. LNCS, vol. 1847, pp. 252–267. Springer, Heidelberg (2000). https://doi.org/10.1007/10722086_21
8. Lane, S.: Categories for the Working Mathematician. Graduate Texts in Mathematics. Springer, New York (1971). https://doi.org/10.1007/978-1-4612-9839-7
9. Patterson, A.: Bisimulation and propositional intuitionistic logic. In: Mazurkiewicz, A., Winkowski, J. (eds.) CONCUR 1997. LNCS, vol. 1243, pp. 347–360. Springer, Heidelberg (1997). https://doi.org/10.1007/3-540-63141-0_24
10. Pinto, L., Uustalu, T.: Proof search and counter-model construction for bi-intuitionistic propositional logic with labelled sequents. In: Giese, M., Waaler, A. (eds.) TABLEAUX 2009. LNCS, vol. 5607, pp. 295–309. Springer, Heidelberg (2009). https://doi.org/10.1007/978-3-642-02716-1_22
11. Priestley, H.A.: Representation of distributive lattices by means of ordered Stone spaces. Bull. Lond. Math. Soc. **2**(2), 186–190 (1970)
12. Rauszer, C.: A formalization of the propositional calculus of H-B logic. Studia Logica **33**(1), 23–34 (1974)
13. Rauszer, C.: Semi-Boolean algebras and their application to intuitionistic logic with dual operations. In: Fundamenta Mathematicae LXXXIII, pp. 219–249 (1974)
14. Rauszer, C.: An algebraic and Kripke-style approach to a certain extension of intuitionistic logic. Dissertationes Mathematicae, Polish Scientific Publishers (1980)
15. Restall, G.: Extending intuitionistic logic with subtraction (1997). http://consequently.org/writing/
16. Tranchini, L.: Natural deduction for bi-intuitionistic logic. J. Appl. Log. **25**, 72–96 (2017)
17. Wolter, F., Zakharyaschev, M.: Intuitionistic modal logic. In: Cantini, A., Casari, E., Minari, P. (eds.) Logic and Foundations of Mathematics. SYLI, vol. 280, pp. 227–238. Springer, Dordrecht (1999). https://doi.org/10.1007/978-94-017-2109-7_17

The McKinsey-Tarski Theorem for Topological Evidence Logics

Alexandru Baltag[1], Nick Bezhanishvili[1], and Saúl Fernández González[2](✉)

[1] ILLC, Universiteit van Amsterdam, Amsterdam, The Netherlands
[2] IRIT, Université de Toulouse, Toulouse, France
saul.fgonzalez@irit.fr

Abstract. We prove an analogue of the McKinsey and Tarski theorem for the recently introduced dense-interior semantics of topological evidence logics. In particular, we show that in this semantics the modal logic $\mathsf{S4.2}$ is sound and complete for any dense-in-itself metrizable space. As a result $\mathsf{S4.2}$ is complete with respect to the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, the Baire space $\mathfrak{B}$, the Cantor space $\mathfrak{C}$, etc. We also show that an extension of this logic with the universal modality is sound and complete for any idempotent dense-in-itself metrizable space, obtaining as a result that this logic is sound and complete with respect to $\mathbb{Q}$, $\mathfrak{B}$, $\mathfrak{C}$, etc.

1 Introduction

Epistemic logics (i.e. the family of modal logics concerned with what an epistemic agent *believes* or *knows*) has by now a well-established semantics in the form of Kripke frames [11]. Hintikka [11] reasonably claims that the accessibility relation encoding knowledge must be minimally reflexive and transitive, which on the syntactic level translates to the corresponding logic of knowledge containing the axioms of $\mathsf{S4}$. This, paired with the fact (proven by McKinsey and Tarski [14]) that $\mathsf{S4}$ is the logic of topological spaces under the *interior semantics*, lays the ground for a topological treatment of knowledge. Moreover, treating the knowledge modality as the topological interior operator, and the open sets as "pieces of evidence" adds an evidential dimension to the notion of knowledge that one cannot obtain within the framework of Kripke frames.

Reading epistemic sentences using the interior semantics might be too simplistic: it equates "knowing" and "having evidence". In addition, the attempts to bring the notion of belief into this framework have not been very successful.

Following [18], a logic that allows us to talk about knowledge, belief and the relation thereof, about evidence (both basic and combined) and justification is introduced in [2]. This is the framework of *topological evidence models* (topo-e-models) and this paper builds on it.

McKinsey and Tarski also proved in [14] a stronger result—their celebrated theorem—namely, that there are single spaces (dense-in-themselves and metrizable) such as the real line, whose logic is $\mathsf{S4}$. The present paper aims to translate the spirit of this theorem to the framework of topo-e-models. To this respect, we

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 177–194, 2019.
https://doi.org/10.1007/978-3-662-59533-6_11

introduce a notion of generic models over a language $\mathcal{L}$, which are topological spaces whose logic is precisely the sound and complete $\mathcal{L}$-logic of topo-e-models, and provide several examples of generic models for the different fragments of the language. More precisely, we show that in this new semantics the modal logic $\mathsf{S4.2}$ is sound and complete for any dense-in-itself metrizable space. As a result $\mathsf{S4.2}$ is complete with respect to the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, the Baire space $\mathfrak{B}$, the Cantor space $\mathfrak{C}$, etc. We also show that extensions of this logic (e.g., with the global modality) are sound and complete for any idempotent dense-in-itsef metrizable space such as $\mathbb{Q}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. Our proofs rely on a recent topological proof of the McKinsey and Tarski theorem [5]. Namely, an open and continuous onto map from any dense-in-itself metrizable space onto a finite rooted $\mathsf{S4}$-frame defined in [5] can be used to define an open and continuous onto map from such a space but now with the dense-interior topology onto a finite rooted $\mathsf{S4.2}$-frame.

This paper is structured as follows: in the present section we show how to use topological spaces to model epistemic sentences and introduce the framework of topological evidence models. In Sect. 2, we explain how McKinsey and Tarski's theorem encodes a notion of *generic model* which we then use to state and prove our main results. These results also include different fragments of the language within the framework of topo-e-models. Finally, we conclude in Sect. 3.

1.1 Logics of Knowledge and Belief

Below we list some logics of belief and knowledge which were mentioned in the introduction and will be used throughout this paper.

The modal logic $\mathsf{S4}$ is the least set of formulas in the language $\mathcal{L}_\Box$ which contains all the propositional tautologies, is closed under uniform substitution and the rules of modus ponens (from ϕ and $\phi \to \psi$ infer ψ) and necessitation (from ϕ infer $\Box\phi$) and contains the axioms:

(K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;
(T) $\Box\phi \to \phi$ (*factivity of knowledge*);
(4) $\Box\phi \to \Box\Box\phi$ (*positive introspection*).

The modal logic $\mathsf{S5}$ contains the axioms and rules of $\mathsf{S4}$ plus the axiom:

(5) $\neg\Box\phi \to \Box\neg\Box\phi$ (*negative introspection*).

$\mathsf{S4.2}$ is $\mathsf{S4}$ plus the axiom:

(.2) $\neg\Box\neg\Box\phi \to \Box\neg\Box\neg\phi$.

$\mathsf{KD45}$ has the (K), (4) and (5) axioms plus:

(D) $\Box\phi \to \neg\Box\neg\phi$.

The logic Stal, with respect to a language with the K and B modalities, adds the axioms in Table 1 to the $\mathsf{S4}$ axioms for K.

Table 1. Extra axioms for Stal

(PI)	$B\phi \to KB\phi$;
(NI)	$\neg B\phi \to K\neg B\phi$;
(KB)	$K\phi \to B\phi$;
(CB)	$B\phi \to \neg B\neg\phi$;
(FB)	$B\phi \to BK\phi$

1.2 The Interior Semantics: The McKinsey-Tarski Theorem

Let Prop be a countable set of propositional variables and consider a modal language $\mathcal{L}_\Box$ defined as follows: $\phi := p \mid \phi \wedge \phi \mid \neg\phi \mid \Box\phi$, with $p \in \mathsf{Prop}$.

A *topological model* is a topological space (X, τ) together with a valuation $V : \mathsf{Prop} \to 2^X$. The semantics of a formula ϕ is defined recursively as follows: $\|p\| = V(p)$; $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$, $\|\neg\phi\| = X \backslash \|\phi\|$, $\|\Box\phi\| = \mathrm{Int}\,\|\phi\|$, where Int is the interior operator of the topology.

We now give some examples of topological spaces (which will be used throughout the remainder of this paper) in which we model epistemic sentences.

Example 1.1 (The real line). Let $\mathbb{R}$ be the set of real numbers. We define the *natural topology* $\tau_\mathbb{R}$ on $\mathbb{R}$, as the topology generated by the basis of open intervals

$$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}.$$

Equivalently, $U \subseteq \mathbb{R}$ is an open set if, for each $x \in U$, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$.

Example 1.2 (The rational numbers). The *natural topology* $\tau_\mathbb{Q}$ on the set of rational numbers $\mathbb{Q}$ is simply the subspace topology[1] $\tau_\mathbb{R}|_\mathbb{Q}$ or, equivalently, the topology generated on $\mathbb{Q}$ by the basis of open intervals $\{(a, b) : a, b \in \mathbb{R}, a < b\}$, where $(a, b) = \{x \in \mathbb{Q} : a < x < b\}$.

Example 1.3 (The Baire space and the Cantor space). Let ω^ω be the set of infinite sequences of natural numbers, and ω^* be the set of finite such sequences. For $s \in \omega^*$ and $\alpha \in \omega^\omega$ we say $s \lhd \alpha$ whenever s is an *initial segment* of α, i.e., whenever $s = \langle s_1, ..., s_n \rangle$ with $s_i = \alpha(i)$ for $1 \leq i \leq n$. For $s \in \omega^*$, let $O(s)$ denote the set of sequences of natural numbers that have s as an initial segment, i.e. $O(s) = \{\alpha \in \omega^\omega : s \lhd \alpha\}$. The *Baire space* $\mathfrak{B} = (\omega^\omega, \tau_\mathfrak{B})$ is the topological space that has ω^ω as its underlying set together with the topology $\tau_\mathfrak{B}$ generated by the basis

$$\mathcal{B}_\mathfrak{B} = \{O(s) : s \in \omega^*\}.$$

[1] Given a topological space (X, τ) and a set $Y \subseteq X$, we can define the *subspace topology* $\tau|_Y$ on Y as the set

$$\tau|_Y := \{U \cap Y : U \in \tau\}.$$

Note that $(Y, \tau|_Y)$ is trivially a topological space.

We can analogously define the *Cantor space* $\mathfrak{C}$ on the set $\mathbf{2}^\omega$ of countable sequences of zeros and ones. The Cantor space has a visual representation in the form of the infinite binary tree. This is a tree whose nodes are the finite sequences of zeros and ones. It has the empty sequence as the root and each node $\langle i_1, ..., i_n \rangle \in \mathbf{2}^*$ has exactly two successors, namely $\langle i_1, ..., i_n, 0 \rangle$ as its left successor and $\langle i_1, ..., i_n, 1 \rangle$ as its right successor. The elements of the Cantor space can be identified with *branches* of this tree, where a branch is a countable collection of nodes $\{s_0, s_1, s_2, ...\}$ such that s_0 is the empty sequence (i.e. the root of the tree) and each s_{k+1} is an immediate successor of s_k. The basic open sets $O(s)$ are identified with "fans", each fan being the subtree that spurs from one node. An open set is any union of some of these fans. $\alpha \in \mathbf{2}^\omega$ is in a basic open set $O(s)$ whenever the corresponding branch "enters" the fan.

Example 1.4 (The binary tree $\mathcal{T}_2$). If we consider the nodes of the infinite binary tree instead of its branches to be the points of our space, we can equip it with a topology by setting the basic open sets to be those of the form $O(s)$, where $s = \langle a_0, ..., a_n \rangle$ and $t \in O(s)$ if and only if t is a finite sequence of length greater than or equal to $n+1$ with its $n+1$ first elements being $a_0, ..., a_n$.

The interior semantics on topological spaces generalises the Kripke semantics on preordered frames[2]. If we are reading $\Box$ as an epistemic operator, we can translate the semantics of [11] into this topological framework, with the addition that having a topological space allows us to have an *evidential* view of knowledge. Indeed, if we read $\Box$ as a knowledge modality, we interpret the open sets in the topology to be pieces of evidence the agent has, and we say that P entails Q whenever $P \subseteq Q$, then the interior semantics defined above gives us that the agent *knows* ϕ whenever she has a piece of evidence which entails ϕ.

Let us revisit some of the examples above in this light.

Example 1.5. An underfunded ornithologist measures the weight of a bird. Her devices of measurement produce results with a margin of error of ± 10 g. Let us code the set of possible worlds with the positive real numbers $(0, \infty)$, where at world x the weight of the bird is precisely x grams. Now, suppose the actual world is $x_0 = 509$ and the ornithologist obtains a measurement of $500\,\text{g} \pm 10\,\text{g}$. Then the open interval $(490, 510)$ is her piece of evidence. With this, there are things she knows and things she does not know. She does not know, for instance, the proposition "the bird is heavier than 500 g" to be true. She knows, however, that the bird is heavier than 400 g. This proposition can be interpreted as the set of worlds $P = (400, \infty)$ and she has a piece of evidence which includes the actual world and entails this proposition: $x_0 \in (490, 510) \subseteq P$.

[2] Given a preordered set $(X, \leq)$, the collection of upwards-closed sets defines an Alexandroff topology on X, i.e., a topology closed under infinite intersections. Conversely, given an Alexandroff topological space (X, τ) the relation $x \leq y$ iff $x \in U$ implies $y \in U$, for all $U \in \tau$, defines a preorder. This correspondence is 1–1 and moreover $x \in \operatorname{Int} P$ iff $y \in P$ for all $y \geq x$. For details, see, e.g., [3].

Example 1.6. Let us equate a world with an infinite stream of data, represented by a sequence of natural numbers. We are thus in our Baire space. Our epistemic agent this time is a scientist, and her evidence comes in the form of *observations*, which are finite streams of data that the scientist is able to grasp. A world is compatible with her observation whenever the stream of data is an initial segment of said world. If she observes $s = \langle a_1, ..., a_n \rangle$, then the set of worlds compatible with it (the corresponding *piece of evidence* in our sense) is precisely the basic open set $O(s)$.

In this setting, open sets correspond to *verifiable* propositions: if P is an open set and the actual world x_0 is in P, then there exist a basic open set $O(s)$ such that $x_0 \in O(s) \subseteq P$. Thus this scientist can potentially make an observation, s, which will allow her to know P. Similarly, closed sets correspond to *refutable* propositions and clopen sets to *decidable* propositions. For more details on this interpretation, see [12].

1.3 McKinsey and Tarski: $\mathsf{S4}$ as a Topological Logic of Knowledge

Modelling knowledge as topological interior gives us an intuitive, evidence-based idea of what knowledge amounts to. Moreover, the interior semantics generalises the Kripke semantics for preorders and:

Theorem 1.7 (McKinsey and Tarski [14]). $\mathsf{S4}$ *is sound and complete with respect to topological spaces under the interior semantics.*

McKinsey and Tarski also proved a stronger result. We do not need to consider the class of all topological spaces to obtain the logic $\mathsf{S4}$. They showed that, instead, we can take some particular, "natural" topological space used to model knowledge, whose logic is $\mathsf{S4}$.

Definition 1.8. *A topological space* (X, τ) *is called* dense-in-itself *if no singleton is an open set, i.e., if* $\{x\} \notin \tau$ *for all* $x \in X$*. We say* (X, τ) *is* metrizable *if there exists a metric*[3] d *on* X *which generates* τ*.*

Remark 1.9. All the spaces presented as examples in Subsect. 1.2 are both dense-in-themselves and metrizable. The corresponding metric for the spaces $\mathbb{R}$ and $\mathbb{Q}$ is $d(x, y) = |x - y|$, and clearly no singleton contains an open interval in these spaces. The binary tree $\mathcal{T}_2$ clearly has no open singletons and it is a regular space with a countable basis and thus metrizable. $\mathfrak{B}$ and $\mathfrak{C}$ are homeomorphic to dense-in-themselves metrizable subspaces of $\mathbb{R}$ (for details on these claims, see [7,15]).

Theorem 1.10 (McKinsey and Tarski [14]). $\mathsf{S4}$ *is the logic of any dense-in-itself metrizable space.*[4]

[3] I.e. a map $d : X \times X \to [0, \infty)$ satisfying for all $x, y, z \in X$: (i.) $d(x, y) = 0$ iff $x = y$; (ii.) $d(x, y) = d(y, x)$; (iii.) $d(x, z) \leq d(x, y) + d(y, z)$. A metric d on X induces a topology τ_d: we say that a set $U \subseteq X$ is open if, for every $x \in U$, there exists some $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies $y \in U$.

[4] The original formulation of this theorem talked about dense-in-itself, metrizable, *separable* spaces. It was shown in [16] that the separability condition can be dropped.

We thus have a semantics based on evidence that allows us to talk about knowledge and whose logic is a philosophically suitable epistemic logic. Moreover, we have some specific spaces which provide "nice" ways to conceptualise knowledge and whose logic is still $\mathsf{S4}$.

This semantics, however, is not the topic of this paper. Instead, we will be working with the *dense interior* semantics. Understanding the conceptual reasons to move away from the interior and introducing this semantics is the aim of the next subsection.

1.4 Dense Interior

The relation between belief and knowledge has historically been a main focus of epistemology. One would want to have a formal system that accounts for knowledge and belief together, which requires careful consideration regarding the way in which they interact. Canonically, knowledge has been thought of as "true, justified belief". However, Gettier's counterexamples of cases of true, justified belief which do not amount to knowledge shattered this paradigm [8].

Stalnaker [18] argues that a relational semantics is insufficient to capture Gettier's considerations in [8] and, trying to stay close to most of the intuitions of Hintikka in [11], provides an axiomatisation for a system of knowledge and belief. This system, Stal, has two modal operators, B and K, and on top of the $\mathsf{S4}$ axioms and rules for K it adds the axioms of Table 1.

In this logic, knowledge is an $\mathsf{S4.2}$ modality, belief is a $\mathsf{KD45}$ modality and the following formulas can be proven: $B\phi \leftrightarrow \neg K \neg K\phi$ and $B\phi \leftrightarrow BK\phi$. "Believing p" is the same as "not knowing you don't know p" and belief becomes "subjective certainty", in the sense that the agent cannot distinguish whether she believes or knows p, and believing amounts to believing that one knows.

Now, modelling epistemic sentences via the interior semantics defined above forces us to equate "knowing" with "having evidence". Moreover, attempts to introduce belief in this framework have had some flagrant issues. To give some examples, the framework considered in [19], in which knowledge is interior and belief is read as the dual of the *derived set operator*[5], makes knowledge amount to true belief, which clearly falls short. [1] takes a Stalnakerian stand but it confines us to work with *hereditarily extremally disconnected spaces* (h.e.d)[6], which seems to be a rather restricted class of spaces. None of the "natural" spaces provided above as examples are h.e.d.

In [2] a new semantics is introduced, building on the idea of *evidence models* of [4] which exploits the notion of evidence-based knowledge allowing to account for notions as diverse as *basic evidence* versus *combined evidence*, *factual, misleading* and *nonmisleading evidence*, etc. It is a semantics whose logic maintains a Stalnakerian spirit with regards to the relation between knowledge and belief,

[5] $BP = \neg d(\neg P)$, where $d(P) = \{x : \forall U \in \tau(x \in U \text{ implies } \exists y \in P \cap U, y \neq x)\}$.

[6] A space is *extremally disconnected* (e.d.) if the closure of an open set is open, and *hereditarily* so if all its subspaces are e.d.

which behaves well dynamically and which does not confine us to work with "strange" classes of spaces.

This is the *dense interior semantics*, defined on *topological evidence models.*

1.5 The Logic of Topological Evidence Models

We briefly present here the framework introduced in [2]. Our language is now $\mathcal{L}_{\forall KB\Box\Box_0}$, which includes the modalities K (knowledge), B (belief), $[\forall]$ (infallible knowledge), $\Box_0$ (basic evidence), $\Box$ (combined evidence).

Definition 1.11 (The dense interior semantics). *We interpret sentences on* topological evidence models *(i.e. tuples (X, τ, E_0, V) where (X, τ, V) is a topological model and E_0 is a subbasis of τ) as follows: $x \in [\![K\phi]\!]$ iff $x \in \mathrm{Int}[\![\phi]\!]$ and $\mathrm{Int}[\![\phi]\!]$ is* dense[7]*; $x \in [\![B\phi]\!]$ iff $\mathrm{Int}[\![\phi]\!]$ is dense; $x \in [\![[\forall]\phi]\!]$ iff $[\![\phi]\!] = X$; $x \in [\![\Box_0\phi]\!]$ iff there is $e \in E_0$ with $x \in e \subseteq [\![\phi]\!]$; $x \in [\![\Box\phi]\!]$ iff $x \in \mathrm{Int}[\![\phi]\!]$. Validity is defined in the standard way.*

We see that "knowing" does not equate "having evidence" in this framework, but it is rather something stronger: in order for the agent to know P, she needs to have a piece of evidence for P which is *dense*, i.e., which has nonempty intersection with (and thus cannot be contradicted by) any other potential piece of evidence she could gather.

Fragments of the Logic. The following logics are obtained by considering certain fragments of the language (i.e. certain subsets of the modalities above).

"K-only", $\mathcal{L}_K$	S4.2.
"Knowledge", $\mathcal{L}_{\forall K}$	S5 axioms and rules for $[\forall]$, plus S4.2 for K, plus $[\forall]\phi \to K\phi$ and $\neg[\forall]\neg K\phi \to [\forall]\neg K\neg\phi$.
"Combined evidence", $\mathcal{L}_{\forall\Box}$	S5 for $[\forall]$, S4 for $\Box$, plus $[\forall]\phi \to \Box\phi$.
"Evidence", $\mathcal{L}_{\forall\Box\Box_0}$	S5 for $[\forall]$, S4 for $\Box$, plus the axioms $\Box_0\phi \to \Box_0\Box_0\phi$, $[\forall]\phi \to \Box_0\phi$, $\Box_0\phi \to \Box\phi$, $(\Box_0\phi \wedge [\forall]\psi) \to \Box_0(\phi \wedge [\forall]\psi)$.

We will refer to these logics respectively as $\mathsf{S4.2}_K$, $\mathsf{Logic}_{\forall K}$, $\mathsf{Logic}_{\forall\Box}$ and $\mathsf{Logic}_{\forall\Box\Box_0}$. K and B are definable in the evidence fragments[8], thus we can think of the logic of $\mathcal{L}_{\forall\Box\Box_0}$ as the "full logic".

2 Generic Spaces for the Logic of Topo-e-models

McKinsey and Tarski's theorem [14] stating that S4 is the logic of any dense-in-itself metrizable space (such as the real line $\mathbb{R}$) under the interior semantics tells us that we have a space which gives a somewhat "natural" way of capturing

7 A set $U \subseteq X$ is dense whenever $\mathrm{Cl}\, U = X$ or equivalently whenever $U \cap V \neq \varnothing$ for all nonempty open set V.

8 $K\phi \equiv \Box\phi \wedge [\forall]\Box\Diamond\phi$ and $B\phi \equiv \neg K\neg K\phi$.

knowledge yet it is "generic" enough so that its logic is precisely the logic of all topological spaces. Whatever is not provable in the logic of knowledge $\mathsf{S4}$ will find a refutation in $\mathbb{R}$ and whatever is true in $\mathsf{S4}$ will hold in every model based on the topology of the real line.

Translating this idea to the framework of topo-e-models is the aim of this paper. We wish to find topological evidence models which capture the logics presented in the preceding chapter, that is, special spaces whose logic under the dense interior semantics is exactly the logic of topo-e-models. We start by formalising the idea of "generic".

Definition 2.1 (Generic models). *Let $\mathcal{L}$ be a language and (X, τ) a topological space. We will say that (X, τ) is a* generic model for $\mathcal{L}$ *if the sound and complete $\mathcal{L}$-logic over the class of all topological evidence models is sound and complete with respect to the family*

$$\{(X, \tau, E_0) : E_0 \text{ is a subbasis of } \tau\}.$$

If $\Box_0$ is not in the language, then a generic model is simply a topological space which is sound and complete with respect to the corresponding $\mathcal{L}$-logic.

Since McKinsey and Tarski's original paper (which appeared in 1944), a number of simplified proofs of this result have been obtained. For an overview, we refer to [3]. Many of these proofs are built on the following idea. It is a well-known fact that $\mathsf{S4}$ is sound and complete with respect to finite rooted preorders (see e.g. [6]). One then constructs an *interior map* (a surjective map which is continuous and open[9]) from a dense-in-itself metrizable space (X, τ) onto any such preorder $(W, \leq)$. It can be proven that given such a map $f : X \to W$ and a valuation V on $(W, \leq)$, if we define $V^f(p) := \{x \in X : fx \in V(p)\}$ it is the case that, for any formula ϕ in the language of $\mathsf{S4}$, $x \models \phi$ in (X, τ, V^f) if and only if $fx \models \phi$ in $(W, \leq, V)$. Completeness is then a straightforward consequence, for if $\mathsf{S4} \not\vdash \phi$, then there is a model based on a finite rooted preorder $(W, \leq, V)$ refuting ϕ and thus we can refute ϕ on (X, τ, V^f). The next subsection builds on a recent proof of the McKinsey-Tarski theorem, contained in [5], which is purely topological.

2.1 S4.2 as the Logic of $\mathbb{R}$

This section is devoted to the proof of our analogue of McKinsey and Tarski's theorem:

Theorem 2.2. $\mathsf{S4.2}_K$ *is the logic of any dense-in-itself metrizable space if we read K as dense interior. That is, for any formula ϕ in the language $\mathcal{L}_K$, and any dense-in-itself metrizable space (X, τ), we have that $\mathsf{S4.2}_K \vdash \phi$ if and only if $(X, \tau) \models \phi$ with the dense interior semantics.*

9 A map $f : (X, \tau) \to (Y, \sigma)$ is *continous* is $U \in \sigma$ implies $f^{-1}[U] \in \tau$ and *open* if $U \in \tau$ implies $f[U] \in \sigma$.

Before tackling this proof, we will need to introduce some auxiliary notions.

Given a topological space (X, τ) define τ^+ to be the collection of dense open sets in (X, τ) plus the empty set:

$$\tau^+ = \{U \in \tau : \mathrm{Cl}\, U = X\} \cup \{\varnothing\}.$$

Recall that a topological space is *extremally disconnected* if the closure of any open set is an open set. The following is straightforward to check.

Lemma 2.3. (X, τ^+) *is an extremally disconnected topological space and, for any valuation* V *and any formula* ϕ *in the modal language* $\mathcal{L}_K$ *we have that* $[\![\phi]\!]^{(X,\tau,V)}$ *under the dense interior semantics coincides with* $\|\phi\|^{(X,\tau^+,V)}$ *under the interior semantics.*

Lemma 2.4. *For any topological space* (X, τ)*, we have that* $(X, \tau^+) \models \mathsf{S4.2}$ *under the interior semantics.*

Proof. Follows from the above together with the soundness and completeness of $\mathsf{S4.2}$ with respect to extremally disconnected spaces (see, e.g., [1,3]).

Now, we will be using the known result that $\mathsf{S4.2}$ is sound and complete with respect to the class of finite rooted frames $(W, \leq)$ in which $\leq$ is a reflexive, transitive and weakly directed[10] relation [6]. Note that if a frame is rooted and weakly directed, for every pair of points $x, y \in W$, and given that $r \leq x, y$ where r is the root of W, weak directedness grants us the existence of some z such that $z \geq x, y$. But this means that, for every pair of points x and y, the set $\uparrow x \cap \uparrow y$ is nonempty, and thus for every pair of nonempty upsets U and V we have that $U \cap V \neq \varnothing$. This means that every nonempty upset is dense in such a frame, and therefore that the topology of upsets $\tau = \mathrm{Up}(W)$ coincides with τ^+. This fact, paired with the previous lemma, immediately gives us the folowing result.

Lemma 2.5. *Let* $\mathfrak{F} = (W, \leq)$ *be a reflexive, transitive and weakly directed rooted frame. Then the dense interior semantics on* $(W, \mathrm{Up}(W))$ *coincides with the interior semantics on it, which in turn coincides with the standard Kripke semantics on* $(W, \leq)$*. In other words, in any model based on such a frame*

$$x \models K\phi \textit{ if and only if } y \models \phi \textit{ for all } y \geq x.$$

Moreover, we have the following:

Lemma 2.6. *Let* (X, τ) *be some topological space and* $(W, \leq, V)$ *be a finite, rooted, reflexive, transitive and weakly directed Kripke model. Let*

$$f : (X, \tau^+) \twoheadrightarrow (W, \mathrm{Up}(W))$$

be an onto interior map and define

$$V^f(p) := \{x \in X : fx \in V(p)\}.$$

Then for every $x \in X$ *we have that* $(X, \tau, V^f), x \models \phi$ *under the dense interior semantics if and only if* $(W, \leq, V), fx \models \phi$ *under the Kripke semantics.*

[10] A relation $\leq$ is *weakly directed* whenever $x \leq y, z$ implies that there exists $t \geq y, z$.

Proof. Straightforward induction on the complexity of ϕ.

Definition 2.7. *Given topological spaces* (X, τ) *and* (Y, σ)*, we will refer to an open* (*resp. continuous, interior*) *map* $f : (X, \tau^+) \to (Y, \sigma)$ *as a* dense-open (*resp.* dense-continuous, dense-interior) *map* $f : (X, \tau) \to (Y, \sigma)$.

Given all the above, in order to prove completeness it suffices to show that there exists a dense-interior map from any dense-in-itself metrizable space (X, τ) onto any finite S4.2 frame. This way, if a formula ϕ is not a theorem of S4.2, then it will be refuted on some such frame and therefore, by using this map plus Lemma 2.6, we can construct a valuation on (X, τ) which refutes ϕ. And indeed:

Theorem 2.8. *Given a dense-in-itself metrizable space* (X, τ) *and a finite rooted* S4.2 *frame* $(W, \leq)$ *there exists an onto dense-interior map* $\bar{f} : (X, \tau) \twoheadrightarrow (W, \leq)$.

Proof. See Appendix A.1.

This finishes the proof of Theorem 2.2.

2.2 Adding Belief

The logic Stal introduced in Sect. 1.1 is the logic of topo-e-models for the belief and knowledge fragment. The formula $B\phi \leftrightarrow \neg K \neg K \phi$ is provable in Stal (see [18]). In particular, for any formula ϕ in the language $\mathcal{L}_{KB}$, there exists a formula ψ in the language $\mathcal{L}_K$ such that $\models_{\mathsf{Stal}} \phi \leftrightarrow \psi$ (indeed, we get ψ by substituting every instance of B in ϕ with $\neg K \neg K$).

And thus we have the following:

Theorem 2.9. Stal *is sound and complete with respect to any dense-in-itself metrizable space with the dense interior semantics.*

Proof. Soundness follows from the fact that Stal is sound with respect to topo-e-models. For completeness, suppose Stal $\nvdash \phi$ and take ψ in the language $\mathcal{L}_K$ such that $\models_{\mathsf{Stal}} \phi \leftrightarrow \psi$. Then S4.2 $\nvdash \psi$, hence by Theorem 2.2, for any dense-in-itself metrizable space (X, τ), there is a point $x \in X$ and valuation V such that $(X, \tau^+, V), x \not\models \psi$. By soundness and the fact that $\models_{\mathsf{Stal}} \phi \leftrightarrow \psi$, we conclude that ϕ is false at x as well.

2.3 The Global Modality $[\forall]$ and the Logic of $\mathbb{Q}$

Three fragments including the global modality $[\forall]$ will be considered in the present subsection: the *knowledge fragment* (the one which includes the K and $[\forall]$ modalities), the *factive evidence fragment* (including $\Box$ and $[\forall]$) and the *evidence fragment* (including $[\forall]$, $\Box$ and $\Box_0$).

First let us concentrate on the factive evidence fragment. Recall that the logic of this fragment, $\mathsf{Logic}_{\forall\Box}$, consists of $\mathsf{S5}_{\forall}$ plus $\mathsf{S4}_{\Box}$ plus the axiom $[\forall]\phi \to \Box\phi$.

This logic is not complete with respect to $\mathbb{R}$. Consider the following formula:

$$[\forall](\Box p \vee \Box\neg p) \to ([\forall]p \vee [\forall]\neg p) \tag{Con}$$

It is the case that (Con) is not derivable in the logic yet it is always true in $\mathbb{R}$. More generally:

Theorem 2.10 (Shehtman [17]). *A topological space (X, τ) satisfies (Con) if and only if it is connected*[11].

Instead of considering connected spaces and adding (Con) as an axiom to our logic (an axiom which would be hard to justify epistemically), we will show completeness of this fragment (plus the other two mentioned above which include the global modality) with respect to a dense-in-itself, metrizable yet disconnected space, namely $\mathbb{Q}$. This parallels a similar result of [17] stating that $\mathbb{Q}$ is sound and complete with respect to $\mathsf{S4}$ with the global modality.

The Knowledge Fragment $\mathcal{L}_{\forall \boldsymbol{K}}$. Similarly to Subsect. 2.1, we will use completeness of the logic with respect to a class of finite frames, namely:

Lemma 2.11 ([9]). $\mathsf{Logic}_{\forall K}$ *is sound and complete with respect to finite models of the form (W, R, V) where W is a finite set, R is a preorder with a final cluster*[12] *and K and $[\forall]$ are respectively interpreted as the Kripke modality for R and the universal modality.*

Once again, we can easily check the following statement.

Lemma 2.12. *Let $\mathfrak{M} = (W, R, V)$ be a finite preordered model with a final cluster, (X, τ) a topological space and $f : X \twoheadrightarrow W$ an onto dense-interior map. Then for any formula ϕ we have $(X, \tau, V_f), x \models \phi$ iff $\mathfrak{M}, fx \models \phi$, where $V_f(p) = f^{-1}[V(p)]$.*

Then, to prove completeness, it suffices to find such a map from $\mathbb{Q}$. And indeed:

Theorem 2.13. *Given a finite preorder with a final cluster (W, R), there exists an onto dense-interior map $f : (\mathbb{Q}, \tau_{\mathbb{Q}}) \twoheadrightarrow (W, R)$.*

Proof. See Appendix A.2.

The Factive Evidence Fragment $\mathcal{L}_{\forall\Box}$. It is proved in [9] that $\mathsf{Logic}_{\forall\Box}$ is sound and complete with respect to finite relational models of the form $(X, \leq, V)$ where $\leq$ is a preorder.

Thus, to prove completeness of this logic with respect to $\mathbb{Q}$ it suffices to find a suitable open and continuous map from $\mathbb{Q}$ onto any such finite frame. And indeed (by a proof similar to the one of Theorem 2.13) we obtain:

[11] A space X is *connected* if there is no proper subset $A \subseteq X$ such that both A and $X \backslash A$ are open. $\mathbb{R}$ is a connected space.

[12] I.e. a set $A \subseteq W$ such that wRa for all $a \in A$ and all $w \in W$.

Theorem 2.14. *Let $(W, \leq)$ be any finite preordered frame. Then there exists an open, continuous and surjective map $f : (\mathbb{Q}, \tau_{\mathbb{Q}}) \twoheadrightarrow (W, \mathrm{Up}_{\leq}(W))$.*

Again, noting that if we define $V^f(p) = \{x \in \mathbb{Q} : fx \in V(p)\}$ we obtain $x \models \phi$ in $(\mathbb{Q}, \tau_{\mathbb{Q}}, V^f)$ if and only if $fx \models \phi$ in $(W, \leq, V)$, completeness follows.

Adding Basic Evidence: The Evidence Fragment $\mathcal{L}_{\forall\Box\Box_0}$. Let us now account for basic evidence. We take the fragment consisting of the modal operators $\Box$, $[\forall]$ and $\Box_0$. Recall that we interpret formulas of this fragment on topo-e-models (X, τ, E_0, V), where E_0 is a subbasis for (X, τ), in the following way: $x \in [\![\Box_0\phi]\!]$ if and only if there exists $e \in E_0$ with $x \in e \subseteq [\![\phi]\!]$.

The logic of this fragment is $\mathsf{Logic}_{\forall\Box\Box_0}$, as discussed in Sect. 1.5. It is proven in [2] that this logic is sound and complete with respect to finite *pseudo-models*, i.e., structures of the form $(X, \leq, E_0^X, V)$, where $\leq$ is a preorder and E_0^X is a subbasis for $\mathrm{Up}(X)$ with $X \in E_0$.

Completeness is an immediate corollary of the following result:

Theorem 2.15. *Let $\mathfrak{M} = (X, \leq, E_0^X, V)$ be a pseudo-model as defined above and $f : \mathbb{Q} \twoheadrightarrow X$ be an onto interior map. Then if we define $V^{\mathbb{Q}}(p) = f^{-1}[V(p)]$ and $E_0^{\mathbb{Q}} := \{e \subseteq \mathbb{Q} : f[e] \in E_0^X\}$, we have that $\mathfrak{N} = (\mathbb{Q}, \tau_{\mathbb{Q}}, E_0^{\mathbb{Q}}, V^{\mathbb{Q}})$ is a topo-e-model and, for every ϕ in the language, $\mathfrak{N}, x \models \phi$ iff $\mathfrak{M}, fx \models \phi$.*

Proof. See Appendix A.3.

To summarise the results in this subsection we obtain:

Theorem 2.16. *$(\mathbb{Q}, \tau_{\mathbb{Q}})$ is a generic model for the fragments $\mathcal{L}_{\forall\Box}$, $\mathcal{L}_{\forall K}$ and $\mathcal{L}_{\forall\Box\Box_0}$.*

Proof. The result follows from Theorems 2.13, 2.14 and 2.15, respectively.

A Condition for Generic Models. We will now generalize the results in the present subsection to a class of spaces. One can easily see that the only part in the proof of Theorem 2.13 which uses a special property of $\mathbb{Q}$ which $\mathbb{R}$ does not have is that we partition $\mathbb{Q}$ in n subspaces which are homeomorphic to $\mathbb{Q}$ itself. Given a dense-in-itself metrizable space which admits such partition, all the proofs in the present subsection will work *mutatis mutandis*. We will now give a necessary and sufficient condition for such a space to have this property.

Definition 2.17 (Idempotent spaces). *A topological space (X, τ) is* idempotent *whenever (X, τ) is homeomorphic to the sum $(X, \tau) \oplus (X, \tau)$.*[13]

Then the following holds:

Lemma 2.18. *A topological space (X, τ) is idempotent if and only if it can be partitioned in n subspaces homeomorphic to itself for each $n \geq 1$.*

[13] $(X, \tau) \oplus (Y, \sigma)$ is the space which has the disjoint union $(X \times \{1\}) \cup (Y \times \{2\})$ as its underlying set and $\tau \oplus \sigma = \{(U \times \{1\}) \cup (V \times \{2\}) : U \in \tau, V \in \sigma\}$ as its topology.

Proof. If (X, τ) admits a partition in two subspaces homeomorphic to itself, since these are disjoint their union (which is X) is homeomorphic to their sum, which is homeomorphic to $X \oplus X$.

Conversely, if (X, τ) is idempotent we can reason recursively to find that X is homeomorphic to the sum $X_1 \oplus ... \oplus X_n$ where each X_i is a copy of X. Let $f : X_1 \oplus ... \oplus X_n \to X$ be a homeomorphism. Then $\{f[X_1], ..., f[X_n]\}$ constitutes a partition of X in n subspaces, each of them homeomorphic to X.

And thus, we have the general result:

Corollary 2.19. *Any dense-in-itself idempotent metrizable space is sound and complete with respect to* $\mathsf{Logic}_{\forall K}$*,* $\mathsf{Logic}_{\forall\Box}$ *and* $\mathsf{Logic}_{\forall\Box\Box_0}$*.*

All the spaces introduced in Sect. 1, except for $\mathbb{R}$ and $\mathcal{T}_2$, are dense-in-themselves, metrizable and idempotent spaces. And thus:

Theorem 2.20. *The rational line* $\mathbb{Q}$*, the Cantor space* $\mathfrak{C}$ *and the Baire space* $\mathfrak{B}$ *are generic spaces for the fragments* $\mathcal{L}_K$*,* $\mathcal{L}_{KB}$*,* $\mathcal{L}_{\forall\Box}$*,* $\mathcal{L}_{\forall K}$ *and* $\mathcal{L}_{\forall\Box\Box_0}$*.*

Completeness of $\mathsf{Logic}_{\forall\Box\Box_0}$ with Respect to $\mathbb{Q}$ with a Particular Subbasis. While so far in the present section we have shown several of the logics in [2] to be sound and complete with respect to single generic models, we failed to provide a single topo-e-model for the fragment involving the basic evidence modality. Instead, we showed that the corresponding logic is sound and complete with respect to the class of topological evidence models based on $(\mathbb{Q}, \tau_{\mathbb{Q}})$ with arbitrary subbases. But can we find one subbasis $\mathcal{S}$ such that the logic of the single space $(\mathbb{Q}, \tau_{\mathbb{Q}}, \mathcal{S})$ is precisely $\mathsf{Logic}_{\forall\Box\Box_0}$?

This would need to be a subbasis which is not a basis (for otherwise $\Box\phi \leftrightarrow \Box_0\phi$ would be a theorem of the logic). One obvious candidate is perhaps the most paradigmatic case of subbasis-which-is-not-a-basis, namely

$$\mathcal{S} = \{(a, \infty), (-\infty, b) : a, b \in \mathbb{Q}\}.$$

We will show that this subbasis does not lead to a complete logic. To show why, consider the following formula, with three propositional variables p_1, p_2, p_3:

$$\gamma = \bigwedge_{i=1,2,3} (\Box_0 p_i \wedge [\exists]\Box_0\neg p_i) \quad \bigwedge_{i \neq j \in \{1,2,3\}} [\exists](\Box_0 p_i \wedge \neg\Box_0 p_j),$$

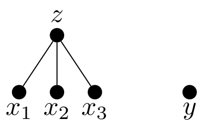
where $[\exists]$ is the dual of $[\forall]$ (i.e. $[\exists]\phi = \neg[\forall]\neg\phi$). Then γ is consistent in the logic yet it cannot be satisfied by any model based on $\mathbb{Q}$ with the aforementioned subbasis.

Indeed, note that, in any topo-e-model, $[\![\Box_0\phi]\!]$ is a union of elements in the subbasis. In particular, with the subbasis $\mathcal{S}$ as defined above, we have that $[\![\Box_0\phi]\!]$ is always of the form $[\![\Box_0\phi]\!] = (-\infty, a) \cup (b, \infty)$ for some $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ (here, we call $(-\infty, -\infty) = (\infty, \infty) = \varnothing$ and $(-\infty, \infty) = \mathbb{Q}$).

Moreover, if the set $[\![\Box_0\phi \wedge [\exists]\Box_0\neg\phi]\!]$ is nonempty, then it is straightforward to see that $[\![\Box_0\phi]\!]$ has to be either of the form (a, ∞) or of the form $(-\infty, a)$ for some $a \in \mathbb{R}$.

By this observation, the first conjunct of γ gives that $[\![\Box_0 p_i]\!]$ is of the form (a, ∞) or $(-\infty, a)$ for some $a \in \mathbb{R}$. By the second conjunct, the sets $[\![\Box_0 p_i]\!]$ and $[\![\Box_0 p_j]\!]$ need to be incomparable for $i \neq j$. But of course, at least two of the sets $[\![\Box_0 p_i]\!]$ have to be of the same form (either $(-\infty, a_i)$ and $(-\infty, a_j)$ or (a_i, ∞) and (a_j, ∞)), and thus it obviously cannot be the case that three such sets are incomparable. Therefore $(\mathbb{Q}, \tau_{\mathbb{Q}}, \mathcal{S}) \models \neg\gamma$.

However, γ is consistent. To show this, we use the fact (see [2]) that the logic is complete with respect to *quasi-models*, i.e. structures of the form $(X, \leq, E_0, V)$, where $\leq$ is a preorder and E_0 is a collection of $\leq$-upsets. $[\forall]$ is interpreted globally, $\Box$ is interpreted as the Kripke modality for $\leq$ and $x \in [\![\Box_0 \phi]\!]$ if and only if there is some $e \in E_0$ with $x \in e \subseteq [\![\phi]\!]$. Let $(X, \leq)$ be the following poset:



and call $e_i = \{x_i, z\}$ for $i = 1, 2, 3$. Let $E_0 = \{e_1, e_2, e_3, \{y\}, X\}$ and $V(p_i) = e_i$ for $i = 1, 2, 3$. It is clear that $(X, \leq, E_0, V)$ is a quasi-model and that $z \models \Box_0 p_i$, $x_i \models \Box_0 p_i \wedge \neg\Box_0 p_j$ for $j \neq i$, and $y \models \Box_0 \neg p_i$. Thus $z \models \gamma$ and γ is therefore consistent in the logic. Since every model based on $\mathbb{Q}$ with E_0 as a subbasis makes $\neg\gamma$ true yet $\neg\gamma \notin \mathsf{Logic}_{\forall\Box\Box_0}$, incompleteness follows.

We conjecture that no particular subbasis will give us completeness. Proving this result, or otherwise finding such a subbasis, constitutes an interesting line of future work.

3 Conclusions and Future Work

We have shown that there are topological spaces which are *generic enough* to capture the logic of topological evidence models, mirroring the McKinsey-Tarski theorem within the framework of topological evidence logics.

A number of questions still remain open. One potential direction for future work is to see whether the completeness results in this paper extend to strong completeness (it is shown in [13] that, under the interior semantics, $\mathsf{S4}$ is strongly complete with respect to any dense-in-itself metrizable space).

It will also be interesting to add a dynamic dimension to this work: one of the advantages of topo-e-models over other topological treatments of evidence logics is how well these models behave dynamically. In [2], dynamic extensions for these logics which include modalities for public announcement or evidence addition are given, along with sound an complete axiomatisations. Thus, one may wonder whether our models are also generic for these logics.

Acknowledgements. We would like to thank Guram Bezhanishvili for helpful discussions and for suggesting the proof of Theorem 2.8. We are also grateful to the reviewers of WoLLIC 2019 for useful comments, which improved the presentation of the paper.

A Appendices

A.1 Proof of Theorem 2.8

Let us take a finite rooted preorder $\mathfrak{F} = (W, \leq)$ and a dense-in-itself metrizable space (X, τ) and construct a dense-interior onto map $\bar{f} : (X, \tau) \twoheadrightarrow (W, \leq)$.[14] For this construction, we will use the following two lemmas. Their proofs[15] can be found respectively in [5, Lemmas 4.13 and 4.22] and [10, Thm. 41].

Lemma A.1. *(i) If $\mathfrak{F} = (W, \leq)$ is a finite rooted preorder, and (G, τ) is a dense-in-itself metrizable space, there exists a continuous, open and surjective map $f : (G, \tau) \to (W, \mathrm{Up}_{\leq}(W))$.*
(ii) (Partition Lemma) *Let X be a dense-in-itself metrizable space and $n \geq 1$. Then there is a partition $\{G, U_1, ..., U_n\}$ of X such that G is a dense-in-itself closed subspace of X with dense complement and each U_i is an open set.*

Lemma A.2. *Given a dense-in-itself metrizable space X and $n \geq 1$, X can be partitioned in n dense sets.*

Note that $\mathfrak{F}$ has a final cluster, i.e., a set $A \subseteq W$ with the property that $w \leq a$ for all $w \in W$ and all $a \in A$. Indeed, let $r \in W$ be the root and let $x, y \in W$ be any two maximal elements (which exist, on account that $\mathfrak{F}$ is finite). Since $r \leq x$ and $r \leq y$, by weak directedness, there is a z such that $x, y \leq z$. But by maximality of x and y, we have that $z \leq x$ and $z \leq y$, hence, by transitivity, $x \leq y$ and $y \leq x$: the maximal elements of $\mathfrak{F}$ form a final cluster. Let this cluster be $A = \{a_1, ..., a_n\}$.

If $W = A$, then we simply partition X in n dense sets $\{A_1, ..., A_n\}$ as per Lemma A.2 and we take $\bar{f}$ to map each $x \in A_i$ to a_i. It is a straightforward check that $\bar{f}$ is dense-open (the image of a dense open set is W) and dense-continuous (the preimage of a nonempty upset is X). Otherwise, let us call $B := W \setminus A$, which is a finite rooted preorder. Let $\{G, U_1, ..., U_n\}$ be a partition of X as given by the Partition Lemma. Since G is a dense-in-itself metrizable space and B is a finite rooted preorder, by Lemma A.1(i), there exists an onto interior map (with respect to the subspace topology of G) $f : G \twoheadrightarrow B$. We extend this map to $\bar{f} : X \twoheadrightarrow W$ by mapping each $x \in U_i$ to a_i.

We now show that $\bar{f}$ is the desired map. It is surjective by construction. It is dense-open, for given a nonempty dense open set $U \subseteq X$, we have that $U \cap G$ is an open set in the subspace topology of G and therefore $\bar{f}[U \cap G] = f[U \cap G]$ is an upset in B. On the other hand, $U \setminus G = U \cap (X \setminus G)$ is the intersection of two dense open sets and therefore is dense open, which means it has nonempty intersection with each of the U_i and hence $\bar{f}[U \setminus G] = A$. Therefore, $\bar{f}[U]$ is the union of an upset in B with A, and thus is an upset in W.

To see that $\bar{f}$ is dense-continuous, take a nonempty upset $U \subseteq W$, which will be a disjoint union $U = B' \cup A$, with B' being an upset in B. Then $\bar{f}^{-1}[B'] =$

[14] We wish to thank Guram Bezhanishvili for the idea of this construction.
[15] Lemma A.1 is a cornerstone of the proof of McKinsey and Tarski's theorem.

$f^{-1}[B']$ is an open set in X and $\bar{f}^{-1}[A] = U_1 \cup ... \cup U_n = X \setminus G$. Therefore, $\bar{f}^{-1}[U]$ is the union of an open set and a dense open set and thus a dense open set. This concludes the proof.

A.2 Proof of Theorem 2.13

Let $(W, \leq)$ be a finite preorder with a final cluster. We have the following:

Lemma A.3. *$(W, \leq)$ is a p-morphic image of a finite disjoint union of finite rooted* S4.2 *frames, via a dense-open and dense-continuous p-morphism.*

Proof. Let $x_1, ..., x_n$ be the minimal elements of W. Now, for $1 \leq i \leq n$ take $W'_i = \uparrow x_i \times \{i\}$. Define an order on $W' = W'_1 \cup ... \cup W'_n$ by: $(x, i) \leq (y, j)$ iff $i = j$ and $x \leq y$. Then $W'_1, ..., W'_n$ are pairwise disjoint finite rooted S4.2 frames (with $A \times \{i\}$ as a final cluster) and $(x, i) \mapsto x$ is a p-morphism from W' onto W. It is easy to see that this mapping is dense-open (for every nonempty open set is dense in W) and dense-continuous (for the preimage of a nonempty W-upset is a W'-upset which contains all the final clusters, and thus is dense).

We can use Lemma A.3 to construct the map: let $W'_1, ..., W'_n$ be the family of pairwise disjoint finite rooted S4.2 frames whose union W' has $(W, \leq)$ as a p-morphic image.

Take $z_1, ..., z_{n-1} \in \mathbb{R} \setminus \mathbb{Q}$ and consider the intervals $A_1 = (-\infty, z_1)$, $A_n = (z_{n-1}, \infty)$ and $A_i = (z_{i-1}, z_i)$ for $1 < i < n$. Now, each A_i, as a subspace, is homeomorphic to $\mathbb{Q}$ (and thus a dense-in-itself metrizable space). From each $(A_i, \tau|_{A_i})$ we can find a dense-open, dense-continuous and surjective map f_i onto W'_i. Then $f = f_1 \cup ... \cup f_n$ is a dense-interior map onto W' which, when composed with the p-morphism in Lemma A.3, gives us the desired map.

A.3 Proof of Theorem 2.15

We show that $E_0^{\mathbb{Q}}$ is a subbasis for $\mathbb{Q}$. First, given that $X \in E_0^X$ and $f[\mathbb{Q}] = X$, we have that $\mathbb{Q} \in E_0^{\mathbb{Q}}$, thus $\bigcup E_0^{\mathbb{Q}} = \mathbb{Q}$.

Now, suppose $p \in U \in \tau_{\mathbb{Q}}$. We show that there exist $e_1^q, ..., e_n^q \in E_0^{\mathbb{Q}}$ such that $p \in e_1^q \cap ... \cap e_n^q \subseteq U$. Note that $fp \in f[U]$ which is an open set. Since E_0^X is a subbasis for $(X, \leq)$ this means that there exist $e_1^x, ..., e_n^x \in E_0^X$ with $fp \in e_1^x \cap ... \cap e_n^x \subseteq f[U]$. Now set

$$e_i^q := f^{-1}[e_i^x] \setminus \{y \notin U : fy \in f[U]\}.$$

The fact that $e_i^q \in E_0^{\mathbb{Q}}$ follows from the fact that $f[e_i^q] = e_i^x$. Indeed, if $y \in f[e_i^q]$ then $y \in ff^{-1}[e_i^x] = e_i^x$ and conversely if $y \in e_i^x$, then either $y \in f[U]$ (in which case $y = fz$ for some $z \in U$ and thus $z \in f^{-1}[e_i^x]$ and therefore $z \notin \{z' \notin U : fz' \in f[U]\}$, which implies $z \in e_i^q$) or $y \notin f[U]$ (in which case $y = fz$ for some z by surjectivity and $z \notin \{z' \notin U : fz' \in f[U]\}$, thus $z \in e_i^q$). In either case, $y \in f[e_i^q]$.

Finally, note that $e_1^q \cap ... \cap e_n^q \subseteq U$. Indeed, for any $x \in e_1^q \cap ... \cap e_n^q$ we have that $fx \in e_1^x \cap ... \cap e_n^x \subseteq f[U]$, and thus by the definition of the e_i^q's it cannot be the case that $x \notin U$.

So for $p \in U \in \tau_{\mathbb{Q}}$ we have found elements $e_1^q, ... e_n^q \in E_0^{\mathbb{Q}}$ such that $p \in e_1^q \cap ... \cap e_n^q \subseteq U$, and therefore $E_0^{\mathbb{Q}}$ is a subbasis.

Now set a valuation $V^{\mathbb{Q}}(p) = \{x \in \mathbb{Q} : fx \in V(p)\}$ and let us show that, for any formula ϕ in the language and any $x \in \mathbb{Q}$, we have that $(\mathbb{Q}, \tau_{\mathbb{Q}}, E_0^{\mathbb{Q}}, V^{\mathbb{Q}}), x \models \phi$ if and only if $(X, \leq, E_0^X, V), fx \models \phi$. This is done by an induction on formulas; the only induction step that requires some attention is the one referring to $\Box_0$.

Let $x \models \Box_0\psi$. This means that there exists some $e \in E_0^{\mathbb{Q}}$ with $x \in e$ and $y \models \psi$ for all $y \in e$. But then $fx \in f[e] \in E_0^X$ and by the induction hypothesis we have $fy \models \psi$ for all $fy \in f[e]$ and thus $fx \models \Box_0\psi$. Conversely, if $fx \in e' \subseteq [\![\psi]\!]^X$ for some $e' \in E_0^X$, we have that $x \in f^{-1}[e'] \in E_0^{\mathbb{Q}}$ and $fy \models \psi$ for each $y \in f^{-1}[e']$ and thus, by induction hypothesis, $y \models \psi$. Therefore $x \models \Box_0\psi$.

References

1. Baltag, A., Bezhanishvili, N., Özgün, A., Smets, S.: The topology of belief, belief revision and defeasible knowledge. In: Grossi, D., Roy, O., Huang, H. (eds.) LORI 2013. LNCS, vol. 8196, pp. 27–40. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-40948-6_3
2. Baltag, A., Bezhanishvili, N., Özgün, A., Smets, S.: Justified belief and the topology of evidence. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 83–103. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_6
3. van Benthem, J., Bezhanishvili, G.: Modal logics of space. In: Aiello, M., Pratt-Hartmann, I., Van Benthem, J. (eds.) Handbook of Spatial Logics, pp. 217–298. Springer, Dordrecht (2007). https://doi.org/10.1007/978-1-4020-5587-4_5
4. van Benthem, J., Pacuit, E.: Dynamic logics of evidence-based beliefs. Stud. Log. **99**(1–3), 61–92 (2011)
5. Bezhanishvili, G., Bezhanishvili, N., Lucero-Bryan, J., van Mill, J.: A new proof of the McKinsey-Tarski theorem. Stud. Log. **106**(6), 1291–1311 (2018)
6. Blackburn, P., De Rijke, M., Venema, Y.: Modal Logic, vol. 53. Cambridge University Press, Cambridge (2001)
7. Engelking, R.: General Topology. Sigma Series in Pure Mathematics. Heldermann Verlag, Berlin (1989)
8. Gettier, E.L.: Is justified true belief knowledge? Analysis **23**(6), 121–123 (1963)
9. Goranko, V., Passy, S.: Using the universal modality: gains and questions. J. Log. Comput. **2**(1), 5–30 (1992)
10. Hewitt, E.: A problem of set-theoretic topology. Duke Math. J. **10**(2), 309–333 (1943)
11. Hintikka, J.: Knowledge and Belief: An Introduction to the Logic of the Two Notions. Contemporary Philosophy. Cornell University Press, Ithaca (1962)
12. Kelly, K.: The Logic of Reliable Inquiry. Logic and Computation in Philosophy. Oxford University Press, Oxford (1996)
13. Kremer, P.: Strong completeness of S4 for any dense-in-itself metric space. Rev. Symb. Log. **6**(3), 545–570 (2013)

14. McKinsey, J.C.C., Tarski, A.: The algebra of topology. Ann. Math. **45**, 141–191 (1944)
15. Munkres, J.R.: Topology. Prentice Hall, Upper Saddle River (2000)
16. Rasiowa, H., Sikorski, R.: The Mathematics of Metamathematics. Institut Mathematyczny, Polskiej Akademii Nauk: Monographie Mathematyczne. PWN-Polish Scientific Publishers, Warszawa (1970)
17. Shehtman, V.: "Everywhere" and "Here". J. Appl. Non-Class. Log. **9**(2–3), 369–379 (1999)
18. Stalnaker, R.: On logics of knowledge and belief. Philos. Stud. **128**(1), 169–199 (2006)
19. Steinsvold, C.: Topological models of belief logics. City University of New York (2006)

A Self-contained Provability Calculus for Γ_0

David Fernández-Duque[1(✉)] and Eduardo Hermo-Reyes[2]

[1] Ghent University, Ghent, Belgium
David.FernandezDuque@UGent.be
[2] University of Barcelona, Barcelona, Spain
ehermo.reyes@ub.edu

Abstract. Beklemishev introduced an ordinal notation system for the Feferman-Schütte ordinal Γ_0 based on the *autonomous expansion* of provability algebras. In this paper we present the logic **BC** (for *Bracket Calculus*). The language of **BC** extends said ordinal notation system to a strictly positive modal language. Thus, unlike other provability logics, **BC** is based on a purely modal signature that gives rise to an ordinal notation system instead of modalities indexed by some ordinal given *a priori*. Moreover, since the order between these notations can be established in terms of derivability within the calculus, the inferences in this system can be carried out without using any external property of ordinals. The presented logic is proven to be equivalent to $\mathbf{RC}_{\Gamma_0}$, that is, to the strictly positive fragment of $\mathbf{GLP}_{\Gamma_0}$.

Keywords: Provability logic · Proof theory · Ordinal analysis

1 Introduction

In view of Gödel's second incompleteness theorem, we know that the consistency of any sufficiently powerful formal theory cannot be established using purely 'finitary' means. Since then, the field of proof theory, and more specifically of ordinal analysis, has been successful in measuring the non-finitary assumptions required to prove consistency assertions via computable ordinals. Among the benefits of this work is the ability to linearly order natural theories of arithmetic with respect to notions such as their 'consistency strength' (e.g., their Π^0_1 ordinal) or their 'computational strength' (their Π^0_2 ordinal). Nevertheless, the assignment of these proof-theoretic ordinals to formal theories depends on a choice of a 'natural' presentation for such ordinals, with well-known pathological examples having been presented by Kreisel [24] and Beklemishev [8].[1] This raises

[1] The Π^1_1 ordinal of a theory is another measure of its strength and does not have such sensitivity to a choice of notation system. However, there are some advantages to considering Π^0_1 ordinals, among others that they give a finer-grained classification of theories.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 195–207, 2019.
https://doi.org/10.1007/978-3-662-59533-6_12

the question of what it means for something to be a natural ordinal notation system, or even if such a notion is meaningful at all.

One possible approach to this problem comes from Beklemishev's ordinal analysis of Peano arithmetic (**PA**) and related theories via their *provability algebras.* Consider the Lindenbaum algebra of the language of arithmetic modulo provability in a finitary theory U such as primitive recursive arithmetic (**PRA**) or the weaker *elementary arithmetic* (**EA**). For each natural number n and each formula φ, the n-consistency of φ is the statement that all Σ_n consequences of $U+\varphi$ are true, formalizable by some arithmetical formula $\langle n\rangle\varphi$ (where φ is identified with its Gödel number). In particular, $\langle 0\rangle\varphi$ states that φ is consistent with U. An *iterated consistency assertion,* also called *worm,* is then an expression of the form $\langle n_1\rangle \dots \langle n_k\rangle\top$, where $\top$ is some fixed tautology.

The operators $\langle n\rangle$ and their duals $[n]$ satisfy Japaridze's provability logic **GLP** [22], a multi-modal extension of the Gödel-Löb provability logic **GL** [12]. As Beklemishev showed, the set of worms is well-ordered by their *consistency strength* $<_0$, where $A <_0 B$ if $A \to \langle 0\rangle B$ is derivable in **GLP**. Moreover, this well-order is of order-type ε_0, which characterizes the proof-theoretical strength of **PA**. This tells us that proof-theoretic ordinals already appear naturally within Lindenbaum algebras of arithmetical theories.

Beklemishev also observed that this process can be extended by considering worms with ordinal entries. Extensions of **GLP**, denoted $\mathbf{GLP}_\Lambda$, have been considered in cases where Λ is an ordinal [3,14,18] or even an arbitrary linear order [6]. Proof-theoretic interpretations for $\mathbf{GLP}_\Lambda$ have been developed by Fernández-Duque and Joosten [17] for the case where Λ is a computable well-order. Nevertheless, we now find ourselves in a situation where an expression $\langle\lambda\rangle\varphi$ requires a system of notation for the ordinal λ. Fortunately we may 'borrow' this notation from finitary worms and represent λ itself as a worm. Iterating this process we obtain the *autonomous worms,* whose order types are exactly the ordinals below the Feferman-Schütte ordinal Γ_0. By iterating this process we obtain a notation system for worms which uses only parentheses, as ordinals (including natural numbers) can be iteratively represented in this fashion. Thus the worm $\langle 0\rangle\top$ becomes `()`, $\langle 1\rangle\top$ becomes `(())`, $\langle\omega\rangle\top$ becomes `((()))`, etc.

These are Beklemishev's *brackets,* which provide a notation system for Γ_0 without any reference to an externally given ordinal [3]. However, it has the drawback that the actual computation of the ordering between different worms is achieved via a translation into a traditional ordinal notation system. Our goal is to remove the need for such an intermediate step by providing an autonomous calculus for determining the ordering relation (and, more generally, the logical consequence relation) between bracket notations. To this end we present the *bracket calculus;* our main result is that our calculus is sound and complete with respect to the intended embedding into $\mathbf{GLP}_{\Gamma_0}$.

2 The Reflection Calculus

Japaridze's logic **GLP** gained much interest due to Beklemishev's proof-theoretic applications [2]; however, from a modal logic point of view, it is not an easy system to work with. To this end, in [4,5,13] Beklemishev and Dashkov introduced the system called *Reflection Calculus*, **RC**, that axiomatizes the fragment of $\mathbf{GLP}_\omega$ consisting of implications of strictly positive formulas. This system is much simpler than $\mathbf{GLP}_\omega$ but yet expressive enough to maintain its main proof-theoretic applications. In this paper we will focus exclusively on reflection calculi, but the interested reader may find more information on the full **GLP** in the references provided.

Similar to $\mathbf{GLP}_\Lambda$, the signature of $\mathbf{RC}_\Lambda$ contains modalities of the form $\langle \alpha \rangle$ for $\alpha \in \Lambda$. However, since this system only considers strictly positive formulas, the signature does not contain negation, disjunction or modalities $[\alpha]$. Thus, the set of formulas in this signature is defined as follows:

Definition 1. *Fix an ordinal Λ. By $\mathbb{F}_\Lambda$ we denote the set of formulas built-up by the following grammar:*

$$\varphi := \top \mid p \mid (\varphi \wedge \psi) \mid \langle \alpha \rangle \varphi \quad \textit{for } \alpha \in \Lambda.$$

Next we define a consequence relation over $\mathbb{F}_\Lambda$. For the purposes of this paper, a *deductive calculus* is a pair $\mathbf{X} = (\mathbb{F}_\mathbf{X}, \vdash_\mathbf{X})$ such that $\mathbb{F}_\mathbf{X}$ is some set, the *language* of $\mathbf{X}$, and $\vdash_\mathbf{X} \subseteq \mathbb{F}_\mathbf{X} \times \mathbb{F}_\mathbf{X}$. We write $\varphi \cong_\mathbf{X} \psi$ for $\varphi \vdash_\mathbf{X} \psi$ and $\psi \vdash_\mathbf{X} \varphi$. We will omit the subscript $\mathbf{X}$ when this does not lead to confusion, including in the definition below, where $\vdash$ denotes $\vdash_{\mathbf{RC}_\Lambda}$.

Definition 2. *Given an ordinal Λ, the calculus $\mathbf{RC}_\Lambda$ over $\mathbb{F}_\Lambda$ is given by the following set of axioms and rules:*

Axioms:

1. $\varphi \vdash \varphi$, $\quad \varphi \vdash \top$;
2. $\varphi \wedge \psi \vdash \varphi$, $\quad \varphi \wedge \psi \vdash \psi$;
3. $\langle \alpha \rangle \langle \alpha \rangle \varphi \vdash \langle \alpha \rangle \varphi$;
4. $\langle \alpha \rangle \varphi \vdash \langle \beta \rangle \varphi$ *for* $\alpha > \beta$;
5. $\langle \alpha \rangle \varphi \wedge \langle \beta \rangle \psi \vdash \langle \alpha \rangle \big(\varphi \wedge \langle \beta \rangle \psi\big)$ *for* $\alpha > \beta$.

Rules:

1. *If* $\varphi \vdash \psi$ *and* $\varphi \vdash \chi$, *then* $\varphi \vdash \psi \wedge \chi$;
2. *If* $\varphi \vdash \psi$ *and* $\psi \vdash \chi$, *then* $\varphi \vdash \chi$;
3. *If* $\varphi \vdash \psi$, *then* $\langle \alpha \rangle \varphi \vdash \langle \alpha \rangle \psi$;

For each $\mathbf{RC}_\Lambda$-formula φ, we can define the signature of φ as the set of ordinals occurring in any of its modalities.

Definition 3. *For any $\varphi \in \mathbb{F}_\Lambda$, we define the* signature *of φ, $\mathcal{S}(\varphi)$, as follows:*

1. $\mathcal{S}(\top) = \mathcal{S}(p) = \varnothing$;
2. $\mathcal{S}(\varphi \wedge \psi) = \mathcal{S}(\varphi) \cup \mathcal{S}(\psi)$;
3. $\mathcal{S}(\langle \alpha \rangle \varphi) = \{\alpha\} \cup \mathcal{S}(\varphi)$.

With the help of this last definition we can make the following observation:

Lemma 1. *For any* φ, $\psi \in \mathbb{F}_\Lambda$*:*

1. *If* $\mathcal{S}(\psi) \neq \varnothing$ *and* $\varphi \vdash \psi$, *then* $\max \mathcal{S}(\varphi) \geq \max \mathcal{S}(\psi)$*;*
2. *If* $\mathcal{S}(\varphi) = \varnothing$ *and* $\varphi \vdash \psi$, *then* $\mathcal{S}(\psi) = \varnothing$.

Proof. By an easy induction on the length of the derivation of $\varphi \vdash \psi$.

The reflection calculus has natural arithmetical [17], Kripke [5,13], algebraic [11] and topological [7,14,20,21] interpretations for which it is sound and complete, but in this paper we will work exclusively with reflection calculi from a syntactical perspective. Other variants of the reflection calculus have been proposed, for example working exclusively with worms [1], admitting the transfinite iteration of modalities [19], or allowing additional conservativity operators [9,10].

3 Worms and the Consistency Ordering

In this section we review the consistency ordering between worms, along with some of their basic properties.

Definition 4. *Fix an ordinal* Λ*. The set of* worms *in* $\mathbb{F}_\Lambda$, $\mathbb{W}_\Lambda$, *is recursively defined as follows: 1.* $\top \in \mathbb{W}_\Lambda$*; 2. If* $A \in \mathbb{W}_\Lambda$ *and* $\alpha < \Lambda$, *then* $\langle \alpha \rangle A \in \mathbb{W}_\Lambda$. *Similarly, we inductively define for each* $\alpha \in \Lambda$ *the set of worms* $\mathbb{W}_\Lambda^{\geq\alpha}$ *where all ordinals are at least* α*: 1.* $\top \in \mathbb{W}_\Lambda^{\geq\alpha}$*; 2. If* $A \in \mathbb{W}_\Lambda^{\geq\alpha}$ *and* $\beta \geq \alpha$, *then* $\langle \beta \rangle A \in \mathbb{W}_\Lambda^{\geq\alpha}$.

Definition 5. *Let* $A = \langle \xi_1 \rangle \ldots \langle \xi_n \rangle \top$ *and* $B = \langle \zeta_1 \rangle \ldots \langle \zeta_m \rangle \top$ *be worms. Then, define* $AB = \langle \xi_1 \rangle \ldots \langle \xi_n \rangle \langle \zeta_1 \rangle \ldots \langle \zeta_m \rangle \top$. *Given an ordinal* λ, *define* $\lambda \uparrow A$ *to be* $\langle \lambda + \xi_1 \rangle \ldots \langle \lambda + \xi_n \rangle \top$.

Often we will want to put an extra ordinal between two worms, and we write $B\langle\lambda\rangle A$ for $B(\langle\lambda\rangle A)$. Next, we define the consistency ordering between worms.

Definition 6. *Given an ordinal* Λ, *we define a relation* $<_0$ *on* $\mathbb{W}_\Lambda$ *by* $B <_0 A$ *if and only if* $A \vdash \langle 0 \rangle B$. *We also define* $B \leq_0 A$ *if* $B <_0 A$ *or* $B \cong A$.

The ordering $\leq_0$ has some nice properties. Recall that if A is a set (or class), a *preorder* on A is a trasitive, reflexive relation $\preccurlyeq \subseteq A \times A$. The preorder $\preccurlyeq$ is *total* if, given $a, b \in A$, we always have that $a \preccurlyeq b$ or $b \preccurlyeq a$, and *antisymmetric* if whenever $a \preccurlyeq b$ and $b \preccurlyeq a$, it follows that $a = b$. A total, antisymmetric preorder is a *linear order.* We say that $\langle A, \preccurlyeq \rangle$ is a *pre-well-order* if $\preccurlyeq$ is a total preorder and every non-empty $B \subseteq A$ has a minimal element (i.e., there is $m \in B$ such that $m \preccurlyeq b$ for all $b \in B$). A *well-order* is a pre-well-order that is also linear. Note that pre-well-orders are not the same as well-quasiorders (the latter need not be total). Pre-well-orders will be convenient to us because, as we will see, worms are pre-well-ordered but not linearly ordered.

Theorem 1. *For any ordinal Λ, the relation $\leq_0$ is a pre-well-order on $\mathbb{W}_\Lambda$.*

Note that $\leq_0$ fails to be a linear order merely because it is not antisymmetric. To get around this, one may instead consider worms modulo provable equivalence. Alternately, as Beklemishev has done [3], one can choose a canonical representative for each worm.

Definition 7 (Beklemishev Normal Form). *A worm $A \in \mathbb{W}$ is defined recursively to be in* BNF *if either*

1. $A = \top$, *or*
2. $A := A_k \langle \alpha \rangle A_{k-1} \langle \alpha \rangle \ldots \langle \alpha \rangle A_0$ *with*
   - $\alpha = \min \mathcal{S}(A)$;
   - $k \geq 1$;
   - $A_i \in \mathbb{W}_\Lambda^{\geq \alpha+1}$, *for* $i \leq k$;

   such that $A_i \in \mathsf{BNF}$ *and* $A_i \vdash_{\mathbf{RC}_{\Gamma_0}} \langle \alpha + 1 \rangle A_{i+1}$ *for each* $i < k$.

This definition essentially mirrors that of Cantor normal forms for ordinals. The following was proven in [3].

Theorem 2. *Given any worm A there is a unique $A' \in \mathsf{BNF}$ such that $A \cong A'$.*

4 Hyperexponential Notation for Γ_0

Ordinal numbers are canonical representatives of well-orders; we assume some basic familiarity with them, but a detailed account can be found in a text such as [23]. In particular, since the set of worms modulo equivalence yields a well-order, we can use ordinal numbers to measure their order-types. More generally, if $\mathfrak{A} = \langle A, \preccurlyeq \rangle$ is any pre-well-order, for $a \in A$ we may define an ordinal $o(a) = \sup_{b \prec a}(o(b) + 1)$, where by convention $\sup \varnothing = 0$, representing the *order-type* of a; this definition is sound since $\mathfrak{A}$ is pre-well-ordered. The rank of $\mathfrak{A}$ is then defined as $\sup_{a \in A}(o(a) + 1)$.

The following lemma is useful in characterizing the rank function [15].

Lemma 2. *Let $\langle A, \preccurlyeq \rangle$ be a well-order. Then $o \colon A \to \mathsf{Ord}$ is the unique function such that*

1. $x \prec y$ *implies that* $o(x) < o(y)$,
2. *if* $\xi < o(x)$ *then* $\xi = o(y)$ *for some* $y \in A$.

In order to compute the ordinals $o(A)$, let us recall a notation system for Γ_0 using *hyperexponentials* [16]. The class of all ordinals will be denoted Ord, and ω denotes the first infinite ordinal. Recall that many number-theoretic operations such as addition, multiplication and exponentiation can be defined on the class of ordinals by transfinite recursion. The ordinal exponential function $\xi \mapsto \omega^\xi$ is of particular importance for representing ordinal numbers. When working with order types derived from reflection calculi, it is convenient to work with a slight variant of this exponential.

Definition 8 (Exponential function). *The* exponential function *is the normal function* $e\colon \mathsf{Ord} \to \mathsf{Ord}$ *given by* $\xi \mapsto -1 + \omega^{\xi}$.

The function e is an example of a *normal function,* i.e. $f\colon \mathsf{Ord} \to \mathsf{Ord}$ which is strictly increasing and *continuous,* in the sense that if λ is a limit then $f(\lambda) = \sup_{\xi<\lambda} f(\xi)$. When $f\colon X \to X$, it is natural and often useful to ask whether f has *fixed points,* i.e., solutions to the equation $x = f(x)$. In particular, normal functions have many fixed points:

Proposition 1. *Every normal function on* Ord *has arbitrarily large fixed points.*

The first ordinal α such that $\alpha = \omega^{\alpha}$ is the limit of the ω-sequence $(\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots)$, and is usually denoted ε_0. Every $\xi < \varepsilon_0$ can be written in terms of 0 using only addition and the function $\omega \mapsto \omega^{\xi}$ via its Cantor normal form. The hyperexponential function is then a natural transfinite iteration of the ordinal exponential which remains normal after each iteration.

Definition 9 (Hyperexponential functions). *The* hyperexponential functions $(e^{\zeta})_{\zeta\in\mathsf{Ord}}$ *are the unique family of normal functions that satisfy*

1. $e^1 = e$,
2. $e^{\alpha+\beta} = e^{\alpha} \circ e^{\beta}$ *for all* α *and* β, *and*
3. *if* $(f^{\xi})_{\xi\in\mathsf{Ord}}$ *is a family of functions satisfying 1 and 2, then for all* $\alpha, \beta \in \mathsf{Ord}$, $e^{\alpha}\beta \leq f^{\alpha}\beta$.

Fernández-Duque and Joosten proved that the hyperexponentials are well-defined [16]. If $\alpha > 0$ then $e^{\alpha}\beta$ is always *additively indecomposable* in the sense that $\xi, \zeta < e^{\alpha}\beta$ implies that $\xi + \zeta < e^{\alpha}\beta$; note that zero is additively indecomposable according to our definition. In [15] it is also shown that the function $\xi \mapsto e^{\xi}1$ is itself a normal function, hence it has a least non-zero fixed point: this fixed point is the Feferman-Schütte ordinal, Γ_0. Just like ordinals below ε_0 may be written using 0, addition, and ω-exponentiation, every ordinal below Γ_0 may be written in terms of 0, 1, addition and the function $(\xi, \zeta) \mapsto e^{\xi}\zeta$.

Theorem 3. *Let* A, B *be worms and* α *be an ordinal. Then,*

1. $o(\top) = 0$,
2. $o(B\langle\, 0 \,\rangle A) = o(A) + 1 + o(B)$, *and*
3. $o(\alpha \uparrow A) = e^{\alpha} o(A)$.

Remark 1. We will not discuss notation systems based on the Veblen hierarchy $(\phi_{\xi})_{\xi\in\mathsf{Ord}}$, but a fairly simple translation from one notation to the other is given in [16]. Beklemishev [3] gives an explicit computation of o in terms of the standard Veblen functions.

Finally we mention a useful property of o proven in [15], where $\max A$ is the greatest ordinal appearing in A.

Lemma 3. *Let* $A \neq \top$ *be a worm and* μ *an ordinal. Then,*

1. *if* $\mu \leq \max A$, *then* $o(\langle\mu\rangle\top) \leq o(A)$, *and*
2. *if* $\max A < \mu$, *then* $o(A) < o(\langle\mu\rangle\top)$.

5 Beklemishev's Bracket Notation System for Γ_0

Before we introduce the full bracket calculus, let us review Beklemishev's notation system from [3].

Definition 10. *By $\mathbb{W}_{()}$ we denote the smallest set such that: 1. $\top \in \mathbb{W}_{()}$; 2. if $a, b \in \mathbb{W}_{()}$, then $(a)b \in \mathbb{W}_{()}$.*

By convention we shall write $()a$, for $a \in \mathbb{W}_{()}$ to the denote $(\top)a \in \mathbb{W}_{()}$.

We can define a translation $* : \mathbb{W}_{()} \to \mathbb{W}$ in such a way that an element $a \in \mathbb{W}_{()}$ will denote the ordinal $o(a^*)$:

1. $\top^* = \top$
2. $\big(\,(a)b\,\big)^* = \langle\, o(a^*)\,\rangle b^*$.

Therefore, we can also define $o^* : \mathbb{W}_{()} \to \Gamma_0$ as $o^*(a) = o(a^*)$.

Next we make some observations about how the ordinals represented by worms in $\mathbb{W}_{()}$ can be bounded in terms of the maximum number of nested brackets occurring in them. With this purpose, we introduce the following two definitions.

Definition 11. *For $a \in \mathbb{W}_{()}$, we define the nesting of a, $\mathsf{N}(a)$, as the maximum number of nested brackets. That is:*

1. $\mathsf{N}(\top) = 0$;
2. $\mathsf{N}(\,(a)b) = \max\big(\mathsf{N}(a)+1,\, \mathsf{N}(b)\big)$.

Definition 12. *We recursively define the function $h : \mathbb{N} \to \Gamma_0$ as follows:*

1. $h(0) = 0$;
2. $h(n+1) = e^{h(n)}1$.

Note that $\lim_{n\to\infty} h(n) = \Gamma_0$. In the following proposition we can find upper and lower bounds for any ordinal $o^*(a)$, with $a \in \mathbb{W}_{()}$, according to the nesting of a.

Proposition 2. *For $a \in \mathbb{W}_{()}$, if $\mathsf{N}(a) = n$, then $h(n) \leq o^*(a) < h(n+1)$.*

Proof. By induction on n. If $\boldsymbol{n = 0}$ then we must have $a = \top$, hence $h(0) = 0 = o^*(a) < 1 = h(1)$.

For $\boldsymbol{n = n' + 1}$, we have that $a = (a_0)\ldots(a_m)$ for some $m \in \omega$. Moreover,

1. $\mathsf{N}(a_i) \leq n'$ for $i,\ 0 \leq i \leq m$;
2. there is a_J such that $\mathsf{N}(a_J) = n'$.

Thus by the I.H. we get that $a^* = \langle\alpha_0\rangle\ldots\langle\alpha_k\rangle\top$ such that:

1. For each i, $\alpha_i < h(n'+1)$;
2. There is $\alpha_J \geq h(n')$.

By Lemma 3,

$$o(\langle h(n')\rangle \top) \leq o(a^*) < o(\langle h(n'+1)\rangle \top);$$

but by Theorem 3 $o(\langle h(n')\rangle \top) = e^{h(n')}1 = h(n)$, while $o(\langle h(n'+1)\rangle \top) = e^{h(n)}1 = h(n+1)$, as needed.

As a consequence of this last proposition, we get the following corollaries.

Corollary 1. *For $a \in \mathbb{W}_{()}$, if $\mathsf{N}(a) = n$, then $a^* \in \mathbb{W}_{h(n)}$.*

Corollary 2. *For $a, b \in \mathbb{W}_{()}$, $o^*(a) \geq o^*(b) \ \Rightarrow\ \mathsf{N}(a) \geq \mathsf{N}(b)$.*

Proof. We reason by contrapositive applying Proposition 2.

6 The Bracket Calculus

In this section we introduce the *Bracket Calculus*, denoted **BC**. This system is analogous to $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$ and, as we will see later, both systems can be shown to be equivalent under a natural translation of **BC**-formulas into $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-formulas.

The main feature of **BC** is that it is based on a signature that uses purely modal notations instead of modalities indexed by ordinals. Moreover, since the order between these notations can be established in terms of derivability within the calculus, the inferences in this system can be carried out without using any external property of ordinals. In this sense, we say that **BC** provides an autonomous provability calculus.

The set of **BC**-formulas, $\mathbb{F}_{()}$, is defined by extending $\mathbb{W}_{()}$ to a strictly positive signature.

Definition 13. *By $\mathbb{F}_{()}$ we denote the set of formulas built-up by the following grammar:*

$$\varphi := \top \mid p \mid \varphi \wedge \psi \mid (a)\,\varphi \quad \text{for } a \in \mathbb{W}_{()}.$$

Similarly to **RC**, **BC** is based on *sequents*, i.e. expressions of the form $\varphi \vdash \psi$, where $\varphi, \psi \in \mathbb{F}_{()}$. In addition to this, we will also use $a \succeq b$, for $a, b \in \mathbb{W}_{()}$, to denote that either $a \vdash ()\,b$ or $a \vdash b$ are derivable. Analogously, we will use $a \succ b$ to denote that the sequent $a \vdash ()b$ is derivable.

Definition 14. ***BC** is given by the following set of axioms and rules:*

Axioms: 1. $\varphi \vdash \varphi, \quad \varphi \vdash \top$; 2. $\varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi$;
Rules:

1. *If $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;*
2. *If $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;*
3. *If $\varphi \vdash \psi$ and $a \succeq b$, then $(a)\,\varphi \vdash (b)\,\psi$ and $(a)\,(b)\,\varphi \vdash (b)\,\psi$;*
4. *If $a \succ b$, then $(a)\,\varphi \wedge (b)\,\psi \vdash (a)\,\big(\varphi \wedge (b)\,\psi\big)$.*

7 Translation and Preservability

In this section we introduce a way of interpreting **BC**-formulas as $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-formulas, and prove that under this translation, both systems can derive exactly the same sequents.

Definition 15. *We define a translation τ between $\mathbb{F}_{()}$ and $\mathbb{F}_{\Gamma_0}$, $\tau : \mathbb{F}_{()} \to \mathbb{F}_{\Gamma_0}$, as follows:*

1. $\top^\tau = \top$;
2. $p^\tau = p$;
3. $(\varphi \wedge \psi)^\tau = (\varphi^\tau \wedge \psi^\tau)$;
4. $(\,(a)\,\varphi)^\tau = \langle\, o^*(a)\,\rangle\varphi^\tau$.

Note that for $a \in \mathbb{W}_{()}$, $a^\tau = a^*$. From this and a routine induction, the following can readily be verified.

Lemma 4. *Given $\varphi \in \mathbb{F}_{()}$ and $\alpha \in \mathcal{S}(\varphi^\tau)$, there is a subformula $a \in \mathbb{W}_{()}$ of φ such that $\alpha = o^*(a)$.*

The following lemma establishes the preservability of **BC** with respect to $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$, under τ.

Lemma 5. *For any $\varphi, \psi \in \mathbb{F}_{()}$: $\varphi \vdash_{\mathbf{BC}} \psi \implies \varphi^\tau \vdash_{\mathbf{RC}_{\boldsymbol{\Gamma_0}}} \psi^\tau$.*

Proof. By induction on the length of the derivation. We can easily check that the set of axioms of **BC** is preserved under τ. Likewise, the cases for a derivation ending on Rules 1 or 2 are straightforward. Thus, we only check Rules 3 and 4.

Regarding Rule 3, we need to prove that if $a \succeq b$ then both sequents $\langle o^*(a)\rangle\varphi^\tau \vdash \langle o^*(b)\rangle\psi^\tau$ and $\langle o^*(a)\rangle\langle o^*(b)\rangle\varphi^\tau \vdash \langle o^*(b)\rangle\psi^\tau$ are derivable in $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$. We can make the following observations by applying the I.H.:

1. Since $a \succeq b$, we have that either $a^\tau \vdash \langle 0\rangle b^\tau$ or $a^\tau \vdash b^\tau$ are derivable in $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$. Therefore, $o(a^\tau) \geq o(b^\tau)$. Since $o^*(a) = o(a^*) = o(a^\tau)$ and the same equality holds for b, we have that $o^*(a) \geq o^*(b)$.
2. We also have that $\varphi^\tau \vdash_{\mathbf{RC}_{\boldsymbol{\Gamma_0}}} \psi^\tau$ and thus, by Rule 3 of $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$ we obtain that $\langle o^*(a)\rangle\varphi^\tau \vdash \langle o^*(a)\rangle\psi^\tau$ and $\langle o^*(a)\rangle\langle o^*(b)\rangle\varphi^\tau \vdash \langle o^*(a)\rangle\langle o^*(b)\rangle\psi^\tau$ are derivable in $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$.

On the one hand, by these two facts together with Axiom 4 we obtain that $\langle o^*(a)\rangle\varphi^\tau \vdash_{\mathbf{RC}_{\boldsymbol{\Gamma_0}}} \langle o^*(b)\rangle\psi^\tau$. On the other hand, we can combine Axioms 4 and 3 to get that $\langle o^*(a)\rangle\langle o^*(b)\rangle\varphi^\tau \vdash_{\mathbf{RC}_{\boldsymbol{\Gamma_0}}} \langle o^*(b)\rangle\psi^\tau$.

We follow an analogous reasoning in the case of Rule 4. By the I.H. we have that $a^\tau \vdash_{\mathbf{RC}_{\boldsymbol{\Gamma_0}}} \langle 0\rangle b^\tau$. Therefore $o^*(a) > o^*(b)$ and by Axiom 5, $\langle o^*(a)\rangle\varphi \wedge \langle o^*(b)\rangle\psi \vdash_{\mathbf{RC}_{\boldsymbol{\Gamma_0}}} \langle o^*(a)\rangle\big(\varphi \wedge \langle o^*(b)\rangle\psi\big)$.

With the following definition we fix a way of translating $\mathbb{F}_{\Gamma_0}$-formulas into formulas in $\mathbb{F}_{()}$. However, since different words in $\mathbb{W}_{()}$ might denote the same ordinal, we need a normal form theorem for $\mathbb{W}_{()}$.

Definition 16. *We define* $\mathsf{NF} \subset \mathbb{W}_{()}$ *to be the smallest set of* $\mathbb{W}_{()}$*-words such that* $\top \in \mathsf{NF}$ *and for any* $(a)b \in \mathbb{W}_{()}$*, if* $a, b \in \mathsf{NF}$ *and* $\big((a)b\big)^* \in \mathsf{BNF}$*, then* $(a)b \in \mathsf{NF}$*.*

Every element of $\mathbb{W}_{()}$ has a unique normal form, as shown by L. Beklemishev in [3].

Theorem 4 (Beklemishev). *For each* $\alpha \in \Gamma_0$ *we can associate a unique* $a_\alpha \in \mathsf{NF}$ *such that* $o^*(a_\alpha) = \alpha$*.*

Proposition 3 (Beklemishev). *The ordering* $\big(\mathsf{NF}, <_0\big)$ *is a well-ordering of order type* Γ_0*.*

Now we are ready to translate $\mathbb{F}_{\Gamma_0}$-formulas into $\mathbb{F}_{()}$-formulas.

Definition 17. *We define a translation* ι *between* $\mathbb{F}_{\Gamma_0}$ *and* $\mathbb{F}_{()}$*,* $\iota : \mathbb{F}_{\Gamma_0} \to \mathbb{F}_{()}$*, as follows:*

1. $\top^\iota = \top$;
2. $p^\iota = p$;
3. $(\varphi \wedge \psi)^\iota = (\varphi^\iota \wedge \psi^\iota)$;
4. $(\langle\alpha\rangle\, \varphi)^\iota = (a_\alpha)\varphi^\iota$.

The following remark follows immediately from the definitions of τ and ι.

Remark 2. For any $\varphi \in \mathbb{F}_{\Gamma_0}$, $(\varphi^\iota)^\tau = \varphi$. In particular, if $A \in \mathbb{W}_{\Gamma_0}$ is a worm then A^ι is a worm and $o^*(A^\iota) = o((A^\iota)^*) = o((A^\iota)^\tau) = o(A)$.

With the next definition, we extend the nesting $\mathsf{N}(a)$ of $a \in \mathbb{W}_{()}$ to $\mathbb{F}_{()}$-formulas.

Definition 18. *For* $\varphi \in \mathbb{F}_{()}$*, we define the nesting of* φ*,* $\mathsf{Nt}(\varphi)$*, as the maximum number of nested brackets. That is:*

1. $\mathsf{Nt}(\top) = \mathsf{Nt}(p) = \mathsf{N}(\top)$;
2. $\mathsf{Nt}(\varphi \wedge \psi) = \max\big(\mathsf{Nt}(\varphi), \mathsf{Nt}(\psi)\big)$;
3. $\mathsf{Nt}((a)\,\varphi) = \max\big(\mathsf{N}((a)), \mathsf{Nt}(\varphi)\big) = \max\big(\mathsf{N}(a) + 1, \mathsf{Nt}(\varphi)\big)$.

The upcoming remark collects a useful observation concerning the nesting $\mathsf{Nt}(\varphi)$ of a formula φ and its subformulas. This fact can be verified by an easy induction.

Remark 3. For any $\varphi \in \mathbb{F}_{()}$ with $\varphi \neq p$, there is a subformula $a \in \mathbb{W}_{()}$ of φ such that $\mathsf{Nt}(\varphi) = \mathsf{Nt}(a)$. Moreover, if $\mathsf{Nt}(\varphi) \geq 1$, there is a subformula $a \in \mathbb{W}_{()}$ of φ such that $\mathsf{Nt}(\varphi) = \mathsf{Nt}(a) + 1$.

The following lemma relates the derivability in $\mathbf{RC}_{\Gamma_0}$ under τ, and the nesting of formulas in $\mathbb{F}_{()}$.

Lemma 6. *For any* $\varphi, \psi \in \mathbb{F}_{()}$*:*

$$\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau \implies \mathsf{Nt}(\varphi) \geq \mathsf{Nt}(\psi).$$

Proof. Suppose that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. If $\mathcal{S}(\psi^\tau) = \varnothing$ then it is easy to check that $\mathsf{Nt}(\psi) = 0$ and there is nothing to prove, so assume otherwise. Then, by Lemma 1.1, $\max \mathcal{S}(\varphi^\tau) \geq \max \mathcal{S}(\psi^\tau)$. Using Lemma 4, let $a \in \mathbb{W}_{()}$ be a subformula of φ such that $o^*(a) = \max \mathcal{S}(\varphi^\tau)$. Moreover, since $\mathcal{S}(\psi^\tau) = \varnothing$, then $\mathsf{Nt}(\psi) \geq 1$. Therefore, with the help of Remark 3 we can consider $b \in \mathbb{W}_{()}$, a subformula of ψ such that $\mathsf{Nt}(\psi) = \mathsf{N}(b)+1$. If we had $\mathsf{N}(a) < \mathsf{N}(b)$ then it would follow from Corollary 2 that $o^*(a) < o^*(b)$, contradicting $\max \mathcal{S}(\varphi^\tau) \geq \max \mathcal{S}(\varphi^\tau)$. Thus $\mathsf{N}(a) \geq \mathsf{N}(b)$ and $\mathsf{Nt}(\varphi) \geq \mathsf{N}(a) + 1 \geq \mathsf{Nt}(\psi)$, as needed.

With the following theorem we conclude the proof of the preservability between **BC** and $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$.

Theorem 5. *For any φ, $\psi \in \mathbb{F}_{()}$:*

$$\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau \iff \varphi \vdash_{\mathbf{BC}} \psi.$$

Proof. The right-to-left direction is given by Lemma 5, so we focus on the other. Proceed by induction on $\mathsf{Nt}(\varphi)$. For the base case, assume $\mathsf{Nt}(\varphi) = 0$ and $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By a subsidiary induction on the length of the derivation of $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$, we set to prove $\varphi \vdash_{\mathbf{BC}} \psi$. If the derivation has length one it suffices to check $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-Axioms 1 and 2, which is immediate. If it has length greater than one it must end in a rule. The case for $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-Rule 1 follows by the I.H.. For $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-Rule 2, we have that there is $\chi \in \mathbb{F}_{\Gamma_0}$ such that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \chi$ and $\chi \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By Remark 2 and Lemma 6, we get that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} (\chi^\iota)^\tau$ and $(\chi^\iota)^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ with $\mathsf{Nt}(\chi^\iota) = 0$. Thus, by the subsidiary I.H., $\varphi \vdash_{\mathbf{BC}} \chi^\iota$ and $\chi^\iota \vdash_{\mathbf{BC}} \psi$ and by **BC**-Rule 2, $\varphi \vdash_{\mathbf{BC}} \psi$.

For the inductive step, let $\mathsf{Nt}(\varphi) = n+1$. We proceed by a subsidiary induction on the length of the derivation. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by means of $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-Axioms 1 and 2, then clearly $\varphi \vdash_{\mathbf{BC}} \psi$. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is an instance of $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-Axiom 3, then we have that $\varphi^\tau := \langle o^*(a) \rangle \langle o^*(b) \rangle \chi^\tau$ and $\psi^\tau := \langle o^*(c) \rangle \chi^\tau$ for some $\chi \in \mathbb{F}_{()}$ and a, b, $c \in \mathbb{W}_{()}$ such that $o^*(a) = o^*(b) = o^*(c)$. Hence, there are A, B, $C \in \mathbb{W}$ such that $a^* = A$, $b^* = B$ and $c^* = C$, and so $A \vdash_{\mathbf{RC}_{\Gamma_0}} B$ and $B \vdash_{\mathbf{RC}_{\Gamma_0}} C$. Since $\mathsf{Nt}(w) < n+1$ for $w \in \{a, b, c\}$, by the main I.H. we have that $a \vdash_{\mathbf{BC}} b$ and $b \vdash_{\mathbf{BC}} c$. Thus, we have the following **BC**-derivation:

$$\dfrac{\dfrac{\dfrac{\chi \vdash \chi \qquad b \vdash c}{(b)\chi \vdash (c)\chi}\,\text{(Rule 3)} \qquad a \vdash b}{(a)\,(b)\chi \vdash (b)\,(c)\chi}\,\text{(Rule 3)} \qquad \dfrac{\chi \vdash \chi \qquad b \vdash c}{(b)\,(c)\chi \vdash (c)\chi}\,\text{(Rule 3)}}{(a)\,(b)\chi \vdash (c)\chi}\,\text{(Rule 2)}$$

If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by using $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-Axiom 4, then $\varphi^\tau := \langle o^*(a) \rangle \chi^\tau$ and $\psi^\tau := \langle o^*(b) \rangle \chi^\tau$. for some $\chi \in \mathbb{F}_{()}$ and $a, b, \in \mathbb{W}_{()}$ with $o^*(a) > o^*(b)$. Therefore, there are A, $B \in \mathbb{W}_{\Gamma_0}$ such that $A \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle B$, $a^* = A$ and $b^* = B$. Since $o^*(a) \geq o^*(()b)$, by Lemma 1, $\mathsf{Nt}(()b) \leq \mathsf{Nt}(a)$ and since $\varphi^\tau := \langle o^*(a) \rangle \chi^\tau$, we have that $\mathsf{Nt}(a) < \mathsf{Nt}(\varphi)$. Thus, by the main I.H. $a \vdash_{\mathbf{BC}} ()b$ and by **BC**-Rule 3, $(a)\chi \vdash_{\mathbf{BC}} (b)\chi$. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is an instance of $\mathbf{RC}_{\boldsymbol{\Gamma_0}}$-Axiom 5, then we have that $\varphi^\tau := \langle o^*(a) \rangle \chi_0^\tau \wedge \langle o^*(b) \rangle \chi_1^\tau$ and $\psi^\tau := \langle o^*(a) \rangle \big(\chi_0^\tau \wedge \langle o^*(b) \rangle \chi_1^\tau \big)$, for some $\chi_0, \chi_1 \in \mathbb{F}_{()}$ and $a, b \in \mathbb{W}_{()}$ with $o^*(a) > o^*(b)$. Therefore, there are $A, B \in \mathbb{W}_{\Gamma_0}$ such that $a^* = A$, $b^* = B$ and $A \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle B$. By Lemma 1 together with the main I.H. we obtain that $a \vdash_{\mathbf{BC}} ()b$ and by applying **BC**-Rule 4, $(a)\chi_0 \wedge (b)\chi_1 \vdash (a)\big(\chi_0 \wedge (b)\chi_1\big)$. Regarding rules, $\mathbf{RC}_{\Gamma_0}$-Rule 1 is immediate and $\mathbf{RC}_{\Gamma_0}$-Rule 3 follows an analogous reasoning to that of Axiom 4. This way, we only check $\mathbf{RC}_{\Gamma_0}$-Rule 2. Assume $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by an application of $\mathbf{RC}_{\Gamma_0}$-Rule 2. Then, there is $\chi \in \mathbb{F}_{\Gamma_0}$ such that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \chi$ and $\chi \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By Remark 2 together with Lemm 6 we obtain that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} (\chi^\iota)^\tau$ and $(\chi^\iota)^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ with $\mathsf{Nt}(\chi) \leq n+1$. By the subsidiary I.H. $\varphi \vdash_{\mathbf{BC}} \chi^\iota$ and $\chi^\iota \vdash_{\mathbf{BC}} \psi$ and hence, by **BC**-Rule 2, $\varphi \vdash_{\mathbf{BC}} \psi$.

With this we obtain our main result: an autonomous calculus for representing ordinals below Γ_0.

Theorem 6. *For $a, b \in \mathsf{NF}$ define $a \lhd b$ if and only if $a \vdash_{\mathbf{BC}} ()b$. Then, $\lhd$ is a strict linear order of order-type Γ_0.*

Proof. By Theorem 5, $a \lhd b$ if and only if $a^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle b^\tau$ if and only if $o^*(a) < o^*(b)$. Moreover if $\xi < o^*(a)$ then by item 2 of Lemma 2 there is some $B <_0 a^\tau$ such that $\xi = o(B)$, hence in view of Remark 2, $\xi = o^*(B^\iota)$. Thus by Lemma 2, o^* is the order-type function on NF. That the range of o^* is Γ_0 follows from Proposition 2 which tells us that $o^*(a) < h(\mathsf{N}(a)+1) < \Gamma_0$ for all $a \in \mathbb{W}_{()}$, while if we define recursively $a_0 = \top$ and $a_{n+1} = (a_n)$, Theorem 3 and an easy induction readily yield $\Gamma_0 = \lim_{n\to\infty} h(n) = \lim_{n\to\infty} o^*(a_n)$.

8 Concluding Remarks

Beklemishev's 'brackets' provided an autonomous notation system for Γ_0 based on worms, but did not provide a method for comparing different worms without first translating into a more traditional notation system. Our calculus **BC** shows that this is not necessary, and indeed all derivations may be carried out entirely within the brackets notation. To the best of our knowledge, this yields the first ordinal notation system presented as a purely modal deductive system.

Our analysis is purely syntactical and leaves room for a semantical treatment of **BC**. As before one may first map **BC** into $\mathbf{RC}_{\Gamma_0}$ and then use the Kripke semantics presented in [5,13], but we leave the question of whether it is possible to define natural semantics that work only with **BC** expressions and do not directly reference ordinals.

Moreover, [15] suggests variants of the brackets notation for representing the Bachmann-Howard ordinal and beyond. Sound and complete calculi for these systems remain to be found.

References

1. de Almeida Borges, A., Joosten, J.: The worm calculus. In: Bezhanishvili, G., D'Agostino, G., Metcalfe, G., Studer, T. (eds.) Advances in Modal Logic, vol. 12. College Publications (2018)

2. Beklemishev, L.D.: Provability algebras and proof-theoretic ordinals, I. Ann. Pure Appl. Log. **128**, 103–124 (2004)
3. Beklemishev, L.D.: Veblen hierarchy in the context of provability algebras. In: Hájek, P., Valdés-Villanueva, L., Westerståhl, D. (eds.) Logic, Methodology and Philosophy of Science, Proceedings of the Twelfth International Congress, pp. 65–78. Kings College Publications (2005)
4. Beklemishev, L.D.: Calibrating provability logic. In: Bolander, T., Braüner, T., Ghilardi, T.S., Moss, L. (eds.) Advances in Modal Logic, vol. 9, pp. 89–94. College Publications, London (2012)
5. Beklemishev, L.D.: Positive provability logic for uniform reflection principles. Ann. Pure Appl. Log. **165**(1), 82–105 (2014)
6. Beklemishev, L.D., Fernández-Duque, D., Joosten, J.J.: On provability logics with linearly ordered modalities. Stud. Log. **102**(3), 541–566 (2014)
7. Beklemishev, L.D., Gabelaia, D.: Topological completeness of the provabilitylogic GLP. Ann. Pure Appl. Log. **164**(12), 1201–1223 (2013)
8. Beklemishev, L.: Another pathological well-ordering. Bull. Symb. Log. **7**(4), 534–534 (2001)
9. Beklemishev, L.D.: On the reflection calculus with partial conservativity operators. In: Kennedy, J., de Queiroz, R.J.G.B. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 48–67. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_4
10. Beklemishev, L.: Reflection calculus and conservativity spectra. Russ. Math. Surv. **73**(4), 569–613 (2018)
11. Beklemishev, L.D.: A universal algebra for the variable-free fragment of RC^∇. In: Artemov, S., Nerode, A. (eds.) LFCS 2018. LNCS, vol. 10703, pp. 91–106. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-72056-2_6
12. Boolos, G.S.: The Logic of Provability. Cambridge University Press, Cambridge (1993)
13. Dashkov, E.V.: On the positive fragment of the polymodal provability logicGLP. Math. Notes **91**(3–4), 318–333 (2012)
14. Fernández-Duque, D.: The polytopologies of transfinite provability logic. Arch. Math. Log. **53**(3–4), 385–431 (2014)
15. Férnandez-Duque, D.: Worms and spiders: reflection calculi and ordinal notation systems. J. Appl. Log. - IfCoLoG J. Log. Appl. **4**(10), 3277–3356 (2017)
16. Fernández-Duque, D., Joosten, J.J.: Hyperations, Veblen progressions and transfinite iteration of ordinal functions. Ann. Pure Appl. Log. **164**(7–8), 785–801 (2013)
17. Fernández-Duque, D., Joosten, J.J.: The omega-rule interpretation of transfinite provability logic. ArXiv:1205.2036 [math.LO] (2013)
18. Fernández-Duque, D., Joosten, J.J.: Well-orders in the transfinite Japaridze algebra. ArXiv:1212.3468 [math.LO] (2013)
19. Hermo-Reyes, E., Joosten, J.J.: Relational semantics for the Turing Schmerl calculus. In: Bezhanishvili, G., D'Agostino, G., Metcalfe, G., Studer, T. (eds.) Advances in Modal Logic, vol. 12, pp. 327–346. College Publications, London (2018)
20. Icard III, T.F.: A topological study of the closed fragment of GLP. J. Log. Comput. **21**, 683–696 (2011)
21. Ignatiev, K.N.: On strong provability predicates and the associated modal logics. J. Symb. Log. **58**, 249–290 (1993)
22. Japaridze, G.K.: The modal logical means of investigation of provability. Ph.D. thesis, Moscow State University (1986). (in Russian)
23. Jech, T.: Set Theory, The Third Millenium Edition, Revised and Expanded. Monographs in Mathematics. Springer, Heidelberg (2002). https://doi.org/10.1007/3-540-44761-X
24. Kreisel, G.: Wie die beweistheorie zu ihren ordinalzahlen kam und kommt. Jahresbericht der Deutschen Mathematiker-Vereinigung **78**, 177–224 (1976/1977)

Descriptive Complexity of Deterministic Polylogarithmic Time

Flavio Ferrarotti[1(✉)], Senén González[1], José María Turull Torres[2], Jan Van den Bussche[3], and Jonni Virtema[3]

[1] Software Competence Center Hagenberg, Hagenberg, Austria
{flavio.ferrarotti,senen.gonzalez}@scch.at
[2] Universidad Nacional de La Matanza, Buenos Aires, Argentina
jturull@unlam.edu.ar
[3] Hasselt University, Hasselt, Belgium
{jan.vandenbussche,jonni.virtema}@uhasselt.be

Abstract. We propose a logical characterization of problems solvable in deterministic polylogarithmic time (PolylogTime). We introduce a novel two-sorted logic that separates the elements of the input domain from the bit positions needed to address these elements. In the course of proving that our logic indeed captures PolylogTime on finite ordered structures, we introduce a variant of random-access Turing machines that can access the relations and functions of the structure directly. We investigate whether an explicit predicate for the ordering of the domain is needed in our logic. Finally, we present the open problem of finding an exact characterization of order-invariant queries in PolylogTime.

1 Introduction

The research area known as Descriptive Complexity [7,11,15] relates computational complexity to logic. For a complexity class of interest, one tries to come up with a natural logic such that a property of inputs can be expressed in the logic if and only if the problem of checking the property belongs to the complexity class. An exemplary result in this vein is that a family $\mathcal{F}$ of finite structures (over some fixed finite vocabulary) is definable in existential second-order logic (ESO), if and only if the membership problem for $\mathcal{F}$ belongs to NP [4]. We also say that ESO *captures* NP. The complexity class P is captured, on ordered finite structures, by a *fixpoint logic*: the extension of first-order logic with least-fixpoints [14,22].

After these two seminal results, many more capturing results have been developed, and the benefits of this enterprise have been well articulated by several

The research reported in this paper results from the project *Higher-Order Logics and Structures* supported by the Austrian Science Fund (FWF: **[I2420-N31]**) and the Research Foundation Flanders (FWO: **[G0G6516N]**). It was further supported by the Austrian Research Promotion Agency (FFG) through the COMET funding for the Software Competence Center Hagenberg.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 208–222, 2019.
https://doi.org/10.1007/978-3-662-59533-6_13

authors in the references given earlier, and others [1]. We just mention here the advantage of being able to specify properties of structures (data structures, databases) in a logical, declarative manner; at the same time, we are guaranteed that our computational power is well delineated.

The focus of the present paper is on computations taking deterministic polylogarithmic time, i.e., time proportional to $\log^k n$ for some arbitrary but fixed k. Such computations are practically relevant and common on ordered structures. Well known examples are binary search in an array or search in a balanced search tree. Another natural example is the computation of $f(x_1, \ldots, x_r)$, where $x_1, \ldots, x_r$ are numbers taken from the input structure and f is a function computable in polynomial time when numbers are represented in binary.

Computations with sublinear time complexity can be formalized in terms of Turing machines with random access to the input [15]. When a family $\mathcal{F}$ of ordered finite structures over some fixed finite vocabulary is defined by some deterministic polylogarithmic-time random-access Turing machine, we say that $\mathcal{F}$ belongs to the complexity class PolylogTime. In this paper, we show how this complexity class can be captured by a new logic which we call *index logic*.

Index logic is two-sorted; variables of the first sort range over the domain of the input structure. Variables of the second sort range over an initial segment of the natural numbers; this segment is bounded by the logarithm of the size of the input structure. Thus, the elements of the second sort represent the bit positions needed to address elements of the first sort. Index logic includes full fixpoint logic on the second sort. Quantification over the first sort, however, is heavily restricted. Specifically, a variable of the first sort can only be bound using an address specified by a subformula that defines the positions of the bits of the address that are set. This "indexing mechanism" lends index logic its name.

In the course of proving our capturing result, we consider a new variant of random-access Turing machines. In the standard variant, the entire input structure is presented as one binary string. In our new variant, the different relations and functions of the structure can be accessed directly. We will show that both variants are equivalent, in the sense that they lead to the same notion of PolylogTime. We note that, in descriptive complexity, it is common practice to work only with relational structures, as functions can be identified with their graphs. In a sublinear-time setting, however, this does not work. Indeed, let f be a function and denote its graph by $\tilde{f}$. If we want to know the value of $f(x)$, we cannot spend the linear time needed to find a y such that $\tilde{f}(x, y)$ holds. Thus, in this work, we allow structures containing functions as well as relations.

At first glance, one might think that a simpler approach to ours, for the characterization of PolylogTime, could be to adapt the construction used by Immerman and Vardi [14,22] to capture P. For instance, by querying binary representations of the indices with Immerman's BIT predicate, where $\mathrm{BIT}(x, i)$ holds iff the i-th bit of x in binary is 1, or avoiding our new variant of random-access Turing machine. In fact, that was our initial approach to the problem. This results, however, in a long and cumbersome characterization proof, mostly due to the need to express arithmetic operations within the logic to access the relevant parts of the input, since in PolylogTime we cannot read it in whole. A challenge, in this sense, was to develop a logic which enables the expression of PolylogTime problems in a relatively clean and natural way. For this, the indexing mechanism in our logic is a key contribution. The alternative of using fixed point operations and BIT to address values of the first sort leads to a logic which is rather awkward to define and to use.

We also devote attention to gaining a detailed understanding of the expressivity of index logic. Specifically, we observe that order comparisons between quantified variables of the first sort can be expressed in terms of their addresses. For constants of the first sort that are directly given by the structure, however, we show that this is not possible. In other words, index logic without an explicit order predicate on the first sort would no longer capture PolylogTime for structures with constants.

Related Work. Many natural fixed point computations, such as transitive closure, converge after a polylogarithmic number of steps. This motivated the study in [10] of a fragment of fixed point logic with counting (FPC) that only allows polylogarithmically many iterations of the fixed point operators (POLYLOG-FPC). They noted that on ordered structures POLYLOG-FPC captures NC, i.e., the class of problems solvable in parallel polylogarithmic time. This holds even in the absence of counting, which on ordered structures can be simulated using fixed point operators. Moreover, an old result in [13] directly implies that POLYLOG-FPC is strictly weaker than FPC with regards to expressive power.

It is well known that the (nondeterministic) logarithmic time hierarchy corresponds exactly to the set of first-order definable Boolean queries (see [15], Theorem 5.30). The relationship between uniform families of circuits within NC^1 and nondeterministic random-access logarithmic time machines was studied in [19]. However, the study of descriptive complexity of classes of problems decidable by *deterministic* formal models of computation in polylogarithmic time, i.e., the topic of this paper, has been overlooked by previous works.

On the other hand, *nondeterministic* polylogarithmic time complexity classes, defined in terms of alternating random-access Turing machines and related families of circuits, have received some attention [5,18]. Recently, a theorem analogous to Fagin's famous theorem [4], was proven for nondeterministic polylogarithmic time [5]. For this task, a restricted second-order logic for finite structures, where second-order quantification ranges over relations of size at most polylogarithmic in the size of the structure, and where first-order universal quantification is bounded to those relations, was exploited. This latter work, is closely related to the work on constant depth quasipolynomial size AND/OR circuits and the corresponding restricted second-order logic in [18]. Both logics capture the full alternating polylogarithmic time hierarchy, but the additional restriction in the first-order univesal quantification in the second-order logic defined in [5], enables a one-to-one correspondence between the levels of the polylogarithmic time hierarchy and the prenex fragments of the logic, in the style of a result of Stockmeyer [21] regarding the polynomial-time hierarchy. Unlike the classical results of Fagin and Stockmeyer [4,21], the results on the descriptive complexity of nondeterministic polylogarithmic time classes only hold over ordered structures.

2 Preliminaries

We allow structures containing functions as well as relations and constants. Unless otherwise stated, we work with finite ordered structures of finite vocabularies. A finite structure $\mathbf{A}$ of vocabulary $\sigma = \{R_1^{r_1}, \ldots, R_p^{r_p}, c_1, \ldots c_q, f_1^{k_1}, \ldots, f_s^{k_s}\}$, where each $R_i^{r_i}$ is an r_i-ary relation symbol, each c_i is a constant symbol, and each $f_i^{k_i}$ is a k_i-ary function symbol, consists of a finite domain A and interpretations for all relation, constant and function symbols in σ. An interpretation of a symbol $R_i^{r_i}$ is a relation $R_i^{\mathbf{A}} \subseteq A^{r_i}$, of a symbol c_i is a value $c_i^{\mathbf{A}} \in A$, and of a symbol $f_i^{k_i}$ is a function $f_i^{\mathbf{A}} : A^{k_i} \to A$. Every finite ordered structure has a corresponding isomorphic structure whose domain is an initial segment of the natural numbers. Thus, we assume as usual that $A = \{0, 1, \ldots, n-1\}$, where n is the cardinality $|A|$ of A.

In this paper, $\log n$ always refers to the binary logarithm of n, i.e., $\log_2 n$. We write $\log^k n$ as a shorthand for $(\lceil \log n \rceil)^k$.

3 Deterministic Polylogarithmic Time

The sequential access that Turing machines have to their tapes makes it impossible to do nontrivial computations in sub-linear time. Therefore, logarithmic time complexity classes are usually studied using models of computation that have random access[1] to their input, i.e., that can access every input address directly. As this also applies to the polylogarithmic complexity classes studied in this paper, we adopt a Turing machine model that has a *random access* read-only input, similar to the logarithmic-time Turing machine in [19].

Our concept of a *random-access Turing machine* is that of a multi-tape Turing machine which consists of: (1) a finite set of states, (2) a read-only random access *input-tape*, (3) a sequential access *address-tape*, and (4) one or more (but a fixed number of) sequential access *work-tapes*. All tapes are divided into cells, each equipped with a *tape head* which scans the cells, and are "semi-infinite" in the sense that they have no rightmost cell, but have a left-most cell. The tape heads of the sequential access address-tape and work-tapes can move left or right. When a head is in the leftmost cell, it is not allowed to move left. The address-tape alphabet only contains symbols 0, 1 and $\sqcup$ (for blank). The position of the input-tape head is determined by the number i stored in binary in between the left-most cell and the first blank cell of the address-tape (if the left-most cell is blank, then i is considered to be 0) as follows: If i is strictly smaller than the length n of the input string, then the input-tape head is in the $(i+1)$-th cell. Otherwise, if $i \geq n$, then the input-tape head is in the $(n+1)$-th cell scanning the special end-marker symbol $\triangleleft$.

[1] The term *random access* refers to the manner how *random-access memory* (RAM) is read and written. In contrast to sequential memory, the time it takes to read or write using RAM is almost independent of the physical location of the data in the memory. We want to emphasise that there is nothing *random* in random access.

Formally, a *random-access Turing machine* M with k work-tapes is a five-tuple $(Q, \Sigma, \delta, q_0, F)$. Here Q is a finite set of *states*; $q_0 \in Q$ is the *initial state*. Σ is a finite set of symbols (the *alphabet* of M). For simplicity, we fix $\Sigma = \{0, 1, \sqcup\}$. $F \subseteq Q$ is the set of *accepting final states*. The *transition* function of M is of the form $\delta : Q \times (\Sigma \cup \{\triangleleft\}) \times \Sigma^{k+1} \to Q \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^{k+1}$. We assume that the tape head directions $\leftarrow$ for "left", $\rightarrow$ for "right" and $-$ for "stay", are not in $Q \cup \Sigma$.

A *configuration* of M on a fixed input w_0 is a $k+2$ tuple $(q, i, w_1, \ldots, w_k)$, where q is the current state of M, $i \in \Sigma^* \# \Sigma^*$ represents the current contents of the index-tape cells, and each $w_j \in \Sigma^* \# \Sigma^*$ represents the current contents of the j-th work-tape cells. We do not include the contents of the input-tape cells in the configuration since they cannot be changed. Further, the position of the input-tape head is uniquely determined by the contents of the index-tape cells. The symbol $\#$ (which we assume is not in Σ) marks the position of the tape head. By convention, the head scans the symbol immediately at the right of $\#$. All symbols in the infinite tapes not appearing in their corresponding strings $i, w_0, \ldots, w_k$ are assumed to be the special symbol blank $\sqcup$.

At the beginning of a computation, all work-tapes are blank except the input-tape, that contains the input string, and the index-tape that contains a 0 (meaning that the input-tape head scans the first cell of the input-tape). Thus, the *initial configuration* of M is $(q_0, \#0, \#, \ldots, \#)$. A *computation* is a sequence of configurations which starts with the initial configuration and ends in a configuration in which no more steps can be performed, and such that each step from a configuration to the next obeys the transition function. An input string is *accepted* if an accepting configuration, i.e., a configuration in which the current state belongs to F, is reached.

Example 1. Following a simple strategy, a random-access Turing machine M can figure out the length n of its input as well as $\lceil \log n \rceil$ in polylogarithmic time. In its initial step, M checks whether the input-tape head scans the end-marker $\triangleleft$. If it does, then the input string is the empty string and its work is done. Otherwise, M writes 1 in the first cell of its address tape and keeps writing 0's in its subsequent cells right up until the input-tape head scans $\triangleleft$. At this point the resulting binary string in the index-tape is of length $\lceil \log n \rceil$. Next, M moves its address-tape head back to the first cell (i.e., to the only cell containing a 1 at this point). From here on, M repeatedly moves the index head one step to the right. Each time it checks whether the index-tape head scans a blank $\sqcup$ or a 0. If $\sqcup$ then M is done. If 0, it writes a 1 and tests whether the input-tape head jumps to the cell with $\triangleleft$; if so, it rewrites a 0, otherwise, it leaves the 1. The binary number left on the index-tape at the end of this process is $n-1$. Adding one in binary is now an easy task. $\square$

The *formal language accepted* by a machine M, denoted $L(M)$, is the set of strings accepted by M. We say that $L(M) \in \mathrm{DTIME}[f(n)]$ if M makes at most $O(f(n))$ steps before accepting or rejecting an input string of length n. We define the class of all formal languages decidable by (deterministic) random-access Turing machines in *polylogarithmic time* as follows:

$$\text{PolylogTime} = \bigcup_{k \in \mathbb{N}} \text{DTIME}[\log^k n]$$

It follows from Example 1 that a PolylogTime random-access Turing machine can check any numerical property that is polynomial time in the size of its input in binary. For instance, it can check whether the length of its input is even, by simply looking at the least-significant bit.

When we want to give a finite structure as an input to a random-access Turing machine, we encode it as a string, adhering to the usual conventions in descriptive complexity theory [15]. Let $\sigma = \{R_1^{r_1}, \ldots, R_p^{r_p}, c_1, \ldots, c_q, f_1^{k_1}, \ldots, f_s^{k_s}\}$ be a vocabulary, and let $\mathbf{A}$ with $A = \{0, 1, \ldots, n-1\}$ be an ordered structure of vocabulary σ. Each relation $R_i^{\mathbf{A}} \subseteq A^{r_i}$ of $\mathbf{A}$ is encoded as a binary string $\text{bin}(R_i^{\mathbf{A}})$ of length n^{r_i}, where 1 in a given position indicates that the corresponding tuple is in $R_i^{\mathbf{A}}$. Likewise, each constant number $c_j^{\mathbf{A}}$ is encoded as a binary string $\text{bin}(c_j^{\mathbf{A}})$ of length $\lceil \log n \rceil$.

We can also encode the functions in a structure. We view k-ary functions as consisting of $\lceil \log n \rceil$ k-ary relations, where the i-th relation indicates whether the i-th bit is 1. Thus, each function $f_i^{\mathbf{A}}$ is encoded as a binary string $\text{bin}(f_i^{\mathbf{A}})$ of length $\lceil \log n \rceil n^{k_i}$.

The encoding of the whole structure $\text{bin}(\mathbf{A})$ is the concatenation of the binary strings encoding its relations, constants and functions. The length $\hat{n} = |\text{bin}(\mathbf{A})|$ of this string is $n^{r_1} + \cdots + n^{r_p} + q\lceil \log n \rceil + \lceil \log n \rceil n^{k_1} + \cdots + \lceil \log n \rceil n^{k_s}$, where $n = |A|$ denotes the size of the input structure $\mathbf{A}$. Note that $\log \hat{n} \in O(\lceil \log n \rceil)$, so $\text{DTIME}[\log^k \hat{n}] = \text{DTIME}[\log^k n]$.

4 Direct-Access Turing Machines

In this section, we propose a new model of random-access Turing machines. In the standard model reviewed above, the entire input structure is assumed to be encoded as one binary string. In our new variant, the different relations and functions of the structure can be accessed directly. We then show that both variants are equivalent, in the sense that they lead to the same notion of PolylogTime. The direct-access model will then be useful to give a transparent proof of our capturing result.

Let our vocabulary $\sigma = \{R_1^{r_1}, \ldots, R_p^{r_p}, c_1, \ldots c_q, f_1^{k_1}, \ldots, f_s^{k_s}\}$. A *direct-access Turing machine that takes σ-structures* $\mathbf{A}$ *as input*, is a multitape Turing machine with $r_1 + \cdots + r_p + k_1 + \cdots + k_s$ distinguished work-tapes, called *address-tapes*, s distinguished read-only (function) *value-tapes*, $q+1$ distinguished read-only *constant-tapes*, and one or more ordinary work-tapes.

Let us define a transition function δ_l for each tape l separately. These transition functions take as an input the current state of the machine, the bit read by each of the heads of the machine, and, for each relation $R_i \in \sigma$, the answer (0 or 1) to the query $(n_1, \ldots, n_{r_i}) \in R_i^{\mathbf{A}}$. Here, n_j denotes the number written in binary in the jth distinguished tape of R_i.

Thus, with m the total number of tapes, the state transition function has the form

$$\delta_Q : Q \times \Sigma^m \times \{0,1\}^p \to Q.$$

If l corresponds to an address-tape or an ordinary work-tape, we get the form

$$\delta_l : Q \times \Sigma^m \times \{0,1\}^p \to \Sigma \times \{\leftarrow, \rightarrow, -\}.$$

If l corresponds to one of the read-only tapes, we have

$$\delta_l : Q \times \Sigma^m \times \{0,1\}^p \to \{\leftarrow, \rightarrow, -\}.$$

Finally we update the contents of the function value-tapes. If l is the function value-tape for a function f_i, then the content of the tape l is updated to $f_i(n_1, \dots n_{k_i})$ written in binary. Here, n_j denotes the number written in binary in the jth distinguished address-tape of f_i *after* the execution of the above transition functions. If one of the n_j is too large, the tape l is updated to contain only blanks. Note that the head of the tape remains in place; it was moved by δ_l already.

In the initial configuration, read-only constant-tapes for the constant symbols $c_1, \dots, c_q$ hold the values in binary of their values in $\mathbf{A}$. One additional constant-tape (there are $q+1$ of them) holds the size n of the domain of $\mathbf{A}$ in binary. Each address-tape, each value-tape, and each ordinary work-tape holds only blanks.

Theorem 2. *A class of finite ordered structures $\mathcal{C}$ of some fixed vocabulary σ is decidable by a random-access Turing machine working in* PolylogTime *with respect to $\hat{n}$, where $\hat{n}$ is the size of the binary encoding of the input structure, iff $\mathcal{C}$ is decidable by a direct-access Turing machine in* PolylogTime *with respect to n, where n is the size of the domain of the input structure.*

The proof (omitted) is based on computing precise locations in which bits can be found, and, for the other direction, on a binary search technique to compute n from $\hat{n}$.

5 Index Logic

In this section we introduce *index logic*, a new logic which over ordered finite structures captures PolylogTime. Our definition of index logic is inspired by the second-order logic in [18], where relation variables are restricted to valuations on the sub-domain $\{0, \dots, \lceil \log n \rceil - 1\}$ (n being the size of the interpreting structure), as well as by the well known counting logics as defined in [9].

Given a vocabulary σ, for every ordered σ-structure $\mathbf{A}$, we define a corresponding set of natural numbers $\mathit{Num}(\mathbf{A}) = \{0, \dots, \lceil \log n \rceil - 1\}$ where $n = |A|$. Note that $\mathit{Num}(\mathbf{A}) \subseteq A$, since we assume that A is an initial segment of the natural numbers. This simplifies the definitions, but it is otherwise unnecessary.

Index logic is a two-sorted logic. Individual variables of the first sort $\mathbf{v}$ range over the domain A of $\mathbf{A}$, while individual variables of the second sort $\mathbf{n}$ range over $Num(\mathbf{A})$. We denote variables of sort $\mathbf{v}$ with $x, y, z, \ldots$, possibly with a subindex such as $x_0, x_1, x_2, \ldots$, and variables of type $\mathbf{n}$ with $\mathtt{x}, \mathtt{y}, \mathtt{z}$, also possibly with a subindex. Relation variables, denoted with uppercase letters $X, Y, Z, \ldots$, are always of sort $\mathbf{n}$, and thus range over relations defined on $Num(\mathbf{A})$.

Definition 3. *Let σ be a vocabulary, we inductively define terms and formulae of index logic as follows:*

- *Each individual variable of sort $\mathbf{v}$ and each constant symbol in σ is a term of sort $\mathbf{v}$.*
- *Each individual variable of sort $\mathbf{n}$ is a term of sort $\mathbf{n}$.*
- *If $t_1, \ldots, t_k$ are terms of sort $\mathbf{v}$ and f is a k-ary function symbol in σ, then $f(t_1, \ldots, t_k)$ is a term of sort $\mathbf{v}$.*
- *If t_1, t_2 are terms of a same sort, then $t_1 = t_2$ and $t_1 \leq t_2$ are (atomic) formulae.*
- *If $t_1, \ldots, t_k$ are terms of sort $\mathbf{v}$ and R is a k-ary relation symbol in σ, then $R(t_1, \ldots, t_k)$ is an (atomic) formula.*
- *If $t_1, \ldots, t_k$ are terms of sort $\mathbf{n}$ and X is a k-ary relation variable, then $X(t_1, \ldots, t_k)$ is an (atomic) formula.*
- *If t is a term of sort $\mathbf{v}$, φ is a formula and $\mathtt{x}$ is an individual variable of sort $\mathbf{n}$, then $t = index\{\mathtt{x} : \varphi(\mathtt{x})\}$ is an (atomic) formula.*
- *If $\bar{t}$ is tuple of terms of sort $\mathbf{n}$, $\bar{\mathtt{x}}$ is tuples of variables also of sort $\mathbf{n}$, X is a relation variable, the lengths of $\bar{t}$ and $\bar{\mathtt{x}}$ are the same and coincide with the arity of X, and φ is a formula, then $[\mathrm{IFP}_{\bar{\mathtt{x}}, X}\varphi]\bar{t}$ is an (atomic) formula.*
- *If φ, ψ are formulae, then $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\neg\psi$ are formulae.*
- *If $\mathtt{x}$ is a variable of type $\mathbf{n}$ and φ is a formula, then $\exists\mathtt{x}(\varphi)$ and $\forall\mathtt{x}(\varphi)$ are formulae.*
- *If $x = index\{\mathtt{x} : \alpha(\mathtt{x})\}$ is an atomic formula such that x does* not *appear free in $\alpha(\mathtt{x})$, and φ is a formula, then $\exists x(x = index\{\mathtt{x} : \alpha(\mathtt{x})\} \wedge \varphi)$ is a formula.*

The concept of a valuation is the standard for a two-sorted logic. Thus, a *valuation* over a structure $\mathbf{A}$ is any total function *val* from the set of all variables of index logic to values satisfying the following constraints:

- If x is a variable of type $\mathbf{v}$, then $val(x) \in A$.
- If $\mathtt{x}$ is a variable of type $\mathbf{n}$, then $val(\mathtt{x}) \in Num(\mathbf{A})$.
- If X is a relation variable with arity r, then $val(X) \subseteq (Num(\mathbf{A}))^r$.

Valuations extend to terms and tuples of terms in the usual way. Further, we say that a valuation *val* is v-equivalent to a valuation val' if $val(v') = val'(v')$ for all variables v' other than v.

Fixed points are defined in the standard way (see [2] and [17] among others). Given an operator $F : \mathcal{P}(B) \to \mathcal{P}(B)$, a set $S \subseteq B$ is a *fixed point* of F if $F(S) = S$. A set $S \subseteq B$ is a *least fixed point* of F if it is a fixed point and for every other fixed point S' of F we have $S \subseteq S'$. We denote the least fixed point of F as $\mathrm{lfp}(F)$. The *inflationary fixed point* of F, denoted by $\mathrm{ifp}(F)$, is the union of all sets S^i where $S^0 = \emptyset$ and $S^{i+1} = S^i \cup F(S^i)$.

Let $\varphi(X, \bar{\mathtt{x}})$ be a formula of vocabulary σ, where X is a relation variable of arity k and $\mathtt{x}$ is a k-tuple of variables of type $\mathbf{n}$. Let $\mathbf{A}$ be a σ-structure. The formula $\varphi(X, \bar{\mathtt{x}})$ gives rise to an operator $F^{\mathbf{A}}_{\varphi,\bar{\mathtt{x}},X} : \mathcal{P}((Num(\mathbf{A}))^k) \to \mathcal{P}((Num(\mathbf{A}))^k)$ defined as follows:

$$F^{\mathbf{A}}_{\varphi,\bar{\mathtt{x}},X}(S) := \{\bar{a} \in (Num(\mathbf{A}))^k \mid \mathbf{A}, val \models \varphi(X, \bar{\mathtt{x}}) \text{ for some valuation } val \text{ with } val(X) = S \text{ and } val(\bar{\mathtt{x}}) = \bar{a}\}.$$

Definition 4. *The formulae of* IFP^{plog} *are interpreted as follows:*

- $\mathbf{A}, val \models t_1 = t_2$ *iff* $val(t_1) = val(t_2)$.
- $\mathbf{A}, val \models t_1 \leq t_2$ *iff* $val(t_1) \leq val(t_2)$.
- $\mathbf{A}, val \models R(t_1, \ldots, t_k)$ *iff* $(val(t_1), \ldots, val(t_k)) \in R^{\mathbf{A}}$.
- $\mathbf{A}, val \models X(t_1, \ldots, t_k)$ *iff* $(val(t_1), \ldots, val(t_k)) \in val(X)$.
- $\mathbf{A}, val \models t = index\{\mathtt{x} : \varphi(\mathtt{x})\}$ *iff* $val(t)$ *in binary is* $b_m b_{m-1} \cdots b_0$, *where* $m = \lceil \log |A| \rceil - 1$ *and* $b_j = 1$ *iff* $\mathbf{A}, val' \models \varphi(\mathtt{x})$ *for* val' $\mathtt{x}$*-equivalent to* val *and* $val'(\mathtt{x}) = j$.
- $\mathbf{A}, val \models [\mathrm{IFP}_{\bar{\mathtt{x}},X}\varphi]\bar{t}$ *iff* $val(\bar{t}) \in \mathrm{ifp}(F^{\mathbf{A}}_{\varphi,\bar{\mathtt{x}},X})$.
- $\mathbf{A}, val \models \neg\varphi$ *iff* $\mathbf{A}, val \not\models \varphi$.
- $\mathbf{A}, val \models \varphi \wedge \psi$ *iff* $\mathbf{A}, val \models \varphi$ *and* $\mathbf{A}, val \models \psi$.
- $\mathbf{A}, val \models \varphi \vee \psi$ *iff* $\mathbf{A}, val \models \varphi$ *or* $\mathbf{A}, val \models \psi$.
- $\mathbf{A}, val \models \exists\mathtt{x}(\varphi)$ *iff there is a* val' $\mathtt{x}$*-equivalent to* val *such that* $\mathbf{A}, val' \models \varphi$.
- $\mathbf{A}, val \models \forall\mathtt{x}(\varphi)$ *iff for all* val' $\mathtt{x}$*-equivalent to* val*, it holds that* $\mathbf{A}, val' \models \varphi$.
- $\mathbf{A}, val \models \exists x(x = index\{\mathtt{x} : \alpha(\mathtt{x})\} \wedge \varphi)$ *iff there is a* val' x*-equivalent to* val *such that* $\mathbf{A}, val' \models x = index\{\mathtt{x} : \alpha(\mathtt{x})\}$ *and* $\mathbf{A}, val' \models \varphi$.

It immediately follows from the famous result by Gurevich and Shelah regarding the equivalence between inflationary and least fixed points [12], that an equivalent index logic can be obtained if we (1) replace $[\mathrm{IFP}_{\bar{\mathtt{x}},X}\varphi]\bar{t}$ by $[\mathrm{LFP}_{\bar{\mathtt{x}},X}\varphi]\bar{t}$ in the formation rule for the fixed point in Definition 3, adding the restriction that every occurrence of X in φ is positive[2], and (2) fix the interpretation $\mathbf{A}, val \models [\mathrm{LFP}_{\bar{\mathtt{x}},X}\varphi]\bar{t}$ iff $val(\bar{t}) \in \mathrm{lfp}(F^{\mathbf{A}}_{\varphi,\bar{\mathtt{x}},X})$.

Moreover, the convenient tool of *simultaneous fixed points*, which allows one to iterate several formulae at once, can still be used here since it does not increase the expressive power of the logic. Following the syntax and semantics proposed by Ebbinghaus and Flum [2], a version of index logic with simultaneous inflationary fixed point can be obtained by replacing the clause corresponding to IFP in Definition 3 by the following:

- If $\bar{t}$ is tuple of terms of sort $\mathbf{n}$, and for $m \geq 0$ and $0 \leq i \leq m$, we have that $\bar{\mathtt{x}}_i$ is a tuple of variables of sort $\mathbf{n}$, X_i is a relation variable whose arity coincides with the length of $\bar{\mathtt{x}}_i$, the lengths of $\bar{t}$ and $\bar{\mathtt{x}}_0$ are the same, and φ_i is a formula, then $[\text{S-IFP}_{\bar{\mathtt{x}}_0,X_0,\ldots,\bar{\mathtt{x}}_m,X_m}\varphi_0, \ldots, \varphi_m]\bar{t}$ is an atomic formula.

[2] This ensures that $F^{\mathbf{A}}_{\varphi,\bar{\mathtt{x}},X}$ is monotonous and thus that the least fixed point $\mathrm{lfp}(F^{\mathbf{A}}_{\varphi,\bar{\mathtt{x}},X})$ is guaranteed to exists.

The interpretation is that $\mathbf{A}, val \models [\text{S-IFP}_{\bar{\mathtt{x}}_0,X_0,\ldots,\bar{\mathtt{x}}_m,X_m}\varphi_0,\ldots,\varphi_m]\bar{t}$ iff $val(\bar{t})$ belongs to the first (here X_0) component of the simultaneous inflationary fixed point.

Thus, we can use index logic with the operators IFP, LFP, S-IFP or S-LFP interchangeably.

The following result confirms that our logic serves our purpose.

Theorem 5. *Over ordered structures, index logic captures* PolylogTime*.*

The proof of the theorem can be found in the full arXiv version of this article [6]; instead we give two worked-out examples illustrating the power of index logic.

5.1 Finding the Binary Representation of a Constant

Assume a constant symbol c of sort $\mathbf{v}$. In this example, we show a formula $\beta_c(\mathtt{x})$ such that the sentence $c = \mathit{index}\{\mathtt{x} : \beta_c\}$ is valid over the class of all finite ordered structures. In other words, β_c defines the binary representation of the number c.

Informally, β_c works by iterating through the bit positions $\mathtt{y}$ from the most significant to the least significant. These bits are accumulated in a relation variable Z. For each $\mathtt{y}$ we set the corresponding bit, on the condition that the resulting number does not exceed c. The set bits are collected in a relation variable Y.

In the formal description of β_c below, we use the following abbreviations. We use M to denote the most significant bit position. Thus, formally, $\mathtt{z} = M$ abbreviates $\forall \mathtt{z}'\, \mathtt{z}' \leq \mathtt{z}$. Furthermore, for a unary relation variable Z, we use $\mathtt{z} = \min Z$ with the obvious meaning. We also use abbreviations such as $\mathtt{z} = \mathtt{z}' - 1$ with the obvious meaning.

Now β_c is a simultaneous fixpoint $[\text{S-IFP}_{\mathtt{y},Y,\mathtt{z},Z}\, \varphi_Y, \varphi_Z](\mathtt{x})$ where

$$\varphi_Z := (Z = \emptyset \wedge \mathtt{z} = M) \vee (Z \neq \emptyset \wedge \mathtt{z} = \min Z - 1)$$
$$\varphi_Y := Z \neq \emptyset \wedge \mathtt{y} = \min Z \wedge \exists x(x = \mathit{index}\{\mathtt{z} : Y(\mathtt{z}) \vee \mathtt{z} = \mathtt{y}\} \wedge c \geq x).$$

5.2 Binary Search in an Array of Key Values

In order to develop insight in how index logic works, we develop in detail an example showing how binary search in an array of key values can be expressed in the logic.

We represent the data structure as an ordered structure $\mathbf{A}$ over the vocabulary consisting of a unary function K, a constant symbol N, a constant symbol T, and a binary relation $\prec$. The domain of $\mathbf{A}$ is an initial segment of the natural numbers. The constant $l := N^{\mathbf{A}}$ indicates the length of the array; the domain elements $0, 1, \ldots, l-1$ represent the cells of the array. The remaining domain elements represent key values. Each array cell holds a key value; the assignment of key values to array cells is given by the function $K^{\mathbf{A}}$.

The simplicity of the above abstraction gives rise to two peculiarities, which, however, pose no problems. First, the array cells belong to the range of the function K. Thus, array cells are allowed to play a double role as key values. Second, the function K is total, so it is also defined on the domain elements that are not array cells. We will simply ignore K on that part of the domain.

We still need to discuss $\prec$ and T. We assume $\prec^{\mathbf{A}}$ to be a total order, used to compare key values. So $\prec^{\mathbf{A}}$ can be different from the built-in order $<^{\mathbf{A}}$. For the binary search procedure to work, the array needs to be sorted, i.e., $\mathbf{A}$ must satisfy $\forall x \forall y(x < y \rightarrow (K(x) \preceq K(y)))$. Finally, the constant $t := T^{\mathbf{A}}$ is the test value. Specifically, we are going to exhibit an index logic formula that expresses that t is a key value stored in the array. In other words, we want to express the condition

$$\exists x(x < N \wedge K(x) = T). \tag{γ}$$

Note that, we express here condition (γ) by a first-order formula that is not an index formula. So, our aim is to show that γ is still expressible, over all sorted arrays, by an index formula.

We recall the procedure for binary search [16] in the following form, using integer variables L, R and I:

$L := 0$
$R := N - 1$
while $L \neq R$ **do**
  $I := \lfloor (L + R)/2 \rfloor$
  if $K(I) \succ T$ **then** $R := I - 1$ **else** $L := I$
od
if $K(L) = T$ **return** 'found' **else return** 'not found'

We are going to express the above procedure as a simultaneous fixpoint, using binary relation variables L and R and a unary relation variable Z. We collect the iteration numbers in Z, thus counting until the logarithm of the size of the structure. Relation variables L and R are used to store the values, in binary representation, of the integer variables L and R during all iterations. Specifically, for each $i \in \mathit{Num}(\mathbf{A})$, the value of the term $\mathit{index}\{\mathtt{x} : L(i, \mathtt{x})\}$ will be the value of the integer variable L before the i-th iteration of the while loop (and similarly for R).

In the formal expression of γ below, we use the formula β_c from Sect. 5.1, with $N - 1$ playing the role of c. We also assume the following formulas:

- A formula avg that expresses, for unary relation variables X and Y and a numeric variable $\mathtt{x}$, that the bit $\mathtt{x}$ is set in the binary representation of $\lfloor x + y \rfloor /2$, where x and y are the numbers represented in binary by X and Y.
- A formula $\mathit{minusone}(X, \mathtt{y})$, expressing that the bit $\mathtt{y}$ is set in the binary representation of $x - 1$, where x is the number represented in binary by X.

These formulas surely exist because index logic includes full fixpoint logic on the numeric sort; fixpoint logic captures PTIME on the numeric sort; and computing the average, or subtracting one, are PTIME operations on binary numbers.

We are going to apply the formula *avg* where X and Y are given by $L(\mathtt{z}, .)$ and $R(\mathtt{z}, .)$. So, formally, below, we use $avg'(\mathtt{z}, \mathtt{x})$ for the formula obtained from formula *avg* by replacing each subformula of the form $X(\mathtt{u})$ by $L(\mathtt{z}, \mathtt{u})$, and $Y(\mathtt{u})$ by $R(\mathtt{z}, \mathtt{u})$.

Furthermore, we are going to apply formula *minusone* where X is given by avg'. So, formally, $minusone'$ will denote the formula obtained from *minusone* by replacing each subformula of the form $X(\mathtt{u})$ by $avg'(\mathtt{z}, \mathtt{u})$.

A last abbreviation we will use is *test*, which will denote the formula $\exists e(e = index\{\mathtt{x} : avg'\} \wedge K(e) \succ T)$.

Now γ is expressed by $\exists x(x = index\{\mathtt{l} : \psi(\mathtt{l})\} \wedge K(x) = T)$, where

$$
\begin{aligned}
\psi(\mathtt{l}) &:= \exists \mathtt{s} \forall \mathtt{s}'(\mathtt{s}' \leq \mathtt{s} \wedge [\text{S-IFP}_{\mathtt{z},\mathtt{x},L,\mathtt{z},\mathtt{x},R,\mathtt{z},Z}\, \varphi_L, \varphi_R, \varphi_Z](\mathtt{s}, \mathtt{l})) \\
\varphi_Z &:= (Z = \emptyset \wedge \mathtt{z} = 0) \vee (Z \neq \emptyset \wedge \mathtt{z} = \max Z + 1) \\
\varphi_L &:= Z \neq \emptyset \wedge \mathtt{z} = \max Z + 1 \wedge \\
&\quad\ \exists \mathtt{z}'(\mathtt{z}' = \max Z \wedge (test \rightarrow L(\mathtt{z}', \mathtt{x})) \wedge (\neg test \rightarrow avg'(\mathtt{z}', \mathtt{x}))) \\
\varphi_R &:= (Z = \emptyset \wedge \mathtt{z} = 0 \wedge \beta_{N-1}(\mathtt{x})) \vee (Z \neq \emptyset \wedge \mathtt{z} = \max Z + 1 \wedge \\
&\quad\ \exists \mathtt{z}'(\mathtt{z}' = \max Z \wedge (test \rightarrow minusone'(\mathtt{z}', \mathtt{x})) \wedge (\neg test \rightarrow R(\mathtt{z}', \mathtt{x}))))
\end{aligned}
$$

6 Definability in Deterministic PolylogTime

We observe here that very simple properties of structures are nondefinable in index logic. Moreover, we provide an answer to a fundamental question on the primitivity of the built-in order predicate (on terms of sort **v**) in our logic. Indeed, we are working with ordered structures, and variables of sort **v** can only be introduced by binding them to an index term. Index terms are based on sets of bit positions which can be compared as binary numbers. Hence, it is plausible to suggest that the built-in order predicate can be removed from our logic without losing expressive power. We prove, however, that this does not work in the presence of constant or function symbols in the vocabulary.

Proposition 6. *Assume that the vocabulary includes a unary relation symbol P. Checking emptiness (or non-emptiness) of $P^{\mathbf{A}}$ in a given structure $\mathbf{A}$ is not computable in* PolylogTime.

Proof. We will show that emptiness is not computable in PolylogTime. For a contradiction, assume that it is. Consider first-order structures over the vocabulary $\{P\}$, where P is a unary relation symbol. Let M be some Turing machine that decides in PolylogTime, given a $\{P\}$-structure $\mathbf{A}$, whether $P^{\mathbf{A}}$ is empty. Let f be a polylogarithmic function that bounds the running time of M. Let n be a natural number such that $f(n) < n$.

Let $\mathbf{A}_\emptyset$ be the $\{P\}$-structure with domain $\{0, \ldots, n-1\}$, where $P^{\mathbf{A}} = \emptyset$. The encoding of $\mathbf{A}_\emptyset$ to the Turing machine M is the sequence $s := \underbrace{0 \ldots 0}_{n \text{ times}}$. Note that the running time of M with input s is strictly less than n. This means that there must exist an index i of s that was not read in the computation $M(s)$. Define

$$s' := \underbrace{0 \ldots 0}_{i \text{ times}} 1 \underbrace{0 \ldots 0}_{n-i-1 \text{ times}} .$$

Clearly the output of the computations $M(s)$ and $M(s')$ are identical, which is a contradiction since s' is an encoding of a $\{P\}$-structure where the interpretation of P is a singleton. □

The technique of the above proof can be adapted to prove a plethora of undefinability results, e.g., it can be shown that k-regularity of directed graphs cannot be decided in PolylogTime, for any fixed k.

We can develop this technique further to show that the order predicate on terms of sort $\mathbf{v}$ is a primitive in the logic. The proof of the following lemma is quite a bit more complicated and can be found in the full arXiv version [6] of this article.

Lemma 7. *Let P and Q be unary relation symbols. There does not exist an index logic formula φ such that for all $\{P, Q\}$-structures $\mathbf{A}$ such that $P^{\mathbf{A}}$ and $Q^{\mathbf{A}}$ are disjoint singleton sets $\{l\}$ and $\{m\}$, respectively, it holds that*

$$\mathbf{A}, val \models \varphi \text{ if and only if } l < m.$$

Theorem 8. *Let c and d be constant symbols in a vocabulary σ. There does not exist an index logic formula φ that does not use the order predicate $\leq$ on terms of sort $\mathbf{v}$ and that is equivalent with the formula $c \leq d$.*

The proof, by contradiction, shows that a formula φ as stated in the theorem would contradict the above lemma. We give the translation in the full arXiv version [6] of this article.

We conclude this section by affirming that, on purely relational vocabularies, the order predicate on sort $\mathbf{v}$ is redundant. The intuition for this result was given in the beginning of this section and we omit the formal proof.

Theorem 9. *Let σ be a vocabulary without constant or function symbols. For every sentence φ of index logic of vocabulary σ there exists an equivalent sentence φ' that does not use the order predicate on terms of sort $\mathbf{v}$.*

7 Discussion

An interesting open question concerns order-invariant queries. Indeed, while index logic is defined to work on ordered structures, it is natural to try to understand which queries about ordered structures that are actually invariant of the order, are computable in PolylogTime. Results of the kind given by Proposition 6 already suggest that very little may be possible. Then again, any polynomial-time numerical property of the size of the domain is clearly computable. We would love to have a logical characterization of the order-invariant queries computable in PolylogTime.

Another natural direction is to get rid of Turing machines altogether and work with a RAM model working directly on structures, as proposed by Grandjean and Olive [8]. Plausibly by restricting their model to numbers bounded in value by a polynomial in n (the size of the structure), we would get an equivalent PolylogTime complexity notion.

In this vein, we would like to note that extending index logic with numeric variables that can hold values up to a polynomial in n, with arbitrary polynomial-time functions on these, would be useful syntactic sugar that would, however, not increase the expressive power.

References

1. Abiteboul, S., Hull, R., Vianu, V.: Foundations of Databases. Addison-Wesley, Boston (1995)
2. Ebbinghaus, H.D., Flum, J.: Finite Model Theory, 2nd edn. Springer, Heidelberg (1999)
3. Fagin, R.: Contributions to model theory of finite structures. Ph.D. thesis, U. C. Berkeley (1973)
4. Fagin, R.: Generalized first-order spectra and polynomial-time recognizable sets. In: Karp, R. (ed.) Complexity of Computation. SIAM-AMS Proceedings, vol. 7, pp. 43–73 (1974)
5. Ferrarotti, F., González, S., Schewe, K.D., Turull Torres, J.M.: The polylog-time hierarchy captured by restricted second-order logic. In: Post-Proceedings of the 20th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing. IEEE Computer Society (2019, to appear)
6. Ferrarotti F., González S., Turull Torres J.M., Van den Bussche J., Virtema J.: Descriptive complexity of deterministic polylogarithmic time. CoRR abs/1903.03413 (2019)
7. Grädel, E., et al.: Finite Model Theory and Its Applications. Springer, Heidelberg (2007). https://doi.org/10.1007/3-540-68804-8
8. Grandjean, E., Olive, F.: Graph properties checkable in linear time in the number of vertices. J. Comput. Syst. Sci. **68**, 546–597 (2004)
9. Grohe, M.: Descriptive Complexity, Canonisation, and Definable Graph Structure Theory. Lecture Notes in Logic. Cambridge University Press, Cambridge (2017)
10. Grohe, M., Pakusa, W.: Descriptive complexity of linear equation systems and applications to propositional proof complexity. In: Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, pp. 1–12. IEEE Computer Society (2017)
11. Gurevich, Y.: Toward logic tailored for computational complexity. In: Börger, E., Oberschelp, W., Richter, M.M., Schinzel, B., Thomas, W. (eds.) Computation and Proof Theory. LNM, vol. 1104, pp. 175–216. Springer, Heidelberg (1984). https://doi.org/10.1007/BFb0099486
12. Gurevich, Y., Shelah, S.: Fixed-point extensions of first-order logic. Ann. Pure Appl. Logic **32**, 265–280 (1986)
13. Immerman, N.: Number of quantifiers is better than number of tape cells. J. Comput. Syst. Sci. **22**(3), 384–406 (1981)

14. Immerman, N.: Relational queries computable in polynomial time. Inf. Control **68**, 86–104 (1986)
15. Immerman, N.: Descriptive Complexity. Springer, New York (1999). https://doi.org/10.1007/978-1-4612-0539-5
16. Knuth, D.: The Art of Computer Programming. Sorting and Searching, vol. 3. Addison-Wesley, Boston (1998)
17. Libkin, L.: Elements of Finite Model Theory. Springer, Heidelberg (2004). https://doi.org/10.1007/978-3-662-07003-1
18. Mix Barrington, D.A.: Quasipolynomial size circuit classes. In: Proceedings of the Seventh Annual Structure in Complexity Theory Conference, Boston, Massachusetts, USA, 22–25 June 1992, pp. 86–93. IEEE Computer Society (1992)
19. Mix Barrington, D.A., Immerman, N., Straubing, H.: On uniformity within NC^1. J. Comput. Syst. Sci. **41**(3), 274–306 (1990)
20. Ramakrishnan, R., Gehrke, J.: Database Management Systems, 3rd edn. McGraw-Hill, Inc., New York (2003)
21. Stockmeyer, L.J.: The polynomial-time hierarchy. Theor. Comput. Sci. **3**(1), 1–22 (1976)
22. Vardi, M.: The complexity of relational query languages. In: Proceedings 14th ACM Symposium on the Theory of Computing, pp. 137–146 (1982)

A Representation Theorem for Finite Gödel Algebras with Operators

Tommaso Flaminio[1(✉)], Lluis Godo[1], and Ricardo O. Rodríguez[2]

[1] Artificial Intelligence Research Institute (IIIA - CSIC), Campus UAB, 08193 Bellaterra, Spain
{tommaso,godo}@iiia.csic.es

[2] FCEyN, Departamento de Computación, CONICET-UBA, Inst. de Invest. en Cs. de la Computación, UBA, Buenos Aires, Argentina
ricardo@dc.uba.ar

Abstract. In this paper we introduce and study *finite Gödel algebras with operators* (GAOs for short) and their dual frames. Taking into account that the category of finite Gödel algebras with homomorphisms is dually equivalent to the category of finite forests with order-preserving open maps, the dual relational frames of GAOs are *forest frames*: finite forests endowed with two binary (crisp) relations satisfying suitable properties. Our main result is a Jónsson-Tarski like representation theorem for these structures. In particular we show that every finite Gödel algebra with operators determines a unique forest frame whose set of subforests, endowed with suitably defined algebraic and modal operators, is a GAO isomorphic to the original one.

Keywords: Finite Gödel algebras · Modal operators · Finite forests · Representation theorem

1 Introduction

Fuzzy modal logic is an active and relatively recent area of research aimed at generalizing classical modal logic to the many-valued or fuzzy framework. This is usually done by considering a Kripke-style relational semantics in which both accessibility relations and evaluations of modal formulas (in each world) are allowed to take values in the real unit interval $[0, 1]$, instead of the classical two-valued set $\{0, 1\}$ (see [4,5,7] for instance).

In this contribution we put forward a new, algebraic-oriented perspective to the area of fuzzy modal logic, and in particular to Gödel modal logic by defining and studying the class of *finite Gödel algebras with operators* (*GAOs* for short). These structures are obtained by expanding the language of Gödel algebras (i.e. prelinear Heyting algebras) by means of two modal operators $\Diamond$ and $\Box$ equationally described by the same axioms used to define these operators in Boolean algebras with operators (BAOs), see [3].

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 223–235, 2019.
https://doi.org/10.1007/978-3-662-59533-6_14

Obviously, while in a BAO the operators $\Diamond$ and $\Box$ are inter-definable, this is not the general case in a GAO since the negation operator in a Gödel algebra is not involutive. Hence, the equation $\Diamond x = \neg\Box\neg x$ does not hold in general in a GAO.

In the same way the dual frames of BAOs are Kripke frames, the duality between finite Gödel algebras and finite forests (see [1]) leads us to introduce the dual structures of GAOs as triples $(\mathbf{F}, R_\Diamond, R_\Box)$, where $\mathbf{F} = (F, \leq)$ is a finite forest, while $R_\Diamond$ and $R_\Box$ are binary (crisp) relations on $\mathbf{F}$ satisfying suitable conditions of (anti-)monotonicity in their first argument.

The main result of this paper is a Jónsson-Tarski like representation theorem for GAOs. In particular we will show how, starting from a Gödel algebra with operators $(\mathbf{A}, \Diamond, \Box)$, one can uniquely define a forest frame $(\mathbf{F}, R_\Diamond, R_\Box)$ such that $(\mathbf{A}, \Diamond, \Box)$ is isomorphic to the GAO whose Gödel reduct is the algebra of subforests of $\mathbf{F}$ and whose modal operators are defined from the binary relations $R_\Diamond$ and $R_\Box$.

Finally, we will discuss the effect of a stronger axiomatization for $\Diamond$ and $\Box$ on the side of the corresponding forest frame. In particular we will see that the equations usually imposed on *positive modal algebras* [6,8] allow for a simpler description of the forest frames needed in the representation theorem.

This paper is organized as follows. After this introduction, in the following Sect. 2 we will recall basic facts on finite Gödel algebras and finite forests. In Sect. 3 we will consider the case of Gödel algebras expanded by the modal operator $\Diamond$, while Gödel algebras with a $\Box$ operator will be studied in Sect. 4. Section 5 is dedicated to introduce Gödel algebras with both $\Diamond$ and $\Box$ and also to discuss the effect of the stronger axiomatization for these modalities obtained by adding the equations of positive modal algebras. We will end this paper in Sect. 6 where we present our future work.

2 Finite Gödel Algebras and Forests

Gödel algebras, the algebraic semantics of infinite-valued Gödel logic [9], are idempotent, bounded, integral, commutative residuated lattices of the form $\mathbf{A} = (A, \wedge, \vee, \rightarrow, \bot, \top)$ satisfying the prelinearity equation: $(a \rightarrow b) \vee (b \rightarrow a) = \top$. In other words, Gödel algebras are *prelinear Heyting algebras*. If not unless specified, all algebras we will consider in this paper are finite.

Let $\mathbf{A}$ be a Gödel algebra and denote by $F_\mathbf{A}$ the set of its prime filters, i.e., filters principally generated by the join-irreducible elements of $\mathbf{A}$. Unlike the case of boolean algebras, prime and maximal filters are not the same for Gödel algebras and indeed $F_\mathbf{A}$ can be ordered in a nontrivial way. In particular, if for $f_1, f_2 \in F_\mathbf{A}$ we define $f_1 \leq f_2$ iff (as prime filters) $f_1 \supseteq f_2$, $\mathbf{F_A} = (F_\mathbf{A}, \leq)$ turns out to be a finite *forest*, i.e., a poset such that the downset of each element is totally ordered.

Finite forests play a crucial role in the theory of Gödel algebras. Indeed, let $\mathbf{F} = (F, \leq)$ be a finite forest, $S_\mathbf{F}$ be the set of all downward closed subsets of F (i.e., the *subforests* of $\mathbf{F}$) and consider the following operations on $S_\mathbf{F}$: for all $x, y \in F$,

1. $x \wedge y = x \cap y$ (the set-theoretic intersection);
2. $x \vee y = x \cup y$ (the set-theoretic union);
3. $x \to y = F \setminus \uparrow(x \setminus y)$ (where $\setminus$ denotes the set-theoretical difference and for every $z \in F$, $\uparrow z = \{k \in F \mid k \geq z\}$).[1]

The algebra $\mathbf{S_F} = (S_\mathbf{F}, \wedge, \vee, \to, \emptyset, F)$ is a Gödel algebra [1, §4.2] and the following is a Stone-like representation theorem for these structures.

Lemma 1 ([1, Theorem 4.2.1]). *Every Gödel algebra* $\mathbf{A}$ *is isomorphic to* $\mathbf{S_{F_A}}$ *through the map* $r : \mathbf{A} \to \mathbf{S_{F_A}}$

$$r : a \in A \mapsto \{f \in F_\mathbf{A} \mid a \in f\}.$$

Example 1. Let free_1 be the 1-generated free Gödel algebra (Fig. 1). Its prime filters, which are all principally generated as upsets of its join-irreducible elements, are $f_1 = \{y \in \mathsf{free}_1 \mid y \geq x\} = \{x, x \vee \neg x, \neg\neg x, \top\}$, $f_2 = \{y \in \mathsf{free}_1 \mid y \geq \neg x\} = \{\neg x, x \vee \neg x, \top\}$, and $f_3 = \{y \in \mathsf{free}_1 \mid y \geq \neg\neg x\} = \{\neg\neg x, \top\}$. The forest $\mathbf{F}_{\mathsf{free}_1}$ is obtained by ordering $\{f_1, f_2, f_3\}$ by reverse inclusion.

Let us consider the set $S_{\mathbf{F}_{\mathsf{free}_1}}$ of subforests of $\mathbf{F}_{\mathsf{free}_1}$:

$$S_{\mathbf{F}_{\mathsf{free}_1}} = \{\emptyset, F_{\mathsf{free}_1}, \{f_2\}, \{f_1\}, \{f_2, f_1\}, \{f_3, f_1\}\}$$

with operations $\wedge, \vee, \to$ as in (1–3) above. Lemma 1 shows that algebra $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ is a Gödel algebra which is isomorphic to free_1.

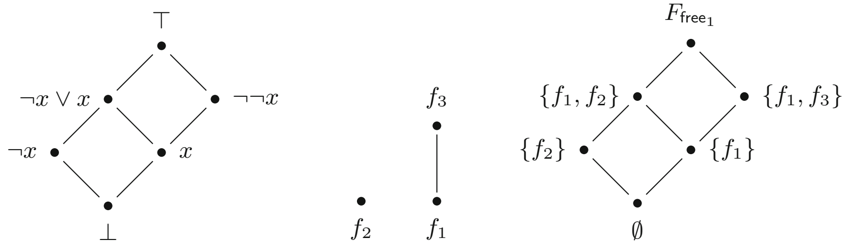


Fig. 1. From left to right: The Hasse diagram of the free Gödel algebra over one generator free_1, the forest $\mathbf{F}_{\mathsf{free}_1}$ of its prime filters, and the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$

[1] Without danger of confusion, and thanks to the following result, we will not distinguish the symbols of a Gödel algebra $\mathbf{A}$ from those of $\mathbf{S_F}$.

3 Gödel Algebras with $\Diamond$-Operators

Definition 1. *A* $\Diamond$-Gödel algebra *is a pair* $(\mathbf{A}, \Diamond)$ *where* $\mathbf{A}$ *is a Gödel algebra and* $\Diamond : A \to A$ *satisfies the following equations:*
($\Diamond$1) $\Diamond(\bot) = \bot$;
($\Diamond$2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

Definition 2. *A* $\Diamond$-forest frame *is a pair* $(\mathbf{F}, R)$ *where* $\mathbf{F} = (F, \leq)$ *is a finite forest and* $R \subseteq F \times F$ *satisfies the following condition:*

(A) for all $x, y, z \in F$, *if* $y \leq x$ *and* $R(x, z)$, *then* $R(y, z)$[2].

For every $\Diamond$-forest frame $(\mathbf{F}, R)$, let $\mathbf{S_F}$ be defined as in the previous section and consider the map $\delta_R : S_\mathbf{F} \to S_\mathbf{F}$ such that, for every $a \in S_\mathbf{F}$

$$\delta_R(a) = \{y \in F \mid \exists z \in a,\ R(y, z)\}. \tag{1}$$

Notice that, for all $a \in S_\mathbf{F}$, $\delta_R(a) \in S_\mathbf{F}$, i.e., $\delta_R(a)$ is a subforest of $\mathbf{F}$. Indeed if $x \in \delta_R(a)$ then there exists $z \in a$ such that $R(x, z)$. Let $y \leq x$ in $\mathbf{F}$. Then (A) of Definition 2 implies $R(y, z)$ as well, that is $y \in \delta_R(a)$ and hence $\delta_R(a)$ is downward closed. Further, the following properties hold.

Proposition 1. *For every* $\Diamond$*-forest frame* $(\mathbf{F}, R)$ *let* $\delta_R : S_\mathbf{F} \to S_\mathbf{F}$ *be defined as in (1). Then:*

1. $\delta_R(\bot) = \bot$;
2. *For all* $b \in S_\mathbf{F}$, $\delta_R(b) = \bigcup\{\delta_R(a) \mid a \leq b$ *and* a *is join-irreducible*$\}$.

Proof. (1) The bottom element of $\mathbf{S_F}$ is the empty forest, whence $\emptyset = \{y \in F \mid \exists z \in \emptyset,\ R(y, z)\} = \delta_R(\bot)$.
(2) The claim is trivial if b is join irreducible. Thus, let $b = a_1 \vee \ldots \vee a_m$ with the a_i's being join irreducible. Therefore, $\delta_R(b) = \{y \in F \mid \exists z \in b,\ R(y, z)\} = \{y \in F \mid \exists z \in a_1 \vee \ldots \vee a_m,\ R(y, z)\} = \{y \in F \mid \exists z \in a_1 \cup \ldots \cup a_m,\ R(y, z)\} = \bigcup_{i=1}^m(\{y \in F \mid \exists z \in a_i,\ R(y, z)\}) = \bigcup_{i=1}^m \delta_R(a_i)$. $\square$

Lemma 2. *For each* $\Diamond$*-forest frame* $(\mathbf{F}, R)$*,* $(\mathbf{S_F}, \delta_R)$ *is a* $\Diamond$*-Gödel algebra.*

Proof. From Lemma 1, $\mathbf{S_F}$ is a Gödel algebra. Equation ($\Diamond$1) holds because of Proposition 1(1). Further, if $a, b \in S_\mathbf{F}$, by Proposition 1(2), $\delta_R(a \vee b) = \delta_R(a) \cup \delta_R(b) = \delta_R(a) \vee \delta_R(b)$ by definition of δ_R. Thus, δ_R satisfies ($\Diamond$2). $\square$

Now, let $(\mathbf{A}, \Diamond)$ be a $\Diamond$-Gödel algebra, let $\mathbf{F_A}$ be as in Sect. 2 and define $Q_\Diamond \subseteq F_\mathbf{A} \times F_\mathbf{A}$ as follows: for all $f_1, f_2 \in F_\mathbf{A}$,

$$Q_\Diamond(f_1, f_2) \text{ iff } \Diamond(f_2) \subseteq f_1, \tag{2}$$

where, for every filter f, $\Diamond(f) = \{\Diamond x \mid x \in f\}$. Then the following holds.

[2] Along this paper we will adopt the notation $R(x, y)$ to denote that the pair (x, y) belongs to the relation R.

Lemma 3. *For each* $\Diamond$*-Gödel algebra* $(\mathbf{A}, \Diamond)$, $(\mathbf{F_A}, Q_\Diamond)$ *is a* $\Diamond$*-forest frame.*

Proof. It is enough to prove that the condition (A) of Definition 2 holds. Let $f_1, f_2, f_3 \in F_\mathbf{A}$ and assume $Q_\Diamond(f_1, f_3)$ (i.e., $\Diamond(f_3) \subseteq f_1$) and $f_1 \geq f_2$, meaning that, as prime filters, $f_1 \subseteq f_2$. Then, $\Diamond(f_3) \subseteq f_1 \subseteq f_2$ and hence $Q_\Diamond(f_2, f_3)$.

Now, our aim is to extend the isomorphism r of Lemma 1 to the case of $\Diamond$-Gödel algebras. Let hence $(\mathbf{A}, \Diamond)$ be a $\Diamond$-Gödel algebra and define, for every $a \in A$,

$$r(\Diamond(a)) = \{f \in F_\mathbf{A} \mid \Diamond(a) \in f\}. \tag{3}$$

Theorem 1. *For every* $\Diamond$*-Gödel algebra* $(\mathbf{A}, \Diamond)$*, the map* $r : (\mathbf{A}, \Diamond) \to (\mathbf{S_{F_A}}, \delta_{Q_\Diamond})$ *is an isomorphism. In particular, for all* $a \in A$*,*

$$r(\Diamond(a)) = \delta_{Q_\Diamond}(r(a)). \tag{4}$$

Proof. We proved in Lemma 1 that $\mathbf{S_{F_A}}$ is a Gödel algebra and the map $r : \mathbf{A} \to \mathbf{S_{F_A}}$ is a Gödel isomorphism. Thus, it remains to show that (4) holds. First of all notice that it is sufficient to prove it for the case of a being a join-irreducible element of $\mathbf{A}$. Indeed, assume that (4) holds for join irreducible elements and let b be not join irreducible. Then b can be displayed as $b = a_1 \vee \ldots \vee a_k$, where the a_i's are join irreducible. By ($\Diamond$2), $\Diamond(b) = \Diamond(a_1) \vee \ldots \vee \Diamond(a_k)$. Therefore, since r is a Gödel algebra isomorphism,

$$r(\Diamond(b)) = r(\Diamond(a_1)) \vee \ldots \vee r(\Diamond(a_k)).$$

By assumption, $r(\Diamond a_i) = \delta_{Q_\Diamond}(r(a_i))$ for all $i = 1, \ldots, k$. Thus, $r(\Diamond(b)) = \delta_{Q_\Diamond}(a_1) \vee \ldots \vee \delta_{Q_\Diamond}(a_k)$ which equals $\delta_{Q_\Diamond}(b)$ by Proposition 1(2).

Let hence a be join irreducible. By Lemma 1, we have:

$$\begin{aligned} \delta_{Q_\Diamond}(r(a)) &= \{f \in F_\mathbf{A} \mid \exists g \in r(a),\ Q_\Diamond(f, g)\} \\ &= \{f \in F_\mathbf{A} \mid \exists g \in F_\mathbf{A},\ (a \in g \ \&\ Q_\Diamond(f, g)\} \\ &= \{f \in F_\mathbf{A} \mid \exists g \in F_\mathbf{A},\ (a \in g \ \&\ \Diamond(g) \subseteq f)\} \end{aligned}$$

Therefore, if $f \in \delta_{Q_\Diamond}(r(a))$, $\Diamond(a) \in f$ and hence $f \in r(\Diamond(a))$.

To prove the other inclusion we have to show that if $f' \in r(\Diamond(a))$, there exists an $f \in F_\mathbf{A}$ such that $a \in f$ and $\Diamond(f) \subseteq f'$. Since a is join irreducible, the filter $f_a = \{b \in A \mid b \geq a\}$ is prime. Let us prove that $\Diamond(f_a) \subseteq f'$.

Claim. $\Diamond(f_a) \subseteq f_{\Diamond(a)} = \{x \in A \mid x \geq \Diamond(a)\}$.

As a matter of fact, if $z \in \Diamond(f_a)$, then there exists $b \geq a$ such that $z = \Diamond(b)$. Since $\Diamond$ is monotone, $\Diamond(b) \geq \Diamond(a)$, whence $z = \Diamond(b) \in f_{\Diamond(a)}$.

Claim. For all $f' \in r(\Diamond(a))$, $f_{\Diamond(a)} \subseteq f'$.

Indeed, if $x \in f_{\Diamond(a)}$, then $x \geq \Diamond(a)$ and hence $x \in f'$ because $\Diamond(a) \in f'$ and f' is upward closed.

By the above claims, for all $f' \in r(\Diamond(a))$, $\Diamond(f_a) \subseteq f'$, whence

$$r(\Diamond(a)) \subseteq \delta_{Q_\Diamond}(r(a)).$$

Thus, for all a, $r(\Diamond(a)) = \delta_{Q_\Diamond}(r(a))$ which settles the claim. □

Example 2. Let free_1 be as in Example 1 and let $\Diamond : \mathsf{free}_1 \to \mathsf{free}_1$ be the following map:

$$\Diamond(\bot) = \bot; \Diamond(x) = \neg x; \Diamond(\neg x) = \neg x \vee x; \Diamond(\neg x \vee x) = \neg x \vee x;$$
$$\Diamond(\neg\neg x) = \top; \Diamond(\top) = \top.$$

It is easy to check that $\Diamond$ satisfies ($\Diamond$1) and ($\Diamond$2) of Definition 1 and hence $(\mathsf{free}_1, \Diamond)$ is a $\Diamond$-Gödel algebra.

Let $\mathbf{F}_{\mathsf{free}_1}$ be the dual forest of free_1 as in Example 1 and let us compute $Q_\Diamond$ according to (2). First: $\Diamond(f_1) = \{\neg x, x \vee \neg x, \top\}$; $\Diamond(f_2) = \{x \vee \neg x, \top\}$ and $\Diamond(f_3) = \{\top\}$. Therefore, (see Fig. 2)

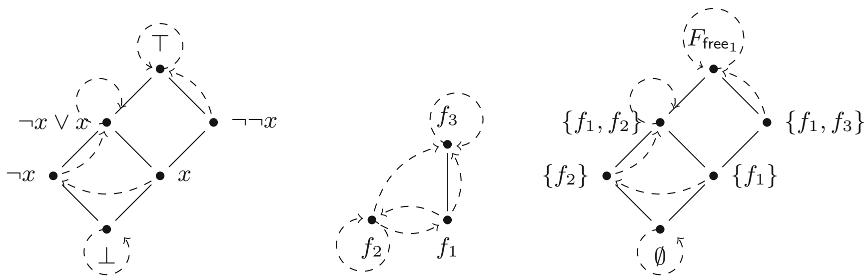


Fig. 2. From left to right: The Hasse diagram of the free Gödel algebra over one generator free_1 and a $\Diamond$ operator (dotted arrows); the forest $\mathbf{F}_{\mathsf{free}_1}$ of its prime filters and the relation $Q_\Diamond$ (dotted arrows); the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ endowed with the operator $\delta_{Q_\Diamond}$ (dotted arrows).

$$Q_\Diamond = \{(f_1, f_2), (f_1, f_3), (f_2, f_2), (f_2, f_1), (f_2, f_3), (f_3, f_3)\}.$$

The relation $Q_\Diamond$ satisfies the property (A) of Definition 2. Indeed, $f_1 \leq f_3$, and for all $f \in F_{\mathsf{free}_1}$, if $Q_\Diamond(f_3, f)$ then $Q_\Diamond(f_1, f)$. Therefore $(F_{\mathsf{free}_1}, Q_\Diamond)$ is a $\Diamond$-forest frame.

Finally, let $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ be the isomorphic copy of free_1 as in Example 1 and let $\delta_{Q_\Diamond} : \mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}} \to \mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ be as in (1):

$\delta_{Q_\Diamond}(\emptyset) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \emptyset,\ Q_\Diamond(f, g)\} = \emptyset$;
$\delta_{Q_\Diamond}(\{f_1\}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \{f_1\},\ Q_\Diamond(f, g)\} = \{f_2\}$;
$\delta_{Q_\Diamond}(\{f_2\}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \{f_2\},\ Q_\Diamond(f, g)\} = \{f_1, f_2\}$;
$\delta_{Q_\Diamond}(\{f_1, f_2\}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \{f_1, f_2\},\ Q_\Diamond(f, g)\} = \{f_1, f_2\}$;
$\delta_{Q_\Diamond}(\{f_1, f_3\}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \{f_1, f_3\},\ Q_\Diamond(f, g)\} = \{f_1, f_2, f_3\} = F_{\mathsf{free}_1}$;
$\delta_{Q_\Diamond}(F_{\mathsf{free}_1}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in F_{\mathsf{free}_1},\ Q_\Diamond(f, g)\} = F_{\mathsf{free}_1}$.

Therefore, $(\mathsf{free}_1, \Diamond)$ and $(\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}, \delta_{Q_\Diamond})$ are isomorphic $\Diamond$-Gödel algebras.

4 Gödel Algebras with $\Box$-Operators

Definition 3. *A $\Box$-Gödel algebra is a pair $(\mathbf{A}, \Box)$ such that $\mathbf{A}$ is a Gödel algebra and $\Box : A \to A$ satisfies the following equalities:*

($\Box$1) $\Box(\top) = \top$;
($\Box$2) $\Box(a \wedge b) = \Box a \wedge \Box b$.

Definition 4. *A* $\Box$-forest frame *is a pair* $(\mathbf{F}, R)$ *where* $\mathbf{F} = (F, \leq)$ *is a finite forest and* $R \subseteq F \times F$ *satisfies the following condition:*

(M) *for all* $x, y, z \in F$*, if* $x \leq y$ *and* $R(x, z)$*, then* $R(y, z)$*.*

For every $\Box$-forest frame $(\mathbf{F}, R)$, let $\beta_R : S_{\mathbf{F}} \to S_{\mathbf{F}}$ be defined as follows: for all $a \in S_{\mathbf{F}}$,

$$\beta_R(a) = \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in a)\}. \tag{5}$$

For all $a \in S_{\mathbf{F}}$, $\beta_R(a)$ is a subforest of $\mathbf{F}$. Indeed, if $x \in \beta_R(a)$ then $\forall z \in F$, $(R(x, z) \Rightarrow z \in a)$. Let $y \leq x$. Thus, for all $z \in F$ either $R(y, z)$ is false (and in this case the condition $R(y, z) \Rightarrow z \in a$ is trivially true), or $R(y, z)$ is true in which case $R(x, z)$ is true as well, because of (M), and hence $z \in a$. Thus $y \in \beta_R(a)$.

Proposition 2. *The following properties hold:*

1. $\beta_R(\top) = \top$;
2. *For all* $b \in A_F$, $\beta_R(b) = \bigcup(\{\beta_R(a) \mid a \leq b$ *and* a *is join irreducible*$\})$.

Proof. (1) Recall from Sect. 2 that the top element of $\mathbf{S_F}$ is F. Thus, $\beta_R(\top) = \beta_R(F) = \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in F)\}$. Obviously, the condition $(R(y, z) \Rightarrow z \in F)$ is true for all $z \in F$ and hence $\beta_R(F) = F$.
(2) Skipping the trivial case in which b is join irreducible, let $b = a_1 \vee \ldots \vee a_m$ with the a_i's join irreducible. Remember that in classical logic, for every finite k, $x \Rightarrow (\exists i \in \{1, \ldots, k\}(y_i)) = \exists i \in \{1, \ldots, k\}\ (x \Rightarrow y_i)$, hence

$$\begin{aligned}
\beta_R(b) &= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in b)\} \\
&= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in \textstyle\bigvee_{i=1}^{m} a_i)\} \\
&= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow (\exists i \in \{1, \ldots, m\}\ (z \in a_i)))\} \\
&= \{y \in F \mid \forall z \in F,\ \exists i \in \{1, \ldots, k\}\ (R(y, z) \Rightarrow z \in a_i)\} \\
&= \textstyle\bigcup_{i=1}^{m} \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in a_i)\} \\
&= \textstyle\bigcup_{i=1}^{m} \beta_R(a_i).
\end{aligned}$$

The claim is hence settled. $\Box$

Lemma 4. *For every* $\Box$*-forest frame* $(\mathbf{F}, R)$*,* $(\mathbf{S_F}, \beta_R)$ *is a* $\Box$*-Gödel algebra.*

Proof. We already showed that $\beta_R(\top) = \top$. If $a, b \in S_{\mathbf{F}}$ and recalling that, as subforests of F, $a \wedge b = a \cap b$, one has

$$\begin{aligned}
\beta_R(a \wedge b) &= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in a \wedge b)\} \\
&= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in a \cap b)\} \\
&= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in a)\} \cap \\
&\quad\ \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in b)\} \\
&= \beta_R(a) \cap \beta_R(b) \\
&= \beta_R(a) \wedge \beta_R(b).
\end{aligned}$$

$\Box$

Let $(\mathbf{A}, \Box)$ be a $\Box$-Gödel algebra and define $Q_\Box$ on $F_\mathbf{A} \times F_\mathbf{A}$ as follows: for each $f_1, f_2 \in F_\mathbf{A}$,

$$Q_\Box(f_1, f_2) \text{ iff } \Box^{-1}(f_1) \subseteq f_2, \tag{6}$$

where, for every filter f, $\Box^{-1}(f) = \{x \in A \mid \Box(x) \in f\}$.

Lemma 5. *For every $\Box$-Gödel algebra $(\mathbf{A}, \Box)$, $(\mathbf{F_A}, Q_\Box)$ is a $\Box$-forest frame.*

Proof. Let $f_1, f_2, f_3 \in F_\mathbf{A}$. If $f_1 \leq f_2$ in the order of $\mathbf{F_A}$, then $f_1 \supseteq f_2$ as prime filters, whence if $\Box^{-1}(f_1) \subseteq f_3$ then $\Box^{-1}(f_2) \subseteq f_3$. Therefore, if $Q_\Box(f_1, f_3)$, then $Q_\Box(f_2, f_3)$ which settles the claim.

The following result is the analogous of Theorem 1 in the case of $\Box$-Gödel algebras where r is the map of Lemma 1 which extends to all elements of a $\Box$-Gödel algebra $(\mathbf{A}, \Box)$ by the following stipulation:

$$r(\Box(a)) = \{f \in F_\mathbf{A} \mid \Box(a) \in f\}. \tag{7}$$

Theorem 2. *For every finite $\Box$-Gödel algebra $(\mathbf{A}, \Box)$, the map $r : (\mathbf{A}, \Box) \to (\mathbf{S}_{\mathbf{F_A}}, \beta_{Q_\Box})$ defined as above is an isomorphism. In particular, for all $a \in A$,*

$$r(\Box(a)) = \beta_{Q_\Box}(r(a)). \tag{8}$$

Proof. Let us start proving that for all $a \in A$, $\beta_{Q_\Box}(r(a)) \subseteq r(\Box(a))$. By definition,

$$\begin{aligned}\beta_{Q_\Box}(r(a)) &= \{f \in F_\mathbf{A} \mid \forall g \in F_\mathbf{A}\ (Q_\Box(f, g) \Rightarrow g \in r(a))\}\\ &= \{f \in F_\mathbf{A} \mid \forall g \in F_\mathbf{A}\ (\Box^{-1}(f) \subseteq g \Rightarrow a \in g)\}.\end{aligned}$$

Let $f \in \beta_{Q_\Box}(r(a))$ and assume, by way of contradiction, that $f \notin r(\Box(a))$, that is to say, $a \notin \Box^{-1}(f)$. Notice that this assumption forces $a \neq \top$.

Claim. $\Box^{-1}(f)$ is a filter of $\mathbf{A}$.

As a matter of facts, $\top \in \Box^{-1}(f)$ because $\top \in f$ and $\Box\top = \top$. Further, if $a, b \in \Box^{-1}(f)$, then $\Box a \in f$ and $\Box b \in f$, whence $\Box(a) \wedge \Box(b) \in f$ since f is a filter. Hence $\Box(a \wedge b) \in f$ by ($\Box$2) showing that $\Box^{-1}(f)$ is $\wedge$-closed. Finally, if $a \in \Box^{-1}(f)$ and $b \geq a$, then by the monotonicity of $\Box$, $\Box(b) \geq \Box(a)$, hence $\Box(b) \in f$ because f us upward closed. Therefore, $\Box^{-1}(f)$ is a filter of $\mathbf{A}$.

Going back to the proof of Theorem 2, if $a \in \Box^{-1}(f)$ and since $a \neq \top$, by [9, Lemma 2.3.15], there exists a prime filter p of $\mathbf{A}$ such that $p \supseteq \Box^{-1}(f)$ and $a \notin p$. On the other hand, $Q_\Box(f, p)$ because p extends $\Box^{-1}(f)$ and $a \notin p$. Thus, $f \notin \beta_{Q_\Box}(r(a))$ and a contradiction has been reached.

For the other inclusion, we have to prove that if $\Box(a) \in f$, then for all $g \in F_\mathbf{A}$, $Q_\Box(f, g) \Rightarrow a \in g$. If $\Box(a) \in f$, then $a \in \Box^{-1}(f)$. Therefore, for all $g \in F_\mathbf{A}$, if $Q_\Box(f, g)$, then $\Box^{-1}(f) \subseteq g$ and hence $a \in g$ which settles the claim. $\Box$

Example 3. As in the previous Examples 1 and 2 let free_1 the free, 1-generated Gödel algebra and consider the map $\Box : \mathsf{free}_1 \to \mathsf{free}_1$ defined as follows (dashed arrows in the leftmost picture of Fig. 3):

$$\Box(\bot) = \bot;\ \Box(x) = x;\ \Box(\neg x) = \bot;\ \Box(\neg x \vee x) = x;\ \Box(\neg\neg x) = \neg\neg x;\ \Box(\top) = \top.$$

That operation makes $(\mathsf{free}_1, \Box)$ into a $\Box$-Gödel algebra.

For the reader convenience, let us compute $\Box^{-1}(f)$ (for $f \in F_{\mathsf{free}_1}$): Adopting the same notation of the previous examples,

$$\Box^{-1}(f_1) = \{\top\};\ \Box^{-1}(f_2) = f_2;\ \Box^{-1}(f_3) = f_3.$$

Therefore, by (6), $Q_\Box \subseteq F_{\mathsf{free}_1} \times F_{\mathsf{free}_1}$ is the following relation (check Fig. 3, central picture):

$$Q_\Box = \{(f_1, f_1), (f_2, f_2), (f_3, f_3), (f_2, f_1), (f_2, f_3), (f_3, f_1)\}.$$

Notice that $(F_{\mathsf{free}_1}, Q_\Box)$ is a $\Box$-forest frame. Indeed, $f_1 \leq f_3$ and for all $f \in F_{\mathsf{free}_1}$, $Q_\Box(f_1, f) \leq Q_\Box(f_3, f)$.

Finally, let $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ be the isomorphic copy of free_1 as in Example 1 and let us define $\beta_{Q_\Box}$ as above, i.e., for all $a \in S_{\mathbf{F}_{\mathsf{free}_1}}$,

$$\beta_{Q_\Box}(a) = \{f \in F_{\mathsf{free}_1} \mid \text{for all } g \in F_{\mathsf{free}_1},\ \text{if } Q_\Box(f, g) \text{ then } g \in a\}.$$

The computation is tedious and we will only show $\beta_{Q_\Box}(\{f_1, f_2\})$. The remaining cases are left to the reader.

$$\beta_{Q_\Box}(\{f_1, f_2\}) = \{f \in F_{\mathsf{free}_1} \mid \text{for all } g \in F_{\mathsf{free}_1},\ \text{if } Q_\Box(f, g) \text{ then } g \in \{f_1, f_2\}\}.$$

Let us enter a case distinction:

- $f_1 \in \beta_{Q_\Box}(\{f_1, f_2\})$. Let g be arbitrary in F_{free_1}. In particular, if $g = f_1$, then $Q_\Box(f_1, f_1)$ and $f_1 \in \{f_1, f_2\}$; if $g = f_2$, we have $Q_\Box(f_1, f_2)$ and again $f_2 \in \{f_1, f_2\}$; if $g = f_3$, $(f_1, f_3) \notin Q_\Box$ whence we conclude that $f_1 \in \beta_{Q_\Box}(\{f_1, f_2\})$.
- $f_2 \in \beta_{Q_\Box}(\{f_1, f_2\})$. Again we distinguish the following cases: for $g = f_1$ or $g = f_2$, $Q_\Box(f_2, g)$ and $g \in \{f_1, f_2\}$; if $g = f_3$, $Q_\Box(f_2, f_3)$ but $f_3 \notin \{f_1, f_2\}$, whence $f_2 \notin \beta_{Q_\Box}(\{f_1, f_2\})$.
- $f_3 \in \beta_{Q_\Box}(\{f_1, f_2\})$. Notice immediately that for $g = f_3$ one has $Q_\Box(f_3, f_3)$ but $f_3 \notin \{f_1, f_2\}$, whence $f_3 \notin \beta_{Q_\Box}(\{f_1, f_2\})$.

Therefore, $\beta_{Q_\Box}(\{f_1, f_2\}) = \{f_1\}$ (see Fig. 3, dashed arrows in the rightmost picture, for the remaining cases).

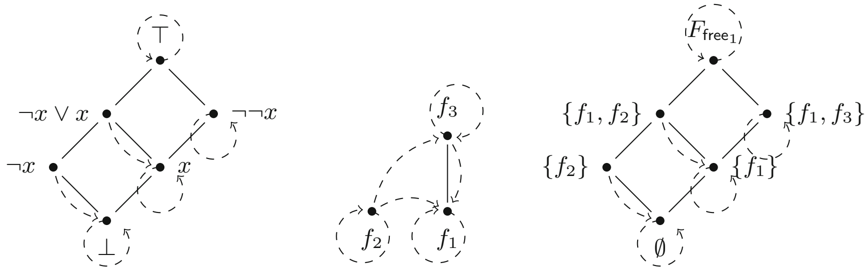


Fig. 3. From left to right: The Hasse diagram of the free Gödel algebra over one generator free_1 and a $\Box$ operator (dotted arrows); the forest $\mathbf{F}_{\mathsf{free}_1}$ of its prime filters and the relation $Q_\Box$ (dotted arrows); the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ endowed with the operator $\beta_{Q_\Box}$ (dotted arrows).

5 Gödel Algebras with $\Diamond$ and $\Box$ Operators

The notions of results provided in the previous sections immediately give us the following

Definition 5. *A* Gödel algebra with operators *(GAO for short) is a triple* $(\mathbf{A}, \Diamond, \Box)$ *where* $\mathbf{A}$ *is a Gödel algebra,* $\Diamond$ *and* $\Box$ *are unary operators of* A *which satisfy the equations* ($\Diamond$1)-($\Diamond$2) *and* ($\Box$1)-($\Box$2) *of Definitions 1 and 3 respectively.*

Let us observe that the equations for $\Diamond$ and $\Box$ are *minimal* in the sense that ($\Diamond$1)-($\Diamond$2) and ($\Box$1)-($\Box$2) are the weakest requirements we may ask the modal operators to satisfy, taking into account that, since the negation operator in Gödel algebras is not involutive, $\Diamond$ and $\Box$ are not inter-definable as in the classical setting. A similar remark concerning the minimality of those equations, but in the more general setting of *Heyting algebras with operators*, can be found in [10].

This remark leads us to the following notion of *frame* for GAOs which, not surprisingly, includes both that of $\Diamond$- and $\Box$-forest frame.

Definition 6. *A* forest frame *is a triple* $(\mathbf{F}, R_\Diamond, R_\Box)$ *such that* $(\mathbf{F}, R_\Diamond)$ *is a* $\Diamond$*-forest frame and* $(\mathbf{F}, R_\Box)$ *is a* $\Box$*-forest frame.*

Given a GAO $(\mathbf{A}, \Diamond, \Box)$ and following exactly the same constructions and results described in the previous Sects. 3 and 4, it is immediate to show that, indeed, $(\mathbf{F_A}, Q_\Diamond, Q_\Box)$ is a forest frame and, vice versa, given any forest frame $(\mathbf{F}, R_\Diamond, R_\Box)$, the algebra $(\mathbf{S_F}, \delta_{R_\Diamond}, \beta_{R_\Box})$ is a GAO. The following result, which is an immediate consequence of Theorems 1 and 2, is a Jónsson-Tarski like representation for GAOs.

Theorem 3. *Let* $(\mathbf{A}, \Diamond, \Box)$ *be a GAO. The map* $r : (\mathbf{A}, \Diamond, \Box) \to (\mathbf{A}_{F_\mathbf{A}}, \delta_{Q_\Diamond}, \beta_{Q_\Box})$*, where* $\delta_{Q_\Diamond}$ *and* $\beta_{Q_\Box}$ *are defined by Eqs. (3) and (7), is an isomorphism. In particular, for all* $a \in A$*,*

$$r(\Diamond(a)) = \delta_{Q_\Diamond}(r(a)) \text{ and } r(\Box(a)) = \beta_{Q_\Box}(r(a)). \tag{9}$$

Following [6,8], let us consider the following equations:
$(D1)$ $\Box(a \vee b) \leq \Box a \vee \Diamond b$;
$(D2)$ $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.

For every Gödel algebra $\mathbf{A}$, let us denote by $\mathbf{A}^{-}$ its $\{\to, \neg\}$-free reduct. Then, if $(\mathbf{A}, \Diamond, \Box)$ satisfies $(D1)$ and $(D2)$, $(\mathbf{A}^{-}, \Diamond, \Box)$ is a *positive modal algebra* in the sense of [6,8]. Since the set of prime filters of $\mathbf{A}$ and that of $\mathbf{A}^{-}$ coincide, $\mathbf{F}_{\mathbf{A}^{-}} = \mathbf{F}_{\mathbf{A}}$ and, following [6], let us define $R_{\mathbf{A}} \subseteq F_{\mathbf{A}^{-}} \times F_{\mathbf{A}^{-}}$ as follows: for all $f_1, f_2 \in F_{\mathbf{A}}$,

$$R_{\mathbf{A}}(f_1, f_2) \text{ iff } \Box^{-1}(f_1) \subseteq f_2 \subseteq \Diamond^{-1}(f_1).$$

Observing that $f_2 \subseteq \Diamond^{-1}(f_1)$ iff $\Diamond(f_2) \subseteq f_1$, by [6, Lemma 2.1(1)], we have that $R_{\mathbf{A}} = Q_{\Diamond} \cap Q_{\Box}$, where $Q_{\Diamond}$ and $Q_{\Box}$ are defined as in (2) and (6) respectively.

Now, let $\mathbf{S}_{\mathbf{F}_{\mathbf{A}^{-}}}$ be the Gödel algebra of subforests of $\mathbf{F}_{\mathbf{A}^{-}}$ and define $\delta_{R_{\mathbf{A}}}$ and $\beta_{R_{\mathbf{A}}}$ on $S_{\mathbf{F}_{\mathbf{A}^{-}}}$ by (1) and (5) respectively. Then the following is an immediate consequence of [6, Theorem 2.2] (see also [8, Theorem 8.1]).

Proposition 3. *Let $(\mathbf{A}, \Diamond, \Box)$ be a GAO which satisfies $(D1)$ and $(D2)$. Then its $\{\to, \neg\}$-free reduct $(\mathbf{A}^{-}, \Diamond, \Box)$ and the positive algebra $((\mathbf{S}_{\mathbf{F}_{\mathbf{A}^{-}}})^{-}, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ are isomorphic (as positive modal algebras).*

Clearly, $S_{\mathbf{F}_{\mathbf{A}^{-}}} = S_{\mathbf{F}_{\mathbf{A}}}$. Now, it is not difficult to extend the above result to GAOs satisfying $(D1)$ and $(D2)$ by expanding the algebra $((\mathbf{S}_{\mathbf{F}_{\mathbf{A}^{-}}})^{-}, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ by the operator $\to$ defined as in Sect. 2: for all $x, y \in S_{\mathbf{F}_{\mathbf{A}}}$,

$$x \to y = F_{\mathbf{A}} \setminus \uparrow(x \setminus y).$$

Then, $(\mathbf{S}_{\mathbf{F}_{\mathbf{A}^{-}}})^{-}$ plus $\to$ and $\neg$ (defined as usual by $\neg x = x \to \emptyset$) is a Gödel algebra isomorphic to $\mathbf{S}_{\mathbf{F}_{\mathbf{A}}}$. Thus, the following holds.

Theorem 4. *Every GAO $(\mathbf{A}, \Diamond, \Box)$ satisfying $(D1)$ and $(D2)$ is isomorphic to $(\mathbf{S}_{\mathbf{F}_{\mathbf{A}}}, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ (as Gödel algebras with operators).*

6 Conclusion and Future Work

In the present paper we have introduced finite Gödel algebras with modal operators and their dual forest frames. Our main result is a Jónsson-Tarski like representation theorem for these structures. Further, we have introduced a proper subclass of Gödel algebras with operators, and we have shown for them a simplified representation which uses, on the dual side of forest frames, only one accessibility relation. It is important to notice that, in contrast with [5] where the authors consider Kripke frames for Gödel modal logic with a unique $[0,1]$-valued accessibility relation, the dual frames of our Gödel algebras with operators have two crisp accessibility relations. This latter observation offers, in our opinion, a fresh new perspective on the semantic approach to fuzzy modal logics which deserves to be further investigated.

As for future work we plan the address the following questions:

(1) To extend the results of this paper to the whole class of Gödel algebras. In this direction we will investigate an extension of Theorem 3 for *general* Gödel algebras with operators. In order to achieve this goal we will take into account that the prime spectrum of a Gödel algebra forms an Esakia space whose underline poset is a forest (see [11, Theorem 2.4]).
(2) The whole class of Gödel algebras with operators forms a variety which determines the equivalent algebraic semantics of a Gödel modal logic. This logic, denoted by $\mathbf{G}_{\Box\Diamond}$, can be regarded as the axiomatic extension of intuitionistic modal logic $\mathbf{IntK}_{\Box\Diamond}$ [12] by the prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$. Our main plans in this direction are to show that $\mathbf{G}_{\Box\Diamond}$ has the finite model property and to compare $\mathbf{G}_{\Box\Diamond}$ with the other approaches to Gödel modal logic existing in the literature, in particular with that of [5]. In this paper the authors introduce a logic with both $\Box$ and $\Diamond$ operators, stronger than $\mathbf{G}_{\Box\Diamond}$[3], that is shown to be complete with respect to the class of Kripke models over the standard Gödel algebra (on the unit real interval $[0, 1]$) where both the accessibility relation and formulas are evaluated on $[0, 1]$.
(3) Finite Nilpotent Minimum (NM) algebras with (or without) a negation fixpoint are dually equivalent to the category of finite forests (and hence categorically equivalent to finite Gödel algebras) [1, Proposition 4.5.4 and §4.5] and [2, Corollary 4.10]. In particular, the connected (disconnected, respectively) rotation of the $\{\bot\}$-free reduct of any finite, directly indecomposable Gödel algebra $\mathbf{A}$ is a finite, directly indecomposable, NM-algebra with (without) negation fixpoint and each directly indecomposable NM-algebra with (without) negation fixpoint arises in this way (see [1, §4.5] and references therein). Taking into account this structural description, we plan to extend the analysis reported in this paper to the classes of NM-algebras with, or without, negation fixpoint.

Acknowledgments. The authors acknowledge partial support by the SYSMICS project (EU H2020-MSCA-RISE-2015 Project 689176). Further, Flaminio acknowledges partial support by the Spanish Ramon y Cajal research program RYC-2016-19799; Flaminio and Godo by the Spanish FEDER/MINECO project TIN2015- 71799-C2-1-P; Rodriguez, by the projects UBA-CyT: 20020150100002BA and PIP 112-2015-0100412 CO.

References

1. Aguzzoli, S., Bova, S., Gerla, B.: Free algebras and functional representation for fuzzy logics. In: Cintula, P., Hájek, P., Noguera, C. (eds.) Handbook of Mathematical Fuzzy Logic - Volume 2, Chap. IX. Studies in Logic, vol. 38, pp. 713–791. College Publications, London (2011)
2. Aguzzoli, S., Flaminio, T., Ugolini, S.: Equivalences between subcategories of MTL-algebras via Boolean algebras and prelinear semihoops. J. Log. Comput. **27**(8), 2525–2549 (2017)

[3] In particular it includes Fisher-Servi connecting axioms $\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$ and $(\Diamond\varphi \to \Box\varphi) \to \Box(\varphi \to \psi)$.

3. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, Cambridge (2001)
4. Bou, F., Esteva, F., Godo, L., Rodriguez, R.: On the minimum many-values modal logic over a finite residuated lattice. J. Log. Comput. **21**(5), 739–790 (2011)
5. Caicedo, X., Rodríguez, R.O.: Bi-modal Gödel logic over [0, 1]-valued Kripke frames. J. Log. Comput. **25**(1), 37–55 (2015)
6. Celani, S., Jansana, R.: Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic. Log. J. IGPL **7**(6), 683–715 (1999)
7. Diaconescu, D., Metcalfe, G., Schnüriger, L.: A real-valued modal logic. Log. Methods Comput. Sci. **14**(1), 1–27 (2018)
8. Dunn, M.: Positive modal logics. Stud. Log. **55**, 301–317 (1995)
9. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer Academic Publishers, Dordrecht (1998)
10. Hasimoto, Y.: Heyting algebras with operators. Math. Log. Q. **47**(2), 187–196 (2001)
11. Horn, A.: Logic with truth values in a linearly ordered Heyting algebra. J. Symb. Log. **34**, 395–405 (1969)
12. Wolter, F., Zakharyaschev, M.: Intuitionistic modal logic. In: Cantini, A., Casari, E., Minari, P. (eds.) Logic and Foundations of Mathematics, pp. 227–238. Kluwer, Dordrecht (1999)

Bar Induction and Restricted Classical Logic

Makoto Fujiwara(✉)

School of Science and Technology, Meiji University,
1-1-1 Higashi-Mita, Tama-ku, Kawasaki-shi, Kanagawa 214-8571, Japan
makotofujiwara@meiji.ac.jp

Abstract. Bar induction is originally discussed by L. E. J. Brouwer under the name of "bar theorem" in his intuitionistic mathematics. Nowadays, there are several formulations of bar induction. Over a well-known classical subsystem RCA_0 of second-order arithmetic, they are equivalent to the full second-order comprehension axiom. However, their interrelation from the purely constructive point of view (in the sense of Bishop) is still unknown. In this paper, we investigate the interrelation between decidable bar induction, monotone bar induction, and bar induction with neither the decidable condition nor the monotonicity condition in the assumptions over an intuitionistic fragment of RCA_0, and show that the third one is equivalent to the second one plus the numerical constant domain axiom which comes from the study of intermediate predicate logics. In addition, we consider the restrictions of bar induction where the side-predicates are of the form $\exists z\, \mathrm{Q_{qf}}(z)$ where $\mathrm{Q_{qf}}(z)$ is quantifier-free. Then we show the close relation between the restrictions of bar induction and the negative translation of a principle classically equivalent to the arithmetical comprehension axiom.

Keywords: Reverse mathematics · Intuitionistic mathematics · Bar induction · Subsystems of second-order arithmetic

Bar induction is originally introduced by L. E. J. Brouwer in his intuitionistic mathematics. As mentioned in [13], however, it is first formulated in a workable form by S. C. Kleene [11]. In this paper, we investigate the logical relationship between a couple of forms of bar induction and their syntactical restrictions in the context of constructive reverse mathematics, which is a research project to reveal the constructive derivability relation between mathematical statements (see [9]). The base theory which we are working on is a subsystem EL_0 of so-called elementary analysis EL ([16, 1.9.10]), which has two-sorted variables in its language. Note that the subscript 0 of EL_0 denotes the restriction of the induction scheme to quantifier-free formulas in this context. We refer the reader to see e.g. [2,3,6] for the details of EL_0. Note that EL_0 is an intuitionistic variant of the most popular base system RCA_0 of classical reverse mathematics [15]. By investigating the derivability relation between mathematical statements over EL_0

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 236–247, 2019.
https://doi.org/10.1007/978-3-662-59533-6_15

rather than $\mathsf{RCA_0}$, we can obtain a sharper classification than classical reverse mathematics (see Corollary 28).

Throughout this paper, we use the lower-case letters x, y, z, i etc. for number (type-0) variables and use the lower-case Greek letters α, β, γ etc. for function (type-1) variables where the arity will be clear from the context. In addition, we suppress a surjective coding of finite sequences of natural numbers onto natural numbers (see e.g. [16, 1.3.9]) for readability and use a lower-case letter s or s' to express a code for a finite sequence of natural numbers, while literally it is just a type-0 variable. We mean by $\langle \cdot \rangle$ a (code for a) finite sequence. For a (code for a) finite sequence s, $|s|$ denotes the length of s, s_i denotes the i-th element of s for $i < |s|$, and $s * \langle y \rangle$ denotes the concatenation of s and $\langle y \rangle$. In addition, $\dot{-}$ is a (primitive recursive) function symbol such that $x \dot{-} 0 = x$ and $x \dot{-} (y+1) = \max\{x - (y+1), 0\}$. For function (type-1) variable β and number (type-0) variable x, $\bar{\beta}x$ denotes the (code of) finite sequence $\langle \beta(0), \beta(1), \ldots, \beta(x-1) \rangle$ for $x > 0$ and $\bar{\beta}0 := \langle\,\rangle$. As usual, $\exists i < x\, A(i)$ is the abbreviation of $\exists i (i < x \wedge A(i))$. For an axiom (scheme), A, $\mathsf{EL_0}$ + A denotes the system obtained from $\mathsf{EL_0}$ by adding also A as its axiom (scheme).

We assume that the reader has some familiarity with (two-sorted) intuitionistic arithmetic. Note that $\neg \exists x\, C$ is intuitionistically equivalent to $\forall x \neg C$ for arbitrary formula C, which is frequently used throughout this paper.

1 Several Formulations of Bar Induction

Definition 1 (Bar Induction). *There are several existing formulations of Brouwer's bar induction. We first present seemingly the most typical formulation from* [11,13]*, which is known as* **monotone bar induction**

$$\mathrm{MBI}^0\text{:}\ \begin{pmatrix} (i)\, \forall \alpha \exists x P(\bar{\alpha}x) \\ (ii)\, \forall s, s'(P(s) \to P(s * s')) \\ (iii)\, \forall s(P(s) \to Q(s)) \\ (iv)\, \forall s(\forall y Q(s * \langle y \rangle) \to Q(s)) \end{pmatrix} \to Q(\langle\,\rangle).$$

Another well-known formulation is so-called **decidable bar induction** DBI^0 *where the monotonicity condition* (ii) *is replaced by the decidable condition* $(ii)'$: $\forall s(P(s) \vee \neg P(s))$*. On the other hand, if one considers bar induction over classical logic, the monotonicity or decidable condition is superfluous (see* [13] *for details). The main principle in this paper is* **unrestricted bar induction** BI^0 *where the second condition is omitted. The unrestricted bar induction has been investigated extensively in a recent paper* [14]*. In addition, we also investigate their special cases where* P *and* Q *are the same formulas and employ notations* MBI^0_*, DBI^0_*, *and* BI^0_* *respectively for them. Such a restriction is already taken into account in* [17]*. For any syntactical class* Γ *of formulas,* (Full, Γ)-BI^0 *and* Γ-BI^0_* *denote the restrictions of* BI^0 *and* BI^0_* *respectively where* Q *is in* Γ*. The same restrictions of* $\mathrm{MBI}^0, \mathrm{DBI}^0, \mathrm{MBI}^0_*$, *and* DBI^0_* *are denoted in the same manner. Note that the superscript* 0 *of our bar induction denotes that they are for type-0 objects, which Brouwer discussed about, while the extended bar induction for finite-type objects has been studied in connection with bar recursion (see e.g.* [4]*).*

Fact 2 **([17, p. 229]).** MBI^0 *is equivalent to* MBI$^0_*$ *over* EL_0.

Proof. We show MBI^0 from MBI$^0_*$. For P and Q in MBI^0, let $P'(s) :\equiv Q'(s) :\equiv \forall s' Q(s * s')$. By applying MBI$^0_*$ to P' and Q', one can easily obtain $Q(\langle\,\rangle)$. $\square$

Remark 3. *The proof above does not work for the syntactical restrictions of* MBI^0 *in general. As we show in Corollary 23, however, the corresponding equivalence holds for Σ^0_1 restrictions (in the presence of Markov's principle* MP*).*

Fact 4. *For any syntactical class* Γ *of formulas,* (Full, Γ)-BI^0 *is equivalent to* Γ-BI$^0_*$ *over* EL_0.

Proof. Apply Γ-BI$^0_*$ to $P'(s) :\equiv Q'(s) :\equiv Q(s)$ for given Q in (Full, Γ)-BI^0. $\square$

Note that MBI^0 is a weakening of BI^0. In addition, as shown in [13, Theorem 4A], DBI^0 is intuitionistically derivable from MBI^0 (without using the continuity principle which is inconsistent with classical logic). Combined with Fact 2, we have that all of BI^0, MBI^0, DBI^0, DBI$^0_*$ are classically equivalent. In fact, simply by imitating the proofs of Propositions 16 and 21 below, one can see that all of those bar induction are equivalent to countable choice AC^{0,0} (see also [13]), and hence also to the full second-order comprehension axiom CA^0 (see [12, Proposition 11.1]) over $\mathsf{RCA}_0(=\mathsf{EL}_0 + \mathrm{LEM})$. Therefore a natural question from the viewpoint of constructive reverse mathematics is how much restricted classical logic grasp the gap between them.

2 Decomposition of BI^0 into MBI^0 and CD^0

In this section, we show that BI^0 can be decomposed into MBI^0 and the so-called **constant domain** axiom

CD^0: $\forall x(C(x) \vee D) \to \forall x\, C(x) \vee D$, where D does not contain x (of type 0) as a free variable.

It is well-known in the study of intermediate logics that the above axiom is not intuitionistically probable and the intermediate logic obtaining by adding the axiom to intuitionistic first-order logic is sound and complete for the subclass of Kripke models with constant domains (see e.g. [1] or [8]). As long as the author knows, this is the first time that the constant domain axiom is taken into account in the context of constructive reverse mathematics.

Proposition 5. LPO *(limited principle of omniscience, sometimes denoted by* Σ^0_1-LEM*):*

$$\forall\alpha(\exists x\, \alpha(x) = 0 \vee \neg\exists x\, \alpha(x) = 0)$$

is provable in $\mathsf{EL}_0 + \mathrm{CD}^0$.

Proof. We reason informally in EL_0+CD^0. Fix α. Since $\exists x\, \alpha(x) = 0 \to \exists x\, \alpha(x) = 0$, we have $\forall y(\alpha(y) = 0 \to \exists x\, \alpha(x) = 0)$. Since $\alpha(y) = 0$ is decidable in the system, we have $\forall y(\neg\alpha(y) = 0 \vee \exists x\, \alpha(x) = 0)$. Applying CD^0, we have $\forall y \neg\alpha(y) = 0 \vee \exists x\, \alpha(x) = 0$, which is equivalent to LPO. $\square$

Lemma 6. CD^0 *is provable in* $\mathsf{EL}_0 + \mathrm{BI}^0$.

Proof. We reason informally in $\mathsf{EL}_0 + \mathrm{BI}^0_*$ (note Fact 4). Assume $\forall x(C(x) \vee D)$ where D does not contain x as a free variable. Let

$$P(s) :\equiv (|s| = 0 \to \forall x\, C(x) \vee D) \wedge (|s| > 0 \to C(s_0)).$$

First, we show the first condition for BI^0_*. Fix α. By our assumption, we have $D \vee C(\alpha(0))$, which is equivalent to

$$\exists x\big((x = 0 \to D) \wedge (x \neq 0 \to C(\alpha(0)))\big).$$

Then we have $\exists x P(\bar{\alpha}(x))$ straightforwardly.

Next, we show the other condition for BI^0_*. Assume $\forall y\, P(s * \langle y \rangle)$. If $|s| > 0$, then $P(s)$ follows immediately. If $|s| = 0$, then by our assumption we have $\forall y\, C(y)$ and hence $P(\langle\,\rangle)$. Thus we have $P(s)$.

Applying BI^0_*, we have $P(\langle\,\rangle)$, and hence, $\forall x\, C(x) \vee D$. □

Remark 7. *By Proposition 5 and Lemma 6, it follows that* LPO *is derived from* BI^0 *(cf.* [17, Chapter 4, Exercise 4.8.11]*).*

Lemma 8. BI^0 *is provable in* $\mathsf{EL}_0 + \mathrm{MBI}^0 + \mathrm{CD}^0$.

Proof. We reason informally in $\mathsf{EL}_0 + \mathrm{MBI}^0 + \mathrm{CD}^0$. By Fact 4, it suffices to show BI^0_*. Assume $\forall \alpha \exists x P(\bar{\alpha}x)$ and $\forall s(\forall y P(s * \langle y \rangle) \to P(s))$.

Let $P'(s) :\equiv \exists s' \sqsubseteq s\, P(s')$, where $s' \sqsubseteq s$ denotes that the finite sequence coded by s' is an initial segment of the one coded by s. Then we have $\forall \alpha \exists x P'(\bar{\alpha}x)$ and $\forall s, s'(P'(s) \to P'(s * s'))$ immediately. For the inductive condition, assume $\forall y\, P'(s * \langle y \rangle)$. By the decidability of the length of finite sequences, we have

$$\forall y(\exists s' \sqsubseteq s\, P(s') \vee P(s * \langle y \rangle)).$$

Since the left-hand side does not contain y as a free variable, by applying CD^0, we have

$$\exists s' \sqsubseteq s\, P(s') \vee \forall y\, P(s * \langle y \rangle).$$

By our assumption, we have $\exists s' \sqsubseteq s\, P(s') \vee P(s)$, and hence $P'(s)$ follows.

Applying MBI^0_*, we have $P'(\langle\,\rangle)$, and hence $P(\langle\,\rangle)$. □

Theorem 9. BI^0 *is equivalent to* MBI^0 *plus* CD^0 *over* EL_0.

Proof. Immediate from Lemmas 6 and 8. □

Remark 10. *The decomposition of* BI^0 *into* MBI^0 *and* CD^0 *in Theorem 9 is proper in the sense that* MBI^0 *is not provable in* $\mathsf{EL} + \mathrm{CD}^0$ *(even in* $\mathsf{EL} + \mathrm{LEM}$*) and* CD^0 *(already* LPO*) is not provable in* $\mathsf{EL} + \mathrm{MBI}^0$ *(see e.g.* [10]*).*

It is known that MBI^0 is intuitionistically derivable from DBI^0 together with some continuity axiom (see [17, Chapter 4, Section 8.13, Proposition (ii)]) which is inconsistent with classical logic. As long as the author knows, however, it is open how much amount of classical logic is required for deriving MBI^0 from DBI^0 (or DBI^0_*). On the other hand, by slightly modifying the proof of Proposition 21 below, we have that DBI^0_* is provable in $\mathsf{EL}_0 + \mathrm{MP} + \mathrm{AC}^{0,0}$.

Remark 11. *The result corresponding to Theorem 9 is briefly mentioned in the context of sheaf models* [5, pp. 287–288].

3 On Σ_1^0 Restrictions of Bar Induction

Definition 12.

– Σ_1^0 *is the class of formulas of the form* $\exists z\, Q_{\mathrm{qf}}$ *where* Q_{qf} *is quantifier-free.*
– ${\Sigma_1^0}^{\mathrm{N}}$ *is the class of formulas of the form* $\neg\neg\exists z\, Q_{\mathrm{qf}}$ *where* Q_{qf} *is quantifier-free.*

Here Q_{qf} *may contain free variables other than* z *as parameters.*

In this section, we consider the restricted bar induction where the side-predicate Q is in Σ_1^0. In his intuitionistic mathematics, Brouwer used bar induction to derive so-called fan theorem, which is exactly the principle used to derive many mathematical theorems. In particular, one can straightforwardly see that Σ_1^0-MBI_*^0 (or even $(\mathrm{Full}, \Sigma_1^0)$-$\mathrm{DBI}^0$) derives fan theorem for binary trees $\mathrm{FAN}_{\mathrm{KL}}$ (see [12, Chapter 12]), which is actually sufficient to derive many mathematical theorems. On the other hand, it is well-known in classical reverse mathematics that subsystem ACA_0 (see [15]) of second-order arithmetic is sufficient to prove many theorems in ordinary (non-set-theoretic) mathematics. As mentioned in [7, Section 1], the negative translation $(\Pi_1^0\text{-AC}^{0,0})^{\mathrm{N}}$ of Π_1^0-$\mathrm{AC}^{0,0}$:

$$\forall \alpha (\forall x \exists y \forall z\, \alpha(x,y,z) = 0 \to \exists \beta \forall x, z\, \alpha(x, \beta(x), z) = 0)$$

is the key principle of the intuitionistic system which interprets ACA_0 by the negative translation (see e.g. [16, Section 1.10] or [12, Chapter 10]). Therefore, it is presumable that there is some connection between $(\Pi_1^0\text{-AC}^{0,0})^{\mathrm{N}}$ and the restrictions of bar induction where the side-predicate Q is in Σ_1^0. In the following, we investigate the precise relation between them. It should be a contribution to a foundational issue on Brouwer's mathematics from the modern viewpoint. As a by-product, one can reduce the consistency of ACA_0 into that of a semi-intuitionistic system with the restricted bar induction where the side-predicate Q is in ${\Sigma_1^0}^{\mathrm{N}}$ (see Corollary 18).

Definition 13. *The following are the principles used in this section.*

$\neg\neg\Pi_1^0$-$\mathrm{AC}^{0,0}$: $\forall \alpha \neg\neg(\forall x \exists y \forall z\, \alpha(x,y,z) = 0 \to \exists \beta \forall x, z\, \alpha(x,\beta(x),z) = 0)$.
Σ_2^0-DNS^0: $\forall \alpha (\forall x \neg\neg\exists y \forall z\, \alpha(x,y,z) = 0 \to \neg\neg\forall x \exists y \forall z\, \alpha(x,y,z) = 0)$.
Σ_1^0-DNS^0: $\forall \alpha (\forall x \neg\neg\exists y\, \alpha(x,y) = 0 \to \neg\neg\forall x \exists y\, \alpha(x,y) = 0)$.
MP $(= \Sigma_1^0$-DNE$)$: $\forall \alpha (\neg\neg\exists x\, \alpha(x) = 0 \to \exists x\, \alpha(x) = 0)$.
Σ_1^0-DNS^1: $\forall \alpha \neg\neg\exists x A_{\mathrm{qf}}(\alpha, x) \to \neg\neg\forall \alpha \exists x A_{\mathrm{qf}}(\alpha, x)$,
where A_{qf} *may contain free variables other than* α *and* x *as parameters.*

Fact 14. *The following hold over* EL_0*:*

1. MP *implies* Σ_1^0-DNS^1.
2. Σ_1^0-DNS^1 *implies* Σ_1^0-DNS^0.

3. Σ^0_2-DNS^0 *implies* Σ^0_1-DNS^0.

We first present a simple decomposition result of the negative translation $(\Pi^0_1\text{-AC}^{0,0})^{\mathrm{N}}$ of Π^0_1-AC^{0,0}, which is classically equivalent to the arithmetical comprehension axiom ACA (see [12, Section 11.3]).

Proposition 15. *The negative translation* $(\Pi^0_1\text{-AC}^{0,0})^{\mathrm{N}}$ *of* Π^0_1-AC^{0,0} *is equivalent to* $\neg\neg\Pi^0_1$-AC^{0,0} *plus* Σ^0_2-DNS^0 *over* $\mathsf{EL_0}$.

Proof. Note that the negative translation $(\Pi^0_1\text{-AC}^{0,0})^{\mathrm{N}}$ of Π^0_1-AC^{0,0} is (intuitionistically equivalent to)

$$\forall \alpha(\forall x \neg\neg\exists y \forall z\, \alpha(x,y,z)=0 \to \neg\neg\exists\beta\forall x,z\,\alpha(x,\beta(x),z)=0).$$

Since $\neg\neg\Pi^0_1$-AC^{0,0} is intuitionistically equivalent to

$$\forall \alpha(\forall x \exists y \forall z\, \alpha(x,y,z)=0 \to \neg\neg\exists\beta\forall x,z\,\alpha(x,\beta(x),z)=0),$$

it is straightforward to see the equivalence. □

In the following, we show some equivalences between $(\Pi^0_1\text{-AC}^{0,0})^{\mathrm{N}}$ and bar induction restricted to Σ^0_1 side-predicates. For one direction, we use the idea of Ferreira's elegant proof of the fact that the numerical double negation shift principle is derivable from MBI^0 together with the characteristic principles of the Dialectica interpretation (see [4]). Another direction is shown by carefully inspecting the proof of bar induction with the use of classical logic.

Proposition 16. $\mathsf{EL_0} + \Sigma^0_1\text{-DNS}^1 + {\Sigma^0_1}^{\mathrm{N}}\text{-MBI}^0_* \vdash (\Pi^0_1\text{-AC}^{0,0})^{\mathrm{N}}$.

Proof. This is exactly an imitation of the essential part of the proof of the main theorem in [4]. Recall that $(\Pi^0_1\text{-AC}^{0,0})^{\mathrm{N}}$ is intuitionistically equivalent to $\forall \alpha(\forall x \neg\neg\exists y \forall z\, \alpha(x,y,z)=0 \to \neg\neg\exists\beta\forall x,z\,\alpha(x,\beta(x),z)=0)$.

We reason informally in $\mathsf{EL_0} + \Sigma^0_1\text{-DNS}^1 + {\Sigma^0_1}^{\mathrm{N}}\text{-MBI}^0_*$. Fix α and assume

$$\forall x \neg\neg\exists y \forall z\, \alpha(x,y,z)=0 \tag{1}$$

and

$$\neg\exists\beta\forall x,z\,\alpha(x,\beta(x),z)=0 \tag{2}$$

to obtain a contradiction. Let $P(s) :\equiv \neg\neg\exists i<|s|\exists z\neg\alpha(i,s_i,z)=0$. Then $P(s)$ is equivalent (over $\mathsf{EL_0}$) to a formula in ${\Sigma^0_1}^{\mathrm{N}}$ and $\neg P(\langle\,\rangle)$ trivially holds. In addition, P is monotone obviously. Therefore, by ${\Sigma^0_1}^{\mathrm{N}}$-MBI$^0_*$, it suffices to show the double negations of the conditions (i) and (iv) (for this P in question) in Definition 1.

We first show $\neg\neg(i)$. By (2), we have $\forall\beta\neg\neg\exists x,z\neg\alpha(x,\beta(x),z)=0$ straightforwardly. Then, applying Σ^0_1-DNS^1, we have $\neg\neg\forall\beta\exists x,z\neg\alpha(x,\beta(x),z)=0$, and hence $\neg\neg\forall\beta\exists x\exists i<x\exists z\neg\alpha(i,\beta(i),z)=0$. Thus obtain $\forall\beta\exists x P(\bar\beta x)$.

In the following, we show (iv). Let $\forall y P(s*\langle y\rangle)$, namely,

$$\forall y\neg\neg(\exists i<|s|\exists z\neg\alpha(i,s_i,z)=0 \vee \exists z\neg\alpha(|s|,y,z)=0) \tag{3}$$

holds. By (1), we have $\neg\neg\exists y \forall z\, \alpha(|s|, y, z) = 0$, which is equivalent to

$$\neg\forall y \neg\neg\exists z \neg\alpha(|s|, y, z) = 0 \tag{4}$$

By (3) and (4), we have $\neg\neg\exists i < |s| \exists z \neg\alpha(i, s_i, z) = 0$, namely, $P(s)$. □

Proposition 17. $\mathsf{EL}_0 + (\mathrm{Full}, \Sigma_1^{0^{\mathrm{N}}})\text{-DBI}^0 \vdash \Sigma_1^{0^{\mathrm{N}}}\text{-MBI}^0_*$.

Proof. Assume $P(s) :\equiv Q(s) :\equiv \neg\neg\exists x P_{\mathrm{qf}}(x, s)$ (where $P_{\mathrm{qf}}(x, s)$ is quantifier-free) in $\Sigma_1^{0^{\mathrm{N}}}$ satisfy $(i), (ii)$, and (iv) in Definition 1. Define $P'(s)$ as

$$\neg\neg\exists x < |s| P_{\mathrm{qf}}(x, s).$$

Since P satisfies the monotonicity condition (ii) as well as the condition (i), it is straightforward to show that P' satisfies (i), namely, $\forall\alpha\exists x P'(\bar{\alpha}x)$. In addition, since the existential quantifier on x is bounded in P',

$$(ii)' : \forall s (P(s') \vee \neg P(s'))$$

is provable in EL_0 (see e.g. [14] or [6]). Furthermore, P' and Q trivially satisfy (iii) in Definition 1. Applying $(\mathrm{Full}, \Sigma_1^{0^{\mathrm{N}}})\text{-DBI}^0$ to P' and $Q \in \Sigma_1^{0^{\mathrm{N}}}$, we have $Q(\langle\rangle)$. □

Corollary 18. *The consistency of* ACA_0 *is reduced to that of* $\mathsf{EL}_0 + \Sigma_1^0\text{-DNS}^1 + (\mathrm{Full}, \Sigma_1^{0^{\mathrm{N}}})\text{-DBI}^0$.

Proof. By Propositions 17 and 16. □

Remark 19. *Recently, Nemoto and Sato showed in* [14] *that the consistency of* ACA_0 *is reduced to that of* $\mathsf{EL}_0^- + (\mathrm{Full}, \Sigma_1^0)\text{-DBI}^0$ *(where* EL_0^- *is a weakening of* EL_0*). On the other hand, our proofs of Propositions 17 and 16 seem to show that it is reduced to* $\mathsf{EL}_0^- + \Sigma_1^0\text{-DNS}^1 + (\mathrm{Full}, \Sigma_1^{0^{\mathrm{N}}})\text{-DBI}^0$ *rather than Corollary 18.*

Lemma 20. $\mathsf{EL}_0 + \Sigma_1^0\text{-DNS}^0 \vdash \Pi_1^0\text{-IND}$ *where* $\Pi_1^0\text{-IND}$ *denotes the induction scheme for formulas of* Π_1^0 *form:*

$$\forall y A_{qf}(0, y) \wedge \forall x (\forall y A_{\mathrm{qf}}(x, y) \to \forall y A_{\mathrm{qf}}(x+1, y)) \to \forall x \forall y A_{\mathrm{qf}}(x, y),$$

where A_{qf} *is a quantifier-free formula possibly containing other free-variables as parameters.*

Proof. We reason informally in $\mathsf{EL}_0 + \Sigma_1^0\text{-DNS}^0$. Assume $\forall y A_{\mathrm{qf}}(0, y)$ and $\forall x (\forall y A_{\mathrm{qf}}(x, y) \to \forall y A_{\mathrm{qf}}(x+1, y))$ to show $\neg\exists x, y \neg A_{\mathrm{qf}}(x, y)$, which is equivalent to $\forall x \forall y A_{\mathrm{qf}}(x, y)$.

Let x', y' satisfy $\neg A_{\mathrm{qf}}(x', y')$. By the second assumption, it is straightforward to have

$$\forall x, y \neg\neg\exists z (\neg A_{\mathrm{qf}}(x+1, y) \to \neg A_{\mathrm{qf}}(x, z)).$$

Using Σ^0_1-DNS^0, we have $\neg\neg\forall x, y\exists z(\neg A_{\mathrm{qf}}(x+1, y) \to \neg A_{\mathrm{qf}}(x, z))$, and hence,

$$\neg\neg\exists\beta\forall x, y(\neg A_{\mathrm{qf}}(x+1, y) \to \neg A_{\mathrm{qf}}(x, \beta(x, y))) \tag{5}$$

by QF-AC^{0,0} (contained in $\mathsf{EL_0}$).

Reason inside the double negations and fix β in (5). Define γ by primitive recursion as

$$\begin{cases} \gamma(0) := y' \\ \gamma(i+1) := \beta(x' \dot{-} (i+1), \gamma(i)). \end{cases}$$

We claim $\forall i \leq x' \neg A_{\mathrm{qf}}(x' \dot{-} i, \gamma(i))$ by (quantifier-free) induction on i. The initial case follows from our definitions. For the induction step, assume $i+1 \leq x'$. By the induction hypothesis, we have $\neg A_{\mathrm{qf}}(x' \dot{-} i, \gamma(i))$. Since $x' \dot{-} (i+1) \geq 0$, by (5), we have $\neg A_{\mathrm{qf}}(x' \dot{-} (i+1), \beta(x' \dot{-} (i+1), \gamma(i)))$, and hence $\neg A_{\mathrm{qf}}(x' \dot{-} (i+1), \gamma(i+1))$. This completes the proof of the claim.

Thus we have $\neg\neg\forall i \leq x' \neg A_{\mathrm{qf}}(x' \dot{-} i, \gamma(i))$. Instantiating i with x', we have $\neg A_{\mathrm{qf}}(0, \gamma(x'))$, which contradicts our first assumption. □

Proposition 21. $\mathsf{EL_0} + \Sigma^0_2$-DNS$^0 + \neg\neg\Pi^0_1$-AC$^{0,0} \vdash {\Sigma^0_1}^{\mathrm{N}}$-BI$^0_*$.

Proof. We reason informally in $\mathsf{EL_0} + \Sigma^0_2$-DNS$^0 + \neg\neg\Pi^0_1$-AC^{0,0}. Fix a predicate $P(s)$ in ${\Sigma^0_1}^{\mathrm{N}}$. Assume $(i), (iv)$ in Definition 1. Since P is in ${\Sigma^0_1}^{\mathrm{N}}$, it suffices to show $\neg\neg P(\langle\,\rangle)$. In the following, we assume $\neg P(\langle\,\rangle)$ and show $\neg(i)$. Since $\neg(i)$ follows from $\neg\forall\alpha\neg\neg\exists x P(\bar{\alpha}x)$ which is intuitionistically equivalent to

$$\neg\neg\exists\alpha\forall x \neg P(\bar{\alpha}x), \tag{6}$$

our goal is to show (6). By (iv), we have $\forall s(\neg P(s) \to \neg\forall x P(s * \langle x\rangle))$, which is equivalent (since P is in ${\Sigma^0_1}^{\mathrm{N}}$) to $\forall s(\neg P(s) \to \neg\neg\exists x \neg P(s * \langle x\rangle))$. Since $\forall s\neg\neg(P(s) \vee \neg P(s))$ is provable in $\mathsf{EL_0}$, we have

$$\forall s\neg\neg\exists x(\neg P(s) \to \neg P(s * \langle x\rangle)). \tag{7}$$

Since $\forall s, x\neg\neg(P(s * \langle x\rangle) \vee \neg P(s * \langle x\rangle))$ is provable in $\mathsf{EL_0}$, it is not hard to see (7) is equivalent to

$$\forall s\neg\neg\exists x\exists y\forall z\, R_{\mathrm{qf}}(s, x, y, z)$$

for some quantifier-free $R_{\mathrm{qf}}(s, x, y, z)$. Then by using Σ^0_2-DNS^0 and $\neg\neg\Pi^0_1$-AC^{0,0}, we have

$$\neg\neg\exists\beta\forall s(\neg P(s) \to \neg P(s * \langle\beta(s)\rangle)). \tag{8}$$

Reason inside the double negations and fix β in (8). Define $\alpha : \mathbb{N} \to \mathbb{N}$(set of the codes for finite sequences) as $\alpha(0) := \langle\,\rangle$ and $\alpha(i+1) := \alpha(i) * \langle\beta(\alpha(i))\rangle$. From our assumption $\neg P(\langle\,\rangle)$ and (8), we have $\forall x \neg P(\bar{\alpha}x)$ by induction on x, which is guaranteed by Lemma 20 and Fact 14.(3) since P is in ${\Sigma^0_1}^{\mathrm{N}}$. Thus we have (6). □

Theorem 22. *The following are pairwise equivalent over* $\mathsf{EL_0} + \Sigma^0_1$-DNS^1*:*

1. $(\text{Full}, {\Sigma_1^0}^{\text{N}})$-$\text{BI}^0$;
2. $(\text{Full}, {\Sigma_1^0}^{\text{N}})$-$\text{MBI}^0$;
3. $(\text{Full}, {\Sigma_1^0}^{\text{N}})$-$\text{DBI}^0$;
4. ${\Sigma_1^0}^{\text{N}}$-BI_*^0;
5. ${\Sigma_1^0}^{\text{N}}$-MBI_*^0;
6. $(\Pi_1^0\text{-AC}^{0,0})^{\text{N}}$;
7. $\neg\neg\Pi_1^0\text{-AC}^{0,0}$ *plus* $\Sigma_2^0\text{-DNS}^0$.

Proof. The equivalence between 1 and 4 is from Fact 4. Obviously, 1 implies 2 and also 3. In addition, 2 implies 5, and 3 implies 5 by Proposition 17. On the other hand, 5 implies 6 by Proposition 16 (using $\Sigma_1^0\text{-DNS}^1$). The equivalence between 6 and 7 is from Proposition 15. Finally, 7 implies 4 by Proposition 21. $\square$

Corollary 23. *All of* $(\text{Full}, \Sigma_1^0)$-$\text{BI}^0$, $(\text{Full}, \Sigma_1^0)$-$\text{MBI}^0$, $(\text{Full}, \Sigma_1^0)$-$\text{DBI}^0$, Σ_1^0-BI_*^0, Σ_1^0-MBI_*^0, $(\Pi_1^0\text{-AC}^{0,0})^{\text{N}}$, *and* $\neg\neg\Pi_1^0\text{-AC}^{0,0}$ *plus* $\Sigma_2^0\text{-DNS}^0$ *are pairwise equivalent over* $\mathsf{EL_0} + \text{MP}$.

Proof. In the presence of MP, every formula in Σ_1^0 is equivalent to its double negation, which is in ${\Sigma_1^0}^{\text{N}}$. On the other hand, $\Sigma_1^0\text{-DNS}^1$ is provable in $\mathsf{EL_0} + \text{MP}$ by Fact 14.(1). Therefore, the corollary follows from Theorem 22. $\square$

Remark 24. *Note that* $\neg\neg\Pi_1^0\text{-AC}^{0,0}$ *is not provable in* $\mathsf{EL} + \text{MP} + \Sigma_2^0\text{-DNS}^0$ *(even in* $\mathsf{EL} + \text{LEM}$*) and* $\Sigma_2^0\text{-DNS}^0$ *is not provable in* $\mathsf{EL} + \text{MP} + \neg\neg\Pi_1^0\text{-AC}^{0,0}$ *(see [7]). That is to say, the decomposition of* $(\Pi_1^0\text{-AC}^{0,0})^{\text{N}}$ *into* $\neg\neg\Pi_1^0\text{-AC}^{0,0}$ *and* $\Sigma_2^0\text{-DNS}^0$ *(Proposition 21) is proper still in the presence of* MP.

A natural question is whether MP is necessary for the equivalence in Corollary 23. The following lemma reveals that MP is necessary for proving Σ_1^0-BI_*^0 from $(\Pi_1^0\text{-AC}^{0,0})^{\text{N}}$ while it is not the case for ${\Sigma_1^0}^{\text{N}}$-BI_*^0 as shown in Proposition 21. The proof is a modification of the discussion in [17, Chapter 4, Section 8.18]

Lemma 25. MP *is provable in* $\mathsf{EL_0} + \Sigma_1^0$-$\text{BI}_*^0$.

Proof. Fix α and assume $\neg\neg\exists x\, \alpha(x) = 0$ (equivalently $\neg\forall x \neg\, \alpha(x) = 0$). Let

$$P(s) :\equiv (|s| > 0 \wedge \neg\, \alpha(s_0) = 0) \vee (|s| = 0 \wedge \exists x\, \alpha(x) = 0)$$

For the condition (i) in Definition 1, fix γ. Now $\alpha(\gamma(0)) = 0$ or $\alpha(\gamma(0)) \neq 0$. In the former case, we have $P(\bar{\gamma}0)$. In the latter case, we have $P(\bar{\gamma}1)$. For the condition (iv) in Definition 1, assume $\forall y\, P(s * \langle y\rangle)$. If $|s| > 0$, $P(s)$ follows immediately. Let $|s| = 0$. Then $\forall y \neg\, \alpha(y) = 0$ follows from $\forall y\, P(s * \langle y\rangle)$, which contradicts our assumption, and hence $\exists x\, \alpha(x) = 0$ follows. Thus we have $P(s)$.

Since $P(s)$ is equivalent to a formula in Σ_1^0, applying Σ_1^0-BI_*^0, we have $P(\langle\,\rangle)$, and hence $\exists x\, \alpha(x) = 0$. $\square$

Theorem 26. *The following are pairwise equivalent over* $\mathsf{EL_0}$*:*

1. $(\mathrm{Full}, \Sigma^0_1)$-$\mathrm{BI}^0$;
2. $(\mathrm{Full}, \Sigma^0_1)$-$\mathrm{MBI}^0$ *plus* MP;
3. $(\mathrm{Full}, \Sigma^0_1)$-$\mathrm{DBI}^0$ *plus* MP;
4. Σ^0_1-BI^0_*;
5. Σ^0_1-MBI^0_* *plus* MP;
6. $(\Pi^0_1\text{-}\mathrm{AC}^{0,0})^{\mathrm{N}}$ *plus* MP;
7. $\neg\neg\Pi^0_1$-$\mathrm{AC}^{0,0}$ *plus* Σ^0_2-DNS^0 *plus* MP.

Proof. Immediate from Corollary 23 and Lemma 25. □

Remark 27. *Since* $\neg\neg\Pi^0_1\text{-}\mathrm{AC}^{0,0} + \Sigma^0_2\text{-}\mathrm{DNS}^0$ *has a modified realizability interpretation in Gödel's* **T** *(verifiably in a classical system), it follows that* MP *is not provable in* $\mathsf{EL} + \neg\neg\Pi^0_1\text{-}\mathrm{AC}^{0,0} + \Sigma^0_2\text{-}\mathrm{DNS}^0$*. Combining this fact with Remark 24, one can see that* $\neg\neg\Pi^0_1$-$\mathrm{AC}^{0,0}$*,* Σ^0_2-DNS^0 *and* MP *are pairwise independent over* EL*.*

Since Π^0_1-$\mathrm{AC}^{0,0}$ is classically equivalent to the arithmetical comprehension axiom ACA (in the function-based language) as mentioned in [12, Section 11.3], by Corollary 23, we obtain the following result for classical reverse mathematics [15].

Corollary 28. *All of* $(\mathrm{Full}, \Sigma^0_1)$-$\mathrm{BI}^0$*,* $(\mathrm{Full}, \Sigma^0_1)$-$\mathrm{MBI}^0$*,* $(\mathrm{Full}, \Sigma^0_1)$-$\mathrm{DBI}^0$*,* Σ^0_1-BI^0_**,* Σ^0_1-MBI^0_**, and* Σ^0_1-DBI^0_* *are equivalent to* ACA *over* RCA_0*.*

The equivalence between 1 and 2 in Theorem 26 can be seen as a result corresponding to Theorem 9 when the side-predicates are restricted to formulas in Σ^0_1. If one wants to show just the equivalence, however, there is a simpler proof without going through $(\Pi^0_1\text{-}\mathrm{AC}^{0,0})^{\mathrm{N}}$:

Proposition 29. $\mathsf{EL}_0 + \Sigma^0_1\text{-}\mathrm{DNS}^0 + {\Sigma^0_1}^{\mathrm{N}}\text{-}\mathrm{MBI}^0_* \vdash {\Sigma^0_1}^{\mathrm{N}}\text{-}\mathrm{BI}^0_*$*.*

Proof. The idea is similar to that for the proof of Lemma 8. We reason informally in $\mathsf{EL}_0 + \Sigma^0_1\text{-}\mathrm{DNS}^0 + {\Sigma^0_1}^{\mathrm{N}}\text{-}\mathrm{MBI}^0_*$. Assume that $P(s)(\equiv Q(s)) :\equiv \neg\neg R(s)$ in ${\Sigma^0_1}^{\mathrm{N}}$ (namely, $R(s)$ is in Σ^0_1) satisfies the conditions (i) and (iv) in Definition 1. Define $P'(s)$ as

$$\neg\neg\exists s' \sqsubseteq s\, R(s').$$

Since R is in Σ^0_1, it is straightforward to show that there is a formula in ${\Sigma^0_1}^{\mathrm{N}}$ which is equivalent (over EL_0) to P'. P' obviously satisfies (ii) in Definition 1. Since P satisfies (i), P' also satisfies (i). In the following, we show that P' satisfies (iv). Assume $\forall y\, P'(s * \langle y\rangle)$, namely, $\forall y \neg\neg\exists s' \sqsubseteq s * \langle y\rangle R(s')$. By the decidability of the length of finite sequences, we have

$$\forall y \neg\neg \left(\exists s' \sqsubseteq s\, R(s') \vee R(s * \langle y\rangle)\right).$$

Since there is a formula in Σ^0_1 which is equivalent (over EL_0) to $\exists s' \sqsubseteq s\, R(s') \vee R(s * \langle y\rangle)$, by Σ^0_1-DNS^0, we have

$$\neg\neg\forall y \left(\exists s' \sqsubseteq s\, R(s') \vee R(s * \langle y\rangle)\right). \tag{9}$$

Since $\forall s \neg\neg (\exists s' \sqsubseteq s\, R(s') \vee \neg\exists s' \sqsubseteq s\, R(s'))$ is provable in $\mathsf{EL_0}$, one can obtain

$$\neg\neg (\exists s' \sqsubseteq s\, R(s') \vee \forall y\, R(s * \langle y \rangle)) .$$

from (9). Since $\neg\neg R(s)$ satisfies (iv), we have

$$\neg\neg (\exists s' \sqsubseteq s\, R(s') \vee \neg\neg R(s)) .$$

Then we have $\neg\neg (\exists s' \sqsubseteq s\, R(s') \vee R(s))$, which is equivalent to $P'(s)$.

Applying ${\Sigma_1^0}^{\mathrm{N}}$-$\mathrm{MBI}_*^0$ to P', we have $P'(\langle\,\rangle)$, and hence, $P(\langle\,\rangle)$. □

Corollary 30. $\mathsf{EL_0} + \mathrm{MP} + \Sigma_1^0$-$\mathrm{MBI}_*^0 \vdash \Sigma_1^0$-$\mathrm{BI}_*^0$.

Proof. In the presence of MP, every formula in Σ_1^0 is equivalent to its double negation, which is in ${\Sigma_1^0}^{\mathrm{N}}$. On the other hand, Σ_1^0-DNS^0 is provable in $\mathsf{EL_0} + \mathrm{MP}$ by Fact 14. Therefore, the corollary follows from Proposition 29. □

Corollary 31. $(\mathrm{Full}, \Sigma_1^0)$-$\mathrm{BI}^0$ *and* $(\mathrm{Full}, \Sigma_1^0)$-$\mathrm{MBI}^0$ *plus* MP *are equivalent over* $\mathsf{EL_0}$.

Proof. By Corollary 30, Fact 4, and Lemma 25. □

Acknowledgment. The author thanks to Tatsuji Kawai for providing an important information on a previous work mentioned in Remark 11. The author is grateful also to Ryota Akiyoshi, Hajime Ishihara, Ulrich Kohlenbach and Takako Nemoto for helpful discussion. A part of this work had been carried out while the author had visited Ludwig-Maximilians-Universität München in February 2017. The visit was hosted by Helmut Schwichtenberg and supported by Core-to-Core Program (A. Advanced Research Networks). Another part of this work had been carried out while the author had visited the Hausdorff Research Institute for Mathematics (HIM), University of Bonn, for their trimester program "Types, Sets and Constructions" in July and August 2018. The author thanks the institute for their support and hospitality. This work is also supported by Waseda University Grant for Special Research Projects 2016S-173 and 2018K-461.

References

1. Ardeshir, M., Ruitenburg, W., Salehi, S.: Intuitionistic axiomatizations for bounded extension kripke models. Ann. Pure Appl. Log. **124**(1), 267–285 (2003). https://doi.org/10.1016/S0168-0072(03)00058-7
2. Berger, J., Ishihara, H., Kihara, T., Nemoto, T.: The binary expansion and the intermediate value theorem in constructive reverse mathematics. Arch. Math. Log. **58**(1), 203–217 (2019). https://doi.org/10.1007/s00153-018-0627-2
3. Dorais, F.G.: Classical consequences of continuous choice principles from intuitionistic analysis. Notre Dame J. Form. Log. **55**(1), 25–39 (2014). https://doi.org/10.1215/00294527-2377860
4. Ferreira, F.: A short note on Spector's proof of consistency of analysis. In: Cooper, S.B., Dawar, A., Löwe, B. (eds.) CiE 2012. LNCS, vol. 7318, pp. 222–227. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-30870-3_22

5. Fourman, M.P., Hyland, J.M.E.: Sheaf models for analysis. In: Fourman, M., Mulvey, C., Scott, D. (eds.) Applications of Sheaves. LNM, vol. 753, pp. 280–301. Springer, Heidelberg (1979). https://doi.org/10.1007/BFb0061823
6. Fujiwara, M.: Intuitionistic and uniform provability in reverse mathematics. Ph.D. thesis, Tohoku University (2015)
7. Fujiwara, M., Kohlenbach, U.: Interrelation between weak fragments of double negation shift and related principles. J. Symb. Log. **83**(3), 991–1012 (2018). https://doi.org/10.1017/jsl.2017.63
8. Görnemann, S.: A logic stronger than intuitionism. J. Symb. Log. **36**(2), 249–261 (1971). https://doi.org/10.2307/2270260
9. Ishihara, H.: Constructive reverse mathematics: compactness properties. In: From Sets and Types to Topology and Analysis. Oxford Logic Guides, vol. 48, pp. 245–267. Oxford University Press, Oxford (2005). https://doi.org/10.1093/acprof:oso/9780198566519.003.0016
10. Ishihara, H., Nemoto, T.: On the independence of premiss axiom and rule, preprint. http://www.jaist.ac.jp/~t-nemoto/ipr2.pdf
11. Kleene, S.C., Vesley, R.E.: The Foundations of Intuitionistic Mathematics. North-Holland Publishing Co., Amsterdam (1965)
12. Kohlenbach, U.: Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. Springer, Berlin (2008). https://doi.org/10.1007/978-3-540-77533-1
13. Kreisel, G., Howard, W.A.: Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis. J. Symb. Log. **31**(3), 325–358 (1966). https://doi.org/10.2307/2270450
14. Nemoto, T., Sato, K.: A marriage of Brouwer's intuitionism and Hilbert's finitism I: Arithmetic. J. Symb. Log. (to appear)
15. Simpson, S.G.: Subsystems of Second Order Arithmetic. Perspectives in Logic, 2nd edn. Cambridge University Press, Cambridge (2009)
16. Troelstra, A.S. (ed.): Metamathematical Investigation of Intuitionistic Arithmetic and Analysis. Lecture Notes in Mathematics, vol. 344. Springer, Berlin (1973)
17. Troelstra, A.S., van Dalen, D.: Constructivism in Mathematics, An Introduction, Vol. I. Studies in Logic and the Foundations of Mathematics, vol. 121. North Holland, Amsterdam (1988)

Uniform Labelled Calculi for Conditional and Counterfactual Logics

Marianna Girlando[1,2](✉), Sara Negri[2], and Giorgio Sbardolini[3]

[1] Aix Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France
marianna.girlando@univ-amu.fr
[2] University of Helsinki, Helsinki, Finland
sara.negri@helsinki.fi
[3] The Ohio State University, Columbus, USA
sbardolini.1@osu.edu

Abstract. Lewis's counterfactual logics are a class of conditional logics that are defined as extensions of classical propositional logic with a two-place modal operator expressing conditionality. Labelled proof systems are proposed here that capture in a modular way Burgess's preferential conditional logic $\mathbb{PCL}$, Lewis's counterfactual logic $\mathbb{V}$, and their extensions. The calculi are based on preferential models, a uniform semantics for conditional logics introduced by Lewis. The calculi are analytic, and their completeness is proved by means of countermodel construction. Due to termination in root-first proof search, the calculi also provide a decision procedure for the logics.

Keywords: Conditional logics · Counterfactual logics · Proof theory · Preferential models · Labelled calculi

1 Introduction

In Stalnaker's and Lewis's approach, conditional logics are defined as extensions of classical propositional logic by means of a two-place modal operator, the conditional, here denoted as $>$. This intensional operator is intended to express a more fine-grained notion of conditionality than material implication.

Lewis introduced counterfactual conditional logics to extend formal reasoning to counterfactual sentences, i.e., statements of the form *If Trump hadn't won the elections, Clinton would have been president.* Other than counterfactual logics (system $\mathbb{V}$ and its extensions) conditional logics include a weaker family of systems: in this paper, we consider *preferential conditional logic* $\mathbb{PCL}$ and all its extensions. These latter systems have received attention in artificial intelligence since the conditional operator can be interpreted as expressing non monotonic

This work was partially supported by the Academy of Finland research project no. 1308664 and by the project TICAMORE ANR-16-CE91-0002-01.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 248–263, 2019.
https://doi.org/10.1007/978-3-662-59533-6_16

inferences, i.e., sentences of the form *Normally, cats are afraid of dogs.* In particular, the fragment of $\mathbb{PCL}$ without nesting of the conditional operator is equivalent to system **P** of [14]. There are other applications of conditional logics in the fields of knowledge base update [13], causality [8] and, in an epistemic setting, belief revision [4,12].

The semantics of conditional logics is defined in terms of possible world structures in which, intuitively, a formula $A > B$ is true at world x if B is true in the set of worlds at which A is true that are more similar, in a sense to be formalized, to x. Preferential models were proposed by Lewis and studied, among others, by Burgess, who proved completeness of $\mathbb{PCL}$ with respect to these models [5]. Halpern and Friedman extended the proof to extensions of $\mathbb{PCL}$ [7]. On a formal level, these models explicitly employ the notion of comparative similarity among worlds: they are defined by adding to a set of possible worlds W a family of subsets W_x for each $x \in W$, representing the worlds accessible from x, and a binary relation $\leqslant_x$, expressing similarity among worlds. Thus, $y \leqslant_x z$ means *world y is at least as similar as z to world x.*

In this article, we define a family of modular labelled calculi **G3P*** for conditional logic $\mathbb{PCL}$ and all its extensions, including counterfactual logics, i.e., $\mathbb{V}$ and its extensions. The calculi are based on preferential semantics: following the well-established methodology proposed by the second author, the calculi import into the sequent calculus the semantic elements of preferential models by means of syntactic elements (labels and relational symbols).

In [20], the second and third author presented a labelled calculus based on ternary relations for a system of Lewis's conditional logic $\mathbb{VC}$ ($\mathbb{V}$ to which the condition of Centering is added). The present article stems form a comment in Weiss's thesis [22]: the author observes, correctly, that Negri and Sbardolini's proof system is actually adequate to capture the stronger system $\mathbb{VCU}$ ($\mathbb{VC}$ + Uniformity). This led us to an analysis of labelled calculi based on preferential models. It turns out that it is possible to define modular proof systems on the basis of these natural classes of models.

The article is organised as follows. In Sect. 2, conditional logics and preferential models are introduced. Section 3 presents the rules of the calculi **G3P*** and Sect. 4 their structural properties. In Sect. 5, we define a proof search strategy that ensures termination in root-first proof search for the systems without the semantic conditions of Uniformity and Absoluteness. This allows to prove completeness of the calculi by extracting a countermodel from failed proof search. The conclusion (Sect. 6) gives a comparison of the calculi presented in this article with other proof systems for conditional logics found in the literature.

2 Conditional Logics and Preferential Models

The language of conditional logics is defined by means of the following grammar, for p propositional variable, $A, B \in \mathcal{L}_{\mathsf{cond}}$, and $>$ the conditional operator:

$$\mathcal{L}_{\mathsf{cond}} = p \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid A > B$$

An axiomatization of $\mathbb{PCL}$ is defined adding to the axioms and inference rules of classical propositional logic the following:

RCEA $\dfrac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)}$ RCK $\dfrac{A \rightarrow B}{(C > A) \rightarrow (C > B)}$

R-And $(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$ ID $A > A$

CM $(A > B) \wedge (A > C) \rightarrow ((A \wedge B) > C)$ RT $(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$

OR $(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$

An axiomatization of Lewis's logic $\mathbb{V}$ is obtained by adding axiom CV to $\mathbb{PCL}$. Extensions of both $\mathbb{PCL}$ and $\mathbb{V}$ are defined by adding the axioms for *normality*, *total reflexivity*, *weak centering*, *centering*, *uniformity* and *absoluteness* (denoted below by the corresponding initials). The family of systems is represented in Fig. 1.

CV $((A > C) \wedge \neg(A > \neg B)) \rightarrow ((A \wedge B) > C)$

N $\neg(\top > \bot)$ T $A \rightarrow \neg(A > \bot)$

W $(A > B) \rightarrow (A \rightarrow B)$ C $(A \wedge B) \rightarrow (A > B)$

U_1 $(\neg A > \bot) \rightarrow \neg(\neg A > \bot) > \bot$ U_2 $\neg(A > \bot) \rightarrow ((A > \bot) > \bot)$

A_1 $(A > B) \rightarrow (C > (A > B))$ A_2 $\neg(A > B) \rightarrow (C > \neg(A > B))$

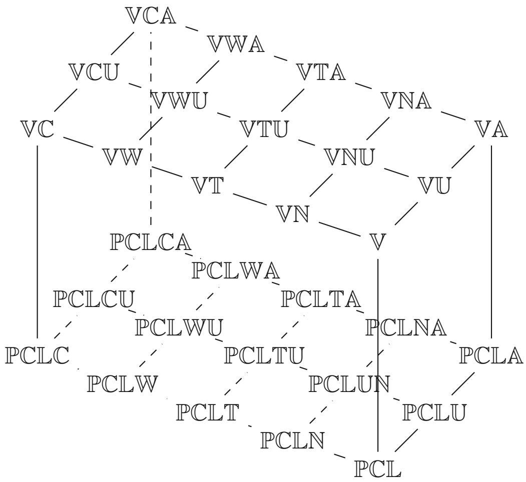


Fig. 1. The conditional logics cube

We use preferential models as the semantics for the conditional logics cube. Preferential models were studied, among others, by Burgess, who proved the adequacy of this class of models with respect to preferential logic [5].

Definition 1. *A* preferential model $\langle W, \{W_x\}_{x \in W}, \{\leqslant_x\}_{x \in W}, [\![\,]\!]\rangle$ *consists of a non-empty set of worlds, W, and for every world $x \in W$, a set W_x of worlds accessible from it, and a binary relation $\leqslant_x$ over W. The propositional evaluation*

$[\![\,]\!] : Atm \to \mathcal{P}(W)$ *selects the worlds at which an atomic formula is true. The relation* $\leqslant_x$ *satisfies the following properties:*

- *Reflexivity, for all* $w \in W$, $w \leqslant_x w$, *and*
- *Transitivity, for all* $w, y, z \in W$, *if* $w \leqslant_x y$ *and* $y \leqslant_x z$ *then* $w \leqslant_x z$.

The truth condition for the conditional operator within preferential models is:

$$x \Vdash A > B \equiv \textit{for all } z \in W_x, \textit{ if } z \Vdash A, \textit{ then there exists } y \in W_x \textit{ such that } y \leqslant_x z,\ y \Vdash A, \textit{ and for all } k \in W_x, \textit{ if } k \leqslant_x y \textit{ then } k \Vdash A \to B.$$

Extensions of preferential models are specified by adding conditions on the relation $\leqslant_x$. These models are adequate for the logics in the conditional cube [7].

Definition 2. *Extensions of preferential models are defined as follows:*

- *Normality: For all* $x \in W$, W_x *is non-empty;*
- *Total reflexivity: For all* $x \in W$ *it holds that* $x \in W_x$*;*
- *Weak centering: For all* $x \in W$, *for all* $y \in W_x$, *it holds that* $x \leqslant_x y$*;*
- *Centering: For all* $x \in W$, *for all* $y \in W_x$, *it holds that* $x \leqslant_x y$ *and if there is* $w \in W_x$ *such that for all* $y \in W_x$, $w \leqslant_x y$, *then* $w = x$*;*
- *Uniformity: For all* $x \in W$, *for all* $y \in W_x$ *it holds that* $W_y = W_x$*;*
- *Absoluteness: Uniformity plus for all* $w_1, w_2 \in W_x$, $w_1 \leqslant_x w_2$ *iff* $w_1 \leqslant_y w_2$*;*
- *Nesting: For all* $x \in W$, *for all* $w_1, w_2 \in W_x$, *either* $w_1 \leqslant_x w_2$ *or* $w_2 \leqslant_x w_1$.

Some of the above conditions are incremental: total reflexivity implies normality, weak centering implies total reflexivity, centering implies weak centering and absoluteness implies uniformity.

3 Labelled Proof Systems

In this section we shall define a family of modular calculi for the conditional cube. To this aim, we enrich our language with a sets of labels $x, y, z, \ldots$ denoting worlds in preferential models. Furthermore, we allow the following expressions to occur in sequents: labelled formulas $x : A$, denoting $x \Vdash A$, and relational atoms $y \in W_x$, $y \leqslant_x z$ and $x = y$, having the same meaning as their semantic counterparts. Following [20], we introduce an indexed modal operator and reformulate the truth condition of $A > B$ in terms of this operator:

$$w \Vdash \Box_x A \equiv \textit{ for all } k \in W_x, \textit{ if } k \leqslant_x w \textit{ then } k \Vdash A$$

$$(*)\, x \Vdash A > B \equiv \textit{ for all } z \in W_x, \textit{ if } z \Vdash A, \textit{ then there exists } y \in W_x \textit{ such that } y \leqslant_x z,\ y \Vdash A, \textit{ and } y \Vdash \Box_x(A \to B).$$

Following [19], we introduce an indexed conditional operator to treat the second disjunct of the truth condition of the conditional operator:

$$C_x^z(A, B) \equiv \textit{ there exists } y \in W_x \textit{ such that } y \leqslant_x z,\ y \Vdash A \textit{ and } y \Vdash \Box_x(A \to B)$$

Thus, the truth condition for the conditional operator can be stated as follows:

$$(**)\, x \Vdash A > B \equiv \textit{ for all } z \in W_x, \textit{ if } z \Vdash A, \textit{ then } C^z_x(A, B).$$

Observe that the extension of the language is only at the level of the labelled rules and produces, in the course of proof search, only formulas of a certain specific form. Formulas containing the new operators never occur as proper subformulas of other formulas and an indexed modality can have only an implication in its scope.

The rules of the labelled proof systems are defined by analysing the truth conditions of the above operators (Fig. 2). Rules Ref and Tr express reflexivity and transitivity of $\leqslant_x$; rule $\mathsf{Ref}_=$ and Repl express reflexivity of equality and the property of replacement of equals. We call **G3P** the calculus for $\mathbb{PCL}$; calculi for extensions are defined in a modular way by adding to $\mathbb{PCL}$ the rules corresponding to the semantic properties of $\leqslant_x$. We denote by **G3P*** the whole family of calculi. The condition of freshness of a variable y in a rule is indicated by $(y!)$.

In order to prove soundness of the rules of the calculus we have to provide a definition of realization in a preferential model. The definition uses the operators of the extended language, and to guarantee its non-circularity we need to define a notion of weight of formulas:

Definition 3. *Given a labelled formula F of the form $x : A$, let the pure part of F be defined as $p(x : A) = A$, and the labelled part as $l(x : A) = l(x : \Box_k A) = x$. The weight of a labelled formula is an ordered pair $\langle \mathsf{w}(p(F)), \mathsf{w}(l(F)) \rangle$ where*

- *for x world label,* $\mathsf{w}(x) = 0$
- $\mathsf{w}(p) = \mathsf{w}(\bot) = 1$*;* $\mathsf{w}(A \circ B) = \mathsf{w}(A) + \mathsf{w}(B) + 1$*, for $\circ$ conjunction, disjunction or implication;* $\mathsf{w}(x : \Box_k A) = \mathsf{w}(A) + 1$*;* $\mathsf{w}(C^z_x(A, B)) = \mathsf{w}(A) + \mathsf{w}(B) + 3$*;* $\mathsf{w}(A > B) = \mathsf{w}(A) + \mathsf{w}(B) + 4$*.*

Definition 4 (Realization). *Given a model $\mathcal{M} = \langle W, \{W_x\}_{x \in W}, \{\leqslant_x\}_{x \in W}, [\![\,]\!] \rangle$, and a set P of world labels, a P-realization over $\mathcal{M}$ is a function $\rho : P \to W$ that assigns to each world label $x \in P$ an element $\rho(x) \in W$. Satisfiability of a formula $F \in \mathcal{L}_{\mathsf{cond}}$ is defined by cases as follows: $\mathcal{M} \vDash_\rho y \in W_x$ if $\rho(y) \in W_{\rho(x)}$; $\mathcal{M} \vDash_\rho y \leqslant_x z$ if $\rho(y) \leqslant_{\rho(x)} \rho(z)$; $\mathcal{M} \vDash_\rho x : p$ if $\rho(x) \in [\![p]\!]$, for p atomic;*[1] *$\mathcal{M} \vDash_\rho w : \Box_x A$ if for all $y \in W_{\rho(x)}$, if $y \leqslant_{\rho(x)} \rho(w)$, then $y \Vdash A$; $\mathcal{M} \vDash_\rho C^z_x(A, B)$ if there exists $y \in W_{\rho(x)}$ such that $y \leqslant_x \rho(z)$, $y \vDash A$ and $y \vDash \Box_x(A \to B)$; $\mathcal{M} \vDash_\rho x : A > B$ if for all $k \in W_{\rho(x)}$, if $k \vDash A$, then $k \vDash C^k_x(A, B)$. A sequent $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$ under the ρ realization iff whenever $\mathcal{M} \vDash_\rho F$ for all $F \in \Gamma$, then $\mathcal{M} \vDash_\rho G$ for some $G \in \Delta$. A sequent is valid in a class of preferential models if it is valid under any realization for any model of that class.*

The above definition immediately yields:

---

[1] The definition can be extended to the propositional formulas of the language in the standard way [17].

Initial sequents

$$x : p, \Gamma \Rightarrow \Delta, x : p \qquad\qquad x : \bot, \Gamma \Rightarrow \Delta$$

Propositional rules (standard)

Conditional rules

$$\frac{k \in W_x, k \leqslant_x w, w : \Box_x A, k : A, \Gamma \Rightarrow \Delta}{k \in W_x, k \leqslant_x w, w : \Box_x A, \Gamma \Rightarrow \Delta}\ \mathsf{L}\Box_{\mathsf{x}} \qquad \frac{k \in W_x, k \leqslant_x w, \Gamma \Rightarrow \Delta, k : A}{\Gamma \Rightarrow \Delta, w : \Box_x A}\ \mathsf{R}\Box_{\mathsf{x}}(\mathsf{k!})$$

$$\frac{y \in W_x, y \leqslant_x z, y : A, y : \Box_x(A \to B), \Gamma \Rightarrow \Delta}{C^z_x(A, B), \Gamma \Rightarrow \Delta}\ \mathsf{LC(y!)}$$

$$\frac{y \leqslant_x z, y \in W_x, z \in W_x, \Gamma \Rightarrow \Delta, C^z_x(A, B), y : A \quad y \leqslant_x z, y \in W_x, z \in W_x, \Gamma \Rightarrow \Delta, C^z_x(A, B), y : \Box_x(A \to B)}{y \leqslant_x z, y \in W_x, z \in W_x, \Gamma \Rightarrow \Delta, C^z_x(A, B)}\ \mathsf{RC}$$

$$\frac{z \in W_x, x : A > B, \Gamma \Rightarrow \Delta, z : A \quad z \in W_x, x : A > B, C^z_x(A, B), \Gamma \Rightarrow \Delta}{z \in W_x, x : A > B, \Gamma \Rightarrow \Delta}\ \mathsf{L}> \qquad \frac{z \in W_x, z : A, \Gamma \Rightarrow \Delta, C^z_x(A, B)}{\Gamma \Rightarrow \Delta, x : A > B}\ \mathsf{R}>\mathsf{(z!)}$$

Relational rules

$$\frac{w \leqslant_x w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{Ref} \qquad \frac{w \leqslant_x z, w \leqslant_x y, y \leqslant_x z, \Gamma \Rightarrow \Delta}{w \leqslant_x y, y \leqslant_x z, \Gamma \Rightarrow \Delta}\ \mathsf{Tr}$$

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{Ref}_= \qquad \frac{x = y, At(x), At(y), \Gamma \Rightarrow \Delta}{x = y, At(y), \Gamma \Rightarrow \Delta}\ \mathsf{Repl}$$

Rules for extensions

$$\frac{y \in W_x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{N(y!)} \qquad \frac{x \in W_x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathsf{T} \qquad \frac{x \leqslant_x y, y \in W_x, \Gamma \Rightarrow \Delta}{y \in W_x, \Gamma \Rightarrow \Delta}\ \mathsf{W}$$

$$\frac{x = y, y \leqslant_x x, y \in W_x, \Gamma \Rightarrow \Delta}{y \leqslant_x x, y \in W_x, \Gamma \Rightarrow \Delta}\ \mathsf{C}$$

$$\frac{z \in W_x, y \in W_x, z \in W_y, \Gamma \Rightarrow \Delta}{y \in W_x, z \in W_y, \Gamma \Rightarrow \Delta}\ \mathsf{U1} \qquad \frac{z \in W_y, y \in W_x, z \in W_x, \Gamma \Rightarrow \Delta}{y \in W_x, z \in W_x, \Gamma \Rightarrow \Delta}\ \mathsf{U2}$$

$$\frac{y \leqslant_x z, y \in W_x, z \in W_x, \Gamma \Rightarrow \Delta \quad z \leqslant_x y, y \in W_x, z \in W_x, \Gamma \Rightarrow \Delta}{y \in W_x, z \in W_x, \Gamma \Rightarrow \Delta}\ \mathsf{Nes} \qquad \frac{y \leqslant_k z, y \leqslant_x z, \Gamma \Rightarrow \Delta}{y \leqslant_x z, \Gamma \Rightarrow \Delta}\ \mathsf{A}$$

$At(y)$ denotes any atoms of the form $y : p, y \in W_x, x \in W_y, y \leqslant_x z, x \leqslant_y z$.

$\mathbf{G3P} =$ Initial sequents, propositional rules, conditional rules, relational rules

$\mathbf{G3P^N} = \mathbf{G3P} + \mathsf{N}$; $\mathbf{G3P^T} = \mathbf{G3P^N} + \mathsf{T}$; $\mathbf{G3P^W} = \mathbf{G3P^T} + \mathsf{W}$; $\mathbf{G3P^C} = \mathbf{G3P^W} + \mathsf{C}$

$\mathbf{G3P^U} = \mathbf{G3P} + \mathsf{U1} + \mathsf{U2}$; $\mathbf{G3P^{NU/TU/WU/CU}} = \mathbf{G3P^{N/T/W/C}} + \mathsf{U1} + \mathsf{U2}$

$\mathbf{G3P^A} = \mathbf{G3P} + \mathsf{A}$; $\mathbf{G3P^{NA/TA/WA/CA}} = \mathbf{G3P^{N/T/W/C}} + \mathsf{A}$

$\mathbf{G3P^V} = \mathbf{G3P} + \mathsf{Nes}$; $\mathbf{G3P^{VN/VT/VW/VCU}} = \mathbf{G3P^{N/T/W/C}} + \mathsf{Nes}$

$\mathbf{G3P^{VNU/VTU/VWU/VCU}} = \mathbf{G3P^{NU/TU/WU/CU}} + \mathsf{Nes}$;

$\mathbf{G3P^{VNA/VTA/VWA/VCA}} = \mathbf{G3P^{NA/TA/WA/CA}} + \mathsf{Nes}$

Fig. 2. Rules of $\mathbf{G3P}^*$

Theorem 1 (Soundness). *If a sequent is derivable in* $\mathbf{G3P}^*$*, then it is valid in the corresponding class of preferential models.*

Remark 1. The sequent calculi of Fig. 2 are fully modular. However, by dropping the requirement of modularity, it is possible to define simpler versions of the calculi for some subfamilies of the logics. In systems with uniformity it holds that for all $x \in W$ and $y \in W_x$, $W_x = W_y$. Thus, we can avoid specifying the relational atoms $y \in W_x$ in the rules (with this reformulation, the rules of uniformity would become superfluous). Similarly, in logics with absoluteness, uniformity holds, and moreover $w_1 \leqslant_x w_2$ iff $w_2 \leqslant_y w_1$. Thus, we can avoid specifying the subscript x in relational atoms $y \leqslant_x z$, and the rule of absoluteness becomes superfluous. Finally, in the presence of nesting the truth condition for the conditional operator can be stated in a simpler way:

$$x \Vdash A > B \equiv \textit{if there exists } z \in W_x \textit{ such that } z \Vdash A, \textit{ then there exists } y \in W_x \textit{ such that } y \Vdash A \textit{ and } y \Vdash \Box_x(A \to B).$$

Rules based on this truth condition, in addition to the simplification explained for uniformity, i.e., no relational atoms $y \in W_x$ and no rules $\mathsf{U1}, \mathsf{U2}$, yield the calculus proposed in [20], a proof system sound and complete with respect to the conditional logic $\mathbb{VCU}$.

4 Structural Properties

The *height* of a derivation is the number of nodes of the longest derivation branch, minus one. We recall that a rule is *height-preserving admissible* if whenever its premiss is derivable, the conclusion is also derivable with no greater derivation height. A rule is *height-preserving invertible* if whenever its conclusion is derivable, the premisses are derivable with no greater derivation height. Derivability with height bounded by n is denoted by $\vdash_n$.

In order to prove admissibility of the structural rules we need a notion of label substitution given by, for instance, $x : A > B[y/x] \equiv y : A > B$ and $w : \Box_x A[y/x] \equiv w : \Box_y A$, extended component-wise to sequents, and a property of height-preserving substitution: If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma[y/x] \Rightarrow \Delta[y/x]$. Admissibility of generalized initial sequents (i.e., sequents of the form $x : A, \Gamma \Rightarrow \Delta, x : A$ in which A is not necessarily atomic) is shown by induction on the weight of A. We omit the routine proofs, the details of which are similar to those in [20].

The *structural rules* of weakening, contraction, and cut of $\mathbf{G3P}^*$ are the following:

$$\frac{\Gamma \Rightarrow \Delta}{\mathcal{F}, \Gamma \Rightarrow \Delta}\ \mathsf{Wk_L} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathcal{F}}\ \mathsf{Wk_R} \qquad \frac{\mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta}{\mathcal{F}, \Gamma \Rightarrow \Delta}\ \mathsf{Ctr_L} \qquad \frac{\Gamma \Rightarrow \Delta, \mathcal{F}, \mathcal{F}}{\Gamma \Rightarrow \Delta, \mathcal{F}}\ \mathsf{Ctr_R}$$

$$\frac{\Gamma \Rightarrow \Delta, \mathcal{F} \quad \mathcal{F}, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathsf{cut}$$

where $\mathcal{F}$ is a relational atom, a labelled formula, or a formula of the form $C_x^z(A, B)$. Observe that for $\mathsf{Wk_R}$, $\mathsf{Ctr_R}$, and cut we can without loss of generality omit the case of relational formulas since they never occur in the right-hand side of sequents. The calculi $\mathbf{G3P}^*$ have the following structural properties:

Theorem 2.

i. All the rules are height-preserving invertible.
ii. The rules of weakening and contraction are height-preserving admissible.
iii. The rule of cut is admissible.

Proof.

i. By induction on the height of the derivation. Invertibility of relational rules, rules for extensions, $\mathsf{L}\Box_\mathsf{x}$, RC and $\mathsf{L}>$ immediately follows from admissibility of weakening. Invertibility of the propositional rules and of $\mathsf{R}\Box_\mathsf{x}$ is proved as in [17]; invertibility of LC is similar to that of the corresponding rule in [19].

ii. By induction on the height n of the derivation. If $n = 0$, the premiss of the contraction rule is an initial sequent, and so is its conclusion. If $n > 0$, we look at the last rule (R) applied. If F is not principal in the rule, it suffices to apply the inductive hypothesis to the premiss of (R), and then (R). If F is the principal formula of R, or was introduced by R, we distinguish two subcases. If (R) is a rule in which the principal formula appears also in the premiss apply the hypothesis to the premiss, and then the rule. If (R) is a rule in which the active formulas are subformulas of the principal formula, apply invertibility to the premiss(es) of the rule, then the inductive hypothesis, and (R).

iii. By primary induction on the weight of the cut formula, and secondary induction on the sum of heights of the derivations of the premisses of cut. As usual, we proceed with a case distinction according to the last rule applied. If at least one of the premisses is an initial sequent, the conclusion of cut is also a sequent, or can be obtained by easy rule permutations. Similarly, if the cut formula is not principal in the last rule R applied to one of the premiss of cut, the conclusion of cut can be obtained by permuting the cut upwards on the premiss of R, and then applying R again. Finally, if the cut formula is principal in both rules applied to the premisses of cut, some more complex permutations are needed. Propositional cases can be found in [20]. We show only the case in which $\mathsf{R}>$ and $\mathsf{L}>$ are the rules applied to the left and right premiss of cut respectively. Consider a derivation ending with

$$\dfrac{\dfrac{y \in W_x, y : A, \Gamma \Rightarrow \Delta, C_x^y(A,B)}{\Gamma \Rightarrow \Delta, x : A > B}\,\mathsf{R}> \quad \dfrac{z \in W_x, x : A > B, \Gamma' \Rightarrow \Delta', z : A \quad z \in W_x, x : A > B, C_x^y(A,B), \Gamma' \Rightarrow \Delta'}{z \in W_x, x : A > B, \Gamma' \Rightarrow \Delta'}\,\mathsf{L}>}{z \in W_x, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathsf{cut}$$

Let $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ be the derivations ending with the topsequents above. The cut is transformed into four cuts of reduced rank as follows. First we have two cuts, to topmost of reduced height, the second of reduced weight, where $\mathcal{D}_1[z/y]$ denotes the derivation resulting from $\mathcal{D}_1$ by application of an height-preserving substitution:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A > B \quad z \in W_x, x : A > B, \Gamma' \Rightarrow \Delta', z : A}{z \in W_x, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : A}\,\mathsf{cut} \quad \mathcal{D}_1[z/y]}{z \in W_x^2, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta', C_x^z(A,B)}\,\mathsf{cut}$$

Second we have the cut of reduced height

$$\dfrac{\Gamma \Rightarrow \Delta, x : A > B \quad \mathcal{D}_3}{z \in W_x, \Gamma, \Gamma', C_x^z(A,B) \Rightarrow \Delta, \Delta'}\,\mathsf{cut}$$

Finally, by cut their conclusions through a fourth cut of reduced weight and obtain the sequent $z \in W_x^3, \Gamma^3, \Gamma'^2 \Rightarrow \Delta^3, \Delta'^2$. Admissible weakening steps give the conclusion of the original cut.

The case of principal cut formula of the form $C_x^z(A,B)$ is reduced in a similar way through four cuts, the uppermost of reduces height, and the lowermost of reduces weight. For principal formula of the form of an indexed modality, the conversion is the standard one for the necessity modality of labelled calculi.

Thanks to admissibility of cut, it is possible to prove the following:

Theorem 3 (Completeness). *If a formula A is valid in preferential models and extensions, then sequent $\Rightarrow x : A$, for an arbitrary label x, is derivable in the corresponding* $\mathbf{G3P}^*$ *calculus.*

Proof. By using the known completeness result for extensions of $\mathbb{PCL}$ w.r.t. preferential models and showing that the inference rules of $\mathbb{PCL}$ are admissible in **G3P**, and that the axioms of $\mathbb{PCL}$ and its extensions are derivable in the corresponding proof system of $\mathbf{G3P}^*$. The proof for $\mathbb{PCL}$ is similar to the proof in [20]. By way of example, we show the derivation of Axiom U_1 in $\mathbf{G3P^{VU}}$, omitting the derivable left premiss of $\mathsf{L}>$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : \bot \cdots \Rightarrow \ldots \qquad \ldots y : \neg A \Rightarrow y : \neg A \ldots}{y \leqslant_x y, y \leqslant_x w, y \in W_x, \ldots, y : \neg A, y : \neg A \to \bot \Rightarrow \ldots}\;\mathsf{L}_{\to}}{k \leqslant_x k, k \leqslant_x w, k \in W_x, w \in W_x, w \in W_z, z \in W_x, w : \neg A, w : \neg A, k : \neg A, k : \Box_x(\neg A \to \bot), x : (\neg A > \bot) \Rightarrow \ldots}\;\mathsf{L}\Box_{\mathsf{x}}}{k \leqslant_x w, k \in W_x, w \in W_x, w \in W_z, z \in W_x, w : \neg A, w : \neg A, k : \neg A, k : \Box_x(\neg A \to \bot), x : (\neg A > \bot) \Rightarrow \ldots}\;\mathsf{Ref}}{w \in W_x, w \in W_z, z \in W_x, w : \neg A, w : \neg A, C_x^w(\neg A, \bot), x : (\neg A > \bot) \Rightarrow \ldots}\;\mathsf{LC}}{w \in W_x, w \in W_z, z \in W_x, w : \neg A, x : (\neg A > \bot) \Rightarrow \ldots}\;\mathsf{L}>}{w \in W_z, z \in W_x, w : \neg A, x : (\neg A > \bot) \Rightarrow C_x^z(\neg(\neg A > \bot), \bot), C_z^w(\neg A, \bot)}\;\mathsf{U1}}{z \in W_x, x : (\neg A > \bot) \Rightarrow z : \neg A > \bot, C_x^z(\neg(\neg A > \bot), \bot)}\;\mathsf{R}>}{z \in W_x, z : \neg(\neg A > \bot), x : (\neg A > \bot) \Rightarrow C_x^z(\neg(\neg A > \bot), \bot)}\;\mathsf{R}\neg}{x : (\neg A > \bot) \Rightarrow x : \neg(\neg A > \bot) > \bot}\;\mathsf{R}>}{\Rightarrow x : (\neg A > \bot) \to \neg(\neg A > \bot) > \bot}\;\mathsf{R}_{\to}$$

By Theorems 1 and 3, and the known completeness results for $\mathbb{PCL}$ and its extensions with respect to preferential models, we have:

Corollary 1. *Formula A is provable in any of the systems of the conditional logics cube if and only if $\Rightarrow x : A$ is derivable in the corresponding labelled system.*

5 Termination and Completeness

In this section, we shall give an alternative direct proof of completeness for the calculi **G3P**, $\mathbf{G3P^{N}}$, $\mathbf{G3P^{T}}$, $\mathbf{G3P^{W}}$, $\mathbf{G3P^{C}}$ and $\mathbf{G3P^{V}}$, $\mathbf{G3P^{VN}}$, $\mathbf{G3P^{VT}}$, $\mathbf{G3P^{VW}}$, $\mathbf{G3P^{VC}}$ (from now on $\mathbf{G3P^{V/N/T/W/C}}$), i.e., the systems without uniformity and absoluteness.[2] The proof proceeds by showing how to construct a countermodel from failed proof search. We first need to prove that root-first proof search, which in general is not terminating because of loops, terminates under the adoption of a suitable strategy.

Example 1. Loop generated by repeated applications of rule $\mathsf{L}>$ and LC (only the right premisses of $\mathsf{L}>$ are shown).

[2] The proofs of termination and completeness for systems with Uniformity and Absoluteness can be given adopting the reformulation of the calculi from Remark 1. The proofs for the current versions of the calculi would be unnecessarily complex.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{y \leqslant_x z, y \in W_x, z \leqslant_x z, z \in W_x, y : A, y : \Box_x(A \to B), x : A > B, x : C > D, C_x^z(C,D), C_x^y(C,D) \Rightarrow}\ \mathsf{L>}}{y \leqslant_x z, y \in W_x, z \leqslant_x z, z \in W_x, y : A, y : \Box_x(A \to B), x : A > B, x : C > D, C_x^z(C,D) \Rightarrow}\ \mathsf{LC}}{z \leqslant_x z, z \in W_x, x : A > B, x : C > D, C_x^z(A,B), C_x^z(C,D) \Rightarrow}\ \mathsf{Ref}}{z \in W_x, x : A > B, x : C > D, C_x^z(A,B), C_x^z(C,D) \Rightarrow}\ \mathsf{L>}}{z \in W_x, x : A > B, x : C > D, C_x^z(A,B) \Rightarrow}\ \mathsf{L>}}{z \in W_x, x : A > B, x : C > D \Rightarrow}
$$

To ensure termination we introduce the notion of *saturated sequent*, a sequent to which all the rules have been applied in a non-redundant way. We then specify a *proof search strategy*, blocking the application of the rules to a saturated sequent.

Definition 5. *Given a* $\mathbf{G3P^{V/N/T/W/C}}$ *derivation, let* $\mathcal{B} = S_0, S_1, \ldots$ *be a derivation branch, with* S_i *sequent* $\Gamma_k \Rightarrow \Delta_k$ *for* $k > 0$, *and* S_0 *sequent* $\Rightarrow x_0 : A_0$. *Let* $\downarrow \Gamma_k / \downarrow \Delta_k$ *be the union of the antecedents/succedents occurring in the derivation from* S_0 *up to* S_k. *A sequent* $\Gamma \Rightarrow \Delta$ *is saturated if it is not an instance of an initial sequent, and the following conditions are satisfied:*

- $(\mathsf{L}_\to)$ *If* $x : A \to B$ *occurs in* $\downarrow \Gamma$, $x : B$ *occurs in* $\downarrow \Gamma$ *or* $x : A$ *occurs in* $\downarrow \Delta$;
- $(\mathsf{R}_\to)$ *If* $x : A \to B$ *occurs in* $\downarrow \Delta$, $x : A$ *occurs in* $\downarrow \Gamma$ *and* $x : B$ *occurs in* $\downarrow \Delta$;[3]
- (Ref) *If* y *occurs in* Γ, *then* $y \leqslant_x y$ *occurs in* Γ;
- (Tr) *If* $y \leqslant_x z$ *and* $z \leqslant_x k$ *occur in* Γ, $y \leqslant_x k$ *occur in* Γ;
- $(\mathsf{L}>)$ *If* $x : A > B$ *and* $z \in W_x$ *occur in* $\downarrow \Gamma$, *then either* $z : A$ *occurs in* $\downarrow \Delta$ *or* $C_x^z(A,B)$ *occurs in* $\downarrow \Gamma$;
- $(\mathsf{R}>)$ *If* $x : A > B$ *occurs in* $\downarrow \Delta$, *then* $z \in W_x$ *and* $z : A$ *occur in* $\downarrow \Gamma$, *for some* z *and* $C_x^z(A,B)$ *occurs in* Δ;
- (LC) *if* $C_x^z(A,B)$ *occurs in* Γ, *then either for some* y $y \leqslant_x z$, $y \in W_x$, $y : \Box_x(A \to B)$ *occur in* $\downarrow \Gamma$, *or for some* w *such that* $z \neq w$, $z \leqslant_x$ *and* $C_x^w(A,B)$ *occur in* $\downarrow \Gamma$;
- (RC) *If* $y \leqslant_x z$, $y \in W_x$, $z \in W_x$ *occur in* $\downarrow \Gamma$ *and* $C_x^z(A,B)$ *occurs in* $\downarrow \Delta$, *then either* $y : A$ *or* $y : \Box_x(A)$ *occurs in* $\downarrow \Delta$;
- $(\mathsf{L}\Box_\mathsf{x})$ *If* $y : \Box_x A$ *occurs in* $\downarrow \Gamma$, $z \leqslant_x y$ *and* $z \in W_x$ *occur in* Γ, *and* $z : A$ *occurs in* $\downarrow \Gamma$;
- $(\mathsf{R}\Box_\mathsf{x})$ *If* $y : \Box_x A$ *occurs in* $\downarrow \Delta$, *then either for some* z, $z \leqslant_x y$ *occurs in* Γ *and* $z : A$ *occurs in* $\downarrow \Delta$, *or for some* $w \neq y$, $y \leqslant_x w$ *occurs in* Γ *and* $w : \Box_x A$ *occurs in* $\downarrow \Delta$;
- (N) *If* x *occurs in* Γ, $y \in W_x$ *occurs in* Γ, *for some* y;
- (T) *If* x *occurs in* Γ, $x \in W_x$ *occurs in* Γ;
- (W) *If* $y \in W_x$ *occurs in* Γ, $x \leqslant_x y$ *occurs in* Γ;
- (C) *If* $y \leqslant_x x$ *and* $y \in W_x$ *occur in* Γ, $y = x$ *occurs in* Γ;
- $(\mathsf{Ref}_=)$ *If* x *occurs in* Γ, *then* $x = x$ *occurs in* Γ;
- (Repl) *If* $y = x$ *occurs in* Γ, *and if some formulas* $At(y)$ *occur in* Γ, *formulas* $At(x)$ *occur in* Γ;
- (Nes) *If* $y \in W_x$ *and* $z \in W_x$ *occur in* Γ, $y \leqslant_x z$ *or* $z \leqslant_x y$ *occur in* Γ.

[3] The saturation conditions for the other propositional rules are standard [20].

In Example 1, the saturation condition (LC) blocks the application of the rule to formula $C_x^y(C, D)$, since $y \leqslant_x z$ and $C_x^z(C, D)$ occur in the antecedent. Intuitively, rule LC is not applied to a formula $C_x^y(C, D)$ if y has been generated by a previous application of LC to the same $C_x(C, D)$, possibly labelled with a different label, i.e., if for some z, formulas $y \leqslant_x z$ and $C_x^z(C, D)$ occur in a lower antecedents. A similar saturation condition is needed for $\mathsf{R}\Box_{\mathsf{x}}$.

Definition 6. *In root-first proof search for $\Rightarrow x_0 : A_0$, apply the following:*

1. *Rules which do not introduce new labels are applied before rules which do introduce new labels;*
2. *A rule R cannot be applied to a sequent if the sequent already satisfies the saturation condition associated to R.*

We need to show that every branch of a derivation starting with $\Rightarrow x_0 : A_0$ and built in accordance with the strategy is finite. Since labels can be attached only to the finitely many subformulas of formula A_0, it suffices to prove that only a finite number of labels can occur in the branch. To this aim, we construct an acyclic graph with the labels occurring in the derivation, and show that the graph is finite: more precisely, that every node of the graph has a finite number of immediate successors, and that each branch of the graph is finite.

Definition 7. *Given a derivation branch as in Definition 5, let x, y be labels occurring in Γ. Let $k(x) = min\{t \mid x$ occurs in $\Gamma_t\}$. We say that "x generates y", in symbols xRy, if for some $t \geqslant k(x)$, $k(y) = t$ and $y \in W_x$ occurs in Γ_t.*

By inspection on the rules of $\mathbf{G3P^{V/N/T/W/C}}$ and by definition of R we have that the relation R does not contain any cycles and forms a graph having at the root label x_0, and that all the labels occurring in the derivation occur in the graph. The notion of conditional degree, needed to prove Lemma 2, corresponds to the level of nesting of the conditional operator $>$.

Definition 8. *The conditional degree of a formula A is defined as: $d(\bot) = d(p) = 0$; $d(A \circ B) = d(C_x^y(A, B)) = max(d(A), d(B))$ for $\circ = \{\wedge, \vee, \rightarrow\}$; $d(\Box_k A) = d(A)$, and $d(A > B) = max(d(A), d(B)) + 1$. For x a label in a derivation, $d(x) = max\{d(C) \mid x : C$ occurs in $\downarrow \Gamma \cup \downarrow \Delta\}$.*

Lemma 1. *Every node in the graph generated by the relation R has a finite number of immediate successors.*

Proof. By definition, label y is generated from x if there exists a t such that y does not occur in Γ_s for any $s < t$, and $y \in W_x$ occurs in Γ_t. We need to prove that only a finite number of formulas $y \in W_x$ can be introduced from x.

Formulas $y \in W_x$ are introduced in root-first proof search by application of rules for semantic conditions, $\mathsf{R}>$, LC or $\mathsf{R}\Box_{\mathsf{x}}$. In the first cases, $y \in W_x$ is introduced by N or T.[4] By the saturation conditions, these rules can be applied

[4] Observe that Repl does not introduce new labels; however, it could introduce new links between the nodes of the graph. In the presence of Repl the structure generated by R is a graph; otherwise, it is a tree.

at most once to a label x; thus, they generate at most 2 new labels. If $y \in W_x$ is introduced by $\mathsf{R}>$, the rule must have been applied to some $x : C > D$ occurring in Δ_{t-1}. By the saturation condition, rule $\mathsf{R}>$ can be applied at most once to each formula $x : C > D$, and the number of such formulas linearly depends on $d(A_0)$, the degree of formula A_0 at the root of the tree. Similarly, rule $\mathsf{R}\Box_{\mathsf{x}}$ is applied to some $w : \Box_A$ in $\downarrow \Gamma_{t-1}$, generating a new $y \in W_x$. Formulas $w : \Box_A$ are introduced by RC and $\mathsf{R}>$, which do not generate loops. The saturation condition $(\mathsf{R}\Box_{\mathsf{x}})$ ensures that no loops arise with formulas $\mathsf{L}\Box_{\mathsf{x}}$. Thus, only a finite number of labels can be introduced. In case $y \in W_x$ is introduced by LC the situation is more complex. As shown in Example 1, rule LC might interact with rule $\mathsf{L}>$ generating a large number of new labels, however, thanks to the proof search strategy, their number is finite. We consider a case of loop more complex than the one in Example 1: suppose formulas $x : E_1 > F_1, \dots, x : E_k > F_k$ occur in the succedent of a sequent. Then, for some $z \in W_x$ in the antecedent, we can apply k times rule $\mathsf{L}>$, generating k formulas $C^z_x(E_1, F_1), \dots, C^z_x(E_k, F_k)$. Then, rule LC can be applied to these formulas, generating k new labels $z_1 \in W_x, \dots, z_k \in W_x$, with $z_1 \leqslant z, \dots, z_k \leqslant_x z$. Moreover, the rule introduces in the antecedent formulas $z_1 : E_1, \dots, z_k : E_k$ and $z_1 : \Box_x(E_1 \to F_1), \dots, z_k : \Box_x(E_k \to F_k)$. Rule $\mathsf{L}>$ can be applied to these labels, generating $k \cdot k$ new formulas:

$$\begin{matrix} C^{z_1}_x(E_1, F_1), & \dots, & C^{z_1}_x(E_k, F_k) \\ \vdots & & \vdots \\ C^{z_k}_x(E_1, F_1), & \dots, & C^{z_k}_x(E_k, F_k) \end{matrix}$$

Application of LC to these formulas would in principle generate $k \cdot k$ new labels; however, the saturation condition (LC) blocks the application of the rule to all formulas. For $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k$, consider formula $C^{z_i}_x(E_j, F_j)$. It holds that formulas $z_i \leqslant_x z$ and $C^z_x(E_j, F_j)$ occur in lower antecedents/succedents, satisfying the saturation condition. Thus, for each $y \in W_x$, and for k $>$-formulas occurring in the antecedent $k \cdot k$ new labels are generated. $\Box$

Lemma 2. *Every branch in the graph generated by the relation R is finite.*

Proof. By induction on $d(x)$, for x a label in the graph. We show that the length of an arbitrary chain starting from x is bounded by the degree of the formula it labels. If $d(x) = 0$, the formulas labelled with x are all atomic or propositional formulas, and no formula $y \in W_x$ needs to be introduced. If $d(x) > 0$, there must be some formula $x : A > B$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$. Thus, there is at least one chain of length greater than zero in the branch, and some label y such that xRy. Observe that y can occur only as label of formulas of smaller degree than the formulas labelled with x. More precisely, for all formulas $x : A > B$ with $d(A > B) \leqslant d(x)$ occurring in $\downarrow \Gamma \cup \downarrow \Delta$, it holds that for all formulas $y : \Box_x(A \to B)$, $d(y) < d(x)$, i.e., all labels introduced by combination of $\mathsf{R}>$, $\mathsf{L}>$, LC, RC and $\mathsf{R}\Box_{\mathsf{x}}$ are label of formulas with a smaller degree than formulas labelled with x. $\Box$

It follows from Lemmas 1 and 2 that the acyclic graph is finite. Since the formulas occurring in a derivation are subformulas of the formula A_0, and since the number of labels occurring in a derivation is finite, proof search terminates.

Theorem 4 (Termination). *Proof search in* $\mathbf{G3P^{V/N/T/W/C}}$ *built in accordance with the proof search strategy for a sequent* $\Rightarrow x_0 : A_0$ *always comes to an end in a finite number of steps, and each sequent occurring as a leaf of the derivation tree is either an initial sequent or a saturated sequent.*

Termination of proof search allows to prove completeness by constructing a countermodel from a saturated sequent.

Theorem 5. *Let* $\Gamma \Rightarrow \Delta$ *be a saturated sequent in a* $\mathbf{G3P^{V/N/T/W/C}}$ *derivation. There exists a finite countermodel* $\mathcal{M}_{\mathcal{B}}$ *satisfying all formulas in* $\downarrow \Gamma$ *and falsifying all formulas in* $\downarrow \Delta$*.*

Proof. The countermodel $\mathcal{M}_{\mathcal{B}}$ is constructed as follows: $W_{\mathcal{B}} = \{x \mid x \text{ occurs in } \downarrow \Gamma \cup \downarrow \Delta\}$; for all $x \in W_{\mathcal{B}}$, $W_x = \{y \mid y \in W_x \text{ occurs in } \Gamma\}$; $\leqslant_x = \{\langle y, z\rangle \mid y \leqslant_x z \text{ occurs in } \Gamma\}$; for p atomic, $[\![p]\!] = \{x \in W_{\mathcal{B}} \mid x : p \text{ occurs in } \Gamma\}$.

It is immediate to verify that the relation $\leqslant_x$ satisfies the properties of reflexivity and transitivity; thus, $\mathcal{M}_{\mathcal{B}}$ is a model for $\mathbb{PCL}$. In the presence of N, T, W, C and Nes, the saturation conditions associated to these rules ensure that the model $\mathcal{M}_{\mathcal{B}}$ is a model for the corresponding logic.[5]

Let ρ be the realization $\rho(x) = x$. We show that 1) if $\mathcal{F}$ occurs in $\downarrow \Gamma$, $\mathcal{M}_{\mathcal{B}} \vDash_\rho \mathcal{F}$, and 2) if $\mathcal{F}$ occurs in $\downarrow \Delta$, $\mathcal{M}_{\mathcal{B}} \nvDash_\rho \mathcal{F}$.[6] The two claims are proved by cases, and by induction on the weight of $\mathcal{F}$. If $\mathcal{F}$ is a relational atom $y \in W_x$ or $y \leqslant_x z$ or a formula $x : p$, claim 1 (and claim 2) hold by definition of the model. The propositional cases and the cases of $\mathcal{F} = x : A > B$ and $\mathcal{F} = y : \Box_x A$ follow applying the inductive hypothesis. By way of example, we prove claim 2 for $\mathcal{F} = x : A > B$. Suppose that formula $x : A > B$ occurs in $\downarrow \Delta$. By the saturation condition associated to $\mathsf{R}>$, for some label z, $z \in W_x$ and $z : A$ occur in $\downarrow \Gamma$ and $C^z_x(A,B)$ occurs in Δ. Thus, by inductive hypothesis, $\mathcal{M}_{\mathcal{B}} \vDash_\rho z : A$ and $\mathcal{M}_{\mathcal{B}} \nvDash_\rho C^z_x(A,B)$, thus by the truth condition for the conditional, $\mathcal{M}_{\mathcal{B}} \nvDash_\rho x : A > B$. □

As a consequence of Theorems 4 and 5 we have that any underivable sequent originates, in a finite number of steps, a saturated sequent which is used to define a countermodel. We therefore have:

Corollary 2 (Strong completeness). *Any sequent* $\Gamma \Rightarrow \Delta$ *is either derivable in* $\mathbf{G3P^{V/N/T/W/C}}$ *or has a (finite) countermodel in the corresponding class of models.*

[5] In case of centering it is convenient to define worlds as equivalence classes, to account for formulas $x = y$. Thus, $[x] = \{y \mid x = y \text{ occurs in } \downarrow \Gamma\}$ and $W^c = \{[x] \mid y \text{ occurs in } \downarrow \Gamma \cup \downarrow \Delta\}$. Centering follows from the saturation condition (C).

[6] In case of centering, we also need to show that if $[x] \vDash_\rho A$ and $y \in [x]$, then $[y] \vDash_\rho A$, and that if $[x] \vDash_\rho A$ then $x : A$ occurs in $\downarrow \Gamma$. The proof follows from admissibility of Repl in its generalized form [20].

Completeness of the proof systems is an obvious consequence:

Theorem 6 (Completeness). *If A is valid in one of the logics without uniformity and absoluteness, sequent $\Rightarrow x : A$ is derivable in the corresponding $\mathbf{G3P^{V/N/T/W/C}}$ calculus.*

Completeness, along with termination, allow to define a decision procedure for the logics based on the labelled calculi. However, the resulting decision procedure would be of at least NEXPTIME complexity - thus, far from the known complexity bounds for the logics.[7]

6 Conclusion and Related Work

In this work, we introduced a family of uniform labelled calculi that capture in a modular way the conditional logic $\mathbb{PCL}$ and its extensions, including Lewis' counterfactual systems. The calculi internalise the semantics of preferential models. This semantics, studied, among others, by Lewis and Burgess [5,15], makes explicit reference to the comparative plausibility ordering among worlds, implicitly assumed in Lewis's sphere models.

Several labelled proof systems for conditional logics have been defined in the literature. A recent approach, based on the methodology of neighbourhood semantics of [18] and [19], is presented in [10] and gives a uniform family of labelled calculi for $\mathbb{PCL}$ and its extensions. Neighbourhood semantics is a generalization of Lewis's sphere semantics; whereas the latter is adequate for $\mathbb{V}$, neighbourhood semantics covers also weaker conditional logics. When compared to labelled calculi based on neighbourhood semantics, the calculi $\mathbf{G3P}^*$ appear to be simpler: they use just one set of labels, whereas the calculi based on neighbourhood semantics need two sets of labels, for worlds and for neighbourhoods. It turns out that the preferential semantics already used to define labelled sequent calculi for Lewis's conditional logic $\mathbb{VC}$ in [20] is sufficiently expressive to treat uniformly also the weaker extensions of $\mathbb{PCL}$.

Preferential models have already been used in [9] to define tableau calculi for $\mathbb{PCL}$ and all its extensions, but with the important difference, with respect to our approach, of the addition of the *Limit Assumption*[8] and the use of a strict relation of comparative similarity. Another semantically inspired approach can be found in [21], that presents a sequent calculus for system $\mathbb{CK}$ and some of its extensions. These logics are the weakest conditional systems, and they are weaker than the logics considered in this article.

Internal calculi (i.e., proof systems in which sequents have a direct formula interpretation) for conditional logics have also been defined: in [2] nested sequent calculi for $\mathbb{CK}$ and some of its extensions are devloped, whereas in [1] a nested and optimal calculus for counterfactual logic $\mathbb{V}$ can be found (refer to [11] for cases of extensions). Finally, display calculi for $\mathbb{CK}$ have been introduced recently in [6].

[7] Refer to [7] for complexity results for conditional logics.

[8] The Limit Assumption states that there are no infinite descending $\leqslant_x$-chains.

With respect to the labelled proof systems $\mathbf{G3P}^*$, the internal calculi are less modular: they capture weaker logics, such as $\mathbb{CK}$, or subfamilies of the logics considered in this article, such as $\mathbb{V}$ and its extensions. In particular, the definition of internal calculi for $\mathbb{PCL}$ seems challenging: up to now, the only internal proof system known for it is the resolution calculus presented in [16]. The labelled approach treats in a modular way both $\mathbb{PCL}$ and $\mathbb{V}$. The challenge of defining labelled calculi on the basis of preferential semantics lies in identifying a decomposition of the conditional operator in terms of simpler operators directly treatable by the sequent calculus rules. Here, this is done by introducing the indexed operator $\Box_x A$, similarly to [9,20], and the binary operator $C_x^z(A, B)$.

As underlined in Remark 1, by dropping the requirement of modularity it is possible to have simpler labelled calculi for sub-families of logics. We plan to define such calculi and analyse their termination in root-first proof search, to investigate the possibility of a better complexity bound for the corresponding logics. Furthermore, following [9], simpler rules could be defined also for **G3P** with the introduction of the Limit Assumption on preferential models. Finally, we plan to study the relationship of $\mathbf{G3P}^*$ with labelled sequent calculi for conditional logics based on neighbourhood models [10]. Via the correspondence between neighbourhood and preferential structures [3], we conjecture that the two families of calculi can be proved equivalent. It would be interesting to know which family of calculi allows for the better decision procedure in terms of complexity.

References

1. Alenda, R., Olivetti, N., Pozzato, G.L.: Nested sequent calculi for conditional logics. In: del Cerro, L.F., Herzig, A., Mengin, J. (eds.) JELIA 2012. LNCS (LNAI), vol. 7519, pp. 14–27. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-33353-8_2
2. Alenda, R., Olivetti, N., Pozzato, G.L.: Nested sequent calculi for normal conditional logics. J. Logic Comput. **26**(1), 7–50 (2013)
3. Alexandroff, P.: Diskrete Räume. Mat.Sb. (NS) **2**(3), 501–519 (1937)
4. Baltag, A., Smets, S.: A qualitative theory of dynamic interactive belief revision. Log. Found. Game Decis. Theory (LOFT 7) **3**, 9–58 (2008)
5. Burgess, J.P.: Quick completeness proofs for some logics of conditionals. Notre Dame J. Formal Log. **22**(1), 76–84 (1981)
6. Chen, J., Greco, G., Palmigiano, A., Tzimoulis, A.: Non normal logics: semantic analysis and proof theory. arXiv preprint arXiv:1903.04868 (2019)
7. Friedman, N., Halpern, J.Y.: On the complexity of conditional logics. In: Doyle, J., Sandewall, E., Torasso, P. (eds.) Principles of knowledge Representation and Reasoning: Proceedings of the Fourth International Conference (KR 1994), pp. 202–213. Morgan Kaufmann Pub. (1994)
8. Galles, D., Pearl, J.: An axiomatic characterization of causal counterfactuals. Found. Sci. **3**(1), 151–182 (1998)
9. Giordano, L., Gliozzi, V., Olivetti, N., Schwind, C.: Tableau calculus for preference-based conditional logics: PCL and its extensions. ACM Trans. Comput. Log. **10**(3), 21 (2009)

10. Girlando, M.: On the proof theory of conditional logics. Ph.D. thesis, University of Helsinki (2019)
11. Girlando, M., Lellmann, B., Olivetti, N., Pozzato, G.L.: Standard sequent calculi for Lewis' logics of counterfactuals. In: Michael, L., Kakas, A. (eds.) JELIA 2016. LNCS (LNAI), vol. 10021, pp. 272–287. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-48758-8_18
12. Girlando, M., Negri, S., Olivetti, N., Risch, V.: Conditional beliefs: from neighbourhood semantics to sequent calculus. Rev. Symb. Log. **11**, 1–44 (2018)
13. Grahne, G.: Updates and counterfactuals. J. Log. Comput. **8**(1), 87–117 (1998)
14. Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics. Artif. Intell. **44**(1–2), 167–207 (1990)
15. Lewis, D.K.: Counterfactuals. Blackwell, Oxford (1973)
16. Nalon, C., Pattinson, D.: A resolution-based calculus for preferential logics. In: Galmiche, D., Schulz, S., Sebastiani, R. (eds.) IJCAR 2018. LNCS (LNAI), vol. 10900, pp. 498–515. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-94205-6_33
17. Negri, S.: Proof analysis in modal logic. J. Philos. Log. **34**(5–6), 507 (2005)
18. Negri, S.: Proof theory for non-normal modal logics: the neighbourhood formalism and basic results. IFCoLog J. Log. Appl. **4**, 1241–1286 (2017)
19. Negri, S., Olivetti, N.: A sequent calculus for preferential conditional logic based on neighbourhood semantics. In: De Nivelle, H. (ed.) TABLEAUX 2015. LNCS (LNAI), vol. 9323, pp. 115–134. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-24312-2_9
20. Negri, S., Sbardolini, G.: Proof analysis for Lewis counterfactuals. Rev. Symb. Log. **9**(1), 44–75 (2016)
21. Poggiolesi, F.: Natural deduction calculi and sequent calculi for counterfactual logics. Studia Logica **104**(5), 1003–1036 (2016)
22. Weiss, Y.: Frontiers of conditional logic. Ph.D. thesis, The Graduate Center, City University of New York (2019)

Bar-Hillel Theorem Mechanization in Coq

Sergey Bozhko[1(✉)], Leyla Khatbullina[2], and Semyon Grigorev[3,4]

[1] Max Planck Institute for Software Systems (MPI-SWS), Saarbrücken, Germany
sbozhko@mpi-sws.com
[2] St. Petersburg Electrotechnical University "LETI", St. Petersburg, Russia
leila.xr@gmail.com
[3] St. Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg, Russia
s.v.grigoriev@spbu.ru
[4] JetBrains Research, Universitetskaya emb., 7-9-11/5A, St. Petersburg, Russia
semen.grigorev@jetbrains.com

Abstract. Formal language theory has a deep connection with such areas as static code analysis, graph database querying, formal verification, and compressed data processing. Many application problems can be formulated in terms of languages intersection. The Bar-Hillel theorem states that context-free languages are closed under intersection with a regular set. This theorem has a constructive proof and thus provides a formal justification of correctness of the algorithms for applications mentioned above. Mechanization of the Bar-Hillel theorem, therefore, is both a fundamental result of formal language theory and a basis for the certified implementation of the algorithms for applications. In this work, we present the mechanized proof of the Bar-Hillel theorem in Coq.

Keywords: Formal languages · Coq · Bar-Hillel theorem · Closure · Intersection · Regular language · Context-free language

1 Introduction

Formal language theory has a deep connection with different areas such as static code analysis [25,29,35,36,39–41], graph database querying [19,20,23,42], formal verification [9,12], and others. One of the most frequent uses is to formulate a problem in terms of languages intersection. In verification, one language can serve as a model of a program and another language describe undesirable behaviors. When the intersection of these two languages is not empty, one can conclude that the program is incorrect. Usually, the only concern is the decidability of the languages intersection emptiness problem. But in some cases, a constructive representation of the intersection may prove useful. This is the case, for example, when the intersection of the languages models graph querying: a language produced by intersection is a query result and to be able to process it, one needs the appropriate representation of the intersection result.

The research was supported by the Russian Science Foundation, grant № 18-11-00100.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 264–281, 2019.
https://doi.org/10.1007/978-3-662-59533-6_17

Let us consider several applications starting with the user input validation. The problem is to check if the input provided by the user is correct with respect to some validation template such as a regular expression for e-mail validation. User input can be represented as a one word language. The intersection of such a language with the language specifying the validation template is either empty or contains the only string: the user input. If the intersection is empty, then the input should be rejected.

Checking that a program is syntactically correct is another example. The AST for the program (or lack thereof) is just a constructive representation of the intersection of the one-word language (the program) and the programming language itself.

Graph database regular querying serves as an example of the intersection of two regular languages [1,2,23]. Next and one of the most comprehensive cases with decidable emptiness problem is an intersection of a regular language with a context-free language. This case is relevant for program analysis [36,39,40], graph analysis [17,20,42], context-free compressed data processing [26], and other areas. The constructive intersection representation in these applications is helpful for further analysis.

The intersection of some classes of languages is not generally decidable. For example, the intersection of the linear conjunctive and the regular languages, used in the static code analysis [41], is undecidable while multiple context-free languages (MCFL) is closed under intersection with regular languages and emptiness problem for MCFLs is decidable [38]. Is it possible to express any useful properties in terms of regular and multiple context-free languages intersection? This question is beyond the scope of this paper but provides a good reason for future research in this area. Moreover, the history of pumping lemma for MCFG shows the necessity to mechanize formal language theory. In this paper, we focus on the intersection of regular and context-free languages.

Some applications mentioned above require certifications. For verification this requirement is evident. For databases it is necessary to reason about security aspects and, thus, we should create certified solutions for query executing. Certified parsing may be critical for secure data loading (for example in Web), as well as certified regular expressions for input validation. As a result, there is a significant number of papers focusing on regular expressions mechanization and certification [14], and a number on certified parsers [5,15,18]. On the other hand, mechanization (formalization) is important by itself as theoretical results mechanization and verification, and there is a lot of work done on formal languages theory mechanization [4,16,32]. Also, it is desirable to have a base to reason about parsing algorithms and other problems of languages intersection.

Context-free languages are closed under intersection with regular languages. It is stated as the Bar-Hillel theorem [3] which provides a constructive proof and construction for the resulting language description. We believe that the mechanization of the Bar-Hillel theorem is a good starting point for certified application development and since it is one of the fundamental theorems, it is an important part of formal language theory mechanization. And this work aims to provide such mechanization in Coq.

Our current work is the first step: we provide mechanization of theoretical results on context-free and regular languages intersection. We choose the result of Jana Hofmann on context-free languages mechanization [21] as a base for our work. The main contribution of this paper is the constructive proof of the Bar-Hillel theorem in Coq. All code is published on GitHub: https://github.com/YaccConstructor/YC_in_Coq.

2 Bar-Hillel Theorem

In this section, we provide the Bar-Hillel theorem and sketch the proof which we use as the base of our work. We also provide some additional lemmas which are used in the proof of the main theorem.

Lemma 1. *If $\boldsymbol{L}$ is a context-free language and $\boldsymbol{\varepsilon} \notin \boldsymbol{L}$ then there is a grammar in Chomsky Normal Form that generates $\boldsymbol{L}$.*

Lemma 2. *If $\boldsymbol{L} \neq \varnothing$ and $\boldsymbol{L}$ is regular then $\boldsymbol{L}$ is the union of regular language $\boldsymbol{A_1}, \dots, \boldsymbol{A_n}$ where each $\boldsymbol{A_i}$ is accepted by a DFA with precisely one final state.*

Theorem 1 (Bar-Hillel). *If $\boldsymbol{L_1}$ is a context-free language and $\boldsymbol{L_2}$ is a regular language, then $\boldsymbol{L_1} \cap \boldsymbol{L_2}$ is context-free.*

Sketch of the proof.

1. By Lemma 1 we can assume that there is a context-free grammar G_{CNF} in Chomsky normal form, such that $L(G_{CNF}) = L_1$
2. By Lemma 2 we can assume that there is a set of regular languages $\{A_1 \dots A_n\}$ where each A_i is recognized by a DFA with precisely one final state and $L_2 = A_1 \cup \dots \cup A_n$
3. For each A_i we can explicitly define a grammar of the $L(G_{\mathrm{CNF}}) \cap A_i$
4. Finally, we join them together with the union operation

As far as Bar-Hillel theorem operates with arbitrary context-free languages and the selected proof requires grammar in CNF, it is necessary to implement a certified algorithm for the conversion of an arbitrary CF grammar to CNF. We wanted to reuse existing mechanized proof for the conversion. We chose the one provided in Smolka's work and discussed it in the context of our work in Sect. 3.1.

3 Bar-Hillel Theorem Mechanization in Coq

In this section, we describe in detail all the fundamental parts of the proof. We also briefly describe the motivation to use the chosen definitions. In addition, we discuss the advantages and disadvantages of using third-party proofs.

The overall goal of this section is to provide a step-by-step algorithm which constructs the context-free grammar of the intersection of two languages. The final formulation of the theorem can be found in the last subsection.

3.1 Hofmann's Results Generalization

A substantial part of this proof relies on the work of Hofmann [21][1] from which many definitions and theorems were taken. Namely, the definition of a grammar, the definitions of a derivation in grammar, some auxiliary lemmas about the decidability of properties of grammar and derivation. We also use the theorem that states that there always exists the transformation from a context-free grammar to a grammar in Chomsky Normal Form.

However, the proof of the existence of the transformation to CNF had one major flaw that we needed to fix: the representation of terminals and nonterminals. In the definition of the grammar, a terminal is an element of the set of terminals—the alphabet of terminals. It is sufficient to represent each terminal by a unique natural number—conceptually, the index of the terminal in the alphabet.

The same observation is correct for nonterminals. Sometimes it is useful when the alphabet of nonterminals bears some structure. For the purposes of our proof, nonterminals are better represented as triples. We decided to make terminals and nonterminals to be polymorphic over the alphabet. We are only concerned that the representation of symbols is a type with decidable relation of equality. Namely, let Tt and Vt be such types, then we can define the types of terminals and nonterminals over Tt and Vt respectively.

Fortunately, the proof of Hofmann has a clear structure, and there was only one aspect of the proof where the use of natural numbers was essential. The grammar transformation which eliminates long rules creates new nonterminals. In the original proof, it was done by taking the maximum of the nonterminals included in the grammar. It is not possible to use the same mechanism for an arbitrary type.

To tackle this problem, we introduced an additional assumption on the alphabet types for terminals and nonterminals. We require the existence of the bijection between natural numbers and the alphabet of terminals as well as nonterminals.

Another difficulty is that the original work defines grammar as a list of rules and does not specify the start nonterminal. Thus, in order to define the language described by a grammar, one needs to specify the start terminal explicitly. It leads to the fact that the theorem about the equivalence of a CF grammar and the corresponding CNF grammar is not formulated in the most general way, namely, it guarantees equivalence only for non-empty words.

The predicate "is grammar in CNF" as defined in Hofmann [21] does not treat the case when the empty word is in the language. That is, with respect to the definition in [21], a grammar cannot have epsilon rules at all.

The question of whether the empty word is derivable is decidable for both the CF grammar and the DFA. Therefore, there is no need to adjust the definition

[1] Jana Hofmann, Verified Algorithms for Context-Free Grammars in Coq. Related sources in Coq: https://www.ps.uni-saarland.de/~hofmann/bachelor/coq_src.zip. Documentation: https://www.ps.uni-saarland.de/~hofmann/bachelor/coq/toc.html. Access date: 10.10.2018.

of the grammar (and subsequently all proofs). It is possible just to consider two cases (1) when the empty word is derivable in the grammar (and acceptable by DFA) and (2) when the empty word is not derivable. We use this feature of CNF definition to prove some of the lemmas presented in this paper.

3.2 Basic Definitions

In this section, we introduce the basic definitions used in the paper, such as alphabets, context-free grammar, and derivation.

We define a symbol as either a terminal or a nonterminal. Next, we define a word and a phrase as lists of terminals and symbols respectively. One can think that word is an element of the language defined by the grammar, and a phrase is an intermediate result of derivation. Also, a right-hand side of any derivation rule is a phrase.

The notion of nonterminal does not make sense for DFA, but in order to construct the derivation in grammar, we need to use nonterminals in intermediate states. For phrases, we introduce a predicate that defines whenever a phrase consists of only terminals. If it is the case, the phrase can be safely converted to the word.

We inherit the definition of CFG from [21]. The rule is defined as a pair of a nonterminal and a phrase, and a grammar is a list of rules. Note, that this definition of a grammar does not include the start nonterminal, and thus does not specify the language by itself.

An important step towards the definition of a language specified by a grammar is the definition of derivability. Proposition $der(G, A, p)$ means that the phrase p is derivable in the grammar G starting from the nonterminal A.

Also, we use the proof of the fact that every grammar is convertible into CNF from [21] because this fact is important for our proof.

We define the language as follows. We say that a phrase (not a word) w belongs to the language generated by a grammar G from a nonterminal A, if w is derivable from nonterminal A in grammar G and w consists only of terminals.

3.3 General Scheme of the Proof

A general scheme of our proof is based on the constructive proof presented in [8]. This proof does not use push-down automata explicitly and operates with grammars, so it is pretty simple to mechanize it. Overall, we will adhere to the following plan.

1. We consider the trivial case when DFA has no states.
2. We state that every CF language can be converted to CNF.
3. We show that every DFA can be presented as a union of DFAs with the single final state.
4. We construct an intersection of grammar in CNF with DFA with one final state.
5. We prove that the union of CF languages is CF language.

6. We putting everything mentioned above together. Additionally, we handle the fact that the initial CF language may contain the ε word. By the definition which we reuse from [21], the grammar in CNF has no epsilon rules, but we still need to consider the case when the empty word is derivable in the grammar. We postpone this consideration to the last step. Only one of the following statements is true: $\varepsilon \in L(G)$ *and* $\varepsilon \in L(dfa)$ or $\neg\varepsilon \in L(G)$ *or* $\neg\varepsilon \in L(dfa)$. So, we should just check emptiness of languages as a separated case.

3.4 Trivial Cases

First, we consider the case when the number of the DFA states is zero. In this case, we immediately derive a contradiction. By definition, any DFA has an initial state. It means that there is at least one state, which contradicts the assumption that the number of states is zero.

It is worth to mention, that in the proof [8] cases when the empty word is derivable in the grammar or a DFA specifies the empty language are discarded as trivial. It is assumed that one can carry out themselves the proof for these cases. In our proof, we include the trivial cases in the corresponding theorems.

3.5 Regular Languages and Automata

In this section, we describe definitions of DFA and DFA with exactly one final state, we also present the function that converts any DFA to a set of DFAs with one final state and lemma that states this split in some sense preserves the language specified.

We assume that a regular language is described by a DFA. We do not impose any restrictions on the type of input symbols and the number of states in DFA. Thus, the DFA is a 5-tuple: (1) a type of states, (2) a type of input symbols, (3) a start state, (4) a transition function, and (5) a list of final states.

Next, we define a function that evaluates the finish state of the automaton if it starts from the state s and receives a word w.

We say that the automaton accepts a word w being in state s if the function $(final\_state\ s\ w)$ returns a final state. Finally, we say that an automaton accepts a word w, if the DFA starts from the initial state and stops in a final state.

The definition of the DFA with exactly one final state differs from the definition of an ordinary DFA in that the list of final states is replaced by one final state. Related definitions such as *accepts* and *dfa_language* are slightly modified.

We define functions *s_accepts* and *s_dfa_language* for DFA with one final state in the same fashion. In the function *s_accepts*, it is enough to check for equality the state in which the automaton stopped with the finite state. Function *s_dfa_language* is the same as *dfa_language* except for that the function for a DFA with one final state should use *s_accepts* instead of *accepts*.

Now we can define a function that converts an ordinary DFA into a set of DFAs with exactly one final state. Let d be a DFA. Then the list of its final states is known. For each such state, one can construct a copy of the original DFA, but with one selected final state.

As a result prove the theorem that the function of splitting preserves the language.

Theorem 2. *Let* $\boldsymbol{dfa}$ *be an arbitrary DFA and* $\boldsymbol{w}$ *be a word. Then the fact that* $\boldsymbol{dfa}$ *accepts* $\boldsymbol{w}$ *implies that there exists a single-state DFA* $\boldsymbol{s\_dfa}$*, such that* $\boldsymbol{s\_dfa} \in \boldsymbol{split\_dfa(dfa)}$ *and* $\boldsymbol{s\_dfa}$ *accepts* $\boldsymbol{w}$*. And vice versa, for any* $\boldsymbol{s\_dfa} \in \boldsymbol{split\_dfa(dfa)}$ *the fact that* $\boldsymbol{s\_dfa}$ *accepts a word* $\boldsymbol{w}$ *implies that* $\boldsymbol{dfa}$ *also accepts* $\boldsymbol{w}$*.*

3.6 Chomsky Induction

Many statements about properties of words in a language can be proved by induction over derivation structure. Although a one can get a phrase as an intermediate step of derivation, DFA only works on words, so we can not simply apply induction over the derivation structure. To tackle this problem, we created a custom induction principle for grammars in CNF.

The current definition of derivability does not imply the ability to "reverse" the derivation back. That is, nothing about the rules of the grammar or properties of derivation follows from the fact that a phrase w is derived from a nonterminal A in a grammar G. Because of this, we introduce an additional assumption on derivations that is similar to the syntactic analysis of words. Namely, we assume that if the phrase w is derived from the nonterminal A in grammar G, then either there is a rule $A \to w \in G$ or there is a rule $A \to rhs \in G$ and w is derivable from rhs.

Any word derivable from a nonterminal A in the grammar in CNF is either a solitary terminal or can be split into two parts, each of which is derived from nonterminals B and C, when the derivation starts with the rule $A \to BC$. Note that if we naively take a step back, we can get a nonterminal which derives some substring in the middle of the word. Such a situation does not make any sense for DFA.

By using induction, we always deal with subtrees that describe a substring of the word.

To put it more formally:

Lemma 3. *Let* $\boldsymbol{G}$ *be a grammar in CNF. Consider an arbitrary nonterminal* $\boldsymbol{N} \in \boldsymbol{G}$ *and phrase which consists only of terminals* $\boldsymbol{w}$*. If* $\boldsymbol{w}$ *is derivable from* $\boldsymbol{N}$ *and* $|\boldsymbol{w}| \geq 2$*, then there exists two nonterminals* $\boldsymbol{N_1}, \boldsymbol{N_2}$ *and two phrases* $\boldsymbol{w_1, w_2}$ *such that:* $\boldsymbol{N \to N_1 N_2 \in G}$*,* $\boldsymbol{der(G, N_1, w_1)}$*,* $\boldsymbol{der(G, N_2, w_2)}$*,* $|\boldsymbol{w_1}| \geq 1$*,* $|\boldsymbol{w_2}| \geq 1$ *and* $\boldsymbol{w_1} \mathbin{++} \boldsymbol{w_2} = \boldsymbol{w}$*.*

Lemma 4. *Let* $\boldsymbol{G}$ *be a grammar in CNF. And* $\boldsymbol{P}$ *be a predicate on nonterminals and phrases (i.e.* $\boldsymbol{P} : \boldsymbol{var} \to \boldsymbol{phrase} \to \boldsymbol{Prop}$*). Let's also assume that the following two hypotheses are satisfied: (1) for every terminal production (i.e. in the form* $\boldsymbol{N} \to \boldsymbol{a}$*) of grammar* $\boldsymbol{G}$*,* $\boldsymbol{P}(\boldsymbol{r}, [\boldsymbol{r}])$ *holds and (2) for every* $\boldsymbol{N}, \boldsymbol{N_1}, \boldsymbol{N_2}$ *such that:* $\boldsymbol{N \to N_1 N_2 \in G}$ *and two phrases that consist only of terminals* $\boldsymbol{w_1}, \boldsymbol{w_2}$*, if* $\boldsymbol{P}(\boldsymbol{N_1}, \boldsymbol{w_1})$*,* $\boldsymbol{P}(\boldsymbol{N_2}, \boldsymbol{w_2})$*,* $\boldsymbol{der}(\boldsymbol{G}, \boldsymbol{N_1}, \boldsymbol{w_1})$ *and* $\boldsymbol{der}(\boldsymbol{G}, \boldsymbol{N_2}, \boldsymbol{w_2})$ *then* $\boldsymbol{P}(\boldsymbol{N}, \boldsymbol{w_1} \mathbin{++} \boldsymbol{w_2})$*. Then for any nonterminal* $\boldsymbol{N}$ *and any phrase consisting only of terminals* $\boldsymbol{w}$*, the fact that* $\boldsymbol{w}$ *is derivable from* $\boldsymbol{N}$ *implies* $\boldsymbol{P}(\boldsymbol{N}, \boldsymbol{w})$*.*

3.7 Intersection of CFG and Automaton

Since we already have lemmas about the transformation of a grammar to CNF and the transformation of a DFA to a DFA into a set of DFA's with exactly one accepting state, further we assume that we only deal with (1) DFA with exactly one final state—*dfa* and (2) grammar in CNF—G. In this section, we describe the proof of the lemma that states that for any grammar in CNF and any automaton with exactly one state there is a grammar for an intersection of the languages.

Construction of Intersection. We present the adaptation of the algorithm given in [8].

Let G_{INT} be the grammar of intersection. In G_{INT}, nonterminals are presented as triples ($from \times var \times to$) where *from* and *to* are states of *dfa*, and *var* is a nonterminal of G.

Since G is a grammar in CNF, it has only two types of productions: (1) $N \to a$ and (2) $N \to N_1 N_2$, where N, N_1, N_2 are nonterminals and a is a terminal.

For every production $N \to N_1 N_2$ in G we generate a set of productions of the form $(from, N, to) \to (from, N_1, m)(m, N_2, to)$ where: *from*, m, *to* enumerate all dfa states.

For every production of the form $N \to a$ we add a set of productions of the form $(from, N, (dfa\_step(from, a))) \to a$ where $from$ enumerates all dfa states and $dfa\_step\,(from, a)$ is the state in which the dfa appears after receiving terminal a in the state $from$.

Next, we join the functions above to get a generic function that works for both types of productions.

Note that at this point we do not conduct any manipulations with the start nonterminal. Nevertheless, the hypothesis of the uniqueness of the final state of the DFA helps to define the start nonterminal of the grammar of intersection unambiguously. The start nonterminal for the intersection grammar is the following nonterminal: *(start, S, final)* where: *start*—the start state of DFA, S—the start nonterminal of the initial grammar, and *final*—the final state of DFA. Without the assumption that the DFA has only one final state it is not clear how to unequivocally define the start nonterminal over the alphabet of triples.

Correctness of Intersection. In this subsection, we present a high-level description of the proof of correctness of the intersection function.

In the interest of clarity of exposition, we skip some auxiliary lemmas and facts like that we can get the initial grammar from the grammar of intersection by projecting the triples back to the corresponding terminals/nonterminals. Also note that grammar remains in CNF after the conversion, since the transformation of rules does not change the structure of them, but only replaces their terminals and nonterminals with attributed ones.

Next, we prove the following lemmas. First, the fact that a word can be derived in the initial grammar and is accepted by *s_dfa* implies it can be derived in the grammar of the intersection. And the other way around, the fact that a word can be derived in the grammar of the intersection implies that it is derived in the initial grammar and is accepted by *s_dfa*.

Let G be a grammar in CNF. In order to use Chomsky Induction, we also assume that syntactic analysis is possible.

Theorem 3. *Let* $\boldsymbol{s\_dfa}$ *be an arbitrary DFA, let* $\boldsymbol{r}$ *be a nonterminal of grammar* $\boldsymbol{G}$*, let* $\boldsymbol{from}$ *and* $\boldsymbol{to}$ *be two states of the DFA. We also pick an arbitrary word—* $\boldsymbol{w}$*. If it is possible to derive* $\boldsymbol{w}$ *from* $\boldsymbol{r}$ *and the* $\boldsymbol{s\_dfa}$ *starting from the state* $\boldsymbol{from}$ *finishes in the state* $\boldsymbol{to}$ *after consuming the word* $\boldsymbol{w}$*, then the word* $\boldsymbol{w}$ *is also derivable in grammar* $(\boldsymbol{convert\_rules}\ \boldsymbol{G}\ \boldsymbol{next})$ *from the nonterminal* $(\boldsymbol{from}, \boldsymbol{r}, \boldsymbol{to})$*.*

On the other side, now we need to prove the theorems of the form "if it is derivable in the grammar of triples, then it is accepted by the automaton and is derivable in the initial grammar".

We start with the DFA.

Theorem 4. *Let* $\boldsymbol{from}$ *and* $\boldsymbol{to}$ *be states of the automaton,* $\boldsymbol{var}$ *be an arbitrary nonterminal of* $\boldsymbol{G}$*. We prove that if a word* $\boldsymbol{w}$ *is derived from the nonterminal* $(\boldsymbol{from}, \boldsymbol{var}, \boldsymbol{to})$ *in the grammar* $(\boldsymbol{convert\_rules}\ \boldsymbol{G})$*, then the automaton starting from the state* $\boldsymbol{from}$ *accepts the word* $\boldsymbol{w}$ *and stops in the state* $\boldsymbol{to}$*.*

Next, we prove the similar theorem for the grammar.

Theorem 5. *Let* $\boldsymbol{from}$ *and* $\boldsymbol{to}$ *be the states of the automaton, let* $\boldsymbol{var}$ *be an arbitrary nonterminal of grammar* $\boldsymbol{G}$*. We prove that if a word* $\boldsymbol{w}$ *is derivable from the nonterminal* $(\boldsymbol{from}, \boldsymbol{var}, \boldsymbol{to})$ *in the grammar* $(\boldsymbol{convert\_rules}\ \boldsymbol{G})$*, then* $\boldsymbol{w}$ *is also derivable in the grammar* $\boldsymbol{G}$ *from the nonterminal* $\boldsymbol{var}$*.*

In the end, one needs to combine both theorems to get a full equivalence. By this, the correctness of the intersection is proved.

3.8 Union of Languages

During the previous step, we constructed a list of context-free grammars. In this section, we provide a function which constructs a grammar for the union of the languages.

First, we need to make sure the sets of nonterminals for each of the grammars under consideration have empty intersections. To achieve this, we label nonterminals. Each grammar of the union receives a unique ID number and all nonterminals within one grammar will have the same ID as the grammar. In addition, it is necessary to introduce a new start nonterminal of the union.

The function that constructs the union grammar takes a list of grammars, then, it (1) splits the list into head [h] and tail [tl], (2) labels [$length\ \ tl$] to h, (3) adds a new rule from the start nonterminal of the union to the start nonterminal of the grammar [h], finally (4) the function is recursively called on the tail [tl] of the list.

Proof of Languages Equivalence. We prove that the function *grammar_union* constructs a correct grammar of the union language. Namely, we prove the following theorem.

Theorem 6. *Let* $\boldsymbol{grammars}$ *be a sequence of pairs of starting nonterminals and grammars. Then for any word* $\boldsymbol{w}$*, the fact that* $\boldsymbol{w}$ *belongs to the language of the union is equivalent to the fact that there exists a grammar* $\boldsymbol{(st, gr)} \in \boldsymbol{grammars}$ *such that* $\boldsymbol{w}$ *belongs to the language generated by* $\boldsymbol{(st, gr)}$*.*

3.9 Putting All Parts Together

Now we can put all previously described lemmas together to prove the main statement of this paper (Fig. 1).

```
Theorem grammar_of_intersection_exists:
  exists
    (NewNonterminal: Type)
    (IntersectionGrammar: @grammar Terminal NewNonterminal) St,
  forall word,
     dfa_language dfa word /\ language G S (to_phrase word) <->
     language IntersectionGrammar St (to_phrase word).
```

Fig. 1. Final theorem

Theorem 7. *Let* $\boldsymbol{Tt}$ *and* $\boldsymbol{Nt}$ *be a decidable types.* $\boldsymbol{Tt}$ *and* $\boldsymbol{Nt}$ *is types of terminals and nonterminals correspondingly. If there exists a bijection from* $\boldsymbol{Nt}$ *to* $\mathbb{N}$ *and syntactic analysis in the sense of definition is possible, then for any DFA* $\boldsymbol{dfa}$ *that define language over* $\boldsymbol{Tt}$ *and any context-free grammar* $\boldsymbol{G}$*, there exists the context-free grammar* $\boldsymbol{G_{INT}}$*, such that* $\boldsymbol{L}(\boldsymbol{G_{INT}}) = \boldsymbol{L}(\boldsymbol{G}) \cap \boldsymbol{L}(\boldsymbol{dfa})$*.*

4 Related Works

There is a large number of contributions in the mechanization of different parts of formal languages theory and certified implementations of parsing algorithms and algorithms for graph database querying. These works use various tools, such as Coq, Agda, HOL4, and are aimed at different problems such as the theory mechanization or executable algorithm certification. We discuss only a small part which is close enough to the scope of this work.

4.1 Formal Language Theory in Coq

The massive amount of work was done by Ruy de Queiroz who formalized different parts of formal language theory, such as pumping lemma [31,33], context-free grammar simplification [34] and closure properties [30] in Coq. The work on closure properties contains mechanization of such properties as closure under union, Kleene star, but it does not contain mechanization of the intersection with a regular language. All these results are summarized in [32].

Gert Smolka et al. also provide a large number of contributions on regular and context-free languages formalization in Coq [10,11,21,22]. The paper [21] describes the certified transformation of an arbitrary context-free grammar to the Chomsky normal form which is required for our proof of the Bar-Hillel theorem. Initially, we hoped to reuse these both parts because the Bar-Hillel theorem is about both context-free and regular languages, and it was the reason to choose results of Gert Smolka as the base for our work. But the works on regular and on context-free languages are independent, and we are faced with the problems of reusing and integration, so in the current proof, we use only results on context-free languages.

4.2 Formal Language Theory in Other Languages

In the parallel with works in Coq there exist works on formal languages mechanization in other languages and tools such as Agda [13] or HOL4 [6].

Firstly, there are works of Denis Firsov who implements some parts of the formal language theory and parsing algorithms in Agda. In particular, Firsov implements CYK parsing algorithm [13,15] and Chomsky Normal Form [16], and some other results on regular languages [14].

There are also works on the formal language theory mechanization in HOL-4 [4,6,7] by Aditi Barthwal and Michael Norrish. This work contains basic definitions and a big number of theoretical results, such as Chomsky normal form and Greibach normal form for context-free grammars. As an application of the mechanized theory authors, provide certified implementation of the SLR parsing algorithm [5].

5 Conclusion

We present mechanized in Coq proof of the Bar-Hillel theorem, the fundamental theorem on the closure of context-free languages under intersection with the regular set. By this, we increase mechanized part of formal language theory and provide a base for reasoning about many applicative algorithms which are based on language intersection. We generalize the results of Gert Smolka and Jana Hofmann: the definition of the terminal and nonterminal alphabets in context-free grammar were made generic, and all related definitions and theorems were adjusted to work with the updated definition. It makes previously existing results more flexible and eases reusing. All results are published at GitHub and are equipped with automatically generated documentation.

The first open question is the integration of our results with other results on formal languages theory mechanization in Coq. There are two independent sets of results in this area: works of Ruy de Queiroz and works of Gert Smolka. We use part of Smolka's results in our work, but even here we do not use existing results on regular languages. We believe that theory mechanization should be unified and results should be generalized. We think that these and other related questions should be discussed in the community.

One direction for future research is mechanization of practical algorithms which are just implementation of the Bar-Hillel theorem. For example, context-free path querying algorithm, based on CYK [20,42] or even on GLL [37] parsing algorithm [17]. Final target here is the certified algorithm for context-free constrained path querying for graph databases.

Another direction is mechanization of other problems on language intersection which can be useful for applications. For example, the intersection of two context-free grammars one of which describes finite language [28]. It may be useful for compressed data processing [24] or speech recognition [27]. And we believe all these works should share the common base of mechanized theoretical results.

A Coq Listing

This listing contains main theorems and definitions from our work.

```
Inductive ter : Type := | T : Tt -> ter.
Inductive var : Type := | V : Vt -> var.

Lemma language_normal_form (G:grammar) (A: var) (u: word):
  u <> [] -> (language G A u <-> language (normalize G) A u).

Inductive symbol : Type :=
  | Ts : ter -> symbol
  | Vs : var -> symbol.

Definition word := list ter.
Definition phrase := list symbol.
Inductive rule : Type := | R : var -> phrase -> rule.
Definition grammar := list rule.

Inductive der (G : grammar) (A : var) : phrase -> Prop :=
  | vDer : der G A [Vs A]
  | rDer l : (R A l) el G -> der G A l
  | replN B u w v :
    der G A (u ++ [Vs B] ++ w) ->
    der G B v -> der G A (u ++ v ++ w).

Definition language (G : grammar) (A : var) (w : phrase) :=
  der G A w /\ terminal w.

Context {State T: Type}.

Record dfa: Type :=
  mkDfa {
    start: State;
    final: list State;
    next: State -> ter T -> State;
  }.

Fixpoint final_state (next_d: dfa_rule) (s: State) (w: word): State :=
  match w with
  | nil => s
  | h :: t => final_state next_d (next_d s h) t
  end.

Record s_dfa : Type :=
  s_mkDfa {
    s_start: State;
    s_final: State;
    s_next: State -> (@ter T) -> State;
  }.

Fixpoint split_dfa_list (st_d : State) (next_d : dfa_rule)
```

```
                        (f_list : list State): list (s_dfa) :=
  match f_list with
  | nil => nil
  | h :: t => (s_mkDfa st_d h next_d) :: split_dfa_list st_d next_d t
  end.

Definition split_dfa (d: dfa) :=
  split_dfa_list (start d) (next d) (final d).

Lemma correct_split:
  forall dfa w,
    dfa_language dfa w <->
    exists sdfa, In sdfa (split_dfa dfa) /\ s_dfa_language sdfa w.

Definition syntactic_analysis_is_possible :=
  forall (G : grammar) (A : var) (w : phrase),
  der G A w -> (R A w \in G) \/ (exists rhs, R A rhs \in G /\ derf G rhs w).

Definition convert_nonterm_rule_2 (r r1 r2: _) (state1 state2 : _) :=
  map (fun s3 => R (V (s1, r, s3))
                   [Vs (V (s1, r1, s2)); Vs (V (s2, r2, s3))])
      list_of_states.

Definition convert_nonterm_rule_1 (r r1 r2: _) (s1 : _) :=
  flat_map (convert_nonterm_rule_2 r r1 r2 s1) list_of_states.

Definition convert_nonterm_rule (r r1 r2: _) :=
  flat_map (convert_nonterm_rule_1 r r1 r2) list_of_states.

Definition convert_terminal_rule
  (next: _) (r: _) (t: _): list TripleRule :=
  map (fun s1 => R (V (s1, r, next s1 t)) [Ts t]) list_of_states.

Definition convert_rule (next: _) (r: _ ) :=
   match r with
   | R r [Vs r1; Vs r2] =>
       convert_nonterm_rule r r1 r2
   | R r [Ts t] =>
       convert_terminal_rule next r t
   | _  => []    (* Never called *)
   end.

Definition convert_rules
  (rules: list rule) (next: _): list rule :=
  flat_map (convert_rule next) rules.
```

```
Definition convert_grammar grammar s_dfa :=
  convert_rules grammar (s_next s_dfa).

Inductive labeled_Vt : Type :=
  | start : labeled_Vt
  | lV : nat -> Vt -> labeled_Vt.

Definition label_var (label: nat) (v: @var Vt): @var labeled_Vt :=
  V (lV label v).

Definition label_grammar_and_add_start_rule label grammar :=
  let '(st, gr) := grammar in
  (R (V start) [Vs (V (lV label st))]) :: label_grammar label gr.

Fixpoint grammar_union (grammars : seq (@var Vt * (@grammar Tt Vt)))
       : @grammar Tt labeled_Vt :=
  match grammars with
  | [] => []
  |  (g::t) => label_grammar_and_add_start_rule (length t)
                                                 g ++ (grammar_union t)
end.

Variable grammars: seq (var * grammar).

Theorem correct_union:
  forall word,
    language (grammar_union grammars)
    (V (start Vt)) (to_phrase word) <->
  exists s_l,
    language (snd s_l) (fst s_l) (to_phrase word) /\ In s_l grammars.

Theorem grammar_of_intersection_exists:
  exists
    (NewNonterminal: Type)
    (IntersectionGrammar: @grammar Terminal NewNonterminal) St,
  forall word,
     dfa_language dfa word /\ language G S (to_phrase word) <->
     language IntersectionGrammar St (to_phrase word).
```

References

1. Abiteboul, S., Vianu, V.: Regular path queries with constraints. J. Comput. Syst. Sci. **58**(3), 428–452 (1999). http://www.sciencedirect.com/science/article/pii/S0022000099916276
2. Alkhateeb, F.: Querying RDF(S) with Regular Expressions. Theses, Université Joseph-Fourier - Grenoble I, June 2008. https://tel.archives-ouvertes.fr/tel-00293206
3. Bar-Hillel, Y., Perles, M., Shamir, E.: On formal properties of simple phrase structure grammars. Sprachtypologie und Universalienforschung **14**, 143–172 (1961)

4. Barthwal, A.: A formalisation of the theory of context-free languages in higher order logic. Ph.D. thesis, College of Engineering & Computer Science, The Australian National University, December 2010
5. Barthwal, A., Norrish, M.: Verified, executable parsing. In: Castagna, G. (ed.) ESOP 2009. LNCS, vol. 5502, pp. 160–174. Springer, Heidelberg (2009). https://doi.org/10.1007/978-3-642-00590-9_12
6. Barthwal, A., Norrish, M.: A formalisation of the normal forms of context-free grammars in HOL4. In: Dawar, A., Veith, H. (eds.) CSL 2010. LNCS, vol. 6247, pp. 95–109. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-15205-4_11
7. Barthwal, A., Norrish, M.: Mechanisation of PDA and grammar equivalence for context-free languages. In: Dawar, A., de Queiroz, R. (eds.) WoLLIC 2010. LNCS (LNAI), vol. 6188, pp. 125–135. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-13824-9_11
8. Beigel, R., Gasarch, W.: A Proof that if $L = L_1 \cap L_2$ where L_1 is CFL and L_2 is Regular then L is Context Free Which Does Not use PDAs. http://www.cs.umd.edu/~gasarch/BLOGPAPERS/cfg.pdf/
9. Bouajjani, A., Esparza, J., Maler, O.: Reachability analysis of pushdown automata: application to model-checking. In: Mazurkiewicz, A., Winkowski, J. (eds.) CONCUR 1997. LNCS, vol. 1243, pp. 135–150. Springer, Heidelberg (1997). https://doi.org/10.1007/3-540-63141-0_10
10. Doczkal, C., Kaiser, J.-O., Smolka, G.: A constructive theory of regular languages in Coq. In: Gonthier, G., Norrish, M. (eds.) CPP 2013. LNCS, vol. 8307, pp. 82–97. Springer, Cham (2013). https://doi.org/10.1007/978-3-319-03545-1_6
11. Doczkal, C., Smolka, G.: Regular language representations in the constructive type theory of Coq. J. Autom. Reason. **61**(1), 521–553 (2018). https://doi.org/10.1007/s10817-018-9460-x
12. Emmi, M., Majumdar, R.: Decision problems for the verification of real-time software. In: Hespanha, J.P., Tiwari, A. (eds.) HSCC 2006. LNCS, vol. 3927, pp. 200–211. Springer, Heidelberg (2006). https://doi.org/10.1007/11730637_17
13. Firsov, D.: Certification of Context-Free Grammar Algorithms (2016)
14. Firsov, D., Uustalu, T.: Certified parsing of regular languages. In: Gonthier, G., Norrish, M. (eds.) CPP 2013. LNCS, vol. 8307, pp. 98–113. Springer, Cham (2013). https://doi.org/10.1007/978-3-319-03545-1_7
15. Firsov, D., Uustalu, T.: Certified CYK parsing of context-free languages. J. Log. Algebraic Methods Program. **83**(5–6), 459–468 (2014)
16. Firsov, D., Uustalu, T.: Certified normalization of context-free grammars. In: Proceedings of the 2015 Conference on Certified Programs and Proofs, pp. 167–174. ACM (2015)
17. Grigorev, S., Ragozina, A.: Context-free path querying with structural representation of result. arXiv preprint arXiv:1612.08872 (2016)
18. Gross, J., Chlipala, A.: Parsing Parses A Pearl of (Dependently Typed) Programming and Proof (2015)
19. Hellings, J.: Conjunctive Context-Free Path Queries (2014)
20. Hellings, J.: Querying for paths in graphs using context-free path queries. arXiv preprint arXiv:1502.02242 (2015)
21. Hofmann, J.: Verified Algorithms for Context-Free Grammars in Coq (2016)
22. Kaiser, J.O.: Constructive formalization of regular languages. Ph.D. thesis, Saarland University (2012)

23. Koschmieder, A., Leser, U.: Regular path queries on large graphs. In: Ailamaki, A., Bowers, S. (eds.) SSDBM 2012. LNCS, vol. 7338, pp. 177–194. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-31235-9_12
24. Lohrey, M.: Algorithmics on SLP-compressed strings: a survey. Groups Complex. Cryptol. **4**, 241–299 (2012)
25. Lu, Y., Shang, L., Xie, X., Xue, J.: An incremental points-to analysis with CFL-reachability. In: Jhala, R., De Bosschere, K. (eds.) CC 2013. LNCS, vol. 7791, pp. 61–81. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-37051-9_4
26. Maneth, S., Peternek, F.: Grammar-based graph compression. Inf. Syst. **76**, 19–45 (2018). http://www.sciencedirect.com/science/article/pii/S0306437917301680
27. Nederhof, M.J., Satta, G.: Parsing non-recursive context-free grammars. In: Proceedings of the 40th Annual Meeting on Association for Computational Linguistics, ACL 2002, pp. 112–119. Association for Computational Linguistics, Stroudsburg (2002). https://doi.org/10.3115/1073083.1073104
28. Nederhof, M.J., Satta, G.: The language intersection problem for non-recursive context-free grammars. Inf. Comput. **192**(2), 172–184 (2004). http://www.sciencedirect.com/science/article/pii/S0890540104000562
29. Pratikakis, P., Foster, J.S., Hicks, M.: Existential label flow inference via CFL reachability. In: Yi, K. (ed.) SAS 2006. LNCS, vol. 4134, pp. 88–106. Springer, Heidelberg (2006). https://doi.org/10.1007/11823230_7
30. Ramos, M.V.M., de Queiroz, R.J.G.B.: Formalization of closure properties for context-free grammars. CoRR abs/1506.03428 (2015). http://arxiv.org/abs/1506.03428
31. Ramos, M.V.M., de Queiroz, R.J.G.B., Moreira, N., Almeida, J.C.B.: Formalization of the pumping lemma for context-free languages. CoRR abs/1510.04748 (2015). http://arxiv.org/abs/1510.04748
32. Ramos, M.V.M., de Queiroz, R.J.G.B., Moreira, N., Almeida, J.C.B.: On the formalization of some results of context-free language theory. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 338–357. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_21
33. Ramos, M.V., Almeida, J.C.B., de Queiroz, R.J., Moreira, N.: Some applications of the formalization of the pumping lemma for context-free languages. In: Proceedings of the 13th Workshop on Logical and Semantic Frameworks with Applications, pp. 43–56 (2018)
34. Ramos, M.V., de Queiroz, R.J.: Formalization of simplification for context-free grammars. arXiv preprint arXiv:1509.02032 (2015)
35. Rehof, J., Fähndrich, M.: Type-base flow analysis: from polymorphic subtyping to CFL-reachability. ACM SIGPLAN Not. **36**(3), 54–66 (2001)
36. Reps, T., Horwitz, S., Sagiv, M.: Precise interprocedural dataflow analysis via graph reachability. In: Proceedings of the 22nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 1995, pp. 49–61. ACM, New York (1995). https://doi.org/10.1145/199448.199462
37. Scott, E., Johnstone, A.: GLL parsing. Electron. Notes Theor. Comput. Sci. **253**(7), 177–189 (2010)
38. Seki, H., Matsumura, T., Fujii, M., Kasami, T.: On multiple context-free grammars. Theor. Comput. Sci. **88**(2), 191–229 (1991). http://www.sciencedirect.com/science/article/pii/030439759190374B
39. Vardoulakis, D., Shivers, O.: CFA2: a context-free approach to control-flow analysis. In: Gordon, A.D. (ed.) ESOP 2010. LNCS, vol. 6012, pp. 570–589. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-11957-6_30

40. Yan, D., Xu, G., Rountev, A.: Demand-driven context-sensitive alias analysis for Java. In: Proceedings of the 2011 International Symposium on Software Testing and Analysis, ISSTA 2011, pp. 155–165. ACM, New York (2011). https://doi.org/10.1145/2001420.2001440
41. Zhang, Q., Su, Z.: Context-sensitive data-dependence analysis via linear conjunctive language reachability. SIGPLAN Not. **52**(1), 344–358 (2017). https://doi.org/10.1145/3093333.3009848
42. Zhang, X., Feng, Z., Wang, X., Rao, G., Wu, W.: Context-free path queries on RDF graphs. In: Groth, P., et al. (eds.) ISWC 2016. LNCS, vol. 9981, pp. 632–648. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-46523-4_38

Proof-Net as Graph, Taylor Expansion as Pullback

Giulio Guerrieri[1], Luc Pellissier[2(✉)], and Lorenzo Tortora de Falco[3]

[1] Department of Computer Science, University of Bath, Bath, UK
g.guerrieri@bath.ac.uk
[2] IRIF, Université Paris Diderot, Paris, France
pellissier@irif.fr
[3] Dipartimento di Matematica e Fisica, Università Roma Tre, Rome, Italy
tortora@uniroma3.it

Abstract. We introduce a new graphical representation for multiplicative and exponential linear logic proof-structures, based only on standard labelled oriented graphs and standard notions of graph theory. The inductive structure of boxes is handled by means of a box-tree. Our proof-structures are canonical and allows for an elegant definition of their Taylor expansion by means of pullbacks.

Keywords: Linear logic · Proof-net · Taylor expansion · Graph

1 Introduction

Linear Logic (LL) [14] has been introduced by Girard as a refinement of intuitionnistic and classical logic that isolates the infinitary parts of reasoning under two modalities: the *exponentials* ! and ?. These modalities give a logical status to the operations of memory/hypothesis management such as *copying*/*contraction* or *erasing*/*weakening*: a linear proof corresponds to a program/proof that uses its arguments/hypotheses *linearly*, i.e. only once, while an exponential proof corresponds to a program/proof that can use its arguments/hypotheses at will.

One of the features of LL is that it allows us to represent its proofs as *proof-nets*, a graphical syntax alternative to sequent calculus. Sequent calculus is a standard formalism for several logical systems. However, sequent calculus forces an order among inference rules even when they are evidently independent, a drawback called *bureaucracy*. Proof-nets, instead, are a geometrical, parallel and bureaucracy-free representation of proofs as labeled oriented *graphs*. In proof-nets deductive rules are disposed on the plane, in parallel, and connected only by their causal relation. Clearly, not all graphs that can be written in the language of LL are proof-nets, i.e. represent a proof in LL sequent calculus. Proof-nets are special inhabitants of the wider land of *proof-structures*: they can be characterized, among proof-structures, by abstract (geometric) conditions called correctness criteria [14]. The procedure of cut-elimination can be applied directly to proof-structures, and proof-nets can also be seen as the proof-structures with a good

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 282–300, 2019.
https://doi.org/10.1007/978-3-662-59533-6_18

behavior with respect to cut-elimination [3]. Cut-elimination defined on proof-structures is more elegant than in sequent calculus because it drastically reduces the need for commutative steps, the non-interesting and bureaucratic burden in every sequent calculus proof of cut-elimination. Indeed, in proof-structures there is no last rule, and so most commutative cut-elimination cases simply disappear.

Unfortunately, this is a faithful picture of the advantages of proof-structures only in the multiplicative fragment of LL (MLL) [9], which does not contain exponentials ! and ?, and so it is not sufficiently expressive to encode classical or intuitionistic logic (or the λ-calculus) in. To handle the exponentials Girard was forced to introduce *boxes*. They come with the black-box principle: "boxes are treated in a perfectly modular way: we can use the box B without knowing its content, i.e., another box B' with exactly the same doors would do as well" [14].

According to this principle, boxes forbid interaction between their content and their outer environment. This is evident in the definition of correctness criteria for MELL (the multiplicative-exponential fragment of LL) and in the definition of cut-elimination steps for MELL. Let us consider cut-elimination. Some cut-elimination steps require us to duplicate or erase whole sub-proofs, typically the steps for the !-modality in MELL. Proofs in sequent calculus are tree-shaped and bear a clear notion of last rule, the root of the tree. This property has an obvious but important consequence: given a !-rule r in a sequent calculus proof, there is an evident sub-proof ending with r, the sub-tree rooted in r. Therefore, non-linear cut-elimination steps can easily be defined by duplicating or erasing sub-trees. Switching to proof structures, the situation radically changes, because a proof structure in general has many last rules, one for each formula in the conclusions. Given a rule r it is not clear how to find a sub-proof-structure ending with r. Thus, in order to define cut-elimination steps for the !-modality in MELL proof-structures—which requires to identify some sub-proof-structure—some information has to been added.

The typical solution is to re-introduce part of the bureaucracy in MELL proof-structures, pairing each !-rule with an *explicit box* containing the sub-proof that can be duplicated or erased during cut-elimination. In some fragments of MELL (for instance the intuitionistic one corresponding to the λ-calculus [24] or more in general the polarized one [1]) where proof-structures still have an implicit tree-like structure (since among the conclusions there is always exactly one distinct output, the analogue of sequent calculus last rule), the explicit box is actually not needed. But here we are interested in the full (classical) MELL fragment, where linear negation is involutive and classical duality can be interpreted as the possibility of juggling between different conclusions. Concretely, in the literature mainly two kinds of solution that make use of explicit boxes can be found:

1. A MELL proof-structure is an oriented graph together with some additional information to identify the content and the border of each box. This additional information can be provided either informally, just drawing the border of each box in the graph [10,14,19], but then the definition of MELL proof-structure

is not rigorous; or in a more formal way [6,7,15], but then the definition is highly technical and *ad hoc*;
2. A MELL proof-structure is an *inductive* oriented graph [8,17,22,26], i.e. an oriented graph where with any vertex v of type ! is associated another oriented graph representing the content of the box of v. This inductive solution can be taken to extremes by representing proof-structures with term-like syntax [12].

The drawback of Item 1 is that the definition of MELL proof-structure is not easily manageable because either it is not precise or it is too tricky. Item 2, instead, provides more manageable definitions of MELL proof-structures, but another drawback arises: they intrinsically are *not canonical*, in that there are different inductive presentations of a MELL proof-structure defined up to associativity and commutativity of contractions, neutrality of weakening with respect to contraction, and permutation of weakenings and contractions with box-border.

Our Contribution. We present here a purely graphical definition of MELL proof-structures (Sect. 3), so as to keep Girard's original intuition of a proof-structure as a graph even in MELL. This definition follows the non-inductive approach seen in Item 1: we use n-ary vertices of type ? collapsing weakening, dereliction and contraction (like in [10]). In this way, we get a *canonical* representation of MELL proof-structures. But our definition is completely based on standard notions (recalled in Sect. 2) coming from the theory of graphs, being formal (with an eye towards complete computer formalization) but avoiding *ad hoc* technicalities to identify the border and the content of a box. The inductive structure of boxes is handled by means of a box-tree: indeed, a MELL proof-structure R is given by an oriented labelled graph $|R|$ plus a tree $\mathcal{A}_R$ (representing the order of the boxes of R) and a graph morphism box_R from $|R|$ to $\mathcal{A}_R$ which allows us to recognize the content and the border of all boxes in R. In this way, our MELL proof-structures are still manageable: sophisticated operations on them, such as the Taylor expansion [11] can be easily defined. As a test of the usability of our MELL proof-structures, we give an elegant definition of their Taylor expansion, by means of *pullbacks* (Sect. 5).

Moreover, our setting allows us to define in a simple way correctness graphs (used to characterize the proof-structures that are proof-nets, i.e. that correspond to proofs in the LL sequent calculus), as we show in Sect. 4 for MLL.

Since the main contribution of our work is to provide a new definition of MELL proof-structures, our paper contains several definitions and no new theorems.

2 Preliminaries on Graphs

Graphs with Half-Edges. There are many formalizations of the familiar notion of graph. Here we adopt the one due to [4]:[1] a graph is still a set of edges and a set of vertices, but edges are now split in halves, allowing some of them to

[1] The folklore attributes the definition of graphs with half-edges to Kontsevitch and Manin, but the idea can actually be traced back to Grothendieck's *dessins d'enfant*.

be hanging. Splitting every edge in two has at least three features that are of particular interest for representing LL proof-structures:

– two half-edges are connected by an involution, thus defining an edge. The fixpoints of this involution are thus dangling edges, connected to only one vertex: they are suited to represent the conclusions of a proof-structure. In this way it is also easy to define some intuitive but formally tricky operations such as the graft or the substitution of a graph for another graph (see Example 1);
– given a graph and any of its vertices, it is natural to define the *corolla* of the vertex, that is the vertex itself with all the half-edges connected to it;
– finally, while studying proof-structures, it is always necessary to treat them both as oriented and unoriented graphs. With this definition of graph, an orientation, so as a labeling and a coloring, are structures on top of the structure of the unoriented graph (see Definition 3).

Definition 1 (graph). *A (finite)* graph τ *is a quadruple* $(F_\tau, V_\tau, \partial_\tau, j_\tau)$*, where*

– F_τ *is a finite set, whose elements are called* flags *of* τ*;*
– V_τ *is a finite set, whose elements are called* vertices *of* τ*;*
– $\partial_\tau : F_\tau \to V_\tau$ *is a function associating with each flag its* boundary*;*
– $j_\tau : F_\tau \to F_\tau$ *is an involution.*

The graph τ *is* empty *if* $V_\tau = \emptyset$.

A flag that is a fixed point of the involution j_F is a *tail* of τ. A two-element orbit $\{f, f'\}$ of j_F is an *edge* of τ between $\partial_\tau(f)$ and $\partial_\tau(f')$, and f and f' are the *halves* of such an edge. The set of edges of τ is denoted by E_τ.

Given two graphs τ and τ', it is always possible to consider their disjoint union $\tau \sqcup \tau'$ defined as the disjoint union of the underlying sets and functions.

A one-vertex graph with set of flags F and involution the identity function id_F on F is called the *corolla* with set of flags F; it is usually denoted by $*_F$.

Given a graph $\tau = (F_\tau, V_\tau, \partial_\tau, j_\tau)$, a vertex v defines a corolla $\tau_v = (F_v, \{v\}, \partial_\tau|_{F_v}, \mathrm{id}_{F_v})$ where $F_v = \partial_\tau^{-1}(v)$. Every graph can be described as the set of corollas of its vertices, together with the involution glueing the flags in edges.

Definition 2 (graph morphism and isomorphism). *Let* τ, σ *be two graphs. A* graph morphism $h\colon \tau \to \sigma$ *from* τ *to* σ *is a couple of functions* $(h_F\colon F_\tau \to F_\sigma, h_V\colon V_\tau \to V_\sigma)$ *such that* $h_V \circ \partial_\tau = \partial_\sigma \circ h_F$ *and* $h_F \circ j_\tau = j_\sigma \circ h_F$.

A graph morphism is injective *if its component functions are. A graph* isomorphism *is a graph morphism whose component functions are bijections.*

The category **Graph** has graphs as objects and morphisms of graphs as morphisms: indeed, graph morphisms compose (by composing the underlying functions) and the couple of identities (on vertices and flags) is neutral. It is a monoidal category, with disjoint union as a monoidal product.

Graphs with Structure. Some structure can be put on top of a graph.

Definition 3 (structured graph). *Let $\tau = (F_\tau, V_\tau, \partial_\tau, j_\tau)$ be a graph.*

- *A* labeled graph *(τ, ℓ_τ) with labels in I is a graph τ and a function $\ell_\tau \colon V_\tau \to I$.*
- *A* colored graph *(τ, c_τ) is a graph τ with a function $\mathsf{c}_\tau \colon F_\tau \to C$ such that $\mathsf{c}_\tau(f) = \mathsf{c}_\tau(f')$ for the two halves f, f' of any edge of τ.*
- *An* oriented graph *(τ, o_τ) is a graph τ with a function $\mathsf{o}_\tau \colon F_\tau \to \{\mathbf{in}, \mathbf{out}\}$ such that $\mathsf{o}_\tau(f) \neq \mathsf{o}_\tau(f')$ for the two halves f, f' of any edge of τ. If $\mathsf{o}_\tau(f) = \mathbf{out}$ and $\mathsf{o}_\tau(f') = \mathbf{in}$, $\{f, f'\}$ is said an* edge *of τ* from *$\partial_\tau(f)$* to *$\partial_\tau(f')$;* **in***-oriented (resp.* **out***-oriented) tails of τ are called* inputs *(resp.* outputs*) of τ; if v is a vertex of τ, its* inputs *(resp. its* outputs*) are the elements of the set $\mathbf{in}_\tau(v) = \partial_\tau^{-1}(v) \cap \mathsf{o}_\tau^{-1}(\mathbf{in})$; (resp. $\mathbf{out}_\tau(v) = \partial_\tau^{-1}(v) \cap \mathsf{o}_\tau^{-1}(\mathbf{out})$);*
- *An* ordered graph *$(\tau, <_\tau)$ is a graph together with an order on the flags.*

Different structures on a graph can combine: for instance, a graph τ can be endowed with both a labeling ℓ_τ and an orientation o_τ.

Graphs can be depicted in diagrammatic form. As a graph is just a disjoint union of corollas glued with the involution, we only need to depict corollas (as in Fig. 1, on the left) and place the two halves of an edge next to each other (as in Fig. 1, on the right). In oriented graphs, inputs of a corolla are depicted above the corolla, outputs are below; arrows also show the orientation. The color of a flag f (if any) is written next to f. The label of a vertex v (if any) is written inside v. If ordered, flags of a corolla are depicted increasing from left to right.

Example 1. The oriented labeled colored ordered corolla $\tau_\mathbf{5} = (*_\mathbf{5}, \mathsf{o}_\mathbf{5}, \ell_\mathbf{5}, \mathsf{c}_\mathbf{5}, <_\mathbf{5})$ depicted in Fig. 1 (on the left) has $*$ as only vertex and $\mathbf{5} = \{0, 1, 2, 3, 4\}$ as set of flags; it is endowed with the order $0 <_\mathbf{5} 4$ and $1 <_\mathbf{5} 2 <_\mathbf{5} 3$, and

- the orientation $\mathsf{o}_\mathbf{5} \colon \mathbf{5} \to \{\mathbf{in}, \mathbf{out}\}$ defined by $\mathsf{o}_\mathbf{5}(0) = \mathsf{o}_\mathbf{5}(4) = \mathbf{out}$ and $\mathsf{o}_\mathbf{5}(1) = \mathsf{o}_\mathbf{5}(2) = \mathsf{o}_\mathbf{5}(3) = \mathbf{in}$,
- the labeling $\ell_\mathbf{5} \colon \{*\} \to \{\maltese\}$ defined by $\ell_\mathbf{5}(*) = \maltese$,
- the coloring $\mathsf{c}_\mathbf{5} \colon \mathbf{5} \to \{a_0, \ldots, a_4\}$ defined by $\mathsf{c}(i) = a_i$ for all $i \in \mathbf{5}$.

Consider also the oriented labeled colored corolla $\sigma_{\mathtt{ax}}$, whose only vertex is labeled by $\mathtt{ax}$, and whose only flags are the outputs 5 (labeled by a_2) and 6 (labeled by a_3). The oriented labeled colored ordered two-vertex graph ρ depicted in Fig. 1 (on the right) is obtained from the corollas $\tau_\mathbf{5}$ and $\sigma_{\mathtt{ax}}$ by defining the involution $j_\rho \colon \{0, \ldots, 6\} \to \{0, \ldots, 6\}$ as $j_\rho(i) = j_{\tau_\mathbf{5}}(i)$ for $i \in \{0, 1, 4\}$, and $j_\rho(i) = i + 3$ for $i \in \{2, 3\}$, and $j_\rho(i) = i - 3$ for $i \in \{5, 6\}$.

Each enrichment of the structure of graphs introduced in Definition 3 induces a notion of morphism that preserves such a structure, and an associated category. For instance, a morphism $h \colon (\tau, \mathsf{o}_\tau) \to (\sigma, \mathsf{o}_\sigma)$ where (τ, o_τ) and $(\sigma, \mathsf{o}_\sigma)$ are oriented graphs, is a graph morphism $h = (h_F, h_V) \colon \tau \to \sigma$ such that $\mathsf{o}_\sigma \circ h_F = \mathsf{o}_\tau$.

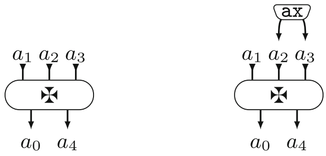


Fig. 1. An oriented labeled colored ordered corolla (on the left), and an oriented labeled colored ordered two-vertex graph (on the right).

Trees and Paths. An *unoriented path* on a graph τ is a finite and even sequence of flags $\varphi = (f_1, \ldots, f_{2n})$ for some $n \in \mathbb{N}$ such that, for all $1 \leqslant i \leqslant n$, $j_\tau(f_{2i-1}) = j_\tau(f_{2i})$ and (if $i \neq n$) $\partial_\tau(f_{2i}) = \partial_\tau(f_{2i+1})$. We say that φ is *between* $\partial_\tau(f_1)$ *and* $\partial_\tau(f_{2n})$ if $n > 0$ (and it is a *cycle* if moreover $\partial_\tau(f_1) = \partial_\tau(f_{2n})$), otherwise it is the *empty* (*unoriented*) *path*, which is between any vertex and itself; the *length* of φ is n. Two vertices are *connected* if there is an unoriented path between them.

Let τ be a graph: τ is *connected* if any vertices $v, v' \in V_\tau$ are connected; a *connected component* of τ is a maximal (with respect the inclusion of flags and vertices) connected sub-graph of τ; τ is *acyclic* (or a *forest*) if it has no cycles; τ is a *tree* if it is a connected forest.

A *rooted tree* τ is an oriented tree such that each vertex has exactly one output. Thus, τ has exactly one output tail: its boundary is called the *root* of τ.

Remark 1. Let τ and τ' be two rooted trees, and $h \colon \tau \to \tau'$ be an oriented graph morphism. As h_F preserves tails and orientation, h_V maps the root of τ to the root of τ'. Rooted trees and oriented graph morphisms form a category **RoTree**.

An *oriented path* on an oriented graph τ is an unoriented path $(f_1, \ldots, f_{2n})$ for some $n \in \mathbb{N}$ such that f_{2i-1} is output and f_{2i} is input for all $1 \leqslant i \leqslant n$. Such a path is said to be *from* $\partial_\tau(f_1)$ *to* $\partial_\tau(f_{2n})$ if $n > 0$, otherwise it is the *empty* (oriented) *path*, which is from any vertex to itself.

The set of oriented paths on an oriented tree is finite. As such, given a tree τ, we define its *reflexive-transitive closure*, or *free category*, $\tau^{\circlearrowleft}$ as the oriented graph with same vertices and same tails as τ, and with an edge from v to v' for any oriented path from v to v' in τ. The operator $(\cdot)^{\circlearrowleft}$ lifts to a functor from the category **RoTree** to the category of oriented graphs.

3 DiLL Proof-Structures

This section is the core of our paper. We define here proof-structures corresponding to some fragments or extension of LL: MELL, DiLL and DiLL_0. Full differential linear logic (DiLL) is an extension of MELL (with the same language as MELL) provided with both promotion rule (i.e. boxes) and co-structural rules (the duals of the structural rules handling the ?-modality) for the !-modality: DiLL_0 and MELL are particular subsystems of DiLL, respectively the promotion-free one (i.e. without boxes) and the one without co-structural rules. As the study of cut-elimination is left to future work, our interest for DiLL is just to have an unitary syntax subsuming both MELL and DiLL_0: this is why, unlike [22,26], our DiLL proof-structures are not allowed to contain a set of DiLL proof-structures inside a box.

Given a countably infinite set of propositional variables $X, Y, Z, \ldots$, (MELL) *formulas* (whose set is denoted by $\mathcal{F}_{\mathsf{MELL}}$) are defined by the following grammar:

$$A, B ::= X \mid X^{\perp} \mid \mathbf{1} \mid \perp \mid A \otimes B \mid A \parr B \mid !A \mid ?A$$

Linear negation $(\cdot)^{\perp}$ is defined via De Morgan laws $\mathbf{1}^{\perp} = \perp$, $(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$ and $(!A)^{\perp} = ?A^{\perp}$, so as to be involutive, i.e. $A^{\perp\perp} = A$ for any formula A. Variables and their negations are *atomic* formulas; $\otimes$ and $\parr$ (resp. ! and ?) are *multiplicative* (resp. *exponential*) *connectives*; $\mathbf{1}$ and $\perp$ are *multiplicative units.*

We equip an oriented graph with labels (specifying the type of the vertices, which is a MELL connective or unit), colors (specifying the type of the flags, which is a MELL formula), and a function that specifies the deepest box each flag or vertex is in; all of them are subject to compatibility conditions.

Definition 4 (module, proof-structure). *A (*DiLL*) module* $M = (|M|, \ell, \mathsf{o}, \mathsf{c}, <)$ *is a labeled (ℓ), oriented (o), colored (c), ordered ($<$) graph $|M|$ such that:*

- $\ell \colon V_{|M|} \to \{\mathtt{ax}, \mathtt{cut}, \mathbf{1}, \perp, \otimes, \parr, ?, !\}$ *associates with each vertex its* type*;*
- $\mathsf{c} \colon F_{|M|} \to \mathcal{F}_{\mathsf{MELL}}$ *associates with each flag its* type*;*
- $<$ *is a strict order on the flags of $|M|$ that is total on the tails of $|M|$ and on the inputs of each vertex labeled by $\parr$ or $\otimes$;*
- *for every vertex $v \in V_{|M|}$,*
  - *if $\ell(v) = \mathtt{cut}$, v has no output and exactly two inputs i_1 and i_2, such that $\mathsf{c}(i_1) = \mathsf{c}(i_2)^{\perp}$;*
  - *if $\ell(v) = \mathtt{ax}$, v has no inputs and exactly two outputs o_1 and o_2, such that $\mathsf{c}(o_1) = \mathsf{c}(o_2)^{\perp}$;*
  - *if $\ell(v) \in \{\mathbf{1}, \perp\}$, v has no inputs and only one output o, with $\mathsf{c}(o) = \ell(v)$;*
  - *if $\ell(v) \in \{\otimes, \parr\}$, v has exactly two inputs $i_1 < i_2$ and one output o, such that $\mathsf{c}(o) = \mathsf{c}(i_1)\, \ell(v)\, \mathsf{c}(i_2)$;*
  - *if $\ell(v) \in \{?, !\}$, v has exactly $n \geqslant 0$ inputs $i_1, \ldots, i_n$ and one output o, such that $\mathsf{c}(o) = \ell(v)\, \mathsf{c}(i_j)$ for all $1 \leqslant j \leqslant n$;*[2]

  In Fig. 2 we depicted the corollas associated with all types of vertices.

*A (*DiLL*)* proof-structure *is a tuple $R = (|R|, \mathcal{A}, \mathsf{box})$, where $|R| = (\|R\|, \ell_R, \mathsf{o}_R, \mathsf{c}_R, <_R)$ is a module with no input tails, called the* structured graph *of R (and $\|R\|$ is the* graph *of R). Moreover, the following hold:*

- *$\mathcal{A}$ is a rooted tree with no input tails, called the* box-tree *of R.*

[2] This implies that $\mathsf{c}(i_j) = \mathsf{c}(i_k)$ for all $1 \leqslant j, k \leqslant n$.

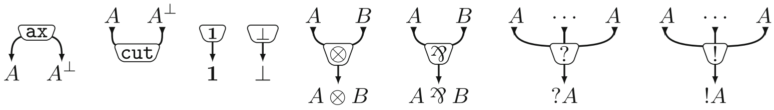


Fig. 2. DiLL cells, with their labels and their typed inputs and outputs.

– $\mathsf{box}\colon |R| \to \mathcal{A}^{\circlearrowleft}$ *is a morphism of oriented graphs,*[3] *the* box-function *of* R, *such that* box_F *induces a partial bijection from* $\bigcup_{v \in V_{\|R\|}, \ell(v)=!} \mathbf{in}_{|R|}(v)$ *to the set of input flags in* $\mathcal{A}$.[4] *Moreover, for any vertex* $v \in V_{\|R\|}$ *with* $f \in \mathbf{in}_{|R|}(v)$, *if* $\mathsf{box}_V(\partial_{\|R\|} \circ j_{\|R\|}(f)) \neq \mathsf{box}_V(\partial_{\|R\|}(f))$ *then* $\ell(v) \in \{!, ?\}$.[5]

A proof-structure is empty *(denoted by* ε*) if its graph is empty.*
A MELL proof-structure *is a proof-structure such that:*

– *for all* $v \in V_{\|R\|}$, *if* $\ell(v) = !$ *then* $\mathsf{card}(\mathbf{in}_{|R|}(v)) = 1$;
– box_F *induces a (total) bijection from* $\bigcup_{v \in V_{\|R\|}, \ell(v)=!} \mathbf{in}_{|R|}(v)$ *to the set of input flags in* $\mathcal{A}$.

A DiLL_0 proof-structure *is a proof-structure whose box-tree contains only its root in the set of vertices. A* MLL *(resp.* MLL^-*)* proof-structure *is a* DiLL_0 *proof-structure whose structured graph has no vertices of type* ! *or* ? *(resp.* $\mathbf{1}$, $\perp$, ! *or* ?*).*

Given a proof-structure $R = (|R|, \mathcal{A}, \mathsf{box})$, the output tails of $|R|$ are the *conclusions* of R. So, if f is the output of the root of $\mathcal{A}$, the pre-images $f_1, \ldots, f_n$ of f via box_F ordered according to $<_{|R|}$ form a finite sequence of the conclusions of R. The *type* of R is the list $(\mathsf{c}_{|R|}(f_1), \ldots, \mathsf{c}_{|R|}(f_n))$ of the types of the conclusions.

Borrowing the terminology of interaction nets [13,16], if R is a proof-structure, we say that the vertices of $|R|$ are the *cells* of R, the flags of $|R|$ are the *ports* of R.

---

[3] The structured graph $|R|$ of R is more structured (it is also labeled, colored, ordered) than an oriented graph such as $\mathcal{A}^{\circlearrowleft}$. When we talk of a morphism between two structured graphs where one of the two, say τ, is less structured than the other, say σ, we mean that τ must be only considered with the same structure as σ. Thus, in this case, box is a morphism from $(\|R\|, \mathsf{o}_R)$—discarding ℓ_R, c_R, $<_R$—to $\mathcal{A}^{\circlearrowleft}$.

[4] This means that for any input flag f' in $\mathcal{A}$ there is exactly one input f of some vertex of type ! in $|R|$ such that $\mathsf{box}_F(f) = f'$; but $\mathsf{box}_F(f)$ need not be an input flag in $\mathcal{A}$ for any input f of some vertex of type ! in $|R|$ (by definition of morphism, $\mathsf{box}_F(f)$ is necessarily an input flag in $\mathcal{A}^{\circlearrowleft}$). Intuitively, a vertex v of type ! represents a generalized co-contraction (in particular, a co-weakening if it has no inputs), and a box is associated with (and only with) each input f of v such that $\mathsf{box}_F(f)$ is an input flag in $\mathcal{A}$ (and not only in $\mathcal{A}^{\circlearrowleft}$): f represents the principal door (in the border) of such a box (note that for $f' \in F_{\|R\|}$, if $f' \neq f$ then $\mathsf{box}_F(f') \neq \mathsf{box}_F(f)$ and that $\mathsf{box}_V(\partial_{\|R\|} \circ j_{\|R\|}(f)) \neq \mathsf{box}_V(\partial_{\|R\|}(f))$ for such a f).

[5] Roughly, it says that the border of a box is made of (inputs of) vertices of type ! or ?.

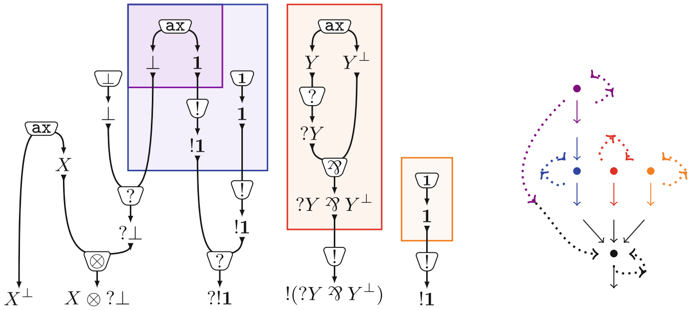


Fig. 3. A MELL proof-structure R with its box-tree $\mathcal{A}_R$. The dotted arrows represent the edges added to $\mathcal{A}_R$ by the reflexive-transitive closure $(\cdot)^{\circlearrowleft}$.

Remark 2 (box). In our syntax, boxes do not have explicit constructors or cells, hence boxes and depth of a proof structure are recovered in a non-inductive way.

Let $R = (|R|, \mathcal{A}_R, \mathsf{box}_R)$ be a proof-structure. With every flag f of $|R|$ such that $\mathsf{box}_{R_F}(f)$ is an input flag of $\mathcal{A}_R$[6] is associated a *box* B_f, that is the subgraph of $|R|$ (which is actually a proof-structure) made up of all the cells v (with their inputs and outputs) such that there is an oriented path on $\mathcal{A}_R$ from $\mathsf{box}_{R_V}(v)$ to $\mathsf{box}_{R_V}(\partial_{\|R\|} \circ j_{\|R\|}(f))$: a *conclusion* of such a box B_f associated with f is every output f' of a vertex v in B_f such that $\partial_{\|R\|} \circ j_{\|R\|}(f')$ is not in B_f. Summing up, every non-root vertex of $\mathcal{A}_R$ represents a box in R, and the root of $\mathcal{A}_R$ represents the parts of R outside all the boxes. The tree-structure of $\mathcal{A}_R$ expresses the nesting condition of boxes.

The *depth* of a cell v of R is the length of the oriented path in $\mathcal{A}_R$ from $\mathsf{box}_R(v)$ to the root of $\mathcal{A}_R$. The *depth* of R is the maximal depth of the cells of R.

Example 2. In Fig. 3 a MELL proof-structure R is depicted: the structured graph $|R|$ of R is on the left; the box-tree $\mathcal{A}_R$ of R is on the right. The box-function box_R is kept implicit by means of colors: the colored areas in $|R|$ represent boxes, and the same color is used on $\mathcal{A}_R$ to show where each box is mapped by box_R.

The proof-structures we have just defined are quite rigid: they depend on their carrier-sets of cells and wires. Nonetheless, a precise answer to the question "When two proof-structures can be considered equal?" requires a notion of isomorphism inherited by the notion of graph isomorphism.

---

[6] According to the constraints on box_R, this condition can be fulfilled only by inputs of a cell of type ! (a !-cell, for short) in $|R|$, and an input of a !-cell need not fulfill it; in particular, if R is a MELL proof-structure, then this condition is fulfilled by all and only the inputs of !-cells (and such an input is unique for any !-cell) in $|R|$; but if R is a DiLL_0 proof-structure, then this condition is not fulfilled by any flag in $|R|$ (since $\mathcal{A}_R$ has no inputs) and so box_R is a graph morphism associating the root of $\mathcal{A}_R$ with any vertex of $|R|$. Therefore, in a DiLL_0 proof-structure $\rho = (|\rho|, \mathcal{A}_\rho, \mathsf{box}_\rho)$, $\mathcal{A}_\rho$ and box_ρ do not induce any structure on $|\rho|$: ρ can be identified with $|\rho|$.

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,(\mathrm{exc}) \qquad \frac{}{\vdash A, A^{\perp}}\,(\mathrm{ax}) \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(\mathrm{cut}) \qquad \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\,(\mathrm{mix})$$

$$\frac{}{\vdash 1}\,(\mathbf{1}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,(\perp) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(\parr) \qquad \frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,(\otimes) \qquad \frac{}{\vdash}\,(\mathrm{emp})$$

Fig. 4. Sequent calculi for MLL^- (all rules but (mix), (emp), ($\mathbf{1}$), ($\perp$)), MLL (all rules).

Definition 5 (isomorphism of proof-structures). *Let* $R = (|R|, \mathcal{A}_R, \mathsf{box}_R)$ *and* $R' = (|R'|, \mathcal{A}_{R'}, \mathsf{box}_{R'})$ *be proof-structures, with* $|R| = (\|R\|, \ell_R, \mathsf{o}_R, \mathsf{c}_R, <_R)$ *and* $|R'| = (\|R'\|, \ell_{R'}, \mathsf{o}_{R'}, \mathsf{c}_{R'}, <_{R'})$. *An* isomorphism of proof-structures $f \colon R \simeq R'$ *is a couple* $f = (f_{|\cdot|}, f_{\mathsf{box}})$ *where:*

- $f_{|\cdot|} \colon |R| \to |R'|$ *is an isomorphism of the structured graphs of* R *and* R',
- $f_{\mathsf{box}} \colon \mathcal{A}_R \to \mathcal{A}_{R'}$ *is an isomorphism of the box-trees of* R *and* R',

such that the following diagram commutes

$$\begin{tikzcd} |R| \arrow[r, "\mathsf{box}_R"] \arrow[d, "f_{|\cdot|}"] & \mathcal{A}_R^{\circlearrowleft} \arrow[d, "f_{\mathsf{box}}^{\circlearrowleft}"] \\ |R'| \arrow[r, "\mathsf{box}_{R'}"] & \mathcal{A}_{R'}^{\circlearrowleft} \end{tikzcd}$$

Note that if R is isomorphic to a proof-structure R', and R is a MELL or DiLL_0 proof-structure, then R' is respectively a MELL or DiLL_0 proof-structure.

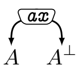
4 Sequent Calculi, Proof-Nets and Correctness for MLL

Every proof in the sequent calculus for LL can be translated in a proof-structure with the same conclusions. Figure 4 gives the rules of the sequent calculi for two multiplicative fragments of LL: MLL^- (without units) and MLL (with units and mix). A MLL^- (resp. MLL) *formula* is a MELL formula without exponential connectives and multiplicative units (resp. without exponential connectives). A *sequent* is a finite sequence of (MLL^- or MLL, depending on the context) formulas $A_1, \dots, A_n$. Capital Greek letters $\Gamma, \Delta, \dots$ range over sequents.

Definition 6 (translation, proof-net). *Let* $\mathsf{X} \in \{\mathsf{MLL}^-, \mathsf{MLL}\}$.

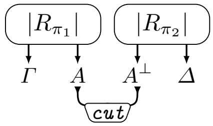
With any proof π *in the sequent calculus for* X *and conclusion* $\vdash \Gamma$ *is associated a* X *proof-structure* R_π *with type* Γ, *called the* translation of π, *defined by induction on the size of* π *as follows:*[7]

$$\pi = \frac{}{\vdash A, A^{\perp}}\,(\mathrm{ax}) \qquad \rightsquigarrow \qquad |R_\pi| =$$

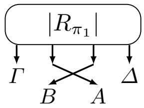


[7] We write only the graph $|R_\pi|$ of R_π, because its box-tree $\mathcal{A}_{R_\pi}$ and its box-function box_{R_π} are trivial (see Footnote 6).

$$\pi = \dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash A^{\perp}, \Delta\end{array}}{\vdash \Gamma, \Delta}\,(\mathrm{cut}) \qquad \rightsquigarrow \qquad |R_\pi| =$$

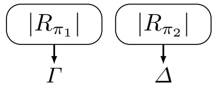


$$\pi = \dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A, B, \Delta\end{array}}{\vdash \Gamma, B, A, \Delta}\,(\mathrm{exc}) \qquad \rightsquigarrow \qquad |R_\pi| =$$

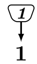


$$\pi = \dfrac{}{\vdash}\,(\mathrm{emp}) \qquad \rightsquigarrow \qquad |R_\pi| = |\varepsilon| \quad (\varepsilon \text{ is the empty proof-structure})$$

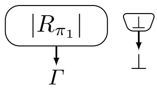
$$\pi = \dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Delta\end{array}}{\vdash \Gamma, \Delta}\,(\mathrm{mix}) \qquad \rightsquigarrow \qquad |R_\pi| =$$



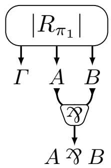
$$\pi = \dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1}) \qquad \rightsquigarrow \qquad |R_\pi| =$$



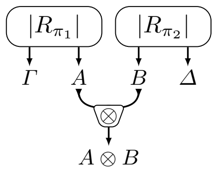
$$\pi = \dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, \bot}\,(\bot) \qquad \rightsquigarrow \qquad |R_\pi| =$$



$$\pi = \dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \parr B}\,(\parr) \qquad \rightsquigarrow \qquad |R_\pi| =$$



$$\pi = \dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash B, \Delta\end{array}}{\vdash \Gamma, A \otimes B, \Delta}\,(\otimes) \qquad \rightsquigarrow \qquad |R_\pi| =$$



A proof-structure R is a X proof-net *(or is* X sequentializable*) if $R = R_\pi$ (i.e. R is the translation of π) for some proof π in the* X *sequent calculus.*

The translation is not surjective (neither injective) over proof-structures, even when we restrict to MLL^- or MLL proof-structures. Purely graph-theoretical conditions, called *correctness criteria*, have been presented in order to characterize the set of sequentializable proof-structures. We give here two among the most celebrated of such correctness criteria, *switching acyclicity* and its variant *switching acyclicity and connectedness*, presented originally in [9]. We define them via the switching operation on a proof-structure R, which roughly consists of "detaching" all inputs but one of every vertex of type $\parr$ in R. This switching can be easily defined in our setting, thanks to modules and involutions.

Definition 7 (switching, correctness graph). *Let R be a* MLL *proof-structure, whose structured graph is $|R|$ and whose (unoriented) graph is $\|R\|$.*

A switching *of R is a function $\mathsf{s}_R \colon \{v \in V_{\|R\|} \mid \ell_{|R|}(v) = \parr\} \to F_{\|R\|}$ such that $\mathsf{s}_R(v)$ is one of the two inputs of v.*

*With every switching s_R of R is associated a s_R-*correctness graph *$\tau(\mathsf{s}_R)$, which is the (unoriented) graph obtained from $\|R\|$ by replacing the involution $j_{\|R\|} \colon F_{\|R\|} \to F_{\|R\|}$ for $\|R\|$ with $j_{\tau(\mathsf{s}_R)} \colon F_{\|R\|} \to F_{\|R\|}$ defined as follows:*

$$j_{\tau(\mathsf{s}_R)}(f) = \begin{cases} j_{\|R\|}(f) & \text{if } f \text{ is an input of a vertex } v \text{ such that either} \\ & \qquad \ell_{|R|}(v) = \parr \text{ and } \mathsf{s}_R(v) = f, \text{ or } \ell_{|R|}(v) \neq \parr; \\ f & \text{otherwise.} \end{cases}$$

A MLL *proof-structure R is* switching acyclic *(resp.* switching acyclic and connected*) if every correctness graph of R is acyclic (resp. acyclic and connected).*

Theorem 1 (Sequentialization, [9]).

1. *A* MLL^- *proof-structure is* MLL^- *sequentializable iff it is switching acyclic and connected.*
2. *A* MLL *proof-structure is* MLL *sequentializable iff it is switching acyclic.*

The definitions and the results of this section can be easily generalized to DiLL_0 and MELL proof-structures.

5 The Taylor Expansion

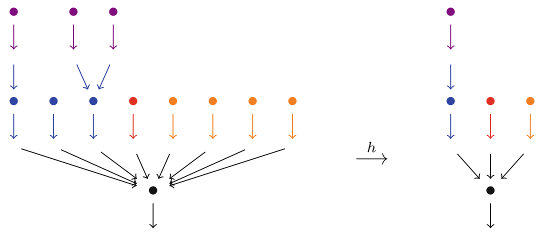
The *Taylor expansion* [11] of a MELL (or more in general a DiLL) proof-structure R is a (usually infinite) set of DiLL_0 proof-structures: roughly speaking, each element of the Taylor expansion of R is obtained from R by replacing each box B in R with n_B copies of its content (for some $n_B \in \mathbb{N}$), recursively on the depth of R. Note that n_B depends not only on B but also on which "copy" of all boxes containing B we are considering. Up to now (with the exception of [15]), the Taylor expansion of MELL proof-structure is defined globally and inductively [19,21]: with every MELL proof-structure R is directly associated its Taylor expansion (the whole set!) by induction on the depth of R.

We adopt an alternative non-inductive approach, which strongly refines [15]: the Taylor expansion is defined pointwise (see Example 3 and Fig. 5). Indeed, proof-structures have a tree structure that is made explicit through their box-function. The definition of the Taylor expansion of a proof-structure uses this tree structure: first we define how to "expand" a tree via the notion of thick subtree [5] (Definition 8), then we take all the expansions of the tree structure of a proof-structure and we *pull* them *back* to the underlying graphs (Definition 9), finally we forget the tree structures associated with them (Definition 10).

Definition 8 (thick subtree [5]). *Let τ be a rooted tree. A* thick subtree *of τ is a pair (σ, h) of a rooted tree σ and a graph morphism $h\colon \sigma \to \tau$.*

As in analysis, an addend of the Taylor expansion of an analytical function f is an approximant of f, here if $\mathcal{A}$ is the box-tree of a proof-structure R, a thick subtree of $\mathcal{A}$ is a sort of approximant of $\mathcal{A}$ taking recursively a number of copies (possibly 0) of each input of the vertices of $\mathcal{A}$, i.e. of each box of R.

Example 3. The following is (a graphical presentation of) a thick subtree (τ, h) of the box-tree $\mathcal{A}_R$ of the proof-structure R in Fig. 3, where the graph morphism $h\colon \tau \to \mathcal{A}_R$ is depicted chromatically (same color means same image via h).



Intuitively, τ is obtained from $\mathcal{A}_R$ by taking 3 copies of the blue box, 1 copy of the red box, 4 copies of the orange box; in the first (resp. second; third) copy of the blue box, 1 copy (resp. 0 copies; 2 copies) of the purple box has been taken.

The crucial point is to pull back the expansion of trees to proof-structures. In Appendix A we recall the definition of pullback in the category of graphs.

Definition 9 (proto-Taylor expansion). *Let $R = (|R|, \mathcal{A}_R, \mathsf{box}_R)$ be a proof-structure. The* proto-Taylor expansion *of R is set $\mathfrak{T}_R^{\mathsf{proto}}$ of thick subtrees of $\mathcal{A}_R$.*

*Let $t = (\tau_t, h_t) \in \mathfrak{T}_R^{\mathsf{proto}}$. The t-*expansion *of R is the pullback (R_t, p_t, p_R):*

$$\begin{tikzcd} R_t \arrow[r, "p_t"] \arrow[d, "p_R"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tau_t^{\circlearrowleft} \arrow[d, "h_t^{\circlearrowleft}"] \\ |R| \arrow[r, "\mathsf{box}_R"] & \mathcal{A}_R^{\circlearrowleft} \end{tikzcd}$$

computed in the category of graphs and graph morphisms.[8]

---

[8] This means that $\tau_t^{\circlearrowleft}$ and $\mathcal{A}_R^{\circlearrowleft}$ are considered as (unoriented) graphs, see also Footnote 3.

Given a proof-structure R and $t = (\tau_t, h_t) \in \mathfrak{T}_R^{\mathsf{proto}}$, the t-expansion (R_t, p_t, p_R) of R is a naked graph. In order for it to be lifted into a DiLL_0 proof-structure, we need to define more structure on it, using either t or R.

- Oriented, labeled and colored structures on the graph $|R|$ are defined through functions defined on the flags and vertices of $|R|$; hence, by precomposing with the graph morphism $p_R = (p_{R_F}, p_{R_V})\colon R_t \to |R|$, this transports to a structure of oriented labeled and colored graph on R_t;
- the order on the flags of R_t is defined as the order induced by their image in $|R|$: $f < f'$ if and only if $p_{R_F}(f) < p_{R_F}(f')$;
- let $[\tau_t]$ be the tree made up only of the root of τ_t and its output and let $\iota\colon \tau_t \to [\tau_t]$ be the graph morphism sending all the vertices of τ_t to the root of τ_t; $\iota^{\circlearrowleft}$ induces by post-composition a morphism $\overline{h_t} = \iota^{\circlearrowleft} \circ p_t \colon R_t \to [\tau_t]^{\circlearrowleft}$.

With its structure of oriented, labeled, ordered and colored graph, the triple $(R_t, [\tau_t], \overline{h_t})$ is a DiLL_0 proof-structure.

Definition 10 (Taylor expansion). *Let R be a proof-structure. The* Taylor expansion *of R is the set* $\mathfrak{T}_R = \{(R_t, [\tau_t], \overline{h_t}) \mid t = (\tau_t, h_t) \in \mathfrak{T}_R^{\mathsf{proto}}\}$.

An element of the Taylor expansion of a proof-structure is thus a DiLL_0 proof-structure. It has much less structure than the pullback (R_t, p_t, p_R), which defines a DiLL_0 proof-structure R_t coming with its projections $p_t\colon R_t \to \tau_t^{\circlearrowleft}$ and $p_R\colon R_t \to |R|$. In particular, a cell in R_t is labelled (through the projections) by the cell of $|R|$ and the branch of the box-tree of R it arose from. But $(R_t, [\tau_t], \overline{h_t})$ where R_t is without its projections p_t and p_R loses the correspondence with $R = (|R|, \mathcal{A}_R, \mathsf{box}_R)$ (see Fig. 5). Reconstructing such projections, starting only from an element of the Taylor expansion, can be seen as the core of the works on the injectivity of the Taylor expansion, see [7,15].

Remark 3. From the definition it follows that each element of the Taylor expansion of a proof-structure R has the *same conclusions* and the *same type* as R. More precisely, let R be a proof-structure and ρ be in the Taylor expansion of R: f

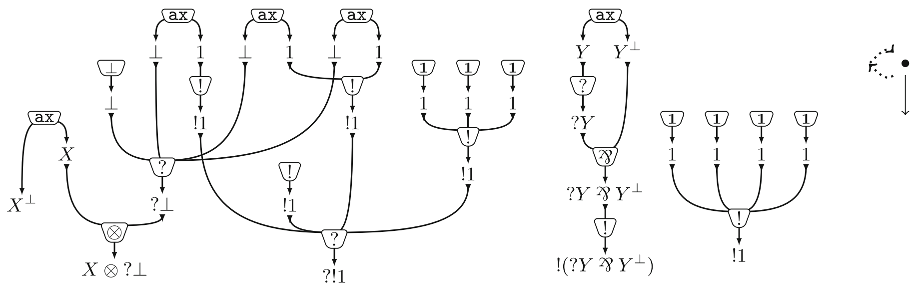


Fig. 5. The element of the Taylor expansion of the MELL proof-structure R in Fig. 3 obtained from the element of $\mathfrak{T}_R^{\mathsf{proto}}$ depicted in Example 3.

is a conclusion of ρ and an output of a cell v of ρ if and only if $p_{R_F}(f)$ is a conclusion of R and an output of a cell $p_{R_V}(v)$ of R. And $\mathsf{c}_{|\rho|}(f) = \mathsf{c}_{|R|}(p_{R_F}(f))$ (i.e. the type of f in ρ is the same as the type of $p_{R_F}(f)$ in R) and $\ell_{|\rho|}(v) = \ell_{|R|}(p_{R_V}(v))$ (i.e. the type of v in ρ is the same as the type of $p_{R_V}(v)$ in R).

Remark 4. One could go further and define an *incomplete Taylor expansion* of a proof-structure, where some boxes are expanded, but not all. This extension fits into this framework: the absence of boxes in an element $(R_t, [\tau_t], \overline{h_t})$ of the Taylor expansion owes only to the fact that $[\tau_t]$ is a root. By replacing this root by a rooted tree, we keep track of which boxes are expanded and which are not.

6 Conclusions

Cut-Elimination. The fact that we get a *canonical* (as explained in Sect. 1) representation of MELL proof-structures is not only an aesthetic matter: it has important consequences on the definition of cut elimination, because it avoids the presence of the bureaucratic commutative-steps. As a notable consequence, as proved in [2], the proof of strong normalization for MELL becomes quite elegant and much easier than with non-canonical MELL proof-structures [23]. The canonicity of our definition of MELL proof-structures paves the way to such a smooth cut elimination; we plan to work out this issue in future work.

Taylor Expansion and Relational Semantics. Relational semantics is the simplest denotational model of LL. It can be seen as a degenerate case of Girard's coherent semantics [14]: formulas are interpreted as sets and proof-structures as relations between them. It is well-known that, given a MELL proof-structure R, there is a correspondence between certain equivalence classes on its relational semantics $[\![R]\!]$ and the elements of its Taylor expansion $\mathfrak{T}_R$: in particular, two cut-free MELL proof-structures with atomic axioms have the same relational semantics if and only if they have the same Taylor expansion. This equivalence, which relates the syntactic notion of Taylor expansion to the semantic notion of relational model, holds only with a *canonical* representation MELL proof-structures, such as ours.

Mix and Forests. In an ongoing work, we are naturally led to consider several proof-structures at the same time and to "mix" them in a single proof-structure. Our definition of proof-structure (Definition 4) is perfectly suited for this purpose. Indeed, the definition of a box-tree lends itself to a generalization: considering not trees, but forests of boxes. The graph morphism condition implies, if its image is a forest, that the proof-structure contains several connected components; and each inverse image of a tree in the forest is actually a proof-structure.

So, by slightly generalizing the definition, we can consider a list of proof-structures as a whole proof-structure, while respecting their individuality, contrarily to all of them having the same image through box. This allows us to mimic the situation of the `mix` rule of sequent calculus: taking two proofs and considering it one can be done by merging two roots of a box-forest.

A Most General Taylor Expansion: Milner's Absorption. It is possible to go farther in the definition of the Taylor expansion and to specify a new box-tree that need not be trivial. This allows for instance to expand some boxes and not all; and even to expand partially a box: copying alongside a box its contents and (co-)contracting the box with the copies.

This is reminiscent of the π-calculus and of Mazza's parsimonious λ-calculus [18], where the exponentials verify the isomorphism $!A \simeq A \otimes !A$.

Other Boxes. Boxes for other connectives of linear logic have been considered in various works: quantifiers (both first-order and second order [14]), fix-points [20] and additives [14, 25]. The boxing tree represents a sequential structure that is added on top of a proof-structure. All these connective share in common to require such a sequentialization.

As the different kind of sequentialization need to merge correctly, we believe this approach to be adapted without problems to other kinds of boxes, paving the way to a unified notion of proof-structures for a richer system than MELL.

Technical Appendix

A Computing a Pullback in the Category of Graphs

The category of graphs has all pullbacks, a fact that we use extensively. We recall here all the definitions and facts that are packed in that affirmation.

Definition 11 (pullback). *Let $\mathcal{C}$ be a category. Let X, Y, and Z be three objects of $\mathcal{C}$ and $f : X \to Z$ and $g : Y \to Z$ be two arrows of $\mathcal{C}$.*

The pullback *of X and Y along f and g is the triple $(A, !_X, !_Y)$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "!_X"] \arrow[d, "!_Y"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

commutes and, for every other $(B, h : B \to X, k : B \to Y)$ making the same diagram commute, there exists a unique arrow $p : B \to A$ such that:

$$\begin{tikzcd} B \arrow[dr, dashed, "p"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\ & A \arrow[r, "!_X"] \arrow[d, "!_Y"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

It is unique (up to unique isomorphism), and it is customary to write $X \times_Z Y$ *a pullback of* X *and* Y *over* Z *(leaving* f *and* g *implicit) and a pullback diagram with a corner:*

$$\begin{tikzcd} A \arrow[r, "!_X"] \arrow[d, "!_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

All pullbacks exist in the category of graphs. Explicitely, let $\tau = (F_\tau, V_\tau, \partial_\tau, j_\tau)$, $\sigma = (F_\sigma, V_\sigma, \partial_\sigma, j_\sigma)$ and $\rho = (F_\rho, V_\rho, \partial_\rho, j_\rho)$ be three graphs and $g : \sigma \to \tau$, $h : \rho \to \tau$ be two graph morphisms. Consider the two sets

$$F = \{(f_1, f_2) \in F_\sigma \times F_\rho \mid g_F(f_1) = h_F(f_2)\}$$
$$V = \{(v_1, v_2) \in V_\sigma \times V_\rho \mid g_V(v_1) = h_V(v_2)\}$$

They are both equiped with two projections, which we will write $\pi^F_\sigma, \pi^F_\rho, \pi^V_\sigma, \pi^V_\rho$. Let $f \in F$.

$$\begin{aligned} g_V \circ \partial_\sigma \circ \pi^F_\sigma(f) &= \partial_\tau \circ g_F \circ \pi^F_\sigma(f), \text{ because } g \text{ is a graph morphism} \\ &= \partial_\tau \circ h_F \circ \pi^F_\rho(f), \text{ by definition of } F \\ &= h_V \circ \partial_\rho \circ \pi^F_\rho(f), \text{ because } h \text{ is a graph morphism} \end{aligned}$$

Hence, we can define $\partial : F \to V$ by $\partial(f) = (\partial_\sigma \circ \pi^F_\sigma(f), \partial_\rho \circ \pi^F_\rho(f))$. In the same way, we define $j : F \to F$ by $j(f) = (j_\sigma \circ \pi^F_\sigma(f), j_\rho \circ \pi^F_\rho(f))$, and check that it is an involution.

Hence $\sigma \times_\tau \rho = (F, V, \partial, j)$ is a graph and $\pi_\sigma = (\pi^F_\sigma, \pi^V_\sigma) : \sigma \times_\tau \rho \to \sigma$ and $\pi_\rho = (\pi^F_\rho, \pi^V_\rho) : \sigma \times_\tau \rho \to \rho$ are graph morphisms.

$$\begin{tikzcd} \sigma \times_\tau \rho \arrow[r, "\pi_\rho"] \arrow[d, "\pi_\sigma"'] & \rho \arrow[d, "h"] \\ \sigma \arrow[r, "g"] & \tau \end{tikzcd}$$

Consider now any $\mu = (F_\mu, V_\mu, \partial_\mu, j_\mu)$ such that the diagram

$$\begin{tikzcd} \mu \arrow[r, "p"] \arrow[d, "q"'] & \rho \arrow[d, "h"] \\ \sigma \arrow[r, "g"] & \tau \end{tikzcd}$$

commutes. For $f \in F_\mu$, let $r_F(f) = (p_F(f), q_F(f))$ and for $v \in V_\mu$, let $r_V(v) = (p_V(v), q_V(v))$. We check that it defines a graph morphism $r : \mu \to \sigma \times_\tau \rho$ and it factorises p and q.

References

1. Accattoli, B.: Compressing polarized boxes. In: 28th Annual Symposium on Logic in Computer Science (LICS 2013), pp. 428–437. IEEE Computer Society (2013). https://doi.org/10.1109/LICS.2013.49
2. Accattoli, B.: Linear logic and strong normalization. In: 24th International Conference on Rewriting Techniques and Applications (RTA 2013). LIPIcs, vol. 21, pp. 39–54. Schloss Dagstuhl (2013). https://doi.org/10.4230/LIPIcs.RTA.2013.39
3. Béchet, D.: Minimality of the correctness criterion for multiplicative proof nets. Math. Struct. Comput. Sci. **8**(6), 543–558 (1998)
4. Borisov, D.V., Manin, Y.I.: Generalized Operads and Their Inner Cohomomorphisms, pp. 247–308. Birkhäuser Basel, Basel (2008). https://doi.org/10.1007/978-3-7643-8608-5_4
5. Boudes, P.: Thick subtrees, games and experiments. In: Curien, P.-L. (ed.) TLCA 2009. LNCS, vol. 5608, pp. 65–79. Springer, Heidelberg (2009). https://doi.org/10.1007/978-3-642-02273-9_7
6. de Carvalho, D., Tortora de Falco, L.: The relational model is injective for multiplicative exponential linear logic (without weakenings). Ann. Pure Appl. Log. **163**(9), 1210–1236 (2012)
7. de Carvalho, D.: Taylor expansion in linear logic is invertible. Log. Methods Comput. Sci. **14**(4), 1–73 (2018). https://doi.org/10.23638/LMCS-14(4:21)2018
8. de Carvalho, D., Pagani, M., Tortora de Falco, L.: A semantic measure of the execution time in linear logic. Theor. Comput. Sci. **412**(20), 1884–1902 (2011). https://doi.org/10.1016/j.tcs.2010.12.017
9. Danos, V., Regnier, L.: The structure of multiplicatives. Arch. Math. Log. **28**(3), 181–203 (1989). https://doi.org/10.1007/BF01622878
10. Danos, V., Regnier, L.: Proof-nets and the Hilbert Space. In: Proceedings of the Workshop on Advances in Linear Logic, pp. 307–328. Cambridge University Press (1995)
11. Ehrhard, T., Regnier, L.: Uniformity and the Taylor expansion of ordinary lambda-terms. Theor. Comput. Sci. **403**(2–3), 347–372 (2008)
12. Ehrhard, T.: A new correctness criterion for MLL proof nets. In: Joint Meeting of the Twenty-Third Conference on Computer Science Logic and the Twenty-Ninth Symposium on Logic in Computer Science (CSL-LICS 2014), pp. 38:1–38:10. ACM (2014). https://doi.org/10.1145/2603088.2603125
13. Ehrhard, T., Regnier, L.: Differential interaction nets. Theor. Comput. Sci. **364**(2), 166–195 (2006). https://doi.org/10.1016/j.tcs.2006.08.003
14. Girard, J.Y.: Linear logic. Theor. Comput. Sci. **50**(1), 1–101 (1987). https://doi.org/10.1016/0304-3975(87)90045-4
15. Guerrieri, G., Pellissier, L., Tortora de Falco, L.: Computing connected proof(-structure)s from their Taylor expansion. In: 1st International Conference on Formal Structures for Computation and Deduction (FSCD 2016). LIPIcs, vol. 52, pp. 20:1–20:18. Schloss Dagstuhl (2016). https://doi.org/10.4230/LIPIcs.FSCD.2016.20
16. Lafont, Y.: Interaction nets. In: Seventeenth Annual ACM Symposium on Principles of Programming Languages (POPL 1990), pp. 95–108. ACM Press (1990). https://doi.org/10.1145/96709.96718
17. Laurent, O.: Polarized proof-nets and lambda-μ-calculus. Theor. Comput. Sci. **290**(1), 161–188 (2003). https://doi.org/10.1016/S0304-3975(01)00297-3
18. Mazza, D.: Simple parsimonious types and logarithmic space. In: 24th Annual Conference on Computer Science Logic (CSL 2015). LIPIcs, vol. 41, pp. 24–40. Schloss Dagstuhl (2015). https://doi.org/10.4230/LIPIcs.CSL.2015.24

19. Mazza, D., Pagani, M.: The separation theorem for differential interaction nets. In: Dershowitz, N., Voronkov, A. (eds.) LPAR 2007. LNCS (LNAI), vol. 4790, pp. 393–407. Springer, Heidelberg (2007). https://doi.org/10.1007/978-3-540-75560-9_29
20. Montelatici, R.: Polarized proof nets with cycles and fixpoints semantics. In: Hofmann, M. (ed.) TLCA 2003. LNCS, vol. 2701, pp. 256–270. Springer, Heidelberg (2003). https://doi.org/10.1007/3-540-44904-3_18
21. Pagani, M., Tasson, C.: The inverse Taylor expansion problem in linear logic. In: Proceedings of the 24th Annual Symposium on Logic in Computer Science (LICS 2009), pp. 222–231. IEEE Computer Society (2009). https://doi.org/10.1109/LICS.2009.35
22. Pagani, M.: The cut-elimination theorem for differential nets with promotion. In: Curien, P.-L. (ed.) TLCA 2009. LNCS, vol. 5608, pp. 219–233. Springer, Heidelberg (2009). https://doi.org/10.1007/978-3-642-02273-9_17
23. Pagani, M., Tortora de Falco, L.: Strong normalization property for second order linear logic. Theor. Comput. Sci. **411**(2), 410–444 (2010). https://doi.org/10.1016/j.tcs.2009.07.053
24. Solieri, M.: Geometry of resource interaction and Taylor-Ehrhard-Regnier expansion: a minimalist approach. Math. Struct. Comput. Sci. **28**(5), 667–709 (2018). https://doi.org/10.1017/S0960129516000311
25. Tortora de Falco, L.: The additive multiboxes. Ann. Pure Appl. Log. **120**(1–3), 65–102 (2003). https://doi.org/10.1016/S0168-0072(02)00042-8
26. Tranquilli, P.: Intuitionistic differential nets and lambda-calculus. Theor. Comput. Sci. **412**(20), 1979–1997 (2011). https://doi.org/10.1016/j.tcs.2010.12.022

Complexity Thresholds in Inclusion Logic

Miika Hannula[1(✉)] and Lauri Hella[2]

[1] University of Helsinki, Helsinki, Finland
miika.hannula@helsinki.fi
[2] Tampere University, Tampere, Finland
lauri.hella@tuni.fi

Abstract. Logics with team semantics provide alternative means for logical characterization of complexity classes. Both dependence and independence logic are known to capture non-deterministic polynomial time, and the frontiers of tractability in these logics are relatively well understood. Inclusion logic is similar to these team-based logical formalisms with the exception that it corresponds to deterministic polynomial time in ordered models. In this article we examine connections between syntactical fragments of inclusion logic and different complexity classes in terms of two computational problems: maximal subteam membership and the model checking problem for a fixed inclusion logic formula. We show that very simple quantifier-free formulae with one or two inclusion atoms generate instances of these problems that are complete for (non-deterministic) logarithmic space and polynomial time. Furthermore, we present a fragment of inclusion logic that captures non-deterministic logarithmic space in ordered models.

Keywords: Team semantics · Inclusion logic · Complexity · Consistent query answering

1 Introduction

In this article we study the computational complexity of inclusion logic. Inclusion logic was introduced by Galliani [9] as a variant of dependence logic, developed by Väänänen in 2007 [26]. Dependence logic is a logical formalism that extends first-order logic with novel atomic formulae $\mathrm{dep}(x_1, \ldots, x_n)$ expressing that a variable x_n depends on variables $x_1, \ldots, x_{n-1}$. One motivation behind dependence logic is to find a unifying logical framework for analyzing dependency notions from different contexts. Since its introduction, versions of dependence logic have been formulated and investigated in a variety of logical environments, including propositional logic [16,29,31], modal logic [7,27], probabilistic logics [5], and two-variable logics [22]. Recent research has also pursued connections and applications of dependence logic to fields such as database theory [14,15],

The first author was supported by grant 308712 of the Academy of Finland.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 301–322, 2019.
https://doi.org/10.1007/978-3-662-59533-6_19

Bayesian networks [4], and social choice theory [24]. A common notion underlying all these endeavours is that of team semantics. Team semantics, introduced by Hodges in [17], is a semantical framework where formulae are evaluated over multitudes instead of singletons of objects as in classical logics. Depending on the application domain these multitudes may then refer to assignment sets, probability distributions, or database tables, each having their characteristic versions of team semantics [5,15,26].

After the introduction of dependence logic Grädel and Väänänen observed that team semantics can be also used to create logics for independence [11]. This was followed by [9] in which Galliani investigated logical languages built upon multiple different dependency notions. Inspired by the inclusion dependencies of database theory, one of the logics introduced was inclusion logic that extends first-order logic with inclusion atoms. Given two sequences of variables $\overline{x}$ and $\overline{y}$ having same length, an inclusion atom $\overline{x} \subseteq \overline{y}$ expresses that the set of values of $\overline{x}$ is included in the set of values of $\overline{y}$. Inclusion logic was shown to be equi-expressive to positive greatest-fixed point logic in [10]. In contrast to dependence logic which is equivalent to existential second-order logic [26], and thus to non-deterministic polynomial time (**NP**), this finding established inclusion logic as the first team-based based logic for polynomial time (**P**). Our focus in this article is to pursue this connection further by investigating the complexity of quantifier-free inclusion logic in terms of two computational problems: maximal subteam membership and model checking problems. In particular, we identify complexity thresholds for these problems in terms of first-order definability, (non-deterministic) logarithmic space, and polynomial time.

The maximal subteam membership problem $\mathsf{MSM}(\phi)$ for a formula ϕ asks whether a given assignment is in the maximal subteam of a given team that satisfies ϕ. This problem is closely related to the notion of a repair of an inconsistent database [2]. A repair of a database instance I w.r.t. some set Σ of constraints is an instance J obtained by deleting and/or adding tuples from/to I such that J satisfies Σ, and the difference between I and J is minimal according to some measure. If only deletion of tuples is allowed, J is called a subset repair. It was observed in [3] that if Σ consists of inclusion dependencies, then for every I there exists a unique subset repair J of I; this was later generalized to arbitrary LAV tgds (local-as-view tuple generating dependencies) in [25].

The research on database repair has been mainly focused on two problems: consistent query answering and repair checking. In the former, given a query Q and a database instance I the problem is to compute the set of tuples that belong to $Q(J)$ for every repair J of I. The latter is the decision problem: is J a repair of I for two given database instances I and J. The complexity of these problems for various classes of dependencies and different types of repairs has been extensively studied in the literature; see e.g. [1,3,23,25]. In this setting, the maximal subteam membership problem can be seen as a variant of the repair checking problem: regarding a team as a (unirelational) database instance I and a formula ϕ of inclusion logic as a constraint, an assignment is a positive instance of $\mathsf{MSM}(\phi)$ just in case it is in the unique subset repair of I. Note however, that in $\mathsf{MSM}(\phi)$, the task is essentially to compute the maximal subteam from a given database instance I, instead of just checking that a given J is the unique subset repair of I. Note further, that using a single formula ϕ as a constraint is actually more general than using a (finite) set Σ of inclusion dependencies. Indeed, as ϕ we can take the conjunction of all inclusions in Σ. Furthermore, using disjunctions and quantifiers, we can form constraints not expressible in the usual formalism with a set of dependencies.

The complexity of model checking in team semantics has been studied in [6,21] for dependence and independence logics. For these logics increase in complexity arises particularly from disjunctions. For example, model checking for a disjunction of three (two, resp.) dependence atoms is complete for **NP** (**NL**, resp.), while a single dependence atom is first-order definable [21]. The results of this paper, in contrast, demonstrate that the complexity of inclusion logic formulae is particularly sensitive to conjunctions. We show that $\mathsf{MSM}(\phi)$ is complete for non-deterministic logarithmic space if ϕ is of the form $x \subseteq y$ or $x \subseteq y \wedge y \subseteq x$; for any other conjunction of (non-trivial) unary inclusion atoms $\mathsf{MSM}(\phi)$ is complete for polynomial time. This result gives a complete characterization of the maximal subteam membership problem for conjunctions of unary inclusion atoms. Based on it we also prove complexity results for model checking of quantifier-free inclusion logic formulae. For instance, for any non-trivial quantifier-free ϕ in which x, y, z do not occur, model checking of $x \subseteq y \vee \phi$ is **NL**-hard, while that of $(x \subseteq z \wedge y \subseteq z) \vee \phi$ is **P**-complete.

We conclude the paper by presenting a fragment of inclusion logic that captures **NL**. Analogous fragments have previously been established at least for dependence logic. By relating to the Horn fragment of existential second-order logic, Ebbing et al. define a fragment of dependence logic that corresponds to **P** [8]. The fragment presented in this paper is constructed by restricting occurrences of inclusion atoms and universal quantifiers, and the correspondence with **NL** is shown by using the well-known characterization of **NL** in terms of transitive closure logic [19,20].

2 Preliminaries

We generally use $x, y, z, \ldots$ for variables and $a, b, c, \ldots$ for elements of models. If $\overline{p}$ and $\overline{q}$ are two tuples, we write $\overline{pq}$ for the concatenation of $\overline{p}$ and $\overline{q}$.

Throughout the paper we assume that the reader has a basic familiarity of computational complexity. We use the notation **L**, **NL**, **P** and **NP** for the classes consisting of all problems computable in logarithmic space, non-deterministic logarithmic space, polynomial time and non-deterministic polynomial time, respectively.

2.1 Team Semantics

As is customary for logics in the team semantics setting, we assume that all formulae are in negation normal form (NNF). Thus, we give the syntax of first-order logic (FO) as follows:

$$\phi ::= t = t' \mid \neg t = t' \mid R\bar{t} \mid \neg R\bar{t} \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\phi \mid \forall x\phi,$$

where t and t' are terms and R is a relation symbol of the underlying vocabulary. For a first-order formula ϕ, we denote by $\mathsf{Fr}(\phi)$ the set of free variables of ϕ, defined in the usual way. The team semantics of FO is given in terms of the notion of a *team*. Let $\mathfrak{A}$ be a model with domain A. An *assignment* s of A is a function from a finite set of variables into A. We write $s(a/x)$ for the assignment that maps all variables according to s, except that it maps x to a. For an assignment $s = \{(x_i, a_i) \mid 1 \leq i \leq n\}$, we may use a shorthand $s = (a_1, \ldots, a_n)$ if the underlying ordering $(x_1, \ldots, x_n)$ of the domain is clear from the context. A *team* X of A with domain $\mathrm{dom}(X) = \{x_1, \ldots, x_n\}$ is a set of assignments from $\mathrm{dom}(X)$ into A. For $V \subseteq \mathrm{dom}(X)$, the *restriction* $X \upharpoonright V$ of a team X is defined as $\{s \upharpoonright V \mid s \in X\}$. If X is a team, $V \subseteq \mathrm{dom}(X)$, and $F : X \to \mathcal{P}(A) \setminus \{\emptyset\}$, then $X[F/x]$ denotes the team $\{s(a/x) \mid s \in X, a \in F(s)\}$. For a set B, $X[B/x]$ is the team $\{s(b/x) \mid s \in X, b \in B\}$. Also, if s is an assignment, then by $\mathfrak{A} \models_s \phi$ we refer to Tarski semantics.

Definition 1. *For a model* $\mathfrak{A}$*, a team* X *and a formula in* FO*, the satisfaction relation* $\mathfrak{A} \models_X \phi$ *is defined as follows:*

- $\mathfrak{A} \models_X \alpha$ *if* $\forall s \in X : \mathfrak{A} \models_s \alpha$*, when* α *is a literal,*
- $\mathfrak{A} \models_X \phi \wedge \psi$ *if* $\mathfrak{A} \models_X \phi$ *and* $\mathfrak{A} \models_X \psi$*,*
- $\mathfrak{A} \models_X \phi \vee \psi$ *if* $\mathfrak{A} \models_Y \phi$ *and* $\mathfrak{A} \models_Z \psi$ *for some* $Y, Z \subseteq X$ *such that* $Y \cup Z = X$*,*
- $\mathfrak{A} \models_X \exists x\phi$ *if* $\mathfrak{A} \models_{X[F/x]} \phi$ *for some* $F : X \to \mathcal{P}(A) \setminus \{\emptyset\}$*,*
- $\mathfrak{A} \models_X \forall x\phi$ *if* $\mathfrak{A} \models_{X[A/x]} \phi$*.*

If $\mathfrak{A} \models_X \phi$, then we say that $\mathfrak{A}$ and X *satisfy* ϕ. If ϕ does not contain quantifiers or symbols from the underlying vocabulary, in which case satisfaction of a formula does not depend on the model $\mathfrak{A}$, we say that X *satisfies* ϕ, written $X \models \phi$, if $\mathfrak{A} \models_X \phi$ for all models $\mathfrak{A}$ with a suitable domain (i.e., a domain that includes all the elements appearing in X). If ϕ is a sentence, that is, a formula without any free variables, then we say that $\mathfrak{A}$ *satisfies* ϕ, and write $\mathfrak{A} \models \phi$, if $\mathfrak{A} \models_{\{\emptyset\}} \phi$, where $\{\emptyset\}$ is the team that consists of the empty assignment $\emptyset$.

We say that two sentences ϕ and ψ are equivalent, written $\phi \equiv \psi$, if $\mathfrak{A} \models \phi \iff \mathfrak{A} \models \psi$ for all models $\mathfrak{A}$. For two logics $\mathcal{L}_1$ and $\mathcal{L}_2$, we write $\mathcal{L}_1 \leq \mathcal{L}_2$ if every $\mathcal{L}_1$-sentence is equivalent to some $\mathcal{L}_2$-sentence; the relations "$\equiv$" and "$<$" for $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined in terms of "$\leq$" in the standard way.

Satisfaction of a first-order formula reduces to Tarski semantics in the following way.

Proposition 2 (Flatness [26]**).** *For all models* $\mathfrak{A}$*, teams* X*, and formulae* $\phi \in$ FO*,*

$$\mathfrak{A} \models_X \phi \textit{ iff } \mathfrak{A} \models_s \phi \textit{ for all } s \in X.$$

A straightforward consequence is that first-order logic is downwards closed.

Corollary 3 (Downward Closure). *For all models* $\mathfrak{A}$*, teams* X*, and formulae* $\phi \in$ FO*,*

$$\textit{If } \mathfrak{A} \models_X \phi \textit{ and } Y \subseteq X\textit{, then } \mathfrak{A} \models_Y \phi.$$

2.2 Inclusion Logic

Inclusion logic (FO($\subseteq$)) is defined as the extension of FO by inclusion atoms.

Inclusion Atom. Let $\overline{x}$ and $\overline{y}$ be two tuples of variables of the same length. Then $\overline{x} \subseteq \overline{y}$ is an *inclusion atom* with the satisfaction relation:

$$\mathfrak{A} \models_X \overline{x} \subseteq \overline{y} \text{ if } \forall s \in X \exists s' \in X : s(\overline{x}) = s'(\overline{y}).$$

Inclusion logic is *local*, meaning that satisfaction of a formula depends only on its free variables. Furthermore, the expressive power of inclusion logic is restricted by its *union closure property* which states that satisfaction of a formula is preserved under taking arbitrary unions of teams.

Proposition 4 (Locality [9]). *Let $\mathfrak{A}$ be a model, X a team, $\phi \in \mathrm{FO}(\subseteq)$ a formula, and V a set of variables such that $\mathrm{Fr}(\phi) \subseteq V \subseteq \mathrm{dom}(X)$. Then*

$$\mathfrak{A} \models_X \phi \iff \mathfrak{A} \models_{X \upharpoonright V} \phi.$$

Proposition 5 (Union Closure [9]). *Let $\mathfrak{A}$ be a model, $\mathcal{X}$ a set of teams, and $\phi \in \mathrm{FO}(\subseteq)$ a formula. Then*

$$\forall X \in \mathcal{X} : \mathfrak{A} \models_X \phi \implies \mathfrak{A} \models_{\bigcup \mathcal{X}} \phi.$$

Note that union closure implies the *empty team property*, that is, $\mathfrak{A} \models_\emptyset \phi$ for all inclusion logic formulae ϕ.

The starting point for our investigations is the result by Galliani and Hella [10] characterizing the expressivity of inclusion logic in terms of positive greatest fixed point logic. The latter logic is obtained from greatest fixed-point logic (the dual of least fixed point logic) by restricting to formulae in which fixed point operators occur only positively, that is, within a scope of an even number of negations. In finite models this positive fragment captures the full fixed point logic (with both least and greatest fixed points), and hence it follows from the famous result of Immerman [18] and Vardi [28] that inclusion logic captures polynomial time in finite ordered models.

Theorem 6 ([10]). *Every inclusion logic sentence is equivalent to some positive greatest fixed point logic sentence, and vice versa.*

Theorem 7 ([10]). *A class $\mathcal{C}$ of finite ordered models is in* **P** *iff it can be defined in* FO($\subseteq$).

2.3 Transitive Closure Logic

In Sect. 6 we relate inclusion logic to transitive closure logic, and hence we next give a short introduction to the latter. A $2k$-ary relation R is said to be *transitive* if $(\overline{a}, \overline{b}) \in R$ and $(\overline{b}, \overline{c}) \in R$ imply $(\overline{a}, \overline{c}) \in R$ for k-tuples $\overline{a}, \overline{b}, \overline{c}$. The *transitive closure* of a $2k$-ary relation R, written $\mathrm{TC}(R)$, is defined as the intersection of all $2k$-ary relations $S \supseteq R$ that are transitive. The transitive closure of R can be alternatively defined as $R_\infty = \bigcup_{i=0}^{\infty} R_i$ for R_i defined recursively as follows: $R_0 = R$ and $R_{i+1} = R \circ R_i$ for $i > 0$; here $A \circ B$ denotes the composition of two relations A and B. Note that $(\overline{a}, \overline{b}) \in R_i$ if and only if there is an R-path of length $i+1$ from $\overline{a}$ to $\overline{b}$.

An assignment s, a model $\mathfrak{A}$, and a formula $\psi(\overline{x}, \overline{y}, \overline{z})$, where $\overline{x}$ and $\overline{y}$ are k-ary, give rise to a $2k$-ary relation defined as follows:

$$R_{\psi,\mathfrak{A},s} = \{\overline{a}\overline{b} \in M^{2k} \mid \mathfrak{A} \models \psi(\overline{a}, \overline{b}, s(\overline{z}))\}.$$

We can now define transitive closure logic. Given a term t, a model $\mathfrak{A}$, and an assignment s, we write $t^{\mathfrak{A},s}$ for the interpretation of t under $\mathfrak{A}, s$, defined in the usual way.

Definition 8 (Transitive Closure Logic). *Transitive closure logic (*TC*) is obtained by extending first-order logic with transitive closure formulae* $[\mathrm{TC}_{\overline{x},\overline{y}}\psi(\overline{x}, \overline{y}, \overline{z})](\overline{t}_0, \overline{t}_1)$ *where* $\overline{t}_0$ *and* $\overline{t}_1$ *are k-tuples of terms, and* $\psi(\overline{x}, \overline{y}, \overline{z})$ *is a formula where* $\overline{x}$ *and* $\overline{y}$ *are k-tuples of variables. The semantics of the transitive closure formula is defined as follows:*

$$\mathfrak{A} \models_s [\mathrm{TC}_{\overline{x},\overline{y}}\psi(\overline{x}, \overline{y}, \overline{z})](\overline{t}_0, \overline{t}_1) \text{ iff } (\overline{t}_0^{\mathfrak{A},s}, \overline{t}_1^{\mathfrak{A},s}) \in \mathrm{TC}(R_{\psi,\mathfrak{A},s}).$$

Thus, $[\mathrm{TC}_{\overline{x},\overline{y}}\psi(\overline{x}, \overline{y}, \overline{z})](\overline{t}_0, \overline{t}_1)$ is true if and only if there is a ψ-path from $\overline{t}_0$ to $\overline{t}_1$. It is well known that transitive closure logic captures non-deterministic logarithmic space in finite ordered models. In particular, this can be achieved by using only one application of the TC operator. We use below the notation min for the least element of the linear order, and $\overline{\min}$ for the tuple $(\min, \ldots, \min)$. Similarly, $\overline{\max}$ denotes the tuple $(\max, \ldots, \max)$, where max is the greatest element.

Theorem 9 ([19,20]). *A class $\mathcal{C}$ of finite ordered models is in* **NL** *iff it can be defined in* TC*. Furthermore, every* TC*-sentence is equivalent in finite ordered models to a sentence of the form*

$$[\mathrm{TC}_{\overline{x},\overline{y}}\alpha(\overline{x}, \overline{y})](\overline{\min}, \overline{\max})$$

where α is first-order.

3 Maximal Subteam Membership

In this section we define the maximal subteam membership problem. We first discuss some of its basic properties and then investigate its complexity over quantifier-free inclusion logic formulae.

3.1 Introduction

For a model $\mathfrak{A}$, a team X, and an inclusion logic formula ϕ, we define $\nu(\mathfrak{A}, X, \phi)$ as the unique subteam $Y \subseteq X$ such that $\mathfrak{A} \models_Y \phi$, and $\mathfrak{A} \not\models_Z \phi$ if $Y \subsetneq Z \subseteq X$. Due to the union closure property $\nu(\mathfrak{A}, X, \phi)$ always exists and it can be alternatively defined as the union of all subteams $Y \subseteq X$ such that $\mathfrak{A} \models_Y \phi$. If ϕ does not contain quantifiers or symbols from the underlying vocabulary, then we may write $\nu(X, \phi)$ instead of $\nu(\mathfrak{A}, X, \phi)$. The maximal subteam membership problem is now given as follows.

Definition 10. *Let* $\phi \in \mathrm{FO}(\subseteq)$. *Then* $\mathsf{MSM}(\phi)$ *is the problem of determining whether* $s \in \nu(\mathfrak{A}, X, \phi)$ *for a given model* $\mathfrak{A}$, *a team* X *and an assignment* $s \in X$.

Grädel proved that for any $\mathrm{FO}(\subseteq)$-formula ϕ, there is a formula ψ of positive greatest fixed point logic such that for any model $\mathfrak{A}$ and assignment s, $\mathfrak{A} \models_s \psi$ if and only if s is in the maximal team of $\mathfrak{A}$ satisfying ϕ (see Theorem 24 in [12]). An easy adaptation of the proof shows that $\nu(\mathfrak{A}, X, \phi)$ is also definable in positive greatest fixed point logic. Thus, it follows that every maximal subteam membership problem is polynomial time computable.

Lemma 11. *For every formula* $\phi \in \mathrm{FO}(\subseteq)$, $\mathsf{MSM}(\phi)$ *is in* $\mathbf{P}$.

In this section we will restrict our attention to maximal subteam problems for quantifier free formulae. Before proceeding to our findings we need to present some auxiliary concepts and results. The following lemmata will be useful below.

Lemma 12. *Let* $\alpha, \beta \in \mathrm{FO}(\subseteq)$, *and let* X *be a team of a model* $\mathfrak{A}$. *Then* $\nu(\mathfrak{A}, X, \alpha \vee \beta) = \nu(\mathfrak{A}, X, \alpha) \cup \nu(\mathfrak{A}, X, \beta)$.

Proof. For "$\subseteq$", note that by definition there are $Y, Z \subseteq X$ such that $Y \cup Z = \nu(\mathfrak{A}, X, \alpha \vee \beta)$, $Y \models \alpha$ and $Z \models \beta$, and hence $Y \subseteq \nu(\mathfrak{A}, X, \alpha)$ and $Z \subseteq \nu(\mathfrak{A}, X, \beta)$. For "$\supseteq$", note that $\nu(\mathfrak{A}, X, \alpha) \cup \nu(\mathfrak{A}, X, \beta)$ satisfies $\alpha \vee \beta$ so it must be contained by $\nu(\mathfrak{A}, X, \alpha \vee \beta)$. $\square$

As an easy corollary we obtain the following lemma.

Lemma 13. *Let* $\alpha, \beta \in \mathrm{FO}(\subseteq)$, *and assume that* $\mathsf{MSM}(\alpha)$ *and* $\mathsf{MSM}(\beta)$ *both belong to a complexity class* $C \in \{\mathbf{L}, \mathbf{NL}\}$. *Then* $\mathsf{MSM}(\alpha \vee \beta)$ *is in* C.

The maximal subteam problem for a single inclusion atom $\overline{x} \subseteq \overline{y}$ can be naturally represented using directed graphs. In this representation each assignment forms a vertex, and an assignment s has an outgoing edge to another assignment s' if $s(\overline{x}) = s'(\overline{y})$. Over finite teams an assignment then belongs to the maximal subteam for $\overline{x} \subseteq \overline{y}$ if and only if it is connected to a cycle.[1]

Lemma 14. *Let* $\mathfrak{A}$ *be a model,* X *a finite team,* $\overline{x}$ *and* $\overline{y}$ *two tuples of the same length from* $\mathrm{dom}(X)$, s *an assignment of* X, *and* α *a first-order formula. Let* $G = (X, E)$ *be a directed graph where* $(s, s') \in E$ *iff* $s(\overline{x}) = s'(\overline{y})$ *and* $\mathfrak{A} \models_{\{s,s'\}} \alpha$. *Then*

(a) $s \in \nu(\mathfrak{A}, X, \overline{x} \subseteq \overline{y} \wedge \alpha) \iff$ *G contains a path from* s *to a cycle,*

[1] We are grateful to Phokion Kolaitis, who pointed out this fact to the second author in a private discussion 2016.

(b) $s \in \nu(\mathfrak{A}, X, \overline{x} \subseteq \overline{y} \wedge \overline{y} \subseteq \overline{x} \wedge \alpha) \iff$ *G contains a path from one cycle to another via s.*

Proof. Assume for the first statement that $s \in \nu(\mathfrak{A}, X, \overline{x} \subseteq \overline{y} \wedge \alpha)$. Then there is a subteam $Y \subseteq X$ such that $s \in Y$ and $\mathfrak{A} \models_Y \overline{x} \subseteq \overline{y} \wedge \alpha$. Thus for each $s' \in Y$ there exists $s'' \in Y$ such that $s''(\overline{y}) = s'(\overline{x})$. Moreover, $\mathfrak{A} \models_{\{s',s''\}} \alpha$, whence $(s', s'') \in E$. In particular there is a non-ending path in G starting from s. Since X is finite, this path necessarily ends in a cycle. Conversely, assume G contains a path from s to a cycle. Then $\mathfrak{A} \models_Y \overline{x} \subseteq \overline{y} \wedge \alpha$ where Y consists of all assignments in the path and cycle. Hence, $s \in \nu(\mathfrak{A}, X, \overline{x} \subseteq \overline{y} \wedge \alpha)$.

For the second statement note that, by the argument above, $s \in \nu(\mathfrak{A}, X, \overline{y} \subseteq \overline{x} \wedge \alpha)$ if and only if $G' = (X, E^{-1})$ contains a path from s to a cycle. But clearly an E^{-1}-path from s to an E^{-1}-cycle is an E-path from an E-cycle to s. □

3.2 Complexity

Next we turn to the computational complexity of maximal subteam membership. In what follows, we give a complete characterization of the maximal subteam problem for arbitrary conjunctions and disjunctions of unary inclusion atoms. A *unary* inclusion atom is an atom of the form $x \subseteq y$ where x and y are single variables. The characterization is given in terms of inclusion graphs.

Definition 15. *Let Σ be a set of unary inclusion atoms over variables in V. Then the* inclusion graph *of Σ is defined as $G_\Sigma = (V, E)$ such that $(x, y) \in E$ iff $x \neq y$ and $x \subseteq y$ appears in Σ.*

We will now prove the following theorem.

Theorem 16. *Let Σ be a finite set of unary inclusion atoms, and let ϕ be the conjunction of all atoms in Σ. Then* $\mathsf{MSM}(\phi)$ *is*

(a) trivially true if G_Σ has no edges,
(b) **NL***-complete if G_Σ has an edge (x, y) and no other edges except possibly for its inverse (y, x),*
(c) **P***-complete otherwise.*

The first statement above follows from the observation that $\mathsf{MSM}(\phi)$ is true for all inputs if ϕ is a conjunction of trivial inclusion atoms $x \subseteq x$. The second statement is shown by relating to graph reachability. Given a directed graph $G = (V, E)$ and two vertices a and b, the problem REACH is to determine whether G contains a path from a to b. This problem is a well-known complete problem for **NL**.

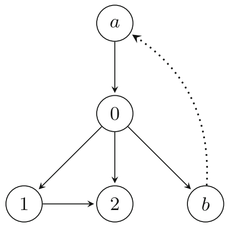
Lemma 17. $\mathsf{MSM}(x \subseteq y)$ *and* $\mathsf{MSM}(x \subseteq y \wedge y \subseteq x)$ *are* **NL***-complete.*

Proof. **Hardness.** We give a logarithmic space many-one reduction from REACH. Let $G = (V, E)$ be a directed graph, and let $a, b \in V$. W.l.o.g. we can assume that G has no cycles. Note that we obtain a directed acyclic graph by replacing nodes v with nodes (v, i) and edges (v, v') with edges $((v, i), (v', j))$, for $i, j \in \{1, \ldots, |V|\}$ such that $i < j$. Then $(b, |V|)$ is reachable from $(a, 1)$ in the acyclic graph if and only if b is reachable from a in the initial graph.

Define E' as the extension of E with an extra edge (b, a). Then b is reachable from a in G if and only if a belongs to a cycle in $G' = (V, E')$. We reduce from (G, a, b) to a team $X = \{s_{c,d} \mid (c, d) \in E'\}$ where $s_{u,v}$ maps (y, x) to (u, v) (see Fig. 1). By Lemma 14, b is reachable from a if and only if $s_{b,a} \in \nu(X, \phi)$, where ϕ is either $x \subseteq y$ or $x \subseteq y \wedge y \subseteq x$.

Membership. By Lemma 14 $\mathsf{MSM}(x \subseteq y)$ and $\mathsf{MSM}(x \subseteq y \wedge y \subseteq x)$ reduce to reachability variants that are clearly in **NL**. □

$$\phi = \begin{cases} x \subseteq y \\ x \subseteq y \wedge y \subseteq x \end{cases} \quad :$$



$\mapsto$

	x	y
$\underline{\boldsymbol{s_{b,a}}}$	a	b
$\boldsymbol{s_{a,0}}$	0	a
$s_{0,1}$	1	0
$s_{0,2}$	2	0
$\boldsymbol{s_{0,b}}$	b	0
$s_{1,2}$	2	1

Fig. 1. Reduction from REACH to $\mathsf{MSM}(\phi)$. The input assignment is underlined and the assignments written in bold form a subteam satisfying ϕ.

Next we turn to the third statement of Theorem 16. Recall that membership in **P** follows directly from Lemma 11. For **P**-hardness we reduce from the monotone circuit value problem (see, e.g., [30]). The proof essentially follows from the following lemma.

Lemma 18. *$\mathsf{MSM}(x \subseteq z \wedge y \subseteq z)$, $\mathsf{MSM}(x \subseteq y \wedge y \subseteq z)$, and $\mathsf{MSM}(x \subseteq y \wedge x \subseteq z)$ are* **P***-complete.*

Proof. Let ϕ be either $x \subseteq z \wedge y \subseteq z$, $x \subseteq y \wedge y \subseteq z$, or $x \subseteq y \wedge x \subseteq z$. We give a logarithmic-space many-one reduction to $\mathsf{MSM}(\phi)$ from the monotone circuit value problem (MCVP). Given a Boolean word $w \in \{\top, \bot\}^n$, and a Boolean circuit C with n inputs, one output, and gates with labels from {AND,OR}, this problem is to determine whether C outputs $\top$. If C outputs $\top$ on w, we say that it *accepts* w. W.l.o.g. we may assume that the in-degree of each AND and OR gate is 2. We annotate each input node by its corresponding input $\top$ or $\bot$, and each gate by some distinct number $i \in \mathbb{N} \setminus \{0\}$. Then each gate has two child nodes i_L, i_R that are either natural numbers or input values from $\{\top, \bot\}$. Next we construct a team X whose values consist of node annotations $i, \top, \bot$ and distinct copies c_i of AND gates i. The team X is constructed by the following rules (see Fig. 2):

– add $s_0 \colon (x, y, z) \mapsto (1, \top, \top)$ where 1 is the output gate,

– for AND gates i, add $s_{i,0}\colon (x,y,z) \mapsto (i_L, i, c_i)$, $s_{i,1}\colon (x,y,z) \mapsto (i_R, c_i, i)$, and $s_i : (x,y,z) \mapsto (c_i, \top, \top)$,
– for OR gates i, add $s_{i,L}\colon (x,y,z) \mapsto (i_L, i, i)$ and $s_{i,R}\colon (x,y,z) \mapsto (i_R, i, i)$.

We will show that C accepts w iff $s_0 \in \nu(X, \phi)$. For the only-if direction we actually show a slightly stronger claim. That is, we show that $s_0 \in \nu(X, \phi)$ is implied even if ϕ is the conjunction of all unary inclusion atoms between x, y, z.

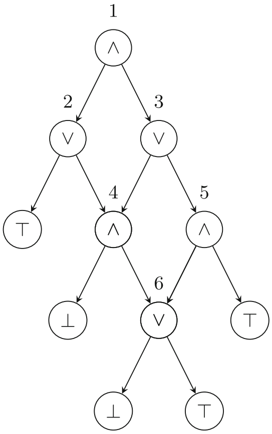
Assume first that C accepts w. We show how to build a subteam $Y \subseteq X$ such that it includes s_0 and satisfies all unary inclusion atoms between x, y, z. First construct a subcircuit C' of C recursively as follows: add the output gate $\top$ to C'; for each added AND gate i, add both child nodes of i; for each added OR gate i, add a child node of i that is evaluated true under w. In other words, C' is a set of paths from the output gate to the input gates that witnesses the assumption that C accepts w. The team Y will now list the auxiliary values c_i and the gates of C' in each column x, y, z. We construct Y by the following rules:

– add s_0,
– for AND gates i in C', add $s_{i,0}$, $s_{i,1}$, and s_i,
– for OR gates i in C', add $s_{i,P}$ iff i_P is in C', for $P = L, R$.

First observe that Y is formed symmetrically in terms of y and z, and thus these columns share the same values. Consider then the symmetric difference between values in columns x and y. Initially, for $Y = \{s_0\}$, this set is $\{1, \top\}$. An inductive argument now shows that, following the partial ordering induced from C', an application of a construction rule to a gate i of C' modifies the symmetric difference by removing i (and also $\top$ if $\top$ is a child of i) and adding any child node of i that is a gate in C'. In the end the symmetric difference is the empty set, and thus we conclude that Y satisfies all unary inclusion atoms between x, y, z.

Vice versa, consider the standard semantic game between Player I and Player II on the given circuit C and input word w. This game starts from the output gate 1, and at each AND (OR, resp.) gate i Player I (Player II, resp.) selects the next node from its two child nodes i_L and i_R. Player II wins iff the game ends at an input node that is true. By the assumption that $s_0 \in \nu(X, \phi)$ we find a team Y that contains s_0 and satisfies ϕ. Note that Y cannot contain any assignment that maps x to $\bot$. For showing that C accepts w it thus suffices to show that Player II has a strategy which imposes the following restriction: for each visited node annotated by i, we have $s(x) = i$ for some $s \in Y$. At start this holds by the assumption that $s_0 \in Y$. Assume that i is any gate with $s \in Y$ such that $s(x) = i$. If ϕ is $x \subseteq z \wedge y \subseteq z$, we have two cases. If i is an OR gate then we find s' from Y with $s'(y) = s'(z) = i$. Then the strategy of Player II is to select the gate $s'(x)$ as her next step. If i is an AND gate, an application of $x \subseteq z$ gives s' from Y with $s'(z) = i$. Then $s'(y) = c_i$, which means that further application of $y \subseteq z$ yields s'' from Y with $s''(z) = c_i$ and hence $s''(y) = i$. Now $\{s'(x), s''(x)\} = \{i_L, i_R\}$, and thus the claim holds for either selection by Player I. The induction step is analogous for the cases where ϕ is $x \subseteq y \wedge y \subseteq z$ or $x \subseteq y \wedge x \subseteq z$. This concludes the proof. $\square$

$$\phi = \begin{cases} x \subseteq z \wedge y \subseteq z \\ x \subseteq y \wedge y \subseteq z \\ x \subseteq y \wedge x \subseteq z \end{cases} :$$



$\mapsto$

	x	y	z
$\boldsymbol{s_0}$	1	$\top$	$\top$
$\boldsymbol{s_{1,0}}$	2	1	c_1
$\boldsymbol{s_{1,1}}$	3	c_1	1
$\boldsymbol{s_{2,L}}$	$\top$	2	2
$s_{2,R}$	2	2	2
$s_{3,L}$	4	3	3
$\boldsymbol{s_{3,R}}$	5	3	3
$s_{4,0}$	$\bot$	4	c_4
$s_{4,1}$	6	c_4	4
$\boldsymbol{s_{5,0}}$	6	5	c_5
$\boldsymbol{s_{5,1}}$	$\top$	c_5	5
$s_{6,L}$	$\bot$	6	6
$\boldsymbol{s_{6,R}}$	$\top$	6	6
$\boldsymbol{s_1}$	c_1	$\top$	$\top$
s_4	c_4	$\top$	$\top$
$\boldsymbol{s_5}$	c_5	$\top$	$\top$

Fig. 2. MCVP and $\mathsf{MSM}(\phi)$

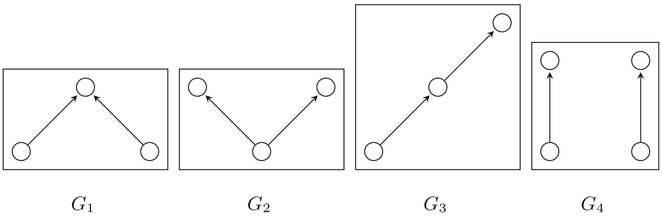


Fig. 3. Subgraphs of G_Σ

The third statement of Theorem 16 now follows. Any G_Σ not covered by the previous lemma has a subgraph of a form depicted in Fig. 3. Of these G_1–G_3 were considered above, and the reduction for G_4 is essentially identical to that for G_1; take a new variable for the new target node and insert values identical to those of z. Additionally, for each node in G_Σ but not in G_i take a copy of any column in the team. That this suffices follows from the arguments of the previous lemmata; in particular, from the fact that any true MCVP instance generates a subteam that satisfies all possible unary inclusion atoms between variables.

Considering disjunctions, observe that MSM over a disjunction of unary inclusion atoms is either trivially true or **NL**-complete. For membership in **NL**, see Lemma 13. For **NL**-hardness of $\mathsf{MSM}(x \subseteq y \vee y \subseteq x)$ we use exactly the same reduction from REACH as in Lemma 17: indeed, by Lemma 12 $s_{b,a} \in \nu(X, x \subseteq y \vee y \subseteq x)$ if and only if $s_{b,a} \in \nu(X, x \subseteq y)$ or $s_{b,a} \in \nu(X, y \subseteq x)$. The first condition holds if and only if a belongs to a cycle in $G' = (V, E')$, which implies that b is reachable from a in G; and the second condition holds if and only if b belongs to a cycle in the graph obtained by inverting the edges of G', which likewise implies that b is reachable from a in G. Provided that another non-trivial inclusion atom $u \subseteq v$ appears in the disjunction, then $\{u, v\} \not\subseteq \{x, y\}$ and the values for u, v can be defined in such a way that the maximal subteam for $u \subseteq v$ is empty.

Corollary 19. *Let Σ be a finite set of unary inclusion atoms, and let ϕ be the disjunction of all atoms in Σ. Then* $\mathsf{MSM}(\phi)$ *is*

(a) trivially true if Σ contains a trivial inclusion atom,
(b) **NL***-complete otherwise.*

Note that the results of this section generalize to inclusion atoms of higher arity, obtained by replacing variables x with tuples $\overline{x}$ such that all pairs of distinct tuples have no common variables. More complex cases arise if the tuples are allowed to overlap. In the full version of the paper [13] we also consider maximal subteam membership over input teams in which the inclusion atoms reference a key. In such cases the complexity of maximal subteam membership collapses to lower levels. For instance, $\mathsf{MSM}(x \subseteq z)$ ($\mathsf{MSM}(x \subseteq z \wedge y \subseteq z)$, resp.) is **L**-complete (**NL**-complete, resp.) over teams in which z is a key.

4 Consistent Query Answering

The maximal subteam membership problem has a close connection to database repairing which provides a framework for managing inconsistency in databases. An inconsistent database is a database that does not satisfy all the integrity constraints that it is supposed to satisfy. Inconsistency may arise, e.g., from data integration where the task is to bring together data from different sources. Often in practise inconsistency is handled through data cleaning which is the process of identifying and correcting inaccurate data records from databases. An inherent limitation of this approach is its inability to avoid arbitrary choices as consistency can usually be restored in a number of ways. The approach of database repair is to tolerate inconsistencies in databases and investigate reliable answers to queries.

A *database* is an interpretation of a relational vocabulary $\sigma = \{R_1, \ldots, R_n\}$ in which each R_i is associated with an arity $\#R_i$. Given a (finite) set Σ of integrity constraints, a database D is called *inconsistent* (w.r.t. to Σ) if $D \not\models \Sigma$, and *consistent* otherwise. Given a partial order $\leq$ on databases over a fixed σ, and a set Σ of integrity constraints, a *repair* of an inconsistent database I is a database D such that it is consistent and all $D' < D$ are inconsistent. The database D is called a $\oplus$-*repair* if the partial order is defined in terms of symmetric difference: $D \leq D'$ if $D \oplus I \subseteq D' \oplus I$. If additionally D is a subset (superset, resp.) of I, then D is called a *subset-repair* (*superset-repair*, resp.). An *answer* to a first-order query $q = \psi(x_1, \ldots, x_n)$ on a database D is any $(a_1, \ldots, a_n)$ such that D satisfies $\psi(a_1, \ldots, a_n)$, and a *consistent answer* on an inconsistent database I is any value $(a_1, \ldots, a_n)$ such that each repair D of I satisfies $\psi(a_1, \ldots, a_n)$.

Let $* \in \{\oplus$, subset, superset$\}$ and let Σ be a set of integrity constraints. The *$*$-repair checking problem w.r.t. Σ* ($*$-$\mathsf{RC}(\Sigma)$) is to determine, given two databases D and I, whether D is a $*$-repair of I. Let also q be a Boolean query. The $*$*-consistent query answering problem w.r.t.* Σ *and* q ($*$-$\mathsf{CQA}(\Sigma, q)$) is to determine, given an inconsistent database I, whether q is true in every $*$-repair of I. *LAV tgds* are first-order formulae of the form

$$\phi = \forall \overline{x}(\psi(\overline{x}) \rightarrow \exists \overline{y}\theta(\overline{x}, \overline{y}))$$

where ψ is a single relational atom and θ is a conjunction of relational atoms, and each variable from $\overline{x}$ occurs in ψ (but not necessarily in θ). *Inclusion dependencies* are the special case of LAV tgds in which also θ is a single relational atom, and no variable occupies two positions in one relational atom. An inclusion dependency is called *unary* if a single variable from $\overline{x}$ appears in exactly one relation position of θ, and it is called *unirelational* if ψ and θ contain the same relation symbol. Note that unary inclusion atoms correspond to unary unirelational inclusion dependencies.

Example 20. Figure 4 depicts a database D consisting of two ternary relations TEACHING and EMPLOYEE. Let Σ consist of a single unary inclusion dependency which states that each `lecturer` in TEACHER is a `name` in EMPLOYEE. The database is inconsistent because Bob is not listed in EMPLOYEE, and it has a unique subset-repair in which (Bob, Mechanics, Spring 2019) is removed from TEACHING. A superset-repair is obtained by adding (Bob, a, b) to EMPLOYEE where a and b are any data values. Such repairs are also $\oplus$-repairs. Consider a query q that returns lecturers located at the Math department. Regardless of the repair type this query has only one consistent answer: Alice.

Consistent query answering and repair checking are known to be tractable for LAV tgds. A *conjunctive query* is a first-order formula of the form $\exists \overline{x}\theta(\overline{x})$ where θ is a conjunction of relational atoms.

Theorem 21 ([25]). *Let* $* \in \{\oplus$, *subset, superset*$\}$, *let* Σ *be a set of LAV tgds, and let* q *be a conjunctive query. The* $*$*-repair checking problem w.r.t.* Σ *and the* $*$*-consistent query answering problem w.r.t.* Σ *and* q *are both solvable in polynomial time.*

TEACHING

lecturer	course	semester
Alice	Analysis	Spring 2019
Alice	Analysis	Fall 2019
Bob	Mechanics	Spring 2019
Carol	Algorithms	Spring 2019

EMPLOYEE

name	department	room
Alice	Math	A321
Carol	CS	B127
Carol	CS	B121

Fig. 4. Database D

Furthermore, it is known that so-called weakly acyclic collections of LAV tgds enjoy subset-repair checking in logarithmic space [1]. Nevertheless, it seems not much attention in general has been devoted to complexity thresholds within polynomial time. Our results can thus be seen as steps toward this direction as the trichotomy in Theorem 16 extends to repair checking and consistent query answering. Let IC be a collection of finite sets of integrity constraints and let C be a complexity class. We say that the $*$-consistent query answering problem is C-complete for IC if for all $\Sigma \in IC$, $*$-$\mathsf{CQA}(\Sigma, q)$ is in C for all Boolean conjunctive queries q and C-complete for some such q.

Theorem 22. *Let $* \in \{\oplus, subset\}$. The subset-repair checking problem and the $*$-consistent query answering problem for finite sets Σ of unary unirelational inclusion dependencies are*

(a) first-order definable if G_Σ has no edges,
(b) **NL***-complete if G_Σ has an edge (x, y) and no other edges except possibly for its inverse (y, x),*
(c) **P***-complete otherwise.*

Proof. Since **NL** and **P** are closed under complement, we may consider the complement of subset-repair checking. For the upper bounds note that D is a repair of I if and only if D satisfies Σ (a first-order property) and no tuple in $I \setminus D$ is in the unique subset-repair of I. That Σ has a unique subset-repair follows already from the union closure property of inclusion logic (Proposition 5) (or that of LAV tgds [25]). For the lower bounds note that in our reductions $s \in \nu(X, \phi)$ if and only if $\nu(X, \phi) \neq \emptyset$.[2]

Consider then subset-consistent query answering over a Boolean conjunctive query $q = \exists \overline{x}(R_{i_1}(\overline{x}_1) \wedge \ldots \wedge R_{i_n}(\overline{x}_n))$ where $\overline{x}_1, \ldots, \overline{x}_n$ are subsequences of $\overline{x}$ (note that q may contain multiple relation symbols even though all constraints are unirelational). Considering first the upper bounds, in case (a) q itself may be used for the first-order definition, and in cases (b) and (c) evaluation of the relational atoms $R_{i_1}(\overline{x}_i)$ may be reduced to the maximal subteam membership problem. For the lower bounds we may simply use atomic queries that describe the input assignment for the maximal subteam membership problem. For instance, in case (b) subset-$\mathsf{CQA}(\Sigma, q)$ is **NL**-hard for $q = R(a, b)$ where a and b are constant values from the reduction in Lemma 17. That the result holds also for $\oplus$-consistent query answering follows from the fact that each set of inclusion dependencies Σ has a unique subset repair which is also the unique universal subset repair and the unique universal $\oplus$-repair [25]. A database U is a *universal $*$-repair* of an inconsistent database I if for each conjunctive query q, a tuple is a consistent answer to q on I if and only if it is an answer to q on U and contains only values that appear in I. That is, it only suffices to consult the universal repair for consistent answers. □

[2] In point of fact, the reduction of Lemma 18 requires slight modification: remove assignments $(c_i, \top, \top)$ and add assignments (c_i, j, k) for each assignment $(i, j, k) \in X$ where i is an AND gate.

5 Model Checking

In this section we discuss the model checking problem for quantifier-free inclusion logic formulae. It turns out that the results of the previous section are now easily adaptable. As above, we herein restrict attention to quantifier-free formulae.

Definition 23. *Let $\phi \in \mathrm{FO}(\subseteq)$. Then $\mathsf{MC}(\phi)$ is the problem of determining whether $\mathfrak{A} \models_X \phi$, given a model $\mathfrak{A}$ and a team X.*

Hardness results for model checking can now be obtained by relating to maximal subteam membership.

Lemma 24. *Let $\alpha, \beta \in \mathrm{FO}(\subseteq)$ be such that*

(i) $\mathsf{Fr}(\alpha) \cap \mathsf{Fr}(\beta) = \emptyset$,
(ii) $\mathsf{MSM}(\alpha)$ *is C-hard for $C \in \{\mathbf{L}, \mathbf{NL}, \mathbf{P}\}$, and*
(iii) There is a team Y of $\mathrm{dom}(\mathfrak{A})$ *with domain* $\mathsf{Fr}(\beta)$ *such that* $\emptyset \neq \nu(\mathfrak{A}, Y, \beta) \subsetneq Y$.

Then $\mathsf{MC}(\alpha \vee \beta)$ is C-hard.

Proof. Let $(\mathfrak{A}, X, s)$ be an instance of $\mathsf{MSM}(\alpha)$, that is, $\mathfrak{A}$ is a model, X a team over $\mathsf{Fr}(\alpha)$ and $s \in X$. It suffices to define a first-order reduction from $(\mathfrak{A}, X, s)$ to a team X' over $\mathsf{Fr}(\alpha) \cup \mathsf{Fr}(\beta)$ such that $s \in \nu(\mathfrak{A}, X, \alpha)$ iff $\mathfrak{A} \models_{X'} \alpha \vee \beta$. Let $Z_0 := \nu(\mathfrak{A}, Y, \beta)$ and $Z_1 := Y \setminus Z_0$. Note that by condition (i), the union of any $t \in X$ and $t' \in Y$ is an assignment over $\mathsf{Fr}(\alpha) \cup \mathsf{Fr}(\beta)$. We define

$$X' := \{s \cup t' \mid t' \in Z_1\} \cup \{t \cup t' \mid t \in X \setminus \{s\}, t' \in Z_0\}.$$

Since Z_0 and Z_1 are fixed, X' is first-order definable from $(\mathfrak{A}, X, s)$. By Locality (Proposition 4), we have $\nu(\mathfrak{A}, X', \alpha) \upharpoonright \mathsf{Fr}(\alpha) = \nu(\mathfrak{A}, X' \upharpoonright \mathsf{Fr}(\alpha), \alpha) = \nu(\mathfrak{A}, X, \alpha)$, and similarly $\nu(\mathfrak{A}, X', \beta) \upharpoonright \mathsf{Fr}(\beta) = \nu(\mathfrak{A}, Y, \beta) = Z_0$. Hence, it follows from Lemma 12 that $\mathfrak{A} \models_{X'} \alpha \vee \beta$ iff for all $t \cup t' \in X' : t \in \nu(\mathfrak{A}, X, \alpha) \vee t' \in \nu(\mathfrak{A}, Y, \beta)$ iff $s \in \nu(\mathfrak{A}, X, \alpha)$. ☐

Note that $\mathfrak{A} \models_X \phi$ if and only if $\nu(\mathfrak{A}, X, \phi) = X$ over inclusion logic formulae ϕ. Hence, model checking can be reduced to maximal subteam membership tests over each individual assignment of a team. In particular, this means that model checking is at most as hard as maximal subteam membership; in some cases, as illustrated in Proposition 26(a), it is strictly less hard. Observe that we may omit the case $C = \mathbf{P}$ because $\mathsf{MC}(\alpha)$ is in $\mathbf{P}$ for any $\alpha \in \mathrm{FO}(\subseteq)$ (Theorem 7).

Lemma 25. *Let $\alpha \in \mathrm{FO}(\subseteq)$ be such that $\mathsf{MSM}(\alpha)$ is in $C \in \{\mathbf{L}, \mathbf{NL}\}$. Then $\mathsf{MC}(\alpha)$ is in C.*

By Lemmata 13, 24, 25, Theorem 7, and the results of the previous section, the computational complexity of model checking for various quantifier-free inclusion formulae directly follows. The following proposition illustrates some examples. Note that the semantics of the inclusion atom is clearly first-order expressible, and the same applies to any conjunction of inclusion atoms.

Proposition 26.

(a) $\mathsf{MC}(x \subseteq y)$ *and* $\mathsf{MC}(x \subseteq y \wedge u \subseteq v)$ *are first-order definable.*
(b) $\mathsf{MC}(x \subseteq y \vee u \subseteq v)$ *and* $\mathsf{MC}(x \subseteq y \vee u = v)$ *are* **NL**-*complete.*
(c) $\mathsf{MC}((x \subseteq z \wedge y \subseteq z) \vee u \subseteq v)$ *and* $\mathsf{MC}((x \subseteq z \wedge y \subseteq z) \vee u = v)$ *are* **P**-*complete.*

6 An NL Fragment of Inclusion Logic

Our aim in this section is to find a natural fragment of inclusion logic that captures the complexity class **NL** over ordered finite models. Our approach is to consider preservation of **NL**-computability under the standard logical operators of FO($\subseteq$). By Lemma 13, we already know that **NL**-computability of maximal subteam membership is preserved under disjunctions. However, Theorem 16 shows that conjunction can increase the complexity of the maximal subteam membership problem from **NL** to **P**-complete, and by Proposition 26, combining a conjunction with a disjunction leads to **P**-complete model checking problems. Thus conjunction cannot be used freely in the fragment we aim for.

The following proposition shows that a single universal quantifier can also increase complexity from **NL** to **P**-complete. In the proof we show **P**-hardness by reduction from the **P**-complete problem GAME. An input to GAME is a DAG (directed acyclic graph) $G = (V, E)$ together with a node $a \in V$. Given such input (V, E, a) we consider the following game $\mathsf{Gm}(V, E, a)$ between two players, I and II. During the game the players move a pebble along the edges of G. In the initial position the pebble is on the node $a_0 = a$. If after $2i$ moves the pebble is on a node a_{2i}, then player I chooses a node a_{2i+1} such that $(a_{2i}, a_{2i+1}) \in E$, and player II responds by choosing a node a_{2i+2} such that $(a_{2i+1}, a_{2i+2}) \in E$. The first player unable to move loses the game, and the other player wins it. Since G is a DAG, every play of the game is finite. In particular, the game is determined, i.e., one of the players has a winning strategy. Now we define (V, E, a) to be a positive instance of GAME if and only if player II has a winning strategy in $\mathsf{Gm}(V, E, a)$.

Note that GAME can be seen as a variation of the monotone circuit value problem MCVP. Indeed, it is straighforward to define for a given monotone circuit C and input word w an input (V, E, a) for GAME such that $\mathsf{Gm}(V, E, a)$ simulates the evaluation game of C on w. Thus MCVP is logarithmic-space reducible to GAME. Conversely, it is also easy to give a logarithmic-space reduction from GAME to MCVP.

Proposition 27. *Let ϕ be the formula $\forall z(\neg Eyz \vee z \subseteq x)$. Then* $\mathsf{MSM}(\phi)$ *is* **P**-*complete. Consequently,* $\mathsf{MC}(\phi \vee Euv)$ *is also* **P**-*complete.*

Proof. We give now a reduction from GAME to $\mathsf{MSM}(\phi)$. Let (V, E, a) be an input to GAME. Without loss of generality we assume that there is $b \in V$ such that $(b, a) \in E$. Now we simply let $\mathfrak{A} = (V, E)$, $X = \{s : \{x, y\} \to V \mid (s(x), s(y)) \in E\}$ and $s_0 = \{(x, b), (y, a)\}$.

We will use below the notation $I = \{c \in V \mid \forall d \in V : (c, d) \notin E)\}$. Thus, I consists of those elements $c \in V$ for which player II wins $\mathsf{Gm}(V, E, c)$ immediately because I cannot move. Furthermore, we denote by W the set of all elements $c \in V$ such that player II has a winning strategy in $\mathsf{Gm}(V, E, c)$.

Let Y be the subteam of X consisting of those assignments $s \in X$ for which $s(y) \in W$. We will show that $Y = \nu(\mathfrak{A}, X, \phi)$. Hence in particular $s_0 \in \nu(\mathfrak{A}, X, \phi)$ if and only if (V, E, a) is a positive instance of GAME, as desired.

To prove that $Y \subseteq \nu(\mathfrak{A}, X, \phi)$ it suffices to show that $\mathfrak{A} \models_Y \phi$. Thus let $Z = Y[A/z]$, $Z' = \{s \in Z \mid (s(y), s(z)) \notin E\}$ and $Z'' = (Z \setminus Z') \cup Z_0$, where $Z_0 = \{s \in Z \mid s(z) = s(x) \text{ and } s(y) \in I\}$ (an example of Z'' is illustrated in Fig. 5). Then clearly $\mathfrak{A} \models_{Z'} \neg Eyz$. To show that $\mathfrak{A} \models_{Z''} z \subseteq x$ assume that $s \in Z''$. If $s \in Z \setminus Z'$, then $(s(y), s(z)) \in E$, and since $s \upharpoonright \{x, y\} \in Y$, player II has an answer c to the move $s(z)$ of player I in $\mathsf{Gm}(V, E, s(y))$ such that $c \in W$. Thus, $s^* = \{(x, s(z)), (y, c)\} \in Y$. If $c \in I$, then $s^*(s^*(x)/z) \in Z_0$. Otherwise there is some $d \in V$ such that $(c, d) \in E$, whence $s^*(d/z) \in Z \setminus Z'$. In both cases, there is $s' \in Z''$ such that $s'(x) = s(z)$. Assume then that $s \in Z_0$. Then by the definition of Z_0 we have $s(x) = s(z)$. Thus we see that for every $s \in Z''$ there is $s' \in Z''$ such that $s'(x) = s(z)$. Now we can conclude that $\mathfrak{A} \models_Z \neg Eyz \vee z \subseteq x$, and hence $\mathfrak{A} \models_Y \phi$.

To prove that $\nu(\mathfrak{A}, X, \phi) \subseteq Y$ it suffices to show that if $\mathfrak{A} \models_{Y'} \phi$ for a team $Y' \subseteq X$, then $s(y) \in W$ for every $s \in Y'$. Thus assume that Y' satisfies ϕ and $s \in Y'$. We describe a winning strategy for player II in $\mathsf{Gm}(V, E, s(y))$. If $s(y) \in I$ she has a trivial winning strategy. Otherwise player I is able to move; let $c \in V$ be his first move. Since $\mathfrak{A} \models_{Y'} \phi$, there are $Z', Z'' \subseteq Y'[A/z]$ such that $Y'[A/z] = Z' \cup Z''$, $\mathfrak{A} \models_{Z'} \neg Eyz$ and $\mathfrak{A} \models_{Z''} z \subseteq x$. Consider the assignment $s' = s(c/z) \in Y'[A/z]$. Since $(s'(y), s'(z)) = (s(y), c) \in E$, it must be the case that $s' \in Z''$. Thus there is $s'' \in Z''$ such that $s''(x) = s'(z) = c$. Then the assignment $s^* = s'' \upharpoonright \{x, y\}$ is in $Y' \subseteq X$, whence $(c, d) \in E$, where $d = s^*(y)$. Let d be the answer of player II for the move c of player I. We observe now that using this strategy player II can find a legal answer from the set $\{s^*(y) \mid s^* \in Y'\}$ to any move of player I, as long as player I is able to move. Since the game is determined, this is indeed a winning strategy.

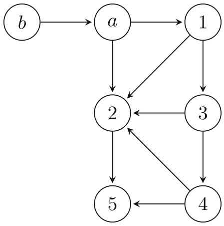
Using Lemma 24, we see that $\mathsf{MC}(\forall z(\neg Eyz \vee z \subseteq x) \vee \beta)$ is **P**-hard for any non-trivial formula β such that $x, y \notin \mathrm{Fr}(\beta)$, in particular for $\beta = Euv$. □

This proposition now demonstrates that, similarly as conjunction, universal quantification cannot be freely used if the goal is to construct a fragment of inclusion logic that captures **NL**. On the positive side, we prove next that existential quantification preserves **NL**-computability. Furthermore, we show that the same holds for conjunction, provided that one of the conjuncts is in FO.

Lemma 28. *Let $\phi \in \mathrm{FO}(\subseteq)$, $\psi \in \mathrm{FO}$, and let X be a team of a model $\mathfrak{A}$. Then*

(a) $\nu(\mathfrak{A}, X, \exists x\phi) = \{s \in X \mid s(a/x) \in X'$ *for some* $a \in A\}$, *where* $X' = \nu(\mathfrak{A}, X[A/x], \phi)$,
(b) $\nu(\mathfrak{A}, X, \phi \wedge \psi) = \nu(\mathfrak{A}, X', \phi)$, *where* $X' = \nu(\mathfrak{A}, X, \psi)$.

Proof. (a) Let $X' = \nu(\mathfrak{A}, X[A/x], \phi)$ and $X'' = \{s \in X \mid s(a/x) \in X'$ for some $a \in A\}$. Assume that $Y \subseteq X$ is a team such that $\mathfrak{A} \models_Y \exists x\phi$.



$\mapsto$

	x	y	z
$\circ$	b	a	$1,2$
	a	1	$2,3$
	a	2	5
	1	2	5
$\circ$	1	3	$2,4$
$\circ$	2	5	2
	3	2	5
	3	4	$2,5$
	4	2	5
$\circ$	4	5	4

Fig. 5. GAME and $\mathsf{MSM}(\forall z(\neg Eyz \vee z \subseteq x))$. The assignments marked by a circle constitute Z''.

Then there is a function $F : X \to \mathcal{P}(A) \setminus \{\emptyset\}$ such that $\mathfrak{A} \models_{Y[F/x]} \phi$, and since clearly $Y[F/x] \subseteq X[A/x]$, we have $Y[F/x] \subseteq X'$. Thus for every $s \in Y$ there is $a \in A$ such that $s(a/x) \in X'$, and hence we see that $Y \subseteq X''$. In particular $\nu(\mathfrak{A}, X, \exists x\phi) \subseteq X''$. To prove the converse inclusion it suffices to show that $\mathfrak{A} \models_{X''} \exists x\phi$. Let $G : X'' \to \mathcal{P}(A) \setminus \{\emptyset\}$ be the function defined by $G(s) = \{a \in A \mid s(a/x) \in X'\}$. By the definition of X'', this function is well-defined and $G(s) \neq \emptyset$ for all $s \in X''$. It is now easy to see that $X''[G/x] = X'$, whence $\mathfrak{A} \models_{X''[G/x]} \phi$, as desired.

(b) Let $X' = \nu(\mathfrak{A}, X, \psi)$ and $X'' = \nu(\mathfrak{A}, X', \psi)$. Assume first that $Y \subseteq X$ is a team such that $\mathfrak{A} \models_Y \phi \wedge \psi$. Then $\mathfrak{A} \models_Y \psi$, whence $Y \subseteq X'$, and furthermore $Y \subseteq X''$, since $\mathfrak{A} \models_Y \phi$. In particular, $\nu(\mathfrak{A}, X, \phi \wedge \psi) \subseteq X''$. On the other hand, by definition $\mathfrak{A} \models_{X''} \phi$. Similarly $\mathfrak{A} \models_{X'} \psi$, whence by downward closure of FO (Corollary 3), $\mathfrak{A} \models_{X''} \psi$. Thus we see that $\mathfrak{A} \models_{X''} \phi \wedge \psi$, which implies that $X'' \subseteq \nu(\mathfrak{A}, X, \phi \wedge \psi)$. $\square$

As a straightforward corollary to this lemma we obtain the following complexity preservation result.

Proposition 29. *Let $\phi \in \mathrm{FO}(\subseteq)$, $\psi \in \mathrm{FO}$, and assume that $\mathsf{MSM}(\phi)$ is in a complexity class $C \in \{\mathbf{L}, \mathbf{NL}\}$. Then*

(a) $\mathsf{MSM}(\exists x\phi)$ is in C, and
(b) $\mathsf{MSM}(\phi \wedge \psi)$ is in C.

Proof. (a) By Lemma 28(a), to check whether a given assignment s is in $\nu(\mathfrak{A}, X, \exists x\phi)$ it suffices to check whether $s(a/x)$ is in $\nu(\mathfrak{A}, X[A/x], \phi)$ for some $a \in A$. Clearly this task can be done in C assuming that $\mathsf{MSM}(\phi)$ is in C.

(b) By Lemma 28(a), it suffices to show that the problem whether an assignment s is in $\nu(\mathfrak{A}, X', \phi)$, where $X' = \nu(\mathfrak{A}, X, \psi)$, can be solved in C with respect to the input $(s, \mathfrak{A}, X)$. Since $\psi \in \mathrm{FO}$, the team X' can be computed in C, whence the claim follows from the assumption that $\mathsf{MSM}(\phi)$ is in C. $\square$

Summarising Lemma 13 and Proposition 29, **NL**-computability of maximal subteam membership is preserved by disjunction, conjunction with first-order formulas, and existential quantification. Since maximal subteam problem is in **NL** for all first-order formulas and, by Lemma 17, for all inclusion atoms, we define a weak fragment $\mathrm{FO}(\subseteq)_w$ of inclusion logic by the following grammar:

$$\phi ::= \alpha \mid \overline{x} \subseteq \overline{y} \mid \phi \vee \phi \mid \phi \wedge \alpha \mid \exists x \phi,$$

where $\alpha \in \mathrm{FO}$.

Theorem 30. *$\mathsf{MC}(\phi)$ is in **NL** for every $\phi \in \mathrm{FO}(\subseteq)_w$.*

Proof. By an easy induction we see that $\mathsf{MSM}(\phi)$ is in **NL** for every $\phi \in \mathrm{FO}(\subseteq)_w$. The claim follows now from Lemma 25.

Vice versa, to show that each **NL** property of ordered models can be expressed in $\mathrm{FO}(\subseteq)_w$, it suffices to show that TC translates to $\mathrm{FO}(\subseteq)_w$ over ordered models.

Theorem 31. *Over finite ordered models,* $\mathrm{TC} \leq \mathrm{FO}(\subseteq)_w$.

Proof. By Theorem 9 we may assume without loss of generality that any TC sentence ϕ is of the form $[\mathrm{TC}_{\overline{x},\overline{y}}\alpha(\overline{x},\overline{y})](\overline{\min},\overline{\max})$ where $\overline{x}$ and $\overline{y}$ are n-ary sequences of variables. We next define an equivalent $\mathrm{FO}(\subseteq)_w$ sentence ϕ'. For two tuples of variables $\overline{x}$ and $\overline{y}$ of the same length, we write $\overline{x} < \overline{y}$ as a shorthand for the formula expressing that $\overline{x}$ is less than $\overline{y}$ in the induced lexicographic ordering, and $\overline{x} = \overline{y}$ for the conjunction expressing that $\overline{x}$ and $\overline{y}$ are pointwise identical. The sentence ϕ' is given as follows:

$$\phi' := \exists \overline{x}\overline{y}\overline{t}_x\overline{t}_y(\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4) \tag{1}$$

where

- $\psi_1 := \overline{y}\overline{t}_y \subseteq \overline{x}\overline{t}_x$,
- $\psi_2 := (\overline{t}_x < \overline{\max} \wedge \overline{t}_x < \overline{t}_y \wedge \alpha(\overline{x},\overline{y})) \vee (\overline{t}_x = \overline{\max} \wedge \overline{t}_y = \overline{\min})$,
- $\psi_3 := \neg\overline{t}_x = \overline{\min} \vee \overline{x} = \overline{\min}$, and
- $\psi_4 := \neg\overline{t}_x = \overline{\max} \vee \overline{x} = \overline{\max}$.

Observe that in (1) the tuple $\overline{t}_x$ can be thought of as a counter which indicates an upper bound for the α-path distance of $\overline{x}$ from $\overline{\min}$.

Assuming $\mathfrak{A} \models \phi'$, we find a non-empty team X such that $\mathfrak{A} \models_X \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4$. Now, $\mathfrak{A} \models_X \psi_1 \wedge \psi_2$ entails that there is an assignment $s \in X$ mapping $\overline{t}_x$ to $\overline{\min}$, and $\mathfrak{A} \models_X \psi_3$ implies that s maps $\overline{x}$ to $\overline{\min}$, too. Then $\mathfrak{A} \models_X \psi_1 \wedge \psi_2$ entails that there is an α-path from $\overline{\min}$ to $s'(\overline{x})$ for some $s' \in X$ with $s'(\overline{t}_x) = \overline{\max}$. Lastly, by $\mathfrak{A} \models_X \psi_4$ it follows that $s'(\overline{x}) = \overline{\max}$ which shows that $[\mathrm{TC}_{\overline{x},\overline{y}}\alpha(\overline{x},\overline{y})](\overline{\min},\overline{\max})$.

Assume then that $[\mathrm{TC}_{\overline{x},\overline{y}}\alpha(\overline{x},\overline{y})](\overline{\min},\overline{\max})$, that is, there is an α-path $\overline{v}_1,\ldots,\overline{v}_k$ where $\overline{v}_1 = \overline{\min}$ and $\overline{v}_k = \overline{\max}$. We may assume that there are no cycles in the path. Let $\overline{a}_i$ denote the ith element in the lexicographic ordering of A^n. We let $X = \{s_1,\ldots,s_k\}$ be such that $(\overline{x},\overline{y},\overline{t}_x,\overline{t}_y)$ is mapped to $(\overline{v}_i,\overline{v}_{i+1},\overline{a}_i,\overline{a}_{i+1})$ by s_i, for $i = 1,\ldots,k-2$, to $(\overline{v}_{k-1},\overline{v}_k,\overline{a}_{k-1},\overline{\max})$ by s_{k-1}, and to $(\overline{v}_k,\overline{v}_1,\overline{\max},\overline{\min})$ by s_k. It is straightforward to verify that $\mathfrak{A} \models_X \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4$ from which it follows that $\mathfrak{A} \models \phi'$. $\square$

It now follows by the above two theorems and Theorem 9 that $\mathrm{FO}(\subseteq)_w$ captures **NL**.

Theorem 32. *A class $\mathcal{C}$ of finite ordered models is in* **NL** *iff it can be defined in* $\mathrm{FO}(\subseteq)_w$.

7 Conclusion

We have studied the complexity of inclusion logic from the vantage point of two computational problems: the maximal subteam membership and the model checking problems for fixed inclusion logic formulae. We gave a complete characterization for the former in terms of arbitrary conjunctions/disjunctions of unary inclusion atoms. In particular, we showed that maximal subteam membership is **P**-complete for any conjunction of unary inclusion atoms, provided that the conjunction contains two non-trivial atoms that are not inverses of each other. Using these results we characterized the complexity of model checking for several quantifier-free inclusion logic formulae. We leave it for future research to address the complexity of quantifier-free inclusion logic formulae that involve inclusion atoms of higher arity and both disjunctions and conjunctions.

Assuming the presence of quantifiers we presented a simple universally quantified formula that has **P**-complete maximal subteam membership problem. Finally, we defined a fragment of inclusion logic, obtained by restricting the scope of conjunction and universal quantification, that captures non-deterministic logarithmic space over finite ordered models.

Acknowledgements. We are grateful to Phokion Kolaitis, who raised the questions on the complexity of quantifier-free formulas of inclusion logic in a private discussion with the second author in 2016.

References

1. Afrati, F.N., Kolaitis, P.G.: Repair checking in inconsistent databases: algorithms and complexity. In: 12th International Conference on Database Theory - ICDT 2009, St. Petersburg, Russia, 23–25 March 2009, pp. 31–41 (2009)
2. Arenas, M., Bertossi, L.E., Chomicki, J.: Consistent query answers in inconsistent databases. In: Proceedings of the Eighteenth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems, Philadelphia, Pennsylvania, USA, 31 May–2 June 1999, pp. 68–79 (1999)
3. Chomicki, J., Marcinkowski, J.: Minimal-change integrity maintenance using tuple deletions. Inf. Comput. **197**(1–2), 90–121 (2005)
4. Corander, J., Hyttinen, A., Kontinen, J., Pensar, J., Väänänen, J.: A logical approach to context-specific independence. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 165–182. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_11
5. Durand, A., Hannula, M., Kontinen, J., Meier, A., Virtema, J.: Approximation and dependence via multiteam semantics. Ann. Math. Artif. Intell. **83**, 297–320 (2018)

6. Durand, A., Kontinen, J., de Rugy-Altherre, N., Väänänen, J.: Tractability frontier of data complexity in team semantics. In: Proceedings Sixth International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2015, Genoa, Italy, 21–22nd September 2015, pp. 73–85 (2015)
7. Ebbing, J., Hella, L., Meier, A., Müller, J.-S., Virtema, J., Vollmer, H.: Extended modal dependence logic $\mathcal{EMDL}$. In: Libkin, L., Kohlenbach, U., de Queiroz, R. (eds.) WoLLIC 2013. LNCS, vol. 8071, pp. 126–137. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-39992-3_13
8. Ebbing, J., Kontinen, J., Müller, J.-S., Vollmer, H.: A fragment of dependence logic capturing polynomial time. Log. Methods Comput. Sci. **10**(3) (2014)
9. Galliani, P.: Inclusion and exclusion dependencies in team semantics: on some logics of imperfect information. Ann. Pure Appl. Log. **163**(1), 68–84 (2012)
10. Galliani, P., Hella, L.: Inclusion logic and fixed point logic. In: Rocca, S.R.D. (ed.) Computer Science Logic 2013 (CSL 2013). Leibniz International Proceedings in Informatics (LIPIcs), Dagstuhl, Germany, vol. 23, pp. 281–295. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik (2013)
11. Grädel, E.: Model-checking games for logics of imperfect information. Theor. Comput. Sci. **493**, 2–14 (2012)
12. Grädel, E.: Games for inclusion logic and fixed-point logic. In: Abramsky, S., Kontinen, J., Väänänen, J., Vollmer, H. (eds.) Dependence Logic, pp. 73–98. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-31803-5_5
13. Hannula, M., Hella, L.: Complexity thresholds in inclusion logic. CoRR, abs/1903.10706 (2019)
14. Hannula, M., Kontinen, J.: A finite axiomatization of conditional independence and inclusion dependencies. Inf. Comput. **249**, 121–137 (2016)
15. Hannula, M., Kontinen, J., Virtema, J.: Polyteam semantics. In: Artemov, S., Nerode, A. (eds.) LFCS 2018. LNCS, vol. 10703, pp. 190–210. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-72056-2_12
16. Hannula, M., Kontinen, J., Virtema, J., Vollmer, H.: Complexity of propositional logics in team semantic. ACM Trans. Comput. Log. **19**(1), 2:1–2:14 (2018)
17. Hodges, W.: Compositional semantics for a language of imperfect information. J. Interest Group Pure Appl. Log. **5**(4), 539–563 (1997)
18. Immerman, N.: Relational queries computable in polynomial time. Inf. Control **68**(1), 86–104 (1986)
19. Immerman, N.: Languages that capture complexity classes. SIAM J. Comput. **16**(4), 760–778 (1987)
20. Immerman, N.: Nondeterministic space is closed under complementation. SIAM J. Comput. **17**(5), 935–938 (1988)
21. Kontinen, J.: Coherence and computational complexity of quantifier-free dependence logic formulas. Studia Logica **101**(2), 267–291 (2013)
22. Kontinen, J., Kuusisto, A., Lohmann, P., Virtema, J.: Complexity of two-variable dependence logic and if-logic. Inf. Comput. **239**, 237–253 (2014)
23. Koutris, P., Wijsen, J.: Consistent query answering for primary keys in logspace. In: 22nd International Conference on Database Theory, ICDT 2019, Lisbon, Portugal, 26–28 March 2019, pp. 23:1–23:19 (2019)
24. Pacuit, E., Yang, F.: Dependence and independence in social choice: Arrow's theorem. In: Abramsky, S., Kontinen, J., Väänänen, J., Vollmer, H. (eds.) Dependence Logic, pp. 235–260. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-31803-5_11
25. ten Cate, B., Fontaine, G., Kolaitis, P.G.: On the data complexity of consistent query answering. Theory Comput. Syst. **57**(4), 843–891 (2015)

26. Väänänen, J.: Dependence Logic. Cambridge University Press, Cambridge (2007)
27. Väänänen, J.: Modal dependence logic. In: Apt, K.R., van Rooij, R. (eds.) New Perspectives on Games and Interaction. Amsterdam University Press, Amsterdam (2008)
28. Vardi, M.Y.: The complexity of relational query languages. In: Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing, pp. 137–146. ACM (1982)
29. Virtema, J.: Complexity of validity for propositional dependence logics. Inf. Comput. **253**, 224–236 (2017)
30. Vollmer, H.: Introduction to Circuit Complexity - A Uniform Approach. EATCS Series. Springer, Heidelberg (1999). https://doi.org/10.1007/978-3-662-03927-4
31. Yang, F., Väänänen, J.: Propositional logics of dependence. Ann. Pure Appl. Logic **167**(7), 557–589 (2016)

The Multiresolution Analysis of Flow Graphs

Steve Huntsman(✉)

BAE Systems FAST Labs, 4301 North Fairfax Drive, Arlington, VA 22203, USA
steve.huntsman@baesystems.com

Abstract. We introduce and prove basic results about several graph-theoretic notions relevant to the multiresolution analysis of flow graphs that represent the transfer of control in computer programs. We take a category-theoretical viewpoint to demonstrate that our definitions are natural and to motivate particular incarnations of related constructions.

Keywords: Program analysis · Flow graph · Program structure tree · Operad · Symmetric monoidal category

1 Introduction

The notion of a "flow graph" is central to the analysis and compilation of computer programs, encompassing constructs that represent the transfer of control and data [7,21,22]. As the complexity of software increases, so does the scale of the corresponding flow graphs: accordingly, a framework for the analysis of flow graphs at multiple resolutions is desirable. Such a framework was originally presented in [15], based on a hierarchical representation of input/output structure called the *program structure tree* (PST).

For an illustration of this framework, consider the simple imperative program "skeleton" and associated control flow graph in Fig. 1. The result of "stretching" it *à la* Sect. B and the PST of the result are shown in Fig. 2. Iterating the process of pruning each leaf of the PST *à la* Sect. 5 leads to "coarsened" control flow graphs such as those in Fig. 3.

The utility of this framework is enhanced by [30], which shows how to restructure the control flow graph of a program in such a way that subroutines can be identified as programs in their own right using the control flow graph alone. This feeds naturally into a "multiresolution analysis" of recursively composing (resp. decomposing) a program from (resp. into) subprograms in a way that can help with building, understanding, and modifying large programs.

This paper extends the work of [15] while correcting both an error of definition (for interiors of single-entry/single-exit regions) found in [3], and another subtler error in the original proof of Theorem 1, by unifying and formalizing several natural concepts relevant to the decomposition and construction of flow graphs. This has several benefits: as the most basic example, we provide a definition of flow graph that is slightly different than its other usual variants but

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 323–341, 2019.
https://doi.org/10.1007/978-3-662-59533-6_20

```
1  START
2  repeat
3      repeat
4          repeat
5              if b goto 7
6              if b
7                  repeat
8                      S
9                  until b
10             endif
11         until b
12         do while b
13             do while b
14                 repeat
15                     S
16                 until b
17             enddo
18         enddo
19     until b
20  until b
21  HALT
```

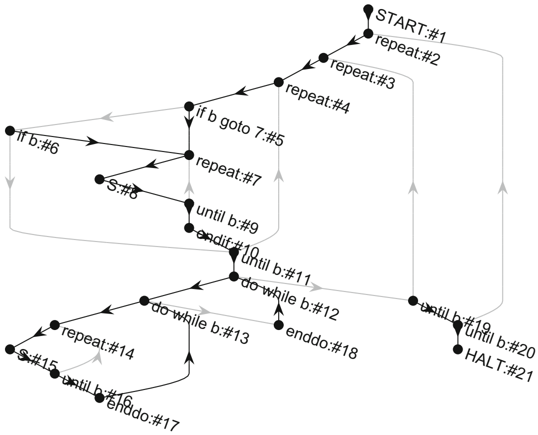


Fig. 1. (L) A simple imperative program. S denotes a generic statement (or subroutine); b denotes a generic Boolean predicate. (R) The corresponding control flow graph: branches are shaded black (resp., gray) if the corresponding b evaluates to $\top$ or $\bot$.

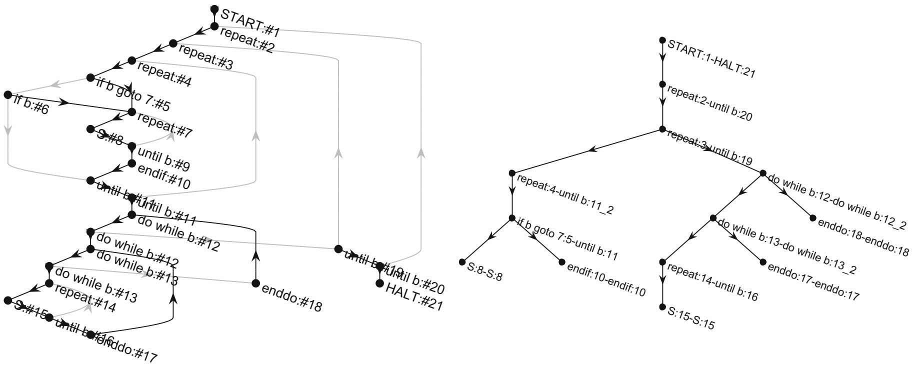


Fig. 2. (L) "Stretching" (*à la* Sect. B) the flow graph of Fig. 1. (R) The resulting PST.

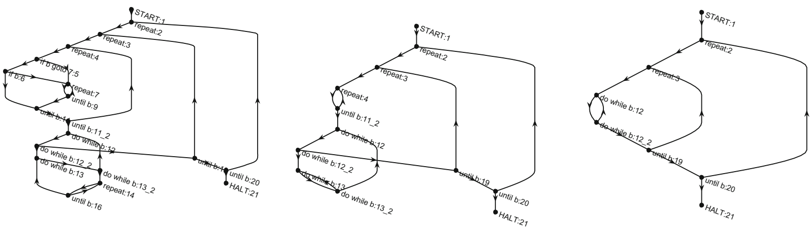


Fig. 3. Succesively coarsening (*à la* Sect. 5) the flow graph of Fig. 2

that is mathematically more natural and well-behaved. This in turn leads to a simpler analogue of the "refined process structure tree" of [23,29] and natural category-theoretic constructions. These include multiresolution operations that approximate and/or refine flow graphs at multiple scales, as well as series and parallel operations that respectively embody sequential execution and if/else constructs in control flow.

While most of the results of this paper are conceptually straightforward and many are at least latent in the literature, few of them have been simultaneously formulated explicitly and mathematically. Indeed, the practical motivation for this paper is simply to show that the "right" definition of a flow graph entails all the obvious desiderata, particularly for treating subroutines as programs in their own right. As Sect. 7 highlights, the precise ability to compose flow graphs in category-theoretically nice ways is novel (though it is obvious that such a thing should be possible somehow): the unit object presents the principal difficulty, and much of our effort is focused on this issue for the case of parallel composition. This compositionality can inform the internal representation of graphical data structures and techniques for their manipulation in binary program analysis platforms such as [5] and program synthesizers [12] as well as compilers.

In particular, constraining the notion of a valid control flow graph to the one considered in the paper could confer an advantage from the point of view of precompilation or reuse/modification: our results give a recipe for inserting and combining precompiled code in a convenient way. In a similar vein, we may want to understand a disassembled binary by synthesizing a similar or equivalent program. After restructuring the control flow graph along the lines of [30] and performing some straightforward normalizations (see Sect. B), we could construct the PST and attempt program synthesis for each of the subroutines corresponding to a leaf node. In particular, we could generate inputs and observe outputs to each of these subroutines, so that program induction is a viable fallback at each point. Recursively going up the PST, we (attempt to) get such a globally synthesized program, and our results indicate precisely how synthesized/induced programs of intermediate scale can be maintained and reasoned over.

In other words, the constructions of the paper can inform tools that blur the lines between compilation and decompilation. In particular, the central results of Sects. 5 and 6 contain the technical details necessary to have confidence that intermediate representations of programs can be (de)composed in a mathematically principled way, offering a firm foundation for future tools. Although superficial errors in [15] and hitherto unrecognized categorical structure in the PST have hindered its use,[1] we believe that tools based on it can and should be built.

[1] To illustrate this point, we quote liberally from [3]: "Unfortunately, we discovered an error in the aforementioned proof regarding SSI [static single information] form...we discovered that this mistake had been made in an earlier paper as well, and that other mistakes had been made in several papers that built on SSI form. The goal of this article, therefore, is to clear up the mistakes to the greatest possible extent...The key mistake was...made by Johnson *et al.* [15], who introduced a data structure called

The paper is organized as follows: we discuss dominance relations in Sect. 2; flow graphs, single-entry/single-exit regions, and the PST in Sect. 3; we introduce the structure of a category on flow graphs in Sect. 4 (this delay is to connect the paper to prior work most clearly); we discuss multiresolution transformations on flow graphs in Sect. 5; and in Sect. 6 we discuss series and parallel composition of flow graphs in the context of formal tensor product structures. Section 7 discusses two-terminal graphs before our concluding remarks in Sect. 8. Section A contains proofs and Sect. B sketches a "stretching" operation that enhances the applicability of our constructions.

We remark at the outset that all graphs (and related objects) are assumed finite throughout this paper. By convention, digraphs are allowed to have loops from a vertex to itself. Given a vertex v in a digraph, let $d_0^+(v)$, $d_0^-(v)$, and $d^0(v)$ respectively denote the number of incoming edges excluding any loop, the number of outgoing edges excluding any loop, and the number (≤ 1) of loops at v. A vertex v is a *source* iff $d_0^+(v) = 0$ and a *target* iff $d_0^-(v) = 0$, i.e., loops have no bearing on these properties.

2 Dominance Relations

Let G be a digraph and $j, k \in V(G)$. We say that j *dominates* k, written j dom k, iff every path from a source s in G to k passes through j [7,21]. Define $D_{jk} = 1$ if j dom k and $D_{jk} = 0$ otherwise. Similarly, let $D^\dagger := D(G^*)$, where G^* is the reversal or adjoint of G with adjacency matrix A^* and corresponding dominance relation dom†. If $D^\dagger_{jk} = 1$, i.e., if j dom† k, write k pdom j and say that k *postdominates* j. Both the dominance and postdominance relations extend to edges. The following two lemmas are straightforward.

Lemma 1. *For distinct edges $\{e_j\}_{j=1}^3$ in a digraph G, if e_1 dom e_3 and e_2 dom e_3, then either e_1 dom e_2 or e_2 dom e_1. Similarly, if e_1 pdom e_2 and e_1 pdom e_3, then either e_2 pdom e_3 or e_3 pdom e_2.* □

Lemma 2. *If e_1 dom e_2 and e_1 pdom e_2 with $e_1 \neq e_2$, then a path from a source to a target that traverses e_2 contains a cycle of the form $(e_1, \dots, e_2, \dots, e_1)$.* □

We use Lemma 2 to fix a subtle (and minor) error in a proof of Theorem 1 that was originally presented by [15]. This helps us to rescue the framework of [15] in its entirety from the problems raised by [3].

3 Flow Graphs, Single-Entry/Single-Exit Regions, and the Program Structure Tree

A *flow graph* G is a digraph with exactly one source and exactly one target, such that there is a unique (entry) edge from the source and a unique (exit) edge to

the program structure tree (PST), which attempted to represent the structure of a control flow graph hierarchically.".

the target, and such that identifying the source of the entry edge with the target of the exit edge yields a strongly connected digraph. (We do not require the entry and exit edges to be distinct, e.g., if $|V(G)| = 2$.)[2]

A *single entry/single exit (SESE) region* in a digraph G is defined as an ordered pair of edges (e_1, e_2) satisfying each of the following conditions [15]: e_1 dom e_2, e_2 pdom e_1, and a cycle in G contains e_1 iff it contains e_2. See Fig. 5 for examples. Note first that (e_1, e_1) is a degenerate SESE region,[3] and second that a nondegenerate SESE region (e_1, e_2) (i.e., a SESE region with $e_1 \neq e_2$) unambiguously corresponds to the ordered vertex pair $(t(e_1), s(e_2))$, where $s(\cdot)$ and $t(\cdot)$ respectively denote the source and target of an edge. We may use either the edge or vertex pairs above to specify a nondegenerate SESE region. Note also that in a DAG the third condition above is trivial. Finally, note that the edges e_s from the source and e_t to the target of a flow graph G together define a SESE region and *vice versa*. With this in mind, write either G or (e_s, e_t) for the flow graph or the equivalent SESE region.

We give a few simple results (the first is straightforward enough that we omit a proof) before moving on to a fundamental theorem.

Lemma 3. *If* (e_1, e_2) *and* (e_2, e_3) *are SESE regions, then so is* (e_1, e_3). □

Lemma 4. *If* (e_1, e_2) *and* (e_1, e_3) *are SESE regions with* $e_2 \neq e_3$ *and* e_2 *dom* e_3*, then* (e_2, e_3) *is a (nondegenerate) SESE region.*

Corollary 1. *If* (e_1, e_2) *is a SESE region with* $e_2 \neq e_3$ *and* e_2 *dom* e_3*, and* (e_2, e_3) *is not a SESE region, then* (e_1, e_3) *is also not a SESE region.*[4] □

The *interior* G° of $G \equiv (e_s, e_t)$ is the set of vertices that are each on at least one path starting from $t(e_s)$ that does not encounter $t(e_t)$. Critically, this definition differs slightly from Definition 6 of [15], wherein the interior of a SESE

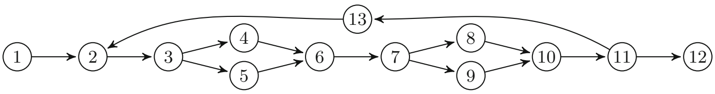


Fig. 4. As [3] points out, the nondegenerate SESE regions $((2, 3), (6, 7))$ and $((6, 7), (10, 11))$ have interiors that intersect at vertex 13 according to the original definition of [15]. Our definition of the interior of a SESE region eliminates such unwanted behavior and allows us to salvage the original attempt to prove Theorem 1

[2] NB. One sometimes sees variants of the definition and naming of this particular sort of concept, for the latter most typically as “flowgraph”, “flowchart”, or “flow chart”. Some concepts with the same name are technically quite different but “spiritually” viewed in a similar context, as, e.g., in the work of Manin [8,19].

[3] NB. Degenerate SESE regions (e_1, e_1) are excluded by the original definition of [15]. We allow such regions to make the series tensor product of Sect. 6.1 work nicely.

[4] A useful restatement of this is that if (e_1, e_2) is a SESE region with $e_2 \neq e_3$ and e_2 dom e_3, then (e_1, e_3) is not a SESE region unless (e_2, e_3) is a SESE region.

region (e_s, e_t) is defined as $\{j \in V : e_s \text{ dom } j \wedge e_t \text{ pdom } j\}$. An example in Sect. 5 of [3] and reproduced in Fig. 4 illustrates the difference between these definitions.

A nondegenerate SESE region (e_1, e_2) is called *canonical* if for any SESE region (e_1, e_2') it is the case that e_2 dom e_2' and if for any SESE region (e_1', e_2) it is the case that e_1 pdom e_1'. Our definition of the interior of a SESE region enables the following corrected version of Theorem 1 of [15] (cf. [3]).

Theorem 1. *Interiors of distinct canonical SESE regions are disjoint or nested.*

Therefore canonical SESE regions are also *minimal*, so we may use the two terms interchangeably: we generally prefer and use the latter. The inclusion relation on minimal SESE regions induces a tree—viz., the PST. An example of this nesting behavior and the corresponding PST are depicted in Fig. 5.

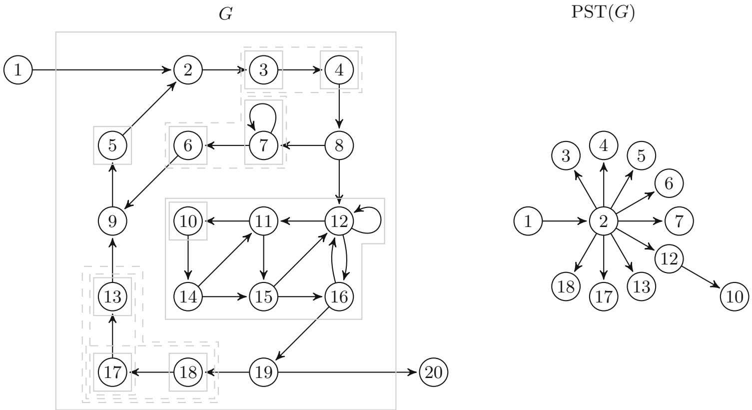


Fig. 5. (L) SESE regions of the flow graph G are outlined in gray: minimal (resp., non-minimal) SESE region outlines are solid (resp., dashed). Locally maximal but not minimal SESE regions are $((2,3),(4,8))$, $((8,7),(6,9))$, and $((19,18),(13,9))$. (R) The PST encodes the nesting of minimal SESE regions. Nodes are labeled by the target of the incoming edge (with a "phantom" edge from $-\infty$ to the source). The sets $\{3,4\}$, $\{6,7\}$, and $\{13,17,18\}$ correspond to locally maximal SESE regions that could sensibly be "aggregated" by identifying the respective vertices and omitting any resulting loops: however, a more mathematically natural variant of this construction is discussed in Sect. 5.

Lemma 5. *A nondegenerate SESE region* (e_0, e_∞) *decomposes as* $(e_0, e_\infty) = \bigcup_{j=1}^{m}(e_{j-1}, e_j)$*, where* $e_m \equiv e_\infty$ *and* (e_{j-1}, e_j) *are minimal SESE regions.*

Define an edge-indexed matrix S by $S_{e_1,e_2} = 1$ if (e_1, e_2) is a nondegenerate SESE region and $S_{e_1,e_2} = 0$ otherwise. Then S is the adjacency matrix of a digraph whose weakly connected components correspond to the situation in Lemma 5. We therefore obtain the following lemma.

Lemma 6. *Each weakly connected component of the digraph corresponding to S is a transitive tournament, hence has a unique source, target, and a path of length 1 from source to target defining a* locally maximal SESE region. □

A closely related construction is the subject of Sect. 5.

4 The Category of Flow Graphs

The principal goal of this section is merely to motivate and justify the details of the sequel. The key points are the introduction of the category **Dgph** of digraphs, and of its full subcategory **Flow** whose objects are flow graphs.

It is natural to attempt to regard transformations of mathematical objects as morphisms in an appropriate category [18]. Unfortunately, in many if not most cases involving digraphs, such an attempt is complicated by technicalities that commonly arise from loops [4]. The basic problem is that while identifying vertices should induce a graph morphism, such a morphism should also preserve edges. In particular, the morphism should preserve any edges between the vertices to be identified, necessarily inducing a loop. Insofar as we want loops in a coarse-grained control flow graph to correspond to actual loops in the atomic control flow, this is highly undesirable.

The common way around this problem is to treat loops on a separate footing. Following [4], define the category **Dgph** as follows. An object of **Dgph** is a *reflexive digraph* $G = (U, \alpha, \omega)$ given by a set U and *head* and *tail* functions $\alpha, \omega : U \to U$ satisfying $\alpha \circ \omega = \omega$ and $\omega \circ \alpha = \alpha$. Meanwhile, for $G' = (U', \alpha', \omega')$, a morphism $f \in \mathbf{Dgph}(G, G')$ is a function $f : U \to U'$ satisfying $f \circ \alpha = \alpha' \circ f$ and $f \circ \omega = \omega' \circ f$.

The *vertices* of $G = (U, \alpha, \omega)$ are the (mutual) image $V \equiv V(G)$ of α and ω; the *loops* are the set $L \equiv L(G) := \{u \in U : \alpha(u) = \omega(u)\}$ (so that $V \subseteq L$), and the *edges* are the set $E \equiv E(G) := U \backslash L$.[5] Thus a morphism $f : U \to U'$ restricts to $f|_V : V \to V'$, $f|_L : L \to L'$, and $f|_E : E \to U'$. In particular, morphisms are only partially specified by their actions on vertices, and the following definition is essentially a convention about how to treat vertex identification by default.

We define **Flow** to be the full subcategory of **Dgph** whose objects are (combinatorially realized as) flow graphs.[6]

5 Coarsening Flow Graphs

We begin this section with intuition: the coarsening of a flow graph G is obtained by taking each leaf of its PST and absorbing the interior of the corresponding sub-flow graph into its source. (See Fig. 6.) The details are below.

[5] The usual notion of a digraph is recovered by considering $\alpha \times \omega$ and its appropriate restrictions on U^2, L^2, and E^2: e.g., we can abusively write $E = (\alpha \times \omega)(E^2)$, where the LHS and RHS respectively refer to usual and reflexive notions of digraph edges.

[6] As pointed out by D. Spivak, it would be desirable to describe flow graphs in terms of **Dgph**, e.g. as algebras for some monad.

For $G \in \mathbf{Dgph}$, define the *absorption* of k into j to be the morphism in $\mathbf{Dgph}$ (or the morphism's image, depending on context) which corresponds to identifying k with j, and in the case $k \neq j$ subsequently annihilating any loop at j (by mapping it to the vertex j). It is clear that first absorbing k and then m into j is equivalent to first absorbing m and then k into j. Consequently, for $U \subseteq V(G)$ we may define the absorption of U into j in the obvious way.[7]

For $G, H \in \mathbf{Flow}$ with $H \subset G$, define the absorption of H to be the result of absorbing the interior of H into its source (considered as a vertex in G). This amounts to replacing H with a single edge between its source and target. Finally, define the *coarsening* $\circledcirc G$ of G to be the result of absorbing all of the sub-flow graphs corresponding to leaves of the program structure tree of G. The fact that $\circledcirc G$ is well-defined follows from [15] (cf. the "prime subprogram parse" of [27]) along with the preceding considerations. In particular, the definitions of absorption and coarsening yield the following technical lemma.

Lemma 7. *Let $G \in \mathbf{Flow}$ and let $\circledcirc G$ result from absorbing the vertex sets L_k into k for all $k \in K$ (so that L_k corresponds to a leaf of the program structure tree and $k \notin L_k$). Let $L := \cup_{k \in K} L_k$ (this set should not be confused with the set of loops in G) and $J := V \backslash (K \cup L)$, so that $V = J \cup K \cup L$ and J, K, L are mutually disjoint. Let $j, j' \in J$; $k, k' \in K$ with $k \neq k'$, and $\ell, \ell' \in L$. Finally, write $L_k^+ := \{k\} \cup L_k$ and let $g \in V$. Then the adjacency matrix of $\circledcirc G$* w.r.t. the vertex set of *G is A', where $A'_{jj'} = A_{jj'}$, $A'_{jk'} = \bigvee_{\ell' \in L_{k'}^+} A_{j\ell'}$, $A'_{kj'} = \bigvee_{\ell \in L_k^+} A_{\ell j'}$, $A'_{kk'} = \bigvee_{\ell \in L_k^+, \ell' \in L_{k'}^+} A_{\ell \ell'}$, and $A'_{kk} = A'_{g\ell'} = A'_{\ell g'} = 0$.* □

The real matter of substance in coarsening a flow graph is producing the sets J, K, and L referred to just above (it turns out to be easier to construct the L_k from L than to go in the opposite direction).

Theorem 2. *Using the notation of the preceding lemma, define a matrix M as follows. For each leaf (e_1, e_2) of the program structure tree, let $(e_1, e_2)^\circ$ denote its interior, and for all $j \in (e_1, e_2)^\circ$ set $M_{j,s(e_1)} = 1$. Then M is the adjacency matrix of a DAG (in fact, a forest) whose weakly connected components have vertex sets L_k^+ and corresponding targets k.*

Having considered coarsening flow graphs, we note that the appropriate mathematical formalization in the opposite direction—i.e., of inserting one flow

[7] Failing to make fixed choices about whether to preserve or annihilate loops from, or formed at, absorbed and absorbing vertices amounts to a context-driven decision about the absorption process that is unlikely to be of any utility and need not be considered. Therefore, we proceed here to consider the space of such possible fixed choices. In the context of control flow graphs, a loop corresponds closely to a do-while construct. With this in mind, preserving such a construction under absorption corresponds to inserting additional computations into a do-while loop, or forming a new do-while loop around existing computations, altering the control flow. Meanwhile, annihilating loops corresponds to embedding the do-while construct within a larger sequence of computations, preserving the control flow. This is *prima facie* cause to restrict consideration to the definition of absorption introduced above.

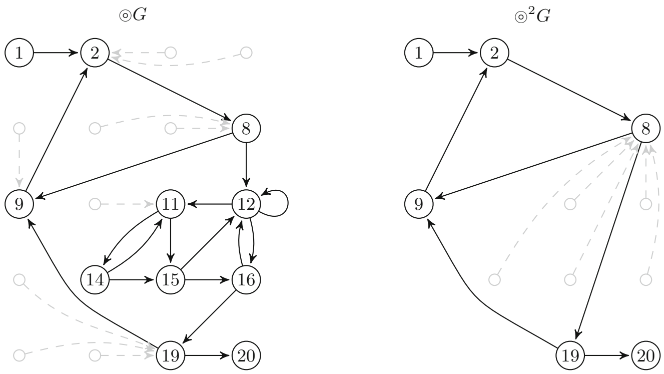


Fig. 6. (L) Coarsening of the flow graph G from Fig. 5. (Note that the pullback of the diagram $a \xrightarrow{g \circ f} c \xleftarrow{g} b$ is $a \xleftarrow{id} a \xrightarrow{f} b$, so that f is the pullback of $g \circ f$ by g. We may therefore think of $\circledcirc G$ somewhat literally as a kind of pullback of G by the leaves of its program structure tree.) (R) Coarsening again. A third coarsening is trivial.

graph into another[8]—is captured by the assertion that flow graphs form a (symmetric) *operad* [17,20,26] (cf. [24,25]). At a high level, an operad is a collection of objects that "plug into each other" like maps $f_{(m)} : X^m \to X$ *à la*

$$f_{(m)} \circ_\ell g_{(n)} := f(\cdot_1, \dots, \cdot_{\ell-1}, g(\cdot_\ell, \dots, \cdot_{\ell+n-1}), \cdot_{\ell+n}, \dots, \cdot_{m+n}).$$

Let $P(n)$ denote the set of flow graphs with n ordered edges and define the following family of maps

$$\begin{aligned} \circ : P(n) \times P(k_1) \times \cdots \times P(k_n) &\to P(k_1 + \cdots + k_n) \\ (G, G_1, \dots, G_n) &\mapsto G \circ (G_1, \dots, G_n) \end{aligned} \tag{1}$$

by replacing, for each $1 \leq j \leq n$, the jth edge in G with G_j in the obvious way. Writing $k_0 \equiv 0$, the edge ordering on $G \circ (G_1, \dots, G_n)$ is obtained by assigning edges $\sum_{i=0}^{j-1} k_i + 1, \dots, \sum_{i=0}^{j} k_i$ to $G_j \hookrightarrow G \circ (G_1, \dots, G_n)$ in the same order as the edges of G_j, i.e., the edge ordering is inherited from its local components.

Definition-checking or direct comparison to other insertion operads (e.g. the little d-disks or d-cubes operads in **Top**) yields the following

Theorem 3. *The triple $\{e, \{P(n)\}_{n=1}^{\infty}, \circ\}$, where e denotes the flow graph with one edge, forms an operad (in* **Set***).* □

Thus the operadic composition $\circ$ and coarsening $\circledcirc$ operations are not only natural, but complementary, and we readily obtain the following lemma.

Lemma 8. *If $G \in P(n)$ and $\circledcirc G_j = e \neq G_j$, then $\circledcirc(G \circ (G_1, \dots G_n)) = G$.* □

[8] Note that we are not explicitly considering the insertion of loops in this setting.

6 Tensoring Flow Graphs

6.1 Tensoring in Series

There is an essentially trivial tensor product on **Flow**. The idea is simply to identify the exit edge of the first flow graph with the entry edge of the second flow graph, i.e., to combine flow graphs in series. The reason that this tensor product structure is interesting and useful is that it allows us a way to model additional structure in an enriched category. Specifically, this leads to the **Flow**-category $\mathbf{SubFlow}_G$ of sub-flow graphs of a flow graph G.

We provide a quick sketch of the details here. Let $f \in \mathbf{Flow}(G, G_f)$ and $f' \in \mathbf{Flow}(G', G'_{f'})$ with $V(G) \cap V(G') = \varnothing$. Define $G \boxtimes G'$ to be the flow graph obtained by identifying the exit edge of G and the entry edge of G', and define $f \boxtimes f'$ to be the morphism in $\mathbf{Flow}(G \boxtimes G', G_f \boxtimes G'_{f'})$ obtained by identifying the output of f on the exit edge of G with that of f' on the entry edge of G'.

The following lemmas are straightforward.

Lemma 9. **Flow** *is a monoidal category with tensor product given by* $\boxtimes$*, and with unit object the flow graph* e *consisting of a single edge.* $\square$

Lemma 10. *For a generic flow graph* G*, we can form a category* $\mathbf{SubFlow}_G$ *enriched* [16] *over* **Flow** *as follows:*

- $\mathbf{SubFlow}_G := E(G)$;[9]
- *for* $e_s, e_t \in \mathbf{SubFlow}_G$*, the hom object* $\mathbf{SubFlow}_G(e_s, e_t) \in \mathbf{Flow}$ *is the (possibly empty) flow graph with entry edge* e_s *and exit edge* e_t*;*
- *the composition morphism is induced by* $\boxtimes$*;*
- *the identity element is determined by the flow graph* e *with one edge.* $\square$

An important advantage of $\mathbf{SubFlow}_G$ over the path category of G is that the former is finite (and the preceding sections essentially detail its construction), whereas the latter is infinite whenever there is a cycle in G.

6.2 Tensoring in Parallel

In this section we show that **Flow** carries a nontrivial monoidal structure (i.e., there is a tensor product operation that coherently combines flow graphs "in parallel" and not merely "in series" [6]). While the concept is rather obvious, the details are technical and we consequently make them explicit. In particular, although **Flow** is conceptually rather similar to the categories of n-cobordisms or tangles, the disjoint union only yields a tensor product in the latter cases: here, it must be modified to account for flow graphs whose entry and exit edges are identical or adjacent.

Let $s(e^+), t(e^+), s(e^-), t(e^-)$ be four fixed distinct points not contained in the vertex set of any graph already under consideration, so that $e^\pm := (s(e^\pm), t(e^\pm))$

[9] In particular, loops and reflexive self-edges are not included here, though the former may be accommodated without substantial changes.

may be regarded as two separated abstract edges. If G is a flow graph with entry edge e_s and (possibly adjacent or even identical) exit edge e_t, define a **Dgph**-morphism (i.e., the image may not be a flow graph) ϕ_G by the vertex/loop map

$$\phi_G(j) := \begin{cases} s(e^+) & \text{if } j = s(e_s) \\ t(e^+) & \text{if } j = t(e_s) \\ & \text{or } t(e_s) = s(e_t) \text{ and } j = t(e_t) \\ s(e^-) & \text{if } t(e_s) \neq s(e_t) \text{ and } e_s \neq e_t \text{ and } j = s(e_t) \\ t(e^-) & \text{if } t(e_s) \neq s(e_t) \text{ and } e_s \neq e_t \text{ and } j = t(e_t) \\ j & \text{otherwise} \end{cases} \tag{2}$$

along with the extension to edges determined by not sending edges to (diagonal/reflexive edges or) loops.

Intuitively, if the entry and exit edges of G are neither identical nor adjacent, then ϕ_G maps them respectively to e^+ and e^-: otherwise, ϕ_G maps the entry edge to e^+ and everything else (vertices and edges to the vertex; loops to a loop) to $t(e^+)$. The rationale for the latter case is that it is the only really generic and consistent way for us to complete the definition of such a nontrivial **Dgph**-morphism from a flow graph, and in fact this sort of definitional guidance is perhaps the primary rationale for invoking category theory *ab initio.*

The following lemma is straightforward.

Lemma 11. *With $j, k \in V(G)$ with $j \neq k$ and $j', k' \in V(G')$ with $j' \neq k'$, $\phi_G(j) = \phi_G(k) \Rightarrow \{j, k\} = \{t(e_s), t(e_t)\}$; similarly, $\phi_{G'}(j') = \phi_{G'}(k') \Rightarrow \{j', k'\} = \{t(e'_s), t(e'_t)\}$.* □

If $V(G) \cap V(G') = \varnothing$, we define

$$G \otimes G' := G \sqcup G' / \sim, \tag{3}$$

where the equivalence relation on the disjoint (graph) union is determined, for $j, k \in V(G)$ with $j \neq k$ and $j', k' \in V(G')$ with $j' \neq k'$, by

$$\begin{aligned} (j,0) &\sim (j,0) \ \forall j; \\ (j',1) &\sim (j',1) \ \forall j'; \\ (j,0) \sim (k,0) &\iff (\phi_G(j) = \phi_G(k)) \wedge \star; \\ (j',1) \sim (k',1) &\iff (\phi_{G'}(j') = \phi_{G'}(k')) \wedge \star'; \\ (j,0) \sim (j',1) &\iff (\phi_G(j) = \phi_{G'}(j')) \wedge \star[j] \wedge \star[j']. \end{aligned} \tag{4}$$

where we use the shorthands $\star := (t(e'_s) \neq s(e'_t)) \wedge (e'_s \neq e'_t)$; $\star' := (t(e_s) \neq s(e_t)) \wedge (e_s \neq e_t)$; $\star[j] := (t(e_s) = s(e_t)) \wedge (j \in \{t(e_s), t(e_t)\}) \Rightarrow \star$, and $\star[j'] := (t(e'_s) = s(e'_t)) \wedge (j' \in \{t(e'_s), t(e'_t)\}) \Rightarrow \star'$, with an obvious extension to edges. (Here e'_s and e'_t denote the entry and exit edges of G'.)

Lemma 12. (4) *indeed defines an equivalence relation.*

The following lemma is straightforward.

Lemma 13. *If G and G' are flow graphs, then so is $G \otimes G'$. Furthermore, if e denotes the flow graph with a single edge, then $G \otimes e \cong e \otimes G \cong G$.* □

Thus in particular we have inclusions $i_G : \phi_G(G) \hookrightarrow G \otimes G'$ and $i'_{G'} : \phi_{G'}(G') \hookrightarrow G \otimes G'$ given respectively by $i_G(\phi_G(j)) = [(j, 0)]$ and $i'_{G'}(\phi_{G'}(j')) = [(j', 1)]$, where as per usual practice $[\cdot]$ indicates an equivalence class under $\sim$. $G \otimes G'$ is a flow graph formed by identifying the entry edges of G and G', and identifying the respective exit edges if this does not affect the interiors of the factors, and otherwise collapsing the smaller factor in a way that sufficiently extends the identification of entry edges.

Meanwhile, for $f \in \mathbf{Flow}(G, G_f)$ and $f' \in \mathbf{Flow}(G', G'_{f'})$, we define $f \otimes f' \in \mathbf{Flow}(G \otimes G', G_f \otimes G'_{f'})$ as follows (see also Fig. 7):

$$(f \otimes f')(k) := \begin{cases} [(f(j), 0)] & \text{if } k = [(j, 0)] \\ [(f'(j'), 1)] & \text{if } k = [(j', 1)] \end{cases} \tag{5}$$

along with the implied extension to edges.

$$\begin{tikzcd}
G \arrow[r, "\phi_G"] \arrow[d, "f"'] & \phi_G(G) \arrow[r, hook, "i_G"] & G \otimes G' \arrow[d, "f \otimes f'"'] & \phi_{G'}(G') \arrow[l, hook', "i'_{G'}"'] & G' \arrow[l, "\phi_{G'}"'] \arrow[d, "f'"] \\
G_f \arrow[r, "\phi_{G_f}"] & \phi_{G_f}(G_f) \arrow[r, hook, "i_{G_f}"] & G_f \otimes G'_{f'} & \phi_{G'_{f'}}(G'_{f'}) \arrow[l, hook', "i'_{G'_{f'}}"'] & G'_{f'} \arrow[l, "\phi_{G'_{f'}}"']
\end{tikzcd}$$

Fig. 7. The tensor product of morphisms in **Flow**.

Lemma 14. (5) *is well-defined.*

Theorem 4. **Flow** *is a monoidal category with tensor product $\otimes$ given by* (3) *and* (5)*, and with unit object the flow graph e consisting of a single edge.*

Corollary 2. $(\mathbf{Flow}, \otimes)$ *is a symmetric monoidal category.* □

6.3 Remarks

The series and parallel tensor operations described above are very similar in spirit to the composition operations encountered in the study of so-called *series-parallel graphs* [2,10] (cf. [9]). While the category-theoretical analysis of series and parallel tensor operations in the context of something like a system or wiring diagram has a very long history [1], a precise treatment appropriate to our development does not appear to be present in the literature.

7 Two-Terminal Graphs

Many of the considerations of the present paper have obvious analogues in the case of *two-terminal graphs* (TTGs). In particular, [27,29] describes a multiresolution decomposition of TTGs (cf. [11]) that is a more granular version of the PST. This *refined process structure tree* reduces via a straightforward graph transformation (similar to that in Sect. B) to constructing the SPQR tree [23].

Unfortunately, computing SPQR trees is a notoriously intricate exercise: indeed, a correct linear time algorithm was not actually implemented until 2000 [13], though an incorrect version of the same algorithm was first described in 1973 [14]. Today there is still not a completely explicit description of the correct linear time algorithm in the literature: for such an account it is necessary to refer to one of the two known publically available implementations in the C++ OGDF[10] and the Java jBPT[11] frameworks.[12,13]

While the computation and properties of the fundamental decomposition for TTGs are more involved than the PST, some analogues of the constructions detailed in this paper are simpler since TTGs are defined to omit loops. On the other hand, one minor complication relative to flow graphs that informs notions of coarsening and inclusion operads for TTGs is that some TTGs can have their sources and targets swapped. A more significant (and perhaps surprising) complication is that it is not clear how to define a canonical parallel tensor operation for TTGs: the principal difficulty is the unit object. Lacking such an operation would be a significant shortcoming relative to the framework for flow graphs, as parallel tensoring in **Flow** corresponds to introducing an if/else statement in control flow.

8 Conclusion

Besides applications to understanding and manipulating programs mentioned in Sect. 1, our particular notion of a flow graph naturally yields an interesting category that readily admits explicit representations and manipulations in (and of!) software. While some of the constructions involved are somewhat delicate and inelegant (for example, much of Sect. 6.2), this is due to properly accounting for degenerate cases that are of little practical concern but that nevertheless constrain practical and principled techniques for representing, reasoning about, and composing program artifacts.

Put another way, requiring that flow graphs exhibit category-theoretical desiderata places strong but satisfiable restrictions on them that can usefully inform the architecture of program analysis platforms, program synthesizers,

[10] http://www.ogdf.net/.

[11] https://code.google.com/archive/p/jbpt/.

[12] An alternative algorithm is in [28], but we are not aware of an implementation.

[13] For acyclic TTGs it is not hard to see that the analogue of SESE regions are vertex pairs (j,k) s.t. $D_{jk}D^{\dagger}_{jk} - \delta_{jk} = 1$, but the cyclic case is much harder.

compilers, etc. More generally, category theory allows us to address corner cases in the construction and manipulation of data structures whose resolution is not obvious.

Acknowledgements. We thank Brendan Fong, Artem Polyvyanyy, and David Spivak for helpful comments.

A Proofs

Proof (Lemma 4). The only thing to show is that e_3 pdom e_2. It must be the case that either e_2 pdom e_3 or e_3 pdom e_2, so assume the former. Since e_2 dom e_3 also, we must have that any source-target path traversing e_3 contains a cycle of the form $(e_2, \dots, e_3, \dots, e_2)$ by Lemma 2; deleting all cycles from this path yields a source-target path traversing e_2 but not e_3. Reversing this path yields a contradiction to the assumption that e_2 pdom e_3. □

Proof (Theorem 1). [Although our definition of the interior of a SESE differs in a slight but critical way from from [15], the proof is a mostly straightforward adaptation of the original attempt. That said, we also fix a minor gap of the original attempt for case (ii).]

Let (e_1, e_2) and (e'_1, e'_2) be distinct canonical SESE regions whose interiors are not disjoint, and let v be in their intersection. Since e_1 dom v and e'_1 dom v, it must be that either e_1 dom e'_1 or e'_1 dom e_1: assume the former w.l.o.g. Similarly, since e_2 pdom v and e'_2 pdom v, either e_2 pdom e'_2 or e'_2 pdom e_2: in the former case, $(e'_1, e'_2) \subset (e_1, e_2)$ and we are done, so assume the latter case. We now have three cases to consider: (i) $e_2 = e'_1$; (ii) $e_2 \neq e'_1$ and e'_1 dom e_2; and (iii) $e_2 \neq e'_1$ and e'_1 does not dominate e_2. We shall show that each case leads to a contradiction.

Case (i). Since in this case $e_2 = e'_1$, we have that e_2 dom v and e_2 pdom v, so it must be that any path from the source to the target that traverses v must contain a cycle of the form $(e_2, \dots, v, \dots, e_2)$ by Lemma 2. But this means that v cannot be in the interior of (e_1, e_2), a contradiction: hence case (i) cannot hold.

Case (ii). Since in this case e'_1 dom e_2 and generically e_1 dom e'_1, we may decompose any path γ_{02} from the source to e_2 (using an obvious notation) as $\gamma_{02} \equiv \gamma_{01}\gamma_{11'}\gamma_{1'2}$. Meanwhile since e_2 pdom e_1, we may decompose any path $\gamma_{1\infty}$ from e_1 to the target as $\gamma_{1\infty} \equiv \gamma_{12}\gamma_{2\infty}$. Taken together, these decompositions imply that we can decompose any path from the source to the target that traverses e'_1 as $\gamma_{01}\gamma_{11'}\gamma_{1'2}\gamma_{2\infty}$, so that e_2 pdom e'_1 and e'_1 pdom e_1.

Moreover, if there is a cycle that traverses e_1, it also traverses e_2 and *vice versa*, so we may write such a cycle as $\omega_{12} \equiv \gamma_{12}\gamma_{21}$, where $\gamma_{12} \equiv \gamma_{11'}\gamma_{1'2}$ as above. Hence such a cycle ω_{12} must traverse e'_1. Similarly, if there is a cycle that traverses e'_1, it also traverses e'_2 and *vice versa*, so we may write such a cycle as $\omega_{1'2'} \equiv \gamma_{1'2'}\gamma_{2'1'}$, where $\gamma_{1'2'}$ traverses e_2 since e'_2 pdom e_2. Hence such a cycle $\omega_{1'2'}$ must traverse e_2. It follows that (e'_1, e_2) is a SESE region.

Since both (e_1, e_2) and (e'_1, e'_2) are canonical SESE regions, we have that e_1 pdom e'_1 and e'_2 dom e_2. At the same time, e'_1 pdom e_1, so it must be that

$e_1 = e_1'$. It follows that (e_1, e_2') is also a SESE region, and therefore also that e_2 dom e_2', so it must be that $e_2 = e_2'$. This contradicts the hypothesis that (e_1, e_2) and (e_1', e_2') are distinct: hence case (ii) cannot hold.

Case (iii). Since in this case e_1' does not dominate e_2, there is a path γ_{02} from the source to e_2 that avoids e_1'. Suppose that e_1' does not postdominate e_2, i.e., suppose that there is a path $\gamma_{2\infty}$ from e_2 to the target that avoids e_1'. Then since e_2' pdom e_2, $\gamma_{2\infty}$ must traverse e_2'. But since e_1' dom e_2' and the concatenated path $\gamma \equiv \gamma_{02}\gamma_{2\infty}$ from the source to the target traverses e_2', it must be that $\gamma_{2\infty}$ traverses e_1', contradicting the assumption that e_1' does not postdominate e_2. Therefore since e_1' pdom e_2 and e_2 pdom v, we have that e_1' pdom v. Moreover, e_1' dom v, so any path from the source to the target that traverses v must contain a cycle of the form $(e_1', \dots, v, \dots, e_1')$ by Lemma 2. But this means that v cannot be in the interior of (e_1', e_2'). By contradiction, case (iii) cannot hold. $\square$

Proof (Lemma 5). Suppose w.l.o.g. that (e_0, e_∞) is not minimal. Then at least one of the following is true: (i) there exists a nondegenerate SESE region (e_0, e_1) such that e_∞ does not dominate e_1; (ii) there exists a nondegenerate SESE region (e_{-1}, e_∞) such that e_0 does not postdominate e_{-1}. Consider case (i), and assume w.l.o.g. that (e_0, e_1) is minimal (otherwise, we have at least one of case (i) or (ii) again). Then e_1 dom e_∞, so (e_1, e_∞) is a nondegenerate SESE region and we can write $(e_0, e_\infty) = (e_0, e_1) \cup (e_1, e_\infty)$. Exactly similar reasoning informs case (ii), and an induction establishes the lemma. $\square$

Proof (Theorem 2). Let (e_1, e_2) be a leaf of the PST. If $s(e_1)$ is in the interior of some other leaf (e_1', e_2') of the PST, then $e_1 = e_2'$. Therefore, $M_{s(e_2'), s(e_1')} = 1$ and any other vertices j with $M_{j, s(e_1')} = 1$ correspond to the remaining elements of $(e_1', e_2')^\circ$, which are leaves in the digraph G_M with adjacency matrix M. On the other hand, if $s(e_1)$ is not in the interior of some other leaf of the PST, then it is a target in G_M. The result follows. $\square$

Proof (Lemma 12). Since it is obvious from the structure of $\star[j]$ and $\star[j']$ that $(j', 1) \sim (j, 0) \iff (j, 0) \sim (j', 1)$, the only thing to show is transitivity. A (perhaps unnecessarily) mechanical proof consists of verifying each of the eight assertions $(\ell_{1,b_1}, b_1) \sim (\ell_{2,b_2}, b_2) \sim (\ell_{3,b_3}, b_3) \Rightarrow (\ell_{1,b_1}, b_1) \sim (\ell_{3,b_3}, b_3)$ for $(b_1, b_2, b_3) \in \{0, 1\}^3$ and $\ell_{1,b_1}, \ell_{2,b_2}, \ell_{3,b_3}$ distinct.

First, consider $(b_1, b_2, b_3) = (0, 0, 0)$: we must show in this case that $(\phi_G(\ell_{10}) = \phi_G(\ell_{20}) = \phi_G(\ell_{30})) \wedge \star$ implies $(\phi_G(\ell_{10}) = \phi_G(\ell_{30})) \wedge \star$, but this is trivial.

Next, consider $(b_1, b_2, b_3) = (0, 0, 1)$. Here we must show that $(\phi_G(\ell_{10}) = \phi_G(\ell_{20}) = \phi_{G'}(\ell_{31})) \wedge \star \wedge \star[\ell_{20}] \wedge \star[\ell_{31}]$ implies $(\phi_G(\ell_{10}) = \phi_{G'}(\ell_{31})) \wedge \star[\ell_{10}] \wedge \star[\ell_{31}]$. By Lemma 11, $\{\ell_{10}, \ell_{20}\} = \{t(e_s), t(e_t)\}$, so $t(e_s) = s(e_t)$ and $\star[\ell_{10}]$ is true, establishing the desired result.

For $(b_1, b_2, b_3) = (0, 1, 0)$, we must show that $(\phi_G(\ell_{10}) = \phi_{G'}(\ell_{21}) = \phi_G(\ell_{30})) \wedge \star[\ell_{10}] \wedge \star[\ell_{21}] \wedge \star[\ell_{30}]$ implies $(\phi_G(\ell_{10}) = \phi_G(\ell_{30})) \wedge \star$. By Lemma 11, $\{\ell_{10}, \ell_{30}\} = \{t(e_s), t(e_t)\}$, so $t(e_s) = s(e_t)$ and $\ell_{10}, \ell_{30} \in \{t(e_s), t(e_t)\}$. Since in the present case both $\star[\ell_{10}]$ and $\star[\ell_{30}]$ are true by assumption and we have just shown their hypotheses true, their mutual conclusion $\star$ is also true here. This yields the desired implication. (NB. Although $\star[\ell_{21}]$ is true in this case, neither its hypothesis nor its conclusion are.)

By symmetry, the last case we need to consider is $(b_1, b_2, b_3) = (0, 1, 1)$: we need to show here that $(\phi_G(\ell_{10}) = \phi_{G'}(\ell_{21}) = \phi_{G'}(\ell_{31})) \wedge \star[\ell_{10}] \wedge \star[\ell_{21}] \wedge \star'$ implies $(\phi_G(\ell_{10}) = \phi_{G'}(\ell_{31})) \wedge \star[\ell_{10}] \wedge \star[\ell_{31}]$. By Lemma 11, $\{\ell_{21}, \ell_{31}\} = \{t(e'_s), t(e'_t)\}$, so $t(e'_s) = s(e'_t)$ and $\star[\ell_{31}]$ is true, so we are done. $\square$

Proof (Lemma 14). We need to show that whenever $[(j, 0)] = [(j', 1)]$ we also have $[(f(j), 0)] = [(f'(j'), 1)]$. An equivalent assertion is that whenever $\phi_G(j) = \phi_{G'}(j')$, we also have $\phi_{G_f}(f(j)) = \phi_{G'_{f'}}(f'(j'))$. There are precisely four cases in which the hypothesis can hold, corresponding to the first four cases of (2) (note that the second case has four subcases). In the first case, both $[(f(j), 0)]$ and $[(f'(j'), 1)]$ must be the source of the entry edge in $G_f \otimes G'_{f'}$ since f and f' are morphisms in **Flow**; similarly, the other cases respectively give that both $[(f(j), 0)]$ and $[(f'(j'), 1)]$ must be the target of the entry edge, the source of the exit edge, and the target of the exit edge. $\square$

Proof (Theorem 4). We must establish two things: that $\otimes$ is a bifunctor, and that it satisfies the necessary coherence conditions.

To see that $\otimes$ is a bifunctor, first note that $(id_G \otimes id_{G'})([(j, 0)]) = [(j, 0)] = id_{G \otimes G'}([(j, 0)])$ by (5), and $(id_G \otimes id_{G'})([(j', 1)]) = [(j', 1)] = id_{G \otimes G'}([(j', 1)])$, so that $id_G \otimes id_{G'} = id_{G \otimes G'}$. Now we must show that $(g \otimes g') \circ (f \otimes f') = (g \circ f) \otimes (g' \circ f')$. But this is easily seen since, again by (5), we have $(g \otimes g')([(f(j), 0)]) = [(g(f(j)), 0)] = ((g \circ f) \otimes (g' \circ f'))([(j, 0)])$ and similarly $(g \otimes g')([(f'(j'), 1)]) = [(g'(f'(j')), 1)] = ((g \circ f) \otimes (g' \circ f'))([(j', 1)])$. Since the action on edges follows trivially, $\otimes$ is indeed a bifunctor.

To see that the putative tensor product is coherent, we first note that the triangle equation turns out to be trivial, so we need only verify the pentagon equation, which we recall in Fig. 8. The associator $\alpha_{G,G',G''} : (G \otimes G') \otimes G'' \to G \otimes (G' \otimes G'')$ is given by

$$\alpha_{G,G',G''} : \begin{cases} [([(j,0)],0)] & \mapsto [(j,0)] \\ [([(j',1)],0)] & \mapsto [([(j',0)],1)] \\ [(j'',1)] & \mapsto [([(j'',1)],1)] \end{cases} \tag{6}$$

along with the implied extension to edges. The explicit form of (6) makes it clear that the associator is bijective, and hence an isomorphism.

For notational convenience, let W, X, Y, Z denote flow graphs with $(w, x, y, z) \in V(W) \times V(X) \times V(Y) \times V(Z)$. The three steps on the top of the pentagon are

$$\begin{matrix} [([([(w,0)],0)],0)] \\ [([([(x,1)],0)],0)] \\ [([(y,1)],0)] \\ [(z,1)] \end{matrix} \mapsto \begin{matrix} [([(w,0)],0)] \\ [([([(x,0)],1)],0)] \\ [([([(y,1)],1)],0)] \\ [(z,1)] \end{matrix} \mapsto \begin{matrix} [(w,0)] \\ [([([(x,0)],0)],1)] \\ [([([(y,1)],0)],1)] \\ [([(z,1)],1)] \end{matrix} \mapsto \begin{matrix} [(w,0)] \\ [([(x,0)],1)] \\ [([([(y,0)],1)],1)] \\ [([([(z,1)],1)],1)] \end{matrix}. \tag{7}$$

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[r, "\alpha_{W,X,Y} \otimes id_Z"] \arrow[d, "\alpha_{W \otimes X,Y,Z}"'] & (W \otimes (X \otimes Y)) \otimes Z \arrow[r, "\alpha_{W,X \otimes Y,Z}"] & W \otimes ((X \otimes Y) \otimes Z) \arrow[d, "id_W \otimes \alpha_{X,Y,Z}"] \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[rr, "\alpha_{W,X,Y \otimes Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

Fig. 8. The pentagon equation.

while the two steps on the bottom of the pentagon are

$$\begin{matrix} [([([(w,0)],0)],0)] \\ [([([(x,1)],0)],0)] \\ [([(y,1)],0)] \\ [(z,1)] \end{matrix} \mapsto \begin{matrix} [([(w,0)],0)] \\ [([(x,1)],0)] \\ [([(y,0)],1)] \\ [([(z,1)],1)] \end{matrix} \mapsto \begin{matrix} [(w,0)] \\ [([(x,0)],1)] \\ [([([(y,0)],1)],1)] \\ [([([(z,1)],1)],1)] \end{matrix}. \tag{8}$$

The pentagon equation follows from the equality of the rightmost parts of (7) and (8), as does the theorem. □

B Stretching Flow Graphs

By inserting new vertices and edges, we can transform many "approximate" flow graphs into *bona fide* sub-flow graphs that can then be captured by the PST.

Lemma 15 ("Sketch of stretch"). *Let G be a flow graph. For each vertex $v \in G^\circ$, perform transformations indicated by the table below. The cumulative result of these transformations is well-defined; repeating them has no effect.* □

$d^+(v) > 1?$	$\perp$	$\perp$	$\perp$	$\perp$	$\top$	$\top$	$\top$	$\top$
$d^-(v) > 1?$	$\perp$	$\perp$	$\top$	$\top$	$\perp$	$\perp$	$\top$	$\top$
$d^0(v) = 1?$	$\perp$	$\top$	$\perp$	$\top$	$\perp$	$\top$	$\perp$	$\top$
old motif								
new motif	same	same	same		same			

Call the result of the process sketched in Lemma 15 the *stretching* of G. This construction is similar to the "normalization" of two-terminal graphs (see Sect. 7).

Corollary 3. *There is a bijective correspondence between induced subgraphs with single sources and targets and SESE regions in a stretching. In particular, any loop corresponds to a minimal SESE region in a stretching.* □

By considering the complete bipartite graph $K_{3,3}$, it is easy to show the following

Lemma 16. *There exists a planar flow graph with a nonplanar stretching.* □

References

1. Bainbridge, E.S.: Feedback and generalized logic. Inf. Control **31**, 75 (1976)
2. Bang-Jensen, J., Gutin, G.: Digraphs: Theory, Algorithms and Applications. Springer, Heidelberg (2009). https://doi.org/10.1007/978-1-84800-998-1
3. Boissinot, B., et al.: SSI properties revisited. ACM TECS **11S**, 21 (2012)
4. Brown, R., et al.: Graphs of morphisms of graphs. Electron. J. Comb. **15**, A1 (2008)
5. Brumley, D., Jager, I., Avgerinos, T., Schwartz, E.J.: BAP: a binary analysis platform. In: Gopalakrishnan, G., Qadeer, S. (eds.) CAV 2011. LNCS, vol. 6806, pp. 463–469. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-22110-1_37
6. Coecke, B. (ed.): New Structures for Physics. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-12821-9
7. Cooper, K.D., Torczon, L.: Engineering a Compiler, 2nd edn. Morgan Kaufmann, Burlington (2012)
8. Delaney, C., Marcolli, M.: Dyson-Schwinger equations in the theory of computation. In: Álvarez-Cónsul, L., Burgos-Gil, J.I., Ebrahimi-Fard, K. (eds.) AMS Feynman Amplitudes, Periods, and Motives (2015)
9. Dougherty, D.J., Gutiérrez, C.: Normal forms for binary relations. Theor. Comput. Sci. **360**, 228 (2006)
10. Duffin, R.J.: Topology of series-parallel networks. J. Math. Anal. Appl. **10**, 303 (1965)
11. Fugishige, S.: Canonical decompositions of symmetric submodular systems. Discret. Appl. Math. **5**, 175 (1983)
12. Gulwani, S.: Dimensions in program synthesis. In: PPDP (2010)
13. Gutwenger, C., Mutzel, P.: A linear time implementation of SPQR-trees. In: Marks, J. (ed.) GD 2000. LNCS, vol. 1984, pp. 77–90. Springer, Heidelberg (2001). https://doi.org/10.1007/3-540-44541-2_8
14. Hopcroft, J., Tarjan, R.: Dividing a graph into triconnected components. SIAM J. Comput. **2**, 135 (1973)
15. Johnson, R., Pearson, D., Pingali, K.: The program structure tree: computing control regions in linear time. In: PLDI (1994)
16. Kelly, G.M.: Basic Concepts of Enriched Category Theory. Cambridge University Press, Cambridge (1982)
17. Leinster, T.: Higher Operads, Higher Categories. Cambridge University Press, Cambridge (2004)
18. MacLane, S.: Categories for the Working Mathematician, 2nd edn. Springer, Heidelberg (2010)
19. Manin, Yu.I.: Renormalization and computation I: motivation and background. arXiv:0904.4921 (2009)
20. Markl, M., Shnider, S., Stasheff, J.: Operads in Algebra, Topology and Physics. AMS (2002)
21. Muchnick, S.S.: Advanced Compiler Design and Implementation. Morgan Kaufmann, Burlington (1997)
22. Nielson, F., Nielson, H.R., Hankin, C.: Principles of Program Analysis. Springer, Heidelberg (2010)
23. Polyvyanyy, A., Vanhatalo, J., Völzer, H.: Simplified computation and generalization of the refined process structure tree. In: Bravetti, M., Bultan, T. (eds.) WS-FM 2010. LNCS, vol. 6551, pp. 25–41. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-19589-1_2

24. Rupel, D., Spivak, D.I.: The operad of temporal wiring diagrams: formalizing a graphical language for discrete-time processes. arXiv:1307.6894 (2013)
25. Spivak, D.I.: The operad of wiring diagrams: formalizing a graphical language for databases, recursion, and plug-and-play circuits. arXiv:1305.0297 (2013)
26. Stasheff, J.: What is an operad? Not. AMS **51**, 630 (2004)
27. Tarjan, R.E., Valdes, J.: Prime subprogram parsing of a program. In: POPL (1980)
28. Tsin, Y.H.: Decomposing a multigraph into split components. In: CATS (2012)
29. Vanhatalo, J., Völzer, H., Koehler, J.: The refined process structure tree. Data Knowl. Eng. **68**, 793 (2009)
30. Zhang, F., D'Hollander, E.H.: Using hammock graphs to structure programs. IEEE Trans. Soft. Eng. **30**, 231 (2004)

An Exponential Lower Bound for Proofs in Focused Calculi

Raheleh Jalali[1,2](✉)

[1] Institute of Mathematics of the Czech Academy of Sciences, Prague, Czech Republic
jalali@math.cas.cz

[2] Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

Abstract. In [7], Iemhoff introduced a special form of sequent-style rules and axioms, which she called focused, and studied the relationship between the focused proof systems, the systems only consisting of this kind of rules and axioms, and the uniform interpolation of the logic that the system captures. Subsequently, as a negative consequence of this relationship, she excludes almost all super-intuitionistic logics from having these focused proof systems. In this paper, we will provide a complexity theoretic analogue of her negative result to show that even in the cases that these systems exist, their proof-length would computationally explode. More precisely, we will first introduce two natural subclasses of focused rules, called PPF and MPF rules. Then, we will introduce some **CPC**-valid (**IPC**-valid) sequents with polynomially short tree-like proofs in the usual Hilbert-style proof system for classical logic, or equivalently **LK** + **Cut**, that have exponentially long proofs in the systems only consisting of PPF (MPF) rules.

Keywords: Focused calculi · Propositional proof complexity · Feasible interpolation · Super-intuitionistic logics

1 Introduction

In the field of proof theory, proof systems, as the main players of the game, deserve to be considered as the objects of the study themselves. Regarding this matter, there are various problems to attack. One of them is investigating whether some special kinds of proof systems exist and if they do, what properties they or their corresponding logics posses, including the Craig or uniform interpolation of the corresponding logic, or the complexity of proofs in the given proof system.

These problems have been studied by many researchers (for instance [3,6,7]). In [6] and [7], Iemhoff inspected the relationship between a specific kind of proof system and the uniform interpolation property of the logic that the proof system captures. She introduced the so-called focused rules and axioms, and studied

Supported by the ERC Advanced Grant 339691 (FEALORA) and by the grant 19-05497S of GA ČR.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 342–355, 2019.
https://doi.org/10.1007/978-3-662-59533-6_21

the sequent calculi only consisting of these rules and axioms, which she named focused calculi. Roughly speaking, a focused axiom is just a modest generalization of the axioms of the classical sequent-style proof system, **LK**. A focused rule is a rule where only one side of its sequents, either left or right, is active in all the premises and in the conclusion and also all the variables in its premises occur in its conclusion. For instance, the usual conjunction and disjunction rules in **LK**, are focused, while the cut rule is not. After her formalization of the focused rules and focused axioms, she provided a method to prove that a super-intuitionistic logic enjoys the uniform interpolation property if it has a terminating focused proof system. Since there are only seven super-intuitionistic logics with the uniform interpolation property, she finally excluded almost all the super-intuitionistic logics (except at most seven of them) from having a focused proof system.

Inspired by Iemhoff's work, in [1] we proposed a generalization of focused rules, called semi-analytic rules, to cover a wider range of proof systems for a wider range of logics. Stated informally, in a semi-analytic rule, the side condition is relaxed and the formulas can appear freely in any side of the sequents in the premises and the conclusion. Iemhoff's results in [6] and [7] are then strengthened to also hold for these rules. It implies that many substructural logics and almost all super-intuitionistic logics (except at most seven of them) do not have a sequent style proof system only consisting of semi-analytic rules and focused axioms.

This paper is a sequel of [1] in its extension of the negative results of [6] and [7] to the remaining cases in which the interpolation property exists. For this purpose, we change our focus from the existence of a proof system of some kind to its efficiency to show an exponential lower bound for the focused proof systems of a certain sort. Beside the clear impacts in the study of focused rules, these lower bounds can also be considered as the basic steps in a universal approach to the proof complexity of the propositional proof systems. In such an approach, we are interested in investigating the proof lengths of a given sequence of tautologies in a generically given proof system with a certain form of axioms and rules. The method we use here is the well-known technique in proof complexity called the feasible interpolation. It reduces a problem in proof complexity to a problem in circuit complexity by extracting a Boolean circuit for an interpolant from a given proof for an implication, where the size of the circuit is polynomially bounded by the size of the proof. The feasible interpolation property for various classical calculi has been studied by Krajíček [9], Pudlák [10], and Pudlák and Sgall [13]. For the intuitionistic calculus, the feasible interpolation theorem was proved by Pudlák in [12] based on the feasible witnessing of the disjunction property developed in [14]. Buss and Pudlák in [15] and Buss and Mints in [14] studied the connection between intuitionistic propositional proof lengths and Boolean circuits. In [5], Hrubeš showed the connection is tighter in the sense that the circuit in question in [15] and [14] is monotone. Here we will use the technique of [5] as we will explain in a moment. For more information on feasible interpolation and its role in proof complexity, the reader is refered to [11].

In this paper, we will prove two lower bounds, one for the classical logic and the other for super-intuitionistic logics. For the first one, we will define a natural subclass of the focused rules, which we will call *polarity preserving focused*, PPF, rules. Then, we show that there are **CPC**-tautologies with exponential proof lengths in any proof system only consisting of PPF rules and focused axioms, which we call PPF calculi, while they have polynomial proof lengths in **LK**. This shows an exponential speed-up of the Frege-style proof system for classical logic with respect to any PPF calculus. To prove the similar exponential lower bound for intuitionistically valid formulas, we first define *monotonicity preserving focused*, MPF, rules and subsequently MPF calculi. Then, we will use the mentioned lower bound technique developed by Hrubeš in [4] and [5] to obtain an exponential lower bound for the lengths of proofs of particular **IPC**-tautologies in MPF calculi, while they have polynomial length proofs in **LK**.

2 Preliminaries

In this section, we will present some definitions and notions that will be needed in the rest of the paper.

Note that any finite object O that we use here, such as a formula or a proof, can be represented by a fixed suitable binary string and by $|O|$ we mean the length of the string representing the object.

In this paper, we work with the usual propositional language $\{\wedge, \vee, \neg, \rightarrow, \bot, \top\}$. By **IPC** and **CPC** we mean intuitionistic and classical propositional logics, respectively. By meta-language, we mean the language in which we define the sequent calculi. A meta-formula is defined inductively; an atom and a formula symbol are meta-formulas and we can construct new meta formulas using the existing ones and the connectives of the language. A meta-multiset is a set of meta-formulas and meta-multiset variables. By $V(A)$, we mean the atoms and meta-formula variables of the meta-formula A.

By a sequent $\Gamma \Rightarrow \Delta$, we mean an expression where Γ and Δ are multisets and it is interpreted as $\bigwedge \Gamma \rightarrow \bigvee \Delta$. A meta-sequent is essentially a sequent defined by meta-multisets. A rule is an expression of the form:

$$\frac{T_1, \cdots, T_n}{T}$$

where T_i's and T are meta-sequents. A sequent calculus is a set of rules.

By monotone **LK**, **mLK**, we mean the sequent calculus consisting of the axioms of **LK**, the structural rules (exchange, weakening, contraction), and its usual conjunction and disjunction rules.

A calculus G is *sound* for logic L, if $G \vdash \Gamma \Rightarrow \Delta$ implies $L \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$. It is called *complete* if $L \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$ implies $G \vdash \Gamma \Rightarrow \Delta$ and *feasibly complete* if the length of the tree-like proof is polynomially bounded by the sequent, i.e., there exists a tree-like proof π of $\Gamma \Rightarrow \Delta$ in G such that $|\pi| \leq |\Gamma \Rightarrow \Delta|^{O(1)}$. We say that logic M is an *extension of logic* L, if $L \vdash A$ implies $M \vdash A$. We say a calculus H is an *extension of a calculus* G, if for any rule of G, if all the premises are provable in H, then the consequence is also provable in H. Moreover, H is called an *axiomatic extension* of G, when all the provable sequents of G are considered as axioms of H, and H can add some rules to them.

A logic L is called *sub-classical* if **CPC** extends L. In the same way, a calculus G is called sub-classical if **LK** extends G.

A logic L (calculus G) has the *Craig interpolation* property when for any formula $\phi \to \psi$ (sequent $\Gamma \Rightarrow \Delta$), if $L \vdash \phi \to \psi$ ($G \vdash \Gamma \Rightarrow \Delta$) then there exists a formula θ such that $V(\theta) \subseteq V(\phi) \cap V(\psi)$ ($V(\theta) \subseteq V(\Gamma) \cap V(\Delta)$) and $L \vdash \phi \to \theta$ and $L \vdash \theta \to \psi$ ($G \vdash \Gamma \Rightarrow \theta$ and $G \vdash \theta \Rightarrow \Delta$). The calculus G has *feasible interpolation* if for any tree-like proof π of $\Gamma \Rightarrow \Delta$, there exists an interpolant θ such that $|\theta| \leq |\pi|^{O(1)}$.

3 Focused Calculi

In this section we will give the definitions of the focused axioms, rules and calculi, which are the building blocks of the rest of the paper.

Definition 3.1. A rule is called *focused* (a left focused rule, L, or a right focused rule, R) if it has one of the following forms:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_{r=1}^{m_i}\rangle_{i=1}^{n}}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n} L \qquad \frac{\langle\langle \Gamma_i \Rightarrow \bar{\phi}_{ir}, \Delta_i \rangle_{r=1}^{m_i}\rangle_{i=1}^{n}}{\Gamma_1, \cdots, \Gamma_n \Rightarrow \phi, \Delta_1, \cdots, \Delta_n} R$$

where Γ_i's and Δ_i's are meta-multiset variables, $\bar{\phi}_{ir}$ is a multi-set of formulas, and $\bigcup_{i,r} V(\phi_{ir}) \subseteq V(\phi)$. By the notation $\langle\langle \cdot \rangle_r \rangle_i$, we mean the sequents first range over $1 \leq r \leq m_i$ and then over $1 \leq i \leq n$.

Example 3.2. The usual conjunction and disjunction rules in **LK** are focused. On the other hand, the implication rules:

$$\frac{\Gamma \Rightarrow \phi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \phi \to \psi \Rightarrow \Delta, \Lambda} \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta}$$

are not focused, simply because both sides of the sequents are active.

Definition 3.3. A sequent is called a *focused axiom* if it is of the following form:

(1) Identity axiom: $(\phi \Rightarrow \phi)$
(2) Context-free right axiom: $(\Rightarrow \bar{\alpha})$
(3) Context-free left axiom: $(\bar{\beta} \Rightarrow)$
(4) Contextual left axiom: $(\Gamma, \bar{\phi} \Rightarrow \Delta)$
(5) Contextual right axiom: $(\Gamma \Rightarrow \bar{\phi}, \Delta)$

where Γ and Δ are meta-multiset variables and in 2–5, the set of the variables of any two elements of $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\phi}$ must be the same.

Example 3.4. It is easy to see that the axioms of **LK**, $(\phi \Rightarrow \phi)$, $(\Gamma \Rightarrow \top, \Delta)$ and $(\Gamma, \bot \Rightarrow \Delta)$ are focused. Here are some more examples which are not in **LK**:

$$\phi, \neg\phi \Rightarrow \quad , \quad \Rightarrow \phi, \neg\phi$$

$$\Gamma, \neg\top \Rightarrow \Delta \quad , \quad \Gamma \Rightarrow \Delta, \neg\bot$$

First let us investigate the power of focused rules and focused axioms. The natural question to ask is whether it is possible to have a calculus consisting only of these rules and axioms, that is complete for some given logic. For **CPC** the answer is yes, and the following theorem can be considered as a witness of the power and naturalness of focused axioms and rules.

Theorem 3.5. **CPC** *has a sequent calculus consisting only of focused rules and focused axioms.*

Proof. Consider a sequent calculus containing the usual axioms of **CPC** and the following axioms:

Axioms:

$$\frac{}{\phi \Rightarrow \phi} \qquad \frac{}{\phi, \neg\phi \Rightarrow} \qquad \frac{}{\Rightarrow \phi, \neg\phi}$$

$$\frac{}{\Gamma \Rightarrow \neg\bot, \Delta} \qquad \frac{}{\Gamma, \neg\top \Rightarrow \Delta}$$

The usual left and right rules for disjunction and conjunction and the following rules for implication:

$$\frac{\Gamma \Rightarrow \neg\phi, \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta} \qquad \frac{\Gamma_1, \neg\phi \Rightarrow \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \phi \to \psi \Rightarrow \Delta_1, \Delta_2}$$

And finally, for any combination $\neg\vee$, $\neg\wedge$, and $\neg\neg$ we have the corresponding right and left rules, using De Morgan's laws. For instance, we have

$$\frac{\Gamma \Rightarrow \neg\phi, \Delta}{\Gamma \Rightarrow \neg(\phi \wedge \psi), \Delta} \; R\neg\wedge$$

It is easy to check that all the rules of this sequent calculus are focused and the system is sound and complete for **CPC**. The proof of the completeness part is based on the observation that if $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is provable in the usual calculus for classical logic, then $\Gamma, \neg\Delta \Rightarrow \neg\Gamma', \Delta'$ is provable in the new calculus. The proof is an easy application of induction on the length of the usual **LK** proof of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. $\square$

So far, we have seen some definitions and a sequent calculus consisting only of focused axioms and rules. Now, it is time to examine how effective such a characterization can be. For this purpose, from now on we will restrict our investigations to two natural sub-classes of focused rules, polarity preserving focused, PPF rules, and monotonicity preserving focused, MPF rules.

Definition 3.6. Let $\mathcal{P}$ be a set of meta-formula variables or atomic constants. A meta-formula ψ is called $\mathcal{P}$*-monotone* if for any $\phi \in \mathcal{P}$, all occurrences of ϕ in ψ are positive, i.e., ϕ does not occur in the scope of negations or in the antecedents of implications. A multiset Γ of meta-formulas is called $\mathcal{P}$-monotone if all of its elements are $\mathcal{P}$-monotone.

A meta-formula is called *monotone* if it is constructed by conjunctions and disjunctions on meta-formula variables, atomic constants and variable-free formulas.

Remark 3.7. Note that since any variable-free formula is classically equivalent to $\top$ or $\bot$, then any monotone formula in our sense is classically equivalent to the usual monotone formulas i.e., the formulas constructed from atomic formulas by applying conjunctions and disjunctions. Therefore, from now on, in the classical settings, we always assume that a monotone formula has the mentioned simpler form.

Definition 3.8. A focused rule is called *polarity preserving*, PPF, if it preserves $\mathcal{P}$-monotonicity backwardly for any $\mathcal{P}$, i.e., if the antecedent of the consequence is $\mathcal{P}$-monotone, then the antecedents of all the premises are also $\mathcal{P}$-monotone. It is *monotonicity preserving*, MPF, if it is focused and preserves monotonicity backwardly, in the same way.

Example 3.9. All analytic focused rules in the language of **CPC**, the focused rules in which any formula in the premises is a subformula of a formula in the consequence, are both PPF and MPF.

3.1 The Classical Case

Let us first see a relationship between focused calculi and the Craig interpolation property.

Theorem 3.10. *Let G be a sequent calculus extending* **mLK** *and only consisting of focused rules and focused axioms. Then, G has feasible interpolation property. Moreover, if the rules are also* PPF *and Γ is $\mathcal{P}$-monotone, then $\Gamma \Rightarrow \Delta$ has a feasible $\mathcal{P}$-monotone interpolant.*

Proof. We need to prove that to any provable sequent $\Gamma \Rightarrow \Delta$, we can assign a formula C such that $G \vdash \Gamma \Rightarrow C$ and $G \vdash C \Rightarrow \Delta$ and $V(C) \subseteq V(\Gamma) \cap V(\Delta)$. Use induction on the length of the proof π of the sequent $\Gamma \Rightarrow \Delta$ in G. If $\Gamma \Rightarrow \Delta$ is a focused axiom, it is easy to see that in different cases of the focused axioms, the interpolant C is either ϕ or $\bot$ or $\top$. We check the case 4 of the focused axioms. The rest are similar. In this case, we have to find C such that $\Gamma, \bar{\phi} \Rightarrow C$ and $C \Rightarrow \Delta$. We claim that $C = \bot$ works here. Note that in the focused axioms, since Γ and Δ are meta-multiset variables, we can substitute anything for them. Hence, we have $\Gamma, \bar{\phi} \Rightarrow \bot$, since it is an instance of the axiom 4 when Δ is substituted by $\bot$. And $\bot \Rightarrow \Delta$ is an instance of the axiom $\bot$ in **mLK** which is weaker than the system G by assumption.

For the rules, suppose the last rule used in the proof π is the following left focused rule:

$$\frac{\langle\langle \Gamma_i, \bar{\phi}_{ir} \Rightarrow \Delta_i \rangle_r\rangle_i}{\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \Delta_1, \cdots, \Delta_n}$$

Then, by induction, there are formulas C_{ir} such that $\Gamma_i, \bar{\phi}_{ir} \Rightarrow C_{ir}$ and $C_{ir} \Rightarrow \Delta_i$. Using the right and left disjunction rules we have $\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bigvee_r C_{ir}$ and $\bigvee_r C_{ir} \Rightarrow \Delta_i$. By the left disjunction rule we have $\bigvee_i \bigvee_r C_{ir} \Rightarrow \Delta_1, \cdots, \Delta_n$. And if we substitute the sequents $\Gamma_i, \bar{\phi}_{ir} \Rightarrow \bigvee_r C_{ir}$ in the original left focused rule we get $\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \bigvee_r C_{1r}, \cdots, \bigvee_r C_{nr}$ and then using the right disjunction rule we get $\Gamma_1, \cdots, \Gamma_n, \phi \Rightarrow \bigvee_i \bigvee_r C_{ir}$.

Note that for any i and r, by induction we have $V(C_{ir}) \subseteq V(\Gamma_i \cup \{\phi_{ir}\}) \cap V(\Delta_i)$. Using this and the fact that for focused rules $\bigcup_{ir} V(\phi_{ir}) \subseteq V(\phi)$, we can easily show that $V(\bigvee_i \bigvee_r C_{ir}) \subseteq V(\Gamma \cup \{\phi\}) \cap V(\Delta)$, where $\Gamma = \Gamma_1, \cdots, \Gamma_n$ and $\Delta = \Delta_1, \cdots, \Delta_n$. Therefore, we have shown that $\bigvee_i \bigvee_r C_{ir}$ is the interpolant.

The case for a right focused rule is dual to the previous case.

The proof for the upper bound for the length of the interpolant goes as follows. We claim that our previously constructed interpolant C has the property $|C| \leq |\pi|^2$ and we will prove it by induction on π.

For the axioms, as we have seen, the interpolant is either ϕ (in the case that the sequent is of the form of the first axiom ($\phi \Rightarrow \phi$)) or $\bot$ or $\top$ (in other cases). In these cases, we have $|C| \leq |\pi|$.

For the left focused rules, we have shown that $C = \bigvee_i \bigvee_r C_{ir}$. Let $N_{\mathcal{R}}$ be the number of the premises of the rule $\mathcal{R}$, which is the last rule used in the proof. We have that the number of the formulas which appear in C, i.e. C_{ir}, is equal to $N_{\mathcal{R}}$. The rest of the symbols appeared in C are connectives, and the number of them is again equal to $N_{\mathcal{R}}$. Since the sequent $\Gamma \Rightarrow \Delta$ is the conclusion of a rule in G, the lengths of the proofs of its premises are less than the length of π and we can use the induction hypothesis for them. Then $|C| \leq \Sigma_{i,r}|C_{ir}| + N_{\mathcal{R}}$. By induction hypothesis we have $|C_{ir}| \leq |\pi_{ir}|^2$, where π_{ir} is the proof of the sequent whose interpolant is C_{ir}. But since the proof is tree-like, we have $\Sigma_{ir}|\pi_{ir}| \leq |\pi|$. It is easy to see that $|C| \leq \Sigma_{i,r}|\pi_{i,r}|^2 + N_{\mathcal{R}} \leq \Sigma_{i,r}|\pi_{i,r}|^2 + \Sigma_{i,r}|\pi_{i,r}| \leq (\Sigma_{i,r}|\pi_{i,r}|)^2 \leq |\pi|^2$, and the claim follows. We have used the fact that $N_{\mathcal{R}} \leq \Sigma_{i,r}|\pi_{i,r}|$. The latter is an easy consequence of the fact that the number of $\pi_{i,r}$ in total is $N_{\mathcal{R}}$.

Finally, for $\mathcal{P}$-monotonicity note that since Γ is $\mathcal{P}$-monotone and all the rules are PPF, all the antecedents in the proof must be $\mathcal{P}$-monotone, as well. Therefore, the interpolants of the axioms are $\mathcal{P}$-monotone. Because, for the axioms, except for the axiom $\phi \Rightarrow \phi$, the interpolants are variable-free and hence $\mathcal{P}$-monotone. And for the identity axiom $\phi \Rightarrow \phi$, the interpolant is ϕ itself which is also $\mathcal{P}$-monotone. Finally, since the interpolants are constructed by the interpolants of the axioms via disjunctions and conjunctions, the interpolant for $\Gamma \Rightarrow \Delta$ is also $\mathcal{P}$-monotone. $\square$

The following theorem is our first example of the mentioned ineffectiveness of the combination of focused axioms and PPF rules. It shows that none of the combinations of focused axioms and PPF rules can simulate the cut rule in a feasible way.

Corollary 3.11. *There is no calculus G consisting of only focused axioms and* PPF *rules, sound and feasibly complete for* **CPC**. *More precisely, if G is a complete calculus for* **CPC**, *then there exists a sequence of* **CPC**-*valid sequents $\phi_n \Rightarrow \psi_n$, with polynomially short tree-like proofs in the Hilbert-style system or equivalently in* $\mathbf{LK}+\mathbf{Cut}$ *such that $||\phi_n \Rightarrow \psi_n||_G$, the length of the shortest tree-like G-proof of $\phi_n \Rightarrow \psi_n$, is exponential in n. Therefore, the* PPF *rules together with focused axioms are either incomplete or feasibly incomplete for* **CPC**.

Proof. Assume that G is a calculus for **CPC** consisting of PPF rules and focused axioms. In the following, we bring the definitions for clique and coloring formulas from [8]. Note that we use $[n]$ to denote $\{1, 2, \cdots, n\}$. Let $Clique^k_n(\bar{p}, \bar{q})$ be the proposition asserting that $\bar{q}$ is a clique of size at least k on a graph with vertices $[n]$. There are $\binom{n}{2}$ atoms p_{ij} where $p_{ij} = 1$ if and only if there is an edge between nodes $\{i, j\} \in \binom{n}{2}$. There are also $k.n$ atoms q_{ui} where their intended meaning is to describe a mapping from $[k]$ to $[n]$. $Clique^k_n(\bar{p}, \bar{q})$ is the conjunction of the following clauses:

- $\bigvee_{i \in [n]} q_{ui}$, all $u \leq k$,
- $\neg q_{ui} \vee \neg q_{uj}$, all $u \in [k]$ and $i \neq j \in [n]$,
- $\neg q_{ui} \vee \neg q_{vi}$, all $u \neq v \in [k]$ and $i \in [n]$,
- $\neg q_{ui} \vee \neg q_{vj} \vee p_{ij}$, all $u \neq v \in [k]$ and $\{i, j\} \in \binom{n}{2}$.

The proposition $Color^m_n(\bar{p}, \bar{r})$ asserts that $\bar{r}$ is an m-coloring of the same graph represented by $\bar{p}$ and also uses $n.m$ atoms r_{ia} where $i \in [n]$ and $a \in [m]$. $Color^m_n(\bar{p}, \bar{r})$ is the conjunction of the following clauses:

- $\bigvee_{a \in [m]} r_{ia}$, all $i \in [n]$,
- $\neg r_{ia} \vee \neg r_{ib}$, all $a \neq b \in [m]$ and $i \in [n]$,
- $\neg r_{ia} \vee \neg r_{ja} \vee \neg p_{ij}$, all $a \in [m]$ and $\{i, j\} \in \binom{n}{2}$.

Note that by the formalization of the Clique formula, every occurrence of $\bar{p}$ in $Clique^k_n(\bar{p}, \bar{q})$ is positive (which means it is monotone in $\bar{p}$). We know that for $m < k$, the formula $\neg Clique^k_n(\bar{p}, \bar{q}) \vee \neg Color^m_n(\bar{p}, \bar{r})$ is a tautology in classical logic which implies that

$$Clique^k_n(\bar{p}, \bar{q}) \Rightarrow \neg Color^m_n(\bar{p}, \bar{r})$$

is **CPC**-valid.

First observe that by the Craig interpolation theorem for **CPC** and the fact that the antecedent is monotone in $\bar{p}$, there exists a monotone interpolant $I(\bar{p})$ such that

$$Clique^k_n(\bar{p}, \bar{q}) \Rightarrow I(\bar{p}) \Rightarrow \neg Color^m_n(\bar{p}, \bar{r})$$

which means that if the graph H represented by $\bar{p}$ has a k-clique then $I(\bar{p}) = 1$ and if H has an m-coloring then $I(\bar{p}) = 0$. In other words, if $I(\bar{p}) \neq 0$ then H does not have an m-coloring and if $I(\bar{p}) \neq 1$ then H does not have a k-clique. By the result in [2], every such monotone interpolant I must have exponential length in n for suitable polynomially bounded choices for k and m.

Secondly, define $\phi_n(\bar{p}, \bar{q}) = Clique^k_n(\bar{p}, \bar{q})$ and $\psi_n(\bar{p}, \bar{r}) = \neg Color^m_n(\bar{p}, \bar{r})$. We will show that this family of sequents, $\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r})$, serve as the **CPC**-valid sequents mentioned in the theorem. The idea is simple. First note that the fact that the sequent

$$Clique^k_n(\bar{p}, \bar{q}) \Rightarrow \neg Color^m_n(\bar{p}, \bar{r})$$

has a tree-like proof of the size $n^{O(1)}$ in the classical Hilbert-style proof system or equivalently $\mathbf{LK} + \mathbf{Cut}$ is a folklore well-known fact in the proof complexity community. Now pick π_n as the shortest tree-like proof of the sequent in G. Note that the antecedent of our sequent, $Clique_n^k(\bar{p}, \bar{q})$, is $\bar{p}$-monotone. Hence, by Theorem 3.10, the interpolant for the sequent $\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r})$ will be $\bar{p}$-monotone, as well. And since $\bar{p}$ are the only common variables and hence the only variables in the interpolant, the interpolant is monotone. However, G captures **CPC**. Therefore, the whole process provides a classical monotone interpolant for the sequent

$$Clique_n^k(\bar{p}, \bar{q}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r})$$

which we will call C_n. By Theorem 3.10, we have $|C_n| \leq |\pi_n|^2$. However, any such C_n should be exponentially long in n as we explained before. Therefore, the shortest proof π_n for our sequent is exponentially long.

□

3.2 The Intuitionistic Case

It is also possible to lower down the previous exponential lower bound to the level of the **IPC**-valid sequents. For that purpose we need a new form of interpolation and its preservation theorem.

Definition 3.12. A sequent is called a *strongly focused* axiom if it has one of the following forms:

(1) $\phi \Rightarrow \phi$
(2) $\Rightarrow \bar{\alpha}$
(3) $\bar{\beta} \Rightarrow$
(4) $\Gamma, \bar{\phi} \Rightarrow \Delta$
(5) $\Gamma \Rightarrow \bar{\phi}, \Delta$

where in (2) and (5), $\bar{\alpha}$ and $\bar{\phi}$ have no variable and Γ and Δ are meta-multiset variables.

Example 3.13. For the strongly focused axioms, note that all the axioms of **LK** are strongly focused. An example of a focused axiom which is not strongly focused is $(\Rightarrow \phi, \neg\phi)$. Since otherwise it would have been an instance of either 2 or 5, which is not possible. The reason is that ϕ can have a variable which must not appear in the right side of the sequent.

Definition 3.14. Let G and H be two sequent calculi. G has *H-monotone feasible interpolation with the exponent $m \geq 1$* if for any k and any sequent $S = (\Sigma \Rightarrow \Lambda_1, \cdots, \Lambda_k)$ if S is provable in G by a tree-like proof π and for any $1 \leq j \leq k$, $\Lambda_j \neq \emptyset$, then there exist formulas $|C_j| \leq |\pi|^m$ for $1 \leq j \leq k$ such that $(\Sigma \Rightarrow C_1, \cdots, C_k)$ and $(C_j \Rightarrow \Lambda_j)$ are provable in H and $V(C_j) \subseteq V(\Sigma) \cap V(\Lambda_j)$, where $V(A)$ is the set of the atoms of A. Moreover, if Σ is monotone, then C_j is also monotone for all $1 \leq j \leq k$. We call C_j's, the interpolants of the partition $\Lambda_1, \cdots, \Lambda_k$ of the succedent of the sequent S. The system G has *H-monotone feasible interpolation* if it has H-monotone feasible interpolation with some exponent $m \geq 1$.

Theorem 3.15. *Let G and H be two sequent calculi such that G is a set of strongly focused axioms, H extends* **mLK** *and any sequent in G is provable in H. Then G has H-monotone feasible interpolation with the exponent one.*

Proof. We will consider the strongly focused axioms one by one:

(1) In this case the sequent S is of the form $(\phi \Rightarrow \phi)$. Therefore, $\Lambda_1 = \phi$. Pick $C_1 = \phi$. It is easy to see that this C_1 works and since ϕ is monotone, C_1 is also monotone.
(2) For the case $(\Rightarrow \bar{\alpha})$, consider C_j to be $\bigvee \Lambda_j$. We can easily see that these C_j's work, using the left and right disjunction rules. For the variables, since $V(\bar{\alpha}) = \emptyset$, we have $V(C_j) \subseteq V(\emptyset) \cap V(\Lambda_j)$. And for the monotonicity, since $V(C_j) = \emptyset$, then C_j is monotone.
(3) The case $(\bar{\beta} \Rightarrow)$ does not happen.
(4) If S is of the form $\Gamma, \bar{\phi} \Rightarrow \Delta$ define $C_j = \bot$. First note that we have $\Gamma, \bar{\phi} \Rightarrow \bot, \bot, \cdots, \bot$ where in the right hand-side we have k many $\bot$'s. The reason is that this sequent is an instance of the axiom (4) itself. Moreover, for every j we have $\bot \Rightarrow \Lambda_j$ since it is an instance of the axiom $\bot$. And again $V(C_j) = \emptyset$.
(5) If S is of the form $(\Gamma \Rightarrow \bar{\phi}, \Delta)$ define $C_j = \bigvee(\Lambda_j \cap \bar{\phi})$. It is easy to see that this C_j works. Because, $C_j \Rightarrow \Lambda_j$ is an instance of an axiom. We also have $\Gamma \Rightarrow C_1, \cdots, C_k$, since in the right hand-side we will have the formula $\bar{\phi}$ (together with some other formulas which we will treat as the context) and it will become an instance of the same axiom. Note that since $V(\bar{\phi}) = \emptyset$, there is nothing to check for the variables. For the monotonicity, note that $V(C_j) = \emptyset$, therefore C_j is monotone.

Note that in all cases and for all $1 \leq j \leq k$, $|C_j| \leq |\pi|$.

The next theorem shows that MPF rules preserve the monotone feasible interpolation property. We will use this theorem later in the lower bound result that we have promised before.

Theorem 3.16. *(monotone feasible interpolation) Let G and H be two sequent calculi such that H extends* **mLK** *and axiomatically extends G by* MPF *rules. Then if G has H-monotone feasible interpolation property, so does H.*

Proof. To prove the theorem, we will prove the following claim:

Claim. Let G and H be two sequent calculi such that H extends **mLK** and axiomatically extend G by MPF rules and G has H-monotone feasible interpolation with the exponent m. Then for any H-provable sequent $\Gamma \Rightarrow \Delta$ and any non-trivial partition of Δ as $\Lambda_1, \cdots, \Lambda_k$ (non-trivial means that none of the Λ_j's are empty), there exist the required interpolants C_j as in the Definition 3.14 such that $\Sigma_j |C_j| \leq |\pi|^M$ where $M = m + 1$.

The proof uses induction on the H-length of π (the number of the rules of H in the proof π). First we will explain how to construct C_j's. Then we will prove the bound for the given construction.

If the H-length of π is zero, it means that the proof is in G. Hence the claim is clear by the assumption. There are two cases to consider based on the last rule of the proof.

∘ If the last rule used in the proof is a right focused one, then it is of the following form:

$$\frac{\langle\langle \Gamma_i \Rightarrow \bar{\phi}_{ir}, \Delta_i \rangle_r \rangle_i}{\Gamma \Rightarrow \phi, \Delta}$$

where $\Gamma = \Gamma_1, \cdots, \Gamma_n$ and $\Delta = \Delta_1, \cdots, \Delta_n$. And, again $\Lambda_1, \cdots, \Lambda_k$ are given such that they are non-empty and $\bigcup_{j=1}^{k} \Lambda_j = \Delta \cup \{\phi\}$. W.l.o.g. suppose $\phi \in \Lambda_1$ and we denote $\Lambda_1 - \{\phi\}$ by Λ'_1. Consider the case that all of the $\Lambda_{ij} = \Delta_i \cap \Lambda_j$ and $\bar{\phi}_{ir} \cup \Lambda'_{i1}$ are non-empty where $\Lambda'_{i1} = \Delta_i \cap \Lambda'_1$. By induction hypothesis for the premises, there exist formulas $D_{ir1}, \cdots, D_{irk}$ such that for every i, r and $j \neq 1$

$$D_{ir1} \Rightarrow \bar{\phi}_{ir}, \Lambda'_{i1} \quad , \quad D_{irj} \Rightarrow \Lambda_{ij} \quad , \quad \Gamma_i \Rightarrow D_{ir1}, \cdots, D_{irk}$$

Again, note that if some of Λ_{ij}'s or $\bar{\phi}_{ir}, \Lambda'_{i1}$ are empty, we can eliminate them from the partition to have a non-trivial partition and hence to apply the IH. Then in these cases, we can simply pick D_{irj} as $\bot$. Now using the rules $(R\vee)$, $(L\vee)$, $(R\wedge)$ and $(L\wedge)$, we get for every i and $j \neq 1$

$$\bigwedge_r D_{ir1} \Rightarrow \bar{\phi}_{ir}, \Lambda'_{i1} \ , \ \bigvee_r D_{irj} \Rightarrow \Lambda_{ij} \ , \ \Gamma_i \Rightarrow \bigwedge_r D_{ir1}, \bigvee_r D_{ir2}, \cdots, \bigvee_r D_{irk}$$

Note that in the right sequent, we first use $(R\vee)$ to get $\Gamma_i \Rightarrow D_{ir1}, \bigvee_r D_{ir2}, \cdots, \bigvee_r D_{irk}$, and then we can use the rule $(R\wedge)$. Now, we can substitute the left sequents in the original rule to get

$$\bigwedge_r D_{ir1} \Rightarrow \phi, \Lambda'_1$$

and using the rule $(L\wedge)$ we have

$$\bigwedge_i \bigwedge_r D_{ir1} \Rightarrow \phi, \Lambda'_1$$

We denote $\bigwedge_i \bigwedge_r D_{ir1}$ by C_1. Using the rule $(L\vee)$ for the sequents $\bigvee_r D_{irj} \Rightarrow \Lambda_{ij}$ we get

$$\bigvee_i \bigvee_r D_{irj} \Rightarrow \Lambda_j$$

and we denote $\bigvee_i \bigvee_r D_{irj}$ by C_j for $j \neq 1$. We can see that first using the rule $(R\vee)$ and after that using the rule $(R\wedge)$ we get

$$\Gamma \Rightarrow \bigwedge_i \bigwedge_r D_{ir1}, \bigvee_i \bigvee_r D_{ir2}, \cdots, \bigvee_i \bigvee_r D_{irk}$$

which is

$$\Gamma \Rightarrow C_1, \cdots, C_k$$

It only remains to check the variables. If a variable is in C_j, then it is in one of D_{irj}'s. By induction hypothesis we have $V(D_{ir1}) \subseteq V(\Gamma_1) \cap V(\{\{\bar{\phi}_{ir}\} \cup \Lambda'_{i1}\}) \subseteq V(\Gamma) \cap V(\{\{\phi\} \cup \Lambda'_1\})$ and $V(D_{irj}) \subseteq V(\{\Gamma_i\}) \cap V(\Lambda_{ij}) \subseteq V(\Gamma) \cap V(\Lambda_j)$, since the rule is occurrence preserving, and this is what we wanted.
∘ The case of the left focused rule is similar to the case for right.

For the monotonicity part, since the extending rules are MPF, it is easy to prove that if the antecedent of the consequence is monotone, then all the antecedents, everywhere in the proof up to the sequents in G, are also monotone. Since G has H-monotone feasible interpolation property, the interpolants in the base case are monotone. Finally, since the conjunctions and disjunctions do not change monotonicity, our constructed interpolants are also monotone.

For the upper bound part, use a similar proof to the corresponding part in Theorem 3.10, this time using the induction on π to show that $\Sigma_j |C_j| \leq |\pi|^M$. For the axioms note $|C_j| \leq |\pi|^m$ for $1 \leq j \leq k$ by the assumption that G has H-monotone feasible interpolation with the exponent m. Since the partition is non-trivial $k \leq |S| \leq |\pi|$, hence $\Sigma_{j=1}^k |C_j| \leq k|\pi|^m \leq |\pi|^{m+1} = |\pi|^M$.

For the rules, define X as the set of all (i, r, j)'s where D_{irj} is $\bot$ coming from handling the empty cases. It is clear that X has at most $N_{\mathcal{R}}$ elements, the number of the premises of the rule $\mathcal{R}$. We have $\Sigma_j |C_j| \leq \Sigma_{(i,r,j) \notin X} |D_{irj}| + |X| + N_{\mathcal{R}} \leq \Sigma_{ir} |\pi_{ir}|^M + 2N_{\mathcal{R}} \leq (\Sigma_{ir} |\pi_{ir}| + 1)^M \leq |\pi|^M$. The second inequality holds using the induction hypothesis and the third inequality holds because $N_{\mathcal{R}} \leq \Sigma_{ir} |\pi_{ir}|$ and $M \geq 2$.

Finally, the theorem is a clear consequence of the Claim. It is enough to apply the Claim to provide the formulas C_j such that $\Sigma_j |C_j| \leq |\pi|^M$ which implies $|C_j| \leq |\pi|^M$. □

Lemma 3.17. [5] *Let $A(\bar{p}, \bar{r_1})$ and $B(\bar{q}, \bar{r_2})$ be propositional formulas and $\bar{p}$, $\bar{q}$, $\bar{r_1}$ and $\bar{r_2}$ be mutually disjoint. Let $\bar{p} = p_1, \cdots, p_n$ and $\bar{q} = q_1, \cdots, q_n$. Assume that A is monotone in $\bar{p}$ or B is monotone in $\bar{q}$ and $A(\bar{p}, \bar{r_1}) \vee B(\neg\bar{p}, \bar{r_2})$ is a classical tautology. Then*

$$\bigwedge_{i=1}^{n} (p_i \vee q_i) \Rightarrow \neg\neg A(\bar{p}, \bar{r_1}), \neg\neg B(\bar{q}, \bar{r_2})$$

*is **IPC**-valid.*

Proof. For the details, the reader is referred to [5]. □

Theorem 3.18. *Let G and H be two sequent calculi such that H is sub-classical, extends* **mLK**, *axiomatically extends G by* MPF *rules and G has H-monotone feasible interpolation property. Then there exists a family of* **IPC***-valid sequents $\phi_n \Rightarrow \psi_n$ with the length of $\phi_n \Rightarrow \psi_n$ bounded by a polynomial in n such that either there exists some n such that $H \nvdash \phi_n \Rightarrow \psi_n$ or $||\phi_n \Rightarrow \psi_n||_H$, the shortest tree-like H-proof of $\phi_n \Rightarrow \psi_n$, is exponential in n. Therefore, the* MPF *rules together with strongly focused axioms are either incomplete or feasibly incomplete for* **IPC**.

Proof. The proof is similar and also inspired by the lower bound proof given in [5]. Similar to the proof of Corollary 3.11, consider the **CPC**-valid sequent

$$Clique_n^k(\bar{p}, \bar{r_2}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r_1})$$

which is equivalent to

$$\Rightarrow \neg Clique_n^k(\bar{p}, \bar{r_2}), \neg Color_n^m(\bar{p}, \bar{r_1})$$

Then, using the Lemma 3.17, if we rewrite $\neg Clique_n^k(\bar{p}, \bar{r_2})$ as $B(\neg\bar{p}, \bar{r_2})$ and $\neg Color_n^m(\bar{p}, \bar{r_1})$ as $A(\bar{p}, \bar{r_1})$, we can easily see that A is monotone in $\bar{p}$ and $A(\bar{p}, \bar{r_1}) \vee B(\neg\bar{p}, \bar{r_2})$ is a classical tautology. Hence, we can transfer the **CPC**-valid sequent

$$\Rightarrow \neg Clique_n^k(\bar{p}, \bar{r_2}), \neg Color_n^m(\bar{p}, \bar{r_1})$$

to a sequent of the form

$$\bigwedge_i (p_i \vee q_i) \Rightarrow \neg\neg A(\bar{p}, \bar{r_1}), \neg\neg B(\bar{q}, \bar{r_2})$$

valid in **IPC**. Now, let

$$\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r_1}), \theta_n(\bar{q}, \bar{r_2})$$

be this sequent. We will show that this family of sequents, $\phi_n(\bar{p}, \bar{q}) \Rightarrow \psi_n(\bar{p}, \bar{r_1}), \theta_n(\bar{q}, \bar{r_2})$, serve as the **IPC**-valid sequents mentioned in the theorem.

If for some n we have $H \nvdash \phi_n \Rightarrow \psi_n, \theta_n$, then the claim follows. Therefore, suppose that for every n we have $H \vdash \phi_n \Rightarrow \psi_n, \theta_n$. Let π_n be the shortest tree-like proof of the sequent $\phi_n \Rightarrow \psi_n, \theta_n$ in H. By Theorem 3.16, for every n, there exist monotone formulas $C_n(\bar{p})$ and $D_n(\bar{q})$ such that $|C_n| \leq |\pi_n|^{O(1)}$ and $|D_n| \leq |\pi_n|^{O(1)}$ and the followings are provable in H: $(\phi_n \Rightarrow C_n, D_n)$, $(C_n \Rightarrow \psi_n)$, $(D_n \Rightarrow \theta_n)$. Since H captures a sub-classical logic we have $(\phi_n \Rightarrow C_n, D_n)$, $(C_n \Rightarrow \psi_n)$, $(D_n \Rightarrow \theta_n)$ in **CPC**. Since $(\phi_n \Rightarrow C_n, D_n)$ is valid in classical logic, we have $C_n(\bar{p}) \vee D_n(\neg\bar{p}) = 1$. On the other hand, since A_n is classically equivalent to ψ_n we know that $C_n(\bar{p}) = 1$ implies $A(\bar{p}, \bar{r_1}) = 1$. Similarly, we have that $D_n(\bar{q}) = 1$ implies $B(\bar{q}, \bar{r_2}) = 1$. We Claim that $C_n(\bar{p})$ interpolates $\neg B(\neg\bar{p}, \bar{r_2}) \Rightarrow A(\bar{p}, \bar{r_1})$. One direction is proved. For the other direction, note that if $B(\neg\bar{p}, \bar{r_2}) = 0$ then $D_n(\neg\bar{p}) = 0$ and since $C_n(\bar{p}) \vee D_n(\neg\bar{p}) = 1$ we have $C_n(\bar{p}) = 1$. Hence the monotone formula C_n interpolates $\neg B(\neg\bar{p}, \bar{r_2}) \Rightarrow A(\bar{p}, \bar{r_1})$ or equivalently the sequent

$$Clique_n^k(\bar{p}, \bar{r_2}) \Rightarrow \neg Color_n^m(\bar{p}, \bar{r_1})$$

However, in the proof of the Corollary 3.11, we mentioned that any such monotone interpolant must have exponential length. Together with the fact that $|C_n(\bar{p})| \leq |\pi_n|^{O(1)}$, we can conclude that $||\phi_n \Rightarrow \psi_n, \theta_n||_H$ is exponential in n which implies the claim.

Corollary 3.19. *There is no calculus consisting only of strongly focused axioms and* MPF *rules, sound and feasibly complete for super-intuitionistic logics.*

Proof. This is an obvious consequence of Theorems 3.15, 3.16 and 3.18. The only point that we have to explain is that if a calculus G consisting only of strongly focused axioms and MPF rules is sound and complete for a super-intuitionistic logic, then G extends **mLK**. The reason is that G is complete for a super-intuitionistic logic and any calculus complete even for **IPC** extends **mLK**.

Aknowlegment. We are thankful to Pavel Pudlák and Amir Akbar Tabatabai for the invaluable discussions that we have had, and their helpful suggestions and comments on the earlier drafts of this paper. We are also thankful to Rosalie Iemhoff for our fruitful discussions on the different aspects of what we call *universal proof theory*.

References

1. Tabatabai, A.A., Jalali, R.: Universal proof theory: semi-analytic rules and interpolation. Manuscript (2019)
2. Alon, N., Boppana, R.: The monotone circuit complexity of boolean functions. Combinatorica **7**(1), 1–22 (1987)
3. Ciabattoni, A., Galatos, N., Terui, K.: Algebraic proof theory for substructural logics: cut-elimination and completions. Ann. Pure Appl. Logic **163**(3), 266–290 (2012)
4. Hrubeš, P.: A lower bound for intuitionistic logic. Ann. Pure Appl. Logic **146**(1), 72–90 (2007)
5. Hrubeš, P.: On lengths of proofs in non-classical logics. Ann. Pure Appl. Logic **157**(2–3), 194–205 (2009)
6. Iemhoff, R.: Uniform interpolation and sequent calculi in modal logic (2016). https://link.springer.com/article/10.1007/s00153-018-0629-0
7. Iemhoff, R.: Uniform interpolation and the existence of sequent calculi (2017)
8. Krajíček, J.: Proof complexity. In: Encyclopaedia of Mathematics and Its Applications, vol. 170, pp. 326–327. Cambridge University Press (2019)
9. Krajíček, J.: Interpolation theorems, lower bounds for proof systems, and independence results for bounded arithmetic. J. Symbolic Logic **62**(2), 457–486 (1997)
10. Pudlák, P.: Lower bounds for resolution and cutting plane proofs and monotone computations. J. Symbolic Logic **62**, 981–998 (1997)
11. Pudlák, P.: The lengths of proofs. In: Buss, S. (ed.) Handbook of Proof Theory, Studies in Logic and the Foundations of Mathematics, vol. 137, pp. 1–78. Elsevier, Amsterdam (1998)
12. Pudlák, P.: On the complexity of propositional calculus, sets and proofs. In: Logic Colloquium 1997, pp. 197–218. Cambridge University Press (1999)
13. Pudlák, P., Sgall, J.: Algebraic models of computation and interpolation for algebraic systems. DIMACS Series in Discrete Math. Theor. Comp. Sci. **39**, 279–295 (1998)
14. Buss, S., Mints, G.: The complexity of the disjunction and existence properties in intuitionistic logic. Ann. Pure Appl. Logic **99**, 93–104 (1999)
15. Buss, S., Pudlák, P.: On the computational content of intuitionistic propositional proofs. Ann. Pure Appl. Logic **109**, 46–94 (2001)

The Complexity of Multiplicative-Additive Lambek Calculus: 25 Years Later

Max Kanovich[1,4], Stepan Kuznetsov[2,4](✉), and Andre Scedrov[3,4]

[1] University College London, London, UK
m.kanovich@ucl.ac.uk
[2] Steklov Mathematical Institute of the RAS, Moscow, Russia
sk@mi-ras.ru
[3] University of Pennsylvania, Philadelphia, USA
scedrov@math.upenn.edu
[4] National Research University Higher School of Economics, Moscow, Russia

Abstract. The Lambek calculus was introduced as a mathematical description of natural languages. The original Lambek calculus is NP-complete (Pentus), while its product-free fragment with only one implication is polynomially decidable (Savateev). We consider Lambek calculus with the additional connectives: conjunction and disjunction. It is known that this system is PSPACE-complete (Kanovich, Kanazawa). We prove, in contrast with the polynomial-time result for the product-free Lambek calculus with one implication, that the derivability problem is still PSPACE-complete even for a very small fragment $(\backslash, \wedge)$, including one implication and conjunction only. PSPACE-completeness is also provided for the $(\backslash, \vee)$ fragment, which includes only one implication and disjunction. Categorial grammars based on the original Lambek calculus generate exactly the class of context-free languages (Gaifman, Pentus). The class of languages generated by Lambek grammars extended with conjunction is known to be closed under intersection (Kanazawa), and therefore includes all finite intersections of context-free languages and, moreover, images of such intersections under alphabetic homomorphisms. We show that the same closure under intersection holds for Lambek grammars extended with disjunction, even for our small $(\backslash, \vee)$ fragment.

Keywords: Lambek calculus · Lambek grammars · Completeness · PSPACE-completeness

1 Introduction

Lambek calculus has been invented to analyze natural and artificial languages by means of categorial grammars [4,17,19,20]. Though the original Lambek calculus can describe only context-free languages [23], it has been proven to be NP-complete [24], even if we confine ourselves to the product-free Lambek calculus

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 356–372, 2019.
https://doi.org/10.1007/978-3-662-59533-6_22

equipped only with the left implication and the right implication [27]. On the contrary, the product-free Lambek calculus, with only one implication, is known to be decidable in polynomial time [26], see also [15]. It is known [2,3] that already the fragment with only one implication is sufficient to generate all context-free languages.

This paper is focused on the complexity issues for Lambek calculus extended with two additional connectives: additive conjunction and disjunction. This calculus is presented on Table 1 in the form of a sequent calculus. Notice that antecedents of sequents are linearly ordered sequences of formulae, not sets or multisets.

Table 1. The Inference rules of Lambek calculus with conjunction and disjunction

I	$\dfrac{}{A \vdash A}$		
L$\backslash$	$\dfrac{\Phi \vdash A \qquad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, \Phi, (A \backslash B), \Sigma_2 \vdash C}$	**R**$\backslash$	$\dfrac{A, \Sigma \vdash B}{\Sigma \vdash (A \backslash B)}$ (Σ is not empty)
L$/$	$\dfrac{\Phi \vdash A \qquad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, (B / A), \Phi, \Sigma_2 \vdash C}$	**R**$/$	$\dfrac{\Sigma, A \vdash B}{\Sigma \vdash (B / A)}$ (Σ is not empty)
L$\cdot$	$\dfrac{\Sigma_1, A, B, \Sigma_2 \vdash C}{\Sigma_1, (A \cdot B), \Sigma_2 \vdash C}$	**R**$\cdot$	$\dfrac{\Sigma_1 \vdash A \qquad \Sigma_2 \vdash B}{\Sigma_1, \Sigma_2 \vdash (A \cdot B)}$
L$\vee$	$\dfrac{\Sigma_1, A, \Sigma_2 \vdash C \qquad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, (A \vee B), \Sigma_2 \vdash C}$	**R**$\vee$	$\dfrac{\Sigma \vdash A}{\Sigma \vdash (A \vee B)} \quad \dfrac{\Sigma \vdash B}{\Sigma \vdash (A \vee B)}$
L$\wedge$	$\dfrac{\Sigma_1, A, \Sigma_2 \vdash C}{\Sigma_1, (A \wedge B), \Sigma_2 \vdash C}$ $\dfrac{\Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, (A \wedge B), \Sigma_2 \vdash C}$	**R**$\wedge$	$\dfrac{\Sigma \vdash A \qquad \Sigma \vdash B}{\Sigma \vdash (A \wedge B)}$

As shown above on the example of the Lambek calculus without additive connectives, there are two different ways of measuring complexity for extensions of the Lambek calculus. The first one is the standard notion of algorithmic complexity of the derivability problem for the calculus in question. For the Lambek calculus with additive connectives, 25 years ago, Kanovich [10] and Kanazawa [9] show that its derivability problem is PSPACE-complete. Here we strengthen this result and prove PSPACE-hardness for the smallest possible fragments, with only two connectives: $\mathcal{L}(\backslash, \wedge)$, with only one implication and additive conjunction, and $\mathcal{L}(\backslash, \vee)$, with one implication and disjunction. The first result is presented in Sect. 2. The second result is similar, so we give only a sketch of the proof, in Appendix A. The upper PSPACE bound is known for the whole Lambek calculus with additive connectives [9,10], [13, Sect. 8] and therefore inherited by its fragments, $\mathcal{L}(\backslash, \wedge)$ and $\mathcal{L}(\backslash, \vee)$.

The other complexity measure is the expressive power of categorial grammars based on a given calculus. A categorial grammar $\mathcal{G}$ is a triple $\langle \Sigma, \triangleright, H \rangle$, where Σ is a finite alphabet, $\triangleright$ is a finite binary correspondence between letters of Σ and Lambek formulae (these formulae could also include additive connectives), and H is a formula. A non-empty word $w = a_1 \dots a_n$ over Σ is accepted by $\mathcal{G}$, if there exist formulae $A_1, \dots, A_n$ such that $a_i \triangleright A_i$ $(i = 1, \dots, n)$ and $A_1, \dots, A_n \vdash H$ is a derivable sequent. The language generated by $\mathcal{G}$ is the set of all accepted words.

For Lambek grammars extended with conjunction, Kanazawa [8] proves that, in addition to context-free languages, they can generate finite intersections of such languages and images of such intersections under alphabetic homomorphisms (*i.e.,* homomorphisms which map letters to letters). In Sect. 3 we prove the dual result, that Lambek grammars enriched with disjunction have the same property. Namely, we show that $\mathcal{L}(\backslash, \vee)$, the product-free fragment with only one implication and disjunction, is already sufficient to generate finite intersections of such languages and images of such intersections under alphabetic homomorphisms.

2 PSPACE-Hardness of the Fragment $\mathcal{L}(\backslash, \wedge)$

Within our fragment $\mathcal{L}(\backslash, \wedge)$, we intend to encode quantified Boolean statements of the form:

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \ldots \exists x_{2n-1} \forall x_{2n} \, (C_1 \vee C_2 \vee \cdots \vee C_m) \tag{1}$$

Here $(C_1 \vee C_2 \vee \cdots \vee C_m)$ is a DNF over the Boolean variables x_1, x_2, $\ldots x_{2n}$.

Definition 1. *We express validity of (1) in terms of* **the winning strategy** *given by a binary tree of height* $2n+1$*, the nodes of which are labelled as follows.*

The root is labelled by "$\exists x_1$*" and has only one outgoing edge the end of which is labelled by "*$\forall x_2$*". In its turn, this node has two outgoing edges the ends of which are labelled by the same "*$\exists x_3$*".*

By induction, for $1 \leq k \leq n$*, each of the nodes on the level* $2k-1$ *is labelled by "*$\exists x_{2k-1}$*", and each of the nodes on the level* $2k$ *is labelled by "*$\forall x_{2k}$*".*

At the node "$\exists x_{2k-1}$*",* **the choice move of the proponent** *is to label the unique outgoing edge either by* t_{2k-1}*,* **meaning** x_{2k-1} **be true***, or by* f_{2k-1}*,* **meaning** x_{2k-1} **be false***. Being at the next node, "*$\forall x_{2k}$*",* **the opponent** *responds by labeling two outgoing edges by* t_{2k} *and* f_{2k}*, resp.*

Lastly, on the final level $2n+1$*, each terminal node* v *is labelled by some* C_ℓ *so that, collecting the sequence of* α_1*,* α_2*,* $\ldots$*,* α_{2n-1}*,* α_{2n} *that label the respective edges along the branch leading from the root "*$\exists x_1$*" to this leaf* v*, we get:*

$$C_\ell(\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, \alpha_{2n}) = \top \tag{2}$$

We illustrate the challenges we have to answer to with Example 1.

Example 1. We consider the following statement (which is invalid):

$$\exists x_1 \forall x_2 \, (C_1 \vee C_2) = \exists x_1 \forall x_2 \, ((x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)) \tag{3}$$

To provide (2), we express C_1 and C_2 as the formulas E_1 and E_2, resp.

$$E_1 = (f_2 \backslash (t_1 \backslash \top)) \;\equiv\; ((t_1 \cdot f_2) \backslash \top) \tag{4}$$

$$E_2 = (t_2 \backslash (f_1 \backslash \top)) \;\equiv\; ((f_1 \cdot t_2) \backslash \top) \tag{5}$$

Following [9,10], we intend to express the "choice move" $\exists x_1$ as $(t_1 \wedge f_1)$, and the "branching move" $\forall x_2$ as $(t_2 \vee f_2)$, resulting in the following encoding sequent:

$$(t_1 \wedge f_1),\ (t_2 \vee f_2),\ (E_1 \wedge E_2) \vdash \top \tag{6}$$

Taking $(t_1 \wedge f_1),\ (t_2 \vee f_2)$ as a sequence, we assume that these formulas should be executed *in the natural order*. Starting with $(t_1 \wedge f_1)$, we have to prove either

$$t_1,\ (t_2 \vee f_2),\ (E_1 \wedge E_2) \vdash \top \tag{7}$$

or

$$f_1,\ (t_2 \vee f_2),\ (E_1 \wedge E_2) \vdash \top \tag{8}$$

Since both sequents are not derivable, we might have concluded that (6) was not derivable and, hence, it was in a proper correlation with the invalid (3).
However, if we first apply $(t_2 \vee f_2)$, the related sequents turn out to be derivable

$$(t_1 \wedge f_1),\ t_2,\ (E_1 \wedge E_2) \vdash \top \tag{9}$$

$$(t_1 \wedge f_1),\ f_2,\ (E_1 \wedge E_2) \vdash \top \tag{10}$$

which shows that in fact (6) is derivable and, hence, fails to express the invalid (3).

The intuitive remedy proposed by [9,10,18] is to force the correct order of actions by means of "leading" q_i. E.g., here we can express the "choice move" $\exists x_1$ and the "branching move" $\forall x_2$ as the following formulas adjusted

$$(q_0 \backslash ((t_1 \cdot q_1) \wedge (f_1 \cdot q_1))) \tag{11}$$

$$(q_1 \backslash ((t_2 \cdot q_2) \vee (f_2 \cdot q_2))) \tag{12}$$

resulting in the correct non-provable encoding sequent, something like that

$$q_0,\ (q_0 \backslash ((t_1 \cdot q_1) \wedge (f_1 \cdot q_1))),\ (q_1 \backslash ((t_2 \cdot q_2) \vee (f_2 \cdot q_2))),\ (q_2 \backslash (E_1 \wedge E_2)) \vdash \top \tag{13}$$

The challenge of implementing this approach within $\mathcal{L}(\backslash, \wedge)$ consists of two parts:

(a) get rid off the disjunctions, in the absence of the full duality of $\wedge$ and $\vee$;
(b) get rid off the positive products of the form $(A \backslash (B_1 \cdot B_2))$

2.1 The Relative Negation and Double Negation (Non-commutative)

Definition 2. *In our encodings we will use the following abbreviation. We fix an atomic proposition b, and define 'relative negation' A^b by:* $A^b = (A \backslash b)$.

Our relative negation can be seen as a non-commutative analogue of the linear logic negation [5], which is defined by $A^\perp = A \multimap \perp$.

As for the relative "double negation", the novelty of our approach is that we are in favour of the "asymmetric" $A^{bb} = ((A \backslash b) \backslash b)$, *because of its nice properties proven in Lemma 1.*

We use also the following notation for the towers of double negations:

$$A^{[0]} = A, \quad A^{[k+1]} = (A^{[k]} \backslash b) \tag{14}$$

Remark 1. The "double negation" in the symmetrical form: ${}^bA^b = (b/(A\backslash b))$, has received recognition as being appropriate and logical within a non-commutative linear logic framework (see [1]).

E.g., the natural $A \vdash {}^bA^b$ is valid, in contrast to our A^{bb}, see Lemma 1(e).

However, the crucial Lemma 1(a) is destroyed with ${}^bA^b$, which is the reason for our "non-logical" choice of A^{bb}. ■

For a sequence $\Gamma = A_1, A_2, .., A_s$, by Γ^{bb} we denote the sequence $A_1^{bb}, A_2^{bb}, .., A_s^{bb}$.

Lemma 1. (a) *The following rules are derivable in Lambek calculus, $s \geq 1$:*

$$\frac{A_1,\ A_2,\ \ldots,\ A_s \vdash C}{A_1^{bb},\ A_2^{bb},\ \ldots,\ A_s^{bb} \vdash C^{bb}} \tag{15}$$

(b) *Though $\vee$ and $\wedge$ are not fully dual: $(A \wedge B)^b \not\vdash (A^b \vee B^b)$, the following equivalence fits our purposes:*

$$A^b \wedge B^b \vdash (A \vee B)^b \quad \textit{and} \quad (A \vee B)^b \vdash A^b \wedge B^b \tag{16}$$

(c) *To simulate branching, we will use the derivable rule:*

$$\frac{\Gamma,\ A,\ \Delta \vdash C \qquad \Gamma,\ B,\ \Delta \vdash C}{\Gamma^{bb},\ (A^b \wedge B^b)^b,\ \Delta^{bb} \vdash C^{bb}} \tag{17}$$

(d) *With $G_i = (q_{i-1}\backslash B)$, the crucial rule of "leading" q_{i-1} is given by:*

$$\frac{\dfrac{\Gamma,\ \gamma,\ B,\ \Delta \vdash C}{\Gamma,\ \gamma,\ q_{i-1},\ G_i,\ \Delta \vdash C}}{\Gamma^{bb},\ (\gamma \cdot q_{i-1})^{bb},\ G_i^{bb},\ \Delta^{bb} \vdash C^{bb}} \tag{18}$$

(e) Essential complications *are caused by the fact that $A \vdash A^{bb}$ is not valid.*

Lemma 2. *If c does not occur in $A_1, \ldots, A_n, B$, then the sequent $A_1^{cc}, \ldots, A_n^{cc} \vdash B^{cc}$ is equiderivable with $A_1, \ldots, A_n \vdash B$.*

Proof. The right-to-left direction is due to Lemma 1(i). For the left-to-right direction, we use the reversibility of $\mathbf{R}\backslash$:

$$B \backslash c, (A_1 \backslash c) \backslash c, \ldots, (A_n \backslash c) \backslash c \vdash c.$$

By induction on k, let us show derivability of

$$A_{n-k}, \ldots, A_n, B \backslash c, (A_1 \backslash c) \backslash c, \ldots, (A_{n-k-1} \backslash c) \backslash c \vdash c.$$

Induction base ($k = 0$) is given above. For the induction step, apply Lemma 11 below, which yields derivability of

$$A_{n-k}, \ldots, A_n, B \backslash c, (A_1 \backslash c) \backslash c, \ldots, (A_{n-k-2} \backslash c) \backslash c \vdash A_{n-k-1} \backslash c$$

and reverse the $\mathbf{R}\backslash$ rule. Finally, we get $A_1, \ldots, A_n, B \backslash c \vdash c$, and one more application of Lemma 11 yields the necessary $A_1, \ldots, A_n \vdash B$. ■

2.2 Complexity of the fragment $\mathcal{L}(\backslash, \wedge)$

Remark 2. Because of Lemma 1, for the sake of readability, here we will conceive of the formula $((A \cdot B)\backslash C)$ as abbreviation for $(B\backslash (A\backslash C))$. In particular, $(A \cdot B)^b$ is abbreviation for $(B\backslash (A\backslash b))$. The formula $(A \vee B)^b$ is conceived of as abbreviation for $((A\backslash b) \wedge (B\backslash b))$.

Theorem 1. *The fragment $\mathcal{L}(\backslash, \wedge)$ is PSPACE-hard.*

Proof. The direction from winning trees to derivable sequents is provided by Corollary 1.

By running from the leaves of the winning tree, labelled by some C_ℓ, to its root "$\exists x_1$", we have to address the following issues:

(a) With one and the same sequent of *polynomial size*, deal with the exponential number of branches and their sequences of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, \alpha_{2n}$ that label the respective edges along the branch leading from the root to some leaf v.
(b) In particular, verify "polynomially" the corresponding equalities (2).

■

Remark 3. To guarantee the proper order of the inference rules applied, we use the "leading" q_0, q_1, $\ldots$, q_{2n-1}, q_{2n}, and $c_{\ell,2n}$, $c_{\ell,2n-1}, \ldots$, $c_{\ell,2}$, $c_{\ell,1}$, $c_{\ell,0}$. The latter $c_{\ell,i}$ is used to keep one and the same C_ℓ in the process of verifying (2).

2.3 Verifying the Equality (2)

We start with (b), assuming that the sequence α_1, α_2, $\ldots$, α_{2n-1}, α_{2n} is *fixed.*

Definition 3. *Let F_ℓ denote: $(q_{2n}\backslash c_{\ell,2n})$, and H_ℓ denote: $(c_{\ell,0}\backslash (e_0\backslash e_0))$. For $1 \le i \le 2n$, let $E_{\ell,i}$ denote the formula: $(c_{\ell,i}\backslash (t_i\backslash c_{\ell,i-1}))$, if the conjunct C_ℓ contains the variable x_i; and $E_{\ell,i}$ denote the formula: $(c_{\ell,i}\backslash (f_i\backslash c_{\ell,i-1}))$, if the conjunct C_ℓ contains the variable $\neg x_i$; and $E_{\ell,i}$ denote the formula: $((c_{\ell,i}\backslash (t_i\backslash c_{\ell,i-1})) \wedge (c_{\ell,i}\backslash (f_i\backslash c_{\ell,i-1})))$, if C_ℓ contains neither x_i, nor $\neg x_i$. We introduce their "closed" versions:*

$$\widetilde{F} = \bigwedge_{\ell=1}^{m} F_\ell, \quad \widetilde{H} = \bigwedge_{\ell=1}^{m} H_\ell, \quad \widetilde{E_i} = \bigwedge_{\ell=1}^{m} E_{\ell,i} \tag{19}$$

Lemma 3. *In case (2) holds, a sequent of the specific form is derivable:*

$$e_0^{bb}, \alpha_1^{bb}, \alpha_2^{bb}, \ldots, \alpha_{2n-2}^{bb}, \alpha_{2n-1}^{bb}, (\alpha_{2n} \cdot q_{2n})^{bb},\ \Delta_n^{bb}\ \vdash\ e_0^{bb} \tag{20}$$

where Δ_n is a sequence of formulas: $\Delta_n = \widetilde{F}, \widetilde{E}_{2n}, \widetilde{E}_{2n-1}, \ldots, \widetilde{E_2}, \widetilde{E_1}, \widetilde{H}$.

NB: Notice that Δ_n *does not depend* on particular α_1, α_2, $\ldots$, α_{2n-1}, α_{2n}.

Proof. Since $\alpha_{i-1}, \alpha_i, c_{\ell,i}, E_{\ell,i,\alpha_i} \vdash (\alpha_{i-1} \cdot c_{\ell,i-1})$, by a simple inverse induction on i, we can "consume" all of the $\alpha_i, c_{\ell,i}$ with getting the sequents derivable:

$$e_0, \alpha_1, \alpha_2, \ldots, \alpha_{2n}, q_{2n}, F_\ell, E_{\ell,2n}, E_{\ell,2n-1}, \ldots E_{\ell,2}, E_{\ell,1}, H_\ell \vdash e_0$$

and (see the rule $\mathbf{L}\wedge$)

$$e_0, \alpha_1, \alpha_2, \ldots, \alpha_{2n-2}, \alpha_{2n-1}, \alpha_{2n}, q_{2n}, \varDelta_n \vdash e_0$$

resulting in (20) with the help of Lemma 1. ■

2.4 Simulating the Opponent's and Proponent's Moves

Now we are ready to simulate the moves in the play.

Lemma 4. *For any sequence $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, \alpha_{2n}$, labeling the branch that leads from the root to "$\forall x_{2n}$", the opponent move at "$\forall x_{2n}$" is to label two outgoing edges by t_{2n} and f_{2n} resp. We simulate the move by the derivable sequent:*

$$e_0^{[6]}, \alpha_1^{[6]}, \alpha_2^{[6]}, \ldots, \alpha_{2n-2}^{[6]}, (\alpha_{2n-1}^{[4]} \cdot q_{2n-1})^{[2]}, G_{2n}^{[2]}, \varDelta_n^{[6]} \vdash e_0^{[6]} \tag{21}$$

where

$$G_{2n} = (q_{2n-1} \backslash \big((t_{2n} \cdot q_{2n})^{[3]} \wedge (f_{2n} \cdot q_{2n})^{[3]}\big)^{[1]}) \tag{22}$$

Proof. Having got two sequences at hand

$$\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, t_{2n},$$

and

$$\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, f_{2n},$$

by Lemma 3 we have

$$e_0^{[2]}, \alpha_1^{[2]}, \alpha_2^{[2]}, \ldots, \alpha_{2n-2}^{[2]}, \alpha_{2n-1}^{[2]}, (t_{2n} \cdot q_{2n})^{[2]}, \varDelta_n^{[2]} \vdash e_0^{[2]} \tag{23}$$

and

$$e_0^{[2]}, \alpha_1^{[2]}, \alpha_2^{[2]}, \ldots, \alpha_{2n-2}^{[2]}, \alpha_{2n-1}^{[2]}, (f_{2n} \cdot q_{2n})^{[2]}, \varDelta_n^{[2]} \vdash e_0^{[2]} \tag{24}$$

by Lemma 1(c) we produce

$$e_0^{[4]}, \alpha_1^{[4]}, \alpha_2^{[4]}, \ldots, \alpha_{2n-2}^{[4]}, \alpha_{2n-1}^{[4]}, \Big((t_{2n} \cdot q_{2n})^{[3]} \wedge (f_{2n} \cdot q_{2n})^{[3]}\Big)^{[1]}, \varDelta_n^{[4]} \vdash e_0^{[4]}$$

and conclude, Lemma 1(d), with the sequent (21) where G_{2n} is given by (22). ■

Lemma 5. *For the shorter sequence $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$, labeling the one-edge shorter branch that leads from the root to "$\exists x_{2n-1}$", the proponent move at "$\exists x_{2n-1}$" is to label the outgoing edge by α_{2n-1}.*
We simulate the move by the derivable sequent:

$$e_0^{[8]}, \alpha_1^{[8]}, \alpha_2^{[8]}, \ldots, \alpha_{2n-3}^{[8]}, (\alpha_{2n-2}^{[6]} \cdot q_{2n-2})^{[2]}, G_{2n-1}^{[2]}, G_{2n}^{[4]}, \varDelta_n^{[8]} \vdash e_0^{[8]} \tag{25}$$

where

$$G_{2n-1} = (q_{2n-2} \backslash \big((t_{2n-1}^{[4]} \cdot q_{2n-1})^{[2]} \wedge (f_{2n-1}^{[4]} \cdot q_{2n-1})^{[2]}\big)) \tag{26}$$

Proof. Lemma 4 provides

$$e_0^{[6]}, \alpha_1^{[6]}, \alpha_2^{[6]}, \ldots, \alpha_{2n-2}^{[6]},\ (\alpha_{2n-1}^{[4]} \cdot q_{2n-1})^{[2]},\ G_{2n}^{[2]},\ \Delta_n^{[6]}\ \vdash\ e_0^{[6]}$$

and, hence,

$$e_0^{[6]}, \alpha_1^{[6]}, \alpha_2^{[6]}, \ldots, \alpha_{2n-2}^{[6]},\ \Big((t_{2n-1}^{[4]} \cdot q_{2n-1})^{[2]} \wedge (f_{2n-1}^{[4]} \cdot q_{2n-1})^{[2]}\Big),\ G_{2n}^{[2]},\ \Delta_n^{[6]}\ \vdash\ e_0^{[6]}$$

By Lemma 1(d) we conclude with the desired (25). ■

Corollary 1. *If the statement (1) is valid then the following sequent is derivable in Lambek:*

$$(e_0^{[6n]} \cdot q_0)^{[2]}, G_1^{[2]}, G_2^{[4]}, \ldots, G_{2n-1}^{[4n-2]}, G_{2n}^{[4n]},\ \Delta_n^{[6n+2]}\ \vdash\ e_0^{[6n+2]} \tag{27}$$

where

$$G_1 = (q_0 \backslash \big((t_1^{[6n-2]} \cdot q_1)^{[2]} \wedge (f_1^{[6n-2]} \cdot q_1)^{[2]}\big)) \tag{28}$$

$$G_2 = (q_1 \backslash \big((t_2^{[6n-6]} \cdot q_2)^{[3]} \wedge (f_2^{[6n-6]} \cdot q_2)^{[3]}\big)^{[1]}) \tag{29}$$

$$\ldots$$

$$G_{2n-1} = (q_{2n-2} \backslash \big((t_{2n-1}^{[4]} \cdot q_{2n-1})^{[2]} \wedge (f_{2n-1}^{[4]} \cdot q_{2n-1})^{[2]}\big)) \tag{30}$$

$$G_{2n} = (q_{2n-1} \backslash \big((t_{2n} \cdot q_{2n})^{[3]} \wedge (f_{2n} \cdot q_{2n})^{[3]}\big)^{[1]}) \tag{31}$$

Proof. By the bottom-up induction following the previous lemmas. ■

The direction from derivable sequents to winning trees is provided by Lemma 6.

Lemma 6. *If the sequent (27) is derivable in Lambek then the statement (1) is valid.*

Proof Sketch. Being derivable in Lambek calculus, the sequent (27) is derivable in linear logic. Replacing b with $\bot$, we get that $A^{bb} \equiv A$, resulting in that we can confine ourselves to Horn-like formulas, similar to (11) and (12), with the leading propositions from Remark 3. In its turn, such a Horn-like derivation can be transformed into a Horn-like tree program (see [11,12,18]), which in fact happens to be a winning strategy for the statement (1).

This concludes the proof of Lemma 6 and thereby the proof of Theorem 1. ■

In fact, we have proved a more general result.

Corollary 2. *Let L be a calculus that includes $\mathcal{L}(\backslash, \wedge)$, with or without Lambek's restriction, and is in turn included in linear logic. Then the fragment of L, which uses only one implication and conjunction, is PSPACE-hard.*

Proof. Given an instance of quantified Boolean formula (1), we take the sequent (27) and prove that there exists a winning tree if and only if (27) is derivable in L. Namely, if there is a winning tree, that sequent is derivable in $\mathcal{L}(\backslash, \wedge)$ with Lambek's restriction, and thereby in the corresponding fragment of L. On the other hand, if that sequent is derivable in L, then, repeating proof of Lemma 6 for the derivation in linear logic, we conclude that there exists a winning tree. ■

We can also modify this technique to establish PSPACE-hardness for the fragment $\mathcal{L}(\backslash, \vee)$, which includes only one implication and disjunction.

Theorem 2. *The fragment $\mathcal{L}(\backslash, \vee)$ is PSPACE-hard.*

We give a proof sketch in Appendix A.

3 Grammars Based on the Lambek Calculus with Disjunction

Theorem 3. *For any fragment of the Lambek calculus with conjunction and disjunction, which includes at least one division operation, $\backslash$, and disjunction, $\vee$, the class of languages generated by categorial grammars based on this calculus (in particular, the class of languages generated by $\mathcal{L}(\backslash, \vee)$-grammars) is closed under finite intersections.*

This theorem immediately yields the following corollary.

Corollary 3. *Grammars based on $\mathcal{L}(\backslash, \vee)$ can generate arbitrary finite intersections of context-free languages.*

Moreover, $\mathcal{L}(\backslash, \vee)$ also captures images of such intersections under *alphabetic homomorphisms.* A alphabetic homomorphism is a mapping $h\colon \Sigma_1^+ \to \Sigma_2^+$ of words over one alphabet to words of another one, such that $h(\Sigma_1) \subseteq \Sigma_2$ and $h(uv) = h(u)\,h(v)$ for any $u, v \in \Sigma_1^+$. The class of languages generated by $\mathcal{L}(\backslash, \vee)$-grammars is closed under alphabetic homomorphisms. Indeed, if the grammar $\mathcal{G} = \langle \Sigma_1, \rhd, H \rangle$ generates language M, then $\mathcal{G}_h = \langle \Sigma_2, \rhd_h, H \rangle$, where $a \rhd_h A$ iff $b \rhd A$ for some $b \in h^{-1}(a)$, generates $h(M)$. This yields the following stronger corollary.

Corollary 4. *Grammars based on $\mathcal{L}(\backslash, \vee)$ can generate all language of the form $h(M_1 \cap \ldots \cap M_k)$, where M_1, $\ldots$, M_k are context-free and h is a alphabetic homomorphism.*

Notice that this extension of Corollary 3 is non-trivial, since $h(M_1 \cap M_2)$ is not always equal to $h(M_1) \cap h(M_2)$. There is an example by Păun [22] of a language which is not a finite intersection of context-free languages, but can be obtained from such an intersection by applying a alphabetic homomorphism: $\{a^{2n^2} \mid n \geq 1\} = h(\{(a^n b^n)^n \mid n \geq 1\})$, where $h(a) = h(b) = a$.

Before proving Theorem 3, we establish several technical lemmata. The first one is a simplified version of Kanazawa's [9] Lemma 13.

Definition 4. *Let the set of variables include two disjoint subsets,* Var_1 *and* Var_2*. A formula is called a P_i-formula if it includes only variables from* Var_i *($i = 1, 2$).*

Lemma 7. *Let Γ and Δ sequences consisting of P_1-formulae and P_2-formulae, in an arbitrary order. Let B be a P_2-formula and C be a P_1-formula. Then the sequent $\Gamma, B, \Delta \vdash C$ is not derivable.*

Proof. Induction on a cut-free derivation. The sequent in question could not be in axiom, because then $B = C$, and P_1-formulae and P_2-formulae do not intersect.

Now consider the last rule applied in the derivation. If it is a one-premise rule, i.e., one of $\mathbf{L}\cdot$, $\mathbf{R}/$, $\mathbf{R}\backslash$, $\mathbf{L}\wedge$, $\mathbf{R}\vee$, then its premise also satisfies the conditions of the lemma, and such a sequent, by induction hypothesis, could not be derivable. Contradiction. The same happens with $\mathbf{L}\vee$ and $\mathbf{R}\wedge$, where both premises are not derivable by induction hypothesis. For $\mathbf{R}\cdot$, induction hypothesis yields non-derivability of the premise into which the B formula goes.

The most tricky cases are $\mathbf{L}\backslash$ and $\mathbf{L}/$. We consider the former; the latter is dual. Recall that $\mathbf{L}\backslash$ is a rule of the form

$$\frac{\Phi \vdash E \quad \Sigma_1, F, \Sigma_2 \vdash C}{\Sigma_1, \Phi, E \setminus F, \Sigma_2 \vdash C}\ \mathbf{L}\backslash$$

Now the question is where comes B. There are three possible cases.

Case 1: B is in Σ_1 or Σ_2. In this case, the right premise satisfies the condition of the lemma, and is therefore not derivable by induction hypothesis.
Case 2: B is in Φ. In this case, let us consider $E \setminus F$, which is either a P_1-formula or a P_2-formula. If $E \setminus F$ is a P_1-formula, then so is E, and the left premise, $\Phi \vdash E$, satisfies the condition of the lemma and is not derivable by induction. If $E \setminus F$ is a P_2-formula, then so is F, and now the right premise $\Sigma_1, F, \Sigma_2 \vdash C$, satisfies the condition of the lemma, and induction hypothesis yields its non-derivability.
Case 3: $B = E \setminus F$. The right premise satisfies the condition of the lemma (F is a P_2-formula and C is a P_1-formula), and is therefore not derivable my induction hypothesis. ∎

The next 4 lemmas are proved by straightforward induction on derivation. We put their proofs in Appendix B.

Definition 5. *Define the notion of* strictly positive *occurrence of a subformula inside a formula:*

- *A is strictly positive in itself;*
- *C occurs strictly positively in $A \setminus B$ if and only if it occurs strictly positively in B; the same for $B \,/\, A$;*
- *C occurs strictly positively in $A \cdot B$ if and only if if occurs strictly positively in A or in B; the same for $A \vee B$ and $A \wedge B$.*

Lemma 8 (Disjunctive Property). *Let F_1 and F_2 be arbitrary formulae, and $E_1, \ldots, E_n$ be formulae without $\wedge$ in which subformulae of the form $A \vee B$ do not occur strictly positively. Then the derivability $E_1, \ldots, E_n \vdash F_1 \vee F_2$ implies the derivability of $E_1, \ldots, E_n \vdash F_i$ for $i = 1$ or 2.*

Lemma 9. *If $F_1, \ldots, F_n$ do not include variable b, then $F_1, \ldots, F_n \vdash b$ is not derivable.*

Lemma 10. *If $F_1, \ldots, F_\ell, E_1 \setminus b, \ldots, E_k \setminus b, b \vdash b$ is derivable and $F_1, \ldots, F_\ell$ do not include b, then $k = \ell = 0$.*

Lemma 11. *If $F_1, \ldots, F_\ell, E_1 \setminus b, \ldots, E_k \setminus b \to b$ is derivable and $F_1, \ldots, F_\ell$ do not include b, then $F_1, \ldots, F_\ell, E_1 \setminus b, \ldots, E_{k-1} \setminus b \to E_k$ is derivable.*

The following lemma is the key one for the proof of Theorem 3.

Lemma 12. *Let $A_1, \ldots, A_n, C$ be P_1-formulae, $B_1, \ldots, B_n, D$ be P_2-formulae, and let b be a fresh variable, $b \notin \mathrm{Var}_1 \cup \mathrm{Var}_2$. Also suppose that no formula of the form $E \vee F$ occurs in A_1, ..., A_n, B_1, ..., or B_n strictly positively. Then the sequent*

$$((A_1 \setminus b) \vee (B_1 \setminus b)) \setminus b, \ldots, ((A_n \setminus b) \vee (B_n \setminus b)) \setminus b \vdash ((C \setminus b) \vee (D \setminus b)) \setminus b$$

is derivable if and only if so are $A_1, \ldots, A_n \vdash C$ and $B_1, \ldots, B_n \vdash D$.

In the notations of Subsect. 2.1, the first sequent of this lemma can be shortly written as $(A_1^b \vee B_1^b)^b, \ldots, (A_n^b \vee B_n^b)^b \vdash (C^b \vee D^b)^b$. Though $(A^b \vee B^b)^b$ is not equivalent to $A \wedge B$, and even not equivalent to $(A \wedge B)^{bb}$, this sequent happens to be equiderivable with $A_1 \wedge B_1, \ldots, A_n \wedge B_n \vdash C \wedge D$, which Kanazawa [9] used for his intersection construction with additive conjunction.

Proof. The **"if"** part is straightforwardly established by direct derivation.

For the **"only if"** part we first use the reversibility of $\mathbf{R}\setminus$ and $\mathbf{L}\vee$, which yields derivability of the following two sequents:

$$C \setminus b, ((A_1 \setminus b) \vee (B_1 \setminus b)) \setminus b, \ldots, ((A_n \setminus b) \vee (B_n \setminus b)) \setminus b \vdash b$$
$$D \setminus b, ((A_1 \setminus b) \vee (B_1 \setminus b)) \setminus b, \ldots, ((A_n \setminus b) \vee (B_n \setminus b)) \setminus b \vdash b.$$

Let us analyze the derivation of the first sequent. We claim derivability of $K_1, \ldots, K_n, C \setminus b \vdash b$, where each K_i is either A_i or B_i. In order to prove it, consider a more general statement, the derivability of

$$K_{n-k}, \ldots, K_n, C \setminus b, (A_1^b \vee B_1^b)^b, \ldots, (A_{n-k-1}^b \vee B_{n-k-1}^b)^b \vdash b.$$

This statement is proved by induction on k. Indeed, for $k = 0$ derivability of this sequent was shown above. For the induction step, suppose that

$$K_{n-k}, \ldots, K_n, C \setminus b, (A_1^b \vee B_1^b)^b, \ldots, (A_{n-k-1}^b \vee B_{n-k-1}^b)^b \vdash b$$

is derivable and apply Lemma 11, which yields derivability of

$$K_{n-k}, \ldots, K_n, C \setminus b, (A_1^b \vee B_1^b)^b, \ldots, (A_{n-k-2}^b \vee B_{n-k-2}^b)^b \vdash A_{n-k-1}^b \vee B_{n-k-1}^b.$$

Now apply the Disjunctive Property (Lemma 8) and obtain derivability of

$$K_{n-k}, \ldots, K_n, C \setminus b, (A_1^b \vee B_1^b)^b, \ldots, (A_{n-k-2}^b \vee B_{n-k-2}^b)^b \vdash K_{n-k-1} \setminus b,$$

where K_{n-k-1} is either A_{n-k-1} or B_{n-k-1}. Reversion of $\mathbf{L}\backslash$ yields the necessary

$$K_{n-(k+1)}, K_{n-k}, \dots, K_n, C \setminus b, (A_1^b \vee B_1^b)^b, \dots, (A_{n-k-2}^b \vee B_{n-k-2}^b)^b \vdash b.$$

In the end of the induction, for $k = n-1$, we get $K_1, \dots, K_n, C \setminus b \vdash b$, and one more application of Lemma 11 yields $K_1, \dots, K_n \vdash C$.

Now recall that C is a P_1-formula, and each of $K_1, \dots, K_n$ is a P_1-formula or a P_2-formula. If $K_i = B_i$ for some i, i.e., it is a P_2-formula, then $K_1, \dots, K_n \vdash C$ is not derivable by Lemma 7. Thus, for all i we have $K_i = A_i$, and obtain the needed sequent $A_1, \dots, A_n \vdash C$.

The same reasoning applied to $D \setminus b, (A_1^b \vee B_1^b)^b, \dots, (A_n^b \vee B_n^b)^b \vdash b$ yields $B_1, \dots, B_n \vdash D$. ∎

Lemma 12, together with Lemma 2 of Subsect. 2.1, yield the following corollary:

Corollary 5. *Let $A_1, \dots, A_n, C$ be P_1-formulae, $B_1, \dots, B_n, D$ be P_2-formulae, and let b and c be fresh variables ($b, c \notin \mathrm{Var}_1 \cup \mathrm{Var}_2$, $b \neq c$). Then the sequent*

$$((A_1^{cc})^b \vee (B_1^{cc})^b)^b, \dots, ((A_n^{cc})^b \vee (B_1^{cc})^b)^b \vdash ((C^{cc})^b \vee (D^{cc})^b)^b$$

is derivable if and only if so are $A_1, \dots, A_n \vdash C$ and $B_1, \dots, B_n \vdash D$.

Proof. The only strictly positive subformula or A_i^{cc} and B_j^{cc} is c. Thus, there is no strictly positive subformula the form $E \vee F$, and we can apply Lemma 12. This lemma yields the fact that

$$((A_1^{cc})^b \vee (B_1^{cc})^b)^b, \dots, ((A_n^{cc})^b \vee (B_1^{cc})^b)^b \vdash ((C^{cc})^b \vee (D^{cc})^b)^b$$

is derivable if and only if so are $A_1^{cc}, \dots, A_n^{cc} \vdash C^{cc}$ and $B_1^{cc}, \dots, B_n^{cc} \vdash D^{cc}$. For these two sequents, we apply Lemma 2 and replace these sequents with equiderivable ones, $A_1, \dots, A_n \vdash C$ and $B_1, \dots, B_n \vdash D$. ∎

Now we are ready to prove the main result of this section.

Proof (of Theorem 3). Consider two categorial grammars over the same alphabet, $\mathcal{G}_1 = \langle \Sigma, \triangleright_1, H_1 \rangle$ and $\mathcal{G}_2 = \langle \Sigma, \triangleright_2, H_2 \rangle$. Without loss of generality we can suppose that all formulae of $\mathcal{G}_i$ are P_i-formulae (otherwise just rename the variables). Construct the new grammar $\mathcal{G} = \langle \Sigma, \triangleright, H \rangle$, where, for each $a \in \Sigma$ we postulate $a \triangleright ((A^{cc})^b \vee (B^{cc})^b)^b$ for any A and B such that $a \triangleright_1 A$ and $a \triangleright_2 A$; $H = ((H_1^{cc})^b \vee (H_2^{cc})^b)^b$. Here b and c are fresh variables: b and c are distinct and do not occur in $\mathcal{G}_1$ or $\mathcal{G}_2$. By Corollary 5 a word $a_1, \dots a_n$ is accepted by $\mathcal{G}$ if and only if it is accepted by both $\mathcal{G}_1$ and $\mathcal{G}_2$. Therefore, the language generated by $\mathcal{G}$ is exactly the intersection of languages generated by $\mathcal{G}_1$ and $\mathcal{G}_2$. ∎

4 Concluding Remarks

In this paper we have proved two refined results on the complexity of the Lambek calculus enriched either with conjunction or disjunction. Namely, we have established PSPACE-completeness for small fragments $\mathcal{L}(\backslash, \wedge)$ and $\mathcal{L}(\backslash, \vee)$. Notice that the encoding used in this paper is more involved than the encodings from [6,8,10,18], because here we were not allowed to use the product (multiplicative conjunction) and one of the divisions. Besides, we have proved that $\mathcal{L}(\backslash, \vee)$-grammars generate all finite intersections of context-free languages and images of such intersections under alphabetic homomorphisms.

There are some questions left for future work. First, we see that in our constructions for proving PSPACE-hardness involve formulae of unbounded implication depth. On the other hand, for the original Lambek calculus without additive connectives, which is NP-complete, Pentus [25], nevertheless, a polynomial time decision procedure for the case where the order (a complexity measure similar to implication depth) of formulae is bounded by a constant d, fixed in advance. The degree of the polynomial, of course, depends on d. For the Lambek calculus with additives, we plan to show that it is not the case. Following the basic ideas of our encoding, with the formulas of the implication nesting depth bounded by some constant, we intend to simulate at least co-NP-hardness of our small fragment $\mathcal{L}(\backslash, \wedge)$ with one implication and conjunction.

Another open question is to describe the class of languages generated by Lambek grammars with additive connectives. In particular, Kuznetsov and Okhotin [14,16] show that such grammars can generate languages described by conjunctive grammars [21]. Such grammars can be quite powerful, for example, can generate $\{a^{4^n} \mid n \geq 1\}$ [7]. It is yet unknown whether all such languages can be generated by $\mathcal{L}(\backslash, \vee)$-grammars.

Acknowledgments. We would like to thank the anonymous referees for their helpful comments.

Financial Support

The work of Max Kanovich and Andre Scedrov was supported by the Russian Science Foundation under grant 17-11-01294 and performed at National Research University Higher School of Economics, Moscow, Russia. The work of Stepan Kuznetsov was supported by the Young Russian Mathematics award, by the grant MK-430.2019.1 of the President of Russia, and by the Russian Foundation for Basic Research grant 18-01-00822. Section 2 was contributed by Kanovich and Scedrov. Section 3 was contributed by Kuznetsov. Sections 1 and 4 were contributed jointly and equally by all co-authors.

A PSPACE-completeness of the fragment $\mathcal{L}(\backslash, \vee)$

In this section we will modify Sect. 2 to establish PSPACE-completeness for the fragment $\mathcal{L}(\backslash, \vee)$, which includes only one implication and disjunction.

Remark 4. For the sake of readability, we conceive of the formula $((A \cdot B) \backslash C)$ as abbreviation for $(B \backslash (A \backslash C))$. In particular, $(A \cdot B)^b$ is abbreviation for

$(B \backslash (A \backslash b))$. Because of Lemma 1, the formula $(A^b \wedge B^b)$ is conceived of as abbreviation for $((A \vee B) \backslash b)$ within this section.

Theorem 4. *The fragment $\mathcal{L}(\backslash, \vee)$ is PSPACE-complete.*

Proof Sketch. We start with the equality (2), assuming that $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$, α_{2n} are given.

To prove Lemma 14, the "disjunction analog" of Lemma 3, we modify the basic material given in the "conjunction" Definition 3 by means of Definitions 6 and 7 working within the $\mathcal{L}(\backslash, \vee)$ fragment.

Definition 6. *For $1 \leq i \leq 2n$, let $E_{\ell,i,\beta}$ denote the formula: $(c_{\ell,i} \backslash (\beta \backslash c_{\ell,i-1}))$. Let F_ℓ denote: $(q_{2n} \backslash c_{\ell,2n})$, and H_ℓ denote: $(c_{\ell,0} \backslash (e_0 \backslash e_0))$.*

Lemma 13. *The following "verifying" sequent is derivable in Lambek calculus*

$$e_0, \alpha_1, \alpha_2, \ldots, \alpha_{2n}, q_{2n}, F_\ell, E_{\ell,2n,\alpha_{2n}}, E_{\ell,2n-1,\alpha_{2n-1}}, \ldots E_{\ell,2,\alpha_2}, E_{\ell,1,\alpha_1}, H_\ell \;\vdash\; e_0$$

Proof. By the inverse induction on i: $\alpha_{i-1}, \alpha_i, c_{\ell,i}, E_{\ell,i,\alpha_i} \;\vdash\; (\alpha_{i-1} \cdot c_{\ell,i-1})$ ■

Definition 7. *We introduce the following formulas:*

$$\begin{cases} \widetilde{F} = \left(\bigvee_{\ell=1}^{m} (F_\ell)^b\right)^b, \quad \widetilde{H} = \left(\bigvee_{\ell=1}^{m} (H_\ell)^b\right)^b \\ \widetilde{E_i} = \left(\bigvee_{1 \leq \ell \leq m,\ E_{\ell,i,\beta} \in \mathcal{E}_{\ell,i}} (E_{\ell,i,\beta})^b\right)^b \end{cases} \tag{32}$$

where a one- or two-element set of formulas, $\mathcal{E}_{\ell,i}$, is defined as follows:

(1) $\mathcal{E}_{\ell,i} = \{ E_{\ell,i,t_i} \}$, if the conjunct C_ℓ contains the variable x_i,
(2) $\mathcal{E}_{\ell,i} = \{ E_{\ell,i,f_i} \}$, if the conjunct C_ℓ contains $\neg x_i$,
(3) $\mathcal{E}_{\ell,i} = \{ E_{\ell,i,t_i},\ E_{\ell,i,f_i} \}$, if C_ℓ contains neither x_i, nor $\neg x_i$.

By applying (2) and Lemma 1, we get the desired verification:

Lemma 14. *The following sequent is derivable in Lambek*

$$e_0^{bb}, \alpha_1^{bb}, \alpha_2^{bb}, \ldots, \alpha_{2n-1}, (\alpha_{2n} \cdot q_{2n})^{bb},\ \Delta_n \;\vdash\; e_0^{bb}$$

where Δ_n is a sequence of formulas: $\Delta_n = \widetilde{F}, \widetilde{E}_{2n}, \widetilde{E}_{2n-1}, \ldots, \widetilde{E_2}, \widetilde{E_1}, \widetilde{H}$

Corollary 6. *It suffices to follow the line of reasoning in Sect. 2 to find appropriate G_1, G_2, $\ldots$, G_{2n-1}, G_{2n}, such that the following sequent is derivable in Lambek calculus if and only if the statement (1) is valid:*

$$(e_0^{[4n]} \cdot q_0)^{[2]}, G_1^{[2]}, G_2^{[4]}, \ldots, G_{2n-1}^{[4n-2]}, G_{2n}^{[4n]},\ \Delta_n^{[4n]} \;\vdash\; e_0^{[4n+2]} \tag{33}$$

B Proofs of Technical Lemmas for Section 3

Proof (of Lemma 8). Induction on derivation. The sequent in question could not be an axiom, since the antecedent of $F_1 \vee F_2 \vdash F_1 \vee F_2$ includes $F_1 \vee F_2$ in a strictly positive position. Consider the last rule applied in the derivation. It could be $\mathbf{L}\backslash$, $\mathbf{L}/$, $\mathbf{L}\cdot$, or $\mathbf{R}\vee$. Rules with $\wedge$ cannot be used, since there are no $\wedge$'s in the antecedent, and the main connective of the succedent is $\vee$. If the last rule is $\mathbf{R}\vee$, we immediately reach our goal.

If the derivation ends with an application of $\mathbf{L}\cdot$:

$$\frac{E_1, \ldots, E_i', E_i'', \ldots, E_n \vdash F_1 \vee F_2}{E_1, \ldots, E_i' \cdot E_i'', \ldots, E_n \vdash F_1 \vee F_2}\ \mathbf{L}\cdot$$

then we apply the induction hypothesis, get $E_1, \ldots, E_i', E_i'', \ldots, E_n \vdash F_i$ ($i = 1$ or 2) and apply $\mathbf{L}\cdot$ to this sequent, which yields our goal.

For $\mathbf{L}\backslash$, we get the following

$$\frac{E_{i+1}, \ldots, E_{j-1} \vdash E_j' \quad E_1, \ldots, E_i, E_j'', \ldots, E_n \vdash F_1 \vee F_2}{E_1, \ldots, E_i, E_{i+1}, \ldots, E_{j-1}, E_j' \backslash E_j'', \ldots, E_n \vdash F_1 \vee F_2}\ \mathbf{L}\backslash$$

and notice that the antecedent of the right premise still satisfies the conditions of the lemma, thus we can apply induction hypothesis. The induction hypothesis yields $E_1, \ldots, E_i, E_j'', \ldots, E_n \vdash F_i$. Applying $\mathbf{L}/$ with the same left premise, $E_{i+1}, \ldots, E_{j-1} \vdash E_j'$, yields our goal.

The $\mathbf{L}/$ case is symmetric. ■

Proof (of Lemma 9). Induction on derivation. The axiom should be of the form $b \vdash b$, which violates the condition. For each inference rule, we apply the induction hypothesis for the premise from which the succedent b comes. ■

Proof (of Lemma 10). Induction on derivation. Induction base is axiom $b \vdash b$. Consider the last rule applied. If it is one of the one-premise rules, then we use the induction hypothesis for the only premise. For applications of $\mathbf{L}/$ or $\mathbf{L}\backslash$, if the rightmost occurrence of b goes to the right premise, we again directly use the induction hypothesis. Notice that for $\mathbf{L}\backslash$ this is always the case. The other rule, $\mathbf{L}/$, however, can decompose one of the F_i and take the rightmost b to the left premise:

$$\frac{F_{i+1}, \ldots, F_\ell, E_1 \backslash b, \ldots, E_k \backslash b, b \vdash F_i'' \quad F_1, \ldots, F_i' \vdash b}{F_1, \ldots, F_i' \,/\, F_i'', F_{i+1}, \ldots, F_\ell, E_1 \backslash b, \ldots, E_k \backslash b, b \vdash b}$$

The right premise, however, now is not derivable by Lemma 9. Contradiction. ■

Proof (of Lemma 11). Induction on derivation again. Any one-premise rule applied for one of the F_i, as well as $\mathbf{L}/$ or $\mathbf{L}\backslash$ which keeps $E_k \backslash b$ in the right premise, is handled by directly using the induction hypothesis and applying the same rule. The situation where $\mathbf{L}/$ takes $E_k \backslash b$ to the left premise leads to contradiction with Lemma 9, exactly as in the proof of the previous lemma. ■

References

1. Abrusci, V.M.: A comparison between Lambek syntactic calculus and intuitionistic linear logic. Zeitschr. Math. Logik Grundl. Math. (Math. Logic Q.) **36**, 11–15 (1990)
2. Bar-Hillel, Y., Gaifman, C., Shamir, E.: On categorial and phrase-structure grammars. Bull. Res. Council Israel **9F**, 1–16 (1960)
3. Buszkowski, W.: The equivalence of unidirectional Lambek categorial grammars and context-free grammars. Zeitschr. Math. Log. Grundl. Math. **31**, 369–384 (1985)
4. Carpenter, B.: Type-Logical Semantics. MIT Press (1998)
5. Girard, J.-Y.: Linear logic. Theor. Comput. Sci. **50**(1), 1–101 (1987)
6. Horčík, R., Terui, K.: Disjunction property and complexity of substructural logics. Theor. Comput. Sci. **412**(31), 3992–4006 (2011)
7. Jeż, A.: Conjunctive grammars can generate non-regular unary languages. Internat. J. Found. Comput. Sci. **19**(3), 597–615 (2008)
8. Kanazawa, M.: The Lambek calculus enriched with additional connectives. J. Log. Lang. Inform. **1**(2), 141–171 (1992)
9. Kanazawa, M.: Lambek calculus: recognizing power and complexity. In: Gerbrandy, J., et al. (eds.) JFAK. Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday. Amsterdam University Press, Vossiuspers (1990)
10. Kanovich, M.I.: Horn fragments of non-commutative logics with additives are PSPACE-complete. In: Proceedings 1994 Annual Conference of the European Association for Computer Science Logic, Kazimierz, Poland (1994)
11. Kanovich, M.: The direct simulation of Minsky machines in linear logic. In: Girard, J.-Y., Lafont, Y., Regnier, L. (eds.) Advances in Linear Logic. London Mathematical Society Lecture Notes, vol. 222, pp. 123–145. Cambridge University Press, Cambridge (1995)
12. Kanovich, M.I.: The undecidability theorem for the Horn-like fragment of linear logic (Revisited). Math. Struct. Comput. Sci. **26**(5), 719–744 (2016)
13. Kanovich, M., Kuznetsov, S., Nigam, V., Scedrov, A.: Subexponentials in non-commutative linear logic. Math. Struct. Comput. Sci. (2018). Part of Dale Miller's Festschrift
14. Kuznetsov, S.: Conjunctive grammars in Greibach normal form and the Lambek calculus with additive connectives. In: Morrill, G., Nederhof, M.-J. (eds.) FG 2012-2013. LNCS, vol. 8036, pp. 242–249. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-39998-5_15
15. Kuznetsov, S.L.: On translating Lambek grammars with one division into context-free grammars. Proc. Steklov Inst. Math. **294**(1), 129–138 (2016)
16. Kuznetsov, S., Okhotin, A.: Conjunctive categorial grammars. In: Proceedings of Mathematics of Language (2017)
17. Lambek, J.: The mathematics of sentence structure. Amer. Math. Monthly **65**, 154–170 (1958)
18. Lincoln, P., Mitchell, J., Scedrov, A., Shankar, N.: Decision problems for propositional linear logic. Ann. Pure Appl. Logic **56**, 239–311 (1992)
19. Moot, R., Retoré, C.: The Logic of Categorial Grammars. A Deductive Account of Natural Language Syntax and Semantics. LNCS, vol. 6850. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-31555-8
20. Morrill, G.V.: Categorial Grammar: Logical Syntax, Semantics, and Processing. Oxford University Press (2011)

21. Okhotin, A.: Conjunctive grammars. J. Autom. Lang. Combin. **6**(4), 519–535 (2001)
22. Păun, G.: A note on the intersection of context-free languages. Fundam. Inform. **3**(2), 135–139 (1980)
23. Pentus, M.: Lambek grammars are context-free. In: Proceedings of the 8th Annual IEEE Symposium on Logic in Computer Science (LICS 1993), pp. 429–433. IEEE Computer Society Press (1993)
24. Pentus, M.: Lambek calculus is NP-complete. Theor. Comput. Sci. **357**(1–3), 186–201 (2006)
25. Pentus, M.: A polynomial-time algorithm for Lambek grammars of bounded order. Linguist. Anal. **36**(1–4), 441–471 (2010)
26. Savateev, Yu.: Unidirectional Lambek grammars in polynomial time. Theory Comput. Syst. **46**(4), 662–672 (2010)
27. Savateev, Yu.: Product-free Lambek calculus is NP-complete. Ann. Pure Appl. Logic **163**(7), 775–788 (2012)

L-Models and R-Models for Lambek Calculus Enriched with Additives and the Multiplicative Unit

Max Kanovich[1,4], Stepan Kuznetsov[2,4](✉), and Andre Scedrov[3,4]

[1] University College London, London, UK
m.kanovich@ucl.ac.uk
[2] Steklov Mathematical Institute of the RAS, Moscow, Russia
sk@mi-ras.ru
[3] University of Pennsylvania, Philadelphia, USA
scedrov@math.upenn.edu
[4] National Research University Higher School of Economics, Moscow, Russia

Abstract. Language and relational models, or L-models and R-models, are two natural classes of models for the Lambek calculus. Completeness w.r.t. L-models was proved by Pentus and completeness w.r.t. R-models by Andréka and Mikulás. It is well known that adding both additive conjunction and disjunction together yields incompleteness, because of the distributive law. The product-free Lambek calculus enriched with conjunction only, however, is complete w.r.t. L-models (Buszkowski) as well as R-models (Andréka and Mikulás). The situation with disjunction turns out to be the opposite: we prove that the product-free Lambek calculus enriched with disjunction only is incomplete w.r.t. L-models as well as R-models. If the empty premises are allowed, the product-free Lambek calculus enriched with conjunction only is still complete w.r.t. L-models but in which the empty word is allowed. Both versions are decidable (PSPACE-complete in fact). Adding the multiplicative unit to represent explicitly the empty word within the L-model paradigm changes the situation in a completely unexpected way. Namely, we prove undecidability for any L-sound extension of the Lambek calculus with conjunction and with the unit, whenever this extension includes certain L-sound rules for the multiplicative unit, to express the natural algebraic properties of the empty word. Moreover, we obtain undecidability for a small fragment with only one implication, conjunction, and the unit, obeying these natural rules. This proof proceeds by the encoding of two-counter Minsky machines.

Keywords: Lambek calculus · Language models · Relational models · Distributive law · Incompleteness · Undecidability

1 Introduction

By $\mathbf{L}{\vee}{\wedge}$ we denote the *Lambek calculus with additive connectives,* disjunction and conjunction. Formulae of $\mathbf{L}{\vee}{\wedge}$ are built from a countable set of variables (which

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 373–391, 2019.
https://doi.org/10.1007/978-3-662-59533-6_23

we denote by $p, q, r, \ldots$) using five binary connectives: $\backslash$ (left implication), $/$ (right implication), $\cdot$ (product, or multiplicative conjunction), $\vee$ (additive disjunction), and $\wedge$ (additive conjunction). The Lambek calculus with additive connectives is formulated as a Gentzen-style sequent calculus of a two-sided (intuitionistic) format. Being a non-commutative substructural logic, however, it has an important difference from traditional sequent calculi. Namely, left-hand sides (antecedents) of $\mathbf{L}\vee\wedge$ sequents are finite *linearly ordered sequences* (not sets or multisets) of formulae. The right-hand side (succedent) of a sequent is a formula. Axioms and inference rules of $\mathbf{L}\vee\wedge$ are presented on Table 1.

Table 1. Lambek calculus with additive connectives

$$\frac{}{A \vdash A}\ \mathrm{Id}$$

$$\frac{\Phi \vdash A \quad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, \Phi, A \backslash B, \Sigma_2 \vdash C}\ \backslash \mathrm{L} \qquad \frac{A, \Sigma \vdash B}{\Sigma \vdash A \backslash B}\ \backslash \mathrm{R},\ \Sigma \text{ is not empty}$$

$$\frac{\Phi \vdash A \quad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, B / A, \Phi, \Sigma_2 \vdash C}\ / \mathrm{L} \qquad \frac{\Sigma, A \vdash B}{\Sigma \vdash B / A}\ / \mathrm{R},\ \Sigma \text{ is not empty}$$

$$\frac{\Sigma_1, A, B, \Sigma_2 \vdash C}{\Sigma_1, A \cdot B, \Sigma_2 \vdash C}\ \cdot \mathrm{L} \qquad \frac{\Sigma_1 \vdash A \quad \Sigma_2 \vdash B}{\Sigma_1, \Sigma_2 \vdash A \cdot B}\ \cdot \mathrm{R}$$

$$\frac{\Sigma_1, A, \Sigma_2 \vdash C \quad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, A \vee B, \Sigma_2 \vdash C}\ \vee \mathrm{L} \qquad \frac{\Sigma \vdash A}{\Sigma \vdash A \vee B} \quad \frac{\Sigma \vdash B}{\Sigma \vdash A \vee B}\ \vee \mathrm{R}$$

$$\frac{\Sigma_1, A, \Sigma_2 \vdash C}{\Sigma_1, A \wedge B, \Sigma_2 \vdash C} \quad \frac{\Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, A \wedge B, \Sigma_2 \vdash C}\ \wedge \mathrm{L} \qquad \frac{\Sigma \vdash A \quad \Sigma \vdash B}{\Sigma \vdash A \wedge B}\ \wedge \mathrm{R}$$

$$\frac{\Phi \vdash A \quad \Sigma_1, \Phi, \Sigma_2 \vdash C}{\Sigma_1, \Phi, \Sigma_2 \vdash C}\ \mathrm{Cut}$$

The first three connectives, namely, two implications, also called divisions (left and right, $\backslash$ and $/$) and product (multiplicative conjunction, $\cdot$), are due to Lambek [15]. These connectives are called *multiplicative.* Two *additive* connectives, $\vee$ and $\wedge$, are added to the Lambek calculus in the spirit of Girard's linear logic [6] (where they are denoted by $\oplus$ and &, respectively). As noticed by Abrusci [1], the Lambek calculus can be considered as a non-commutative variant of linear logic. A specific feature of the Lambek calculus, however, is the so-called Lambek's non-emptiness restriction: as one can see from the form of the rules, left-hand sides of sequents are required to be non-empty. This restriction is motivated by linguistic applications of the Lambek calculus [17, Sect. 2.5].

The cut rule is eliminable by a standard argument. Cut elimination yields the subformula property and makes it easy to formulate elementary fragments. If one takes a subset of the set of connectives, and leaves only the corresponding rules of inference, the calculus obtained is a conservative fragment of $\mathbf{L}\vee\wedge$. The fragment without additive connectives ($\vee$ and $\wedge$) is the original Lambek calculus denoted by $\mathbf{L}$. Fragments with only one additive connective are denoted by $\mathbf{L}\vee$

and $\mathbf{L}\wedge$. We also consider product-free fragments with conjunction, $\mathbf{L}(\backslash, /, \wedge)$ and $\mathbf{L}(\backslash, \wedge)$, which include, respectively, only $\backslash$, $/$, $\wedge$ and only $\backslash$ and $\wedge$.

From the point of view of semantics, there exist many classes of models for the Lambek calculus. We consider two natural ones, language and relational ones. Language models, or *L-models,* are inspired by linguistic motivation and applications of the Lambek calculus. An L-model is defined on $\mathcal{P}(\Sigma^+)$, the set of all languages over an alphabet Σ without the empty word, by an interpretation function w which maps Lambek formulae to languages from $\mathcal{P}(\Sigma^+)$. The interpretation function is defined arbitrarily on variables, and should commute with Lambek connectives in the following way:

$$\begin{aligned}
w(A \setminus B) &= w(A) \setminus w(B) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\, vu \in w(B)\};\\
w(B \,/\, A) &= w(B) \,/\, w(A) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\, uv \in w(B)\};\\
w(A \cdot B) &= w(A) \cdot w(B) = \{uv \mid u \in w(A), v \in w(B)\}.
\end{aligned}$$

A sequent $A_1, \dots, A_n \vdash B$ is considered true in such a model, if and only if $w(A_1) \cdot \ldots \cdot w(A_n) \subseteq w(B)$.

Notice that the empty word, ε, is not allowed due to Lambek's restriction. The empty *set,* however, could appear as a result of division, and this is absolutely acceptable.

For a relational model, or *R-model,* the base set is the set of all subrelations of a fixed transitive binary relation $W \subseteq U \times U$, *i.e.,* $\mathcal{P}(W)$. The interpretation function now maps Lambek formulae to subsets of W, and should obey the following commutation rules:

$$\begin{aligned}
w(A \setminus B) &= w(A) \setminus w(B) = \{\langle y, z\rangle \in W \mid (\forall \langle x, y\rangle \in w(A))\, \langle x, z\rangle \in w(B)\};\\
w(B \,/\, A) &= w(B) \,/\, w(A) = \{\langle x, y\rangle \in W \mid (\forall \langle y, z\rangle \in w(A))\, \langle x, z\rangle \in w(B)\};\\
w(A \cdot B) &= w(A) \circ w(B) = \{\langle x, z\rangle \mid (\exists y \in U)\, \langle x, y\rangle \in w(A), \langle y, z\rangle \in w(B)\}.
\end{aligned}$$

Truth conditions for R-models are exactly the same as in L-models: $A_1, \dots, A_n \vdash B$ is true, iff $w(A_1) \cdot \ldots \cdot w(A_n) \subseteq w(B)$.

Additive connectives, both in L-models and R-models, are interpreted as set-theoretical union and intersection:

$$\begin{aligned}
w(A \vee B) &= w(A) \cup w(B);\\
w(A \wedge B) &= w(A) \cap w(B).
\end{aligned}$$

Both L-models and R-models provide sound semantics for $\mathbf{L}\vee\wedge$ (and, therefore, all its elementary fragments): if a sequent is derivable, then it is true in all models. Completeness (the reverse implication), however, is a more subtle issue.

There is a folklore fact that $\mathbf{L}\vee\wedge$ is incomplete both w.r.t. both L-interpretation and R-interpretation, due to the distibutivity law

$$(A \vee C) \wedge (B \vee C) \vdash (A \wedge B) \vee C.$$

The distributivity law is true for set-theoretic interpretation of $\vee$ and $\wedge$—in particular, in all L-models and all R-models—but is not provable in $\mathbf{L}\vee\wedge$. The failure to derive distributivity is a common feature of several substructural logics, as noticed by Ono and Komori [19].

L-models and R-models are both specific subclasses of general algebraic models for $\mathbf{L}\vee\wedge$, *residuated lattices* [5,22]. A residuated lattice is a lattice equipped with a monoidal structure (multiplication and the unit) and division operations, obeying the natural condition: $a \preceq c \,/\, b \iff a \cdot b \preceq c \iff b \preceq a \,\backslash\, c$ (where $\preceq$ is the lattice preorder). Residuated lattices in general, as opposed to lattices of formal languages or binary relations, are not required to be distributive. This removes the incompleteness issue mentioned above; in fact, $\mathbf{L}\vee\wedge$ is complete w.r.t. interpretations on arbitrary residuated lattices, which is proved by an argument in the style of Lindenbaum and Tarski. Moreover, there is a more specific completeness result for $\mathbf{L}\vee\wedge$ w.r.t. so-called *syntactic concept lattices,* introduced by Wurm [23] as a modification of L-models without the distributivity constraint.

By $\mathbf{L}\vee\wedge + \mathrm{distrib}$ we denote $\mathbf{L}\vee\wedge$ with the distributivity principle,

$$(A \vee C) \wedge (B \vee C) \vdash (A \wedge B) \vee C,$$

added as an extra axiom (Cut is kept as an official rule of the system, since it becomes non-eliminable after adding extra axioms). It looks natural to conjecture completeness of $\mathbf{L}\vee\wedge + \mathrm{distrib}$ w.r.t. L-models and/or R-models. However, these are both open questions.

Some fragments of $\mathbf{L}\vee\wedge$, however, are still complete w.r.t L-models and R-models. Algebraically this means that, in particular, distributivity cannot be expressed in the weaker languages of these fragments. Namely, the Lambek calculus extended with conjunction only, $\mathbf{L}\wedge$, is R-complete, as shown by Andréka and Mikulás [2]. For L-completeness, the question about $\mathbf{L}\wedge$, which includes both divisions, product, and conjunction, is still open. For the Lambek calculus without additives, however, L-completeness was shown by Pentus [21], and for $\mathbf{L}(\backslash, /, \wedge)$ L-completeness was shown by Buszkowski [3].

In this paper we emphasize $\mathbf{L}\vee$, the disjunction-only fragment of $\mathbf{L}\vee\wedge$. The situation with disjunction turns out to be the opposite: in Sect. 2 we prove that the product-free Lambek calculus enriched with disjunction only is incomplete w.r.t. L-models as well as R-models—in fact, w.r.t. any class of distributive residuated lattices.

If one abolishes Lambek's restriction, *i.e.* allows the use of empty premises, the product-free Lambek calculus enriched with conjunction only is still complete w.r.t. L-models in which the empty word is allowed. Both versions are decidable (PSPACE-complete in fact [11]).

Adding the multiplicative unit to represent explicitly the empty word within the L-model paradigm changes the situation in a completely unexpected way. Even the product-free fragment with only one implication, conjunction, and the unit cannot be extended to a decidable system complete with respect to L-models. This proof proceeds by the encoding of two-counter Minsky machines with the help of certain simple rules for the multiplicative unit, caused by the empty word.

Let us focus on L-models. The unit in L-models is necessarily interpreted as $\{\varepsilon\}$, where ε is the empty word. In particular, adding the unit forces us to allow the empty word in L-models.

An attempt to axiomatise the unit constant by the rules for multiplicative unit taken from linear logic [16] results in an L-sound, but not L-complete system [4,12]. Unfortunately, no L-complete recursively enumerable axiomatisation for the Lambek calculus with the unit constant is known. In Sect. 3, we present an extension of the Lambek calculus that respects the most natural peculiarities of the empty word ε in L-models, such as: $\varepsilon \cdot \varepsilon = \varepsilon$ and $x \cdot \varepsilon = \varepsilon \cdot x$. Our main result is that this system, which we denote by $\mathbf{L}^{+\varepsilon}$, is undecidable. Moreover, we get undecidability for any L-sound calculus that includes $\mathbf{L}^{+\varepsilon}$.

2 Incompleteness of $\mathbf{L}\vee$ w.r.t. L-Models and R-Models

We show that $\mathbf{L}\vee$ is incomplete w.r.t. language and relational models by presenting a concrete example of a sequent true in all such models, but not derivable in $\mathbf{L}\vee$.

Theorem 1. *The sequent*

$$(((x \,/\, y) \vee x) \,/((x \,/\, y) \vee (x \,/\, z) \vee x)) \cdot ((x \,/\, y) \vee x) \cdot (((x \,/\, y) \vee x) \backslash ((x \,/\, z) \vee x)) \\ \vdash (x \,/(y \vee z)) \vee x$$

is not derivable in $\mathbf{L}\vee$*, but is derivable in* $\mathbf{L}\vee\wedge + \mathrm{distrib}$ *and, therefore, true in all L-models and all R-models.*

Before going into the detailed proof of this theorem, let us show the ideas behind it. The monstruous sequent which we use as our counter-example comes from the *diamond construction* originally due to Lambek [15]. For two formulae A and B let C be their *meeting* formula, if both $C \vdash A$ and $C \vdash B$ are derivable, and let D be their *joining* formula, if both $A \vdash D$ and $B \vdash D$. (Meeting and joining formulae are of course not unique.) In $\mathbf{L}\vee\wedge$, constructing meeting and joining formulae is trivial, since one just takes $C = A \wedge B$ and $D = A \vee B$. Moreover, this gives the maximum meeting and the minimum joining formula: for any other meeting formula C' and any other joining formula D' we have $C' \vdash A \wedge B$ and $A \vee B \vdash D'$.

In $\mathbf{L}\vee$, however, only the joining formula, $A \vee B$, is explicitly given. Wishing to encode distributivity, we need some meeting formula to use it *in lieu* of $A \wedge B$. Such a formula is given by the following lemma, which is a variation of the diamond constructions of Lambek [15] and Pentus [20].

Lemma 1. *For any calculus extending* $\mathbf{L}$*, if D is a joining formula for A and B, then $(A \,/\, D) \cdot A \cdot (A \setminus B)$ is a meeting formula for A and B. In particular, in* $\mathbf{L}\vee$ *for any two formulae A and B we have a meeting formula,* $(A \,/(A \vee B)) \cdot A \cdot (A \setminus B)$.

Proof. For $C = (A \,/\, D) \cdot A \cdot (A \backslash B)$ the necessary sequents $C \vdash A$ and $C \vdash B$ are derived as follows:

$$\dfrac{\dfrac{A \vdash A \qquad \dfrac{B \vdash D \quad A \vdash A}{A \,/\, D, B \vdash A}}{A \,/\, D, A, A \backslash B \vdash A}}{(A \,/\, D) \cdot A \cdot (A \backslash B) \vdash A} \qquad \dfrac{\dfrac{A \vdash D \qquad \dfrac{A \vdash A \quad B \vdash B}{A, A \backslash B \vdash B}}{A \,/\, D, A, A \backslash B \vdash B}}{(A \,/\, D) \cdot A \cdot (A \backslash B) \vdash B}$$

Now we are ready to explain the construction in Theorem 1 and prove the theorem. Take $A = (x \,/\, y) \vee x$ and $B = (x \,/\, z) \vee x$. Then $D = A \vee B$ is equivalent to $(x \,/\, y) \vee (x \,/\, z) \vee x$, and by Lemma 1 the left-hand side of the sequent in Theorem 1 is exactly the meeting formula for A and B, which we denote by C. Thus, $C \vdash A$ and $C \vdash B$ are derivable in $\mathbf{L}\vee$ and therefore in $\mathbf{L}\vee\wedge$, and so is $C \vdash A \wedge B$.

Recall that $A \wedge B = ((x \,/\, y) \vee x) \wedge ((x \,/\, z) \vee x)$ and apply distributivity:

$$((x \,/\, y) \vee x) \wedge ((x \,/\, z) \vee x) \vdash ((x \,/\, y) \wedge (x \,/\, z)) \vee x.$$

Finally, recall that $(x \,/\, y) \wedge (x \,/\, z)$ is equivalent to $x \,/ (y \vee z)$ in $\mathbf{L}\vee\wedge$,[1] which allows us to get rid of $\wedge$ in the right-hand side:

$$((x \,/\, y) \vee x) \wedge ((x \,/\, z) \vee x) \vdash ((x \,/ (y \vee z)) \vee x.$$

Now cut with $C \vdash ((x \,/\, y) \vee x) \wedge ((x \,/\, z) \vee x)$ (*i.e.*, $C \vdash A \wedge B$) yields the needed sequent in Theorem 1.

The second statement, that this sequent is not derivable in $\mathbf{L}\vee$ without distributivity, does not follow automatically from the fact that distributivity is not provable in $\mathbf{L}\vee\wedge$. This is because the formula C constructed using the diamond construction is a stronger meeting formula than $A \wedge B$: $C \vdash A \wedge B$, but not $A \wedge B \vdash C$. Thus, we still have to prove that the sequent in Theorem 1 is not derivable in $\mathbf{L}\vee$ or, equivalently, in $\mathbf{L}\vee\wedge$.

Such a non-derivability proof can be performed, as suggested by one of the anonymous reviewers, by presenting an algebraic counter-model, *i.e.*, an interpretation over a residuated lattice which falsifies the sequent in question. (This lattice should be necessarily non-distributive, thus, it is neither an L-model nor an R-model.) Another, purely syntactic strategy is to apply the proof search algorithm directly (recall that the derivability problem in $\mathbf{L}\vee\wedge$ is decidable). Following this line, we used an automatic theorem-prover for $\mathbf{L}\vee\wedge$, and

[1] The derivations establishing equivalence are as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{y \vdash y \quad x \vdash x}{x \,/\, y, y \vdash x}}{(x \,/\, y) \wedge (x \,/\, z), y \vdash x} \qquad \dfrac{\dfrac{z \vdash z \quad x \vdash x}{x \,/\, z, z \vdash x}}{(x \,/\, y) \wedge (x \,/\, z), z \vdash x}}{(x \,/\, y) \wedge (x \,/\, z), y \vee z \vdash x}}{(x \,/\, y) \wedge (x \,/\, z) \vdash x \,/ (y \vee z)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{y \vdash y}{y \vdash y \vee z} \quad x \vdash x}{x \,/ (y \vee z), y \vdash x}}{x \,/ (y \vee z) \vdash x \,/\, y} \qquad \dfrac{\dfrac{\dfrac{z \vdash z}{z \vdash y \vee z} \quad x \vdash x}{x \,/ (y \vee z), z \vdash x}}{x \,/ (y \vee z) \vdash x \,/\, z}}{x \,/ (y \vee z) \vdash (x \,/\, y) \wedge (x \,/\, z)} \quad .$$

its extension with Kleene star, implemented by Jipsen, based on [7,8,18,19]. Jipsen's theorem-prover is avaliable online: http://www1.chapman.edu/~jipsen/kleene/. The algorithm performs exhaustive proof search and thus establishes non-derivability. In order to make this paper self-contained and independent from external derivability-checking software, in the Appendix we represent the execution of the proof search algorithm (and, thus, the proof of non-derivability), with some simplifications, in a human-readable form.

3 Undecidability of the Fragment $(\backslash, \wedge, \mathbf{1})$

In this section we consider the extension of the Lambek calculus with the multiplicative unit constant. In L-models, because of the principle $A \cdot \mathbf{1} \vdash A$, the constant $\mathbf{1}$ is necessarily interpreted as the singleton set $\{\varepsilon\}$, where ε is the empty word. In particular, introducing the unit constant requires modification of the definition of L-models by allowing the empty word to belong to our languages. For the same reason, we have to abolish Lambek's non-emptiness restriction. Because of this specific interpretation of the unit constant, we introduce principles connected with this particular interpretation of the unit. Such principles include $A \cdot \{\varepsilon\} = \{\varepsilon\} \cdot A$ and $\{\varepsilon\} \cdot \{\varepsilon\} = \{\varepsilon\}$. On Table 2, we present a calculus, denoted by $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$, which reflects these two principles as sequential rules.

Table 2. Axioms and inference rules for a minimal $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{}{A, \mathbf{1} \vdash A}\,\mathbf{1}$$

$$\frac{\Phi \vdash A \quad \Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, \Phi, A \backslash B, \Sigma_2 \vdash C}\,\backslash\mathrm{L} \qquad \frac{A, \Sigma \vdash B}{\Sigma \vdash A \backslash B}\,\backslash\mathrm{R}$$

$$\frac{\Sigma_1, A, \Sigma_2 \vdash C}{\Sigma_1, A \wedge B, \Sigma_2 \vdash C} \quad \frac{\Sigma_1, B, \Sigma_2 \vdash C}{\Sigma_1, A \wedge B, \Sigma_2 \vdash C}\,\wedge\mathrm{L} \qquad \frac{\Sigma \vdash A \quad \Sigma \vdash B}{\Sigma \vdash A \wedge B}\,\wedge\mathrm{R}$$

$$\frac{\Sigma_1, A, (\mathbf{1} \wedge G), \Sigma_2 \vdash C}{\Sigma_1, (\mathbf{1} \wedge G), A, \Sigma_2 \vdash C}\,\mathrm{L}\varepsilon \qquad \frac{\Sigma_1, (\mathbf{1} \wedge G), A, \Sigma_2 \vdash C}{\Sigma_1, A, (\mathbf{1} \wedge G), \Sigma_2 \vdash C}\,\mathrm{R}\varepsilon$$

$$\frac{\Sigma_1, (\mathbf{1} \wedge G), (\mathbf{1} \wedge G), \Sigma_2 \vdash C}{\Sigma_1, (\mathbf{1} \wedge G), \Sigma_2 \vdash C}\,\mathrm{D}\varepsilon$$

The "commuting" rules Lε and Rε are caused by the fact that, for any set X,

$$X \cdot \{\varepsilon\} = \{\varepsilon\} \cdot X, \quad \varnothing \cdot X = X \cdot \varnothing,$$

whereas the "doubling" rule Dε is caused by

$$\{\varepsilon\} \cdot \{\varepsilon\} = \{\varepsilon\}, \quad \varnothing \cdot \varnothing = \varnothing.$$

Thus, these rules express the natural algebraic properties of the empty word, ε. However, we do not claim that we get an L-complete system. Indeed, the L-complete extension happens to be quite involved (cf. [12]). In particular, it is still an open problem whether it is recursively enumerable.

We emphasize that our rules Lε, Rε, and Dε are *not* derivable in the multiplicative-additive Lambek calculus, that is, non-commutative multiplicative-additive linear logic (cf. [10,16]).

The cut rule is not included in the system, so that all our derivations will be cut-free.

Theorem 2. *The derivability problem for* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$ *is undecidable. Moreover, any L-sound system which includes* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$*, i.e., rules of Table 2, is undecidable.*

We prove undecidability by encoding of two-counter Minsky machines (cf. [9]).

In the forward encoding, from computations to derivations, we present explicit derivations in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

For the backwards direction, from derivations to computations, we use a semantic approach by constructing an appropriate L-model for the sequent in question (cf. [13,14,18], where phase semantics is used for similar purposes).

Definition 1. *In our encoding, we use the following construction.*
We fix an atomic proposition b*, and define 'relative negation'* A^b *by:*

$$A^b = (A \backslash b)$$

Our relative negation can be seen as a non-commutative analogue of the linear logic negation, which is defined by $A^{\perp} = A \multimap \perp$.
As for the relative "double negation," the novelty of our approach is that we are in favour of the "asymmetric"

$$A^{bb} = ((A \backslash b) \backslash b)$$

For the sake of readability of product-free formulas,

(a) *Here we will conceive of the formula* $((A \cdot B) \backslash C)$ *as abbreviation for* $(B \backslash (A \backslash C))$*. In particular,* $(A \cdot B)^b$ *is abbreviation for* $(B \backslash (A \backslash b))$*.*

(b) *Given a sequence of formulas* α*:*

$$\alpha = \alpha_1, \alpha_2, \ldots \alpha_{m-1}, \alpha_m$$

we will conceive of the expression $(\alpha \backslash C)$ *as abbreviation for the following product-free formula*

$$(\alpha \backslash C) = (\alpha_m \backslash (\alpha_{m-1} \backslash (\ldots \backslash (\ldots \backslash (\alpha_2 \backslash (\alpha_1 \backslash C))))))$$

In particular,

$$\alpha^b = (\alpha_m \backslash (\alpha_{m-1} \backslash (\ldots \backslash (\ldots \backslash (\alpha_2 \backslash (\alpha_1 \backslash b))))))$$

Lemma 2. *Given a sequence of formulas* α *and a sequence of formulas* β*, let the following sequent be derivable in* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$*, i.e., by the rules from Table 2:*

$$(1 \wedge G),\ \alpha,\ \Delta \vdash\ b \tag{1}$$

Let $g_{\alpha,\beta}$ *be defined as:*

$$g_{\alpha,\beta} = (\beta \backslash\, \alpha^{bb}) = (\beta \backslash\, ((\alpha \backslash\, b) \backslash\, b)) \tag{2}$$

Then the sequent

$$(1 \wedge G \wedge g_{\alpha,\beta}),\ \Delta,\ \beta \vdash\ b \tag{3}$$

is also (cut-free) derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

Proof. We develop a chain of derivable sequents:

$$\begin{array}{rl}
(1 \wedge G),\ \alpha,\ \Delta \vdash b & \\
\alpha,\ (1 \wedge G),\ \Delta \vdash b & \quad \text{“}\varepsilon \cdot \alpha \Rightarrow \alpha \cdot \varepsilon\text{”} \\
(1 \wedge G),\ \Delta \vdash (\alpha \backslash\, b) & \\
(1 \wedge G),\ \Delta,\ ((\alpha \backslash\, b) \backslash\, b) \vdash b & \\
(1 \wedge G),\ \Delta,\ \beta,\ (\beta \backslash\, ((\alpha \backslash\, b) \backslash\, b)) \vdash b & \\
(1 \wedge G),\ \Delta,\ \beta,\ g_{\alpha,\beta} \vdash b & \\
(1 \wedge G),\ \Delta,\ \beta,\ (1 \wedge g_{\alpha,\beta}) \vdash b & \\
(1 \wedge G),\ (1 \wedge g_{\alpha,\beta}),\ \Delta,\ \beta \vdash b & \quad \text{“}\delta \cdot \varepsilon \Rightarrow \varepsilon \cdot \delta\text{”} \\
(1 \wedge G \wedge g_{\alpha,\beta}),\ (1 \wedge G \wedge g_{\alpha,\beta}),\ \Delta,\ \beta \vdash b & \\
(1 \wedge G \wedge g_{\alpha,\beta}),\ \Delta,\ \beta \vdash b & \quad \text{“}\varepsilon \cdot \varepsilon = \varepsilon\text{”}
\end{array} \tag{4}$$

which concludes the proof. ■

Corollary 1. (“Post-ish productions”). *Let* $\xi_1,\ \xi_2,\ \ldots,\ \xi_n$ *be a list of all atomic propositions in question. Let* Δ_1 *and* Δ_2 *be sequences made from the above atomic propositions (repetitions are allowed).*
Let G *be of the form*

$$G\ \equiv\ G' \wedge \bigwedge_{i=1}^{n} g_{\xi_i,\xi_i}\ \equiv\ G' \wedge \bigwedge_{i=1}^{n} (\xi_i \backslash\, \xi_i^{bb}) \tag{5}$$

Then a sequent of the form

$$(1 \wedge G),\ \Delta_1,\ \Delta_2\ \vdash\ b \tag{6}$$

is cut-free derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$ *(i.e., by the rules from Table 2) if and only if the following sequent is cut-free derivable in* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

$$(1 \wedge G),\ \Delta_2,\ \Delta_1\ \vdash\ b \tag{7}$$

Proof. By induction with the help of $g_{\xi,\xi}$ of the “trivial” form, $g_{\xi,\xi} = (\xi \backslash\, \xi^{bb})$.

3.1 From Computations to Derivations

Definition 2 (Machine encoding). *Here e_1, e_2, p_1, p_2, l_0, l_1, l_2, ... are distinct atomic propositions: e_1 and e_2 serve as "end markers," p_1 and p_2 are used to represent the counters c_1 and c_2, respectively, l_0, l_1, l_2, ... represent "states."*

Taking advantage of the fact that the number of counters is no more than 2, so that one and the same l_i is able of controlling the "left part" and the "right part" simultaneously, we represent a configuration (L_i, k_1, k_2) of our Minsky machine in the state L_i, in which the value of c_1 is k_1, and the value of c_2 is k_2, as the following sequence of atomic propositions:

$$e_1, \underbrace{p_1, p_1, \ldots, p_1}_{k_1 \ times}, l_i, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \ times}, e_2 \tag{8}$$

The final configuration $(L_0, 0, 0)$ is represented as

$$e_1, l_0, e_2 \tag{9}$$

Definition 3. *The Minsky instructions are encoded as follows*

(a) *An instruction I of the form: "L_i: $inc(c_1)$; goto L_j;" will be encoded in the "reverse" form as the product-free formula (see Definition 1)*

$$A_I = (l_i \backslash (p_1 \cdot l_j)^{bb}) \tag{10}$$

It is worth noting that $A_I = g_{\alpha,\beta}$, where $\alpha = p_1, l_j$, and $\beta = l_i$.

(b) *An instruction I of the form "L_i: $inc(c_2)$; goto L_j;" will be encoded in the "reverse" form as:*

$$A_I = (l_i \backslash (l_j \cdot p_2)^{bb}) \tag{11}$$

(c) *An instruction I of the form "L_i: $dec(c_1)$; goto L_j;" will be encoded in the "reverse" form as:*

$$A_I = ((p_1 \cdot l_i) \backslash l_j^{bb}) \tag{12}$$

(d) *An instruction I of the form "L_i: $dec(c_2)$; goto L_j;" will be encoded in the "reverse" form as:*

$$A_I = ((l_i \cdot p_2) \backslash l_j^{bb}) \tag{13}$$

(e) The most challenging issues *to be addressed to is our encoding of the zero-tests.*

A zero-test with the c_1 counter of the form "L_i: if $(c_1 = 0)$ goto L_j;" will be encoded by

$$A_I = ((e_1 \cdot l_i) \backslash (e_1 \cdot l_j)^{bb}) \tag{14}$$

(f) *A zero-test with the c_2 counter of the form "L_i: if $(c_2 = 0)$ goto L_j;" will be encoded by*

$$A_I = ((l_i \cdot e_2) \backslash (l_j \cdot e_2)^{bb}) \tag{15}$$

Lemma 3. *A move by instruction of Case (a) from a configuration with L_i to the configuration with L_j is simulated as follows. Taking $\alpha = p_1, l_j$, and $\beta = l_i$, let G be of the form*

$$G \;\equiv\; G' \wedge A_I \wedge \bigwedge_{i=1}^{n} g_{\xi_i,\xi_i} \;\equiv\; G' \wedge g_{\alpha,\beta} \wedge \bigwedge_{i=1}^{n} (\xi_i \backslash \xi_i^{bb}) \tag{16}$$

Let a sequent (representing a Minsky configuration) be cut-free derivable

$$(1 \wedge G),\; e_1, \underbrace{p_1, p_1, \ldots, p_1}_{k_1+1 \;\; times}, l_j, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;\; times}, e_2 \;\vdash\; b \tag{17}$$

Then the following sequent is also cut-free derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

$$(1 \wedge G),\; e_1, \underbrace{p_1, p_1, \ldots, p_1}_{k_1 \;\; times}, l_i, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;\; times}, e_2 \;\vdash\; b \tag{18}$$

Proof. According to Corollary 1, the sequent (17) can be transformed into a cut-free derivable sequent of the form

$$(1 \wedge G),\; p_1, l_j,\; \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;\; times}, e_2, e_1,\; \underbrace{p_1, p_1, \ldots, p_1}_{k_1 \;\; times} \;\vdash\; b \tag{19}$$

By Lemma 2, we get the following

$$(1 \wedge G),\; \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;\; times}, e_2, e_1,\; \underbrace{p_1, p_1, \ldots, p_1}_{k_1 \;\; times},\; l_i \;\vdash\; b \tag{20}$$

and, applying Corollary 1 once more, we conclude with (18).

Lemma 4. *A move by instruction of Case (e) from a configuration with L_i to the configuration with L_j is simulated as follows. (Here we have to answer to the challenge of the zero-tests.) Taking $\alpha = e_1, l_j$, and $\beta = e_1, l_i$, let G be of the form*

$$G \;\equiv\; G' \wedge A_I \wedge \bigwedge_{i=1}^{n} g_{\xi_i,\xi_i} \;\equiv\; G' \wedge g_{\alpha,\beta} \wedge \bigwedge_{i=1}^{n} (\xi_i \backslash \xi_i^{bb}) \tag{21}$$

Let a sequent (representing a Minsky configuration) be cut-free derivable

$$(1 \wedge G),\; e_1, l_j, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;\; times}, e_2 \;\vdash\; b \tag{22}$$

Then the following sequent is also cut-free derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$.

$$(1 \wedge G),\; e_1, l_i, \underbrace{p_2, p_2, \ldots, p_2}_{k_2 \;\; times}, e_2 \;\vdash\; b \tag{23}$$

Proof. By Lemma 2, applied to (22), we get the following

$$(1 \wedge G),\ \underbrace{p_2, p_2, \ldots, p_2}_{k_2\ times}, e_2,\ e_1, l_i\ \vdash\ b \tag{24}$$

and, applying Corollary 1, we conclude with (23). ■

The other cases are considered in a similar fashion.

Corollary 2. *With a configuration* (L_i, k_1, k_2)*, let* M *terminate in* $(L_0, 0, 0)$*. Then the following sequent is cut-free derivable in* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$*, i.e., by the rules from Table 2:*

$$(1 \wedge G),\ e_1, \underbrace{p_1, p_1, \ldots, p_1}_{k_1\ times}, l_i, \underbrace{p_2, p_2, \ldots, p_2}_{k_2\ times}, e_2\ \vdash\ b \tag{25}$$

where G *is of the form:*

$$G = ((e_1 \cdot l_0 \cdot e_2) \backslash b) \wedge \bigwedge_{i=1}^{n} g_{\xi_i, \xi_i} \wedge \bigwedge_{over\ instructions\ I} A_I \tag{26}$$

Proof. By induction on the length of a terminating sequence of configurations. ■

3.2 From Derivations to Computations

We prove that our encoding is faithful:

Lemma 5. *Let the sequent (25) be derivable in* $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$*. Then, with the configuration* (L_i, k_1, k_2)*,* M *terminates in* $(L_0, 0, 0)$*.*

Proof. By interpretation with the help of L-models.

Each of the atomic propositions a, save b, is interpreted by "itself":

$$w(a) = \{a\} \tag{27}$$

Our specific b is interpreted as

$$w(b) = \{\, xy \mid x \text{ and } y \text{ are words such that } yx \in B_M \,\} \tag{28}$$

where the set of "terminating strings," B_M, is defined as

$$B_M = \{\, e_1 \underbrace{p_1 p_1 \ldots p_1}_{k_1\ times} l_i \underbrace{p_2 p_2 \ldots p_2}_{k_2\ times} e_2 \mid \text{from } (L_i, k_1, k_2),\ M \text{ goes to } (L_0, 0, 0) \,\} \tag{29}$$

Lemma 6. $w(1 \wedge G) = \{\varepsilon\}$.

Proof. Assume A_I be of the form (see Definition 3)

$$A_I = (l_i \backslash (p_1 \cdot l_j)^{bb})$$

To show that $\varepsilon \in w(A_I)$, we prove that for any word x, the following holds:

$$p_1 l_j \cdot x \in w(b) \quad \Longrightarrow \quad x \cdot l_i \in w(b) \tag{30}$$

If $p_1 l_j \cdot x \in w(b)$ then the word x is of the form

$$x = \underbrace{p_2 p_2 \ldots p_2}_{k_2 \ times} e_2 e_1 \underbrace{p_1 p_1 \ldots p_1}_{k_1 \ times} \tag{31}$$

with M going from $(L_j, k_1 + 1, k_2)$ to $(L_0, 0, 0)$. Then, by applying this instruction I, with (L_i, k_1, k_2), M terminates in $(L_0, 0, 0)$. Hence

$$e_1 \underbrace{p_1 p_1 \ldots p_1}_{k_1 \ times} l_i \underbrace{p_2 p_2 \ldots p_2}_{k_2 \ times} e_2 \ \in \ B_M$$

which results in the desired $x \cdot l_i \in w(b)$.

The other cases should be considered in a similar fashion. ■

If the sequent (25) is derivable in $\mathbf{L}^{+\varepsilon}(\backslash, \wedge, \mathbf{1})$, then

$$w(1 \wedge G) \cdot e_1 \underbrace{p_1 p_1 \ldots p_1}_{k_1 \ times} l_i \underbrace{p_2 p_2 \ldots p_2}_{k_2 \ times} e_2 \ \in \ w(b)$$

and, hence, with the configuration (L_i, k_1, k_2), M terminates in $(L_0, 0, 0)$.

Now Theorem 2 follows from Corollary 1 and Lemma 5.

4 Concluding Remarks

In the present paper we have proved two main results.

First, the Lambek calculus extended with additive disjunction is not complete w.r.t. L-models and R-models.

Second, any extension of the Lambek calculus with one implication, conjunction, and the multiplicative unit turns out to be undecidable, if we enrich this calculus with the natural rules, representing the basic properties of the empty word, ε, in L-models.

Namely, the "commuting" rules Lε and Rε are caused by that, for any word x and set X, $\varepsilon \cdot x = x \cdot \varepsilon$, $\varnothing \cdot X = X \cdot \varnothing$, whereas the "doubling" rule Dε is caused by $\varepsilon \cdot \varepsilon = \varepsilon$, $\varnothing \cdot \varnothing = \varnothing$.

There are several questions left open. One open question is, whether the Lambek calculus with product and both implications enriched with additive conjunction is L-complete. Another open question is whether there is a recursively enumerable extension of the Lambek calculus with the unit, which is L-complete; the same question for R-completeness. Notice that some of our rules motivated by the L-sound behaviour of ε are not valid in R-models, where the unit is interpreted as the diagonal relation. More precisely, the "doubling" rule is valid in R-models, while the "commuting" rule is not.

Acknowledgments. We would like to thank the anonymous referees for their helpful comments. We also thank the participants of the Logical Problems in Computer Science seminar at the Lomonosov Moscow State University for fruitful discussions and helpful remarks.

Financial Support
The work of Max Kanovich and Andre Scedrov was supported by the Russian Science Foundation under grant 17-11-01294 and performed at National Research University Higher School of Economics, Moscow, Russia. The work of Stepan Kuznetsov was supported by the Young Russian Mathematics award, by the grant MK-430.2019.1 of the President of Russia, and by the Russian Foundation for Basic Research grant 18-01-00822. Section 2 was contributed by Kuznetsov. Section 3 was contributed by Kanovich and Scedrov. Sections 1 and 4 were contributed jointly and equally by all co-authors.

Appendix

In this Appendix, we give a complete proof of the fact that the sequent from Theorem 1, namely

$$(((x \,/\, y) \vee x) \,/\, ((x \,/\, y) \vee (x \,/\, z) \vee x)) \cdot ((x \,/\, y) \vee x) \cdot (((x \,/\, y) \vee x) \setminus ((x \,/\, z) \vee x)) \\ \vdash (x \,/\, (y \vee z)) \vee x,$$

is not derivable in $\mathbf{L}\vee$.

Our argument is based on brute-force proof search; however, some of the sequents proven to be non-derivable get marked (numbered) and then referred to, if they appear in the proof search again. This makes our proof a bit shorter.

Due to cut elimination, we seek only for a cut-free proof. We start with the well-known fact that rules $\cdot$L, $/$ R, $\setminus$ R, and $\vee$L are invertible: derivability of their conclusion yields derivability of their premise(s). Thus, in our proof search, if such a rule is applicable, we can always suppose that it was applied immediately, as the last (lowermost) rule in the derivation. Moreover, $\vee$L has two premises, and they should be *both* derivable if so is the goal sequent. Hence, when trying to prove non-derivability of the goal sequent we can *choose* one of the premises of $\vee$L and prove that it is not derivable.

Now, we are ready to prove that the sequent of Theorem 1 is not derivable in $\mathbf{L}\vee$. First, by invertibility of $\cdot$L we replace all $\cdot$'s by commas in the left-hand side of the sequent:

$$((x \,/\, y) \vee x) \,/\, ((x \,/\, y) \vee (x \,/\, z) \vee x), (x \,/\, y) \vee x, ((x \,/\, y) \vee x) \setminus ((x \,/\, z) \vee x) \\ \vdash (x \,/\, (y \vee z)) \vee x$$

Next, we apply invertibility of $\vee$L to $(x \,/\, y) \vee x$. The sequent should be derivable with both $x \,/\, y$ and x in this place; we choose $x \,/\, y$:

$$((x \,/\, y) \vee x) \,/ ((x \,/\, y) \vee (x \,/\, z) \vee x), x \,/\, y, ((x \,/\, y) \vee x) \,\backslash ((x \,/\, z) \vee x) \\ \vdash (x \,/ (y \vee z)) \vee x$$

The lowermost rule in the definition introduces one of the main connectives. There are now 4 of them:

$$((x \,/\, y) \vee x) \overset{1}{/} ((x \,/\, y) \vee (x \,/\, z) \vee x), x \overset{2}{/} y, ((x \,/\, y) \vee x) \overset{3}{\backslash} ((x \,/\, z) \vee x) \\ \vdash (x \,/ (y \vee z)) \overset{4}{\vee} x$$

Now we consider all possible cases. The enumeration of cases is as follows: for $/$ and $\backslash$ connectives, the case number is of the form n–m, where n is the number of the connective (as shown above) and m is the number of formulae that are sent to Φ by the $/$L or $\backslash$L rule.[2] For the 4th connective, $\vee$, we have cases 4a and 4b, for choosing $x \,/ (y \vee z)$ or x, respectively.

Case 1–1. In this case we have $x \,/\, y \vdash (x \,/\, y) \vee (x \,/\, z) \vee x$ (fine) and

$$(x \,/\, y) \vee x, ((x \,/\, y) \vee x) \,\backslash ((x \,/\, z) \vee x) \vdash (x \,/ (y \vee z)) \vee x.$$

Invert $\vee$L and choose $x \,/\, y$. Now we have 3 options:

$$x \overset{1}{/} y, ((x \,/\, y) \vee x) \overset{2}{\backslash} ((x \,/\, z) \vee x) \vdash (x \,/ (y \vee z)) \overset{3}{\vee} x.$$

Notice that here Φ in $/$L or $\backslash$L is determined in a unique way.

Subcase 1. We get

$$((x \,/\, y) \vee x) \,\backslash ((x \,/\, z) \vee x) \vdash y \tag{32}$$

as the left premise. This sequent is not derivable ($\backslash$ cannot be decomposed, since there is nothing to the left of the formula).

Subcase 2. Here we have $x \,/\, y \vdash (x \,/\, y) \vee x$ (fine) as the left premise and

$$(x \,/\, z) \vee x \vdash (x \,/ (y \vee z)) \vee x \tag{33}$$

as the right one. We show that (33) is not derivable. Inverting $\vee$L and choosing $x \,/\, z$ yields $x \,/\, z \vdash (x \,/ (y \vee z)) \vee x$, and now either $x \,/\, z \vdash x \,/ (y \vee z)$ or $x \,/\, z \vdash x$ should be derivable. The latter is trivially not. For the former, inverting $/$R and $\vee$L, choosing y, gives $x \,/\, z, y \vdash x$, which is also not derivable.

Subcase 3a. Here we get

$$x \,/\, y, ((x \,/\, y) \vee x) \,\backslash ((x \,/\, z) \vee x) \vdash x \,/ (y \vee z).$$

Inverting $/$R and $\vee$L, choosing y, gives

$$x/y, ((x \,/\, y) \vee x)\backslash((x \,/\, z) \vee x), y \vdash x.$$

[2] Due to Lambek's restriction, Φ should be non-empty, *i.e.*, $m > 0$.

Decomposing $/$ with $\Phi = ((x\,/\,y) \vee x)\,\backslash((x\,/\,z) \vee x)$ gives the left premise $((x\,/\,y) \vee x)\,\backslash((x\,/\,z) \vee x) \vdash y$, which is already shown to be non-derivable (32). Decomposing $/$ with $\Phi = ((x\,/\,y) \vee x)\,\backslash((x\,/\,z) \vee x), y$ gives a non-derivable right premise $x\,/\,y \vdash x$. Finally, decomposing $\backslash$ gives $(x\,/\,z) \vee x, y \vdash x$, which is also not derivable: inverting $\vee$L and choosing x gives $x, y \vdash x$.

Subcase 3b. In this case we have

$$x/y, ((x\,/\,y) \vee x)\backslash((x\,/\,z) \vee x) \vdash x.$$

Decomposing $/$ yields (32), which is not derivable. Decomposing $\backslash$ yields $((x\,/\,z)\vee x \vdash x$, which is shown to be non-derivable by inverting $\vee$L and choosing $x\,/\,z$.

Case 1–2. The right premise now is

$$(x\,/\,y) \vee x \vdash (x\,/(y \vee z)) \vee x, \tag{34}$$

which is shown to be non-derivable exactly as (33).

Case 2–1. The left premise here is (32), which is not derivable.

Case 3–1. Here the left premise is fine, and the right one is

$$((x\,/\,y) \vee x)\,/((x\,/\,y) \vee (x\,/\,z) \vee x), (x\,/\,z) \vee x \vdash (x\,/(y \vee z)) \vee x.$$

Invert $\vee$L and choose $x\,/\,z$:

$$((x\,/\,y) \vee x)/((x\,/\,y) \vee (x\,/\,z) \vee x), x/z \vdash (x\,/(y \vee z)) \vee x.$$

Decomposing the left $/$ yields (34), which is not derivable, as the right premise. Decomposing the right $/$ is impossible, since Φ should be non-empty. Finally, decomposing $\vee$ on the right yields two subcases.

Subcase a.

$$((x\,/\,y) \vee x)\,/((x\,/\,y) \vee (x\,/\,z) \vee x), x\,/\,z \vdash x\,/(y \vee z).$$

Inverting $/$R and $\vee$L, choosing y, yields

$$((x\,/\,y) \vee x)/((x\,/\,y) \vee (x\,/\,z) \vee x), x/z, y \vdash x. \tag{35}$$

Decomposing the right $/$ would yield $y \vdash z$, which is not derivable. So the only option is decomposing the left $/$. This gives two possible situations, depending on how many formulae go to Φ. If Φ takes one formula, then the right premise of $/$L is

$$(x\,/\,y) \vee x, y \vdash x.$$

The choice of x in inverting $\vee$L gives $x, y \vdash x$, which is not derivable. If Φ takes two formulae, then we have the right premise of the form

$$(x\,/\,y) \vee x \vdash x,$$

which is also not derivable, now by choosing $x\,/\,y$.

Subcase b.

$$((x / y) \vee x)/((x / y) \vee (x / z) \vee x), x / z \vdash x.$$

Now we can only decompose the left /, which yields

$$(x / y) \vee x, x / z \vdash x$$

as the right premise. Both choices in inverting $\vee$L fail: neither $x / y, x / z \vdash x$, nor $x, x / z \vdash x$ is derivable.

Case 3–2. Here the right premise is (33), which is not derivable.

Case 4a. Here we again invert $/$R and $\vee$L, choosing y:

$$((x / y) \vee x) \overset{1}{/} ((x / y) \vee (x / z) \vee x), x \overset{2}{/} y, ((x / y) \vee x) \overset{3}{\backslash} ((x / z) \vee x), y \vdash x$$

Again, as in the top-level analysis, we consider several cases.

Subcase 1–1. The right premise is of the form

$$(x / y) \vee x, x / y, ((x / y) \vee x) \backslash ((x / z) \vee x), y \vdash x$$

Invert $\vee$L choosing x:

$$x, x \overset{1}{/} y, ((x / y) \vee x) \overset{2}{\backslash} ((x / z) \vee x), y \vdash x$$

Here decomposition 1–1 yields non-derivable (32); 1–2 yields

$$((x / y) \vee x) \backslash ((x / z) \vee x), y \vdash y, \tag{36}$$

which is also not derivable (no decomposition possible). Decomposition 2–1 gives right premise

$$x, (x / z) \vee x, y \vdash x,$$

and choosing x in the inversion of $\vee$L gives non-derivable $x, x, y \vdash x$. Finally, decomposition 2–2 gives

$$(x / z) \vee x, y \vdash x \tag{37}$$

as the right premise, and choosing x also makes it non-derivable: $x, y \vdash x$.

Subcase 1–2. This gives a non-derivable right premise (37).

Subcase 1–3. Here the right premise is $(x / y) \vee x \vdash x$, which is invalidated by choosing x / y in the inversion of $\vee$L.

Subcases 2–1 and 2–2 give non-derivable left premises: (32) and (36) respectively.

Subcase 3–1. Here we get

$$((x / y) \vee x) /((x / y) \vee (x / z) \vee x), (x / z) \vee x, y \vdash x.$$

Inverting $\vee$L, choosing x / z, yields (35), which is not derivable.

Subcase 3–2. Here the right premise is

$$(x \,/\, z) \vee x, y \vdash x,$$

which is not derivable (37).

Case 4b. In this case we have

$$((x \,/\, y) \vee x) \overset{1}{/} ((x \,/\, y) \vee (x \,/\, z) \vee x), x \overset{2}{/} y, ((x \,/\, y) \vee x) \overset{3}{\backslash} ((x \,/\, z) \vee x) \vdash x.$$

Here we return to the beginning of the proof and consider the same cases 1–1, 1–2, 2–1, 3–1, and 3–2. Each of these cases decomposes / or \, with the same left premise. The right premises are here are of the form $\Gamma \vdash x$. We suppose that such a sequent is derivable. Then, by application of $\vee$L, we get $\Gamma \vdash (x \,/ (y \vee z)) \vee x$. Now we are *exactly* in the situation of one of the cases from 1–1 to 3–2, and can use the argumentation above "as is."

This finishes our case analysis and thus the proof of Theorem 1.

References

1. Abrusci, V.M.: A comparison between Lambek syntactic calculus and intuitionistic linear logic. Zeitschr. math. Logik Grundl. Math. (Math. Logic Q.) **36**, 11–15 (1990)
2. Andréka, H., Mikulás, Sz.: Lambek calculus and its relational semantics: completeness and incompleteness. J. Log. Lang. Inform. **3**(1), 1–37 (1994)
3. Buszkowski, W.: Compatibility of a categorial grammar with an associated category system. Zeitschr. math. Log. Grundl. Math. **28**, 229–238 (1982)
4. Buszkowski, W.: On the complexity of the equational theory of relational action algebras. In: Schmidt, R.A. (ed.) RelMiCS 2006. LNCS, vol. 4136, pp. 106–119. Springer, Heidelberg (2006). https://doi.org/10.1007/11828563_7
5. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated lattices. An algebraic Glimpse at Substructural Logics. Elsevier, Amsterdam (2007)
6. Girard, J.-Y.: Linear logic. Theor. Comput. Sci. **50**(1), 1–101 (1987)
7. Jipsen, P., Tsinakis, C.: A survey of residuated lattices. In: Martinez, J. (ed.) Ordered Algebraic Structures, pp. 19–56. Kluwer Academic Publishers, Dordrecht (2002)
8. Jipsen, P.: From semirings to residuated Kleene lattices. Stud. Logica **76**(2), 291–303 (2004)
9. Kanovich, M.: The direct simulation of Minsky machines in linear logic. In: Girard, J.-Y., Lafont, Y., Regnier, L. (eds.) Advances in Linear Logic, London Mathematical Society Lecture Notes, vol. 222, pp. 123–145. Cambridge University Press, Cambridge (1995)
10. Kanovich, M., Kuznetsov, S., Nigam, V., Scedrov, A.: Subexponentials in non-commutative linear logic. Math. Struct. Comput. Sci. (2018). https://doi.org/10.1017/S0960129518000117. FirstView
11. Kanovich, M., Kuznetsov, S., Scedrov, A.: The complexity of multiplicative-additive Lambek calculus: 25 years later. In: Iemhoff, R. et al. (eds.) WoLLIC 2019. LNCS, vol. 11541, pp. 356–372, Springer, Heidelberg (2019)
12. Kuznetsov, S.L.: Trivalent logics arising from L-models for the Lambek calculus with constants. J. Appl. Non-Class. Log. **4**(1–2), 132–137 (2014)

13. Lafont, Y.: The undecidability of second order linear logic without exponentials. J. Symb. Log. **61**(2), 541–548 (1996)
14. Lafont, Y., Scedrov, A.: The undecidability of second order multiplicative linear logic. Inf. Comput. **125**(1), 46–51 (1996)
15. Lambek, J.: The mathematics of sentence structure. Amer. Math. Monthly **65**, 154–170 (1958)
16. Lambek, J.: Deductive systems and categories II: standard constructions and closed categories. In: Hilton, P. (ed.) Category Theory, Homology Theory and Their Applications I. LNM, vol. 86, pp. 76–122. Springer, Berlin (1969)
17. Moot, R., Retoré, C.: The Logic of Categorial Grammars. A Deductive Account of Natural Language Syntax and Semantics. LNCS, vol. 6850. Springer, Heidelberg (2012)
18. Okada, M., Terui, K.: The finite model property for various fragments of intuitionistic linear logic. J. Symb. Log. **64**(2), 790–802 (1999)
19. Ono, H., Komori, Y.: Logics without contraction rule. J. Symb. Log. **50**(1), 169–201 (1985)
20. Pentus, M.: The conjoinability relation in Lambek calculus and linear logic. J. Log. Lang. Inform. **3**(2), 121–140 (1994)
21. Pentus, M.: Models for the Lambek calculus. Ann. Pure Appl. Log. **75**(1–2), 179–213 (1995)
22. Wald, M., Dilworth, R.P.: Residuated lattices. Trans. Amer. Math. Soc. **45**, 335–354 (1939)
23. Wurm, C.: Language-theoretic and finite relation models for the (full) Lambek calculus. J. Log. Lang. Inform. **26**(2), 179–214 (2017)

Logics for First-Order Team Properties

Juha Kontinen and Fan Yang(✉)

University of Helsinki, Helsinki, Finland
{juha.kontinen,fan.yang}@helsinki.fi

Abstract. In this paper, we introduce a logic based on team semantics, called **FOT**, whose expressive power coincides with first-order logic both on the level of sentences and (open) formulas, and we also show that a sublogic of **FOT**, called $\mathbf{FOT}^{\downarrow}$, captures exactly downward closed first-order team properties. We axiomatize completely the logic **FOT**, and also extend the known partial axiomatization of dependence logic to dependence logic enriched with the logical constants in $\mathbf{FOT}^{\downarrow}$.

Keywords: Dependence logic · Team semantics · First-order logic

1 Introduction

In this paper, we define logics based on team semantics for characterizing first-order team properties, and we also study the axiomatization problem of these logics.

Team semantics is a semantical framework originally introduced by Hodges [21], and later systematically developed by Väänänen with the introduction of *dependence logic* [31], which extends first-order logic with *dependence atoms*. Other notable logics based on team semantics include *independence logic* introduced by Grädel and Väänänen [16] (which is first-order logic extended with independence atoms), and *inclusion logic* introduced by Galliani [11] (which is first-order logic extended with inclusion atoms). In team semantics formulas are evaluated in a model over *sets* of assignments for the free variables (called *teams*) rather than single assignments as in the usual first-order logic. Teams X with the domain $\{v_1, \ldots, v_k\}$ are essentially k-ary relations $rel(X) = \{(s(v_1), \ldots, s(v_k)) \mid s \in X\}$, and thus open formulas define team properties. In general, knowing the expressive power of a logic for sentences (with no free variables) does not automatically give a characterization for the expressive power of open formulas of the same logic. Such a peculiar phenomenon has sparked several studies on the expressive power of logics based on team semantics. In particular, while it follows straightforwardly from the earlier known results of Henkin, Enderton, Walkoe, and Hodges [8,20,22,32] that dependence logic (**D**) and independence logic (**Ind**) are both equivalent to existential second-order logic (**ESO**) on the level of sentences, it turns out that open formulas of **D** have different expressive power from open formulas of **Ind**: The latter characterize all **ESO** team properties [11], whereas the former characterize only downward closed **ESO** team

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 392–414, 2019.
https://doi.org/10.1007/978-3-662-59533-6_24

properties [26]. Along the same line, a later breakthrough showed that inclusion logic corresponds, over sentences, to *positive greatest fixed-point logic* [15], which is strictly more expressive than first-order logic as well. In this paper we define a team-based logic, called **FOT**, whose expressive power coincides with first-order logic (**FO**) both on the level of sentences and open formulas, in the sense that **FOT**-formulas characterize (modulo the empty team) exactly team properties definable by first-order sentences with an extra relation symbol R. To the best of the authors' knowledge, no such logic has been defined previously.

In related previous work, it was shown in [10,13,28] that first-order logic extended with *constancy atoms* $=\!\!(x)$ and **FO** extended with classical negation $\sim$ are both equivalent to **FO** over sentences, whereas on the level of formulas they are both strictly less expressive than **FO**, and thus fail to capture all first-order team properties. It was also illustrated in [24] that a certain simple disjunction of dependence atoms already defines an NP-complete team property. Therefore, any logic based on team semantics having the disjunction $\vee$ inherited from first-order logic and in which dependence atoms are expressible will be able to express NP-complete team properties indicating that $\vee$ is too expressive connective to be added to **FOT**. The logic **FOT** we define in this paper has weaker version of disjunction $\mathbin{\dot\vee}$ and classical negation $\dot\sim$ as well as weaker quantifiers $\forall^1, \exists^1$. We prove, in Sect. 3, that our logic **FOT** captures first-order team properties (modulo the empty team) and we also show, as an application of Lyndon's Interpolation Theorem of first-order logic, that a sublogic of **FOT**, denoted as **FOT**$^{\downarrow}$, captures exactly downward closed first-order team properties (modulo the empty team).

In the second part of this paper study the axiomatization problem of our logics **FOT** and **FOT**$^{\downarrow}$. In Sect. 4 we introduce a sound and complete system of natural deduction for **FOT** that on one hand behaves like the system of **FO** to a certain extend (in the sense of Lemma 12), while on the other hand incorporates natural and interesting rules for inclusion atoms and their interaction with the weak logical constants.

In Sect. 5, we apply our results to the problem of finding axiomatizations for larger and larger fragments of dependence logic and its variants by extending the known partial axiomatization of dependence logic to **D** enriched with the logical constants in **FOT**$^{\downarrow}$ (denoted as **D** $\oplus$ **FOT**$^{\downarrow}$), which, by our result in the first part of this paper, is expressively equivalent to **D**. While **D** is not effectively axiomatizable (for it is equivalent to **ESO**), a complete axiomatization for first-order consequences of **D**-sentences has been given in [27]. More precisely, a system of natural deduction for dependence logic was introduced in [27] for which the completeness theorem.

$$\Gamma \vdash \theta \iff \Gamma \models \theta \tag{1}$$

holds whenever Γ is a set of **D**-sentences and θ is an **FO**-sentence. This result has been, subsequently, generalized to e.g., allows also open formulas [25], and treats also independence logic [17] or dependence logic with generalized quantifiers [9]. A recent new generalization given in [33] extends the known systems for **D** and

Ind to cover the case when θ in (1) is not necessarily an **FO**-formula but merely a formula defining a first-order team property. However, since the problem of whether a **D**- or **Ind**-formula defines a first-order team property is undecidable, the extension of [33] is not effectively represented. Motivated by the results of [33], we give an effective extension of (1), in which θ is a sentence of our compositionally defined logic $\mathbf{FOT}^{\downarrow}$ and Γ is a set of $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$-sentences. Finding an effective axiomatization in the more general case of **Ind** enriched with the logical constants in **FOT** is left as future work.

Apart from theoretical significance, our results also provide new logical tools for the applications of team-based logics in other related areas; such applications have been studied in recent years, e.g., in database theory [19], formal semantics of natural language [4,5], Bayesian statistics [7,18], social choice theory [30], and quantum information theory [23]. In particular, inquisitive logic [6] adopts, independently, also the team semantics to provide formal semantics of questions in natural language, and the first-order version of inquisitive logic can be viewed as a team-based logic (in a slightly different setting) with the weak disjunction $\mathbin{\vee\!\!\!\vee}$ and the weak quantifiers $\forall^1, \exists^1$. The study we provide in this paper for the expressive power and axiomatization problem of these weaker logical constants will potentially help clarifying properties of first-order inquisitive logic. In the recent formalization of Arrow's Theorem [2] in social choice in independence logic [30], the weak disjunction $\mathbin{\vee\!\!\!\vee}$ plays a natural role, and the completeness theorem of the type (1) was crucial for deriving Arrow's Theorem formally. The axiomatization results we obtained in this paper are then expected to contribute to the formal analysis of Arrow's Theorem and other impossibility theorems in social choice.

2 Preliminaries

We consider first-order vocabularies $\mathcal{L}$ with an equality symbol $=$. An $\mathcal{L}$-term t is defined inductively as usual, and formulas of *First-order Logic* (**FO**) are defined as:

$$\alpha ::= t_1 = t_2 \mid Rt_1 \dots t_k \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \exists x\alpha \mid \forall x\alpha.$$

Throughout the paper, we reserve the first Greek letters $\alpha, \beta, \gamma, \delta$ for first-order formulas. As usual, define $\alpha \to \beta := \neg\alpha \vee \beta$. We use the letters $\mathsf{v}, \mathsf{x}, \mathsf{y}, \mathsf{z}, \dots$ in sans-serif face to stand for sequences of variables, and sequences of terms are denoted as $\mathsf{t}, \mathsf{t}', \dots$ We write $\mathsf{Fv}(\alpha)$ for the set of free variables of α, and write $\alpha(\mathsf{x})$ to indicate that the free variables of α are among $\mathsf{x} = \langle x_1, \dots, x_n \rangle$. A formula with no free variables is called a *sentence*.

For any $\mathcal{L}$-model M, we use the same notation M also to denote its domain. We write $\mathcal{L}(R)$ for the vocabulary expanded from $\mathcal{L}$ by adding a fresh relation symbol R, and write (M, R^M) for the $\mathcal{L}(R)$-expansion of M in which the k-ary relation symbol R is interpreted as $R^M \subseteq M^k$. We sometimes write $\alpha(R)$ to emphasize that the first-order formula α is in the vocabulary $\mathcal{L}(R)$ for some $\mathcal{L}$.

We assume that the reader is familiar with the usual Tarskian semantics of first-order logic. In this paper, we consider logics with *team semantics*. A *team*

X of M over a set V of variables is a set of assignments $s : V \to M$, where V is called the domain of X, denoted $\mathsf{dom}(X)$. Given a first-order formula α, given any $\mathcal{L}$-model M and any team X over $V \supseteq \mathsf{Fv}(\alpha)$, we define the *satisfaction* relation $M \models_X \alpha$ inductively as follows:

- $M \models_X \lambda$ for λ a first-order atom iff for all $s \in X$, $M \models_s \lambda$ in the usual sense.
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_{\{s\}} \alpha$.
- $M \models_X \alpha \wedge \beta$ iff $M \models_X \alpha$ and $M \models_X \beta$.
- $M \models_X \alpha \vee \beta$ iff there are $Y, Z \subseteq X$ such that $X = Y \cup Z$, $M \models_Y \alpha$ and $M \models_Z \beta$.
- $M \models_X \exists x\alpha$ iff $M \models_{X(F/x)} \alpha$ for some $F : X \to \wp(M) \setminus \{\emptyset\}$, where $X(F/x) = \{s(a/x) \mid s \in X,\ a \in F(s)\}$.
- $M \models_X \forall x\alpha$ iff $M \models_{X(M/x)} \alpha$, where $X(M/x) = \{s(a/x) \mid s \in X,\ a \in M\}$.

It is easy to verify that first-order formulas have the following properties:

Empty team property: $M \models_\emptyset \phi$.
Downward closure: $[\,M \models_X \phi$ and $Y \subseteq X\,] \Longrightarrow M \models_Y \phi$.
Union closure: $[\,M \models_X \phi$ and $M \models_Y \phi\,] \Longrightarrow M \models_{X\cup Y} \phi$.

Downward closure property together with union closure property are equivalent to

Flatness property: $M \models_X \phi \iff M \models_{\{s\}} \phi$ for all $s \in X$.

Logics based on team semantics do not in general have the flatness property. For instance, *dependence logic* [31], which is first-order logic extended with dependence atoms $=\!\!(t_1, \ldots, t_n, t)$, is downward closed but not flat; and *inclusion logic* [11], which is first-order logic extended with inclusion atoms $t_1, \ldots, t_n \subseteq t'_1, \ldots, t'_n$, is union closed but not flat. Especially, dependence atoms and inclusion atoms are not flat. We recall their semantics below:

- $M \models_X =\!\!(\mathsf{t}, t')$ iff for all $s, s' \in X$, $s(\mathsf{t}) = s'(\mathsf{t})$ implies $s(t') = s'(t')$.
- $M \models_X \mathsf{t} \subseteq \mathsf{t}'$ iff for all $s \in X$, there is $s' \in X$ such that $s(\mathsf{t}) = s'(\mathsf{t}')$.

In this paper, we study two (non-flat) logics based on team semantics, called **FOT** and **FOT**$^\downarrow$, whose formulas are built from a different (yet similar) set of connectives and quantifiers than those in first-order logic as follows:

$$(\mathbf{FOT}) \qquad \phi ::= \lambda \mid \mathsf{x} \subseteq \mathsf{y} \mid \dot\sim \phi \mid \phi \wedge \phi \mid \phi \mathbin{\mathpalette\make@circled\vee} \phi \mid \exists^1 x\phi \mid \forall^1 x\phi$$
$$(\mathbf{FOT}^\downarrow) \qquad \phi ::= \lambda \mid \neg\delta \mid \phi \wedge \phi \mid \phi \mathbin{\mathpalette\make@circled\vee} \phi \mid \exists^1 x\phi \mid \forall^1 x\phi$$

where λ is an arbitrary first-order atomic formula, x and y are two sequences of variables of the same length, and δ is a quantifier-free and disjunction-free formula (i.e., $\delta ::= \lambda \mid \neg\delta \mid \delta \wedge \delta$). We call the logical constants $\dot\sim$, $\mathbin{\mathpalette\make@circled\vee}$, $\exists^1, \forall^1$, which were introduced in [1,26,33], *weak classical negation*, *weak disjunction*, *weak existential quantifier* and *weak universal quantifier*, respectively. Their team semantics are defined as:

- $M \models_X \dot{\sim}\phi$ iff $X = \emptyset$ or $M \not\models_X \phi$.
- $M \models_X \phi \mathbin{\dot\vee} \psi$ iff $M \models_X \phi$ or $M \models_X \psi$.
- $M \models_X \exists^1 x\phi$ iff $M \models_{X(a/x)} \phi$ for some $a \in M$, where $X(a/x) = \{s(a/x) \mid s \in X\}$.
- $M \models_X \forall^1 x\phi$ iff $M \models_{X(a/x)} \phi$ for all $a \in M$.

It is easy to verify that formulas of **FOT** and **FOT**$^{\downarrow}$ have the empty team property, and **FOT**$^{\downarrow}$ formulas have the downward closure property.

In **FOT** we adopt the usual convention for classical implication, and write $\phi \rightarrowtriangle \psi$ for $\dot{\sim}\phi \mathbin{\dot\vee} \psi$ and $\phi \leftrightarrowtriangle \psi$ for $(\phi \rightarrowtriangle \psi) \wedge (\psi \rightarrowtriangle \phi)$. With the help of the classical implication $\rightarrowtriangle$, one can easily express $\neg\delta$ for δ being quantifier-free and disjunction free (or flat) as

$$\neg\delta(\mathsf{x}) \equiv \forall^1 \mathsf{y}(\mathsf{y} \subseteq \mathsf{x} \rightarrowtriangle \dot{\sim}\delta(\mathsf{y}/\mathsf{x})).$$

Also, dependence atoms $=\!\!(\mathsf{x}, y)$ with variables as arguments are definable in **FOT**:

$$=\!\!(\mathsf{x}, y) \equiv \forall^1 \mathsf{u}_0 v_0 \mathsf{u}_1 v_1((\mathsf{u}_0 v_0 \subseteq \mathsf{x}y \wedge \mathsf{u}_1 v_1 \subseteq \mathsf{x}y \wedge \mathsf{u}_0 = \mathsf{u}_1) \rightarrowtriangle v_0 = v_1).$$

Recall that the *constancy atom* $=\!\!(x)$ abbreviates the dependence atom $=\!\!(\langle\,\rangle, x)$ with the first argument being the empty sequence $\langle\,\rangle$, and its semantics reduces to

- $M \models_X =\!\!(x)$ iff for all $s, s' \in X$, $s(x) = s'(x)$.

Constancy atoms can be defined alternatively as $=\!\!(x) \equiv \exists^1 y(y = x)$.

3 Characterizing First-Order Team Properties

In this section, we prove that **FOT**-formulas characterize first-order team properties (modulo the empty team), and **FOT**$^{\downarrow}$-formulas characterize downward closed first-order team properties (modulo the empty team).

Let us first define formally the relevant notions. Observe that a team X of an $\mathcal{L}$-model M over a domain $\{v_1, \ldots, v_k\}$ can also be viewed as a k-ary relation $rel(X) \subseteq M^k$ defined as $rel(X) = \{(s(v_1), \ldots, s(v_k)) \mid s \in X\}$. We call a collection $\mathcal{P}_M \subseteq \wp(M^k)$ of k-ary relations (or teams) of an $\mathcal{L}$-model M a *local team property*; and a *(global) team property* is a class $\mathcal{P}$ of local team properties $\mathcal{P}_M$ for all $\mathcal{L}$-models M. A formula $\phi(\mathsf{v})$ of a logic based on team semantics clearly defines a team property $\mathcal{P}^\phi$ such that for all M, $\mathcal{P}^\phi_M = \{rel(X) \mid M \models_X \phi(\mathsf{v})\}$. Clearly, the team properties $\mathcal{P}^\phi$ defined by **FOT**$^{\downarrow}$-formulas ϕ are *downward closed*, that is, $A \subseteq B \in \mathcal{P}^\phi_M$ implies $A \in \mathcal{P}^\phi_M$ for any M. Note also that while the empty relation $\emptyset$ may not be contained in a team property $\mathcal{P}_M$, since all team-based logics considered in this paper have the empty team property (i.e., $\emptyset$ is in $\mathcal{P}^\phi_M$ for all ϕ), we will confine ourselves only to those team properties $\mathcal{P}$ with the empty relation $\emptyset$ contained in each local property $\mathcal{P}_M$.

We call a team property $\mathcal{P}$ *first-order* if there is a first-order $\mathcal{L}(R)$-sentence $\alpha(R)$ such that $(M, A) \models \alpha(R)$ iff $A \in \mathcal{P}_M$ for all M and all nonempty relations A. It is worth noting that we are using the terminology "definability" in two different semantic settings: Even though every first-order team property $\mathcal{P}$ is (trivially) defined by some first-order $\mathcal{L}(R)$-sentence $\alpha(R)$ with an extra relation symbol R (in the sense of the usual semantics of first-order logic), it does not follow that each such first-order team property $\mathcal{P}$ is definable by some first-order $\mathcal{L}$-formula $\beta(\mathsf{v})$ in the team semantics sense (i.e., $\mathcal{P} = \mathcal{P}^{\beta}_{M}$). As a simple illustration, in view of the flatness property of first-order formulas, the following very simple team property (of the empty vocabulary $\mathcal{L}_0$)

$$\mathcal{P}_{\leq 1} = \{(M, rel(X)) \mid M \text{ an } \mathcal{L}_0\text{-model and } |X| \leq 1\}$$

cannot be defined by any first-order formula $\beta(\mathsf{v})$ of the empty vocabulary $\mathcal{L}_0$.

We now show that the team properties defined by formulas of **FOT** and $\mathbf{FOT}^{\downarrow}$ are first-order.

Theorem 1. *For any $\mathcal{L}$-formula $\phi(v_1, \ldots, v_k)$ of* **FOT** *or* $\mathbf{FOT}^{\downarrow}$*, there exists a first-order $\mathcal{L}(R)$-sentence $\gamma_\phi(R)$ with a fresh k-ary relation symbol R such that for any $\mathcal{L}$-model M and any team X over $\{v_1, \ldots, v_k\}$,*

$$M \models_X \phi \iff (M, rel(X)) \models \gamma_\phi(R). \tag{2}$$

Proof. We prove the theorem by proving a slightly more general claim: For any subformula $\theta(\mathsf{v}, \mathsf{x})$ of $\phi(\mathsf{v})$, there exists a first-order $\mathcal{L}(R)$-formula $\gamma_\theta(R, \mathsf{x})$ such that for all $\mathcal{L}$-models M, teams X and sequences a of elements in M,

$$M \models_{X(\mathsf{a}/\mathsf{x})} \theta(\mathsf{v}, \mathsf{x}) \iff (M, rel(X)) \models \gamma_\theta(R, \mathsf{x})(\mathsf{a}/\mathsf{x}).$$

It is easy to verify that the formula γ_θ defined inductively as follows (and found essentially in, e.g., [11,33]) will work:

- If $\theta(\mathsf{v}, \mathsf{x}) = \delta(\mathsf{v}, \mathsf{x})$ for some quantifier-free and disjunction-free first-order formula δ, let $\gamma_\theta(R, \mathsf{x}) = \forall \mathsf{u}(R\mathsf{u} \to \delta(\mathsf{u}/\mathsf{v}, \mathsf{x}))$.
- If $\theta(\mathsf{v}, \mathsf{x}) = \rho(\mathsf{vx}) \subseteq \sigma(\mathsf{vx})$, where $\rho(\mathsf{vx})$ and $\sigma(\mathsf{vx})$ are two sequences of variables from vx, let $\gamma_\theta(R, \mathsf{x}) = \forall \mathsf{u} \exists \mathsf{w}\big(R\mathsf{u} \to (R\mathsf{w} \wedge \rho(\mathsf{ux}) = \sigma(\mathsf{wx}))\big)$.
- If $\theta(\mathsf{v}, \mathsf{x}) = \theta_0(\mathsf{v}, \mathsf{x}) \wedge \theta_1(\mathsf{v}, \mathsf{x})$ and does not belong to the case of the first item, let $\gamma_\theta(R, \mathsf{x}) = \gamma_{\theta_0}(R, \mathsf{x}) \wedge \gamma_{\theta_1}(R, \mathsf{x})$.
- If $\theta(\mathsf{v}, \mathsf{x}) = \theta_0(\mathsf{v}, \mathsf{x}) \mathbin{\vee\!\!\!\vee} \theta_1(\mathsf{v}, \mathsf{x})$, let $\gamma_\theta(R, \mathsf{x}) = \gamma_{\theta_0}(R, \mathsf{x}) \vee \gamma_{\theta_1}(R, \mathsf{x})$.
- If $\theta(\mathsf{v}, \mathsf{x}) = \dot{\sim} \theta_0(\mathsf{v}, \mathsf{x})$, let $\gamma_\theta(R, \mathsf{x}) = \forall \mathsf{u} \neg R\mathsf{u} \vee \neg \gamma_{\theta_0}(R, \mathsf{x})$.
- If $\theta(\mathsf{v}, \mathsf{x}) = \exists^1 y \theta_0(\mathsf{v}, y\mathsf{x})$, let $\gamma_\theta(R, \mathsf{x}) = \exists y \gamma_{\theta_0}(R, y\mathsf{x})$.
- If $\theta(\mathsf{v}, \mathsf{x}) = \forall^1 y \theta_0(\mathsf{v}, y\mathsf{x})$, let $\gamma_\theta(R, \mathsf{x}) = \forall y \gamma_{\theta_0}(R, y\mathsf{x})$. $\square$

Next we prove the reverse direction of Theorem 1, from which we can conclude that **FOT**-formulas characterize exactly first-order team properties (modulo the empty team).

Theorem 2. *For any first-order $\mathcal{L}(R)$-sentence $\gamma(R)$ with a k-ary relation symbol R, there exists an $\mathcal{L}$-formula $\phi_\gamma(v_1,\dots,v_k)$ of* **FOT** *such that for any $\mathcal{L}$-model M and any nonempty team X over $\{v_1,\dots,v_k\}$,*

$$M \models_X \phi_\gamma(\mathsf{v}) \iff (M, rel(X)) \models \gamma(R).$$

Moreover, if R occurs in $\gamma(R)$ only negatively (i.e., every occurrence of R is in the scope of an odd number of nested negation symbols), ϕ_γ can be also chosen to be an $\mathbf{FOT}^{\downarrow}$*-formula.*

Proof. We may assume w.l.o.g. that the first-order sentence $\gamma(R)$ is in prenex normal form $Q_1x_1\dots Q_nx_n\theta(\mathsf{x})$, where $Q_i \in \{\forall,\exists\}$, θ is quantifier-free and in negation normal form (i.e., negations occur only in front of atomic formulas), and every occurrence of R is of the form $R\mathsf{x}_i$ for some sequence x_i of bound variables (for $R\mathsf{t} \equiv \exists\mathsf{y}(\mathsf{y}=\mathsf{t}\wedge R\mathsf{y})$).

Define the translation $\phi_\gamma(\mathsf{v}) := Q_1^1x_1\dots Q_n^1x_n\phi_\theta(\mathsf{x},\mathsf{v})$ in **FOT**, where $Q_i^1 = \forall^1$ if $Q_i=\forall$, $Q_i^1=\exists^1$ if $Q_i=\exists$, and $\phi_\theta(\mathsf{x},\mathsf{v})$ is defined inductively as follows:

– if $\theta=\lambda(\mathsf{x})$ is an atomic formula in which R does not occur, then $\phi_\lambda(\mathsf{x},\mathsf{v}) = \lambda(\mathsf{x})$;
– if $\theta = R\mathsf{x}_i$, then $\phi_\theta(\mathsf{x},\mathsf{v}) = \mathsf{x}_i \subseteq \mathsf{v}$;
– if $\theta=\neg\lambda$ for some (atomic) formula λ, then $\phi_\theta = \dot\sim \phi_\lambda$;
– if $\theta=\theta_0\wedge\theta_1$, then $\phi_\theta=\phi_{\theta_0}\wedge\phi_{\theta_1}$;
– if $\theta=\theta_0\vee\theta_1$, then $\phi_\theta=\phi_{\theta_0}\mathbin{\vee\!\!\!\vee}\phi_{\theta_1}$.

Next, we show by induction that for each quantifier-free formula $\theta(\mathsf{x})$, for any nonempty team X over $\{v_1,\dots,v_k\}$ and $a_1,\dots,a_n\in M$,

$$M \models_{X(\mathsf{a}/\mathsf{x})} \phi_\theta(\mathsf{x},\mathsf{v}) \iff (M, rel(X)) \models \theta(\mathsf{a}/\mathsf{x}). \tag{3}$$

If $\theta=\lambda(\mathsf{x})$ is an atomic formula in which R does not occur, then $\phi_\lambda=\lambda(\mathsf{x})$ and

$$\begin{aligned} M \models_{X(\mathsf{a}/\mathsf{x})} \lambda(\mathsf{x}) &\iff M \models_X \lambda(\mathsf{x})(\mathsf{a}/\mathsf{x})\\ &\iff M\models_s \lambda(\mathsf{x})(\mathsf{a}/\mathsf{x}) \text{ for all } s\in X\\ &\iff (M, rel(X))\models \lambda(\mathsf{a}/\mathsf{x}). \qquad \text{(since } R \text{ does not occur in } \lambda) \end{aligned}$$

If $\theta = R\mathsf{x}_i$, then $\phi_\theta = \mathsf{x}_i\subseteq\mathsf{v}$ and

$$\begin{aligned} M \models_{X(\mathsf{a}/\mathsf{x})} \mathsf{x}_i\subseteq\mathsf{v} &\iff \text{For all } s\in X(\mathsf{a}/\mathsf{x}),\text{ there exists } s'\in X(\mathsf{a}/\mathsf{x}) \text{ s.t. } s(\mathsf{x}_i)=s'(\mathsf{v})\\ &\iff \mathsf{a}_i\in rel(X)=\{s'(\mathsf{v})\mid s'\in X\}\\ &\iff (M, rel(X))\models R\mathsf{x}_i(\mathsf{a}/\mathsf{x}). \end{aligned}$$

If $\theta=\neg\lambda(\mathsf{x})$, then

$$\begin{aligned} M \models_{X(\mathsf{a}/\mathsf{x})} \dot\sim\phi_\lambda(\mathsf{x},\mathsf{v}) &\iff M \not\models_{X(\mathsf{a}/\mathsf{x})} \phi_\lambda(\mathsf{x},\mathsf{v}) && \text{(since)} X(\mathsf{a}/\mathsf{x})\neq\emptyset\\ &\iff (M, rel(X))\not\models \lambda(\mathsf{a}/\mathsf{x}) && \text{(by induction hypothesis)}\\ &\iff (M, rel(X))\models \neg\lambda(\mathsf{a}/\mathsf{x}). \end{aligned}$$

The cases when $\theta = \theta_0(\mathsf{x}) \vee \theta_1(\mathsf{x})$ and $\theta = \theta_0(\mathsf{x}) \wedge \theta_1(\mathsf{x})$ follow easily from the induction hypothesis.

Finally, we have

$$\begin{aligned} M \models_X Q_1^1 x_1 \dots Q_n^1 x_n \phi_\theta(\mathsf{x}, \mathsf{v}) &\iff Q_1 a_1 \in M \dots Q_n a_n \in M : \ M \models_{X(\mathsf{a}/\mathsf{x})} \phi_\theta(\mathsf{x}, \mathsf{v}) \\ &\iff Q_1 a_1 \in M \dots Q_n a_n \in M : \ (M, rel(X)) \models \theta(\mathsf{a}/\mathsf{x}) \\ &\iff (M, rel(X)) \models Q_1 x_1 \dots Q_n x_n \theta(\mathsf{x}). \end{aligned}$$

This completes the proof for the translation into **FOT**. Now, if R occurs only negatively in γ (thus also in θ), we can define alternatively the translation into $\mathbf{FOT}^{\downarrow}$ as: If $\theta = \neg R\mathsf{x}_i$, define $\phi_\theta = \neg \bigwedge_j x_{ij} = v_j$; if $\theta = \alpha(\mathsf{x})$ is a literal in which R does not occur, define $\phi_\alpha = \alpha(\mathsf{x})$. It is easy to verify that (3) still holds for these two cases. □

To conclude from the above theorems that $\mathbf{FOT}^{\downarrow}$-formulas characterize downward closed first-order team properties (modulo the empty team), we now prove a characterization theorem for first-order sentences $\alpha(R)$ that define downward closed team properties, by applying Lyndon's Interpolation Theorem of first-order logic, which we recall below.

Theorem 3 (Lyndon's Interpolation [29]). *Let α be a first-order $\mathcal{L}_0$-formula and β a first-order $\mathcal{L}_1$-formula. If $\alpha \models \beta$, then there is a first-order $\mathcal{L}_0 \cap \mathcal{L}_1$-formula δ such that $\alpha \models \delta$ and $\delta \models \beta$, and moreover a predicate symbol has a positive (resp. negative) occurrence in δ only if it has a positive (resp. negative) occurrence in both α and β.*

Proposition 4. *A first-order $\mathcal{L}(R)$-sentence $\alpha(R)$ defines a downward closed team property with respect to R if and only if there is a first-order $\mathcal{L}(R)$-sentence $\beta(R)$ such that $\alpha \equiv \beta$ and R occurs only negatively in β.*

Proof. "$\Longleftarrow$": Suppose α is a first-order $\mathcal{L}(R)$-sentence in which the k-ary predicate R occurs only negatively, and we assume w.l.o.g. that α is in negation normal form. We can show by induction that α is downwards closed with respect to R. The only nontrivial case is when $\alpha = \neg R\mathsf{t}$. In this case, for any model M, any $A \subseteq B \subseteq M^k$, and any assignment s, $(M, B) \models_s \neg R\mathsf{t} \Longrightarrow \mathsf{t}^M\langle s\rangle \notin B \Longrightarrow \mathsf{t}^M\langle s\rangle \notin A \Longrightarrow (M, A) \models_s \neg R\mathsf{t}$.

"$\Longrightarrow$": Suppose that α is a first-order $\mathcal{L}(R)$-sentence that is downwards closed with respect to R. It is easy to see that $\alpha \equiv \exists S(\alpha(S/R) \wedge \forall \mathsf{x}(R\mathsf{x} \to S\mathsf{x}))$, where $\alpha(S/R)$ is obtained from α by replacing every occurrence of R by S. Put $\gamma = \alpha(S/R) \wedge \forall \mathsf{x}(R\mathsf{x} \to S\mathsf{x})$, and note that R occurs only negatively in γ. Then, $\gamma \models \alpha$, since for any $\mathcal{L}(R, S)$-model (M, A, B) such that $(M, A, B) \models \gamma(R, S)$, we have $(M, A) \models \exists S\gamma(R, S)$, which implies $(M, A) \models \alpha(R)$.

Now, by Lyndon's Interpolation Theorem, there is a first-order $\mathcal{L}(R)$-sentence β such that $\gamma(R, S) \models \beta(R)$ and $\beta(R) \models \alpha(R)$, and moreover, R occurs only negatively in β. It remains to show $\alpha \models \beta$. For any $\mathcal{L}(R)$-model (M, A) such that $(M, A) \models \alpha(R)$. Clearly the $\mathcal{L}(R, S)$-model (M, A, A) satisfies $(M, A, A) \models \alpha(S/R) \wedge \forall \mathsf{x}(R\mathsf{x} \to S\mathsf{x})$. Since $\gamma \models \beta$, we have $(M, A, A) \models \beta(R)$, thereby $(M, A) \models \beta(R)$. □

Corollary 5. For any $\mathcal{L}$-formula $\phi(\mathsf{v})$ of $\mathbf{FOT}^{\downarrow}$, there exists a first-order $\mathcal{L}(R)$-sentence $\gamma_\phi(R)$ with R occurring only negatively such that (2) holds, and vice versa. In particular, $\mathbf{FOT}^{\downarrow}$-formulas characterize exactly downward closed first-order team properties (modulo the empty team).

4 Axiomatizing FOT

In this section, we introduce a system of natural deduction for **FOT**, and prove the soundness and completeness theorem. For the convenience of our proofs, we present our system of natural deduction in sequent style.

Table 1. Introduction and elimination rules for the weak logical constants

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\,\wedge\mathsf{I} \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}\,\wedge\mathsf{E} \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}\,\wedge\mathsf{E}$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \mathbin{\mathbb{V}} \psi}\,\mathbb{V}\mathsf{I} \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \psi \mathbin{\mathbb{V}} \phi}\,\mathbb{V}\mathsf{I} \qquad \frac{\Gamma \vdash \phi \mathbin{\mathbb{V}} \psi \quad \Gamma, \phi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}\,\mathbb{V}\mathsf{E}$$

$$\frac{\Gamma \vdash \phi(t/x) \qquad \Gamma \vdash {=}(t)}{\Gamma \vdash \exists^1 x\phi}\,\exists^1\mathsf{I} \qquad \frac{\Gamma \vdash \exists^1 x\phi \qquad \Gamma, \phi(v/x), {=}(v) \vdash \psi}{\Gamma \vdash \psi}\,\exists^1\mathsf{E}\ \text{(a)}$$

$$\frac{\Gamma \vdash \exists^1 x\phi \qquad \Gamma, \phi(c/x) \vdash \psi}{\Gamma \vdash \psi}\,\exists^1\mathsf{E}\ \text{(b)}$$

$$\frac{\Gamma \vdash \phi(v/x) \qquad \Gamma \vdash {=}(v)}{\Gamma \vdash \forall^1 x\phi}\,\forall^1\mathsf{I}\ \text{(c)} \qquad \frac{\Gamma \vdash \forall^1 x\phi \qquad \Gamma \vdash {=}(t)}{\Gamma \vdash \phi(t/x)}\,\forall^1\mathsf{E}$$

$$\frac{\Gamma \vdash \phi(c/x)}{\Gamma \vdash \forall^1 x\phi}\,\forall^1\mathsf{I}\ \text{(d)}$$

(a). $v \notin \mathsf{Fv}(\Gamma \cup \{\phi, \psi\})$ (b). c does not occur in Γ, ϕ, ψ
(c). $v \notin \mathsf{Fv}(\Gamma \cup \{\phi\})$ (d). c does not occur in Γ, ϕ

Definition 6. The system of natural deduction for **FOT** consists of all rules for identity, all rules in Table 1, and the following rules, where letters in sans-serif face (such as x, y) stand for sequences of variables, c is a constant symbol, ${=}(t)$ is short for $\exists^1 x(x = t)$, $\mathsf{con}(\mathsf{t})$ is short for $\bigwedge_i {=}(t_i)$, and $c\mathsf{x} \subseteq v\mathsf{y}$ is short for $\exists^1 u(u = c \wedge u\mathsf{x} \subseteq v\mathsf{y})$:

$$\frac{\phi \in \Gamma}{\Gamma \vdash \phi}\,\mathsf{AssmI}$$

$$\frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \dot{\sim}\phi}\,\dot{\sim}\mathsf{I} \qquad \frac{\Gamma \vdash \phi \qquad \Gamma \vdash \dot{\sim}\phi}{\Gamma \vdash \psi}\,\dot{\sim}\mathsf{E} \qquad \frac{\Gamma, \dot{\sim}\phi \vdash \bot}{\Gamma \vdash \phi}\,\mathsf{RAA}$$

$$\frac{}{\Gamma \vdash =\!\!(c)}\,\mathsf{conI} \qquad \frac{\Gamma \vdash \mathsf{con}(\mathsf{t})}{\Gamma \vdash =\!\!(f\mathsf{t})}\,\mathsf{conI}$$

$$\frac{}{\Gamma \vdash \mathsf{x} \subseteq \mathsf{x}}\,\subseteq\!\mathsf{Id} \qquad \frac{\Gamma \vdash x_1 \dots x_n \subseteq y_1 \dots y_n}{\Gamma \vdash x_{i_1} \dots x_{i_k} \subseteq y_{i_1} \dots y_{i_k}}\,\subseteq\!\mathsf{Pro}\ \text{(a)}$$

$$\frac{\Gamma \vdash \mathsf{x} \subseteq \mathsf{y} \qquad \Gamma \vdash \mathsf{y} \subseteq \mathsf{z}}{\Gamma \vdash \mathsf{x} \subseteq \mathsf{z}}\,\subseteq\!\mathsf{Tr} \qquad \frac{\Gamma \vdash \mathsf{x} \subseteq \mathsf{y} \qquad \Gamma \vdash \alpha(\mathsf{y})}{\Gamma \vdash \alpha(\mathsf{x})}\,\subseteq\!\mathsf{Cmp}\ \text{(b)}$$

$$\frac{\Gamma \vdash \mathsf{con}(\mathsf{x}) \qquad \Gamma \vdash \mathsf{y} \subseteq \mathsf{z}}{\Gamma \vdash \mathsf{xy} \subseteq \mathsf{xz}}\,\subseteq\!\mathsf{W}_{\mathsf{con}} \qquad \frac{\Gamma \vdash \mathsf{con}(\mathsf{x}) \qquad \Gamma \vdash \mathsf{x} \subseteq \mathsf{y}}{\Gamma \vdash \exists^1 \mathsf{z}(\mathsf{zx} \subseteq \mathsf{wy})}\,\subseteq\!\mathsf{W}_{\exists^1}$$

$$\frac{\Gamma \vdash \dot{\sim}\mathsf{x} \subseteq \mathsf{y} \qquad \Gamma, \mathsf{c} \subseteq \mathsf{x}, \dot{\sim}\mathsf{c} \subseteq \mathsf{y} \vdash \phi}{\Gamma \vdash \phi}\,\dot{\sim}\!\subseteq\!\mathsf{E}\ \text{(c)}$$

$$\frac{\Gamma \vdash \dot{\sim}\lambda(\mathsf{x}) \qquad \Gamma, \mathsf{c} \subseteq \mathsf{x}, \dot{\sim}\lambda(\mathsf{c}) \vdash \phi}{\Gamma \vdash \phi}\,\dot{\sim}\lambda\mathsf{E}\ \text{(c)}$$

$$\frac{\Gamma, \exists^1 \mathsf{z} R\mathsf{z}, \phi(R) \vdash \bot}{\Gamma, \phi(\mathsf{v}) \vdash \bot}\,\subseteq\!\mathsf{wI}_R\ \text{(d)}$$

(a). $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ (b). α is $\dot{\sim}$ and inclusion atom-free.
(c). c is a sequence of constant symbols that do not occur in Γ or ϕ, and λ is a first-order atom. (d). Γ is a set of sentences in which R does not occur, $\phi(R)$ is an inclusion atom-free sentence in which the relation symbol R occurs only in the form $R\mathsf{x}$, and $\phi(\mathsf{v})$ is a formula with free variables v obtained from $\phi(R)$ by replacing every $R\mathsf{x}$ by $\mathsf{x} \subseteq \mathsf{v}$.

We write $\Gamma \vdash_{\mathbf{FOT}} \phi$ or simply $\Gamma \vdash \phi$, if the sequent $\Gamma \vdash \phi$ is derivable in the system. Write $\phi \dashv\vdash \psi$ if $\phi \vdash \psi$ and $\psi \vdash \phi$.

The weak disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ admits the usual introduction and elimination rule, and note that the usual elimination rule is not sound for the other disjunction $\vee$. The soundness of the introduction and elimination rule for $\exists^1$ follows from the equivalence $\exists^1 x \equiv \exists x(=\!\!(x) \wedge \phi)$, and the introduction and elimination rule for $\forall^1$ have a similar flavor. The rules $\subseteq\!\mathsf{Id}$, $\subseteq\!\mathsf{Pro}$, $\subseteq\!\mathsf{Tr}$ and $\subseteq\!\mathsf{Cmp}$ for inclusion atoms were introduced in [17], and the first three rules completely axiomatize the implication problem of inclusion dependencies in database theory [3]. The two weakening rules $\subseteq\!\mathsf{W}_{\mathsf{con}}$ and $\subseteq\!\mathsf{W}_{\exists^1}$ for inclusion atoms extend the length of an inclusion atom. We leave it for the reader to verify that these rules for inclusion atoms are also sound and derivable if constants are allowed to occur as arguments in inclusion atoms (i.e., to allow inclusion atoms, e.g., of the form $c\mathsf{x} \subseteq v\mathsf{y}$). The rules $\dot{\sim} \subseteq \mathsf{E}$ and $\dot{\sim}\lambda\mathsf{E}$ in a sense describe the meanings of a negated inclusion atom $\dot{\sim}\mathsf{x} \subseteq \mathsf{y}$ and a negated first-order atom $\lambda(\mathsf{x})$ by providing a witness c. These two rules are designed for deriving Proposition 9(ii)(v) (which is crucial for the normal form lemma, Lemma 11, leading to the completeness theorem), and they can also be formulated, in a more complex form, without any mention of the constant symbols. The rule $\subseteq \mathsf{wI}_R$ simulates the transformation in Theorem 2, and it will be applied in the proof of the completeness theorem (Theorem 13) in a reverse manner with respect to a fresh relation symbol R, which is assumed to be always available. How to simplify this rule $\subseteq \mathsf{wI}_R$ is left as future work.

Theorem 7 (Soundness). *$\Gamma \vdash_{\mathbf{FOT}} \phi \Longrightarrow \Gamma \models \phi$.*

Proof. We only verify the soundness of $\dot{\sim}\subseteq\mathsf{E}$ and $\subseteq\mathsf{wI}_R$.

$\dot{\sim}\subseteq\mathsf{E}$: Suppose $\Gamma \models \dot{\sim}\mathsf{x} \subseteq \mathsf{y}$ and $\Gamma, \mathsf{c} \subseteq \mathsf{x}, \dot{\sim}\mathsf{c} \subseteq \mathsf{y} \models \phi$, and suppose that for some $\mathcal{L}$-model M and team X, $M \models_X \Gamma$. Then we have $M \models_X \dot{\sim}\mathsf{x} \subseteq \mathsf{y}$, which implies that there exists $s \in X$ such that for the $\mathcal{L}(\mathsf{c})$-model $(M, s(\mathsf{x}))$, we have $(M, s(\mathsf{x})) \models_X \mathsf{c} \subseteq \mathsf{x} \wedge \dot{\sim}\mathsf{c} \subseteq \mathsf{y}$. Thus, by the assumption, $(M, s(\mathsf{x})) \models_X \phi$, which gives $M \models_X \phi$ since c do not occur in ϕ.

$\subseteq\mathsf{wI}_R$: Suppose $\Gamma, \phi(\mathsf{v}) \not\models \bot$. Clearly, every **FOT**-formula can be turned into a (semantically) equivalent formula in prenex and negation normal form (cf. Proposition 8(ii)(iii)(iv)). We may then w.l.o.g. assume that $\phi(\mathsf{v})$ is in prenex and negation normal form. Then there exist a model M and a nonempty team X such that $M \models_X \Gamma$ and $M \models_X \phi(\mathsf{v})$. By (the proof of) Theorem 2, $(M, rel(X)) \models \phi_*(R)$ in **FO**, where $\phi_*(R)$ is an **FO**-sentence obtained from the inclusion atom-free **FOT**-sentence $\phi(R)$ by replacing every logical constant in **FOT** by its counterpart in **FO**, i.e., by replacing $\dot{\sim}$ by $\neg$, $\mathbin{\vee\!\!\!\vee}$ by $\vee$, $\forall^1$ by $\forall$, and $\exists^1$ by $\exists$. It is not hard to prove that $(M, rel(X)) \models_{\{\emptyset\}} \phi(R)$ in **FOT** follows . Since Γ is a set of sentences in which R does not occur, we also have $(M, rel(X)) \models_{\{\emptyset\}} \Gamma$. Also, since $X \neq \emptyset$, $(M, rel(X)) \models \exists^1 \mathsf{z} R \mathsf{z}$. Hence, we conclude $\Gamma, \exists^1 \mathsf{z} R \mathsf{z}, \phi(R) \not\models \bot$. □

We collect the basic facts concerning the logical constants in **FOT** in the following proposition. The proofs are standard and left to the reader.

Proposition 8. *(i) $\Gamma, \forall^1 x\phi \vdash \phi(c/x)$ and $\Gamma, \phi(c/x) \vdash \exists^1 x\phi$.*
(ii) $Q^1 x\phi \wedge \psi \dashv\vdash Q^1 x(\phi \wedge \psi)$ and $Q^1 x\phi \mathbin{\vee\!\!\!\vee} \psi \dashv\vdash Q^1 x(\phi \mathbin{\vee\!\!\!\vee} \psi)$, whenever $x \notin \mathsf{Fv}(\psi)$.
(iii) $\Gamma, \phi \vdash \psi$ iff $\Gamma, \dot{\sim}\psi \vdash \dot{\sim}\phi$, and $\Gamma, \dot{\sim}\dot{\sim}\phi \vdash \phi$.
(iv) $\dot{\sim}\forall^1 x\phi \dashv\vdash \exists^1 x \dot{\sim}\phi$, $\dot{\sim}\exists^1 x\phi \dashv\vdash \forall^1 x \dot{\sim}\phi$, $\dot{\sim}(\phi \mathbin{\vee\!\!\!\vee} \psi) \dashv\vdash \dot{\sim}\phi \wedge \dot{\sim}\psi$ and $\dot{\sim}(\phi \wedge \psi) \dashv\vdash \dot{\sim}\phi \mathbin{\vee\!\!\!\vee} \dot{\sim}\psi$.

A routine inductive proof that uses Proposition 8(i) shows that the usual Replacement Lemma holds for our logic, that is, if $\theta \dashv\vdash \chi$, then $\phi \dashv\vdash \phi(\chi/\theta)$, where $\phi(\chi/\theta)$ is obtained from ϕ by replacing an occurrence of θ in ϕ by χ.

It is easy to prove that $\Gamma, \phi \vdash \psi$ iff $\Gamma \vdash \phi \rightarrow \psi$. In the following proposition, we list some derivable technical clauses that will be used in the proof of the completeness theorem. See Appendix for the proof.

Proposition 9. *Let ξ and η be two sequences of variables of the same length.*

(i) $\mathsf{xy}\xi \subseteq \mathsf{vv}\eta \dashv\vdash \mathsf{x} = \mathsf{y} \wedge \mathsf{x}\xi \subseteq \mathsf{v}\eta$.
(ii) $\xi \subseteq \eta \dashv\vdash \forall^1 \mathsf{x}(\mathsf{x} \subseteq \xi \multimap \mathsf{x} \subseteq \eta)$.
(iii) $\mathit{con}(\mathsf{z}) \vdash \mathsf{w}\xi \subseteq \mathsf{z}\eta \multimapboth (\mathsf{w} = \mathsf{z} \wedge \xi \subseteq \eta)$.
(iv) $\mathit{con}(\mathsf{x}) \vdash \mathsf{x} \subseteq \mathsf{v} \multimapboth \exists^1 \mathsf{y}(\mathsf{xy} \subseteq \mathsf{vu})$.
(v) If $\lambda(\mathsf{z})$ is a first-order atom, then $\lambda(\mathsf{z}) \dashv\vdash \forall^1 \mathsf{w}(\mathsf{w} \subseteq \mathsf{z} \multimap \lambda(\mathsf{w}))$.

To prove the completeness theorem, we also need the following three lemmas. The first lemma emphasizes the fact that all variables quantified by the weak quantifiers have constant values, the second lemma proves a normal form for **FOT**-formulas, and the third lemma shows that derivations in the system of **FO** can be simulated in the system of **FOT**.

Lemma 10. *Let $\phi(\mathsf{v}) = Q^1\mathsf{x}\theta(\mathsf{x}, \mathsf{v})$ be a formula in prenex and negation normal form. Then $\phi \dashv\vdash \phi_{\mathit{con}}$, where ϕ_{con} is the formula obtained from ϕ by replacing every (first-order or inclusion) literal $\mu(\mathsf{x}, \mathsf{v})$ (i.e., an atom or negated atom) by $\mu \wedge \mathit{con}(\mathsf{x})$.*

Proof. By applying Proposition 8(ii), $Q^1\mathsf{I}$ and $Q^1\mathsf{E}$, it is easy to prove that $Q^1\mathsf{x}\theta(\mathsf{x}, \mathsf{v}) \dashv\vdash Q^1\mathsf{x}(\theta(\mathsf{x}, \mathsf{v}) \wedge \mathsf{con}(\mathsf{x}))$. Next we push the formula $\mathsf{con}(\mathsf{x})$ inside the quantifier-free formula θ in negation normal form all the way to the front of literals by using Replacement Lemma and the standard equivalences $(\theta_0 \wedge \theta_1) \wedge \mathsf{con}(\mathsf{x}) \dashv\vdash (\theta_0 \wedge \mathsf{con}(\mathsf{x})) \wedge (\theta_1 \wedge \mathsf{con}(\mathsf{x}))$ and $(\theta_0 \mathbin{\mathord{\vee}\mkern-11mu\mathord{\vee}} \theta_1) \wedge \mathsf{con}(\mathsf{x}) \dashv\vdash (\theta_0 \wedge \mathsf{con}(\mathsf{x})) \mathbin{\mathord{\vee}\mkern-11mu\mathord{\vee}} (\theta_1 \wedge \mathsf{con}(\mathsf{x}))$. □

Lemma 11. *For every* **FOT**-*formula ϕ, we have $\phi(\mathsf{v}) \dashv\vdash Q^1\mathsf{x}\theta(\mathsf{x}, \mathsf{v})$, where $\theta(\mathsf{x}, \mathsf{v})$ is a quantifier-free formula in negation normal form in which first-order atoms are of the form $\lambda(\mathsf{x})$, and inclusion atoms are of the form $\mathsf{x}_i \subseteq \mathsf{v}$ for some variables x_i from x.*

Proof. We first turn $\phi(\mathsf{v})$ into an equivalent formula in prenex and negation normal form by exhaustively applying Proposition 8(ii)(iii)(iv). Assume that the bound variables of $\phi(\mathsf{v})$ are among x. By Lemma 10 we may also assume that every literal $\mu(\mathsf{x}, \mathsf{v})$ in ϕ is replaced by $\mu(\mathsf{x}, \mathsf{v}) \wedge \mathsf{con}(\mathsf{x})$ (call such a formula a formula in *constant normal form*). Observe that now in $\phi(\mathsf{v})$ a generic first-order atom is of the form $\lambda(\mathsf{x}, \mathsf{v})$, and a generic inclusion atom is of the form $\eta\xi\rho\sigma \subseteq \eta'\xi'\rho'\sigma'$ (modulo permutation by $\subseteq\mathsf{Pro}$), where $|\eta| = |\eta'| \geq 0$, $|\xi| = |\xi'| \geq 0$, $|\rho| = |\rho'| \geq 0$ and $|\sigma| = |\sigma'| \geq 0$,

- $(\eta, \eta') = (\mathsf{x}_i, \mathsf{x}_j)$ for some bound variables $\mathsf{x}_i, \mathsf{x}_j$ from x;
- $(\xi, \xi') = (\mathsf{x}_i, \mathsf{v}_i)$ for some bound variables x_i from x, and free variables v_i from v;
- $(\rho, \rho') = (\mathsf{v}_i, \mathsf{x}_i)$ for some free variables v_i from v, and bound variables x_i from x;
- $(\sigma, \sigma') = (\mathsf{v}_i, \mathsf{v}_j)$ for some free variables $\mathsf{v}_i, \mathsf{v}_j$ from v.

To obtain the required normal form we have to transform every (first-order or inclusion) atom in ϕ in the required form. We achieve this in several steps.

In Step 1 of our transformation, we replace in $\phi(\mathsf{v})$ every inclusion atom $\eta\xi\mathsf{v}_i\sigma \subseteq \eta'\xi'\mathsf{x}_i\sigma'$ by $\mathsf{v}_i = \mathsf{x}_i \wedge \eta\xi\sigma \subseteq \eta'\xi'\sigma'$. Note that by Proposition 9(iii), we have

$$\mathsf{con}(\mathsf{x}_i) \wedge \eta\xi\mathsf{v}_i\sigma \subseteq \eta'\xi'\mathsf{x}_i\sigma' \dashv\vdash \mathsf{con}(\mathsf{x}_i) \wedge (\mathsf{v}_i = \mathsf{x}_i \wedge \eta\xi\sigma \subseteq \eta'\xi'\sigma')$$

$$\text{and } \mathsf{con}(\mathsf{x}_i) \wedge \dot{\sim}\, \eta\xi\mathsf{v}_i\sigma \subseteq \eta'\xi'\mathsf{x}_i\sigma' \dashv\vdash \mathsf{con}(\mathsf{x}_i) \wedge \dot{\sim}(\mathsf{v}_i = \mathsf{x}_i \wedge \eta\xi\sigma \subseteq \eta'\xi'\sigma').$$

Hence, by Replacement Lemma, the resulting formula $\phi_1(\mathsf{v})$ is provably equivalent to ϕ. We assume further (here and also in the other steps) that $\phi_1(\mathsf{v})$ is turned into prenex, negation and constant normal form by applying Proposition 8(ii)(iii)(iv) and Lemma 10.

In Step 2, we replace in $\phi_1(\mathsf{v})$ every first-order atom $\lambda(\mathsf{x},\mathsf{v})$ by $\forall^1\mathsf{yz}(\mathsf{yz} \subseteq \mathsf{xv} \multimap \lambda(\mathsf{y},\mathsf{z}))$. By Proposition 9(v), the resulting formula $\phi_2(\mathsf{v})$ is provably equivalent to $\phi_1(\mathsf{v})$. Up to now, every first-order atom in the formula is transformed to the required form, and the steps afterwards will not generate first-order atoms in non-normal form.

In Step 3, we apply Proposition 9(ii) to replace in $\phi_2(\mathsf{v})$ every inclusion atom $\eta\xi\mathsf{v}_i \subseteq \eta'\xi'\mathsf{v}_j$ by $\forall^1\mathsf{wyz}(\mathsf{wyz} \subseteq \eta\xi\mathsf{v}_i \multimap \mathsf{wyz} \subseteq \eta'\xi'\mathsf{v}_j)$, and denote the resulting formula by $\phi_3(\mathsf{v})$. In Step 4, we apply Proposition 9(iii) to replace in $\phi_3(\mathsf{v})$ every inclusion atom $\mathsf{x}_i\mathsf{x}_k \subseteq \mathsf{x}_j\mathsf{v}_k$ by $\mathsf{x}_i = \mathsf{x}_j \wedge \mathsf{x}_k \subseteq \mathsf{v}_k$, and denote the resulting formula by $\phi_4(\mathsf{v})$. Up to now every inclusion atom in the formula is transformed to the form $\mathsf{x}_i \subseteq \mathsf{v}_i$, where x_i are bound variables and v_i are free variables in $\phi_4(\mathsf{v})$. Yet, v_i may contain repetitions, and it may also be only a subsequence of v. Handling these requires two additional steps.

In Step 5, we remove repetitions on the right side of the inclusion atoms, by applying Proposition 9(i) to replace in $\phi_4(\mathsf{v})$ every inclusion atom of the form $\mathsf{x}_i\mathsf{x}_j\mathsf{x}_k \subseteq \mathsf{v}_i\mathsf{v}_i\mathsf{v}_j$ by $\mathsf{x}_i = \mathsf{x}_j \wedge \mathsf{x}_i\mathsf{x}_k \subseteq \mathsf{v}_i\mathsf{v}_j$. Denote the resulting formula by $\phi_5(\mathsf{v})$. In Step 6, we extend the length of those shorter inclusion atoms. Assuming $\mathsf{v} = \mathsf{v}_i\mathsf{v}_j$, we apply Proposition 9(iv) to replace in $\phi_5(\mathsf{v})$ every inclusion atom of the form $\mathsf{x}_i \subseteq \mathsf{v}_i$ by $\exists^1\mathsf{y}(\mathsf{x}_i\mathsf{y} \subseteq \mathsf{v}_i\mathsf{v}_j)$. Denote the resulting formula by $\phi_6(\mathsf{v})$.

As before we assume that $\phi_6(\mathsf{v})$ is turned into prenex and negation normal form, but now we shall apply Lemma 10 in a reverse manner to remove the added constancy atoms for each literal in ϕ_6. Finally, the resulting provably equivalent formula is in the required normal form. Note that our transformation clearly terminates, because we have performed the steps in the transformation in such an order that each step will not generate new formulas for which the transformations in the previous steps apply. □

Lemma 12. *Let $\Delta \cup \{\delta\}$ be a set of* **FO***-formulas whose free variables are among* x. *If $\Delta \vdash_{\mathbf{FO}} \delta$, then $\Delta^*(\mathsf{c}/\mathsf{x}) \vdash_{\mathbf{FOT}} \delta^*(\mathsf{c}/\mathsf{x})$, where $*$ is the operation that replaces every logical constant in* **FO** *by its counterpart in* **FOT**, *and* c *is a sequence of fresh constant symbols. In particular, if $\Delta \cup \{\delta\}$ is a set of* **FO***-sentences, then $\Delta \vdash_{\mathbf{FO}} \delta$ implies $\Delta^* \vdash_{\mathbf{FOT}} \delta^*$.*

Proof. See Appendix.

Finally, we are in a position to prove the completeness theorem of our system.

Theorem 13 (Completeness). $\Gamma \models \phi \Longrightarrow \Gamma \vdash_{\mathbf{FOT}} \phi$.

Proof. Since **FOT** (being expressively equivalent to **FO**) is compact, we may assume that Γ is finite. Now, suppose $\Gamma \nvdash_{\mathbf{FOT}} \phi$. By RAA we derive $\Gamma, \dot{\sim}\phi \nvdash_{\mathbf{FOT}} \bot$, which is equivalent to $Q^1\mathsf{x}\theta(\mathsf{x},\mathsf{v}) \nvdash_{\mathbf{FOT}} \bot$, where $\psi(\mathsf{v}) = Q^1\mathsf{x}\theta(\mathsf{x},\mathsf{v})$ is the normal form of the formula $\bigwedge\Gamma \wedge \dot{\sim}\phi$ given by Lemma 11. By applying $\subseteq\mathsf{wI}_R$ we obtain $\exists^1\mathsf{z}R\mathsf{z}, \psi(R) \nvdash_{\mathbf{FOT}} \bot$, where R is a fresh relation symbol, and $\psi(R)$ is the inclusion atom-free sentence obtained from $\psi(\mathsf{v})$ by replacing every inclusion atom $\mathsf{x}_i \subseteq \mathsf{v}$ by $R\mathsf{x}_i$. It follows from Lemma 12 that $\exists\mathsf{z}R\mathsf{z}, \psi_*(R) \nvdash_{\mathbf{FO}} \bot$, where ψ_* is the **FO**-formula obtained from the **FOT**-formula ψ by replacing every logical constant in **FOT** by its counterpart in **FO**. Now, by the completeness theorem of **FO**, there exists a model (M, R^M) such that $R^M \neq \emptyset$ and $(M, R^M) \models \psi_*(R)$. It then follows from (the proof of) Theorem 2 that $M \models_{X_R} \psi(\mathsf{v})$, where $X_R = \{s : \{v_1, \ldots, v_k\} \to M \mid s(\mathsf{v}) \in R^M\}$ is the (nonempty) team associated with R. Hence $\psi \not\models \bot$. □

5 Axiomatizing FOT$^{\downarrow}$ Consequences in D $\oplus$ FOT$^{\downarrow}$

Let $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$ denote dependence logic (**D**) extended with the syntax of $\mathbf{FOT}^{\downarrow}$, that is, formulas of $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$ are defined by the grammar:

$$\phi ::= \lambda \mid =\!\!(t_1, \ldots, t_n, t) \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \mathbin{\vee\!\!\!\vee} \phi \mid \exists x\phi \mid \forall x\phi \mid \exists^1 x\phi \mid \forall^1 x\phi$$

where λ is a first-order atom and α is first-order. Recall that **D** captures downward closed **ESO** team properties [26]. It can be easily seen from the proof of Theorem 1 that enriching the syntax of **D** with the weak connective $\mathbin{\vee\!\!\!\vee}$ and quantifiers $\exists^1, \forall^1$ from $\mathbf{FOT}^{\downarrow}$ does not increase the expressive power of the logic; in other words, $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$ has the same expressive power as **D**. In this section, we introduce a system of natural deduction for the logic $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$ by extending the systems of [27] and [33] for **D** so that this new system is sound and complete for $\mathbf{FOT}^{\downarrow}$ consequences in the sense that

$$\Gamma \models \theta \iff \Gamma \vdash \theta$$

whenever Γ is a set of sentences in $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$, and θ is a sentence in $\mathbf{FOT}^{\downarrow}$.

One crucial step in our argument for the completeness theorem involves an application of the rule

$$\frac{\Gamma, \dot{\sim}\phi \vdash \bot}{\Gamma \vdash \phi}\,\mathsf{RAA},$$

where, as in [33], the formula $\dot{\sim}\phi$ should be read as a shorthand for the defining formula of $\dot{\sim}\phi$ in the language of the logic in question. In our system this rule will only be applied for ϕ being an $\mathbf{FOT}^{\downarrow}$-sentence, and in this case $\dot{\sim}\phi \equiv \neg\phi$, where $\neg\phi$ is the syntactic negation obtained by pushing negation to the very front of first-order atoms in ϕ using the definitions: $\neg\neg\lambda := \lambda$, $\neg(\psi \wedge \chi) := \neg\psi \mathbin{\vee\!\!\!\vee} \neg\chi$, $\neg(\psi \mathbin{\vee\!\!\!\vee} \chi) := \neg\psi \wedge \neg\chi$, $\neg\forall^1 x\psi := \exists^1 x\neg\psi$, $\neg\exists^1 x\psi := \forall^1 x\neg\psi$.

Definition 14. The system of natural deduction for $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$ consists of all rules of the system of $\mathbf{D}$ defined in [27] (including rules of identity, and particularly those rules in Table 2), the rule RAA for ϕ in the rule being an $\mathbf{FOT}^{\downarrow}$-sentence, all rules in Table 1 from Sect. 4, and the following rules, where α ranges over first-order formulas only:

$$\frac{}{\Gamma \vdash \exists x \exists y (x \neq y)}\ \mathsf{Dom}$$

$$\frac{\Gamma \vdash \phi \vee \bot}{\Gamma \vdash \phi}\ \bot\vee\mathsf{E} \qquad \frac{\Gamma \vdash \mathsf{con}(\mathsf{x})}{\Gamma \vdash \alpha(\mathsf{x}) \mathbin{\mathpalette\make@circled\vee} \neg\alpha(\mathsf{x})}\ \mathbin{\vee\!\!\!\vee}\,\mathsf{wI} \qquad \frac{\Gamma \vdash \forall^1 x \alpha}{\Gamma \vdash \forall x \alpha}\ \forall^1\forall\mathsf{Trs}$$

$$\frac{\Gamma \vdash \forall^1 x \phi \vee \psi}{\Gamma \vdash \exists y z \forall^1 x ((y = z \wedge \phi) \vee (y \neq z \wedge \psi))}\ \forall^1\mathsf{Ext}\ [x \notin \mathsf{Fv}(\psi),\ y, z \text{ are fresh}]$$

$$\frac{\Gamma, \mathsf{con}(\mathsf{x}) \vdash =\!\!(y)}{\Gamma \vdash =\!\!(\mathsf{x}, y)}\ =\!\!(\cdot)\mathsf{wI} \qquad \frac{\Gamma \vdash =\!\!(\mathsf{x}, y) \qquad \Gamma \vdash \mathsf{con}(\mathsf{x})}{\Gamma \vdash =\!\!(y)}\ =\!\!(\cdot)\mathsf{wE}$$

Table 2. Some rules from the system [27] of $\mathbf{D}$

$$\frac{\Gamma \vdash \exists x \forall y \phi(x, y, \mathsf{z})}{\Gamma \vdash \forall y \exists x (=\!\!(\mathsf{z}, x) \wedge \phi)}\ =\!\!(\cdot)\mathsf{I} \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Gamma, \phi \vdash \chi}{\Gamma \vdash \chi \vee \psi}\ \vee\mathsf{Sub}$$

$$\frac{\Gamma \vdash \phi(t/x)}{\Gamma \vdash \exists x \phi}\ \exists\mathsf{I} \qquad \frac{\Gamma \vdash \exists x \phi \qquad \Gamma, \phi(v/x) \vdash \psi}{\Gamma \vdash \psi}\ \exists\mathsf{E}\ [v \notin \mathsf{Fv}(\Gamma \cup \{\phi, \psi\})]$$

The axiom Dom stipulates that the domain of a model has at least two elements, which we assume throughout this section. This domain assumption is often postulated in the literature on dependence logic, especially because over models with singleton domain all dependence atoms become trivially true (as there is only one single assignment over such a domain). In our setting, the axiom Dom is required for Proposition 16(v), which shows that the weak disjunction $\mathbin{\vee\!\!\!\vee}$ is definable in terms of the other disjunction $\vee$ in $\mathbf{D}$ (as long as the domain has more than one elements). The rules $\bot\vee\mathsf{E}$, $\mathbin{\vee\!\!\!\vee}\,\mathsf{wI}$ and $\forall^1\forall\mathsf{Trs}$ are evident. The invertible rule $\forall^1\mathsf{Ext}$ is an adaption of a similar rule in the system of $\mathbf{D}$ in [27], and it is inspired also by a similar equivalence given in [14]. The rules $=\!\!(\cdot)\mathsf{wI}$ and $=\!\!(\cdot)\mathsf{wE}$ for dependence atoms were introduced in [34] in the propositional context.

Theorem 15 (Soundness). $\Gamma \vdash \phi \Longrightarrow \Gamma \models \phi$.

Proof. See Appendix. □

In the following proposition we list some technical clauses that will be used in our proof of the completeness theorem. See Appendix for the proof. In addition, Proposition 8(i)(ii) are still derivable in the system of $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$ by the same derivation.

Proposition 16. *(i)* $\vdash {=\!\!(}\mathsf{z}, c)$*, and in particular* $\vdash {=\!\!(}c)$ *for any constant symbol* c*.*

(ii) ${=\!\!(}c\mathsf{x}, y) \dashv\vdash {=\!\!(}\mathsf{x}, y)$ *for any constant symbol* c*.*

(iii) $\forall^1 vQ\mathsf{u}(\phi \wedge {=\!\!(}v\mathsf{x}, y)) \dashv\vdash \forall^1 vQ\mathsf{u}(\phi \wedge {=\!\!(}\mathsf{x}, y))$ *and* $\forall^1 vQ\mathsf{u}(\phi \wedge {=\!\!(}\mathsf{x}, v)) \dashv\vdash \forall^1 vQ\mathsf{u}\phi$*.*

(iv) $\exists^1 x\phi \dashv\vdash \exists x({=\!\!(}x) \wedge \phi)$*.*

(v) $\phi \mathbin{\mathpalette\make@circled\vee} \psi \dashv\vdash \exists x\exists y\big({=\!\!(}x) \wedge {=\!\!(}y) \wedge \big((x = y \wedge \phi) \vee (x \neq y \wedge \psi)\big)\big)$*, where* x, y *are fresh.*

(vi) $\exists x\forall^1 y\phi(x, y, \mathsf{z}) \vdash \forall^1 y\exists x({=\!\!(}\mathsf{z}, x) \wedge \phi)$ *and* $\forall x\forall^1 y\phi \dashv\vdash \forall^1 y\forall x\phi$*.*

Recall from [31] that every **D**-formula $\phi(\mathsf{z})$ is logically equivalent to a formula of the form

$$\forall \mathsf{x}\exists \mathsf{y}\big(\bigwedge_{i\in I} {=\!\!(}\mathsf{x}_i, y_i) \wedge \alpha(\mathsf{x}, \mathsf{y}, \mathsf{z})\big), \tag{4}$$

where each x_i are from x, each y_i is from y, and α is first-order. In the next theorem we derive a similar normal form for formulas in $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$. See Appendix for the proof.

Theorem 17. *Every* $\mathbf{D}\oplus\mathbf{FOT}^{\downarrow}$*-formula* $\phi(\mathsf{z})$ *is semantically equivalent to, and provably implies a formula of the form*

$$\forall^1 \mathsf{v}\forall \mathsf{x}\exists \mathsf{y}\big(\bigwedge_{i\in I} {=\!\!(}\mathsf{x}_i, y_i) \wedge \alpha(\mathsf{v}, \mathsf{x}, \mathsf{y}, \mathsf{z})\big), \tag{5}$$

where each x_i *are from* x*, each* y_i *is from* y*, and* α *is first-order.*

Recall also from [27] that for every **D**-sentence ψ in normal form (4), there is a first-order sentence Ψ of infinite length (called the *game expression* of ψ) such that for any countable model M, $M \models \psi$ iff $M \models \Psi$. Moreover, the infinitary first-order sentence Ψ can be approximated by some first-order sentences Ψ_n ($n \in \mathbb{N}$) of finite length in the sense that for any recursively saturated (or finite) model M, $M \models \Psi$ iff $M \models \Psi_n$ for all $n \in \mathbb{N}$. Also, in the system of **D** one derives $\psi \vdash_{\mathbf{D}} \Psi_n$ for any $n \in \mathbb{N}$. Now, for any $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$-sentence $\phi = \forall^1 \mathsf{v}\psi$ of the form (5) with ψ a **D**-sentence, it is not hard to show (by the same argument as in [27]) that the game expression Φ^* of ϕ can be defined as $\forall \mathsf{v}\Psi$, and the n-approximation Φ^*_n can be defined as $\forall \mathsf{v}\Psi_n$ for each $n \in \mathbb{N}$. Next, as in [27], we show that every n-approximation Φ^*_n can be derived from the formula ϕ.

Theorem 18. *For any* $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$*-sentence* ϕ *and any* $n \in \mathbb{N}$*,* $\phi \vdash \Phi^*_n$*.*

Proof. W.l.o.g. we may assume that the $\mathcal{L}$-sentence $\phi = \forall^1 \mathsf{v}\psi(\mathsf{v})$ is of the form (5) with $\psi(\mathsf{v})$ a **D**-sentence. Let c be a sequence of new constant symbols. Observe that the $\mathcal{L}(\mathsf{c})$-sentence $\psi(\mathsf{c}/\mathsf{v})$ is in the normal form (4) for **D**-formulas. By the result in [27] we have $\psi(\mathsf{c}/\mathsf{v}) \vdash \Psi_n(\mathsf{c}/\mathsf{v})$. Thus, by Proposition 8(i), $\forall^1\mathsf{I}$ and $\forall^1\forall\mathsf{Trs}$, we derive $\forall^1 \mathsf{v}\psi(\mathsf{v}) \vdash \psi(\mathsf{c}/\mathsf{v}) \vdash \Psi_n(\mathsf{c}/\mathsf{v}) \vdash \forall^1 \mathsf{v}\Psi_n(\mathsf{v}) \vdash \forall \mathsf{v}\Psi_n(\mathsf{v})$, thereby $\phi \vdash \Phi^*_n$. $\square$

Theorem 19 (Completeness). *For any set* Γ *of* $\mathbf{D}\oplus\mathbf{FOT}^{\downarrow}$*-sentences and an* $\mathbf{FOT}^{\downarrow}$*-sentence* θ*,* $\Gamma \models \theta \Longrightarrow \Gamma \vdash \theta$*.*

Proof. We only provide a sketch of the proof, which combines the arguments in [27] and in [33]. Suppose $\Gamma \nvdash \theta$. Then $\Gamma, \dot\sim\theta \nvdash \bot$ by RAA, where $\dot\sim\theta = \neg\theta$ as θ is an $\mathbf{FOT}^{\downarrow}$-sentence. Let Γ^* be the set of all approximations of sentences in $\Gamma \cup \{\neg\theta\}$. By Theorem 18, we have $\Gamma^* \nvdash \bot$. Since restricted to first-order formulas our extended system (or the system of $\mathbf{D}$ as defined in [27]) has the same rules as the deduction system of the usual first-order logic, we derive $\Gamma^* \nvdash_{\mathbf{FO}} \bot$. From this point on we follow exactly the argument in [27] to find a model M for $\Gamma \cup \{\neg\theta\}$. Thus, $M \models \Gamma$ and $M \not\models \theta$. $\square$

Acknowledgements. The authors would like to thank two anonymous referees for helpful suggestions concerning the presentation of this paper. Both authors were supported by grant 308712 of the Academy of Finland, and the second author was supported also by Research Funds of the University of Helsinki.

Appendix

Proof of Proposition 9

Item (i): The right to left direction follows from $\subseteq\mathsf{Pro}$ (applied to repeated arguments in the inclusion atom) and rules of identity. For the left to right direction, $\mathsf{xy}\xi \subseteq \mathsf{vv}\eta \vdash \mathsf{x}\xi \subseteq \mathsf{v}\eta$ follows from $\subseteq\mathsf{Pro}$. Next, by $\subseteq\mathsf{Pro}$, rules of identity and $\subseteq\mathsf{Cmp}$ we have $\mathsf{xy}\xi \subseteq \mathsf{vv}\eta \vdash \mathsf{xy} \subseteq \mathsf{vv} \vdash \mathsf{xy} \subseteq \mathsf{vv} \wedge \mathsf{v} = \mathsf{v} \vdash \mathsf{x} = \mathsf{y}$.

Item (ii): For the right to left direction, by Proposition 8(iii)(iv), it suffices to show the contrapositive $\dot\sim \xi \subseteq \eta \vdash \exists^1\mathsf{x}(\mathsf{x} \subseteq \xi \wedge \dot\sim \mathsf{x} \subseteq \eta)$. For any sequence c of fresh constant symbols, we have $\mathsf{c} \subseteq \xi, \dot\sim \mathsf{c} \subseteq \eta \vdash \exists^1\mathsf{x}(\mathsf{x} \subseteq \xi \wedge \dot\sim \mathsf{x} \subseteq \eta)$ by Proposition 8(i). Then the desired clause follows from $\dot\sim\subseteq\mathsf{E}$. For the other direction, by $\forall^1\mathsf{I}$, it suffices to show that $\xi \subseteq \eta \vdash \mathsf{c} \subseteq \xi \multimap \mathsf{c} \subseteq \eta$ for c a sequence of fresh constant symbols, which is further reduced to showing that $\xi \subseteq \eta, \mathsf{c} \subseteq \xi \vdash \mathsf{c} \subseteq \eta$. But this follows from $\subseteq\mathsf{Tr}$.

Item (iii): We first show $\mathsf{con}(\mathsf{z}) \vdash \mathsf{w}\xi \subseteq \mathsf{z}\eta \multimap (\mathsf{w} = \mathsf{z} \wedge \xi \subseteq \eta)$, which is equivalent to $\mathsf{con}(\mathsf{z}), \mathsf{w}\xi \subseteq \mathsf{z}\eta \vdash \mathsf{w} = \mathsf{z} \wedge \xi \subseteq \eta$. By $\subseteq\mathsf{W}_{\mathsf{con}}$ we have $\mathsf{con}(\mathsf{z}), \mathsf{w} \subseteq \mathsf{z} \vdash \mathsf{wz} \subseteq \mathsf{zz}$. By item (i), $\mathsf{wz} \subseteq \mathsf{zz} \vdash \mathsf{w} = \mathsf{z}$. Hence, by $\subseteq\mathsf{Pro}$ the desired clause follows. Next, we show $\mathsf{con}(\mathsf{z}), \mathsf{w} = \mathsf{z}, \xi \subseteq \eta \vdash \mathsf{w}\xi \subseteq \mathsf{z}\eta$. Again by $\subseteq\mathsf{W}_{\mathsf{con}}$ we have that $\xi \subseteq \eta, \mathsf{con}(\mathsf{z}) \vdash \mathsf{z}\xi \subseteq \mathsf{z}\eta$, and thus the desired clause follows from rules of identity.

Item (iv): The direction $\mathsf{con}(\mathsf{x}), \mathsf{x} \subseteq \mathsf{v} \vdash \exists^1\mathsf{y}(\mathsf{xy} \subseteq \mathsf{vu})$ is given by $\subseteq\mathsf{W}_{\exists^1}$, and the other direction $\mathsf{con}(\mathsf{x}), \exists^1\mathsf{y}(\mathsf{xy} \subseteq \mathsf{vu}) \vdash \mathsf{x} \subseteq \mathsf{v}$ follows easily from $\subseteq\mathsf{Pro}$.

Item (v): We first show the left to right direction, which, by $\forall^1\mathsf{I}$, is reduced to showing that $\lambda(\mathsf{z}) \vdash \mathsf{c} \subseteq \mathsf{z} \multimap \lambda(\mathsf{c})$ for c a sequence of fresh constant symbols. But this follows from $\subseteq\mathsf{Cmp}$. Next, we show the other direction, which is equivalent to the contrapositive $\dot\sim \lambda(\mathsf{z}) \vdash \exists^1\mathsf{w}(\mathsf{w} \subseteq \mathsf{z} \wedge \dot\sim \lambda(\mathsf{w}))$. For any sequence c of fresh constant symbols, we have $\mathsf{c} \subseteq \mathsf{z}, \dot\sim \lambda(\mathsf{c}) \vdash \exists^1\mathsf{w}(\mathsf{w} \subseteq \mathsf{z} \wedge \dot\sim \lambda(\mathsf{w}))$ by Proposition 8(i). Then the desired clause follows from $\dot\sim\lambda\mathsf{E}$. $\square$

Proof of Lemma 12

We prove that $\Delta \vdash_{\mathbf{FO}} \delta$ implies $\Delta^*(\mathsf{c}/\mathsf{x}) \vdash_{\mathbf{FOT}} \delta^*(\mathsf{c}/\mathsf{x})$ by induction on the depth of the proof tree of $\Delta \vdash_{\mathbf{FO}} \delta$. If the proof tree has depth 1, then either $\delta \in \Delta$ or δ is the identity axiom $t = t$. In both cases $\Delta^*(\mathsf{c}/\mathsf{x}) \vdash_{\mathbf{FOT}} \delta^*(\mathsf{c}/\mathsf{x})$ trivially follows in our system.

If the proof tree has depth > 1, and the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of a rule for $\neg$ or $\wedge$ or $\vee$ in **FO**, then we derive $\Delta^*(\mathsf{c}/\mathsf{x}) \vdash_{\mathbf{FOT}} \delta^*(\mathsf{c}/\mathsf{x})$ by applying the induction hypothesis and the corresponding (classical) rules for $\dot{\sim}$ or $\wedge$ or $\mathbin{\rotatebox[origin=c]{180}{$\wedge$}}$ in our system for **FOT**.

If the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of the $\exists\mathsf{I}$ rule:

$$\dfrac{\begin{array}{c}\vdots\pi\\ \Delta \vdash \alpha(t/x)\end{array}}{\Delta \vdash \exists x\alpha}\ \exists\mathsf{I}$$

where the variables and constant symbols occurring in the term t are, respectively, among vy and d (denoted as $t(\mathsf{vy}, \mathsf{d})$), then the corresponding derivation in **FOT** is:

$$\dfrac{\begin{array}{c}\vdots\pi^*\\ \Delta^*(\mathsf{c}/\mathsf{y}) \vdash \alpha^*(t(\mathsf{c}'/\mathsf{v}, \mathsf{c}/\mathsf{y})/x, \mathsf{c}/\mathsf{y})\end{array} \qquad \dfrac{\dfrac{}{\Delta^*(\mathsf{c}/\mathsf{y}) \vdash \mathsf{con}(\mathsf{c}'\mathsf{cd})}\ \mathsf{conI}}{\Delta^*(\mathsf{c}/\mathsf{y}) \vdash {=\!\!(}t(\mathsf{c}'/\mathsf{v}, \mathsf{c}/\mathsf{y}, \mathsf{d}))}\ \mathsf{conI}}{\Delta^*(\mathsf{c}/\mathsf{y}) \vdash \exists^1 x\alpha^*(x, \mathsf{c}/\mathsf{y})}\ \exists^1\mathsf{I}$$

where π^* is a derivation corresponding to π given by the induction hypothesis.

If the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of the $\exists\mathsf{E}$ rule:

$$\dfrac{\begin{array}{c}\vdots\pi_1\\ \Delta \vdash \exists x\alpha\end{array} \qquad \begin{array}{c}\vdots\pi_2\\ \Delta, \alpha(v/x) \vdash \delta\end{array}}{\Delta \vdash \delta}\ \exists\mathsf{E}$$

where $v \notin \mathsf{Fv}(\Delta \cup \{\alpha, \delta\})$, then the corresponding derivation in **FOT** is:

$$\dfrac{\begin{array}{c}\vdots\pi_1^*\\ \Delta^*(\mathsf{c}/\mathsf{y}) \vdash \exists^1 x\alpha^*(x, \mathsf{c}/\mathsf{y}, \mathsf{c}'/\mathsf{u})\end{array} \qquad \begin{array}{c}\vdots\pi_2^*\\ \Delta^*(\mathsf{c}/\mathsf{y}), (\alpha(v/x))^*(d/v, \mathsf{c}/\mathsf{y}, \mathsf{c}'/\mathsf{u}) \vdash \delta^*(\mathsf{c}/\mathsf{y})\end{array}}{\Delta^*(\mathsf{c}/\mathsf{y}) \vdash \delta^*(\mathsf{c}/\mathsf{y})}\ \exists^1\mathsf{E}$$

where π_1^*, π_2^* are, respectively, derivations corresponding to π_1, π_2 given by the induction hypothesis, and d is a fresh constant symbol.

If the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of the $\forall\mathsf{I}$ rule:

$$\dfrac{\begin{array}{c}\vdots\pi\\ \Delta \vdash \alpha(v/x)\end{array}}{\Delta \vdash \forall x\alpha}\ \forall\mathsf{I}$$

where $v \notin \mathsf{Fv}(\Delta \cup \{\alpha\})$, then the corresponding derivation in **FOT** is:

$$\dfrac{\begin{array}{c}\vdots\pi^*\\ \Delta^*(\mathsf{c}/\mathsf{y}) \vdash (\alpha(v/x))^*(d/v, \mathsf{c}/\mathsf{y})\end{array}}{\Delta^*(\mathsf{c}/\mathsf{y}) \vdash \forall^1 x\alpha^*(x, \mathsf{c}/\mathsf{y})}\ \forall^1\mathsf{I}$$

If the last step of the derivation of $\Delta \vdash_{\mathbf{FO}} \delta$ is an application of the $\forall\mathsf{E}$ rule:

$$\dfrac{\begin{array}{c}\vdots\pi\\ \Delta \vdash \forall x\alpha\end{array}}{\Delta \vdash \alpha(t/x)}\ \forall\mathsf{E}$$

where $t = t(\mathsf{vy}, \mathsf{d})$, then the corresponding derivation in **FOT** is:

$$\dfrac{\begin{array}{c}\vdots\pi^*\\ \Delta^*(\mathsf{c}/\mathsf{y}) \vdash \forall^1 x\alpha^*(x, \mathsf{c}/\mathsf{y})\end{array} \qquad \dfrac{\dfrac{}{\Delta^*(\mathsf{c}/\mathsf{y}) \vdash \mathsf{con}(\mathsf{c'cd})}\ \mathsf{conI}}{\Delta^*(\mathsf{c}/\mathsf{y}) \vdash =\!\!(t(\mathsf{c}'/\mathsf{v}, \mathsf{c}/\mathsf{y}, \mathsf{d}))}\ \mathsf{conI}}{\Delta^*(\mathsf{c}/\mathsf{y}) \vdash \alpha^*(t(\mathsf{c}'/\mathsf{v}, \mathsf{c}/\mathsf{y})/x, \mathsf{c}/\mathsf{y})}\ \forall^1\mathsf{E}$$

□

Proof of Theorem 15 (Soundness Theorem of $\mathbf{D} \oplus \mathbf{FOT}^{\downarrow}$)

We only verify the soundness of $\forall^1\mathsf{Ext}$, by showing that $\forall^1 x\phi \vee \psi \equiv \exists yz\forall^1 x((y = z\wedge\phi)\vee(y \neq z\wedge\psi))$. For the left to right direction, suppose $M \models_X \forall^1 x\phi\vee\psi$, and we may w.l.o.g. also assume that $x, y, z \notin \mathsf{dom}(X)$. Then there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $M \models_Y \forall^1 x\phi$ and $M \models_Z \psi$. Let a, b be two distinct elements in M. Define $F : X \to \wp(M) \setminus \{\emptyset\}$ as $F(s) = \{a\}$, and define $G : X(F/y) \to M$ by taking

$$a \in G(s) \iff s \upharpoonright \mathsf{dom}(X) \in Y, \quad \text{and} \quad b \in G(s) \iff s \upharpoonright \mathsf{dom}(X) \in Z.$$

Putting $X' = X(F/y)(G/z)$ we show that $M \models_{X'(c/x)} (y = z \wedge \phi) \vee (y \neq z \wedge \psi)$ for arbitrary $c \in M$. Define $Y' = \{s \in X'(c/x) \mid s(z) = a\}$ and $Z' = X'(c/x)\setminus Y'$. Clearly, $Y' \cup Z' = X'(c/x)$, $M \models_{Y'} y = z$ and $M \models_{Z'} y \neq z$. Since $M \models_Z \psi$ and $x, y, z \notin \mathsf{dom}(Z)$, we have $M \models_{Z'} \psi$. Also, since $M \models_Y \forall x\phi$, we have $M \models_{Y(c/x)} \phi$, which implies $M \models_{Y'} \phi$.

For the right to left direction, suppose $M \models_X \exists yz\forall^1 x((y = z \wedge \phi) \vee (y \neq z \wedge \psi))$. Then there exist appropriate functions F, G s.t. for any $a \in M$, there exists $Y_a \subseteq X(F/y)(G/z)(a/x) = X'(a/x)$ s.t. $M \models_{Y_a} y = z \wedge \phi$ and $M \models_{X'(a/x)\setminus Y_a} y \neq z \wedge \psi$.

Claim: For any $a, b \in M$, $Y_a \upharpoonright \mathsf{dom}(X) = Y_b \upharpoonright \mathsf{dom}(X)$. Indeed, for any $s \in Y_a \subseteq X'(a/x)$, we have $s(y) = s(z)$. For $s' = s(b/x) \in X'(b/x)$, we must also have $s'(y) = s'(z)$, thus $s' \in Y_b$. Hence, $s \upharpoonright \mathsf{dom}(X) = s' \upharpoonright \mathsf{dom}(X) \in Y_b \upharpoonright \mathsf{dom}(X)$. This shows that $Y_a \upharpoonright \mathsf{dom}(X) \subseteq Y_b \upharpoonright \mathsf{dom}(X)$. The other inclusion is proved similarly.

Now, to show $M \models_X \forall^1 x\phi \vee \psi$, let $Y = Y_a \upharpoonright \mathsf{dom}(X)$ and $Z = X \setminus Y$ for any $a \in M$. Since $M \models_{X'(a/x)\setminus Y_a} \psi$ and $Z = X \setminus (Y_a \upharpoonright \mathsf{dom}(X)) = (X'(a/x) \setminus Y_a) \upharpoonright \mathsf{dom}(X)$, we obtain $M \models_Z \psi$. Meanwhile, for any $b \in M$, by the claim, $Y = Y_b \upharpoonright \mathsf{dom}(X)$. Since $M \models_{Y_b} \phi$ and $Y_b \upharpoonright (\mathsf{dom}(X) \cup \{x\}) = Y(b/x)$, we obtain $M \models_{Y(b/x)} \phi$. □

Proof of Proposition 16

Item (i): By rules of identity we have $\vdash c = c \wedge \mathsf{z} = \mathsf{z}$, which implies $\vdash \exists x \forall y (x = c \wedge \mathsf{z} = \mathsf{z})$. Now, by $=\!\!(\cdot)\mathsf{I}$ we derive $\vdash \forall y \exists x (=\!\!(\mathsf{z}, x) \wedge x = c \wedge \mathsf{z} = \mathsf{z})$, which yields $\vdash =\!\!(\mathsf{z}, c)$.

Item (ii): For the direction $=\!\!(c\mathsf{x}, y) \vdash =\!\!(\mathsf{x}, y)$, by applying $=\!\!(\cdot)\mathsf{wE}$ we derive that $=\!\!(c\mathsf{x}, y), =\!\!(c), \mathsf{con}(\mathsf{x}) \vdash =\!\!(y)$. Since $\vdash =\!\!(c)$ by item (i), we conclude by $=\!\!(\cdot)\mathsf{wI}$ that $=\!\!(c\mathsf{x}, y) \vdash =\!\!(\mathsf{x}, y)$. The other direction is derived similarly by applying $=\!\!(\cdot)\mathsf{wE}$, $=\!\!(\cdot)\mathsf{wI}$ and item (i).

Items (iii): By Proposition 8(i) and $\forall^1\,\mathsf{I}$ it suffices to show $Q\mathsf{u}(\phi(c/v) \wedge =\!\!(c\mathsf{x}, y)) \dashv\vdash Q\mathsf{u}\phi(c/v) \wedge =\!\!(\mathsf{x}, y)$ and $Q\mathsf{u}(\phi(c/v) \wedge =\!\!(\mathsf{x}, c)) \dashv\vdash Q\mathsf{u}\phi(c/v)$. But these follow from items (i) and (ii).

Item (iv): The direction $\exists x (=\!\!(x) \wedge \phi) \vdash \exists^1 x \phi$ follows easily from $\exists\mathsf{E}$ and $\exists^1\,\mathsf{I}$. For the other direction, by Proposition 16(i), we have $\exists^1 x \phi \vdash \phi(c/x)$. Moreover, by rules of identity, $\vdash =\!\!(c)$ and $\exists\mathsf{I}$, we have $\phi(c/x) \vdash \exists x (x = c \wedge =\!\!(c) \wedge \phi(c/x)) \vdash \exists x (=\!\!(x) \wedge \phi)$. Putting these together we obtain $\exists^1 x \phi \vdash \exists x (=\!\!(x) \wedge \phi)$.

Item (v): We first prove the right to left direction. By $\exists\mathsf{E}$ it suffices to prove that $=\!\!(x), =\!\!(y), (x = y \wedge \phi) \vee (x \neq y \wedge \psi) \vdash \phi \mathbin{\vee\!\!\!\vee} \psi$. We first derive $=\!\!(x), =\!\!(y) \vdash x = y \mathbin{\vee\!\!\!\vee} x \neq y$ by $\mathbin{\vee\!\!\!\vee}\mathsf{wI}$. Next, by applying $\vee\mathsf{Sub}$, $\bot\vee\mathsf{E}$ and $\mathbin{\vee\!\!\!\vee}\mathsf{I}$, we derive

$$x = y, (x = y \wedge \phi) \vee (x \neq y \wedge \psi) \vdash \phi \vee (x = y \wedge x \neq y) \vdash \phi \vee \bot \vdash \phi \vdash \phi \mathbin{\vee\!\!\!\vee} \psi$$

and similarly $x \neq y, (x = y \wedge \phi) \vee (x \neq y \wedge \psi) \vdash \phi \mathbin{\vee\!\!\!\vee} \psi$. Hence, we conclude by $\mathbin{\vee\!\!\!\vee}\mathsf{E}$ that $x = y \mathbin{\vee\!\!\!\vee} x \neq y, (x = y \wedge \phi) \vee (x \neq y \wedge \psi) \vdash \phi \mathbin{\vee\!\!\!\vee} \psi$, from which the desired clause follows.

For the left to right direction, by $\mathbin{\vee\!\!\!\vee}\mathsf{I}$, it suffices to prove that the right formula is derivable from both ϕ and ψ. We now first derive the right hand side from ϕ. By the rules of identity, $\vdash \exists x \forall z (x = x)$ for some fresh variables x, z. Thus, we conclude by applying $=\!\!(\cdot)\mathsf{I}$ that $\vdash \forall z \exists x (=\!\!(x) \wedge x = x)$, which reduces to $\vdash \exists x =\!\!(x)$. Next, we derive by rules of identity that $\vdash \exists x (=\!\!(x) \wedge \exists y (x = y))$ and thus $\vdash \exists x \exists y (=\!\!(x) \wedge =\!\!(y) \wedge x = y)$. Lastly, we conclude by the introduction rule of $\vee$ that

$$\phi \vdash \exists x \exists y (=\!\!(x) \wedge =\!\!(y) \wedge x = y \wedge \phi) \vdash \exists x \exists y (=\!\!(x) \wedge =\!\!(y) \wedge ((x = y \wedge \phi) \vee (x \neq y \wedge \psi))).$$

Similarly, to derive the right hand side from ψ, first note that by Dom we have $\vdash \exists x \exists y (x \neq y)$ for some fresh variables x, y, which then yields $\vdash \exists x \forall z \exists y (x \neq y)$. Then, by a similar argument as above, we derive by applying $=\!\!(\cdot)\mathsf{I}$ that $\vdash \exists x (=\!\!(x) \wedge \exists y (x \neq y))$, and that $\vdash \exists y (=\!\!(y) \wedge \exists x (=\!\!(x) \wedge (x \neq y)))$, from which the required clause follows.

Item (vi): $\forall x \forall^1 y \phi \dashv\vdash \forall^1 y \forall x \phi$ follows easily from Proposition 8(i). For the other clause, by $\exists\mathsf{E}$, it suffices to prove $\forall^1 y \phi(x, y, \mathsf{z}) \vdash \forall^1 y \exists x (=\!\!(\mathsf{z}, x) \wedge \phi)$. We derive by Proposition 8(i) and the rules for $\forall$,$\exists$ that $\forall^1 y \phi(x, y, \mathsf{z}) \vdash \phi(x, c, \mathsf{z}) \vdash \exists x \forall w \phi(x, c, \mathsf{z})$ for some fresh constant symbol c and variable w. Moreover, by $=\!\!(\cdot)\mathsf{I}$ we derive that $\exists x \forall w \phi(x, c, \mathsf{z}) \vdash \forall w \exists x (=\!\!(\mathsf{z}, x) \wedge \phi(x, c, \mathsf{z})) \vdash \exists x (=\!\!(\mathsf{z}, x) \wedge \phi(x, c, \mathsf{z}))$. Putting these together and by applying $\forall^1\mathsf{I}$ we conclude that $\forall^1 y \phi(x, y, \mathsf{z}) \vdash \exists x (=\!\!(\mathsf{z}, x) \wedge \phi(x, c, \mathsf{z})) \vdash \forall^1 y \exists x (=\!\!(\mathsf{z}, x) \wedge \phi)$. $\square$

Proof of Theorem 17

We adapt the argument for the normal form proof in [27]. We give the semantic and syntactic proof at the same time, and the semantic equivalence clearly follows from the syntactic equivalence (by soundness theorem) whenever the latter is available in the following steps of the proof. First, rewrite every occurrence of $\theta \vee\!\!\!\vee \chi$ and $\exists^1 x\eta$ in ϕ using the equivalent formulas with logical constants from $\mathbf{D}$ given by Proposition 16(iv)(v), and denote the resulting provably equivalent formula (which is $\vee\!\!\!\vee$ and $\exists^1$-free) by ϕ'. Next, turn the formula ϕ' into a formula $Q_1 x \dots Q_n \theta$ in prenex normal form, where each $Q_i \in \{\forall, \exists, \forall^1\}$ and θ is quantifier-free. This step is done, as in [27], by induction on the complexity of the formula ϕ', where the inductive steps with $\forall^1$ follow from the provable equivalences given by Proposition 8(ii) and the rule $\forall^1\mathsf{Ext}$.

Now, since $\theta(\mathsf{x})$ is a formula of $\mathbf{D}$, we proceed in the same way as in [27] to turn θ into a formula of the form $\forall \mathsf{y}\exists \mathsf{z}(\bigwedge_{i\in I} d_i(\mathsf{x},\mathsf{y},\mathsf{z}) \wedge \alpha(\mathsf{x},\mathsf{y},\mathsf{z})) = \forall \mathsf{y}\exists \mathsf{z}\theta'$, where each d_i is a dependence atom and α is first-order. The formula θ is semantically equivalent to $\forall \mathsf{y}\exists \mathsf{z}\theta'$, and in the deduction system of $\mathbf{D}$ we can prove (as is done in [27]) that $\theta \vdash \forall \mathsf{y}\exists \mathsf{z}\theta'$. Thus, altogether we now have $\phi \equiv Q\mathsf{x}\forall \mathsf{y}\exists \mathsf{z}\theta'$ and $\phi \vdash Q\mathsf{x}\forall \mathsf{y}\exists \mathsf{z}\theta'$.

To turn the formula $Q\mathsf{x}\forall \mathsf{y}\exists \mathsf{z}\theta'$ finally into the required normal form (5), we swap the order of the quantifiers using an inductive argument similar to that in [27], where the inductive step for $\forall^1$ is taken care of in the deduction system by applying Proposition 16(vi), and on the semantic side by using the equivalences $\exists x\forall^1 y\psi(x,y,\mathsf{z}) \equiv \forall^1 y\exists x(=\!\!(\mathsf{z},x) \wedge \psi)$ and $\forall x\forall^1 y\psi \equiv \forall^1 y\forall x\psi$ (we leave it for the reader to verify). To conclude the proof, we apply Proposition 16(iii) to remove variables quantified by $\forall^1$ in dependence atoms $d_i(\mathsf{x},\mathsf{y},\mathsf{z})$ in the first conjunct of the quantifier-free formula. □

References

1. Abramsky, S., Väänänen, J.: From IF to BI. Synthese **167**(2), 207–230 (2009)
2. Arrow, K.: A difficulty in the concept of social welfare. J. Polit. Econ. **58**(4), 328–346 (1950)
3. Casanova, M.A., Fagin, R., Papadimitriou, C.H.: Inclusion dependencies and their interaction with functional dependencies. J. Comp. System Sci. **28**(1), 29–59 (1984)
4. Ciardelli, I.: Dependency as question entailment. In: Abramsky, S., Kontinen, J., Väänänen, J., Vollmer, H. (eds.) Dependence Logic, pp. 129–181. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-31803-5_8
5. Ciardelli, I., Iemhoff, R., Yang, F.: Questions and dependency in intuitionistic logic. Notre Dame J. Formal Logic (2019, to appear). arXiv:1704.01866
6. Ciardelli, I., Roelofsen, F.: Inquisitive logic. J. Philos. Logic **40**(1), 55–94 (2011)
7. Corander, J., Hyttinen, A., Kontinen, J., Pensar, J., Väänänen, J.: A logical approach to context-specific independence. Ann. Pure Appl. Logic (2019). https://doi.org/10.1016/j.apal.2019.04.004
8. Enderton, H.: Finite partially-ordered quantifiers. Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik **16**, 393–397 (1970)

9. Engström, F., Kontinen, J., Väänänen, J.: Dependence logic with generalized quantifiers: axiomatizations. J. Comput. Syst. Sci. **88**, 90–102 (2017)
10. Galliani, P.: The dynamics of imperfect information. Ph.D. thesis, University of Amsterdam (2012)
11. Galliani, P.: Inclusion and exclusion in team semantics: on some logics of imperfect information. Ann. Pure Appl. Logic **163**(1), 68–84 (2012)
12. Galliani, P.: Epistemic operators in dependence logic. Stud. Logica **101**(2), 367–397 (2013)
13. Galliani, P.: On strongly first-order dependencies. In: Abramsky, S., Kontinen, J., Väänänen, J., Vollmer, H. (eds.) Dependence Logic, pp. 53–71. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-31803-5_4
14. Galliani, P., Hannula, M., Kontinen, J.: Hierarchies in independence logic. In Proceedings of Computer Science Logic 2013, vol. 23 of LIPIcs, pp. 263–280 (2013)
15. Galliani, P., Hella, L.: Inclusion logic and fixed point logic. In: Computer Science Logic 2013, vol. 23 of LIPIcs, pp. 281–295 (2013)
16. Grädel, E., Väänänen, J.: Dependence and independence. Stud. Logica **101**(2), 399–410 (2013)
17. Hannula, M.: Axiomatizing first-order consequences in independence logic. Ann. Pure Appl. Logic **166**(1), 61–91 (2015)
18. Hannula, M., Hirvonen, Å., Kontinen, J., Kulikov, V., Virtema, J.: Facets of distribution identities in probabilistic team semantics. CoRR abs/1812.05873 (2018)
19. Hannula, M., Kontinen, J.: A finite axiomatization of conditional independence and inclusion dependencies. Inf. Comput. **249**, 121–137 (2016)
20. Henkin, L.: Some remarks on infinitely long formulas. In: Proceedings Symposium Foundations of Mathematics Infinitistic Methods, Warsaw, Pergamon, pp. 167–183 (1961)
21. Hodges, W.: Compositional semantics for a language of imperfect information. Logic J. IGPL **5**, 539–563 (1997)
22. Hodges, W.: Some strange quantifiers. In: Mycielski, J., Rozenberg, G., Salomaa, A. (eds.) Structures in Logic and Computer Science. LNCS, vol. 1261, pp. 51–65. Springer, Heidelberg (1997). https://doi.org/10.1007/3-540-63246-8_4
23. Hyttinen, T., Paolini, G., Väänänen, J.: Quantum team logic and bell's inequalities. Rev. Symbolic Logic **8**(4), 722–742 (2015)
24. Kontinen, J.: Coherence and complexity of quantifier-free dependence logic formulas. Stud. Logica **101**(2), 267–291 (2013)
25. Kontinen, J.: On natural deduction in dependence logic. In: Logic Without Borders, pp. 297–304. De Gruyter (2015)
26. Kontinen, J., Väänänen, J.: On definability in dependence logic. J. Logic Lang. Inf. **18**(3), 317–332 (2009)
27. Kontinen, J., Väänänen, J.: Axiomatizing first-order consequences in dependence logic. Ann. Pure Appl. Logic **164**, 11 (2013)
28. Lück, M.: Axiomatizations of team logics. Ann. Pure Appl. Logic **169**(9), 928–969 (2018)
29. Lyndon, R.C.: An interpolation theorem in the predicate calculus. Pacific J. Math. **9**(1), 129–142 (1959)
30. Pacuit, E., Yang, F.: Dependence and independence in social choice: arrow's theorem. In: Abramsky, S., Kontinen, J., Väänänen, J. (eds.) Dependence Logic, pp. 235–260. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-31803-5_11
31. Väänänen, J.: Dependence Logic: A New Approach to Independence Friendly Logic. Cambridge University Press, Cambridge (2007)

32. Walkoe, W.: Finite partially-ordered quantification. J. Symbolic Logic **35**, 535–555 (1970)
33. Yang, F.: Negation and partial axiomatizations of dependence and independence logic revisited. Ann. Pure Appl. Logic (2019). https://doi.org/10.1016/j.apal.2019.04.010
34. Yang, F., Väänänen, J.: Propositional logics of dependence. Ann. Pure Appl. Logic **167**(7), 557–589 (2016)

Modal Auxiliaries and Negation: A Type-Logical Account

Yusuke Kubota[1(✉)] and Robert Levine[2]

[1] National Institute for Japanese Language and Linguistics, Tachikawa, Japan
kubota@ninjal.ac.jp
[2] Ohio State University, Columbus, USA

Abstract. This paper proposes an analysis of modal auxiliaries in English in Type-Logical Grammar. The proposed analysis captures the scopal interactions between different types of modal auxiliaries and negation by incorporating the key analytic idea of Iatridou and Zeijlstra [6], who classify English modal auxiliaries into PPI and NPI types. In order to technically implement this analysis, we build on Kubota and Levine's [8,10] treatment of modal auxiliaries as higher-order operators that take scope at the clausal level. The proposed extension of the Kubota/Levine analysis is shown to have several interesting consequences, including a formal derivability relation from the higher-order entry for auxiliaries to a lower-order VP/VP entry traditionally recognized in categorial grammar (CG) research. The systematic analysis of the scopal properties of auxiliaries and the somewhat more abstract meta-comparison between 'transformational' and 'non-transformational' analytic ideas that become possible in a type-logical setup highlight the value of taking a logical perspective on the syntax of natural language embodied in Type-Logical Grammar research.

Keywords: Modal auxiliary · Negation · Scope · Type-logical grammar

1 Introduction

Modal auxiliaries in English exhibit a somewhat puzzling patterns in terms of their scopal interactions with negation. So far as we are aware, this particular empirical domain has not been explored in detail in the literature of Type-Logical Grammar (TLG). In this paper, we show that by extending the analysis of auxiliary verbs as semantically higher-order operators proposed by Kubota and Levine [8,10], a relatively simple analysis of the modal-negation scopal interaction becomes available.

The proposed analysis builds on the classification of English modal auxiliaries into two different types based on the polarity distinction proposed by Iatridou and Zeijlstra [6], and can be thought of as a precise logical formalization of the core ideas behind the reconstruction-based analysis by Iatridou and Zeijlstra in minimalist syntax. We show that our logical reconceptualization of Iatridou and Zeijlstra's configurational analysis has several interesting consequences. In particular, our type-logical account illuminates the relationship

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 415–432, 2019.
https://doi.org/10.1007/978-3-662-59533-6_25

between the configurational analysis standard in the mainstream syntax and the lexicalist alternative familiar in the G/HPSG and CG literature more clearly than previous proposals in the respective traditions of generative grammar. We formulate our analysis in Hybrid Type-Logical Grammar (Hybrid TLG) [8,9], but the main results of the present paper are largely neutral to the particular variant of TLG, and can be translated to other variants of CG.[1]

2 Modals and Negation: The Empirical Landscape

It has long been noted that the scopal relationship between modals and negation is essentially unpredictable, though there are certain semantic aspects of modal operators which appear to be relevant.

(1) a. John should not criticize Mary. $(\Box \neg \mathbf{criticize}(\mathbf{m})(\mathbf{j}))$

b. John need not criticize Mary. $(\neg\Box\, \mathbf{criticize}(\mathbf{m})(\mathbf{j}))$

c. John may not criticize Mary. $(\Diamond \neg \mathbf{criticize}(\mathbf{m})(\mathbf{j}), \neg\Diamond\, \mathbf{criticize}(\mathbf{m})(\mathbf{j}))$

It is generally agreed that these variations in scope behavior do not admit of any purely semantic solution following from the meanings of the modals: both *should* and *need* denote (different flavors of) universal quantification over the relevant possible worlds, but have opposite scoping vis-à-vis negation. Similarly, *may* and *might* are both arguably variants of existential quantification over possible worlds, but the former can scope either way so far as negation is concerned, whereas the latter is necessarily wide-scoping. The following table lists the relevant patterns for the major familiar modal auxiliaries:

(2)

modal	scopal pattern
will	$\mathsf{F} > \neg$
would	$\mathfrak{W} > \neg$
shall	$\mathsf{F} > \neg$
should	$\Box > \neg$
ought	$\Box > \neg$
might	$\Diamond > \neg$
must	$\Box > \neg$
may	$\Diamond < > \neg$
can	$\Diamond < > \neg$
could	$\Diamond < > \neg$
need	$\neg > \Box$

[1] As a reviewer notes, transportability of the analysis depends significantly what is common between the two frameworks. Since the Displacement Calculus [14] is largely similar to Hybrid TLG, translation of the present analysis to the Displacement Calculus should for the most part be straightforward (see Morrill and Valentín [15] in this connection). Lowering to VP/VP is of course not available in Linear Categorial Grammar [12] and Abstract Categorial Grammar [4], but lowering to (NP $\multimap$ S) $\multimap$ (NP $\multimap$ S) should be possible.

The patterns reflected in this table can be summarized as follows. The great majority of modals outscope negation that they are syntactically associated with. The striking exception is the deontic necessity modal *need*, which invariably outscopes negation, and the three 'possibility' modals *can*, *could* and *may*, which appear to be neutral.

Iatridou and Zeijlstra [6] argue that a principled account of the patterns in (2) can be given directly in terms of the sensitivity displayed by the individual modals to the scope of negation. *Need* is a known negative polarity item (NPI; see Levine [11] for discussion of the somewhat unusual behavior of this NPI), and hence when it appears with a local negator, such as *not* or *never*, it always scopes under negation. Iatridou and Zeijlstra propose that the invariably wide scope of *must*, *should*, *ought*, etc., with respect to local negation reflects their status as *positive* polarity items (PPIs). On their account, the different scopal relations between different types of modals and negation is a consequence of the 'reconstruction' possibilities of modals depending on their polarity statuses—NPI, PPI or neutral modals—as summarized in the following table:[2]

(3)

	PPI modals	Neutral modals	NPI modals
Universal	*must*, *should*, *ought to*, *be to*	*have to*, *need to*	*need*
Existential	—	*can*, *may*	—

On Iatridou and Zeijlstra's account, the auxiliaries are raised to the head of TP, and hence above negP. In the case of a sentence such as *John need not worry*, *need* cannot be licensed unless it is reconstructed back under negP, due to its NPI status. By contrast, PPI modals such as *must*, *should* and *ought* are prohibited from reconstruction, again due to their lexical property as PPIs. Neutral modals such as *can* and *may* optionally reconstruct to their original sites, giving rise to scope ambiguity with negation.

---

[2] One might wonder about the classification of *must* and *should* as PPIs, given that they can appear unproblematically in the scope of negation in sentences such as *I don't think that John should be even one little bit nice to anyone in that room*, where the NPIs *even*, *anyone* and *one little bit* appear with no hint of ill-formedness. But here it is crucial to bear in mind that polarity items as a broad class are known to be sensitive to not only semantic scope effects but syntactic contexts as well; see Richter and Soehn [19] for a survey of syntactic conditions on a range of NPIs in German. Iatridou and Zeijlstra argue that the same syntactic sensitivity holds for PPIs, and note that

> PPIs... are fine in the scope of negation or any other context that is known to ban PPIs if this context is clause-external (Szabolcsi 2004:24–27), as illustrated in (i)–(iv):
>
> (i) I don't think that John called someone. not > $[_{\mathrm{CP/IP}}$ some
> (ii) No one thinks/says that John called someone. no one > $[_{\mathrm{CP/IP}}$ some
> (iii) I regret that John called someone. regret > $[_{\mathrm{CP/IP}}$ some
> (iv) Every boy who called someone got help. every $[_{\mathrm{CP/IP}}$ some

What seems to hold for the PPI modals, then, is that they cannot be in the scope of negation that originates in syntactically *local* operators.

3 Higher Order Negation: The Formal Analysis

In this section, we present our type-logical analysis of modal-negation scope interaction. After reviewing Kubota and Levine's [8,10] analysis of modal auxiliaries in Hybrid Type-Logical Grammar (Hybrid TLG) in Sect. 3.1, we present our extended fragment that takes into account the scopal interactions with negation in Sect. 3.2. Sections 3.3, 3.4 and 3.5 discuss some consequences of our proposal that help clarify the relationship between the higher-order operator analysis we propose and alternative approaches in the literature.

3.1 Higher-Order Modals

Kubota and Levine [8,10] posit the following type of lexical entries for modal auxiliaries in English (where $\mathsf{id}_{et} = \lambda P_{et}.P$ and VP_f, VP_b are abbreviations of $\mathrm{NP}\backslash \mathrm{S}_f$, $\mathrm{NP}\backslash \mathrm{S}_b$):

(4) $\lambda\sigma.\sigma(\mathsf{can't});\ \lambda\mathscr{F}.\neg\Diamond\mathscr{F}(\mathsf{id}_{et});\ \mathrm{S}_f{\upharpoonright}(\mathrm{S}_f{\upharpoonright}(\mathrm{VP}_f/\mathrm{VP}_b))$

The following derivation illustrates how the $\Box > \exists$ reading for *Someone must be present (at the meeting)* is captured in this analysis (see Appendix A for a formal fragment of Hybrid TLG):

(5)

$$
\dfrac{
\begin{array}{l}\lambda\sigma.\sigma(\mathsf{must});\\ \lambda\mathscr{F}.\Box\mathscr{F}(\mathsf{id}_{et});\\ \mathrm{S}_f{\upharpoonright}(\mathrm{S}_f{\upharpoonright}(\mathrm{VP}_f/\mathrm{VP}_b))\end{array}
\quad
\dfrac{
\dfrac{
\begin{array}{l}\lambda\sigma.\sigma(\mathsf{someone});\\ \mathbf{\exists_{person}};\\ \mathrm{S}_f{\upharpoonright}(\mathrm{S}_f{\upharpoonright}\mathrm{NP})\end{array}
\quad
\dfrac{
\dfrac{
\begin{bmatrix}\varphi_2;\\ x;\\ \mathrm{NP}\end{bmatrix}^2
\quad
\dfrac{
\begin{bmatrix}\varphi_1;\\ f;\mathrm{VP}_f/\mathrm{VP}_b\end{bmatrix}^1
\quad
\begin{array}{l}\mathsf{be}\circ\mathsf{present};\\ \mathbf{present};\mathrm{VP}_b\end{array}
}{\varphi_1\circ\mathsf{be}\circ\mathsf{present};\ f(\mathbf{present});\ \mathrm{VP}_f}\,/\mathrm{E}
}{\varphi_2\circ\varphi_1\circ\mathsf{be}\circ\mathsf{present};\ f(\mathbf{present})(x);\ \mathrm{S}_f}\,\backslash\mathrm{E}
}{\begin{array}{l}\lambda\varphi_2.\varphi_2\circ\varphi_1\circ\mathsf{be}\circ\mathsf{present};\\ \lambda x.f(\mathbf{present})(x);\mathrm{S}_f{\upharpoonright}\mathrm{NP}\end{array}}\,{\upharpoonright}\mathrm{I}^2
}{\begin{array}{l}\mathsf{someone}\circ\varphi_1\circ\mathsf{be}\circ\mathsf{present};\\ \mathbf{\exists_{person}}(\lambda x.f(\mathbf{present})(x));\mathrm{S}_f\end{array}}\,{\upharpoonright}\mathrm{E}
}{\begin{array}{l}\lambda\varphi_1.\mathsf{someone}\circ\varphi_1\circ\mathsf{be}\circ\mathsf{present};\\ \lambda f.\mathbf{\exists_{person}}(\lambda x.f(\mathbf{present})(x));\mathrm{S}_f{\upharpoonright}(\mathrm{VP}_f/\mathrm{VP}_b)\end{array}}\,{\upharpoonright}\mathrm{I}^1
}{\mathsf{someone}\circ\mathsf{must}\circ\mathsf{be}\circ\mathsf{present};\ \Box\mathbf{\exists_{person}}(\lambda x.\mathbf{present}(x));\ \mathrm{S}_f}\,{\upharpoonright}\mathrm{E}
$$

Here, the hypothetical reasoning for the NP hypothesis (indexed 2) is for the subject quantifier *someone*, which enters into the derivation once the whole clause is built (semantically scoping over it and 'lowering' its prosody in the gap position corresponding to the λ-bound prosodic variable φ_2). The derivation involves another set of steps of hypothetical reasoning, with the $\mathrm{VP}_f/\mathrm{VP}_b$ hypothesis (indexed 1). This lets the modal semantically take scope above the subject quantifier (with prosodic lowering similar to the case of quantifiers). We thus obtain the result in which the modal auxiliary and the subject quantifier appear in their respective surface positions but in which the modal outscopes the quantifier.

The key idea behind this analysis is that auxiliaries are treated like generalized quantifiers (which are of type $\mathrm{S}{\upharpoonright}(\mathrm{S}{\upharpoonright}\mathrm{NP})$ in Hybrid TLG) except that they 'quantify over' VP/VP type expressions rather than NPs. The meaning contribution of the modal is the propositional modal operator, so, on this analysis (unlike the VP/VP analysis more familiar in the CG literature), the semantic scope and the 'syntactic position' at which the modal is introduced in the derivation correspond to each other straightforwardly. The features f and b abbreviate the 'VFORM' features (in G/HPSG terms) *fin* and *bse* that mark finite and base forms of verbs respectively. This ensures that modals can only combine with base forms of verbs and after the modal is combined with the verb, the result is finite, and no other modal can stack on top of the resultant VP.

The main empirical motivation for this 'quantificational' analysis of modal auxiliaries comes from the famous scope anomaly in Gapping sentences noted by Siegel [20] and Oehrle [16], as in examples such as (6).

(6) John can't eat steak and Mary just (eat) pizza!
$\neg\Diamond \mathbf{eat}(\mathbf{steak})(\mathbf{j}) \wedge \mathbf{eat}(\mathbf{pizza})(\mathbf{m})$

We do not repeat the argument here, but refer the reader to Kubota and Levine [8,10] for a detailed discussion. The key point is that the ordinary VP/VP analysis has difficulty in accounting for the wide scope interpretation of modals in examples like (6) in any straightforward manner (relatedly, assigning the semantic translation $\lambda\mathscr{K}.\mathscr{K}(\lambda g \lambda x.\Box g(x))$, which would correspond to the semantic translation of a syntactically type-raised entry of the lower-order VP/VP entry, would fail to capture the wide-scope reading in (6)).

Puthawala [18] has recently shown that the same type of scope anomaly is observed in Stripping as well, and that the Kubota/Levine analysis can be straightforwardly extended to the Stripping cases in (7) as well:

(7) a. John won't apply for the job, or Mary either.
$\neg(\mathsf{F}\,\mathbf{apply\text{-}for}(\iota(\mathbf{job}))(\mathbf{j}) \vee \mathsf{F}\,\mathbf{apply\text{-}for}(\iota(\mathbf{job}))(\mathbf{m}))$
b. Mary can't testify for the defense and John also!
$\neg\Diamond(\mathbf{testify\text{-}for}(\mathbf{defense})(\mathbf{m}) \wedge \mathbf{testify\text{-}for}(\mathbf{defense})(\mathbf{j}))$

As noted by Kubota and Levine [8,10], an interesting consequence of the higher-order analysis of modal auxiliaries in TLG outlined above is that the more familiar VP/VP sign for the modal auxiliary standardly assumed in the CG literature is immediately derivable via hypothetical reasoning from the higher-order one posited in the lexicon. The proof goes as follows:

(8)
$$\dfrac{\dfrac{\dfrac{\begin{array}{l}\lambda\sigma.\sigma(\mathsf{can't});\\ \lambda\mathscr{F}.\neg\Diamond\mathscr{F}(\mathsf{id}_{et});\\ \mathrm{S}_f{\upharpoonright}(\mathrm{S}_f{\upharpoonright}(\mathrm{VP}_f/\mathrm{VP}_b))\end{array} \quad \dfrac{\dfrac{[\varphi_1; x; \mathrm{NP}]^1 \quad \dfrac{[\varphi_2; g; \mathrm{VP}_f/\mathrm{VP}_b]^2 \quad [\varphi_3; f; \mathrm{VP}_b]^3}{\varphi_2 \circ \varphi_3;\ g(f);\ \mathrm{VP}_f}\,/\mathrm{E}}{\varphi_1 \circ \varphi_2 \circ \varphi_3;\ g(f)(x);\ \mathrm{S}_f}\,\backslash\mathrm{E}}{\lambda\varphi_2.\varphi_1 \circ \varphi_2 \circ \varphi_3;\ \lambda g.G(f)(x);\ \mathrm{S}_f{\upharpoonright}(\mathrm{VP}_f/\mathrm{VP}_b)}\,{\upharpoonright}\mathrm{I}^2}{\varphi_1 \circ \mathsf{can't} \circ \varphi_3;\ \neg\Diamond f(x);\ \mathrm{S}_f}\,{\upharpoonright}\mathrm{E}}{\mathsf{can't} \circ \varphi_3;\ \lambda x.\neg\Diamond f(x);\ \mathrm{VP}_f}\,\backslash\mathrm{I}^1}{\mathsf{can't};\ \lambda f\lambda x.\neg\Diamond f(x);\ \mathrm{VP}_f/\mathrm{VP}_b}\,/\mathrm{I}^3$$

This is essentially a case of lowering in the sense of Hendriks [5] in a system that extends the Lambek calculus with a discontinuous connective (in our case, $\upharpoonleft$). Ignoring directionality, it corresponds to the elementary theorem $(((\phi \to \psi) \to \varrho) \to \varrho) \to \zeta \vdash (\phi \to \psi) \to \zeta$ in standard propositional logic.

We call the family of theorems of which (8) is an instance 'slanting'. In slanting derivations, the vertical slash $\upharpoonleft$ is eliminated from the lexical specification of a scopal operator 'slanting'. In addition to clarifying the relationship between the higher-order and more familiar type assignments for scopal operators (see Sect. 3.3), slanting is useful in ensuring the correct scoping relations between multiple operators in certain cases, as discussed in Kubota and Levine [9] with respect to the analysis of quantifier-coordination interaction and as we show below in connection to modal auxiliary scope (Sects. 3.4 and 3.5).

3.2 Capturing the Modal/Negation Scope Interaction

In order to capture the polarity sensitivity of different types of modal auxiliaries in English, we posit a syntactic feature *pol* for category S that takes one of the three values $+$, $-$ and $\varnothing$.[3] The treatment of polarity here follows the general approach to polarity marking in the CG literature by Dowty [3], Bernardi [2] and Steedman [21], but differs from them in some specific details. Intuitively, S_{pol+} and S_{pol-} are positively and negatively marked clauses respectively, and $\mathrm{S}_{pol\varnothing}$ is a 'smaller' clause that isn't yet assigned polarity marking. To avoid cluttering the notation, we suppress the feature name *pol* in what follows and write S_{pol+}, S_{pol-} and $\mathrm{S}_{pol\varnothing}$ simply as S_{+}, S_{-} and $\mathrm{S}_{\varnothing}$, respectively. Positive-polarity modals are then lexically specified to obligatorily take scope at the level of S_{+}. Negative-polarity modals on the other hand are lexically specified to take scope at the level of $\mathrm{S}_{\varnothing}$, before negation turns an 'unmarked' clause to a negatively marked clause. We assume further that complete sentences in English are marked either *pol*+ or *pol*−; thus, $\mathrm{S}_{\varnothing}$ does not count as a stand-alone sentence.

The analysis of PPI and NPI modals outlined above can be technically implemented by positing the following lexical entries for the modals and the negation morpheme (where $\alpha, \beta \in \{\varnothing, -\}$ and $\gamma \in \{bse, fin\}$):

(9) a. $\lambda\sigma.\sigma(\mathsf{should})$; $\lambda\mathscr{G}.\Box\mathscr{G}(\mathsf{id}_{et})$; $\mathrm{S}_{f,+}\upharpoonleft(\mathrm{S}_{f,\beta}\upharpoonleft(\mathrm{VP}_{f,\alpha}/\mathrm{VP}_{b,\alpha}))$

b. $\lambda\sigma.\sigma(\mathsf{need})$; $\lambda\mathscr{G}.\Box\mathscr{G}(\mathsf{id}_{et})$; $\mathrm{S}_{f,\varnothing}\upharpoonleft(\mathrm{S}_{f,\varnothing}\upharpoonleft(\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}))$

c. $\lambda\sigma.\sigma(\mathsf{not})$; $\lambda\mathscr{G}.\neg\mathscr{G}(\mathsf{id}_{et})$; $\mathrm{S}_{\gamma,-}\upharpoonleft(\mathrm{S}_{\gamma,\varnothing}\upharpoonleft(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}))$

We assume that different modals are assigned the following syntactic categories, depending on their polarity sensitivity:

[3] We remain agnostic about the exact formal implementation of syntactic features in the present paper. This could be done, for example, via some mechanism of unification as in HPSG. Another approach would involve the use of dependent types, along lines suggested by Morrill [13] and worked out in some detail by Pompigne [17]. So far as we can tell, the results of the current paper does not hinge on the specific choice on this matter.

(10)

PPI $\mathbf{S}_{f,+}\upharpoonright(\mathbf{S}_{f,\beta}\upharpoonright(\mathbf{VP}_{f,\alpha}/\mathbf{VP}_{b,\alpha}))$	**NPI** $\mathbf{S}_{f,\varnothing}\upharpoonright(\mathbf{S}_{f,\varnothing}\upharpoonright(\mathbf{VP}_{f,\varnothing}/\mathbf{VP}_{b,\varnothing}))$
should	*need*
must	*dare*
ought	
might	
can	*can*
could	*could*
may	*may*
will	*will*
would	*would*

We now illustrate the working of this fragment with the analyses for (11a) (which involves a PPI modal) and (11b) (which involves an NPI modal).

(11) a. John should not come.
b. John need not come.

The derivation for (11a) goes as follows:

(12) a.

$$\dfrac{\begin{matrix}\text{john};\\ \mathbf{j};\mathrm{NP}\end{matrix} \quad \dfrac{\begin{bmatrix}\varphi_4;\\ h;\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}\end{bmatrix}^4 \quad \dfrac{[\varphi_1; f; \mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}]^1 \quad \text{come}; \mathbf{come}; \mathrm{VP}_{b,\varnothing}}{\varphi_1\circ\text{come}; f(\mathbf{come}); \mathrm{VP}_{b,\varnothing}}/\mathrm{E}}{\varphi_4\circ\varphi_1\circ\text{come}; h(f(\mathbf{come})); \mathrm{VP}_{f,\varnothing}}/\mathrm{E}}{\text{john}\circ\varphi_4\circ\varphi_1\circ\text{come}; h(f(\mathbf{come}))(\mathbf{j}); \mathrm{S}_{f,\varnothing}}/\mathrm{E}$$

b.

$$\dfrac{\begin{matrix}\lambda\sigma.\sigma(\text{should});\\ \lambda\mathscr{G}.\Box\mathscr{G}(\text{id}_{et});\\ \mathrm{S}_{f,+}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright(\mathrm{VP}_{f,\alpha}/\mathrm{VP}_{b,\alpha}))\end{matrix} \quad \dfrac{\dfrac{\begin{matrix}\lambda\sigma.\sigma(\text{not});\\ \lambda\mathscr{G}.\neg\mathscr{G}(\text{id}_{et});\\ \mathrm{S}_{\gamma,-}\upharpoonright(\mathrm{S}_{\gamma,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{matrix} \quad \dfrac{\begin{matrix}(12\text{a})\\ \text{john}\circ\varphi_4\circ\varphi_1\circ\text{come};\\ h(f(\mathbf{come}))(\mathbf{j});\mathrm{S}_{f,\varnothing}\end{matrix}}{\begin{matrix}\lambda\varphi_1.\text{john}\circ\varphi_4\circ\varphi_1\circ\text{come};\\ \lambda f.h(f(\mathbf{come}))(\mathbf{j});\\ \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing})\end{matrix}}\upharpoonright\mathrm{I}^1}{\text{john}\circ\varphi_4\circ\text{not}\circ\text{come}; \neg h(\mathbf{come})(\mathbf{j}); \mathrm{S}_{f,-}}\upharpoonright\mathrm{E}}{\begin{matrix}\lambda\varphi_4.\text{john}\circ\varphi_4\circ\text{not}\circ\text{come};\\ \lambda h.\neg h(\mathbf{come})(\mathbf{j});\mathrm{S}_{f,-}\upharpoonright(\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing})\end{matrix}}\upharpoonright\mathrm{I}^4}{\text{john}\circ\text{should}\circ\text{not}\circ\text{come}; \Box\neg\mathbf{come}(\mathbf{j}); \mathrm{S}_{f,+}}\upharpoonright\mathrm{E}$$

The key point here is that although both *should* and *not* are lexically specified to take scope at the clausal level, their scopal relation is fixed. Specifically, once *should* takes scope, the resultant clause is S_+, which is incompatible with the specification on the argument category for *not*. This means that negation is forced to take scope before the PPI modal does.

Exactly the opposite relation holds between the NPI modal *need* and negation. Here, after negation takes scope, we have S_-, but this specification is incompatible with the argument category for the NPI modal, which requires the clause it scopes over to be $\mathrm{S}_\varnothing$. Thus, as in (13), the only possibility is to have *need* take scope before the negation does, which gives us the $\neg > \Box$ Scopal relation.[4]

[4] Extending the present analysis to cases involving negative quantifiers (e.g. *Nothing need be said about this*) is a task that we leave for future work.

(13) a.

$$\dfrac{\begin{array}{l}\mathsf{john};\\ \mathbf{j};\mathrm{NP}\end{array} \quad \dfrac{\left[\begin{array}{l}\varphi_4;\\ h;\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}\end{array}\right]^4 \quad \dfrac{[\varphi_1; f; \mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}]^1 \quad \mathsf{come}; \mathbf{come}; \mathrm{VP}_{b,\varnothing}}{\varphi_1 \circ \mathsf{come};\ f(\mathbf{come});\ \mathrm{VP}_{b,\varnothing}}/\mathrm{E}}{\varphi_4 \circ \varphi_1 \circ \mathsf{come};\ h(f(\mathbf{come}));\ \mathrm{VP}_{f,\varnothing}}/\mathrm{E}}{\mathsf{john} \circ \varphi_4 \circ \varphi_1 \circ \mathsf{come};\ h(f(\mathbf{come}))(\mathbf{j});\ \mathrm{S}_{f,\varnothing}}/\mathrm{E}$$

b.

$$\dfrac{\begin{array}{l}\lambda\sigma.\sigma(\mathsf{not});\\ \lambda\mathscr{G}.\neg\mathscr{G}(\mathsf{id}_{et});\\ \mathrm{S}_{\gamma,-}\upharpoonright(\mathrm{S}_{\gamma,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{array} \quad \dfrac{\dfrac{\begin{array}{l}\lambda\sigma.\sigma(\mathsf{need});\\ \lambda\mathscr{G}.\Box\mathscr{G}(\mathsf{id}_{et});\\ \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{array} \quad \dfrac{\begin{array}{c}(13\mathrm{a})\\ \mathsf{john} \circ \varphi_4 \circ \varphi_1 \circ \mathsf{come};\\ h(f(\mathbf{come}))(\mathbf{j}); \mathrm{S}_{f,\varnothing}\end{array}}{\begin{array}{l}\lambda\varphi_4.\mathsf{john} \circ \varphi_4 \circ \varphi_1 \circ \mathsf{come};\\ \lambda h.h(f(\mathbf{come}))(\mathbf{j});\\ \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing})\end{array}}\upharpoonright\mathrm{I}^4}{\mathsf{john} \circ \mathsf{need} \circ \varphi_1 \circ \mathsf{come};\ \Box f(\mathbf{come})(\mathbf{j});\ \mathrm{S}_{f,\varnothing}}\upharpoonright\mathrm{E}}{\begin{array}{l}\lambda\varphi_1.\mathsf{john} \circ \mathsf{need} \circ \varphi_1 \circ \mathsf{come};\\ \lambda f.\Box f(\mathbf{come})(\mathbf{j}); \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing})\end{array}}\upharpoonright\mathrm{I}^1}{\mathsf{john} \circ \mathsf{need} \circ \mathsf{not} \circ \mathsf{come};\ \neg\Box\mathbf{come}(\mathbf{j});\ \mathrm{S}_{f,-}}\upharpoonright\mathrm{E}$$

We assume that modals that give rise to scope ambiguity with negation are simply ambiguous between PPI and NPI variants, as in (10). This accounts for the scope ambiguity of examples such as (1c).[5]

3.3 Slanting and the VP/VP Analysis of Auxiliaries

The analysis of modal scope presented above can, in a sense, be thought of as a logical reconceptualization of the configurational account proposed by Iatridou and Zeijlstra. Instead of relying on reconstruction and movement, our analysis simply regulates the relative scope relations between the auxiliary and negation via the three-way distinction of the polarity-marking feature *pol*, but aside from this technical difference, the essential analytic idea is the same: the semantic scope of the modal and negation operators transparently reflects the form of the abstract combinatoric structure that is not directly visible from surface

[5] Though we have chosen to posit two distinct lexical entries for the 'neutral' modals (*can*, *could* and *may*) for high and low scoping possibilities with respect to negation, corresponding respectively to the scoping properties of the unambiguous modals, it is easy to collapse these two entries for these modals by making the polarity features for the two S's and two VPs in the complex higher-order category for the modal totally underspecified and unconstrained (except for one constraint $\langle\alpha,\beta\rangle \neq \langle\varnothing,-\rangle$, to exclude the possibility of double negation marking **can not not*), along the following lines:

(i) $\lambda\sigma.\sigma(\mathsf{can});\ \lambda\mathscr{G}.\Diamond\mathscr{G}(\mathsf{id}_{et});\ \mathrm{S}_{f,\alpha}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright(\mathrm{VP}_{f,\delta}/\mathrm{VP}_{b,\zeta}))$

By (partially) resolving underspecification, we can derive both the 'PPI' and 'NPI' variants of the modal lexical entry in (10) from (i), thus capturing scope ambiguity via a single lexical entry. (i) allows for other instantiations of feature specification, but these are either redundant (yielding either high or low scope that are already derivable with the PPI and NPI instantiations in (10)), or useless (i.e. cannot be used in any well-formed syntactic derivation), and hence harmless. Thus, if desired, the lexical ambiguity we have tentatively assumed in the main text can be eliminated by adopting the more general lexical entry along the lines of (i) without the danger of overgeneration.

constituency, be it a level of syntactic representation (i.e. LF, as in Iatridou and Zeijlstra's account), or the structure of the proof that yields the pairing of surface string semantic translation (as in our approach, and more generally, in CG-based theories of natural language syntax/semantics).

One might then wonder whether the two analyses are mere notational variants or if there is any advantage gained by recasting the LF-based analysis in a type-logical setup. We do think that our approach has the advantage of being fully explicit, without relying on the notions of reconstruction and movement whose exact details remain somewhat elusive. However, rather than dwelling on this point, we would like to point out an interesting consequence that immediately follows from our account and which illuminates the relationship between the 'transformational' analysis of auxiliaries (of the sort embodied in our analysis of modal auxiliaries as 'VP-modifier quantifiers') and the lexicalist alternatives in the tradition of non-transformational syntax (such as G/HPSG and CG).

To see the relevant point, note first that PPI modals such as *should* can be derived in the lower-order category $\mathrm{VP}_{f,+}/\mathrm{VP}_{b,\delta}$ as follows (here, $\alpha, \beta, \delta \in \{\varnothing, -\}$):

(14)

$$\dfrac{\dfrac{\begin{array}{l}\lambda\sigma.\sigma(\mathsf{should});\\ \lambda\mathscr{G}.\Box\mathscr{G}(\mathsf{id}_{et});\\ \mathrm{S}_{f,+}\restriction(\mathrm{S}_{f,\beta}\restriction(\mathrm{VP}_{f,\alpha}/\mathrm{VP}_{b,\alpha}))\end{array} \quad \dfrac{\dfrac{\begin{bmatrix}\varphi_3;\\ x;\mathrm{NP}\end{bmatrix}^3 \quad \dfrac{\begin{bmatrix}\varphi_1;\\ f;\mathrm{VP}_{f,\delta}/\mathrm{VP}_{b,\delta}\end{bmatrix}^1 \quad \begin{bmatrix}\varphi_2;\\ g;\mathrm{VP}_{b,\delta}\end{bmatrix}^2}{\varphi_1 \circ \varphi_2;\ f(g);\ \mathrm{VP}_{f,\delta}}/\mathrm{E}}{\varphi_3 \circ \varphi_1 \circ \varphi_2;\ f(g)(x);\ \mathrm{S}_{f,\delta}}\backslash\mathrm{E}}{\lambda\varphi_1.\varphi_3 \circ \varphi_1 \circ \varphi_2;\ \lambda f.f(g)(x);\ \mathrm{S}_{f,\delta}\restriction(\mathrm{VP}_{f,\delta}/\mathrm{VP}_{b,\delta})}\restriction\mathrm{I}^1}{\dfrac{\dfrac{\varphi_3 \circ \mathsf{should} \circ \varphi_2;\ \Box g(x);\ \mathrm{S}_{f,+}}{\mathsf{should} \circ \varphi_2;\ \lambda x.\Box g(x);\ \mathrm{VP}_{f,+}}\backslash\mathrm{I}^3}{\mathsf{should};\ \lambda g\lambda x.\Box g(x);\ \mathrm{VP}_{f,+}/\mathrm{VP}_{b,\delta}}/\mathrm{I}^2}\restriction\mathrm{E}$$

Similarly, the negation morpheme *not* can be slanted to the $\mathrm{VP}_{b,-}/\mathrm{VP}_{b,\varnothing}$ category:

(15)

$$\dfrac{\dfrac{\begin{array}{l}\lambda\sigma.\sigma(\mathsf{not});\\ \lambda\mathscr{G}.\neg\mathscr{G}(\mathsf{id}_{et});\\ \mathrm{S}_{\gamma,-}\restriction(\mathrm{S}_{\gamma,\varnothing}\restriction(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{array} \quad \dfrac{\dfrac{\begin{bmatrix}\varphi_3;\\ x;\mathrm{NP}\end{bmatrix}^3 \quad \dfrac{\begin{bmatrix}\varphi_1;\\ f;\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}\end{bmatrix}^1 \quad \begin{bmatrix}\varphi_2;\\ g;\mathrm{VP}_{b,\varnothing}\end{bmatrix}^2}{\varphi_1 \circ \varphi_2;\ f(g);\ \mathrm{VP}_{b,\varnothing}}/\mathrm{E}}{\varphi_3 \circ \varphi_1 \circ \varphi_2;\ f(g)(x);\ \mathrm{S}_{b,\varnothing}}\backslash\mathrm{E}}{\lambda\varphi_1.\varphi_3 \circ \varphi_1 \circ \varphi_2;\ \lambda f.f(g)(x);\ \mathrm{S}_{b,\varnothing}\restriction(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing})}\restriction\mathrm{I}^1}{\dfrac{\dfrac{\varphi_3 \circ \mathsf{not} \circ \varphi_2;\ \neg g(x);\ \mathrm{S}_{b,-}}{\mathsf{not} \circ \varphi_2;\ \lambda x.\neg g(x);\ \mathrm{VP}_{b,-}}\backslash\mathrm{I}^3}{\mathsf{not};\ \lambda g\lambda x.\neg g(x);\ \mathrm{VP}_{b,-}/\mathrm{VP}_{b,\varnothing}}/\mathrm{I}^2}\restriction\mathrm{E}$$

These two lowered categories can be combined to produce the following sign:

(16)

$$\dfrac{\dfrac{\mathsf{should};\ \lambda g\lambda x.\Box g(x);\ \mathrm{VP}_{f,+}/\mathrm{VP}_{b,\delta} \quad \dfrac{\mathsf{not};\ \lambda g\lambda x.\neg g(x);\ \mathrm{VP}_{b,-}/\mathrm{VP}_{b,\varnothing} \quad [\varphi_1; g; \mathrm{VP}_{b,\varnothing}]^1}{\mathsf{not} \circ \varphi_1;\ \lambda x.\neg g(x);\ \mathrm{VP}_{b,-}}/\mathrm{E}}{\mathsf{should} \circ \mathsf{not} \circ \varphi_1;\ \lambda x.\Box\neg g(x);\ \mathrm{VP}_{f,+}}/\mathrm{E}}{\mathsf{should} \circ \mathsf{not};\ \lambda g\lambda x.\Box\neg g(x);\ \mathrm{VP}_{f,+}/\mathrm{VP}_{b,\varnothing}}/\mathrm{I}^1$$

Slanting the NPI modal *need*, on the other hand, yields the following result:

(17)

$$
\dfrac{\dfrac{\dfrac{
\begin{array}{l}\lambda\sigma.\sigma(\mathsf{need});\\ \lambda\mathscr{G}.\Box\mathscr{G}(\mathsf{id}_{et});\\ \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{array}
\quad
\dfrac{\dfrac{\begin{bmatrix}\varphi_3;\\ x;\mathrm{NP}\end{bmatrix}^3 \quad \dfrac{\begin{bmatrix}\varphi_1;\\ f;\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}\end{bmatrix}^1 \quad \begin{bmatrix}\varphi_2;\\ g;\mathrm{VP}_{b,\varnothing}\end{bmatrix}^2}{\varphi_1\circ\varphi_2;\ f(g);\ \mathrm{VP}_{f,\varnothing}}/\mathrm{E}}{\varphi_3\circ\varphi_1\circ\varphi_2;\ f(g)(x);\ \mathrm{S}_{f,\varnothing}}\backslash\mathrm{E}}{\lambda\varphi_1.\varphi_3\circ\varphi_1\circ\varphi_2;\ \lambda f.f(g)(x);\ \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing})}{\upharpoonright}\mathrm{I}^1
}{\varphi_3\circ\mathsf{need}\circ\varphi_2;\ \Box g(x);\ \mathrm{S}_{f,\varnothing}}{\upharpoonright}\mathrm{E}}{\mathsf{need}\circ\varphi_2;\ \lambda x.\Box g(x);\ \mathrm{VP}_{f,\varnothing}}\backslash\mathrm{I}^3}{\mathsf{need};\ \lambda g\lambda x.\Box g(x);\ \mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}}/\mathrm{I}^2
$$

Note that this resultant category cannot be combined with the lowered negation category in (15) due to feature mismatch (*need* requires its argument to be $\mathrm{VP}_{b,\varnothing}$, but *not* marks the VP as $\mathrm{VP}_{b,-}$). Thus, the lowered *need* is correctly prevented from outscoping negation.

It is however possible to derive *need not* as a complex auxiliary with the correct negation-outscoping semantics:

(18) a.

$$
\dfrac{\begin{bmatrix}\varphi_3;\\ x;\mathrm{NP}\end{bmatrix}^3 \quad \dfrac{\begin{bmatrix}\varphi_4;\\ h;\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}\end{bmatrix}^4 \quad \dfrac{\begin{bmatrix}\varphi_1;\\ f;\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}\end{bmatrix}^1 \quad \begin{bmatrix}\varphi_2;\\ g;\mathrm{VP}_{b,\varnothing}\end{bmatrix}^2}{\varphi_1\circ\varphi_2;\ f(g);\ \mathrm{VP}_{b,\varnothing}}/\mathrm{E}}{\varphi_4\circ\varphi_1\circ\varphi_2;\ h(f(g));\ \mathrm{VP}_{f,\varnothing}}/\mathrm{E}}{\varphi_3\circ\varphi_4\circ\varphi_1\circ\varphi_2;\ h(f(g))(x);\ \mathrm{S}_{f,\varnothing}}\backslash\mathrm{E}
$$

b.

$$
\dfrac{\dfrac{
\begin{array}{l}\lambda\sigma.\sigma(\mathsf{not});\\ \lambda\mathscr{G}.\neg\mathscr{G}(\mathsf{id}_{et});\\ \mathrm{S}_{\gamma,-}\upharpoonright(\mathrm{S}_{\gamma,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{array}
\quad
\dfrac{\dfrac{
\begin{array}{l}\lambda\sigma.\sigma(\mathsf{need});\\ \lambda\mathscr{G}.\Box\mathscr{G}(\mathsf{id}_{et});\\ \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{array}
\quad
\dfrac{\begin{array}{c}(18\mathrm{a})\\ \varphi_3\circ\varphi_4\circ\varphi_1\circ\varphi_2;\\ h(f(g))(x);\mathrm{S}_{f,\varnothing}\end{array}}{\begin{array}{l}\lambda\varphi_4.\varphi_3\circ\varphi_4\circ\varphi_1\circ\varphi_2;\\ \lambda h.h(f(g))(x);\\ \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{f,\varnothing}/\mathrm{VP}_{b,\varnothing})\end{array}}{\upharpoonright}\mathrm{I}^4
}{\varphi_3\circ\mathsf{need}\circ\varphi_1\circ\varphi_2;\ \Box f(g)(x);\ \mathrm{S}_{f,\varnothing}}{\upharpoonright}\mathrm{E}}{\begin{array}{l}\lambda\varphi_1.\varphi_3\circ\mathsf{need}\circ\varphi_1\circ\varphi_2;\\ \lambda f.\Box f(g)(x);\mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing})\end{array}}{\upharpoonright}\mathrm{I}^1
}{\dfrac{\varphi_3\circ\mathsf{need}\circ\mathsf{not}\circ\varphi_2;\ \neg\Box g(x);\ \mathrm{S}_{f,-}}{\mathsf{need}\circ\mathsf{not}\circ\varphi_2;\ \lambda x.\neg\Box g(x);\ \mathrm{VP}_{f,-}}\backslash\mathrm{I}^3}{\upharpoonright}\mathrm{E}
$$

$$
\dfrac{\mathsf{need}\circ\mathsf{not}\circ\varphi_2;\ \lambda x.\neg\Box g(x);\ \mathrm{VP}_{f,-}}{\mathsf{need}\circ\mathsf{not};\ \lambda g\lambda x.\neg\Box g(x);\ \mathrm{VP}_{f,-}/\mathrm{VP}_{b,\varnothing}}/\mathrm{I}^2
$$

Note also that we can derive string-level signs for modals that mimic the higher order version in their ability to outscope generalized quantifiers:

(19)

$$
\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{[\varphi_3;\mathscr{P};\mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha}]^3 \quad \dfrac{[\varphi_2;f;\mathrm{VP}_{f,\alpha}/\mathrm{VP}_{b,\alpha}]^2 \quad [\varphi_1;P;\mathrm{VP}_{b,\alpha}]^1}{\varphi_2\circ\varphi_1;\ f(P);\ \mathrm{VP}_{f,\alpha}}}{\varphi_3\circ\varphi_2\circ\varphi_1;\ \mathscr{P}(f(P));\ \mathrm{S}_{f,\alpha}}}{\lambda\varphi_2.\varphi_3\circ\varphi_2\circ\varphi_1;\ \lambda f.\mathscr{P}(P);\ \mathrm{S}_{f,\alpha}\upharpoonright(\mathrm{VP}_{f,\alpha}/\mathrm{VP}_{b,\alpha})}{\upharpoonright}\mathrm{I}^2
\quad
\begin{array}{l}\lambda\sigma.\sigma(\mathsf{can});\\ \lambda\mathscr{F}.\Diamond\mathscr{F}(\mathsf{id}_{et});\\ \mathrm{S}_{f,+}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright(\mathrm{VP}_{f,\alpha}/\mathrm{VP}_{b,\alpha}))\end{array}
}{\varphi_3\circ\mathsf{can}\circ\varphi_1;\ \Diamond\,\mathscr{P}(P);\ \mathrm{S}_{f,+}}}{\mathsf{can}\circ\varphi_1;\ \lambda\mathscr{P}.\Diamond\,\mathscr{P}(P);\ (\mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha})\backslash\mathrm{S}_{f,+}}{\upharpoonright}\mathrm{I}^3}{\mathsf{can};\ \lambda P\lambda\mathscr{P}.\Diamond\,\mathscr{P}(P);\ ((\mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha})\backslash\mathrm{S}_{f,+})/\mathrm{VP}_{b,\alpha}}{\upharpoonright}\mathrm{I}^1
$$

In short, in our type-logical setup, alternative lexical signs that correspond to the lexical entries for the relevant expressions that are directly specified in the lexicon in lexicalist theories of syntax are all derivable as theorems from the more abstract, higher-order entries we have posited above. This is essentially the consequence of the slanting lemma (whose basic form is shown in (8) in Appendix B) in the revised system augmented with the polarity markings. Significantly, the polarity markings ensure that slanting of the higher-order modals and negation preserves the correct scope relations between these operators.

The formal derivability of the lower-order entry from the higher-order entry is an interesting and useful result, as it potentially illuminates the deeper relationship between the 'transformational' and 'lexicalist' analyses of auxiliaries in the different traditions of the generative grammar literature. The two approaches have tended to be seen as reflecting fundamentally incompatible assumptions about the basic architecture of grammar, but if a formal connection can be established between the two at an abstract level by making certain (not totally implausible) assumptions, then the two may not be as different from each other as they have appeared to be throughout the whole history of the controversy between the transformational and non-transformational approaches to syntax. In any event, we take our result above to indicate that the logic-based setup of Type-Logical Grammar can be fruitfully employed for the purpose of meta-comparison of different approaches to grammatical phenomena in the syntactic literature.

3.4 Slanting and Coordination

The slanting lemma moreover plays a crucial role in deriving the correct scope relations in certain examples involving coordination of higher-order operators. For example, consider the conjunction of modals in (20).

(20) Every physicist can and should learn how to teach quantum mechanics to the undergratuate literature majors.

There is a reading for this sentence in which the two modals outscope the subject universal quantifier in each conjunct ('it is possible that every physicist learns …and it is deontically necessary that every physicist learns…').

Assuming that *and* is of type (X\X)/X, combining only expressions whose prosodies are strings, it may appear impossible to derive (20) on the relevant reading, since the modals in (20) must be higher-order to outscope the subject quantifier, and therefore must have functional prosodies. In fact, however, a straightforward derivation is available with no additional assumptions or machinery. Note first that the modal auxiliary can be derived in the ((S/VP)\S)/VP Type (see the discussion in Sect. 3.3; the complete derivation is given in (19) in Appendix B):

(21) can; $\lambda P \lambda \mathscr{P}. \Diamond \mathscr{P}(P)$; $((\mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha})\backslash \mathrm{S}_{f,+})/\mathrm{VP}_{b,\alpha}$

By conjoining two such modals via generalized conjunction, we obtain:

(22) $\mathsf{can} \circ \mathsf{and} \circ \mathsf{should}$; $\lambda R \lambda \mathscr{R}. \Diamond \mathscr{R}(R) \wedge \Box \mathscr{R}(R)$; $((\mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha})\backslash \mathrm{S}_{f,+})/\mathrm{VP}_{b,\alpha}$

We apply this functor first to the sign with VP type derived for *learn how to teach quantum mechanics to the undergratuate literature majors*, and finally to the slanted version of the quantified subject *every physicist*, derivable as in (23):

(23)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi_1; y; \mathrm{NP}]^1 \quad [\varphi_2; P; \mathrm{VP}_{f,\alpha}]^2}{\varphi_1 \circ \varphi_2;\ P(y);\ \mathrm{S}_{f,\alpha}}}{\lambda\varphi_1.\varphi_1 \circ \varphi_2;\ \lambda y.P(y);\ \mathrm{S}_{f,\alpha}{\upharpoonright}\mathrm{NP}}{\upharpoonright}\mathrm{I}^1 \quad \begin{array}{l}\lambda\sigma_1.\sigma_1(\textsf{every} \circ \textsf{physicist});\\ \boldsymbol{\forall}_{\mathbf{phys}};\mathrm{S}_{f,\alpha}{\upharpoonright}(\mathrm{S}_{f,\alpha}{\upharpoonright}\mathrm{NP})\end{array}}{\textsf{every} \circ \textsf{physicist} \circ \varphi_2;\ \boldsymbol{\forall}_{\mathbf{phys}}(\lambda y.P(y));\ \mathrm{S}_{f,\alpha}}}{\textsf{every} \circ \textsf{physicist};\ \lambda P.\boldsymbol{\forall}_{\mathbf{phys}}(\lambda y.P(y));\ \mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha}}{\upharpoonright}\mathrm{I}^2$$

This yields the following result, with the correct semantic translation for (20):

(24)

$$\dfrac{\begin{array}{c}(23)\\ \begin{array}{l}\textsf{every} \circ \textsf{physicist};\\ \boldsymbol{\forall}_{\mathbf{phys}};\mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha}\end{array}\end{array} \quad \dfrac{\begin{array}{c}(22)\\ \begin{array}{l}\textsf{can} \circ \textsf{and} \circ \textsf{should};\\ \lambda R\lambda\mathscr{R}.\Diamond\mathscr{R}(R) \wedge \Box\mathscr{R}(R);\\ ((\mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha})\backslash\mathrm{S}_{f,+})/\mathrm{VP}_{b,\alpha}\end{array}\end{array} \quad \begin{array}{c}\vdots\\ \textsf{learn}\ldots;\ \mathbf{LHT};\ \mathrm{VP}_{b,\alpha}\end{array}}{\begin{array}{l}\textsf{can} \circ \textsf{and} \circ \textsf{should} \circ \textsf{learn} \circ \ldots;\\ \lambda\mathscr{R}.\Diamond\mathscr{R}(\mathbf{LHT}) \wedge \Box\mathscr{R}(\mathbf{LHT});(\mathrm{S}_{f,\alpha}/\mathrm{VP}_{f,\alpha})\backslash\mathrm{S}_{f,+}\end{array}}}{\begin{array}{l}\textsf{every} \circ \textsf{physicist} \circ \textsf{can} \circ \textsf{and} \circ \textsf{should} \circ \textsf{learn} \circ \ldots;\\ \Diamond\boldsymbol{\forall}_{\mathbf{phys}}(\mathbf{LHT}) \wedge \Box\boldsymbol{\forall}_{\mathbf{phys}}(\mathbf{LHT});\mathrm{S}_{f,+}\end{array}}$$

3.5 VP Fronting

Work in phrase-structure-theoretic approaches to the syntax/semantics interface has tended to follow the treatment of negation in Kim and Sag [7], which distinguishes *not* (and possibly *never*) as complements of auxiliaries from *not* as adjuncts to the auxiliaries' VP complements. This approach is supposedly motivated by the ambiguity of sentences with *could not/never* sequences, where both $\neg > \Diamond$ and $\Diamond > \neg$ readings are available.

There is, in fact, a very sparse empirical base in English for this phrase structure-based analysis of modal/negation scoping relations, a fact that Kim and Sag [7] themselves tacitly acknowledge. One of the few lines of argument that Kim and Sag [7] appeal to is the fact that fronted VPs containing *not* adjuncts are always interpreted with narrowly scoping negation, as illustrated in (25):

(25) ... and NOT vote, you certainly can __, if the nominees are all second-rate.

Data of this sort are intended to provide empirical support for the putative correlation of phrase structural position with the scope of negation, and the particular empirical fact about fronted VP with negation exemplified by (25) needs to be accounted for in any approach to modal/negation interaction in any theoretical framework. But there seems no strong reason to prefer the phrase structural account to any of a number of alternatives.

Indeed, we can readily capture the pattern in (25) in our approach by requiring that topicalization clauses are subject to polarity requirements which entail narrow scope for the negation within the fronted VP. We start by presenting the topicalization operator in (26a) (with the polymorphic syntactic type X), illustrating its ordinary operation to produce (26b) (where the semantics is simply an identity function, since we ignore the pragmatic effects of topicalization):[6]

[6] Here and below, ϵ denotes the null string.

(26) a. $\lambda\varphi\lambda\sigma.\varphi\circ\sigma(\boldsymbol{\epsilon});\ \lambda P\lambda\mathscr{C}.\mathscr{C}(P);\ (\mathrm{S}_{f,\beta}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright\mathrm{X}))\upharpoonright\mathrm{X}$ where $\beta\in\{+,-\}$

b. ...and vote, John can __.

c. #...and not vote, John can __. ($\neg > \Diamond$)

The derivation for (26b) is given in (27).

(27)

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{can};\ \lambda P\lambda y.\Diamond P(y);\ \mathrm{VP}_{f,+}/\mathrm{VP}_{b,\alpha} \quad \begin{bmatrix}\varphi_1;\\ Q;\ \mathrm{VP}_{b,\alpha}\end{bmatrix}^1}{\mathsf{can}\circ\varphi_1;\ \lambda y.\Diamond Q(y);\ \mathrm{VP}_{f,+}} \quad \mathsf{john};\ \mathbf{j};\ \mathrm{NP}}{\mathsf{john}\circ\mathsf{can}\circ\varphi_1;\ \Diamond Q(\mathbf{j});\ \mathrm{S}_{f,+}}}{\lambda\varphi_1.\mathsf{john}\circ\mathsf{can}\circ\varphi_1;\ \lambda Q.\Diamond Q(\mathbf{j});\ \mathrm{S}_{f,+}\upharpoonright\mathrm{VP}_{b,\alpha}}{\upharpoonright}\mathrm{I}^1 \quad \dfrac{\begin{matrix}\lambda\varphi\lambda\sigma.\varphi\circ\sigma(\boldsymbol{\epsilon});\\ \lambda P\lambda\mathscr{C}.\mathscr{C}(P);\\ (\mathrm{S}_{f,\beta}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright\mathrm{X}))\upharpoonright\mathrm{X}\end{matrix} \quad \begin{matrix}\mathsf{vote};\\ \mathbf{vote};\\ \mathrm{VP}_{b,\alpha}\end{matrix}}{\begin{matrix}\lambda\sigma.\mathsf{vote}\circ\sigma(\boldsymbol{\epsilon});\\ \lambda\mathscr{C}.\mathscr{C}(\mathbf{vote});\\ \mathrm{S}_{f,\beta}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright\mathrm{VP}_{b,\alpha})\end{matrix}}}{\mathsf{vote}\circ\mathsf{john}\circ\mathsf{can}\circ\boldsymbol{\epsilon};\ \Diamond\mathbf{vote}(\mathbf{j});\ \mathrm{S}_{f,+}.}$$

The requirement on the topicalization operator in (26a) effectively means that $\mathrm{S}_{\varnothing}$ is 'too small' to host a topicalized phrase. That is, in order to license topicalization, the clause needs to have already 'fixed' the polarity value to either $+$ or $-$. This condition turns out to have the immediate effect or enforcing narrow scope on negation in fronted VPs.

To see how this condition works, let's suppose it did not hold; that is, suppose that β could take any of the three polarity values. Then the following would be one way in which *not* inside a topicalized phrase would outscope the modal.

(28) a.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{bmatrix}\varphi_4;\\ Q;\ \mathrm{VP}_{b,\varnothing}\end{bmatrix}^1 \quad \begin{bmatrix}\varphi_5;\\ g;\ \mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}\end{bmatrix}^2}{\varphi_5\circ\varphi_4;\ g(Q);\ \mathrm{VP}_{b,\varnothing}} \quad \begin{matrix}\mathsf{john};\\ \mathbf{j};\\ \mathrm{NP}\end{matrix}}{\mathsf{john}\circ\varphi_5\circ\varphi_4;\ g(Q)(\mathbf{j});\ \mathrm{S}_{b,\varnothing}}}{\begin{matrix}\lambda\varphi_5.\mathsf{john}\circ\varphi_5\circ\varphi_4;\\ \lambda g.g(Q)(\mathbf{j});\mathrm{S}_{b,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing})\end{matrix}}{\upharpoonright}\mathrm{I}^2 \quad \begin{matrix}\lambda\sigma_0.\sigma_0(\mathsf{can});\\ \lambda\mathscr{F}.\Diamond\mathscr{F}(\mathsf{id}_{et});\\ \mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{S}_{b,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{matrix}}{\mathsf{john}\circ\mathsf{can}\circ\varphi_4;\ \Diamond Q(\mathbf{j});\ \mathrm{S}_{f,\varnothing}}}{\lambda\varphi_4.\mathsf{john}\circ\mathsf{can}\circ\varphi_4;\ \Diamond Q(\mathbf{j});\ \mathrm{S}_{f,\varnothing}\upharpoonright\mathrm{VP}_{b,\varnothing}}{\upharpoonright}\mathrm{I}^1$$

b.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}(28\mathrm{a})\\ \lambda\varphi_4.\mathsf{john}\circ\\ \mathsf{can}\circ\varphi_4;\\ \Diamond Q(\mathbf{j});\\ \mathrm{S}_{f,\varnothing}\upharpoonright\mathrm{VP}_{b,\varnothing}\end{matrix} \quad \dfrac{\dfrac{\begin{matrix}\mathsf{vote};\\ \mathbf{vote};\\ \mathrm{VP}_{b,\varnothing}\end{matrix} \quad \begin{bmatrix}\varphi_1;\\ f;\\ \mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}\end{bmatrix}^3}{\begin{matrix}\varphi_1\circ\mathsf{vote};\\ f(\mathbf{vote});\mathrm{VP}_{b,\varnothing}\end{matrix}} \quad \begin{matrix}\lambda\varphi_2\lambda\sigma_1.\\ \varphi_2\circ\sigma_1(\boldsymbol{\epsilon});\\ \lambda\alpha\lambda\mathscr{C}.\mathscr{C}(\alpha);\\ (\mathrm{S}_{f,\beta}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright\mathrm{X}))\upharpoonright\mathrm{X}\end{matrix}}{\begin{matrix}\lambda\sigma_1.\varphi_1\circ\mathsf{vote}\circ\sigma_1(\boldsymbol{\epsilon});\\ \lambda\mathscr{C}.\mathscr{C}(f(\mathbf{vote}));\mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{S}_{f,\varnothing}\upharpoonright\mathrm{VP}_{b,\varnothing})\end{matrix}}}{\varphi_1\circ\mathsf{vote}\circ\mathsf{john}\circ\mathsf{can}\circ\boldsymbol{\epsilon};\ \Diamond f(\mathbf{vote})(\mathbf{j});\ \mathrm{S}_{f,\varnothing}}}{\begin{matrix}\lambda\varphi_1.\varphi_1\circ\mathsf{vote}\circ\mathsf{john}\circ\mathsf{can}\circ\boldsymbol{\epsilon};\\ \lambda f.\Diamond f(\mathbf{vote})(\mathbf{j});\mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing})\end{matrix}}{\upharpoonright}\mathrm{I}^3 \quad \begin{matrix}\lambda\sigma.\sigma(\mathsf{not});\\ \lambda\mathscr{G}.\neg\mathscr{G}(\mathsf{id}_{et});\\ \mathrm{S}_{f,-}\upharpoonright(\mathrm{S}_{f,\varnothing}\upharpoonright(\mathrm{VP}_{b,\varnothing}/\mathrm{VP}_{b,\varnothing}))\end{matrix}}{\mathsf{not}\circ\mathsf{vote}\circ\mathsf{john}\circ\mathsf{can};\ \neg\Diamond\mathbf{vote}(\mathbf{j});\ \mathrm{S}_{f,-}}$$

Here, the derivation uses the NPI version of *can*, in order to license the negation wide scope reading. Since the negation is inside the topicalized phrase rather than the main clause, topicalization needs to be hosted by a clause to which negation hasn't yet combined. But this is precisely the possibility that the restriction $\beta\in\{+,-\}$ excludes (note the conflict in the greyed-in expressions). Using the other version of *can* will only produce the other scopal relation (one in which the modal outscopes negation), so, this option is not available for licensing the reading in question. Thus neither version of *can* admits a derivation resulting in wide scope for topicalized negation, and the same result holds for all NPI (i.e. narrow-scoping) modals.

There is in contrast no difficulty in obtaining the narrow scope interpretation of negation, as shown in (26c), with α and $\delta = -$, and $\beta = +$.

(29) a.

$$\dfrac{\dfrac{\begin{array}{l}\mathsf{vote};\\ \mathbf{vote};\\ \mathrm{VP}_{b,\varnothing}\end{array} \quad \mathsf{not};\ \lambda Q\lambda y.\neg Q(y);\ \mathrm{VP}_{b,-}/\mathrm{VP}_{b,\varnothing}}{\mathsf{not}\circ\mathsf{vote};\ \lambda y.\neg\mathbf{vote}(y);\ \mathrm{VP}_{b,-}} \quad \begin{array}{l}(15)\\ \lambda\varphi_2\lambda\sigma_1.\varphi_2\circ\sigma_1(\boldsymbol{\epsilon});\\ \lambda\alpha\lambda\mathscr{C}.\mathscr{C}(\alpha);\ (\mathrm{S}_{f,\beta}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright \mathrm{X}))\upharpoonright \mathrm{X}\end{array}}{\lambda\sigma_1.\mathsf{not}\circ\mathsf{vote}\circ\sigma_1(\boldsymbol{\epsilon});\ \lambda\mathscr{C}.\mathscr{C}(\lambda y.\neg\mathbf{vote}(y));\ \mathrm{S}_{f,\beta}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright\mathrm{VP}_{b,-})}$$

b.

$$\dfrac{\begin{array}{l}(29a)\\ \lambda\sigma_1.\mathsf{not}\circ\mathsf{vote}\circ\sigma_1(\boldsymbol{\epsilon});\\ \lambda\mathscr{C}.\mathscr{C}(\lambda y.\neg\mathbf{vote}(y));\\ \mathrm{S}_{f,\beta}\upharpoonright(\mathrm{S}_{f,\beta}\upharpoonright\mathrm{VP}_{b,-})\end{array} \quad \dfrac{\dfrac{\begin{array}{l}\lambda\sigma_0.\sigma_0(\mathsf{can});\\ \lambda\mathscr{F}.\Diamond\mathscr{F}(\mathsf{id}_{et});\\ \mathrm{S}_{f,+}\upharpoonright(\mathrm{S}_{b,-}\upharpoonright(\mathrm{VP}_{b,\alpha}/\mathrm{VP}_{b,\alpha}))\end{array} \quad \dfrac{\dfrac{\dfrac{\begin{bmatrix}\varphi_1;\\ P;\\ \mathrm{VP}_{b,-}\end{bmatrix}^1 \quad \begin{bmatrix}\varphi_3;\\ f;\\ \mathrm{VP}_{b,-}/\mathrm{VP}_{b,-}\end{bmatrix}^3}{\varphi_3\circ\varphi_1;\ f(P);\ \mathrm{VP}_{b,-}} \quad \begin{array}{l}\mathsf{john};\\ \mathbf{j};\\ \mathrm{NP}\end{array}}{\begin{array}{c}\lambda\varphi_3.\mathsf{john}\circ\varphi_3\circ\varphi_1;\\ \lambda P.f(P)(\mathbf{j});\mathrm{S}_{f,+}\end{array}}}{\begin{array}{c}\lambda\varphi_3.\mathsf{john}\circ\varphi_3\circ\varphi_1;\\ \lambda P.f(P)(\mathbf{j});\mathrm{S}_{f,+}\upharpoonright(\mathrm{VP}_{b,-}/\mathrm{VP}_{b,-})\end{array}}\upharpoonright\mathrm{I}^3}{\mathsf{john}\circ\mathsf{can}\circ\varphi_1;\ \Diamond P(\mathbf{j});\ \mathrm{S}_{f,+}}}{\lambda\varphi_1.\mathsf{john}\circ\mathsf{can}\circ\varphi_1;\ \lambda P.\Diamond P(\mathbf{j});\ \mathrm{S}_{f,+}\upharpoonright\mathrm{VP}_{b,-}}\upharpoonright\mathrm{I}^1}{\mathsf{not}\circ\mathsf{vote}\circ\mathsf{john}\circ\mathsf{can}\circ\boldsymbol{\epsilon};\ \Diamond\neg\mathbf{vote}(\mathbf{j});\ \mathrm{S}_{f,+}}$$

The slanted version of *not* combines freely with its VP argument to yield a topicalized VP_{-}, but the type of the mother—in particular, its polarity specification—is determined by the highest scoping operator, *can*, which yields a positive polarity clause.

4 Conclusion

In this paper, we proposed an explicit analysis of scope interactions between modal auxiliaries and negation in English in Type-Logical Grammar. The proposed analysis builds on two previous works in somewhat different research traditions: (i) Iatridou and Zeijlstra's [6] configurational analysis of modal auxiliaries that captures their scopal properties in terms of the distinction between PPI and NPI modals; (ii) Kubota and Levine's Kubota and Levine's [8,10] analysis of modal auxiliaries in Type-Logical Grammar as higher-order operators that take clausal scope (unlike the more traditional VP/VP analysis in lexicalist theories such as CG and G/HPSG). Our analysis captures the different scoping patterns of different types of modals via the polarity-marking distinction, whose core analytic idea is due to Iatridou and Zeijlstra, but it does so without making recourse to the notion of reconstruction, which is a type of lowering movement whose exact formal implementation in minimalist syntax is somewhat unclear. Our analysis moreover clarifies the relationship between configurational (or transformational) and non-transformational analyses of modal auxiliaries by showing precisely how the latter type of analysis can be thought of as a derivative of the former type of analysis when both are recast within a logical calculus that allows one to *derive* (in the literal sense of 'derive' in formal logic) certain types of lexical descriptions from more abstract and seemingly unrelated lexical descriptions. We take this result to be highly illuminating, as it helps clarify a deeper connection between different stripes of syntactic research that is in no sense obvious unless one takes a logical perspective on grammatical composition.

Acknowledgments. We would like to thank two anonymous reviewers for helpful comments. This work was supported by JSPS KAKENHI JP15K16732 and the NINJAL collaborative research project 'Cross-linguistic Studies of Japanese Prosody and Grammar'.

A Hybrid Type-Logical Grammar

A.1 Syntactic Types

(30)
$\mathcal{A} := \{\ \mathrm{S}, \mathrm{NP}, \mathrm{N}, \ldots \}$ (atomic type)
$\mathcal{D} := \mathcal{A} \mid \mathcal{D}\backslash\mathcal{D} \mid \mathcal{D}/\mathcal{D}$ (directional type)
$\mathcal{T} := \mathcal{D} \mid \mathcal{T}{\upharpoonright}\mathcal{T}$ (type)

Note: The algebra of syntactic types is *not* a free algebra generated over the set of atomic types with the three binary connectives /, \, and $\upharpoonright$. Specifically, given the definitions in (30), in Hybrid TLG, a vertical slash cannot occur 'under' a directional slash. Thus, $\mathrm{S}/(\mathrm{S}{\upharpoonright}\mathrm{NP})$ is not a well-formed syntactic type. This is a deliberate design, and Hybrid TLCG differs from closely related variants of TLG (such as the Displacement Calculus Morrill [14] and NL_λ Barker and Shan [1]) in this respect.

A.2 Mapping from Syntactic Types to Semantic Types

(31)
a. $\mathsf{Sem}(\mathrm{NP}) = e$
b. $\mathsf{Sem}(\mathrm{S}) = t$
c. $\mathsf{Sem}(\mathrm{N}) = e \to t$

(32) For any complex syntactic category of the form α/β (or $\beta\backslash\alpha$, $\alpha{\upharpoonright}\beta$), $\mathsf{Sem}(\alpha/\beta)$ $(= \mathsf{Sem}(\beta\backslash\alpha) = \mathsf{Sem}(\alpha{\upharpoonright}\beta)) = \mathsf{Sem}(\beta) \to \mathsf{Sem}(\alpha)$

A.3 Mapping from Syntactic Types to Prosodic Types

(33) For any directional type $\mathcal{D}$, $\mathsf{Pros}(\mathcal{D}) = \textbf{st}$ (with **st** for 'strings').

(34) For any complex syntactic type $A{\upharpoonright}B$ involving the vertical slash $\upharpoonright$, $\mathsf{Pros}(A{\upharpoonright}B) = \mathsf{Pros}(B) \to \mathsf{Pros}(A)$.

A.4 Deductive Rules

(35) **Connective** **Introduction** **Elimination**

$/$

$$\cfrac{\cfrac{\begin{array}{ccc} \vdots & \underline{[\varphi;\, x;\, A]^n} & \vdots \\ \vdots & \vdots & \vdots \end{array}}{b \circ \varphi;\, \boldsymbol{\mathcal{F}};\, B}}{b;\, \lambda x.\boldsymbol{\mathcal{F}};\, B/A}\;/\mathrm{I}^n \qquad \cfrac{a;\, \boldsymbol{\mathcal{F}};\, A/B \quad b;\, \boldsymbol{\mathcal{G}};\, B}{a \circ b;\, \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{G}});\, A}\;/\mathrm{E}$$

$\backslash$

$$\cfrac{\cfrac{\begin{array}{ccc} \vdots & \underline{[\varphi;\, x;\, A]^n} & \vdots \\ \vdots & \vdots & \vdots \end{array}}{\varphi \circ b;\, \boldsymbol{\mathcal{F}};\, B}}{b;\, \lambda x.\boldsymbol{\mathcal{F}};\, A\backslash B}\;\backslash\mathrm{I}^n \qquad \cfrac{b;\, \boldsymbol{\mathcal{G}};\, B \quad a;\, \boldsymbol{\mathcal{F}};\, B\backslash A}{b \circ a;\, \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{G}});\, A}\;\backslash\mathrm{E}$$

$\upharpoonright$

$$\cfrac{\cfrac{\begin{array}{ccc} \vdots & \underline{[\varphi;\, x;\, A]^n} & \vdots \\ \vdots & \vdots & \vdots \end{array}}{b;\, \boldsymbol{\mathcal{F}};\, B}}{\lambda\varphi.b;\, \lambda x.\boldsymbol{\mathcal{F}};\, B\upharpoonright A}\;\upharpoonright\mathrm{I}^n \qquad \cfrac{a;\, \boldsymbol{\mathcal{F}};\, A\upharpoonright B \quad b;\, \boldsymbol{\mathcal{G}};\, B}{a(b);\, \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{G}});\, A}\;\upharpoonright\mathrm{E}$$

Notes: Corresponding to the asymmetry in the status of the directional slashes ($/$, $\backslash$) and the vertical slash ($\upharpoonright$) in the definitions of syntactic types, there is an asymmetry in the definitions of the deductive rules for the two types of slashes.

Note in particular that in the Introduction rules for $/$ ($\backslash$), instead of lambda binding, the prosodic variable of the hypothesis that is withdrawn is *removed* from the prosodic term on the condition that it appears on the right (left) edge of the prosody of the expression that feeds into the rule. (One way to make sense of this is to take the $/,\backslash$ Introduction rules as abbreviations of theorems in which the variable is first bound by left and right lambda abstraction as usual [23], immediately followed by a step of feeding an empty string to the prosodic function thus obtained.)

So far as we can tell, fixing the prosodic type to be **st** for directional (i.e. Lambek) syntactic types is crucial for ensuring the particular way in which the directional and vertical slashes interact with one another in the various Slanting lemma and related results (which play important roles in the linguistic analyses we have presented above).

References

1. Barker, C., Shan, C.: Continuations and Natural Language. OUP, Oxford (2015)
2. Bernardi, R.: Reasoning with polarity in categorial type logic. Ph.D. thesis, University of Utrecht (2002)
3. Dowty, D.: The role of negative polarity and concord marking in natural language reasoning. In: Harvey, M., Santelmann, L. (eds.) Proceedings from Semantics and Linguistic Theory IV, pp. 114–144. Cornell University, Ithaca (1994)

4. de Groote, P.: Towards abstract categorial grammars. In: Association for Computational Linguistics, 39th Annual Meeting and 10th Conference of the European Chapter, pp. 148–155 (2001)
5. Hendriks, H.: Studied flexibility. Ph.D. thesis, University of Amsterdam, Amsterdam (1993)
6. Iatridou, S., Zeijlstra, H.: Negation, polarity and deontic modals. Linguist. Inq. **44**, 529–568 (2013)
7. Kim, J.B., Sag, I.: Negation without head movement. Nat. Lang. Linguist. Theory **20**, 339–412 (2002)
8. Kubota, Y., Levine, R.: Gapping as like-category coordination. In: Béchet, D., Dikovsky, A. (eds.) Logical Aspects of Computational Linguistics 2012, vol. 7351, pp. 135–150. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-31262-5_9
9. Kubota, Y., Levine, R.: Against ellipsis: arguments for the direct licensing of 'non-canonical' coordinations. Linguist. Philos. **38**(6), 521–576 (2015)
10. Kubota, Y., Levine, R.: Gapping as hypothetical reasoning. Nat. Lang. Linguist. Theory **34**(1), 107–156 (2016)
11. Levine, R.: The modal need VP gap (non)anomaly. In: Csipak, E., Eckardt, R., Liu, M., Sailer, M. (eds.) Beyond 'Any' and 'Ever': New Perspectives on Negative Polarity Sensitivity, pp. 241–265. Mouton de Gruyter, Berlin (2013)
12. Martin, S., Pollard, C.: A dynamic categorial grammar. In: Morrill, G., Muskens, R., Osswald, R., Richter, F. (eds.) Formal Grammar 2014. LNCS, vol. 8612, pp. 138–154. Springer, Heidelberg (2014). https://doi.org/10.1007/978-3-662-44121-3_9
13. Morrill, G.: Type Logical Grammar: Categorial Logic of Signs. Kluwer, Dordrecht (1994)
14. Morrill, G.: Categorial Grammar: Logical Syntax, Semantics, and Processing. OUP, Oxford (2010)
15. Morrill, G., Valentín, O.: A reply to Kubota and Levine on gapping. Nat. Lang. Linguist. Theory **35**(1), 257–270 (2017)
16. Oehrle, R.T.: Boolean properties in the analysis of gapping. In: Huck, G.J., Ojeda, A.E. (eds.) Syntax and Semantics: Discontinuous Constituency, vol. 20, pp. 203–240. Academic Press, Cambridge (1987)
17. Pogodalla, S., Pompigne, F.: Controlling extraction in abstract categorial grammars. In: de Groote, P., Nederhof, M.J. (eds.) FG 2010, FG 2011. LNCS, pp. 162–177. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-32024-8_11
18. Puthawala, D.: Stripping isn't so mysterious, or anomalous scope, either. In: Foret, A., Kobele, G., Pogodalla, S. (eds.) FG 2018. LNCS, vol. 10950, pp. 102–120. Springer, Heidelberg (2018). https://doi.org/10.1007/978-3-662-57784-4_6
19. Richter, F., Soehn, J.P.: Braucht niemanden zu scheren: a survey of NPI licensing in German. In: Müller, S. (ed.) The Proceedings of the 13th International Conference on Head-Driven Phrase Structure Grammar, pp. 421–440. CSLI Publications, Stanford (2006)
20. Siegel, M.A.: Compositionality, case, and the scope of auxiliaries. Linguist. Philos. **10**(1), 53–75 (1987)
21. Steedman, M.: Taking Scope. MIT Press, Cambridge (2012)

22. Szabolcsi, A.: Positive polarity - negative polarity. Nat. Lang. Linguist. Theory **22**, 409–452 (2004)
23. Wansing, H.: Formulas-as-types for a hierarchy of sublogics of intuitionistic propositional logic. In: Pearce, D., Wansing, H. (eds.) All-Berlin 1990. LNCS, vol. 619, pp. 125–145. Springer, Heidelberg (1992). https://doi.org/10.1007/BFb0031928

Subset Models for Justification Logic

Eveline Lehmann and Thomas Studer(✉)

Institute of Computer Science, University of Bern, Bern, Switzerland
{lehmann,tstuder}@inf.unibe.ch

Abstract. We introduce a new semantics for justification logic based on subset relations. Instead of using the established and more symbolic interpretation of justifications, we model justifications as sets of possible worlds. We introduce a new justification logic that is sound and complete with respect to our semantics. Moreover, we present another variant of our semantics that corresponds to traditional justification logic.

These types of models offer us a versatile tool to work with justifications, e.g. by extending them with a probability measure to capture uncertain justifications. Following this strategy we will show that they subsume Artemov's approach to aggregating probabilistic evidence.

Keywords: Justification logic · Semantics · Probabilistic evidence

1 Introduction

Justification logic is a variant of modal logic that includes terms representing explicit evidence. A formula of the form $t : A$ means that t *justifies* A (or t *represents evidence for* A, or t *is a proof of* A). Justification logic has been introduced by Artemov [3,4] to give a classical provability interpretation to $\mathsf{S4}$. Later it turned out that this approach is not only useful in proof theory [4,15] but also in epistemic logic [5,6,11,12]. For a general overview on justification logic, we refer to [2,8,16].

There are various kinds of semantics available for justification logic. Most of them interpret justification terms in a symbolic way. In provability interpretations [4,15], terms represent (codes of) proofs in formal system like Peano arithmetic. In Mkrtychev models [18], which are used to obtain decidability, terms are represented as sets of formulas. In Fitting models [13], the evidence relation maps pairs of terms and possible worlds to sets of formulas. In modular models [7,14], the logical type a justification is a set of formulas, too. Notable exceptions are [1,9] where terms are interpreted as sets of possible worlds. However, these papers do not consider the usual term structure of justification logics. Also note that there are topological approaches to evidence available [10,21,22], which, however, do not feature justifications explicitly in their language.

This work was supported by the Swiss National Science Foundation grant 200021_165549.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 433–449, 2019.
https://doi.org/10.1007/978-3-662-59533-6_26

It is the aim of this paper to provide a new semantics, called *subset semantics* for justification logic that interprets terms as sets of possible worlds and operations on terms as operations on sets of possible worlds. We will then say that $t : A$ is true if A is true in all worlds belonging to the interpretation of t. We give a systematic study of this new semantics including soundness and completeness results and we show that the approach of [1] can be seen as a special case of our semantics.

Usually, justification logic includes an application operator that represents modus ponens (MP) on the level of terms. We provide two approaches to handle this operator in our semantics. The first is to include a new constant $\mathsf{c}^\star$, which is interpreted as the set of all worlds closed under (MP) and then use this new constant to define an application operator. Unlike in traditional justification logic, this application operator will be commutative. The second way is to include a (non-commutative) application operator directly. However, this leads to some quite cumbersome definitions.

Another difference between our semantics and many other semantics for justification logic is that we allow non-normal (impossible) worlds. They are usually needed to model the fact that agents are not omniscient and that they do not see all consequences of the facts they are already aware of. In an impossible world both A and $\neg A$ may be true or none of them. This way of using impossible worlds was investigated by Rantala [19,20].

We start with presenting the $\mathsf{c}^\star$-subset models with the corresponding syntax, axioms and semantics and proving soundness and completeness. In a second part we will present the alternative approach, i.e. keeping the (j)-axiom and dealing with some cumbersome definitions within the semantics. It will be shown that the corresponding models are sound and complete as well. In a last section we will show that $\mathsf{c}^\star$-subset models can be used to reason about uncertain knowledge by referring to Artemov's work on aggregating probabilistic evidence.

2 $\mathsf{L}^\star_{\mathsf{CS}}$-Subset Models

2.1 Syntax

Justification terms are built from countably many constants c_i and variables x_i and the special and unique constant $\mathsf{c}^\star$ according to the following grammar:

$$t ::= c_i \mid x_i \mid \mathsf{c}^\star \mid (t + t) \mid !t$$

The set of terms is denoted by Tm. The operation $+$ is left-associative.

Formulas are built from countably many atomic propositions p_i and the symbol $\perp$ according to the following grammar:

$$F ::= p_i \mid \perp \mid F \to F \mid t : F$$

The set of atomic propositions is denoted by Prop and the set of all formulas is denoted by $\mathcal{L}_J$. The other classical Boolean connectives $\neg, \top, \wedge, \vee, \leftrightarrow$ are defined as usual.

We investigate a family of justification logics that differ in their axioms and how the axioms are justified. We have two sets of axioms, the first axioms are:

cl all axioms of classical propositional logic;
j+ $s : A \vee t : A \rightarrow (s + t) : A$;
jc* $\mathsf{c}^\star : A \wedge \mathsf{c}^\star : (A \rightarrow B) \rightarrow \mathsf{c}^\star : B$.

The set of these axioms is denoted by $\mathsf{L}^\star_\alpha$.
There is another set of axioms:

j4 $t : A \rightarrow !t : (t : A)$;
jd $t : \bot \rightarrow \bot$;
jt $t : A \rightarrow A$.

This set is denoted by $\mathsf{L}^\star_\beta$. It is easy to see that **jd** is a special case of **jt**. By $\mathsf{L}^\star$ we denote all logics that are composed from the whole set $\mathsf{L}^\star_\alpha$ and some subset of $\mathsf{L}^\star_\beta$. Moreover, a justification logic $\mathsf{L}^\star$ is defined by the set of axioms and its constant specification CS that determines which constant justifies which axiom. So the constant specification is a set

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom of } \mathsf{L}^\star\}$$

In this sense $\mathsf{L}^\star_{\mathsf{CS}}$ denotes the logic $\mathsf{L}^\star$ with the constant specification CS. To deduce formulas in $\mathsf{L}^\star_{\mathsf{CS}}$ we use a Hilbert system given by $\mathsf{L}^\star$ and the rules modus ponens:

$$\frac{A \qquad A \rightarrow B}{B}\ (\text{MP})$$

and axiom necessitation

$$\frac{}{\underbrace{!\ldots!}_{n} : \underbrace{!\ldots!}_{n-1} : \ \ldots : !!c : !c : c : A}\ (\text{AN!}) \quad \forall n \in \mathbb{N}, \text{ where } (c, A) \in \mathsf{CS}$$

2.2 Semantics

Definition 1 ($\mathsf{L}^\star_{\mathsf{CS}}$-subset models). *Given some logic $\mathsf{L}^\star$ and some constant specification CS, then an $\mathsf{L}^\star_{\mathsf{CS}}$-subset model $\mathcal{M} = (W, W_0, V, E)$ is defined by:*

- *W is a set of objects called worlds.*
- *$W_0 \subseteq W$ and $W_0 \neq \emptyset$.*
- *$V : W \times \mathcal{L}_J \rightarrow \{0, 1\}$ such that for all $\omega \in W_0$, $t \in \mathsf{Tm}$, $F, G \in \mathcal{L}_J$:*
  - *$V(\omega, \bot) = 0$;*
  - *$V(\omega, F \rightarrow G) = 1$ iff $V(\omega, F) = 0$ or $V(\omega, G) = 1$;*
  - *$V(\omega, t : F) = 1$ iff $E(\omega, t) \subseteq \{ \upsilon \in W \mid V(\upsilon, F) = 1 \}$.*
- *$E : W \times \mathsf{Tm} \rightarrow \mathcal{P}(W)$ that meets the following conditions where we use*

$$[A] := \{\omega \in W \mid V(\omega, A) = 1\}. \qquad (1)$$

For all $\omega \in W_0$, and for all $s, t \in \mathsf{Tm}$:

- *$E(\omega, s + t) \subseteq E(\omega, s) \cap E(\omega, t)$;*
- *$E(\omega, \mathsf{c}^\star) \subseteq W_{MP}$ where W_{MP} is the set of deductively closed worlds, see below;*
- *if **jd** $\in \mathsf{L}^\star$, then $\exists \upsilon \in W_0$ with $\upsilon \in E(\omega, t)$;*
- *if **jt** $\in \mathsf{L}^\star$, then $\omega \in E(\omega, t)$;*
- *if **j4** $\in \mathsf{L}^\star$, then*

$$E(\omega, !t) \subseteq \{ \upsilon \in W \mid \forall F \in \mathcal{L}_J \ (V(\omega, t : F) = 1 \Rightarrow V(\upsilon, t : F) = 1) \};$$

- *for all $n \in \mathbb{N}$ and for all $(c, A) \in \mathsf{CS} : E(\omega, c) \subseteq [A]$ and*

$$E(\omega, \underbrace{!...!}_{n} c) \subseteq [\underbrace{!...!}_{n-1} c : ....!c : c : A].$$

The set W_{MP} is formally defined as follows:

$$W_{MP} := \{\omega \in W \mid \forall A, B \in \mathcal{L}_J \ ((V(\omega, A) = 1 \text{ and } V(\omega, A \to B) = 1) \text{ implies } V(\omega, B) = 1)\}.$$

So W_{MP} collects all the worlds where the valuation function is closed under modus ponens. W_0 is the set of *normal* worlds. The set $W \setminus W_0$ consists of the *non-normal* worlds. Moreover, using the notation introduced by (1), we can read the condition on V for justification terms $t : F$ as:

$$V(\omega, t : F) = 1 \quad \text{iff} \quad E(\omega, t) \subseteq [F]$$

Since the valuation function V is defined on worlds and formulas, the definition of truth is pretty simple:

Definition 2 (Truth in $\mathsf{L}^\star_{\mathsf{CS}}$-subset models). *Let $\mathcal{M} = (W, W_0, V, E)$ be an $\mathsf{L}^\star_{\mathsf{CS}}$-subset model, $\omega \in W$ and $F \in \mathcal{L}_J$. We define the relation $\Vdash$ as follows:*

$$\mathcal{M}, \omega \Vdash F \quad \textit{iff} \quad V(\omega, F) = 1$$

2.3 Soundness

Since non-normal worlds will not be sound even with respect to the axioms of classical logic, we only have soundness within W_0.

Theorem 3 (Soundness of $\mathsf{L}^\star_{\mathsf{CS}}$-subset models). *For any justification logic $\mathsf{L}^\star_{\mathsf{CS}}$ and any formula $F \in \mathcal{L}_J$:*

$$\mathsf{L}^\star_{\mathsf{CS}} \vdash F \quad \Rightarrow \quad \mathcal{M}, \omega \Vdash F \quad \textit{for all } \mathsf{L}^\star_{\mathsf{CS}}\textit{-subset models } \mathcal{M} \textit{ and all } \omega \in W_0$$

The proof straight forward is by induction on the length on the derivation of F and can be found in [17].

The **j**-axiom $s : (A \to B) \to (t : A \to s \cdot t : B)$ is not part of our logic. Using the $(\mathsf{c}^\star)$-axiom, we can define an application operation such that the **j**-axiom is valid.

Definition 4 (Application). *We introduce a new abbreviation $\cdot$ on terms by:*

$$s \cdot t := s + t + \mathsf{c}^{\star}$$

Lemma 5 (The "j-axiom" follows). *For all $\mathcal{M} = (W, W_0, V, E)$, $\omega \in W_0$, $A, B \in \mathcal{L}_J$ and $s, t \in \mathsf{Tm}$:*

$$\mathcal{M}, \omega \Vdash s : (A \to B) \to (t : A \to s \cdot t : B)$$

The proof is straight forward and can be found in Appendix A.

2.4 Completeness

To prove completeness we will construct a canonical model and then show that for every formula F that is not derivable in $\mathsf{L}^{\star}_{\mathsf{CS}}$, there is a model $\mathcal{M}^C$ with a world $\Gamma \in W_0^C$ s.t. $\mathcal{M}^C, \Gamma \Vdash \neg F$. Before we start with the definition of the canonical model, we must do some preliminary work. We will first prove that our logics are conservative extensions of classical logic. With this result we can argue, that the empty set is consistent and hence can be extended to so-called maximal $\mathsf{L}^{\star}_{\mathsf{CS}}$-consistent sets of formulas. These sets will be used to build the W_0-worlds in the canonical model.

Theorem 6 (Conservativity). *All logics $\mathsf{L}^{\star}$ presented are conservative extensions of the classical logic CL, i.e. for any formula $F \in \mathsf{L}_{\mathsf{cp}}$:*

$$\mathsf{L}^{\star} \vdash F \quad \Leftrightarrow \quad \mathsf{CL} \vdash F$$

The proof is standard and can be found in [17].

Definition 7 (Consistency). *A logical theory L is called consistent, if $\mathsf{L} \nvdash \bot$. A set of formulas $\Gamma \subset \mathcal{L}_J$ is called L-consistent if $\mathsf{L} \nvdash \bigwedge \Sigma \to \bot$ for every finite $\Sigma \subseteq \Gamma$. A set of formulas Γ is called maximal L-consistent, if it is L-consistent and none of its proper supersets is.*

Since all presented logics are conservative extensions of CL , we have the following consistency result.

Lemma 8 (Consistency of the logics). *All presented logics are consistent.*

As usual, we have a Lindenbaum lemma and the usual properties of maximal consistent sets hold, see, e.g., [16].

Lemma 9 (Lindenbaum Lemma). *Given some logic L, then for each L- consistent set of formulas $\Gamma \subset \mathcal{L}_J$ there exists a maximal consistent set Γ' such that $\Gamma \subseteq \Gamma'$.*

Lemma 10 (Properties of maximal consistent sets). *Given some logic L and its language $\mathcal{L}_J$. If Γ is a maximal L-consistent set, then for all $F, G \in \mathcal{L}_J$:*

(1) if $\mathsf{L} \vdash F$, then $F \in \Gamma$;

(2) $F \in \Gamma$ *if and only if* $\neg F \notin \Gamma$*;*
(3) $F \to G \in \Gamma$ *if and only if* $F \notin \Gamma$ *or* $G \in \Gamma$*;*
(4) $F \in \Gamma$ *and* $F \to G \in \Gamma$ *imply* $G \in \Gamma$*.*

Definition 11 (Canonical Model). *For a given logic* $\mathsf{L}^{\star}_{\mathsf{CS}}$ *we define the canonical model* $\mathcal{M}^C = (W^C, W^C_0, V^C, E^C)$ *by:*

- $W^C = \mathcal{P}(\mathcal{L}_J)$.
- $W^C_0 = \{\, \Gamma \in W^C \mid \Gamma$ *is maximal* $\mathsf{L}^{\star}_{\mathsf{CS}}$ *– consistent set of formulas* $\}$.
- $V^C : V^C(\Gamma, F) = 1 \quad$ *iff* $\quad F \in \Gamma$*;*
- E^C *: With* $\Gamma/t := \{F \in \mathcal{L}_J \mid t : F \in \Gamma\}$ *and*

$$W^C_{MP} := \left\{\, \Gamma \in W^C \;\middle|\; \forall A, B \in \mathcal{L}_J : \text{ if } A \to B \in \Gamma \text{ and } A \in \Gamma \text{ then } B \in \Gamma \,\right\}$$

we define :

$$E^C(\Gamma, t) = \left\{\, \Delta \in W^C \;\middle|\; \Delta \supseteq \Gamma/t \,\right\} \text{ for } t \neq \mathsf{c}^{\star};$$
$$E^C(\Gamma, \mathsf{c}^{\star}) = \left\{\, \Delta \in W^C_{MP} \;\middle|\; \Delta \supseteq \Gamma/\mathsf{c}^{\star} \,\right\}.$$

Now we must show that the canonical model is indeed an $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset model.

Lemma 12. *The canonical model* $\mathcal{M}^C$ *is an* $\mathsf{L}^{\star}_{\mathsf{CS}}$*-subset model.*

The proof can be found in Appendix B.

The Truth Lemma follows very closely:

Lemma 13 (Truth Lemma). *Let* $\mathcal{M}^C = (W^C, W^C_0, E^C, V^C)$ *be a canonical model, then for any* $\Gamma \in W^C_0$*:*

$$\mathcal{M}^C, \Gamma \Vdash F \text{ if and only if } F \in \Gamma.$$

Proof.

$$\mathcal{M}^C, \Gamma \Vdash F \overset{\text{Def. 2}}{\Longleftrightarrow} V^C(\Gamma, F) = 1 \overset{\text{Def. 12}}{\Longleftrightarrow} F \in \Gamma.$$

Hence each maximal $\mathsf{L}^{\star}_{\mathsf{CS}}$-consistent set is represented by some world in the canonical model and thus completeness follows directly:

Theorem 14 (Completeness). *Given some logic* $\mathsf{L}^{\star}_{\mathsf{CS}}$*, then*

$$\mathcal{M}, \Gamma \Vdash F \quad \text{for all } \mathsf{L}^{\star}_{\mathsf{CS}}\text{-subset models } \mathcal{M} \text{ and for all } \Gamma \in W_0 \Longrightarrow \mathsf{L}^{\star}_{\mathsf{CS}} \vdash F.$$

Proof. The proof works with contraposition: Assume that $\mathsf{L}^{\star}_{\mathsf{CS}} \nvdash F$. Then $\{\neg F\}$ is $\mathsf{L}^{\star}_{\mathsf{CS}}$-consistent and by the Lindenbaum Lemma contained in some maximal $\mathsf{L}^{\star}_{\mathsf{CS}}$-consistent world Γ of the canonical model $\mathcal{M}^C$. Then $\mathcal{M}^C, \Gamma \nVdash F$.

3 $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-Subset Models

In this part we present an alternative definition of subset models for justification logic that directly interprets the application operator. Hence we work with the standard language of justification logic and we consider the **j**-axiom instead of the axiom $(\mathsf{c}^{\star})$.

3.1 Syntax

In this section, justification terms are built from constants c_i and variables x_i according to the following grammar:

$$t ::= c_i \mid x_i \mid (t \cdot t) \mid (t + t) \mid !t$$

This set of terms is denoted by Tm^{A}. The operations $\cdot$ and $+$ are left-associative and ! binds stronger than anything else. Formulas are built from atomic propositions p_i and the following grammar:

$$F ::= p_i \mid \perp \mid F \to F \mid t : F$$

The set of atomic propositions is denoted by Prop and the set of all formulas is denoted by $\mathcal{L}_J^{\mathsf{A}}$. Again we use the other logical connectives as abbreviations.

As in the first section, we investigate again a whole family of logics. They are arranged in two sets of axioms. The first set, denoted by $\mathsf{L}_\alpha^{\mathsf{A}}$ contains the following axioms:

cl all axioms of classical propositional logic;
j $s : (A \to B) \to (t : A \to s \cdot t : B)$;
j+ $s : A \vee t : A \to (s + t) : A$.

The other is identical to $\mathsf{L}_\beta^\star$ (modulo the different language) and contains:

j4 $t : A \to !t : (t : A)$;
jd $t : \perp \to \perp$;
jt $t : A \to A$.

For the sake of uniformity we denote this set of axioms by $\mathsf{L}_\beta^{\mathsf{A}}$. By L^{A} we denote all logics that are composed from the whole set $\mathsf{L}_\alpha^{\mathsf{A}}$ and some subset of $\mathsf{L}_\beta^{\mathsf{A}}$.

There are no differences between these logics and the ones of the former section except in case of application. Therefore we skip all the details already mentioned and proved before.
CS and $\mathsf{L}_{\mathsf{CS}}^{\mathsf{A}}$ are defined as before except that the corresponding logic has changed as mentioned. And deducing formulas in $\mathsf{L}_{\mathsf{CS}}^{\mathsf{A}}$ works the same as in the previous section.

3.2 Semantics

Definition 15 **($\mathsf{L}_{\mathsf{CS}}^{\mathsf{A}}$-subset models).** *Given some logic $\mathsf{L}_{\mathsf{CS}}^{\mathsf{A}}$ then an $\mathsf{L}_{\mathsf{CS}}^{\mathsf{A}}$-subset model $\mathcal{M} = (W, W_0, V, E)$ is defined like an $\mathsf{L}_{\mathsf{CS}}^\star$-subset model where*

$$E : W \times \mathsf{Tm}^{\mathsf{A}} \to \mathcal{P}(W)$$

meets the following condition for terms of the form $s \cdot t$:

$$E(\omega, s \cdot t) \subseteq \{\upsilon \in W \mid \forall F \in \mathsf{APP}_\omega(s,t)(\upsilon \in [F])\},$$

where we use

$$\mathsf{APP}_\omega(s,t) := \{F \in \mathcal{L}_J^{\mathsf{A}} \mid \exists H \in \mathcal{L}_J^{\mathsf{A}} \textit{ s.t. } E(\omega, s) \subseteq [H \to F] \textit{ and } E(\omega, t) \subseteq [H]\}.$$

The set $\mathsf{APP}_\omega(s,t)$ contains all formulas that are colloquially said derivable by applying modus ponens to a formula justified by s and a formula justified by t.

Truth in an $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-subset models is defined as before.

Definition 16 (Truth in $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-subset models). *For an $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-subset model $\mathcal{M} = (W, W_0, V, E)$ and a world $\omega \in W$ and a formula F we define the relation $\Vdash$ as follows:*

$$\mathcal{M}, \omega \Vdash F \quad \textit{iff} \quad V(\omega, F) = 1.$$

3.3 Soundness

Theorem 17 (Soundness of $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-subset models). *For any justification logic L^{A}, any constant specification CS and any formula F:*

$$\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}} \vdash F \quad \Rightarrow \quad \mathcal{M}, \omega \Vdash F \quad \textit{for all } \mathsf{L}^{\mathsf{A}}_{\mathsf{CS}} - \textit{subset models } \mathcal{M} \textit{ and all } \omega \in W_0.$$

The proof is straight forward by induction on the length of the derivation and can be found in [17].

3.4 Completeness

Before we start defining a canonical model, we have to do the same preliminary work for $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$ as we had to do in the previous section for $\mathsf{L}^{\star}_{\mathsf{CS}}$. Since the logics $\mathsf{L}^{\star}_{\mathsf{CS}}$ from the former section differ only in one axiom, i.e. **j** replaces **jc***, we skip all the parts that are already done and focus on the changes that it brings about.

As before, we have a conservativity and consistency result.

Theorem 18 (Conservativity). *All logics L^{A} presented are conservative extensions of the classical logic CL, i.e. for any formula $F \in \mathsf{L}_{\mathsf{cp}}$:*

$$\mathsf{L}^{\mathsf{A}} \vdash F \quad \Leftrightarrow \quad \mathsf{CL} \vdash F.$$

Lemma 19 (Consistency of L^{A}). *All logics in L^{A} are consistent.*

All the other ingredients we needed in the former section to define and further develop the canonical model were generally defined and proven and can be adopted without additional effort.

To prove completeness we define a canonical model as follows:

Definition 20 (Canonical Model). *For a given logic L^{A} and a constant specification CS we define the canonical model $\mathcal{M}^C = (W^C, W_0^C, V^C, E^C)$ by:*

- $W^C = \mathcal{P}(\mathcal{L}^{\mathsf{A}}_J)$;
- $W_0^C = \{\Gamma \in W^C \mid \Gamma \textit{ is maximal } \mathsf{L}^{\mathsf{A}}_{\mathsf{CS}} - \textit{consistent set of formulas}\}$;
- $V^C : V^C(\Gamma, F) = 1 \quad \textit{iff} \quad F \in \Gamma$;
- $E^C : E^C(\Gamma, t) = \{\Delta \in W \mid \Delta \supseteq \Gamma/t\}$.

Now we must show that such a canonical model is in fact a subset model.

Lemma 21. *The canonical model $\mathcal{M}^C$ is an $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-subset model.*

The proof is very similar to the proof of Lemma 12 and can be found in Appendix C

Lemma 22 (Truth Lemma). *Let $\mathcal{M}^C = (W^C, W^C_0, E^C, V^C)$ be some canonical $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-subset model, then for all $\Gamma \in W_0$:*

$$\mathcal{M}^C, \Gamma \Vdash F \textit{ if and only if } F \in \Gamma.$$

Proof.

$$\mathcal{M}^C, \Gamma \Vdash F \overset{\text{Def. 17}}{\Longleftrightarrow} V^C(\Gamma, F) = 1 \overset{\text{Def. 21}}{\Longleftrightarrow} F \in \Gamma.$$

Theorem 23 (Completeness). *Given some constant specification* CS *then*

$$\mathcal{M}, \Gamma \Vdash F \quad \textit{for all models } \mathcal{M} \textit{ and for all } \Gamma \in W_0 \Longrightarrow \mathsf{L}^{\mathsf{A}}_{\mathsf{CS}} \vdash F.$$

Proof. The proof is analogue to the one of Theorem 14.

4 Artemov's Aggregated Evidence and $\mathsf{L}^{\star}_{\mathsf{CS}}$-Subset Models

Artemov [1] considers the case in which we have a database, i.e. a set of propositions $\Gamma = \{F_1, \dots F_n\}$ with some kind of probability estimates and in which we also have some proposition X that logically follows from Γ. Then we can search for the best justified lower bound for the probability of X. He presents us a nice way to find this lower bound. To find it, he assumes probability events $u_1, \dots, u_n$, each of them supporting some proposition in Γ, i.e. $u_i : F_i$, and calculates some aggregated evidence $e(u_1, \dots, u_n)$ for X with them. The probability of e then provides a tight lower bound for the probability of X.

The trick he uses is the following:

(1) First he collects all subsets Δ_i of Γ which support X, i.e. $\Delta_i \vdash X$, and creates a new evidence t_i from all the corresponding u_{i_j} s.t. $u_{i_j} : F_{i_j}$ for each $F_{i_j} \in \Delta_i$.
(2) In the second step he combines all these new pieces of evidence to a new evidence (the so-called aggregated evidence) that actually is the greatest evidence supporting X.

The model he has in mind contains some evaluation in a probability space $(\Omega, \mathcal{F}, P)$ with a mapping $\star$ from propositions to Ω and evidence terms to $\mathcal{F}$ that meets some restrictions (for more details on this see [1]). Step (1) is to create a new evidence t_i for each Δ_i described above, which consists of the intersection of the corresponding u_{i_j}'s.

$$t_i := \bigcap \{u_{i_j} \mid u_{i_j} \subseteq F^{\star}_{i_j} \text{ for some } F_{i_j} \in \Delta_i\}.$$

Step (2) then is to union all these pieces of evidence to a new so-called aggregated evidence:

$$\mathsf{AE}^{\Gamma}(\mathsf{X}) := \bigcup \{t_i \mid t_i \text{ is an evidence for } X \text{ obtained by step (1)}\}.$$

On the syntactic side evidence terms are built from variables $u_1, \ldots, u_n$, constants 0 and 1 and operations $\cap$ and $\cup$, where st is used as an abbreviation for $s \cap t$. With this we can built a free distributive lattice $\mathcal{L}_n$ where st is the meet and $s \cup t$ is the join of s and t, 0 is the bottom and 1 the top element of this lattice. Moreover Artemov defines formulas in a usual way from propositional letters $p, q, r, \ldots$ by the usual connectives and adds formulas of the kind $t : F$ where t is an evidence term and F a purely propositional formula.

The logical postulates of the logic of Probabilistic Evidence PE are:

(1) axioms and rules of classical logic in the language of PE;
(2) $s : (A \to B) \to (t : A \to [st] : B)$;
(3) $(s : A \wedge t : A) \to [s \cup t] : A$;
(4) $1 : A$, where A is a propositional tautology,
$0 : F$, where F is a propositional formula;
(5) $t : X \to s : X$, for any evidence terms s and t such that $s \preceq t$ in $\mathcal{L}_n$.

Artemov presents Soundness and Completeness proofs connecting PE with the presented semantic, for more details see [1].

Before we can start adapting Artemovs approach to our models, we have to point out some differences between the semantics and syntax used. First, contrary to the models of Artemov, subset models may contain inconsistent worlds, but this does not significantly affect the applicability of Artemov's approach on them.

Another difference is that our evidence function has another domain. In Artemov's models the evidence functions is $E : \mathsf{Tm} \to \mathcal{P}(\Omega)$ while in our models it is $E : W \times \mathsf{Tm} \to \mathcal{P}(W)$. This difference is due to the fact that we allow terms to justify non-purely propositional formulas. Although we need to adapt Artemov's definitions, these adaptations will maintain the essential characteristics. So let's adapt the $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset models to aggregated $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset models by first describing the new syntax for the terms:

Definition 24 (Justification Terms). *Justification terms are built from constants* $0, 1, c_i$ *and variables* x_i *and the special and unique constant* $\mathsf{c}^{\star}$ *according to the following grammar:*

$$t ::= 0 \mid 1 \mid c_i \mid x_i \mid \mathsf{c}^{\star} \mid (t + t) \mid (t \cup t) \mid !t$$

This set of terms is denoted by Tm^{P}. As before, we introduce the abbreviation $st := s + t + \mathsf{c}^{\star}$.

Even though we have other operators as well, we can construct a free distributive lattice where we take $s + t$ as the meet of s and t, $s \cup t$ as the join of them, 0 as the bottom element of the lattice. Note, that st then is the meet of s, t, and $\mathsf{c}^{\star}$. Moreover, 1 and $!t$ are treated like constants.[1] As usual, we have

$$s \preceq t \quad \text{iff} \quad s \cup t = t \tag{2}$$

[1] We do not claim that 1 is the top element since some set $E(\omega, t)$ for a world $\omega \in W_0$ and $t \in \mathsf{Tm}^{\mathsf{P}}$ may contain non-normal worlds. If we claimed that 1 was the top element we would obtain $t \preceq 1$ and furthermore the set $E(\omega, 1)$ would contain non-normal worlds as well. But since in non-normal worlds axioms may not be true, $E(\omega, 1) \nsubseteq [A]$ for some axiom A may be the case and therefore axiom (4) would fail.

So not all pairs of terms are comparable. But that has no consequences so far.

There is no difference to our subset models regarding the rules for forming formulas except that the terms are contained in Tm^{P}, of course. The set of formulas built according to these grammar and rules is denoted by $\mathsf{L}_{\mathsf{prob}}$.

In the definition of $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset models we only change the conditions on the evidence function and the domain of V.

Definition 25 (PE-adapted subset models). *An $\mathsf{L}^{\star}_{\mathsf{CS}}$- subset model is called a PE-adapted $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset model if the valuation function and the evidence function meet the additional conditions respectively are redefined as follows:*

– $V : W \times \mathsf{L}_{\mathsf{prob}} \to \{0, 1\}$ *where all conditions listed in Definition 1 remain the same.*
– *For all $\omega \in W_0$ and for all $s, t \in \mathsf{Tm}^{\mathsf{P}}$:*
  - $E(\omega, 1) = W_0$;
  - $E(\omega, 0) = \emptyset$;
  - $E(\omega, s \cup t) = E(\omega, s) \cup E(\omega, t)$.

And in fact, such an PE-adapted $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset model is a model of probabilistic evidence PE.

Theorem 26 (Soundness). *PE-adapted $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset models $\mathcal{M}$ are sound with respect to probabilistic evidence PE, i.e. for all $F \in \mathsf{L}_{\mathsf{prob}}$*

$$\mathsf{PE} \vdash F \;\;\Rightarrow\;\; \mathcal{M}, \omega \Vdash F \quad \textit{for all } \mathsf{PE}\textit{-adapted } \mathsf{L}^{\star}_{\mathsf{CS}}\textit{-subset models and all } \omega \in W_0.$$

The proof is by induction on the length of the derivation of F and can be found in Appendix D.

Theorem 27 (model existence). *There exists a PE-adapted $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset model.*

Proof. We construct a model $\mathcal{M} = \{W, W_0, V, E\}$ as follows:

– $W = W_0 = \{\omega\}$.
– The valuation function is built bottom up:
  (1) $V(\omega, \perp) = 0$;
  (2) $V(\omega, P) = 1$, for all $P \in \mathsf{Prop}$;
  (3) $V(\omega, A \to B) = 1$ iff $V(\omega, A) = 0$ or $V(\omega, B) = 1$;
  (4) $V(\omega, t : F) = 1$ iff $t \not\geq 1$ or if $t \geq 1$ and $V(\omega, F) = 1$.
– $E(\omega, t) = \begin{cases} \{\omega\} & \text{if } t \geq 1 \\ \emptyset & \text{otherwise.} \end{cases}$

It is straightforward to show that $\mathcal{M}$ is indeed a PE-adapted $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset model. Let us only show the condition $E(\omega, s \cup t) = E(\omega, s) \cup E(\omega, t)$.

Suppose first $s, t \not\geq 1$, Then $E(\omega, s \cup t) = \emptyset = E(\omega, s) = E(\omega, t)$ and hence the claim follows immediately.

Suppose at least one term of s and t is in greater than 1, then $E(\omega, s) = \{\omega\}$ or $E(\omega, t) = \{\omega\}$ and hence $E(\omega, s) \cup E(\omega, t) = \{\omega\}$ and since $s \leq s \cup t$ and $t \leq s \cup t$ we obtain $s \cup t \geq 1$ and therefore $E(\omega, s \cup t) = \{\omega\}$, so the claim holds.

Note that we cannot use the canonical model to show that adapted subset models exists since in the canonical model

$$E(\Gamma, s \cup t) \not\subseteq E(\Gamma, s) \cup E(\Gamma, t).$$

However, in an adapted model we need these sets to be equal (see Definition 25) since otherwise axioms (3) and (5) would not be sound.

5 Conclusion

We introduced a new semantics, called subset semantics, for justifications. So far, often a symbolic approach was used to interpret justifications. In our semantics, justifications are modeled as sets of possible worlds. We also presented a new justification logic that is sound and complete with respect to our semantics. Moreover, we studied a variant of subset models that corresponds to traditional justification logic.

Subset models provide a versatile tool to work with justifications. In particular, we can naturally extend them with probability measures to capture uncertain justifications. In the last part of the paper, we showed that subset models subsume Artemov's approach to aggregating probabilistic evidence.

A The "j-axiom" Follows (Lemma 5)

For all $\mathcal{M} = (W, W_0, V, E)$, $\omega \in W_0$, $A, B \in \mathcal{L}_J$ and $s, t \in \mathsf{Tm}$:

$$\mathcal{M}, \omega \Vdash s : (A \rightarrow B) \rightarrow (t : A \rightarrow s \cdot t : B)$$

Proof. Suppose $\mathcal{M}, \omega \Vdash s : (A \rightarrow B)$ and $\mathcal{M}, \omega \Vdash t : A$. Thus $E(\omega, s) \subseteq [A \rightarrow B]$ and $E(\omega, t) \subseteq [A]$. We find

$$E(\omega, s \cdot t) = E(\omega, s + t + \mathsf{c}^\star) \subseteq \\ E(\omega, s) \cap E(\omega, t) \cap E(\omega, \mathsf{c}^\star) \subseteq [A \rightarrow B] \cap [A] \cap E(\omega, \mathsf{c}^\star).$$

Hence for all $\upsilon \in E(\omega, s \cdot t)$ we have $V(\upsilon, A \rightarrow B) = 1$ and $V(\upsilon, A) = 1$ and $\upsilon \in E(\omega, \mathsf{c}^\star)$ and therefore $V(\upsilon, B) = 1$. Hence $E(\omega, s \cdot t) \subseteq [B]$ and we obtain $\mathcal{M}, \omega \Vdash s \cdot t : B$.

B The Canonical Model $\mathcal{M}^C$ Defined in Definition 11 is an $\mathsf{L}^\star_{\mathsf{CS}}$-Subset Model (Lemma 12)

Proof. In order to prove this, we have to show that $\mathcal{M}^C$ meets all the conditions we made for the valuation and evidence function and the constant specification i.e.:

(1) $W_0^C \neq \emptyset$.

(2) For all $\Gamma \in W_0^C$:
   (a) $V^C(\Gamma, \bot) = 0$;
   (b) $V^C(\Gamma, F \to G) = 1 \quad \text{iff} \quad V^C(\Gamma, F) = 0$ or $V^C(\Gamma, G) = 1$;
   (c) $V^C(\Gamma, t : F) = 1 \quad \text{iff} \quad E(\Gamma, t) \subseteq [F]$.
(3) For all $\Gamma \in W_0^C, F \in \mathcal{L}_J, s, t \in \mathsf{Tm}$:
   (a) $E^C(\Gamma, s + t) \subseteq E^C(\Gamma, s) \cap E^C(\Gamma, t)$;
   (b) $E^C(\Gamma, \mathsf{c}^\star) \subseteq W_{MP}^C$;
   (c) if **jd** in $\mathsf{L}^\star$: $\forall \Gamma \in W_0^C$ and $\forall t \in \mathsf{Tm} : \exists v \in W_0^C$ s.t. $v \in E^C(\Gamma, t)$;
   (d) if **jt** in $\mathsf{L}^\star$: $\forall \Gamma \in W_0^C$ and $\forall t \in \mathsf{Tm} : \Gamma \in E^C(\Gamma, t)$;
   (e) if **j4** in $\mathsf{L}^\star$:

$$E^C(\Gamma, !t) \subseteq \left\{ \Delta \in W^C \;\middle|\; \forall F \in \mathcal{L}_J (V^C(\Gamma, t : F) = 1 \Rightarrow V^C(\Delta, t : F) = 1) \right\};$$

   (f) for all $(c, A) \in \mathsf{CS}$ and for all $\Gamma \in W_0^C$: $E^C(\Gamma, c) \subseteq [A]$ and

$$E(\Gamma, \underbrace{!\ldots!}_{n} c) \subseteq [\underbrace{!\ldots!}_{n-1} c : \ldots.!c : c : A] \text{ for all } n \in \mathbb{N}.$$

So the proofs are here:

(1) Since the empty set is proven to be $\mathsf{L}^\star_{\mathsf{CS}}$-consistent (see Lemma 8) it can be extended by the Lindenbaum Lemma to a maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistent set of formulas Γ with $\Gamma \in W_0^C$.
(2) Suppose $\Gamma \in W_0^C$:
   (a) We claim $V^C(\Gamma, \bot) = 0$: Suppose the opposite, then $V^C(\Gamma, \bot) = 1$ hence by the definition of V^C follows that $\bot \in \Gamma$. But this is a contradiction to the fact that Γ is consistent.
   (b) From left to right: Suppose $V^C(\Gamma, F \to G) = 1$, then by the definition of $V^C, F \to G \in \Gamma$. Since Γ is maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistent this implies by Lemma 10 (3) that $F \notin \Gamma$ or $G \in \Gamma$. Hence again by the definition of $V^C, V^C(\Gamma, F) = 0$ or $V^C(\Gamma, G) = 1$.
   From right to left: Suppose $V^C(\Gamma, F) = 0$ or $V^C(\Gamma, G) = 1$, then by the definition of V^C either $F \notin \Gamma$ or $G \in \Gamma$. Since $\Gamma \in W_0^C, \Gamma$ is maximal $\mathsf{L}^\star$-consistent and hence in both cases by Lemma 10 (3) $F \to G \in \Gamma$. But this means again by the definition of V^C that $V(\Gamma, F \to G) = 1$.
   (c) From left to right: Suppose $V^C(\Gamma, t : F) = 1$, then by Definition 11 $t : F \in \Gamma$. Hence with the definition of Γ/t we obtain $F \in \Gamma/t$. So for each $\Delta \in E^C(\Gamma, t), F \in \Delta$ (again by Definition 11). Hence for these Δ it follows by the definition of V^C that $V^C(\Delta, F) = 1$ and therefore $\Delta \in [F]$. Since this is true for all $\Delta \in E^C(\Gamma, t)$ we obtain $E^C(\Gamma, t) \subseteq [F]$.
   From right to left: The proof is by contraposition.
   Suppose $V^C(\Gamma, t : F) \neq 1$, then by the definition of $V^C \quad t : F \notin \Gamma$. We define a world Δ by $\Delta := \Gamma/t$. Since $\Delta \in \mathcal{P}(\mathcal{L}_J)$ we can be sure that Δ exists, i.e. $\Delta \in W$. Since $t : F \notin \Gamma$ it follows that $F \notin \Gamma/t$ and therefore $F \notin \Delta$. But obviously $\Delta \supseteq \Gamma/t$ hence $\Delta \in E^C(\Gamma, t)$. So we conclude

$E^C(\Gamma, t) \not\subseteq [F]$.
It remains to show that in case of $t = \mathsf{c}^\star$, $\Delta := \Gamma/t \in W^C_{MP}$ since otherwise $\Delta \notin E^C(\Gamma, \mathsf{c}^\star)$. In fact this is the case. Since $\Gamma \in W^C_0$ we obtain that Γ is a maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistent set of formulas and hence, whenever $\mathsf{c}^\star : A, \mathsf{c}^\star : (A \to B) \in \Gamma$ then by **jc*** we obtain $\mathsf{c}^\star : B \in \Gamma$. This means that whenever $A \in \Delta$ and $A \to B \in \Delta$ then $B \in \Delta$. Hence $\Delta = \Gamma/\mathsf{c}^\star$ is closed under modus ponens and therefore $\Delta \in W^C_{MP}$. So together with the former reasoning $\Delta \in E(\Gamma, \mathsf{c}^\star)$.

(3) Suppose $\Gamma \in W^C_0$:

(a) Given some $F \in \mathcal{L}_J, s, t \in \mathsf{Tm}$: We start by an observation on the relation between the sets $\Gamma/(s+t)$ and Γ/s for $\Gamma \in W^C_0$. If $s : A \in \Gamma$ then since Γ is maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistent $s + t : A \in \Gamma$ and therefore $\Gamma/s \subseteq \Gamma/(s+t)$. With the same reasoning $\Gamma/t \subseteq \Gamma/(s+t)$. Therefore if $\Delta \supseteq \Gamma/(s+t)$ then $\Delta \supseteq \Gamma/s$ and $\Delta \supseteq \Gamma/t$. This means that $E^C(\Gamma, s+t) \subseteq E^C(\Gamma, s)$ and $E^C(\Gamma, s+t) \subseteq E^C(\Gamma, t)$ and therefore $E^C(\Gamma, s+t) \subseteq E^C(\Gamma, s) \cap E^C(\Gamma, t)$.

(b) This follows directly from the definition of $E^C(\Gamma, \mathsf{c}^\star)$.

(c) If **jd** in $\mathsf{L}^\star$, then for any $\Gamma \in W^C_0$ we obtain $\neg(t : \bot) \in \Gamma$. Hence $\bot \notin \Gamma/t$. Therefore Γ/t is $\mathsf{L}^\star_{\mathsf{CS}}$-consistent and can be expanded by the Lindenbaum Lemma to a maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistent set $\Delta \supseteq \Gamma/t$ with $\Delta \in W^C_0$ and $\Delta \in E^C(\Gamma, t)$.

(d) Assume for some $F \in \mathcal{L}_J, \Gamma \in W^C_0$, $t \in \mathsf{Tm}$ that $F \in \Gamma/t$, i.e. $t : F \in \Gamma$, since Γ is maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistent and $t : F \to F$ is an instance of the **jt**-axiom, we conclude that $F \in \Gamma$. Since F was arbitrary we obtain $\Gamma \supseteq \Gamma/t$ and hence $\Gamma \in E^C(\Gamma, t)$.

(e) Suppose for some $\Delta \in E^C(\Gamma, !t)$, hence $\Delta \supseteq \Gamma/!t$. Then assume for some arbitrary $F \in \mathcal{L}_J$, $V(\Gamma, t : F) = 1$ i.e. by Definition 11 $t : F \in \Gamma$. Since Γ is maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistent and $t : F \to !t : (t : F)$ is an instance of the **j4**-axiom we obtain $!t : (t : F) \in \Gamma$ and hence $t : F \in \Gamma/!t$. But then $t : F \in \Delta$ and by Definition 11 it follows that $V^C(\Delta, t : F) = 1$. Since F was an arbitrary formula and Δ an arbitrary world of $E^C(\Gamma, !t)$ we conclude that the condition holds.

(f) Suppose $(c, A) \in \mathsf{CS}$, maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistency implies for all $\Gamma \in W^C_0$ that $c : A \in \Gamma$. Hence $A \in \Gamma/c$ and for all $\Delta \in E^C(\Gamma, c)$ we obtain $A \in \Delta$ and therefore $E^C(\Gamma, c) \subseteq [A]$.
Furthermore maximal $\mathsf{L}^\star_{\mathsf{CS}}$-consistency implies for all $\Gamma \in W_0$ by axiom necessitation that

$$\underbrace{!...!}_{n} c : ... :!c : c : A \in \Gamma$$

. Hence

$$\underbrace{!...!}_{n-1} c : ... :!c : c : A \in \Gamma / \underbrace{!...!}_{n} c$$

and for all $\Delta \in E^C(\Gamma, \underbrace{!...!}_{n} c)$ we obtain

$$\underbrace{!...!}_{n-1} c : ... :!c : c : A \in \Delta$$

and therefore

$$E^C(\Gamma, \underbrace{!...!}_{n} c) \subseteq [\underbrace{!...!}_{n-1} c : ... :!c : c : A]$$

.

C The Canonical Model Defined in Definition 20 Is an $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-Subset Model (Lemma 21)

Proof. In order to prove that, we have to proceed in the same way as in the previous section, i.e. showing that $\mathcal{M}^C$ meets all the conditions we made for the valuation and the evidence function as well as the constant specification.

Since the canonical model is defined in the same way as the one of $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset models, the corresponding proofs can be reused (see Lemma 12). Nevertheless, there is some difference. Instead of showing that $E^C(\Gamma, \mathsf{c}^\star) \subseteq W^C_{MP}$ we have to show that $E^C(\Gamma, s \cdot t) \subseteq \{\Delta \in W^C \mid \forall F \in \mathsf{APP}_\Gamma(s,t)(\Delta \in [F])\}$. Assume that we are given $\Gamma \in W^C_0$, $F \in \mathcal{L}^{\mathsf{A}}_J$, $s, t \in \mathsf{Tm}^{\mathsf{A}}$. Take any $\Delta \in E^C(\Gamma, s \cdot t)$, i.e. $\Delta \supseteq \Gamma/(s \cdot t)$. Hence for all F s.t. $s \cdot t : F \in \Gamma$ we know that $F \in \Delta$. Hence by the definition of V^C, we have $V(\Delta, F) = 1$ and therefore $\Delta \in [F]$.

It remains to show: if $F \in \mathsf{APP}_\Gamma(s,t)$ then $s \cdot t : F \in \Gamma$. Suppose for some formula F that $F \in \mathsf{APP}_\Gamma(s,t)$ then by definition of $\mathsf{APP}_\Gamma(s,t)$ we know that there is a formula H s.t. $E^C(\Gamma, s) \subseteq [H \to F]$ and $E^C(\Gamma, t) \subseteq [H]$. By using Lemma 21 (the part that corresponds to Lemma 12 (2c)) we conclude $V^C(\Gamma, s : (H \to F)) = 1$ and $V^C(\Gamma, t : H) = 1$. Hence by the definition of V^C we obtain $s : (H \to F) \in \Gamma$ and $t : H \in \Gamma$ and since Γ is maximal $\mathsf{L}^{\mathsf{A}}_{\mathsf{CS}}$-consistent and $s : (H \to F) \to (t : H \to s \cdot t : F)$ is an instance of the j-axiom we conclude that $s \cdot t : F \in \Gamma$.

D Soundness of PE-adapted $\mathsf{L}^{\star}$-Subset Models (Theorem 26)

PE-adapted $\mathsf{L}^{\star}_{\mathsf{CS}}$-subset models $\mathcal{M}$ are sound with respect to probabilistic evidence PE, i.e. for all $F \in \mathsf{L}_{\mathsf{prob}}$

$$\mathsf{PE} \vdash F \ \Rightarrow \ \mathcal{M}, \omega \Vdash F \quad \text{for all PE-adapted } \mathsf{L}^{\star}_{\mathsf{CS}}\text{-subset models and all } \omega \in W_0.$$

Proof. The proof is by induction on the length of the derivation of F:

– If F is derived by axiom necessitation or modus ponens or is an instance of axiom (1), then the proof is the analogue as in Theorem 3 since the relevant definitions have remained the same.
– If F is an instance of axiom (2) the proof is analogue to the proof of Lemma 5: Suppose $\mathcal{M}, \omega \Vdash s : (A \to B)$ and $\mathcal{M}, \omega \Vdash t : A$ then $E(\omega, s) \subseteq [A \to B]$ and $E(\omega, t) \subseteq [A]$.

$$E(\omega, st) = E(\omega, s + t + \mathsf{c}^\star) \subseteq$$
$$E(\omega, s) \cap E(\omega, t) \cap E(\omega, \mathsf{c}^\star) \subseteq [A \to B] \cap [A] \cap E(\omega, \mathsf{c}^\star).$$

Hence for all $v \in E(\omega, st)$ we have $V(v, A \to B) = 1$ and $V(v, A) = 1$ and $v \in E(\omega, \mathsf{c}^\star)$ and therefore $V(v, B) = 1$. Hence $E(\omega, st) \subseteq [B]$ and we obtain $\mathcal{M}, \omega \Vdash st : B$.

– If F is an instance of axiom (3) then $F = (s : A \wedge t : A) \to [s \cup t : A]$ for some $A \in \mathsf{L}_{\mathsf{prob}}$, $s, t \in \mathsf{Tm}^{\mathsf{P}}$. Suppose $\mathcal{M}, \omega \Vdash s : A \wedge t : A$ hence $E(\omega, s) \subseteq [A]$ and $E(\omega, t) \subseteq [A]$. Therefore $E(\omega, s \cup t) \subseteq E(\omega, s) \cup E(\omega, t) \subseteq [A]$ and since $\omega \in W_0$ we obtain $\mathcal{M}, \omega \Vdash s \cup t : A$.
– If F is an instance of axiom (4) then either $F = 1 : A$ for some axiom A or $0 : G$ for some formula G.
Suppose $F = 1 : A$ for some axiom A. We assume that $\mathcal{M}, \omega \Vdash A$ for all $\omega \in W_0$, hence $E(\omega, 1) = W_0 \subseteq [A]$ and therefore $\mathcal{M}, \omega \Vdash 1 : A$ for all $\omega \in W_0$.
Suppose $F = 0 : G$: For any $\omega \in W_0$ we have $E(\omega, 0) = \emptyset$ by Definition 25. Since $\emptyset$ is a subset of any subset of W, we obtain $E(\omega, 0) = \emptyset \subseteq [G]$ for any formula $G \in \mathsf{L}_{\mathsf{prob}}$.
– F is an instance of axiom (5). Assume $\mathcal{M}, \omega \Vdash t : X$ for some term t and some formula X and let $s \preceq t$. By (2) we find $t = s \cup t$. Thus

$$E(\omega, t) = E(\omega, s \cup t) = E(\omega, s) \cup E(\omega, t)$$

and therefore $E(\omega, s) \subseteq E(\omega, t)$. The assumption $\mathcal{M}, \omega \Vdash t : X$ means that $E(\omega, t) \subseteq [X]$. Hence we also get $E(\omega, s) \subseteq [X]$ and conclude $\mathcal{M}, \omega \Vdash s : X$.

References

1. Artemov, S.: On aggregating probabilistic evidence. In: Artemov, S., Nerode, A. (eds.) LFCS 2016. LNCS, vol. 9537, pp. 27–42. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-27683-0_3
2. Artemov, S., Fitting, M.: Justification Logic: Reasoning with Reasons. Cambridge University Press (in preparation)
3. Artemov, S.N.: Operational modal logic. Technical Report MSI 95–29, Cornell University, December 1995
4. Artemov, S.N.: Explicit provability and constructive semantics. Bull. Symb. Logic **7**(1), 1–36 (2001)
5. Artemov, S.N.: Justified common knowledge. TCS **357**(1–3), 4–22 (2006)
6. Artemov, S.N.: The logic of justification. RSL **1**(4), 477–513 (2008)
7. Artemov, S.N.: The ontology of justifications in the logical setting. Studia Logica **100**(1–2), 17–30 (2012)
8. Artemov, S.N., Fitting, M.: Justification logic. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy. Fall 2012 edition (2012)
9. Artemov, S., Nogina, E.: Topological semantics of justification logic. In: Hirsch, E.A., Razborov, A.A., Semenov, A., Slissenko, A. (eds.) CSR 2008. LNCS, vol. 5010, pp. 30–39. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-79709-8_7
10. Baltag, A., Bezhanishvili, N., Özgün, A., Smets, S.: Justified belief and the topology of evidence. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 83–103. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_6

11. Bucheli, S., Kuznets, R., Studer, T.: Justifications for common knowledge. Appl. Non-Class. Logics **21**(1), 35–60 (2011)
12. Bucheli, S., Kuznets, R., Studer, T.: Realizing public announcements by justifications. J. Comput. Syst. Sci. **80**(6), 1046–1066 (2014)
13. Fitting, M.: The logic of proofs, semantically. APAL **132**(1), 1–25 (2005)
14. Kuznets, R., Studer, T.: Justifications, ontology, and conservativity. In: Bolander, T. Braüner, T., Ghilardi, S., Moss, L. (eds.) Advances in Modal Logic, vol. 9, pp. 437–458. CollegePublications (2012)
15. Kuznets, R., Studer, T.: Weak arithmetical interpretations for the logic of proofs. Logic J. IGPL **24**(3), 424–440 (2016)
16. Kuznets, R., Studer, T.: Logics of Proofs and Justifications. College Publications (in preparation)
17. Lehmann, E., Studer, T.: Subset models for justification logic. E-print 1902.02707. arXiv.org (2019)
18. Mkrtychev, A.: Models for the logic of proofs. In: Adian, S., Nerode, A. (eds.) LFCS 1997. LNCS, vol. 1234, pp. 266–275. Springer, Heidelberg (1997). https://doi.org/10.1007/3-540-63045-7_27
19. Rantala, V.: Impossible worlds semantics and logical omniscience. Acta Philosophica Fennica **35**, 106–115 (1982). Cited By 35
20. Rantala, V.: Quantified modal logic: non-normal worlds and propositional attitudes. Studia Logica **41**(1), 41–65 (1982)
21. van Benthem, J., Duque, D.F., Pacuit, E.: Evidence logic: a new look at neighborhood structures. In: Advances in Modal Logic (2012)
22. van Benthem, J., Duque, D.F., Pacuit, E.: Evidence and plausibility in neighborhood structures. CoRR, abs/1307.1277 (2014)

Algebraic Semantics for Quasi-Nelson Logic

Fei Liang[1(✉)] and Thiago Nascimento[2]

[1] School of Philosophy and Social Development, Shandong University, Jinan, China
f.liang@sdu.edu.cn

[2] Programa de Pós-graduação em Sistemas e Computação, UFRN, Natal, Brazil
thiagnascsilva@gmail.com

Abstract. Quasi-Nelson logic is a generalization of Nelson logic in the sense that the negation is not necessary involutive. In this paper, we give a Hilbert-style presentation **QN** of quasi-Nelson logic, and show that **QN** is regularly BP-algebraizable with respect to its algebraic counterpart obtained by the Blok-Pigozzi algorithm, namely the class of Q-algebras. Finally, we show that the class of Q-algebras coincides with the class of quasi-Nelson algebras.

Keywords: Quasi-Nelson logic · Algebraizable logics · Quasi-Nelson algebras

1 Introduction

Nelson logic $\mathcal{N}_3$, introduced in [10], is a conservative expansion of the negation-free fragment of intuitionistic propositional logic by an unary logical connective $\sim$ of strong negation (which is involutive). The logic $\mathcal{N}_3$ is by now well studied, both from a proof-theoretic view and from an algebraic view. In particular, *Nelson algebras* (the algebraic counterpart of $\mathcal{N}_3$) can be represented as twist-structures over (i.e., special powers of) Heyting algebras [14,16]. Moreover, the variety of Nelson algebras is term-equivalent to the variety of compatibly involuted commutative integral bounded residuated lattices satisfying the *Nelson axiom* (called *Nelson residuated lattices*, [15]).

Rivieccio and Spinks [12] introduced *quasi-Nelson algebras* as a natural generalization of Nelson algebras in the sense that the negation $\sim$ is not involutive. Similar to Nelson algebras, quasi-Nelson algebras can be regarded as models of non-involutive Nelson logic, which is an expansion of the negation-free fragment of intuitionistic propositional logic; moreover, they can be represented as twist-structures over Heyting algebras (Definition 2). Furthermore, [12] proved that

The first author is supported by "The Fundamental Research Funds of Shandong University" (11090079614065). The second author was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 450–466, 2019.
https://doi.org/10.1007/978-3-662-59533-6_27

the class of quasi-Nelson algebras is term-equivalent to the variety of commutative integral bounded residuated lattices satisfying the Nelson axiom (called *quasi-Nelson residuated lattices* therein). Like Nelson algebras, the class of quasi-Nelson algebras forms a "quasivariety of logic" in the sense of Blok and La Falce [1]; however, no axiomatization of the inherent logic of quasi-Nelson algebras has yet been presented in the literature. Continuing the work above, in this paper we shall first introduce a Hilbert-style calculus **QN** for quasi-Nelson logic, and show that **QN** is algebraizable in the sense of Blok and Pigozzi. We shall further prove that the algebraic counterpart of **QN**, *viz.*, its equivalent variety semantics, is equivalent to the class of quasi-Nelson algebras.

Although the results in this paper clearly pertain in universal algebra and algebraic logic, they are potentially relevant to algebraic proof theory. Specifically, the term-equivalence results can throw light, as an interesting case study, on some unsolved issues that have recently cropped up about how to characterize the existence of (multi-type) analytic calculi for logical systems. In this respect, the present paper can also be regarded as a continuation of [9]. As in the cases of semi De Morgan logic and bilattice logic [6,7,9], the term-equivalent facts for quasi-Nelson algebras could also pave the way for designing analytic calculi for logics which are axiomatically presented by axioms which are not all analytic inductive in the sense of [5].

The paper is organized as follows. In Sect. 2 we recall some basic definitions and results about quasi-Nelson algebras. Section 3 gives a Hilbert-style presentation **QN** of quasi-Nelson logic. In Sect. 4, we prove that **QN** is regularly BP-algebraizable, and show that the algebraic counterpart of **QN** is equivalent to the class of quasi-Nelson algebras. Finally, we mention some prospects for future work in Sect. 5.

2 Preliminaries

In this section, we recall two equivalent presentations of quasi-Nelson algebras. They will be used to establish the equivalence between differing algebraic semantics for quasi-Nelson logic in Sect. 4.

Definition 1 ([12, Definition 4.1]). *A quasi-Nelson algebra is an algebra* $\mathbf{A} = (A; \wedge, \vee, \sim, \to, 0, 1)$ *having the following properties:*

(SN1) *The reduct* $(A; \wedge, \vee, 0, 1)$ *is a bounded distributive lattice with lattice order* $\leq$.

(SN2) *The relation* $\preceq$ *on* A *defined for all* $a, b \in A$ *by* $a \preceq b$ *iff* $a \to b = 1$ *is a quasiorder on* A.

(SN3) *The relation* $\equiv := \preceq \cap (\preceq)^{-1}$ *is a congruence on the reduct* $(A; \wedge, \vee, \to, 0, 1)$ *and the quotient algebra* $\mathbf{A}_+ = (A; \wedge, \vee, \to, 0, 1)/\equiv$ *is a Heyting algebra.*

(SN4) *For all* $a, b \in A$*, it holds that* $\sim(a \to b) \equiv \sim\sim(a \wedge \sim b)$.

(SN5) *For all* $a, b \in A$*, it holds that* $a \leq b$ *iff* $a \preceq b$ *and* $\sim b \preceq \sim a$.

(SN6) *For all* $a, b \in A$,

(SN6.1) $\sim\sim(\sim a \to \sim b) \equiv (\sim a \to \sim b)$.
(SN6.2) $\sim a \wedge \sim b \equiv \sim(a \vee b)$.
(SN6.3) $\sim\sim a \wedge \sim\sim b \equiv \sim\sim(a \wedge b)$.
(SN6.4) $\sim\sim\sim a \equiv \sim a$.
(SN6.5) $a \preceq \sim\sim a$.
(SN6.6) $a \wedge \sim a \preceq 0$.

Let π_1 and π_2 denote the first and second projection functions respectively.

Definition 2 ([12, Definition 3.1]). *Let $\mathbf{H}_+ = \langle H_+, \wedge_+, \vee_+, \to_+, 0_+, 1_+\rangle$ and $\mathbf{H}_- = \langle H_-, \wedge_-, \vee_-, \to_-, 0_-, 1_-\rangle$ be Heyting algebras and $n\colon H_+ \to H_-$ and $p\colon H_- \to H_+$ be maps satisfying the following conditions:*

(1) n preserves finite meets, joins and the bounds (i.e., one has $n(x \wedge_+ y) = n(x) \wedge_- n(y)$, $n(x \vee_+ y) = n(x) \vee_- n(y)$, $n(1_+) = 1_-$ and $n(0_+) = 0_-$),
(2) p preserves meets and the bounds (i.e., one has $p(x \wedge_- y) = p(x) \wedge_+ p(y)$, $p(1_-) = 1_+$ and $p(0_-) = 0_+$),
(3) $n \cdot p = Id_{H_-}$ and $Id_{H_+} \leq_+ p \cdot n$.

The algebra $\mathbf{H}_+ \bowtie \mathbf{H}_- = \langle H_+ \times H_-, \wedge, \vee, \to, \sim, 0, 1\rangle$ is defined as follows. For all $\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in H_+ \times H_-$,

$$
\begin{aligned}
1 &= \langle 1_+, 0_-\rangle \\
0 &= \langle 0_+, 1_-\rangle \\
\sim\langle a_+, a_-\rangle &= \langle p(a_-), n(a_+)\rangle \\
\langle a_+, a_-\rangle \wedge \langle b_+, b_-\rangle &= \langle a_+ \wedge_+ b_+, a_- \vee_- b_-\rangle \\
\langle a_+, a_-\rangle \vee \langle b_+, b_-\rangle &= \langle a_+ \vee_+ b_+, a_- \wedge_- b_-\rangle \\
\langle a_+, a_-\rangle \to \langle b_+, b_-\rangle &= \langle a_+ \to_+ b_+, n(a_+) \wedge_- b_-\rangle\rangle.
\end{aligned}
$$

A twist-structure $\mathbf{A}$ *over $\mathbf{H}_+ \bowtie \mathbf{H}_-$ is a $\{\wedge, \vee, \to, \sim, 0, 1\}$-subalgebra of $\mathbf{H}_+ \bowtie \mathbf{H}_-$ with carrier set A such that for all $\langle a_+, a_-\rangle \in A$, $a_+ \wedge_+ p(a_-) = 0_+$ and $n(a_+) \wedge_- a_- = 0_-$.*

Lemma 1 of [13] shows that (1)–(3) implies that p also preserves $\to$, i.e. $p(x \to_- y) = p(x) \to_+ p(y)$. By (SN3) in Definition 1, a quasi-Nelson algebras has the global outline of a Heyting algebra. Moreover, let $A_- := \{[\sim a] \mid a \in A\} \subseteq A_+$, $n([a]) := [\sim\sim a]$ and $p([a]) := [a]$, where $[.]$ is the equivalence class defined by $\equiv$ in Definition 1. By Proposition 4.2 in [12], the following theorem holds:

Theorem 1. *Every quasi-Nelson algebra $\mathbf{A}$ is isomorphic to a twist-structure over $\mathbf{A}_+, \mathbf{A}_-$ by the map $\iota(a) := \langle [a], [\sim a]\rangle$.*

3 A Hilbert System for Quasi-Nelson Logic

In this section, we give a Hilbert-style presentation **QN** of quasi-Nelson logic and highlight some theorems and derivations of **QN** that will be used to prove its algebraizability in subsequent sections.

Fix a denumerable set Atprop of propositional variables, and let p denote an element in Atprop. The language $\mathcal{L}$ of quasi-Nelson logic over Atprop is defined recursively as follows:

$$\varphi ::= p \mid {\sim}\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi)$$

To simplify the notation, in what follows, we omit the outmost parenthesis. Let $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. We use Fm to denote the set of all formulas. A *logic* is then defined as a finitary and substitution-invariant consequence relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$.

The Hilbert-system for **QN** of quasi-Nelson logic consists of the following axiom schemes:

AX1 $\varphi \to (\psi \to \varphi)$
AX2 $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
AX3 $(\varphi \wedge \psi) \to \varphi$
AX4 $(\varphi \wedge \psi) \to \psi$
AX5 $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi)))$
AX6 $\varphi \to (\varphi \vee \psi)$
AX7 $\psi \to (\varphi \vee \psi)$
AX8 $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$
AX9 ${\sim}{\sim}({\sim}\varphi \to {\sim}\psi) \to ({\sim}\varphi \to {\sim}\psi)$
AX10 $({\sim}\varphi \wedge {\sim}\psi) \leftrightarrow {\sim}(\varphi \vee \psi)$
AX11 $({\sim}{\sim}\varphi \wedge {\sim}{\sim}\psi) \leftrightarrow {\sim}{\sim}(\varphi \wedge \psi)$
AX12 ${\sim}{\sim}{\sim}\varphi \to {\sim}\varphi$
AX13 ${\sim}(\varphi \to \psi) \leftrightarrow {\sim}{\sim}(\varphi \wedge {\sim}\psi)$
AX14 $\varphi \to {\sim}{\sim}\varphi$
AX15 $(\varphi \to \psi) \to ({\sim}{\sim}\varphi \to {\sim}{\sim}\psi)$
AX16 ${\sim}\varphi \to {\sim}(\varphi \wedge \psi)$
AX17 ${\sim}(\varphi \wedge \psi) \to {\sim}(\psi \wedge \varphi)$
AX18 ${\sim}(\varphi \wedge (\psi \wedge \chi)) \leftrightarrow {\sim}((\varphi \wedge \psi) \wedge \chi)$
AX19 ${\sim}\varphi \to {\sim}(\varphi \wedge (\psi \vee \varphi))$
AX20 ${\sim}\varphi \to {\sim}(\varphi \wedge (\varphi \vee \psi))$
AX21 ${\sim}(\varphi \wedge (\psi \vee \chi)) \leftrightarrow {\sim}((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
AX22 ${\sim}(\varphi \vee (\psi \wedge \chi)) \leftrightarrow {\sim}((\varphi \vee \psi) \wedge (\varphi \vee \chi))$
AX23 ${\sim}\varphi \leftrightarrow {\sim}(\varphi \wedge (\psi \to \psi))$
AX24 ${\sim}(\varphi \to \varphi) \to \psi$
AX25 $({\sim}\varphi \to {\sim}\psi) \to ({\sim}(\varphi \wedge \psi) \to {\sim}\psi)$
AX26 $({\sim}\varphi \to {\sim}\psi) \to (({\sim}\chi \to {\sim}\gamma) \to ({\sim}(\varphi \wedge \chi) \to {\sim}(\psi \wedge \gamma)))$

together with the single inference rule of *modus ponens* (MP): $\varphi, \varphi \to \psi \vdash \psi$.

Remark 1. Notice that since the inter-derivability relation $\dashv\vdash$ does not realize a congruence on the formula algebra, **QN** is not selfextensional [17], and hence does not fall within the setting of [5]. Therefore, the analytic calculus for quasi-Nelson logic is challenge. However, the term-equivalent facts in Sect. 4 make it possible to solve this problem.

AX1–AX8 together with (MP) provide an axiomatization of the negation-free fragment of intuitionistic propositional logic, while AX9-AX14 are the logical analogues of (SN4) and (SN6) respectively. It is not difficult to see that intuitionistic propositional logic is a strengthening of **QN**.

By the usual inductive argument on the length of derivations, it is not difficult to prove that the deduction theorem holds for **QN**.

Theorem 2 (Deduction Theorem). *If* $\Phi \cup \{\varphi\} \vdash \psi$*, then* $\Phi \vdash \varphi \to \psi$*.*

In what follows, we prove some theorems and derivations which will be used in the next section.

Corollary 1. *(1)* $\varphi \to \varphi$
(2) $\varphi \to (\psi \to (\varphi \wedge \psi))$
(3) $(\varphi \wedge {\sim}\varphi) \to \psi$
(4) ${\sim}(\varphi \wedge \varphi) \to {\sim}\varphi$
(5) $\{\varphi \to \psi, \psi \to \chi\} \vdash \varphi \to \chi$

Proof. The proofs for (1) and (5) are same as the proofs in classical propositional logic [8, Chap. 2] and hence are omitted.

As to (2), we have:

1. φ	assumption
2. $\varphi \to (\psi \to \varphi)$	AX1
3. $\psi \to \varphi$	1, 2, MP
4. $\psi \to \psi$	Corollary 1.1
5. $\psi \to \varphi \wedge \psi$	3, 4, AX5, MP

and hence $\varphi \to (\psi \to (\varphi \wedge \psi))$ is derivable by the deduction theorem.

As to (3), we have:

1. $\varphi \wedge {\sim}\varphi$	assumption
2. ${\sim}{\sim}(\varphi \wedge {\sim}\varphi)$	AX14
3. ${\sim}(\varphi \to \varphi)$	2, AX13, MP
4. ψ	3, AX24, MP

and hence $(\varphi \wedge {\sim}\varphi) \to \psi$ is derivable by the deduction theorem.

As to (4), we have:

1. $({\sim}\varphi \to {\sim}\varphi) \to ({\sim}(\varphi \wedge \varphi) \to {\sim}\varphi)$	AX25
2. ${\sim}\varphi \to {\sim}\varphi$	Corollary 1.1
3. ${\sim}(\varphi \wedge \varphi) \to {\sim}\varphi$	1, 2, MP

4 QN Is Regularly BP-Algebraizable

In this section, we prove that **QN** is regularly BP-algebraizable. We give an algebraic semantics (called Q-algebras) for it via the algorithm of [2, Theorem

2.17]. Furthermore, we show that Q-algebras coincides with quasi-Nelson algebras defined as in Sect. 2. Combining with Theorem 4.4 in [12], we arrive at four equivalent characterizations of quasi-Nelson logic.

Before proving **QN** is regularly BP-algebraizable, we first recall some relevant definitions from [4]. Let Fm be a set of formulas, henceforth the set of *equations* of the language $\mathbf{L}$ is denoted by Eq and is defined as $\mathsf{Eq} := Fm \times Fm$. We write $\varphi \approx \psi$ rather than (φ, ψ).

Definition 3. *A logic* $\mathbf{L}$ *is algebraizable if and only if there are equations* $E(\varphi) \subseteq \mathsf{Eq}$ *and a transform* $\rho : \mathsf{Eq} \to 2^{Fm}$, *denoted by* $\Delta(\varphi, \psi) := \rho(\varphi \approx \psi)$, *such that* $\mathbf{L}$ *respects the following conditions:*

(**Alg**) $\varphi \dashv\vdash_{\mathbf{L}} \Delta(\mathsf{E}(\varphi))$
(**Ref**) $\vdash_{\mathbf{L}} \Delta(\varphi, \varphi)$
(**Sym**) $\Delta(\varphi, \psi) \vdash_{\mathbf{L}} \Delta(\psi, \varphi)$
(**Trans**) $\Delta(\varphi, \psi) \cup \Delta(\psi, \gamma) \vdash_{\mathbf{L}} \Delta(\varphi, \gamma)$
(**Cong**) *for each* n*-ary operator* $\bullet$, $\bigcup_{i=1}^{n} \Delta(\varphi_i, \psi_i) \vdash_{\mathbf{L}} \Delta(\bullet(\varphi_1, \ldots, \varphi_n), \bullet(\psi_1, \ldots, \psi_n))$

We call any such $\mathsf{E}(\varphi)$ *the set of defining equations and any such* $\Delta(\varphi, \psi)$ *the set of equivalence formulas of* $\mathbf{L}$.

Definition 4. *Let* $\mathbf{L}$ *be algebraizable. We say* $\mathbf{L}$ *is* finitely algebraizable *when the set of equivalence formulas is finite. We say* $\mathbf{L}$ *is* BP-algebraizable *when it is finitely algebraizable and the set of defining equations is finite.*

Definition 5. *A logic* $\mathbf{L}$ *is regularly BP-algebraizable when it is BP-algebraizable and satisfies:*

(G) $\varphi, \psi \vdash_{\mathbf{L}} \Delta(\varphi, \psi)$

for any nom-empty set $\Delta(\varphi, \psi)$ *of equivalence formulas.*

Let $\mathsf{E}(\varphi) := \{\varphi \approx \varphi \to \varphi\}$, and $\Delta(\varphi, \psi) := \{\varphi \to \psi, \psi \to \varphi, {\sim}\varphi \to {\sim}\psi, {\sim}\psi \to {\sim}\varphi\}$. In what follows, we prove in the Appendix that **QN** is regularly BP-algebraizable.

Proposition 1. QN *is regularly BP-algebraizable.*

By the algorithm in [4, Proposition 3.41], we can obtain the corresponding algebras for **QN**:

Definition 6. *An* Q*-algebra is a structure* $\mathbf{A} = (A; \wedge, \vee, {\sim}, \to)$ *which satisfies the following equations and quasiequations:*

(1) $\mathsf{E}(\varphi)$ *for each* $\varphi \in \mathsf{AX}$.
(2) $\mathsf{E}(\Delta(\varphi, \varphi))$.
(3) $\mathsf{E}(\Delta(\varphi, \psi))$ *implies* $\varphi \approx \psi$.
(4) $\mathsf{E}(\varphi)$ *and* $\mathsf{E}(\varphi \to \psi)$ *implies* $\mathsf{E}(\psi)$.

We will introduce below a class of algebras that thanks to Proposition 1 is equivalent to the class given in Definition 6, as can be seen in [3, Theorem 30].

Definition 7. *Let* $\mathbf{L}$ *be a logic with a set* $\boldsymbol{Ax}$ *of axioms and a set* $\boldsymbol{Ru}$ *of proper inference rules. Assume* $\mathbf{L}$ *is regularly algebraizable with finite equivalence system* $\Delta(\varphi,\psi) = \{\varepsilon_0(\varphi,\psi), \cdots, \varepsilon_{n-1}(\varphi,\psi)\}$. *Let* $\top$ *be a fixed but arbitrary theorem of* $\mathbf{L}$. *Then the unique equivalent quasivariety of* $\mathbf{L}$ *is defined by the identities:*

(1) $\varphi \approx \top$ *for each* $\varphi \in \boldsymbol{Ax}$
(2) $(\psi_0 \approx \top, \cdots, \psi_{p-1} \approx \top)$ *implies* $\varphi \approx \top$, *for each inference rule in* $\boldsymbol{Ru}$.
(3) $\Delta(\varphi,\psi) \approx \top$ *implies* $\varphi \approx \psi$.

In what follows, we show that the class given in Definition 7 is term-equivalent to the class of quasi-Nelson algebras and as it is the unique equivalent quasivariety of $\mathbf{L}$, this class must be Q-algebras.

Proposition 2. *Every* Q*-algebra is a quasi-Nelson algebra.*

Proof. As to (SN1), let $1 := \varphi \to \varphi$ and $0 := {\sim}(\varphi \to \varphi)$, in order to prove that $(A; \wedge, \vee, 0, 1)$ is a bounded distributive lattice with lattice order $\leq$, it suffices to show that it satisfies the following properties: (i) idempotence, the difficult part is ${\sim}(\varphi \wedge \varphi) \to {\sim}\varphi$ and ${\sim}\varphi \to {\sim}(\varphi \wedge \varphi)$, which follow from Corollary 1.3 and AX16; (ii) commutativity: the difficult part is ${\sim}(\varphi \wedge \psi) \to {\sim}(\psi \wedge \varphi)$ which is AX17; (iii) associativity: the difficult part is ${\sim}(\varphi \wedge (\psi \wedge \chi)) \leftrightarrow {\sim}((\varphi \wedge \psi) \wedge \chi)$ which is AX18; (iv) absorption: the difficult part is ${\sim}(\varphi \wedge (\psi \vee \varphi)) \leftrightarrow {\sim}\varphi$ and ${\sim}(\varphi \wedge (\varphi \vee \psi)) \leftrightarrow {\sim}\varphi$ which are AX19 and AX20 respectively; (v) distributivity: the difficult part is ${\sim}(\varphi \wedge (\psi \vee \chi)) \leftrightarrow {\sim}((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$ and ${\sim}(\varphi \vee (\psi \wedge \chi)) \leftrightarrow {\sim}((\varphi \vee \psi) \wedge (\varphi \vee \chi))$ which are AX21 and AX22 respectively. Hence, $(A; \wedge, \vee, 0, 1)$ is a distributive lattice, it is bounded by AX23, AX24 and Corollary 1.1.

As to (SN2), it suffices to show that the relation satisfies reflexivity and transitivity. In order to prove them, it is useful to show that $\varphi \to \psi \approx 1$ iff $\vdash \varphi \to \psi$. The right to left direction follows from the definition of E. The left to right direction follows from the definition of Δ. Hence, the reflexivity follows from Corollary 1.1, and transitivity follows from Corollary 1.3.

As to (SN3), by (SN2) and Definition 6(3), the relation $\equiv$ is a equivalent relation. By the same proof as in intuitionistic propositional logic, we can show that $\equiv$ is closed under $\vee, \wedge, 0, 1, \to$ and hence it is a congruence on $(A; \wedge, \vee, \to, 0, 1)$. To prove $\mathbf{A}_+ = (A; \wedge, \vee, \to, 0, 1)/{\equiv}$ is a Heyting algebra, it suffices to show that $[\varphi] \wedge [\psi] \leq_\equiv [\chi]$ iff $[\varphi] \leq_\equiv [\psi] \to [\chi]$ where $[.]$ means the equivalence class defined by $\equiv$. It is equivalent to show that $((\varphi \wedge \psi) \wedge \chi) \leftrightarrow (\varphi \wedge \psi)$ iff $(\varphi \wedge (\psi \to \chi)) \leftrightarrow \varphi$, which follows from Theorem 2, Corollary 1.2, AX3, and AX4.

As to (SN5), it suffices to prove that $(\varphi \wedge \psi) \to \varphi$, $\varphi \to (\varphi \wedge \psi)$, ${\sim}(\varphi \wedge \psi) \to {\sim}\varphi$ and ${\sim}\varphi \to {\sim}(\varphi \wedge \psi)$ iff $\varphi \to \psi$ and ${\sim}\psi \to {\sim}\varphi$. From right to left, ${\sim}(\varphi \wedge \psi) \to {\sim}\varphi$ follows from ${\sim}\psi \to {\sim}\varphi$ and AX25, others are obvious. From left to right, $\varphi \to \psi$ follows from $\varphi \to (\varphi \wedge \psi)$, AX4 and Corollary 1.5. ${\sim}\psi \to {\sim}\varphi$ follows from ${\sim}(\varphi \wedge \psi) \to {\sim}\varphi$, AX16, AX17 and Corollary 1.5.

(SN4) and (SN6) follows from AX9–AX15.

Corollary 2. *Given a quasi-Nelson algebra* $\mathbf{A}$*, for any* $a, b, c \in A$*, we have:*

(1) $a \wedge (a \to b) \preceq b$*.*
(2) $a \wedge b \preceq c$ *iff* $a \preceq b \to c$*.*

Proof. (1) By (SN2), we need to prove $(a \wedge (a \to b)) \to b = 1$. Hence, it suffices to show $\langle [(a \wedge (a \to b)) \to b], [\sim((a \wedge (a \to b)) \to b)] \rangle = \langle [1], [0] \rangle$ by Theorem 1. By (SN3), we have $(a \wedge (a \to b)) \to b \equiv 1$. Moreover, $\sim((a \wedge (a \to b)) \to b) \equiv (\sim\sim a \wedge \sim\sim(a \to b)) \wedge \sim b \equiv \sim(a \to b) \wedge \sim\sim(a \to b) \preceq 0$ by (SN4), (SN6.3), (SN6.4) and (SN6.6). We also have $0 \preceq \sim((a \wedge (a \to b)) \to b)$ since 0 is the least element and (SN5). Therefore, $\sim((a \wedge (a \to b)) \to b) \equiv 0$.

(2) From left to right, we only need to show that: if $(a \wedge b) \to c = 1$ then $a \to (b \to c) = 1$ by (SN2). Hence, by Theorem 1, it suffices to show that if $\langle [(a \wedge b) \to c], [\sim(a \wedge b) \to c)] \rangle = \langle [1], [0] \rangle$, then $\langle [a \to (b \to c)], [\sim(a \to (b \to c))] \rangle = \langle [1], [0] \rangle$. The assumption implies that: (i) $(a \wedge b) \to c \equiv 1$ and (ii) $\sim((a \wedge b) \to c) \equiv 0$. (i) implies $a \to (b \to c) \equiv (a \wedge b) \to c \equiv 1$ by (SN3). Since $\sim((a \wedge b) \to c) \equiv (\sim\sim a \wedge \sim\sim b) \wedge \sim c$ by (SN4), (SN6.3) and (SN6.4), (ii) implies $(\sim\sim a \wedge \sim\sim b) \wedge \sim c \equiv 0$. Therefore, by the same argument, $\sim(a \to (b \to c)) \equiv (\sim\sim a \wedge \sim\sim b) \wedge \sim c \equiv 0$. The argument for the right to left direction is quite similar and hence omitted.

Proposition 3. *Every quasi-Nelson algebra is a* Q*-algebra.*

Combining Theorem 4.4 in [12] with Propositions 2 and 3, we have:

Theorem 3. *The following algebras are term-equivalent:*

(1) Quasi-Nelson residuated lattices ([12][Definition 2.3]);
(2) Twist-structures over pairs of Heyting algebras (Definition 2);
(3) Quasi-Nelson algebras (Definition 1);
(4) Q*-algebras (Definition 6).*

5 Future Work

Since its introduction in [12], many questions regarding the class of quasi-Nelson algebras have been proposed and answered. This paper is the first attempt to introduce a Hilbert-style axiomatization of the inherent logic of quasi-Nelson algebras. There are some directions for future work based on the results in the present paper. Given that intuitionistic propositional logic is an extension of quasi-Nelson logic, a natural further direction of research is to investigate the position of quasi-Nelson logic in the hierarchy of subintuitionistic logics. In [6,7,9], the equivalence established between semi De Morgan algebras (resp. bilattices) and their heterogeneous counterparts has made it possible to introduce proper display (hence analytic) calculi for semi-De Morgan logic and bilattice logic. Interestingly, in the case of semi De Morgan logic, this equivalence result is very similar to the term-equivalence result with which Palma [11] proved that the variety of semi De morgan algebras is closed under canonical extensions.

A natural question is then whether this strategy can be systematically extended so as to to design analytic calculi for logics which are axiomatically presented by axioms which are not all analytic inductive, as is also the case of quasi-Nelson logic.

Appendix: Proofs of the Main Results

Proposition 1: QN is regularly BP-algebraizable.

Proof. As to (**Alg**), it suffices to prove that:

$$\varphi \dashv\vdash \{\varphi \to (\varphi \to \varphi), (\varphi \to \varphi) \to \varphi, {\sim}\varphi \to {\sim}(\varphi \to \varphi), {\sim}(\varphi \to \varphi) \to {\sim}\varphi\}$$

The right to left direction can be proved by Theorem 2 and MP. From left to right, $\varphi \vdash \varphi \to (\varphi \to \varphi)$ immediately follows from Theorem 2, and hence the proof is omitted. We only prove the last two items: (i) $\varphi \vdash {\sim}\varphi \to {\sim}(\varphi \to \varphi)$ and (ii) $\varphi \vdash {\sim}(\varphi \to \varphi) \to {\sim}\varphi$. For (i),

1. φ	assumption
2. ${\sim}\varphi \to (\varphi \wedge {\sim}\varphi)$	Corollary 1.2, 1, MP
3. ${\sim}\varphi \to {\sim}(\varphi \to \varphi)$	Corollary 1.3, 2, Corollary 1.5

(ii) follows from AX24, Corollary 1.3, and Corollary 1.5. (**Ref**) immediately follows from Corollary 1.1. (**Sym**) is a straightforward consequence of the definition of Δ.

As to (**Trans**), we need to prove that: (i)

$$\{\varphi \leftrightarrow \psi, {\sim}\varphi \leftrightarrow {\sim}\psi\} \cup \{\psi \leftrightarrow \chi, {\sim}\psi \leftrightarrow {\sim}\chi\} \vdash \varphi \leftrightarrow \chi$$

and (ii)

$$\{\varphi \leftrightarrow \psi, {\sim}\varphi \leftrightarrow {\sim}\psi\} \cup \{\psi \leftrightarrow \chi, {\sim}\psi \leftrightarrow {\sim}\chi\} \vdash {\sim}\varphi \leftrightarrow {\sim}\chi$$

For (i), this is an immediate consequence of Corollary 1.5. For (ii), we show $\vdash {\sim}(\varphi \to \psi) \to ({\sim}(\psi \to \chi) \to \gamma)$, which implies (ii).

1. ${\sim}(\varphi \to \psi)$	assumption
2. ${\sim}(\psi \to \chi)$	assumption
3. ${\sim}{\sim}(\varphi \wedge {\sim}\psi)$	1, AX13, MP
4. ${\sim}{\sim}\varphi \wedge {\sim}{\sim}{\sim}\psi$	3, AX11, MP
5. ${\sim}{\sim}\psi \wedge {\sim}{\sim}{\sim}\chi$	same as 1, 3, 4 above
6. ${\sim}{\sim}(\psi \wedge {\sim}\psi) \to \gamma$	AX13, AX24, Corollary 1.5
7. $({\sim}{\sim}\psi \wedge {\sim}{\sim}{\sim}\psi) \to \gamma$	AX11, 6, Corollary 1.5
8. $((({\sim}{\sim}\varphi \wedge {\sim}{\sim}{\sim}\psi) \wedge {\sim}{\sim}\psi) \wedge {\sim}{\sim}{\sim}\chi) \to \gamma$	AX3, AX4, 7, Corollary 1.5
9. γ	4, 5, 8, MP

and hence we have $\vdash {\sim}(\varphi \to \psi) \to ({\sim}(\psi \to \chi) \to \gamma)$ by the deduction theorem.

As to (**Cong**), we need to prove that $\to$ respects (**Alg**) for each connective $\bullet \in \{\wedge, \vee, \to, \sim\}$.

For ($\sim$), we need to prove that: (i)

$$\{\varphi \to \psi, \psi \to \varphi, \sim\varphi \to \sim\psi, \sim\psi \to \sim\varphi\} \vdash \sim\varphi \to \sim\psi$$

and

$$\{\varphi \to \psi, \psi \to \varphi, \sim\varphi \to \sim\psi, \sim\psi \to \sim\varphi\} \vdash \sim\psi \to \sim\varphi$$

It follows by hypothesises. And we need to prove (ii)

$$\{\varphi \to \psi, \psi \to \varphi, \sim\varphi \to \sim\psi, \sim\psi \to \sim\varphi\} \vdash \sim\sim\varphi \to \sim\sim\psi$$

and

$$\{\varphi \to \psi, \psi \to \varphi, \sim\varphi \to \sim\psi, \sim\psi \to \sim\varphi\} \vdash \sim\sim\psi \to \sim\sim\varphi$$

They are shown by AX15, hypothesises and MP.

For ($\wedge$), we need to prove that: (i)

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash (\varphi_1 \wedge \varphi_2) \to (\psi_1 \wedge \psi_2)$$

and

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\varphi_2 \leftrightarrow \psi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash (\psi_1 \wedge \psi_2) \to (\varphi_1 \wedge \varphi_2)$$

They are shown as follows:

1. $\varphi_1 \to \psi_1$	assumption
2. $\varphi_2 \to \psi_2$	assumption
3. $(\varphi_1 \wedge \varphi_2) \to \varphi_1$	AX3
4. $(\varphi_1 \wedge \varphi_2) \to \psi_1$	1, 3, Corollary 1.5
5. $(\varphi_1 \wedge \varphi_2) \to \psi_2$	same as 1, 3, 4 above
6. $(\varphi_1 \wedge \varphi_2) \to (\psi_1 \wedge \psi_2)$	AX5, 4, 5, MP

The other proof is similar; And we need to prove (ii)

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash \sim(\varphi_1 \wedge \varphi_2) \to \sim(\psi_1 \wedge \psi_2)$$

and

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\varphi_2 \leftrightarrow \psi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash \sim(\psi_1 \wedge \psi_2) \to \sim(\varphi_1 \wedge \varphi_2)$$

They can be proved by AX 26, hypothesises and MP.

For ($\vee$), we need to prove that: (i)

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash (\varphi_1 \vee \varphi_2) \to (\psi_1 \vee \psi_2)$$

and

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash (\psi_1 \vee \psi_2) \to (\varphi_1 \vee \varphi_2)$$

They are shown as follows, By AX6, AX7, hypothesis and corollary 1.5 we have the two following derivations:

1. $\varphi_1 \to \psi_1$	assumption
2. $\varphi_2 \to \psi_2$	assumption
3. $\psi_1 \to (\psi_1 \vee \psi_2)$	AX6
4. $\varphi_1 \to (\psi_1 \vee \psi_2)$	1, 3, Corollary 1.5
5. $\varphi_2 \to (\psi_1 \vee \psi_2)$	same as 1, 3, 4 above
6. $(\varphi_1 \vee \varphi_2) \to (\psi_1 \vee \psi_2)$	AX8, 4, 5, MP

the other proof is similar; And we need to prove (ii)

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash \sim(\varphi_1 \vee \varphi_2) \to \sim(\psi_1 \vee \psi_2)$$

and

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash \sim(\psi_1 \vee \psi_2) \to \sim(\varphi_1 \vee \varphi_2)$$

We only prove the first one, the other proof is similar and hence omitted.

1. $\sim\varphi_1 \to \sim\psi_1$	assumption
2. $\sim\varphi_2 \to \sim\psi_2$	assumption
3. $(\sim\varphi_1 \wedge \sim\varphi_2) \to \sim\psi_1$	1, AX3, Corollary 1.5
4. $(\sim\varphi_1 \wedge \sim\varphi_2) \to \sim\psi_2$	2, AX4, Corollary 1.5
5. $(\sim\varphi_1 \wedge \sim\varphi_2) \to (\sim\psi_1 \wedge \sim\psi_2)$	3, 4, AX5, Corollary 1.5
6. $\sim(\varphi_1 \vee \varphi_2) \to \sim(\psi_1 \vee \psi_2)$	AX10, 5, Corollary 1.5

For $(\to)$, we need to prove that: (i)

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash (\varphi_1 \to \varphi_2) \to (\psi_1 \to \psi_2)$$

and

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash (\psi_1 \to \psi_2) \to (\varphi_1 \to \varphi_2)$$

They can be shown by Corollary 1.5; And we need to prove (ii)

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash \sim(\varphi_1 \to \varphi_2) \to \sim(\psi_1 \to \psi_2)$$

and

$$\{\varphi_1 \leftrightarrow \psi_1, \sim\varphi_1 \leftrightarrow \sim\psi_1\} \cup \{\psi_2 \leftrightarrow \varphi_2, \sim\varphi_2 \leftrightarrow \sim\psi_2\} \vdash \sim(\psi_1 \to \psi_2) \to \sim(\varphi_1 \to \varphi_2)$$

We only prove the first one, the other proof is similar and hence omitted.

1. $\varphi_1 \to \psi_1$	assumption
2. $\sim\varphi_2 \to \sim\psi_2$	assumption
3. $(\varphi_1 \to \psi_1) \to (\sim\sim\varphi_1 \to \sim\sim\psi_1)$	AX15
4. $\sim\sim\varphi_1 \to \sim\sim\psi_1$	1,3, MP
5. $(\sim\sim\varphi_1 \wedge \sim\sim\sim\varphi_2) \to \sim\sim\psi_1$	4, AX3, Corollary 1.5
6. $(\sim\sim\varphi_1 \wedge \sim\sim\sim\varphi_2) \to \sim\sim\sim\psi_2$	same as 1, 3, 4, 5. above
7. $(\sim\sim\varphi_1 \wedge \sim\sim\sim\varphi_2) \to (\sim\sim\psi_1 \wedge \sim\sim\sim\psi_2)$	AX5, 6, 7, MP
8. $\sim\sim(\varphi_1 \wedge \sim\varphi_2) \to \sim\sim(\psi_1 \wedge \sim\psi_2)$	AX11, 7, Corollary 1.5
9. $\sim(\varphi_1 \to \varphi_2) \to \sim(\psi_1 \to \psi_2)$	AX13, 8, Corollary 1.5

Therefore, **QN** is algebraizable, and it is regularly BP-algebraizable since the set of equivalence (defined by Δ) is finite. In order to prove that **QN** is regularly BP-algebraizable see that $\varphi, \psi \vdash_{\mathbf{L}} \varphi \to \psi$ and $\varphi, \psi \vdash_{\mathbf{L}} \psi \to \varphi$ follow from deduction theorem and that $\varphi, \psi \vdash_{\mathbf{L}} \sim \varphi \to \sim \psi$ and $\varphi, \psi \vdash_{\mathbf{L}} \sim \psi \to \sim \varphi$ follow from deduction theorem and Corollary 1.4.

Proposition 3. Every quasi-Nelson algebra is a Q-algebra.

Proof. We need to prove that a quasi-Nelson algebra satisfies all equations and quasi-equations in Definition 7. Henceforward, see that being $\preceq$ a quasiorder (SN2), it holds the following equation: $\varphi \to \varphi \approx 1$.

1. $\varphi \approx 1$ for each $\varphi \in \mathsf{AX}$.

E(**AX1**). We need to show that $\varphi \to (\psi \to \varphi) \approx 1$. By (SN2), it suffices to show $\varphi \preceq \psi \to \varphi$. By Corollary 2.2, it is equivalent to show $\varphi \wedge \psi \preceq \varphi$, which can be proved by (SN1) and (SN5).

E(**AX2**). We need to show that $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \approx 1$. By Theorem 1, it suffices to show that

$$(((\langle[\varphi], [\sim\varphi]\rangle \to (\langle[\psi], [\sim\psi]\rangle \to \langle[\chi], [\sim\chi]\rangle)) \to (((\langle[\varphi], [\sim\varphi]\rangle \to \langle[\psi], [\sim\psi]\rangle) \to (\langle[\varphi], [\sim\varphi]\rangle \to \langle[\chi], [\sim\chi]\rangle)) = \langle[1], [0]\rangle$$

By Definition 2, it is equivalent to show:

$$\begin{aligned}&(\langle[\varphi], [\sim\varphi]\rangle \to (\langle[\psi \to \chi], n[\psi] \wedge [\sim s\chi]\rangle)) \to \\ &\quad (((\langle[\varphi \to \psi], n[\varphi] \wedge [\sim \psi]\rangle) \to \langle[\varphi \to \chi], n[\varphi] \wedge [\sim \chi]\rangle) \\ &\quad = (\langle[\varphi \to (\psi \to \chi)], n[\varphi] \wedge n[\psi] \wedge [\sim \chi]\rangle) \to \\ &\quad (\langle[(\varphi \to \psi) \to (\varphi \to \chi)], n[\varphi \to \psi] \wedge n[\varphi] \wedge [\sim \chi]\rangle) \\ &\quad = \langle[(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))], \\ &\quad n[\varphi \to (\psi \to \chi)] \wedge n[\varphi \to \psi] \wedge n[\varphi] \wedge [\sim \chi]\rangle = \langle[1], [0]\rangle\end{aligned}$$

Since $\mathbf{A}_+$ is a Heyting algebra, $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi) \equiv 1$. Moreover, by the fact that n preserves meet and bounds, and $\mathbf{A}_-$ is a Heyting algebra, and the definition of n, we obtain that

$$\begin{aligned}&n[\varphi \to (\psi \to \chi)] \wedge n[\varphi \to \psi] \wedge n[\varphi] \wedge [\sim \chi]\rangle \\ &\quad = n[\varphi \to (\psi \to \chi) \wedge (\varphi \to \psi) \wedge \varphi] \wedge [\sim \chi] \leq_{\equiv} n[\chi] \wedge [\sim \chi] \\ &\quad = [\sim \sim \chi \wedge \sim \chi] = [0]\end{aligned}$$

where $\leq_{\equiv}$ is the lattice order in $\mathbf{A}_-$, and hence $n[\varphi \to (\psi \to \chi)] \wedge n[\varphi \to \psi] \wedge n[\varphi] \wedge [\sim \chi] = [0]$ since $[0]$ is the least element in $\mathbf{A}_-$.

E(**AX3**) and E(**AX4**). They are immediate consequences of (SN1) and (SN5).

E(**AX5**). We need to show that $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi))) \approx 1$. By Theorem 1, it suffices to show that

$$(\langle[\varphi], [\sim \varphi]\rangle \to \langle[\psi], [\sim \psi]\rangle) \to (((\langle[\varphi], [\sim \varphi]\rangle \to$$

$$\langle[\chi],[\sim\chi]\rangle) \to (\langle[\varphi],[\sim\varphi]\rangle \to (\langle[\psi],[\sim\psi]\rangle \wedge \langle[\chi],[\sim\chi]\rangle)) = \langle[1],[0]\rangle$$

By Definition 2, it is equivalent to show:

$$\begin{aligned}
&(\langle[\varphi],[\sim\varphi]\rangle \to \langle[\psi],[\sim\psi]\rangle) \to \\
&((\langle[\varphi],[\sim\varphi]\rangle \to \langle[\chi],[\sim\chi]\rangle) \to (\langle[\varphi],[\sim\varphi]\rangle \to (\langle[\psi],[\sim\psi]\rangle \wedge \langle[\chi],[\sim\chi]\rangle)))\\
&= \langle[\varphi\to\psi], n[\varphi]\wedge[\sim\psi]\rangle \to \\
&(\langle[\varphi\to\chi], n[\varphi]\wedge[\sim\chi]\rangle \to (\langle[\varphi],[\sim\varphi]\rangle \to \langle[\psi\wedge\chi],[\sim\psi\vee\sim\chi]\rangle))\\
&= \langle[\varphi\to\psi], n[\varphi]\wedge[\sim\psi]\rangle \to \\
&(\langle[\varphi\to\chi], n[\varphi]\wedge[\sim\chi]\rangle \to (\langle[\varphi\to\psi\wedge\chi], n[\varphi]\wedge\langle[\sim\psi\vee\sim\chi]\rangle))\\
&= \langle[\varphi\to\psi], n[\varphi]\wedge[\sim\psi]\rangle \to \\
&(\langle[(\varphi\to\chi)\to(\varphi\to\psi\wedge\chi)], n[\varphi\to\chi]\wedge n[\varphi]\wedge\langle[\sim\psi\vee\sim\chi]\rangle))\\
&= \langle[(\varphi\to\psi)\to((\varphi\to\chi)\to(\varphi\to\psi\wedge\chi))],\\
&n[\varphi\to\psi]\wedge n[\varphi\to\chi]\wedge n[\varphi]\wedge\langle[\sim\psi\vee\sim\chi]\rangle)\\
&= \langle[1],[0]\rangle
\end{aligned}$$

Since $\mathbf{A}_+$ is a Heyting algebra, $(\varphi\to\psi)\to((\varphi\to\chi)\to(\varphi\to\psi\wedge\chi)) \equiv 1$. Moreover,

$$\begin{aligned}
&n[\varphi\to\psi]\wedge n[\varphi\to\chi]\wedge n[\varphi]\wedge\langle[\sim\psi\vee\sim\chi]\rangle\\
&= n[(\varphi\to\psi)\wedge(\varphi\to\chi)\wedge\varphi]\wedge\langle[\sim\psi\vee\sim\chi]\rangle \leq_{\equiv}\\
&n[\psi]\wedge n[\chi]\wedge\langle[\sim\psi\vee\sim\chi]\rangle = [\sim\psi]\wedge[\sim\sim\chi]\wedge\langle[\sim\psi\vee\sim\chi]\rangle = [0]
\end{aligned}$$

By the fact that n preserves meet and bounds, and $\mathbf{A}_-$ is a Heyting algebra, and the definition of n. Hence, $n[\varphi\to\psi]\wedge n[\varphi\to\chi]\wedge n[\varphi]\wedge\langle[\sim\psi\vee\sim\chi]\rangle = [0]$ since $[0]$ is the least element in $\mathbf{A}_-$.

E(AX6) and E(AX7). They are immediate consequences of (SN1) and (SN5).

E(AX8). We need to show that $(\varphi\to\chi)\to((\psi\to\chi)\to((\varphi\vee\psi)\to\chi)) \approx 1$. By Theorem 1, it suffices to show that

$$\begin{aligned}
&(\langle[\varphi],[\sim\varphi]\rangle \to \langle[\chi],[\sim\chi]\rangle) \to \\
&(\langle[\psi],[\sim\psi]\rangle \to \langle[\chi],[\sim\chi]\rangle) \to ((\langle[\varphi],[\sim\varphi]\rangle \vee \langle[\psi],[\sim\psi]\rangle) \to \langle[\chi],[\sim\chi]\rangle))\\
&= \langle[1],[0]\rangle
\end{aligned}$$

By Definition 2, it is equivalent to show:

$$\begin{aligned}
&(\langle[\varphi],[\sim\varphi]\rangle \to \langle[\chi],[\sim\chi]\rangle) \to \\
&((\langle[\psi],[\sim\psi]\rangle \to \langle[\chi],[\sim\chi]\rangle) \to ((\langle[\varphi],[\sim\varphi]\rangle \vee \langle[\psi],[\sim\psi]\rangle) \to \langle[\chi],[\sim\chi]\rangle))\\
&= \langle[\varphi\to\chi], n[\varphi]\wedge[\sim\chi]\rangle \to \\
&(\langle[\psi\to\chi], n[\psi]\wedge[\sim\chi]\rangle \to (\langle[\varphi\vee\psi],[\sim\varphi\wedge\sim\psi]\rangle \to \langle[\chi],[\sim\chi]\rangle))\\
&= \langle[\varphi\to\chi], n[\varphi]\wedge[\sim\chi]\rangle \to \\
&(\langle[\psi\to\chi], n[\psi]\wedge[\sim\chi]\rangle \to (\langle[\varphi\vee\psi\to\chi], n[\varphi\vee\psi]\wedge[\sim\chi]\rangle))\\
&= \langle[\varphi\to\chi], n[\varphi]\wedge[\sim\chi]\rangle \to
\end{aligned}$$

$$
\begin{aligned}
&(\langle[(\psi \to \chi) \to (\varphi \vee \psi \to \chi)], n[\psi \to \chi] \wedge n[\varphi \vee \psi] \wedge [\sim \chi]\rangle))\\
&\quad = \langle[(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))],\\
&\quad n[\varphi \to \chi] \wedge n[\psi \to \chi] \wedge n[\varphi \vee \psi] \wedge [\sim \chi]\rangle\\
&\qquad\qquad = \langle[1], [0]\rangle
\end{aligned}
$$

Since $\mathbf{A}_+$ is a Heyting algebra, $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi)) \equiv 1$. Moreover,

$$
\begin{aligned}
&n[\varphi \to \chi] \wedge n[\psi \to \chi] \wedge n[\varphi \vee \psi] \wedge [\sim \chi]\\
&\quad = n[(\varphi \to \chi) \wedge (\psi \to \chi) \wedge (\varphi \vee \psi)] \wedge [\sim \chi]\rangle \leq_{\equiv}\\
&\qquad n[\chi] \wedge [\sim \chi] = [\sim \sim \chi] \wedge [\sim \chi] = [0]
\end{aligned}
$$

By the fact that n preserves meet and bounds, and $\mathbf{A}_-$ is a Heyting algebra, and the definition of n. Hence, $n[\varphi \to \chi] \wedge n[\psi \to \chi] \wedge n[\varphi \vee \psi] \wedge [\sim \chi] = [0]$ since $[0]$ is the least element in $\mathbf{A}_-$.

E(**AX9**)–E(**AX14**). They are immediate consequences of (SN6.1), (SN6.2), (SN6.3), (SN6.4), (SN4) and (SN6.5) respectively.

E(**AX15**). We need to show that $(\varphi \to \psi) \to (\sim\sim\varphi \to \sim\sim\psi) \approx 1$. By Theorem 1, it suffices to show that

$$(\langle[\varphi], [\sim \varphi]\rangle \to \langle[\psi], [\sim \psi]\rangle) \to (\sim\sim\langle[\varphi], [\sim \varphi]\rangle \to \sim\sim\langle[\psi], [\sim \psi]\rangle = \langle[1], [0]\rangle$$

By Definition 2, it is equivalent to show:

$$
\begin{aligned}
&\langle[\varphi \to \psi], n[\varphi] \wedge [\sim \psi]\rangle \to (\langle pn[\varphi], np[\sim \varphi]\rangle \to \langle pn[\psi], np[\sim \psi]\rangle)\\
&\quad = \langle[\varphi \to \psi], n[\varphi] \wedge [\sim \psi]\rangle \to \langle pn[\varphi] \to pn[\psi], npn[\varphi] \wedge np[\sim \psi]\rangle\\
&\quad = \langle[\varphi \to \psi] \to (pn[\varphi] \to pn[\psi]), n[\varphi \to \psi] \wedge npn[\varphi] \wedge np[\sim \psi]\rangle\\
&\quad = \langle[\varphi \to \psi] \to (pn[\varphi] \to pn[\psi]), n[\varphi \to \psi] \wedge n[\varphi] \wedge [\sim \psi]\rangle = \langle[1], [0]\rangle.
\end{aligned}
$$

Since $\mathbf{A}_+$ is a Heyting algebra and n preserves meet, $n[\varphi] \wedge n[(\varphi \to \psi)] = n[\varphi \wedge (\varphi \to \psi)] \leq_{\equiv} n[\psi]$. Hence, $pn[\varphi] \wedge pn[\varphi \to \psi] = pn[\varphi \wedge (\varphi \to \psi)] \leq_{\equiv} pn[\psi]$ by p is order-preserving and preserve meet. By residuation law, we obtain that $pn[\varphi \to \psi] \leq_{\equiv} pn[\varphi] \to pn[\psi]$, and hence $[\varphi \to \psi] \leq pn[\varphi] \to pn[\psi]$ since $Id_{\mathbf{A}_+} \leq_{\equiv} pn$, that is, $[\varphi \to \psi] \to (pn[\varphi] \to pn[\psi]) = [1]$. For the other part, since $n[\varphi \to \psi] \wedge n[\varphi] \wedge [\sim \psi] = n[\varphi \wedge (\varphi \to \psi)] \wedge [\sim \psi] \leq_{\equiv} n[\psi] \wedge [\sim \psi] = [\sim \psi] \wedge [\sim \psi] = [0]$ by the fact that n preserves meet and the definition of n. Therefore, $n[\varphi \to \psi] \wedge n[\varphi] \wedge [\sim \psi] = [0]$ since $[0]$ is the least element in $\mathbf{A}_-$.

E(**AX16**). Since $\varphi \wedge \psi \leq \varphi$ by (SN1), we have $\sim \varphi \preceq \sim(\varphi \wedge \psi)$ by (SN5), which is equivalent to $\sim \varphi \to \sim(\varphi \wedge \psi) \approx 1$ by (SN2).

E(**AX17**)–E(**AX22**). The arguments are similar as above. All of them are verified by (SN1), (SN5) and (SN2).

E(**AX23**). Since $\varphi \wedge (\psi \to \psi) \leq \varphi$, thanks (SN5) we have $\sim \varphi \preceq \sim (\varphi \wedge (\psi \to \psi))$ and by (SN2) follows $\sim \varphi \to \sim (\varphi \wedge (\psi \to \psi))$. The other way around is the same idea, given that $\psi \to \psi \approx 1$ and $\varphi \leq \varphi \wedge (\psi \to \psi)$.

E(**AX24).** We want to prove that $\sim(\varphi \to \varphi) \to \psi \approx 1$. By Theorem 1, it is equivalent to prove that $\langle[\sim(\varphi \to \varphi) \to \psi], [\sim(\sim(\varphi \to \varphi) \to \psi)]\rangle = \langle[1],[0]\rangle$. Thanks (SN3) we know that $\sim(\varphi \to \varphi) \to \psi \equiv 1$. Regarding $\sim(\sim(\varphi \to \varphi) \to \psi)$, observe that $\sim(\sim(\varphi \to \varphi) \to \psi) \equiv \sim\sim\varphi \wedge \sim\varphi \wedge \sim\psi$ by (SN4), (SN6.3), (SN6.4) and (SN6.6). Finally $\sim\sim\varphi \wedge \sim\varphi \wedge \sim\psi \equiv 0$ by (SN6.6) and the fact that 0 is the least element.

E(**AX25).** We need to show that $(\sim\varphi \to \sim\psi) \to (\sim(\varphi \wedge \psi) \to \sim\psi) \approx 1$. By Theorem 1 it suffices to show that

$$(\sim\langle[\varphi],[\sim\varphi]\rangle \to \sim\langle[\psi],[\sim\psi]\rangle) \to \\ (\sim(\langle[\varphi],[\sim\varphi]\rangle \wedge \langle[\psi],[\sim\psi]\rangle) \to \sim\langle[\psi],[\sim\psi]\rangle) = \langle[1],[0]\rangle$$

By Definition 2, it is equivalent to show:

$$\begin{aligned}
&(\sim\langle[\varphi],[\sim\varphi]\rangle \to \sim\langle[\psi],[\sim\psi]\rangle) \to \\
&\quad(\sim(\langle[\varphi],[\sim\varphi]\rangle \wedge \langle[\psi],[\sim\psi]\rangle) \to \sim\langle[\psi],[\sim\psi]\rangle) \\
&= (\langle p[\sim\varphi], n[\varphi]\rangle \to \langle p[\sim\psi], n[\psi]\rangle) \to \\
&\quad(\sim(\langle[\varphi \wedge \psi],[\sim\varphi \vee \sim\psi]\rangle) \to \langle p[\sim\psi], n[\psi]\rangle) \\
&= \langle p[\sim\varphi] \to p[\sim\psi], np[\sim\varphi] \wedge n[\psi]\rangle \to \\
&\quad(\langle p[\sim\varphi \vee \sim\psi], n[\varphi \wedge \psi]\rangle \to \langle p[\sim\psi], n[\psi]\rangle) \\
&= \langle p[\sim\varphi] \to p[\sim\psi], np[\sim\varphi] \wedge n[\psi]\rangle \to \\
&\quad\langle p[\sim\varphi \vee \sim\psi] \to p[\sim\psi], np[\sim\varphi \vee \sim\psi] \wedge n[\psi]\rangle \\
&= \langle(p[\sim\varphi] \to p[\sim\psi]) \to \\
&(p[\sim\varphi \vee \sim\psi] \to p[\sim\psi]), n(p[\sim\varphi] \to p[\sim\psi]) \wedge np[\sim\varphi \vee \sim\psi] \wedge n[\psi]\rangle = \langle[1],[0]\rangle
\end{aligned}$$

Since $\mathbf{A}_{-}$ is a Heyting algebra and p preserves $\to$ and bounds, $(p[\sim\varphi] \to p[\sim\psi]) \to (p[\sim\varphi \vee \sim\psi] \to p[\sim\psi]) = p(([\sim\varphi] \to [\sim\psi]) \to ([\sim\varphi \vee \sim\psi] \to [\sim\psi])) = p[1] = [1]$. Moreover,

$$\begin{aligned}
&n(p[\sim\varphi] \to p[\sim\psi]) \wedge np[\sim\varphi \vee \sim\psi] \wedge n[\psi] \\
&= np([\sim\varphi] \to [\sim\psi]) \wedge np[\sim\varphi \vee \sim\psi] \wedge n[\psi] \\
&= ([\sim\varphi] \to [\sim\psi]) \wedge [\sim\varphi \vee \sim\psi] \wedge n[\psi] \\
&= (([\sim\varphi] \to [\sim\psi]) \wedge [\sim\varphi] \wedge n[\psi]) \vee (([\sim\varphi] \to [\sim\psi]) \wedge [\sim\psi] \wedge n[\psi]) \leq_{\equiv} [0]
\end{aligned}$$

since $np = Id_{\mathbf{A}_{-}}$ and $\mathbf{A}_{-}$ is a Heyting algebra. Therefore, $n(p[\sim\varphi] \to p[\sim\psi]) \wedge np[\sim\varphi \vee \sim\psi] \wedge n[\psi] = [0]$ since $[0]$ is the least element in $\mathbf{A}_{-}$.

E(**AX26**). We need to show that $(\sim\varphi \to \sim\psi) \to ((\sim\chi \to \sim\gamma) \to (\sim(\varphi \wedge \chi) \to \sim(\psi \wedge \gamma))) \approx 1$. By Theorem 1 it suffices to show that

$$(\sim\langle[\varphi],[\sim\varphi]\rangle \to \sim\langle[\psi],[\sim\psi]\rangle) \to \\ ((\sim\langle[\chi],[\sim\chi]\rangle \to \sim\langle[\gamma],[\sim\gamma]\rangle)) \to \\ (\sim(\langle[\varphi],[\sim\varphi]\rangle \wedge \langle[\chi],[\sim\chi]\rangle) \to \sim(\langle[\psi],[\sim\psi]\rangle \wedge \langle[\gamma],[\sim\gamma]\rangle))) = \langle[1],[0]\rangle$$

By Definition 2, it is equivalent to show:

$$
\begin{aligned}
&(\sim\langle[\varphi],[\sim\varphi]\rangle \to \sim\langle[\psi],[\sim\psi]\rangle) \to \\
&\qquad((\sim\langle[\chi],[\sim\chi]\rangle \to \sim\langle[\gamma],[\sim\gamma]\rangle)) \to \\
&\qquad(\sim(\langle[\varphi],[\sim\varphi]\rangle \wedge \langle[\chi],[\sim\chi]\rangle) \to \sim(\langle[\psi],[\sim\psi]\rangle \wedge \langle[\gamma],[\sim\gamma]\rangle)))) \\
&= (\langle p[\sim\varphi], n[\varphi]\rangle \to \langle p[\sim\psi], n[\psi]\rangle) \to \\
&\qquad(((\langle p[\sim\chi], n[\chi]\rangle \to \langle p[\sim\gamma], n[\gamma]\rangle)) \to \\
&\qquad(\sim\langle[\varphi\wedge\chi],[\sim\varphi\vee\sim\chi]\rangle \to \sim\langle[\psi\wedge\gamma],[\sim\psi\vee\sim\gamma]\rangle)) \\
&= (\langle p[\sim\varphi] \to p[\sim\psi], np[\sim\varphi \wedge n[\psi]\rangle) \to \\
&\qquad(((\langle p[\sim\chi] \to p[\sim\gamma], np[\sim\chi] \wedge n[\gamma]\rangle)) \to \\
&\qquad(\langle p[\sim\varphi\vee\sim\chi], n[\varphi\wedge\chi]\rangle \to \langle p[\sim\psi\vee\sim\gamma], n[\psi\wedge\gamma]\rangle)) \\
&= (\langle p[\sim\varphi] \to p[\sim\psi], np[\sim\varphi \wedge n[\psi]\rangle) \to \\
&\qquad(((\langle p[\sim\chi] \to p[\sim\gamma], np[\sim\chi] \wedge n[\gamma]\rangle)) \to \\
&\qquad(\langle p[\sim\varphi\vee\sim\chi] \to p[\sim\psi\vee\sim\gamma], np[\sim\varphi\vee\sim\chi] \wedge n[\psi\wedge\gamma]\rangle)) \\
&= \langle (p[\sim\varphi] \to p[\sim\psi]) \to ((p[\sim\chi] \to p[\sim\gamma]) \to \\
&\qquad(p[\sim\varphi\vee\sim\chi] \to p[\sim\psi\vee\sim\gamma])), \\
&n(p[\sim\varphi]\to p[\sim\psi])\wedge n(p[\sim\chi]\to p[\sim\gamma])\wedge np[\sim\varphi\vee\sim\chi]\wedge n[\psi\wedge\gamma]\rangle = \langle[1],[0]\rangle
\end{aligned}
$$

Since $\mathbf{A}_-$ is a Heyting algebra and p preserves $\to$ and bounds, $(p[\sim\varphi] \to p[\sim\psi]) \to ((p[\sim\chi] \to p[\sim\gamma]) \to (p[\sim\varphi\vee\sim\chi] \to p[\sim\psi\vee\sim\gamma])) = p(([\sim\varphi] \to [\sim\psi]) \to (([\sim\chi] \to [\sim\gamma]) \to ([\sim\varphi\vee\sim\chi] \to [\sim\psi\vee\sim\gamma])) = p[1] = [1]$. Moreover,

$$
\begin{aligned}
&n(p[\sim\varphi] \to p[\sim\psi]) \wedge n(p[\sim\chi] \to p[\sim\gamma]) \wedge np[\sim\varphi\vee\sim\chi] \wedge n[\psi\wedge\gamma] \\
&= np([\sim\varphi] \to [\sim\psi]) \wedge np([\sim\chi] \to [\sim\gamma]) \wedge np[\sim\varphi\vee\sim\chi] \wedge n[\psi\wedge\gamma] \\
&= ([\sim\varphi] \to [\sim\psi]) \wedge ([\sim\chi] \to [\sim\gamma]) \wedge [\sim\varphi\vee\sim\chi] \wedge n[\psi\wedge\gamma] \\
&= (([\sim\varphi] \to [\sim\psi]) \wedge ([\sim\chi] \to [\sim\gamma]) \wedge [\sim\varphi] \wedge n[\psi\wedge\gamma]) \\
&\vee(([\sim\varphi] \to [\sim\psi]) \wedge ([\sim\chi] \to [\sim\gamma]) \wedge [\sim\chi] \wedge n[\psi\wedge\gamma]) \leq_{\equiv} \\
&\qquad([\sim\psi] \wedge n[\psi\wedge\gamma]) \vee ([\sim\chi] \wedge n[\psi\wedge\gamma]) = [0]
\end{aligned}
$$

since $np = Id_{\mathbf{A}_-}$, n preserves meet and $\mathbf{A}_-$ is a Heyting algebra. Therefore, $n(p[\sim\varphi] \to p[\sim\psi]) \wedge n(p[\sim\chi] \to p[\sim\gamma]) \wedge np[\sim\varphi\vee\sim\chi] \wedge n[\psi\wedge\gamma] = [0]$ since $[0]$ is the least element in $\mathbf{A}_-$.

2. We have only an inference rule in **QN**, *modus ponens*. We need to prove that if $\varphi \approx 1$ and $\varphi \to \psi \approx 1$, then $\psi \approx 1$ and it follows from transitivity of $\preceq$.

3. We shall prove that if $\varphi \to \psi \approx 1$, $\psi \to \varphi \approx 1$, $\sim\varphi \to \sim\psi \approx 1$, $\sim\psi \to \sim\varphi \approx 1$, then $\varphi = \psi$. Thanks (SN2) we have $\varphi \preceq \psi$ and $\sim\psi \preceq \sim\varphi$ and therefore by (SN5) follows that $\varphi \leq \psi$. Following the same idea we have $\psi \leq \varphi$ and being $\leq$ the order relation on the lattice we have $\varphi \approx \psi$.

References

1. Blok, W.J., La Falce, S.B.: Komori identities in algebraic logic. Rep. Math. Logic **34**, 79–106 (2000)
2. Blok, W.J., Pigozzi, D.: Algebraizable Logic, vol. 396. Memoirs of the American Mathematical Society, Providence (1989)
3. Czelakowski, J., Pigozzi, D.: Fregean logics. Ann. Pure Appl. Log. **127**, 17–76 (2004)
4. Font, J.M.: Abstract Algebraic Logic: An Introductory Textbook. College Publications (2016)
5. Greco, G., Ma, M., Palmigiano, A., Tzimoulis, A., Zhao, Z.: Unified correspondence as a proof-theoretic tool. J. Log. Comput. **28**(7), 1367–1442 (2016)
6. Greco, G., Liang, F., Palmigiano, A., Rivieccio, U.: Bilattice logic properly displayed. Fuzzy Sets Syst. **363**, 138–155 (2019)
7. Greco, G., Liang, F., Moshier, M.A., Palmigiano, A.: Multi-type display calculus for semi De Morgan logic. In: Kennedy, J., de Queiroz, R.J.G.B. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 199–215. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_14
8. Hamilton, A.G.: Logic for Mathematicians. Cambridge University Press, Cambridge (1978)
9. Liang, F.: Multi-type algebraic proof theory. Dissertation, TU Delft (2018)
10. Nelson, D.: Constructible falsity. J. Symb. Log. **14**, 16–26 (1949)
11. Palma, C.: Semi De Morgan algebras. Dissertation, The University of Lisbon (2005)
12. Rivieccio, U., Spinks, M.: Quasi-Nelson algebras. In: Proceedings of LSFA 2018, Fortaleza, Brazil, 26–28 September 2018. Universidade Federal do Ceará (2018)
13. Rivieccio, U., Spinks, M.: Quasi-Nelson algebras; or, non-involutive Nelson algebras. Manuscript
14. Sendlewski, A.: Nelson algebras through Heyting ones: I. Stud. Log. **49**, 105–126 (1990)
15. Spinks, M., Rivieccio, U., Nascimento, T.: Compatibly involutive residuated lattices and the Nelson identity. Soft Comput. (2018). https://doi.org/10.1007/s00500-018-3588-9
16. Vakarelov, D.: Notes on N-lattices and constructive logic with strong negation. Stud. Log. **36**, 109–125 (1977)
17. Wójcicki, R.: Referential semantics. In: Wójcicki, R. (ed.) Theory of Logical Calculi. SYLI, vol. 199, pp. 341–401. Springer, Dordrecht (1988). https://doi.org/10.1007/978-94-015-6942-2_6

A Case for Property-Type Semantics

Kristina Liefke(✉)

Institute for Linguistics, Goethe University Frankfurt, Frankfurt, Germany
Liefke@lingua.uni-frankfurt.de

Abstract. In linguistic semantics, propositionalism is the view that all intensional constructions (esp. attitude reports) can be interpreted as relations to truth-evaluable propositional contents. Propositionalism has been adopted for its uniformity and ontological parsimony, and for its ability to capture natural language reasoning. These merits notwithstanding, propositionalism has been challenged by the observation that some intensional constructions (incl. objectual and *de se*-attitude reports, 'know how'-sentences, and *de dicto*-readings of depiction reports) resist a propositionalist analysis. This paper reconciles the merits of propositionalism with its empirical challenges. To this aim, it replaces *propositions* by *properties* as uniform objects of the attitudes. This replacement is motivated by the observation that all non-propositional attitudinal objects can be coded as properties through established type-shifts. It is supported by the ability of the resulting semantics to distinguish truth-evaluable from non-truth-evaluable attitude complements, to capture cross-attitudinal co-predication, and to explain differences w.r.t. the acceptability of different kinds of co-predication. At the same time, it gives a sense of what a propositionalist semantics – if successful – might look like and which requirements it must meet.

1 Introduction

Propositionalism (see [6,22,35,39]; cf. [55], [10, pp. 148–149]) is an approach to the semantics of intensional constructions that analyzes these constructions as cases of propositional[1] embedding.[2] Propositionalism has been adopted for its

I wish to thank two anonymous referees for WoLLIC 2019 for valuable comments on an earlier version of this paper. The paper has profited from discussions with David Boylan, Daniel Gutzmann, and Ede Zimmermann. The research for this paper is supported by the German Research Foundation (via Ede Zimmermann's grant ZI 683/13-1).

[1] We hereafter identify propositions with sets of possible worlds or of world/time-pairs (i.e. with sets of *indices*). However, the results of this paper also apply to 'dynamic' sentence-contents (i.e. relations to discourse referents of non-canonical arity) and to more fine-grained candidates for propositions (e.g. sets of partial or impossible worlds, sets of sets of worlds, or semantically primitive propositions).

[2] In this paper, we assume a liberal version of propositionalism (called *P̲ropositionalism* in [55]) that regards the general construal of information content in terms of

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 467–485, 2019.
https://doi.org/10.1007/978-3-662-59533-6_28

uniformity and ontological parsimony (see [22]; cf. [23,26]) and for its ability to capture natural language reasoning (see [17,46]). Propositionalism is often illustrated on the example of intensional transitive verbs like want, need, and seek. While these verbs superficially combine with an individual-denoting direct object (in (1a): with the DP a laptop), they are commonly analyzed as taking a proposition-denoting clausal complement with phonologically null elements and hidden structure (in (1b): as the CP FOR PRO to HAVE a laptop; see [6,22]). The analyses of (1a) (in (1b)) and (3a) (in (3b))[3] then have a similar form to propositional attitude reports (here: (2) resp. (4)).

(1) a. Bill wants/needs [$_{\text{DP}}$a laptop].
≡ b. Bill wants/needs [$_{\text{CP}}$FOR [$_{\text{TP}}$PRO to HAVE [$_{\text{DP}}$a laptop]]].

(2) Bill wants/expects [$_{\text{CP}}$that he has/will get [$_{\text{DP}}$a laptop]].

(3) a. Bill seeks [$_{\text{DP}}$a unicorn].
≡ b. Bill seeks (or strives) [$_{\text{CP}}$FOR [$_{\text{TP}}$PRO to FIND [$_{\text{DP}}$a unicorn]]].

(4) Bill strives [$_{\text{CP}}$that he finds [$_{\text{DP}}$a unicorn]].

The presence of the hidden predicate HAVE in (1a) is supported by the ability of the lower clause to be modified by temporal adverbials (see (5b); cf. [22,27]), to satisfy the presupposition of too and again (cf. [45, pp. 262–264]), and to antecede ellipsis (see (6); cf. [5, p. 35]).

(5) Bill will need [$_{\text{DP}}$a laptop] tomorrow.
≡ a. Tomorrow is the time of Bill's need (when Bill's need will arise)
≡ b. Tomorrow is when Bill needs *to have* the laptop

(6) A: Do you want [$_{\text{DP}}$another sausage]?
B: I can't ___. I'm on a diet.

The structural similarity between (1b) and (2) enables a uniform analysis of different occurrences of DP/CP complement-neutral verbs (see (7); cf. [12,26]), facilitates an easy analysis of DP/CP coordinations (see (8); cf. [2,43]), and explains the validity of inferences from propositional to 'objectual' attitudes (see (9); cf. [15,26]):

(7) a. Bill wants [$_{\text{DP}}$a laptop].
b. Bill wants [$_{\text{CP}}$that he gets a laptop soon].

(8) Bill wants [[$_{\text{DP}}$a laptop] and [$_{\text{CP}}$that he gets it soon]].

(9) a. Bill expects [$_{\text{CP}}$that he will get a laptop].
⇒ b. Bill expects [$_{\text{DP}}$something] (viz. a laptop/that he will get a laptop).

truth-conditions. According to this version, the 'propositional' reduction of intensional complements proceeds by semantic representation (or *coding*) of the complements' original denotation. Propositionalism (with a capital 'P') differs from better-known versions of propositionalism, according to which this reduction proceeds by syntactic means (i.e. by restructuring into a clause-embedding structure; see [6,22]) or by lexical decomposition (see [32,39]). We will discuss a variant of Propositionalism in Sect. 6.

[3] This analysis of seek is due to Quine [39] and can be found in [31, pp. 264, 267]. For arguments against the propositionalist analysis of seek, see [45,52,53].

2 Empirical Challenges for Propositionalism

Its empirical merits notwithstanding, Propositionalism has recently[4] come under empirical pressure. This pressure stems from intensional constructions that have been shown to resist a propositionalist analysis. These include the following:

2.1 'want'/'need'-Constructions Without HAVE

A first challenge for Propositionalism comes from want- and need-constructions (e.g. (10)) whose analysis does not contain the implicit predicate HAVE. In particular, the DP a marathon in (10) cannot be analyzed as the CP 'FOR PRO to HAVE a marathon' (analogous to (1b); see (10b)). Rather, it suggests an analysis as 'FOR PRO to *run/participate in* a marathon':

(10) John needs [$_{\text{DP}}$a marathon]. (said by John's coach)
$\equiv$ a. John needs [$_{\text{CP}}$FOR [$_{\text{TP}}$PRO to *run* [$_{\text{DP}}$a marathon]]].
$\not\equiv$ b. #John needs [$_{\text{CP}}$FOR [$_{\text{TP}}$PRO to HAVE [$_{\text{DP}}$a marathon]]].

In contrast to 'typical' want/need-constructions (e.g. (1a)), (10) lacks evidence for a hidden clausal structure (see [45, pp. 271–275]). In particular, the contextually supplied predicate run in (10a) resists modification by most temporal adverbs[5] (see (11b)).

(11) John needs [$_{\text{DP}}$a marathon] in two hours.
$\equiv$ a. John's need for a marathon will arise/be present in two hours
$\not\equiv$ b. John will need to *run* the marathon in two hours

To capture the difference in modificational behavior between (5) and (11), Deal [5, pp. 34–37] has proposed to analyze the non-specific object of need in (11) as a property (see [52]; cf. [13,45]). (10) then receives the analysis in (12), where BE is Partee's [36] type-shifter from existential generalized quantifiers to properties. For perspicuity, we hereafter mark complements in boldface. Below, i, j, k are variables over indices (type s); x, y, z are variables over individuals (type e).

(12) $[\![\text{John needs } [BE \text{ [a marathon]]}]\!]^{i,g}$
$= \mathit{need}(i)(\boldsymbol{\lambda j \lambda y (\exists x)[\mathit{marathon}(j)(x) \wedge x = y]})(\mathit{john})$
$\equiv \mathit{need}(i)(\boldsymbol{\mathit{marathon}})(\mathit{john})$

[4] The empirical pressure on propositionalism (in some form) is not a recent phenomenon. (For some early challenges, see [3,23,38].) What is new is the *explicit* collection of *diverse* natural language constructions that challenge Propositionalism (see [11,17,40,51]).

[5] These include all durational adverbs, but exclude the adverbial before (see [45]).

Note that Deal's analysis of (10) gives rise to a (type-)ambiguity in the interpretation of need between proposition- and property-complemented occurrences. We will see below that our proposed property-type semantics avoids this ambiguity.

We close this subsection with a note on typing and type notation: for reasons that will become clear below,[6] we adopt Tichý's [50] rule for the formation of multiary function types (see also [30,34]). This rule associates types of the form $(\alpha_1 \times \ldots \times \alpha_n) \to \alpha_{n+1}$ with functions from n-tuples of objects of the types $\alpha_1, \ldots, \alpha_n$ to objects of type α_{n+1}. Following Tichý, we write $(\alpha_1 \times \ldots \times \alpha_n) \to \alpha_{n+1}$ as $(\alpha_1 \ldots \alpha_n; \alpha_{n+1})$ and identify the type (α) with α.

This completes our presentation of the first challenge for Propositionalism.

2.2 *De se*-Reports and 'know how'-Sentences

A second set of challenges for Propositionalism comes from *de se*-readings of attitude reports like (13a) (see [4,23,37]) and from 'know how'-sentences like (14a) (see [40,51]). These constructions are usually analyzed as (13b) resp. (14b), where PRO_s is a silent pronoun that is controlled by the subject of the higher clause:

(13) a. Bill believes (of himself as himself) [$_{\text{CP}}$that he is a coffee addict].
$\equiv$ b. Bill believes [$_{\text{CP}}$that PRO_s is a coffee addict].

(14) a. Bill knows [$_{\text{CP}}$how to brew coffee].
$\equiv$ b. Bill knows [$_{\text{CP}}$how PRO_s to brew coffee].

The complements of the above constructions are typically interpreted as *centered propositions* (i.e. as sets of centered worlds, type-$(se; t)$; see (15a)) or, equivalently, as self-ascribed properties. The latter are properties that are true in all centered worlds that are compatible with what the property-ascriber believes (see (15b), where $\text{Dox}_{bill,\ i}$ is the set of Bill's doxastic alternatives in i). Below, $\langle j, x\rangle$ is a variable over pairs of indices (j) and individuals (x).

(15) $[\![\text{Bill believes [}_{\text{CP}}\text{that PRO}_s \text{ is a coffee addict]}]\!]^{i,g}$
a. $= \ \mathit{believe}(i)(\boldsymbol{\lambda\langle j, x\rangle[\mathit{coffee\text{-}addict}(j)(x)]})(\mathit{bill})$
b. $= \ 1 \ \Leftrightarrow \ (\forall\langle j, x\rangle)[\text{Dox}_{\mathit{bill},\,i}(j, x) \to \mathit{coffee\text{-}addict}(j)(x)]$

The type-$(se; t)$ interpretation of the complements of *de se*-reports is motivated by the need to capture the attitude holder's self-identification with the subject of the attitude complement (in (13), (14): Bill's self-characterization as a coffee addict/as an able coffee brewer). The absence of such self-identification results in situations that intuitively do *not* support the *de se*-reading of the associated sentence. For (13a), such situations include the situation where Bill is watching a man binging on coffee and concludes that this man is a coffee addict,

[6] Tichý-style types will allow us to distinguish between centered propositions (i.e. characteristic functions of sets of ordered index/individual-pairs; type $(se; t)$) and properties (i.e. functions from indices to characteristic functions of sets of individuals; type $(s; (e; t))$).

without realizing that the man is his mirror image. The *de se*-interpretation of 'know how'-sentences has an analogous motivation.

Note that, like PRO_s in (13b) and (14b), the silent pronoun PRO in (1b) is today often[7] taken to be obligatorily controlled by the matrix subject (see [1,4,48]). The different readings of (1b) are then interpreted as structures (here: (16a), (16b)) that also have centered propositions as their complements:

(16) a. $[\![\text{Bill wants } [_{\text{CP}}\text{FOR } [_{\text{TP}}\text{PRO}_s \text{ to HAVE } [_{\text{DP}}\text{a laptop}]]]]\!]^{i,g}$
$= \mathit{want}(i)(\boldsymbol{\lambda\langle j,y\rangle(\exists x)[\mathit{laptop}(j)(x) \wedge \mathit{have}(j)(x)(y)]})(\mathit{bill})$

b. $[\![[_{\text{DP}}\text{a laptop}]\ [\lambda_1\ [\text{Bill wants } [_{\text{CP}}\text{FOR } [_{\text{TP}}\text{PRO}_s \text{ to HAVE } t_1]]]]]\!]^{i,g}$
$= (\exists x)\big[\mathit{laptop}(i)(x) \wedge \mathit{want}(i)(\boldsymbol{\lambda\langle j,y\rangle[\mathit{have}(j)(x)(y)]})(\mathit{bill})\big]$

We will take the control-view of want/need as evidence *against* Propositionalism and *for* our proposed property-type semantics.

2.3 Objectual Attitude Reports

Propositionalism is further challenged by objectual attitude reports like (17a) and (17b) that contain verbs such as love, adore, worship, and fear. These reports intuitively express relations to individuals/individual objects (in (17a): Emilie).

(17) a. Klimt adored $[_{\text{DP}}$Emilie$]$. b. Klimt adored $[_{\text{DP}}$a woman$]$.

In contrast to the complements of *de se*-reports, the direct objects in objectual attitude reports often resist the extension to a full CP. Syntactically, this is due to the DP-bias of verbs like adore, s.t. sentences like (18) are ungrammatical:

(18) *Klimt adored $[_{\text{CP}}$that Emilie was beautiful$]$.

On the level of semantics, this is due to the fact that many objectual attitude reports are intuitively not equivalent to the result of supplementing their direct object with the infinitive to be (or to be there) (see (19)[8]; cf. [11,55], *pace* [35]) or with a contextually determined VP (see (20); cf. [49,52], *pace* [45]):

(19) Klimt adored $[_{\text{DP}}$the fact $[_{\text{CP}}$that Emilie was there (with him)$]]$.

(20) Klimt adored $[_{\text{DP}}$the fact $[_{\text{CP}}$that Emilie was beautiful$]]$.

For example, Klimt might not have adored Emilie's exemplifying any particular property (incl. her being there), but only Emilie *herself* (see [15, p. 829], [49]).[9] In this situation, (19) and (20) are false and are, hence, not equivalent to (17a).

[7] Exceptions are Larson et al. [6,22] and Stanley [47], who assume that PRO inherits its reference from its antecedent (cf. [51, p. 3]). In contrast to the 'subject control'-view, this view does not require a property-interpretation of attitude complements.

[8] To compensate for the DP-bias of adore, we prefix the CP with the DP shell the fact. This move is justified in [25].

[9] Further arguments against propositional(ist) accounts of objectual attitude reports are presented in [55, pp. 434–435] and [11, pp. 62–63].

In view of the above, the complements of objectual attitude reports are usually interpreted as (type-$(s; e)$) *individual concepts* (see (21a), where *emilie* is a constant of type $(s; e)$;[10] cf. [11,15]) or as (type-$(s; ((s; (e; t)); t))$) *intensional generalized quantifiers* (see (21b), (22); cf. [32, p. 394 ff.], [28]). Below, P is a variable over type-$(s; (e; t))$ properties.

(21) $[\![$Klimt adored $[_{\text{DP}}$Emilie$]]\!]^{i,g}$
 $=$ a. $adore'(i)(\boldsymbol{emilie})(klimt)$
 $=$ b. $adore\,(i)(\boldsymbol{\lambda j \lambda P\,[P\,(j)(emilie(j))]})(klimt)$

(22) a. $[\![$Klimt adored $[_{\text{DP}}$a woman$]]\!]^{i,g}$
 $= adore\,(i)(\boldsymbol{\lambda j \lambda P\,(\exists x)\,[woman\,(j)(x) \wedge P\,(j)(x)]})(klimt)$

 b. $[\![[_{\text{DP}}$a woman$]$ $[\lambda_1$ [Klimt adored $t_1]]]\!]^{i,g}$
 $= (\exists x)\big[woman(i)(x) \wedge adore\,(i)(\boldsymbol{\lambda j \lambda P\,[P\,(j)(x)]})(klimt)\big]$

2.4 Depiction and Resemblance Reports

The interpretation of objectual attitude reports suggests that the non-specific objects of depiction verbs (see (23); cf. [11, pp. 37, 130–150], [55]), of verbs of resemblance (see (24); cf. [28,52]), and of verbs of absence (e.g. seek/look for, owe, (be) missing; cf. [42,54]) are also interpreted as intensional (generalized) quantifiers.

(23) Uli is painting (/imagining) $[_{\text{DP}}$a unicorn].

(24) Paul resembles $[_{\text{DP}}$a penguin].

However, Zimmermann [52,53] has argued that this interpretation faces two problems: firstly, the interpretation of the VP in such reports (in (23): paints a penguin) cannot be systematically obtained from the interpretation of the verb (paint) and its direct object (a unicorn), s.t. the truth-conditions of these reports are underspecified (see [52, pp. 157–160]).[11] Secondly, the interpretation of the direct object in depiction and resemblance reports as an intensional quantifier wrongly predicts the availability of non-specific readings of strong quantificational objects (e.g. of the DP every penguin in (25), see (25b); cf. [52, pp. 160–161]).

(25) Uli is painting every penguin.
 $=$ a. *specific:* For every particular penguin in a given domain, Uli is painting it.
 $\neq$ b. *non-specific:* $^{?}$Uli is painting an image of all penguins.

[10] For convenience, we interpret attitude subjects (e.g. the name Klimt in (17a)) as individuals, rather than as individual concepts. In Sect. 5, we will outline a strategy that enables the extensional (i.e. type-e) interpretation of *all* occurrences of proper names.

[11] In contrast to verbs like seek (which allow for a propositional paraphrase; see (3) and our fn. 3), most depiction reports do not reduce to an attitude towards a proposition that is definable in terms of the intensional quantifier (see [52, p. 159]).

To avoid these problems, Zimmermann has proposed to interpret the direct objects in depiction and resemblance reports as (type-$(s;(e;t))$) *properties* (see [52]; cf. [13, 45]). The property-interpretation of the different readings of (23) is given in (26):[12]

(26) a. $[\![\text{Uli paints } [_{\text{DP}}\text{a unicorn}]]\!]^{i,g} = \mathit{paint}(i)(\boldsymbol{unicorn})(\mathit{uli})$
b. $[\![[_{\text{DP}}\text{a unicorn}]\ [\lambda_1\ [\text{Uli paints } t_1]]]\!]^{i,g}$
$= (\exists x)\big[\mathit{unicorn}(i)(x) \wedge \mathit{paint}(i)(\boldsymbol{\lambda j \lambda y\, [x = y]})(\mathit{uli})\big]$

The rationale behind Zimmermann's property-analysis lies in the one-to-one correspondence between existential quantifiers and their restrictors (i.e. properties), and in the attendant exclusion[13] of non-specific readings of depiction and resemblance reports with non-existential intensional quantifiers (see (25b)). Specific readings of such reports can still be obtained by quantifying-in (see (26b)).

The different (types of) complements of the intensional verbs from this section are summarized in Table 1. (We temporarily neglect the rightmost column.)

3 Strategy

To preserve the merits of the propositionalist analysis (see Sect. 4), we propose to interpret the complements of the constructions from the previous section as

---

[12] To avoid unwarranted inferences to a common objective (see $(*)$), Zimmermann [53] has proposed to interpret the non-specific objects of depiction and resemblance reports as existentially quantified sub-properties of the properties that are denoted by the reports' direct objects. On this account, (23) is then interpreted as $(\circledast)$, where '$P \sqsubseteq Q$' $\big(:= (\forall i)(\forall x)[P(i)(x) \rightarrow Q(i)(x)]\big)$ asserts that Q is more general than P:

$(*)$ a. Uli is painting [$_{\text{DP}}$a unicorn]. b. Penny is painting [$_{\text{DP}}$a panther].
$\not\Rightarrow$ c. Uli is painting [$_{\text{DP}}$something Penny is painting].

$(\circledast)$ $(\exists P)[P \sqsubseteq \mathit{unicorn} \wedge \mathit{paint}(i)(P)(\mathit{uli})]$

However, this move places overly strong demands on the truth of co-predication reports like († a) (in († b)). This interpretation requires that Penny stands in the admiring- and the painting-relation to the same sub-property of being an emu.

(†) a. Penny [[admires and paints] [$_{\text{DP}}$an emu]].
b. $(\exists P)\big[P \sqsubseteq \mathit{emu} \wedge (\mathit{admire} \wedge \mathit{paint})(i)(P)(\mathit{penny})\big]$

(‡) Uli imagines [[$_{\text{DP}}$a unicorn] and [$_{\text{CP}}$that a boy is petting it]].

Since Zimmermann's account further does not allow the straightforward modelling of embedded DP/CP-coordinations like (‡), we adopt instead the simpler account from [52]. We will suggest a strategy for blocking inferences like $(*)$ in Sect. 6.

[13] This exclusion results from the restriction of paint, resemble, etc. to the restrictors B $(:= \lambda j \lambda y[\mathcal{Q}(j)(\lambda k \lambda z.\, z = y)])$ of existential quantifiers $\mathcal{Q}$ (see [52, p. 164]).

Table 1. Intensional verbs and (the types of) their complements.

VERB	COMPLEMENT	TYPE	TYPE-SHIFTER
want/need	centered proposition	$(se;t)$	CURRY
believe (*de se*)	centered proposition	$(se;t)$	CURRY
believe (non-*de se*)	proposition	$(s;t)$	EGN + CURRY
adore/love/fear	intensional quantifier	$(s;((s;(e;t));t))$	BE
	individual concept	$(s;e)$	LIFT + BE
paint/resemble	property	$(s;(e;t))$	(BE)
need – HAVE	property	$(s;(e;t))$	(BE)

properties of individuals (type $(s;(e;t))$). All intensional verbs are then interpreted in the type $(s;((s;(e;t));(e;t)))$.[14] This interpretation is motivated by the possibility of coding (centered and uncentered) propositions, intensional quantifiers, and individual concepts as properties through established type-shifts. It is supported by the ability of the resulting semantics to avoid Zimmermann's problems from Sect. 2.4, to distinguish truth-evaluable from non-truth-evaluable complements, and to capture cross-attitudinal coordination and quantification.

3.1 Centered Propositions

The equivalence of centered propositions and self-ascribed properties (see Sect. 2.2) already presupposes the possibility of representing objects of type $(se;t)$ in the type $(s;(e;t))$. This representation exploits the one-to-one correspondence between multiary functions and certain unary functions of higher type (see [44]). It is achieved by the familiar operation of Schönfinkelization, or *currying* (see [33, p. 8 ff.]). The relevant instance of currying, i.e. the type-shifter CURRY, is given below, where p^* is a variable over centered propositions (type $(se;t)$):

(27) $\text{CURRY} := \lambda p^* \lambda j \lambda y [p^*(j,y)]$

To enable a property-type interpretation of (1b) (in (30)), we incorporate CURRY into the semantics of want. The resulting interpretation of want (in (28)) applies to an index i, a centered proposition, and an individual to assert the obtaining-in-i of the wanting-relation between the individual and the property that results from currying the centered proposition. *De se*-occurrences of propo-

[14] This strategy is already anticipated by Zimmermann [52, pp. 167–168, fn. 30]: "Lewis [...] argues that, for reasons of theoretical homogeneity, the objects of intentional attitudes should all be of the same type, so that preference may seem to be at odds with propositional attitudes like belief. However, a *de se* account would get the two closer to each other: both belief and (a *de se* version of) preference are relations.".

sitional attitude verbs like believe are interpreted analogously (see (29)). Below, want and believe are constants of type $(s;((s;(e;t));(e;t)))$:[15]

(28) $[\![\text{want}]\!]_{\text{control}} = \lambda j \lambda p^* \lambda x\, [\mathtt{want}(j)(\textsc{curry}(p^*))(x)]$

(29) $[\![\text{believe}]\!]_{de\ se} = \lambda j \lambda p^* \lambda x\, [\mathtt{believe}(j)(\textsc{curry}(p^*))(x)]$

(30) $[\![\text{Bill wants } [_{\text{CP}}\text{FOR } [_{\text{TP}}\text{PRO}_s \text{ to HAVE } [_{\text{DP}}\text{a laptop}]]]]\!]^{i,g}$
$= \mathtt{want}(i)\big(\textsc{curry}\big(\lambda\langle j,y\rangle(\exists x)[\mathit{laptop}(j)(x) \wedge \mathit{have}(j)(x)(y)]\big)\big)(\mathit{bill})$
$\equiv \mathtt{want}(i)\big(\lambda p^* \lambda k \lambda z\, [p^*(k,z)]\big(\lambda\langle j,y\rangle(\exists x)[\mathit{laptop}(j)(x) \wedge$
$\mathit{have}(j)(x)(y)]\big)\big)(\mathit{bill})$
$\equiv \mathtt{want}(i)(\lambda j \lambda y\, (\exists x)[\mathit{laptop}(j)(x) \wedge \mathit{have}(j)(x)(y)])(\mathit{bill})$

The above suggests that propositional attitude and control verbs are ambiguous between centered proposition-taking occurrences (see (28)) and uncentered proposition-taking occurrences (see (31), where want' has type $(s;((s;t);(e;t)))$):

(31) $[\![\text{want}]\!]_{\text{non-control}} = \lambda j \lambda p \lambda x\, [\mathit{want}'(j)(p)(x)]$

To avoid this ambiguity, we lift non-control/non-*de se* occurrences of attitude verbs to centered proposition-taking occurrences. This is accomplished through the type-shifter EGN:

(32) $\textsc{egn} := \lambda p \lambda\langle j,y\rangle[p(j)]$

EGN sends propositions to centered propositions that are invariant under different individual centers, i.e. to 'boring' centered propositions. The latter are propositions p^* such that, for any index w and inhabitants x, y of w, $p^*(w,x)$ iff $p^*(w,y)$ (see [7, p. 107]). In combination with CURRY, EGN enables a property-type interpretation of the complement of (2):

(33) $[\![\text{Bill wants } [_{\text{CP}}\text{that he/Bill has } [_{\text{DP}}\text{a laptop}]]]\!]^{i,g}$
$= \mathtt{want}(i)\big(\textsc{curry}\big(\textsc{egn}\big(\lambda j(\exists x)[\mathit{laptop}(j)(x) \wedge \mathit{have}(j)(x)(\mathit{bill})]\big)\big)\big)(\mathit{bill})$
$\equiv \mathtt{want}(i)(\lambda j \lambda y\, (\exists x)[\mathit{laptop}(j)(x) \wedge \mathit{have}(j)(x)(\mathit{bill})])(\mathit{bill})$

To ensure that want preserves the truth-conditional contribution of *want*, respectively of want', we posit the following axioms: (Analogous axioms also hold for all other (subject-)control verbs and DP/CP-neutral attitude verbs.)

(**Ax1.i**) $(\forall j)(\forall p^*)(\forall x)\big[\mathit{want}(j)(p^*)(x) \Leftrightarrow \mathtt{want}(j)(\textsc{curry}(p^*))(x)\big]$

(**Ax1.ii**) $(\forall j)(\forall p)(\forall x)\big[\mathit{want}'(j)(p)(x) \Leftrightarrow \mathtt{want}(j)(\textsc{curry}(\textsc{egn}(p)))(x)\big]$

The above suggests that there exist semantic relations between the complements of control- and non-control-uses of want, as evidenced by (8) (see (38) in Sect. 4).

[15] Since these constants have a different type from the (type-$(s;((se;t);(e;t)))$) translations of want and need in (16) resp. (12), we here use different(-font) constants.

3.2 Individual Concepts and Intensional Quantifiers

To enable a property-type interpretation of the direct objects in objectual attitude reports (e.g. (17a), (17b)), we use the type-shifters KAP and BE from [20] and [36], respectively. Below, c is a variable over individual concepts (type $(s; e)$); $\mathcal{Q}$ is a variable over intensional generalized quantifiers (type $(s; ((s; (e; t)); t))$):

(34) $\textsc{kap} := \lambda c \lambda j \lambda y [c(j) = y]$ $\qquad$ $\textsc{be} := \lambda \mathcal{Q} \lambda j \lambda y [\mathcal{Q}(j)(\lambda k \lambda z.\, z = y)]$

KAP is a particular instance of a variant of Kaplan's coding strategy for intensional objects[16] that shifts individual concepts to properties. BE is an intensional version of the operation of *Existential Lowering* from [53, p. 736]. This operation obtains properties from intensional quantifiers. KAP and BE enable the property-type interpretation of (17a) and (17b) as follows:

(35) $[\![\text{Klimt adored } [_{\text{DP}}\text{Emilie}]]\!]^{i,g}$
$= \texttt{adore}(i)(\textsc{kap}(\mathit{emilie}))(\mathit{klimt})$
$\equiv \texttt{adore}(i)(\lambda j \lambda y.\, \mathit{emilie}(j) = y)(\mathit{klimt})$
$\equiv \texttt{adore}(i)\big(\textsc{be}\big(\lambda j \lambda P [P(j)(\mathit{emilie}(j))]\big)\big)(\mathit{klimt})$

(36) a. $[\![\text{Klimt adored } [_{\text{DP}}\text{a woman}]]\!]^{i,g}$
$= \texttt{adore}(i)\big(\textsc{be}\big(\lambda j \lambda P (\exists x)[\mathit{woman}(j)(x) \wedge P(j)(x)]\big)\big)(\mathit{klimt})$
$\equiv \texttt{adore}(i)(\lambda j \lambda y (\exists x)[\mathit{woman}(j)(x) \wedge x = y])(\mathit{klimt})$

b. $[\![[_{\text{DP}}\text{a woman}]\ [\lambda_1 [\text{Klimt adored } t_1]]]\!]^{i,g}$
$= (\exists x)\big[\mathit{woman}(i)(x) \wedge \texttt{adore}(i)\big(\textsc{be}\big(\lambda j \lambda P [P(j)(x)]\big)\big)(\mathit{klimt})\big]$
$\equiv (\exists x)\big[\mathit{woman}(i)(x) \wedge \texttt{adore}(i)(\lambda j \lambda y.\, x = y)(\mathit{klimt})\big]$

To ensure that `adore` preserves the truth-conditional contribution of *adore′*, respectively of *adore*, we posit the following axioms:

(Ax2.i) $(\forall j)(\forall c)(\forall x)\big[\mathit{adore}'(j)(c)(x) \Leftrightarrow \texttt{adore}(j)(\textsc{kap}(c))(x)\big]$
(Ax2.ii) $(\forall j)(\forall \mathcal{Q})(\forall x)\big[\mathit{adore}(j)(\mathcal{Q})(x) \Leftrightarrow \texttt{adore}(j)(\textsc{be}(\mathcal{Q}))(x)\big]$

3.3 Existential Quantifiers

We have suggested in Sect. 2.4 that non-specific readings of (the direct objects in) depiction and resemblance reports are best interpreted as relations to properties (see (26)). To obtain properties from the standard interpretation of referential DPs (i.e. intensional generalized quantifiers), we again use Partee's type-shifter BE. The non-specific reading of (23) is then interpreted as follows:

(37) $[\![\text{Uli paints } [_{\text{DP}}\text{a unicorn}]]\!]^{i,g}$
$= \texttt{paint}(i)\big(\textsc{be}\big(\lambda j \lambda P (\exists x)[\mathit{unicorn}(j)(x) \wedge P(j)(x)]\big)\big)(\mathit{uli})$
$\equiv \texttt{paint}(i)(\lambda j \lambda y (\exists x)[\mathit{unicorn}(j)(x) \wedge x = y])(\mathit{uli})$
$\equiv \texttt{paint}(i)(\mathit{unicorn})(\mathit{uli})$

[16] This strategy represents objects A of type $(s; \alpha)$ by type-$(\alpha; (s; t))$ functions, $\lambda a^{\alpha} \lambda i [A(i) = a]$, from objects a to the set of indices at which the extension of A is a.

4 Support for Our Strategy

Our proposal to replace propositions by properties as uniform objects of the attitudes is supported by the ability of property-type semantics to straightforwardly accommodate 'unlike' coordinations in attitude complements (see (38))[17] and to account for cross-attitudinal coordination (see (39)–(40)) and quantification (see (41)):

(38) $[\![\text{Bill wants } [[_{\text{DP}}\text{a laptop}] \text{ and } [_{\text{CP}}\text{that Mary stops whining}]]]\!]^{i,g}$
$\equiv [\![\text{Bill wants } [[\text{FOR PRO}_s \text{ to HAVE a laptop}] \text{ and } [\text{that Mary stops whining}]]]\!]^{i,g}$
$= \texttt{want}\,(i)\big(\lambda j \lambda y\,(\exists x)[(\mathit{laptop}(j)(x) \wedge \mathit{have}(j)(x)(y)) \wedge (\mathit{stop}\,(\mathit{whine}))(j)(\mathit{mary})]\big)(\mathit{bill})$

(39) a. $[\![[_{\text{DP}}\text{a woman}]\ [\lambda_1\,[\text{Klimt } [\text{adored and painted}]\ t_1]]]\!]^{i,g}$
$= \big(\lambda P\,(\exists x)[\mathit{woman}(i)(x) \wedge P(i)(x)]\big)\big(\lambda j \lambda y\big[(\texttt{adore} \wedge \mathit{paint})(j)(\lambda k \lambda z.\, y = z)(\mathit{klimt})\big]\big)$
$\equiv (\exists x)\big[\mathit{woman}(i)(x) \wedge (\texttt{adore} \wedge \mathit{paint})(i)(\lambda j \lambda y.\, x = y)(\mathit{klimt})\big]$

b. $[\![\text{Klimt } [[\text{adored and painted}]\ [_{\text{DP}}\text{a woman}]]]\!]^{i,g}$
$= (\texttt{adore} \wedge \mathit{paint})(i)\big(\text{BE}\big(\lambda j \lambda P\,(\exists x)[\mathit{woman}(j)(x) \wedge P(j)(x)]\big)\big)(\mathit{klimt})$
$\equiv (\texttt{adore} \wedge \mathit{paint})(i)(\mathit{woman})(\mathit{klimt})$

(40) $[\![\text{Klimt } [\text{wanted and sought}]\ [_{\text{DP}}\text{Emilie's attention}]]\!]^{i,g}$
$\equiv [\![[\text{Emilie's attention}]\ [\text{Klimt } [\lambda_1\,[\text{wants FOR PRO}_s \text{ to HAVE } t_1] \text{ and } [\text{seeks } t_1]]]]\!]^{i,g}$
$= \texttt{want}\,(i)(\lambda j \lambda y\,[\mathit{have}(j)(\mathit{emilies\text{-}attention}(j))(y)])(\mathit{klimt}) \wedge \texttt{seek}\,(i)(\lambda k \lambda z\,[\mathit{emilies\text{-}attention}(k) = z])(\mathit{klimt})$

In particular, since the different complements of want from (38) are traditionally assigned different types (i.e. centered resp. uncentered propositions), (38) cannot be easily captured by classical Montagovian or propositional semantic accounts. Similar observations hold for (39) and (40), whose coordinated verbs traditionally take intensional quantifiers and properties, respectively centered propositions and properties as their complements.

The above suggests that the proposed semantics also enables an easy interpretation of instances of cross-attitudinal quantification. This is indeed the case. The property-type interpretation of the non-specific reading of (41) is given in (42):

(41) Bill wants $[_{\text{DP}}\text{something that Mary fears}]$.

(42) $\texttt{want}\,(i)\big(\lambda j \lambda y\,(\exists x)\big[\texttt{fear}\,(j)(\lambda k \lambda z.\, z = x)(\mathit{mary}) \wedge \mathit{have}(j)(x)(y)\big]\big)(\mathit{bill})$

[17] Liefke and Werning [26, p. 647] have suggested that the naturalness (or unnaturalness) of these coordinations depends on the overlap (resp. disjointness) of the world-parts with respect to which the conjuncts of such coordinations are evaluated.

Note that – the uniform interpretation of attitudinal objects notwithstanding – the proposed property-type semantics can still distinguish truth-evaluable attitude complements (i.e. coded centered and uncentered propositions; e.g. (43b), (43a)) from non-truth-evaluable attitude complements (i.e. coded individuals, properties, and intensional quantifiers; e.g. (44b), (44a)).

(43) a. $\lambda j \lambda y (\exists x)[laptop(j)(x) \wedge have(j)(x)(bill)]$
b. $\lambda j \lambda y (\exists x)[laptop(j)(x) \wedge have(j)(x)(y)]$

(44) a. $woman \ (\equiv \ \lambda j \lambda y [woman(j)(y)])$ b. $\lambda j \lambda y [x = y]$

For example, while the DP Emilie's attention has clear truth- (or accuracy-) conditions in the complement of the first coordinated verb, want, in (40),[18] it lacks such conditions in the complement of the second verb, seek. Since Propositionalists propose to analyze all attitude complements as truth-evaluable propositional contents, they cannot easily explain this observation. The difference between truth-evaluable and non-truth-evaluable attitude complements also explains the different modificational behavior of (5) and (11). This is achieved by stipulating that temporal adverbials (e.g. tomorrow) can only modify truth-evaluable complements (hence, the lower clause in (1b), but not in (10a)).

In addition to the above, the proposed property-type semantics is also able to explain certain differences w.r.t. the acceptability of different kinds of copredication. It has been reported that speakers intuitively judge (45) to be more natural than (46), which is, in turn, judged to be more natural than (47) (see [29,45]):

(45) Bill [[wants and needs] [$_{\text{DP}}$a laptop]].

(46) John [[needed and was looking for] [$_{\text{DP}}$a hammer]].

(47) John [[needed and crafted] [$_{\text{DP}}$a birdhouse]].

We propose to explain this difference in acceptability or naturalness through the identity (resp. difference) between the type-shifters that are used in the interpretation of each of the two attitude verbs. Coordinations of verbs whose interpretations use the same type-shifter are thus more acceptable (or natural) than coordinations of verbs whose interpretations use different type-shifters. The high degree of naturalness of (45) is then explained by the fact that want and need both have their control-use in (45), such that (45) can receive a 'non-distributive' interpretation involving the type-shifter CURRY (see (48)). The lower degree of naturalness of (47) (in comparison to (45)) is explained by the fact that the verbs need and craft assume different type-shifters (viz. CURRY and BE), such that (47) can only receive a distributive interpretation (see (50)).[19] The intermediate degree of naturalness of (46) is explained by the fact that the verbs

[18] This conjunct is true at all indices whose agentive center has Emilie's attention.

[19] Daniel Gutzmann (p.c.) has pointed out that the distributive interpretation of (47) (in ($\star$)) is equally natural/acceptable as (45):

($\star$) John [[needed [$_{\text{DP}}$a birdhouse]] and [crafted [$_{\text{DP}}$a birdhouse]]].

need and look for only assume the same type-shifter (viz. BE) since the relevant occurrence of need does not have a control-use, such that it directly receives a property-type interpretation. The report (46) can thus receive a non-distributive interpretation (in (49)).

(48) $[\![\text{Bill [[wants and needs] [}_{\text{CP}}\text{FOR [}_{\text{TP}}\text{PRO to HAVE [}_{\text{DP}}\text{a laptop]]]]}]\!]^{i,g}$
$= (\mathtt{want} \wedge \mathtt{need})(i)\big(\textsc{curry}\big(\lambda\langle j, y\rangle(\exists x)[\mathit{laptop}(j)(x) \wedge \mathit{have}(j)(x)(y)]\big)\big)(\mathit{bill})$
$\equiv (\mathtt{want} \wedge \mathtt{need})(i)(\lambda j \lambda y(\exists x)[\mathit{laptop}(j)(x) \wedge \mathit{have}(j)(x)(y)])(\mathit{bill})$

(49) $[\![\text{John [[needed to USE and was looking for] [}_{\text{DP}}\text{a hammer]]}]\!]^{i,g}$
$= (\mathtt{need} \wedge \mathtt{look\text{-}for})(i)\big(\textsc{be}\big(\lambda j \lambda P(\exists x)[\mathit{hammer}(j)(x) \wedge P(j)(x)]\big)\big)(\mathit{john})$
$\equiv (\mathtt{need} \wedge \mathtt{look\text{-}for})(i)(\mathit{hammer})(\mathit{john})$

(50) $[\![\text{John [[needed [}_{\text{CP}}\text{for PRO to HAVE [}_{\text{DP}}\text{a birdhouse]]] and [crafted [}_{\text{DP}}\text{a birdhouse]]]]}]\!]^{i,g}$
$= \big(\lambda x.\, \mathtt{need}(i)\big(\textsc{curry}\big(\lambda\langle j, y\rangle(\exists z)[\mathit{birdhouse}(j)(z) \wedge \mathit{have}(j)(z)(y)]\big)\big)(x) \wedge \mathit{craft}(i)\big(\textsc{be}\big(\lambda k \lambda P(\exists u[\mathit{birdhouse}(k)(u) \wedge P(k)(u)])\big)\big)(x)\big)(\mathit{john})$
$\equiv \big(\mathtt{need}(i)\big(\lambda j \lambda y(\exists z)[\mathit{birdhouse}(j)(z) \wedge \mathit{have}(j)(z)(y)]\big)(\mathit{john}) \wedge \mathit{craft}(i)(\mathit{birdhouse})(\mathit{john})\big)$

5 Achieving Ontological Parsimony

In the introduction to this paper, we have suggested that our proposed property-type semantics shares the two main lines of support for Propositionalism: empirical support (see Sect. 4) and methodological support. Methodological support for Propositionalism most saliently lies in its ontological parsimony. However, we have seen in Sect. 3 that a compositional property-type semantics for the featured constructions still requires a large number of different intensional objects, including centered and uncentered propositions, individual concepts, and intensional quantifiers, next to properties. This is due to the need to obtain intensional complements compositionally from the standard denotations of finite and infinitival clauses and of referential and quantificational DPs. The proposed version of property-type semantics is thus ontologically *lavish*, rather than parsimonious.

There are two ways of answering the objection from ontological lavishness: by restricting the domain of evaluation for ontological parsimony/lavishness to the semantic complements of intensional verbs (alternative 1) or by further restricting the intensional objects that are assumed by compositional property-type semantics (alternative 2). The first alternative justifies the ontological parsimony of property-type semantics by restricting the domain of evaluation for parsimony or lavishness to the types of intensional objects that can serve as the denotation of attitude complements. Since property-type semantics assumes a

---

We explain this observation through the fact that – in contrast to (45) – the possibility of interpreting a single occurrence of the DP a birdhouse as the common object of need and craft is not available in (47).

uniform(-type) interpretation of all such complements, it is more parsimonious than Montague-style/intensionalist semantics (see Sect. 2; cf. [22]).

The second alternative assumes the possibility of restricting intensional objects to the denotations of attitude verbs and attitudinal modifiers. All other natural language expressions (including attitude complements) receive an extensional interpretation. To get a match between the type of attitude verbs (here: $((s;(e;t));(e;t)))$ and the type of their complements, we use Heim and Kratzer's rule of Intensional Function Application ([19, p. 186]; see [9, p. 11]):[20]

Definition 1 (Intensional Function Application (IFA)). *If α is a branching node whose daughters are β, γ, and $[\![\beta]\!]^{i,g}$ is a function whose domain contains $(\lambda j. [\![\gamma]\!]^{j,g})$, then $[\![\alpha]\!]^{i,g} = [\![\beta]\!]^{i,g}\big(\lambda j. [\![\gamma]\!]^{j,g}\big)$.*

In particular, IFA enables the formation of (type-$(s;(e;t))$) properties from sets of individuals (type $(e;t)$) that are parametrized by indices.[21]

The extensional interpretation of the complements in the constructions from Sect. 2 and the shifting of these interpretations by extensional variants of EGN, KAP, and BE (see (51)–(53)) then enable the compositional, ontologically parsimonious interpretation of all intensional constructions from Sect. 2. Below, ξ, T, and O are variables of types t, $(e;t)$, and $((e;t);t)$, respectively:

(51) $\textit{ext}\text{-}\textsc{egn} := \lambda\xi\lambda x\,[\xi]$

(52) $\textit{ext}\text{-}\textsc{kap} := \lambda y\lambda x\,[x = y]$

(53) $\textit{ext}\text{-}\textsc{be} := \lambda O\lambda x\,[O(\lambda y.\, x = y)]$

The interpretations of (16), (33), (17a), and (23) are given below, where the extensional correlates of intensional non-logical constants are written in roman font:

(54) a. $[\![\text{Bill wants } [_{\text{CP}}\text{FOR } [_{\text{TP}}\text{PRO}_s \text{ to HAVE } [_{\text{DP}}\text{a laptop}]]]]\!]^{i,g}$
$= \big([\![\text{want}]\!]^{i,g}\big(\lambda j.[\![\lambda y(\exists x)[\text{laptop}(x) \wedge \text{have}(x)(y)]]\!]^{j,g}\big)\big)\big([\![\text{Bill}]\!]^{i,g}\big)$

b. $[\![[_{\text{DP}}\text{a laptop}]\ [\lambda_1 [\text{Bill wants } [_{\text{CP}}\text{FOR } [_{\text{TP}}\text{PRO}_s \text{ to HAVE } t_1]]]]]\!]^{i,g}$
$= \Big(\big([\![\lambda T(\exists x)[\text{laptop}(x) \wedge T(x)]]\!]^{i,g}\big)\big(\lambda x.[\![\text{want}]\!]^{i,g}\big(\lambda j.[\![\lambda y.\,\text{have}(x)(y)]\!]^{j,x,g}\big)\big)\Big)\big([\![\text{Bill}]\!]^{i,g}\big)$

[20] De Groote and Kanazawa [14] (see [8, pp. 195–203]) propose a generalization of IFA that sends expressions of extensional type to expressions of intensional type through an intensionalization operation *int*. This operation improves upon IFA by allowing for the 'intensional lifting' of all argument- (and value-)types in an expression, rather than only of the complete expression. For example, *int* sends expressions of type $(e;t)$ to expressions of type $((s;e);(s;t))$ (equivalent to type $(s;((s;e);t))$), rather than to expressions of type $(s;(e;t))$, as does IFA. However, since we are presently only interested in lifting type-$(e;t)$ to type-$(s;(e;t))$ objects, we restrict ourselves to IFA.

[21] The use of index-parameters is motivated by the core idea behind IFA, viz. the possibility of constructing an intensional model from a class of extensional models viewed as possible worlds or indices.

(55) $[\![\text{Bill wants } [_{\text{CP}}\text{that he/Bill has } [_{\text{DP}}\text{a laptop}]]]\!]^{i,g}_{\text{non-control}}$
$$= \big([\![\text{want}]\!]^{i,g}\big(\lambda j.\,[\![\mathit{ext}\text{-EGN}\big((\exists x)[\text{laptop}(x) \wedge \text{have}(x)(\mathit{bill})]\big)]\!]^{j,g}\big)\big)\big([\![\text{Bill}]\!]^{i,g}\big)$$
$$\equiv \big([\![\text{want}]\!]^{i,g}\big(\lambda j.\,[\![\lambda y(\exists x)[\text{laptop}(x) \wedge \text{have}(x)(\mathit{bill})]]\!]^{j,g}\big)\big)\big([\![\text{Bill}]\!]^{i,g}\big)$$

(56) $[\![\text{Klimt adored } [_{\text{DP}}\text{Emilie}]]\!]^{i,g}$
$$= \big([\![\text{adore}']\!]^{i,g}\big(\lambda j.\,[\![\mathit{ext}\text{-KAP}(\text{emilie})]\!]^{j,g}\big)\big)\big([\![\text{Klimt}]\!]^{i,g}\big)$$
$$\equiv \big([\![\text{adore}']\!]^{i,g}\big(\lambda j.\,[\![\lambda x.\, x = \text{emilie}]\!]^{j,g}\big)\big)\big([\![\text{Klimt}]\!]^{i,g}\big)$$
$$\equiv \big([\![\text{adore}]\!]^{i,g}\big(\lambda j.\,[\![\mathit{ext}\text{-BE}\big(\lambda T\,[T(\text{emilie})]\big)]\!]^{j,g}\big)\big)\big([\![\text{Klimt}]\!]^{i,g}\big)$$

(57) $[\![\text{Uli paints } [_{\text{DP}}\text{a unicorn}]]\!]^{i,g}$
$$= \big([\![\text{paint}]\!]^{i,g}\big(\lambda j.\,[\![\mathit{ext}\text{-BE}\big(\lambda T(\exists x)[\text{unicorn}(x) \wedge T(x)]\big)]\!]^{j,g}\big)\big)\big([\![\text{Uli}]\!]^{i,g}\big)$$
$$\equiv \big([\![\text{paint}]\!]^{i,g}\big(\lambda j.\,[\![\lambda y(\exists x)[\text{unicorn}(x) \wedge y = x]]\!]^{j,g}\big)\big)\big([\![\text{Uli}]\!]^{i,g}\big)$$

Note that – contrary to its ‘intensional’ counterpart (30) – the IFA-interpretation of (16) (in (54)) does not require an extensional variant of CURRY. This is due to the fact that IFA shifts parametrized objects of type $(\alpha;\beta)$ to unary functions of type $(s;(\alpha;\beta))$, rather than to multiary functions of type $(s\alpha;\beta)$. For $(\alpha;\beta) := (e;t)$, the instances of this shift then already have the desired complement type.

6 Conclusion and Future Work

In this paper, we have presented an alternative to Propositionalism, viz. property-type semantics. We have shown that this semantics preserves the merits of Propositionalism (esp. the uniform interpretation of attitude complements and the parsimony of the associated intensional ontology), while avoiding its empirical shortcomings (i.e. the inability to interpret objectual and *de se*-attitude reports, ‘know how’-sentences, and non-specific readings of depiction/resemblance reports). This is achieved by incorporating, into the semantics of attitude verbs, type-shifters from the familiar intensional complements to properties. As a result of this incorporation, all attitude verbs can be interpreted as relations to properties.

We close this paper with three pointers to future work. These regard an answer to the monotonicity problem from [53] (see (i), below), the formulation of a logic for relations between syntactically different attitude complements (see (ii)), and the development of a ‘coding’-version of Propositionalism (along the lines of property-type semantics; see (iii)):

(i) Zimmermann [53] has observed that property-type semantics like [52] and the one proposed in the present paper wrongly predict the validity of inferences to a common objective (see (59); cf. $(*)$ in fn. 12, where the direct objects in (59a) and (59b) have a non-specific reading and where the direct object in (59c) is interpreted as a quantifier over non-specific objects). This prediction is based on the possibility of quantifying over the non-specific objects in (59a) and (59b) (see (58)) and on the observation that the quantifier $(\exists P)$ in (59c) has the same witness for Uli’s as for Penny’s painting.

$$
\begin{array}{llll}
(58) & \text{a.} & [\![\text{Uli paints } [_{\text{DP}}\text{a unicorn}]]\!]^{i,g} & = \mathit{paint}(i)(\mathit{unicorn})(\mathit{uli}) \\
\hline
\Rightarrow & \text{b.} & [\![\text{Uli paints } [_{\text{DP}}\text{some-thing}]]\!]^{i,g} & = (\exists P)[\mathit{paint}(i)(P)(\mathit{uli})]
\end{array}
$$

$$
\begin{array}{llll}
(59) & \text{a.} & [\![\text{Uli paints } [_{\text{DP}}\text{a unicorn}]]\!]^{i,g} & = \mathit{paint}(i)(\mathit{unicorn})(\mathit{uli}) \\
 & \text{b.} & [\![\text{Penny paints } [_{\text{DP}}\text{a panther}]]\!]^{i,g} & = \mathit{paint}(i)(\mathit{panther})(\mathit{penny}) \\
\hline
\Rightarrow & \text{c.} & \multicolumn{2}{l}{[\![\text{Uli paints } [_{\text{DP}}\text{something Penny is painting}]]\!]^{i,g}} \\
 & & \multicolumn{2}{l}{= (\exists P)[\mathit{paint}(i)(P)(\mathit{uli}) \wedge \mathit{paint}(i)(P)(\mathit{penny})]}
\end{array}
$$

To block such intuitively invalid inferences, one could deny that the conclusion of such inferences (above, (59c)) involves unrestricted quantification over properties. Relevant work (see [18])[22] then needs to identify a suitable mechanism for contextual domain restriction.

(ii) We have suggested at the end of Sect. 4 that the uniform interpretation of attitude complements enables the identification of semantic relations between attitude complements of different syntactic categories. This is due to the partial ordering on the domain of properties, which is induced by the ordering on the set of truth-values. The resulting inclusion relations between properties may be used to provide a logic for the relations between different (traditionally, different-type) attitudinal objects. Such a logic has recently been demanded in [17, p. 16].

(iii) In linguistic semantics, propositionalism is often identified with one of two variants: *sententialism* (see [6,22]; cf. [35]) or *weak propositionalism* (see [39]; cf. [31, pp. 264, 267]). Respectively, these variants assume that the reduction of intensionality to clausal embedding proceeds by a syntactic (i.e. restructuring- or ellipsis-)analysis (i.e. sententialism) or by lexical decomposition/paraphrase (i.e. weak propositionalism). Our considerations from this paper suggest a third version of propositionalism – reminiscent of Zimmermann's [55] *P̲ropositionalism* (see fn. 2) – that is weaker than these variants.

The suggested version of propositionalism does not require that each intensional construction be truth-conditionally equivalent to some instance of clausal embedding. Rather, it only demands that all semantic attitude complements can be *coded* (via semantic representation, or type-shift) as (centered or uncentered) propositions. We expect that this version of propositionalism will be able to interpret some intensional constructions (incl. constructions with DP-biased attitude verbs) that resist a clausal analysis or paraphrase. We leave the development of this version of propositionalism as a topic for future research.

References

1. Anand, P., Nevins, A.: Shifty operators in changing contexts. In: Young, R.B. (ed.) Proceedings of SALT, vol. XIV (2004)
2. Bayer, S.: The coordination of unlike categories. Language **72**(3), 579–616 (1996)
3. Castañeda, H.-N.: 'He': a study in the logic of self-consciousness. Ratio **8**, 130–157 (1966)

[22] A similar strategy is pursued in unpublished lecture notes by Ede Zimmermann (based on joint work with Magdalena Kaufmann).

4. Chierchia, G.: Anaphora and attitudes de se. In: Bartsch, R., van Benthem, J.F.A.K., van Emde Boas, P. (eds.) Semantics and Contextual Expression, pp. 1–11. Foris Publications, Dordrecht (1989)
5. Deal, A.R.: Property-type objects and modal embedding. In: Proceedings of Sinn und Bedeutung 12 (2008)
6. den Dikken, M., Larson, R., Ludlow, P.: Intensional transitive verbs and abstract clausal complementation. In: Grzankowski, A., Montague, M. (eds.) Non-Propositional Intentionality, pp. 46–94. Oxford University Press, Oxford (2018)
7. Egan, A.: Secondary qualities and self-location. Philos. Phenomenol. Res. **72**, 97–119 (2006)
8. van Eijck, J., Unger, C.: Computational Semantics with Functional Programming. Cambridge University Press, Cambridge (2010)
9. von Fintel, K., Heim, I.: Intensional Semantics: Lecture Notes, MIT, Cambridge (2011)
10. Forbes, G.: Objectual attitudes. Linguist. Philos. **23**(2), 141–183 (2000)
11. Forbes, G.: Attitude Problems: An Essay on Linguistic Intensionality. Oxford University Press, Oxford (2006)
12. Forbes, G.: Content and Theme in Attitude Ascriptions. In: Grzankowski, A., Montague, M. (eds.) Non-Propositional Intentionality, pp. 114–133. Oxford University Press, Oxford (2018)
13. van Geenhoven, V., McNally, L.: On the property analysis of opaque complements. Lingua **115**, 885–914 (2005)
14. de Groote, P., Kanazawa, M.: A note on intensionalization. J. Log. Lang. Inf. **22**(2), 173–194 (2013)
15. Grzankowski, A.: Limits of propositionalism. Inquiry **57**(7–8), 819–838 (2016)
16. Grzankowski, A.: A relational theory of non-propositional attitudes. In: Grzankowski, A., Montague, M. (eds.) Non-Propositional Intentionality, pp. 134–151. Oxford University Press, Oxford (2018)
17. Grzankowski, A., Montague, M.: Non-propositional intentionality: an introduction. In: Grzankowski, A., Montague, M. (eds.) Non-Propositional Intentionality, pp. 1–18. Oxford University Press, Oxford (2018)
18. Haslinger, N.: Quantificational arguments of opaque verbs in German: disentangling monotonicity and context dependency. Master's thesis, University of Vienna (in progress)
19. Heim, I., Kratzer, A.: Semantics in Generative Grammar, Blackwell Textbooks in Linguistics, vol. 13. Blackwell, Malden (1998)
20. Kaplan, D.: How to Russell a Frege-Church. J. Philos. **72**(19), 716–729 (1975)
21. Kratzer, A.: Decomposing attitude verbs: handout from a talk in honor of Anita Mittwoch on her 80th birthday. Hebrew University, Jerusalem (2006)
22. Larson, R.: The grammar of intensionality. In: Preyer, G., Peter, G. (eds.) Logical Form and Language, pp. 228–262. Oxford University Press, Oxford, (2002)
23. Lewis, D.: Attitudes de dicto and de se. Philos. Rev. **88**(4), 513–543 (1979)
24. Liefke, K.: A single-type semantics for natural language. Dissertation, Tilburg Center for Logic and Philosophy of Science, Tilburg University (2014)
25. Liefke, K.: A 'situated' solution to Prior's substitution problem. In: Espinal, M.T., Castroviejo, E., Leonetti, M., McNally, L. (eds.) Proceedings of Sinn und Bedeutung, vol. 23. Semantics Archive (to appear)
26. Liefke, K., Werning, M.: Evidence for single-type semantics - an alternative to e/t-based dual-type semantics. J. Semant. **35**(4), 639–685 (2018)
27. McCawley, J.: On identifying the remains of deceased clauses. Lang. Res. **9**, 73–85 (1974)

28. Moltmann, F.: Intensional verbs and quantifiers. Nat. Lang. Semant. **5**(1), 1–52 (1997)
29. Moltmann, F.: Intensional verbs and their intentional objects. Nat. Lang. Semant. **16**, 239–270 (2008)
30. Montague, R.: English as a formal language. In: Thomason, R.H. (ed.) Formal Philosophy: Selected Papers of Richard Montague, pp. 188–221. Yale University Press, New Haven (1976)
31. Montague, R.: The proper treatment of quantification in ordinary English. In: Thomason, R.H. (ed.) Formal Philosophy: Selected Papers of Richard Montague, pp. 247–270. Yale University Press, New Haven (1976)
32. Montague, R.: Universal grammar. Theoria **36**(3), 373–398 (1970)
33. Muskens, R.: Meaning and Partiality. CSLI Lecture Notes. CSLI Publications, Stanford (1995)
34. Orey, S.: Model theory for the higher order predicate calculus. Trans. Am. Math. Soc. **92**(1), 72–84 (1959)
35. Parsons, T.: Meaning sensitivity and grammatical structure. In: Chiara, M.L., et al. (eds.) Structures and Norms in Science, pp. 369–383. Kluwer Academic Publishers, Dordrecht (1997)
36. Partee, B.: Noun phrase interpretation and type-shifting principles. In: Groenendijk, J., de Jong, D., Stokhof, M. (eds.) Studies in Discourse Representation Theory and the Theory of Generalized Quantifiers, pp. 115–143. Foris Publications, Dordrecht (1987)
37. Percus, O., Sauerland, U.: On the LFs of attitude reports. In: Weisgerber, M. (ed.) Proceedings of Sinn und Bedeutung 7. Arbeitspapiere des FB Sprachwissenschaft, vol. 114. University of Konstanz, Konstanz (2003)
38. Perry, J.: The problem of the essential indexical. Noûs **13**(1), 3–21 (1979)
39. Quine, W.V.: Quantifiers and propositional attitudes. J. Philos. **53**(5), 177–187 (1956)
40. Roberts, C.: Know-how: a compositional approach. In: Tor, E., Itor, E. (eds.) Theory and Evidence, pp. 1–31. CSLI, Stanford (2009)
41. Russell, B.: On denoting. Mind **14**(56), 479–493 (1905)
42. Sæbø, K.J.: Do you know what it means to miss New Orleans?: more on missing. In: Approaches to Meaning, pp. 105–127. Brill (2014)
43. Sag, I., Gazdar, G., Wasow, T., Weisler, S.: Coordination and how to distinguish categories. Nat. Lang. Linguist. Theory **3**(2), 117–171 (1985)
44. Schönfinkel, M.: Über die Bausteine der mathematischen Logik. Math. Ann. **92**, 305–316 (1924)
45. Schwarz, F.: On needing propositions and looking for properties. In: Gibson, M., Howell, J. (eds.) Proceedings of SALT, vol. XVI, pp. 259–276. Cornell University, Ithaca, NY (2006)
46. Sinhababu, N.: Advantages of propositionalism. Pac. Philos. Q. **96**(2), 165–180 (2015)
47. Stanley, J.: Know How. Oxford University Press, Oxford (2011)
48. Stephenson, T.: Control in centred worlds. J. Semant. **27**(4), 409–436 (2010)
49. Szabó, Z.G.: Sententialism and Berkeley's master argument. Philos. Q. **55**(220), 462–474 (2005)
50. Tichý, P.: Foundations of partial type theory. Rep. Math. Log. **14**, 59–72 (1982)
51. Yalcin, S.: Stanley on the de se. Handout from a talk at the Pacific APA (2012)
52. Zimmermann, T.E.: On the proper treatment of opacity in certain verbs. Nat. Lang. Semant. **1**(2), 149–179 (1993)

53. Zimmermann, T.E.: Monotonicity in opaque verbs. Linguist. Philos. **29**(6), 715–761 (2006)
54. Zimmermann, T.E.: What it takes to be missing. In: Hanneforth, T., Fanselow, G. (eds.) Language and Logos: Studies in Theoretical and Computational Linguistics, vol. 72, pp. 255–265. Walter de Gruyter (2012)
55. Zimmermann, T.E.: Painting and opacity. In: Freitag, W., Rott, H., Sturm, H., Zinke, A. (eds.) Von Rang und Namen: Philosophical Essays in Honour of Wolfgang Spohn, pp. 427–453. Mentis, Münster (2016)

Note on Globally Sound Analytic Calculi for Quantifier Macros

Matthias Baaz[1(✉)] and Anela Lolic[2(✉)]

[1] Institute of Discrete Mathematics and Geometry, TU Wien, Vienna, Austria
baaz@logic.at
[2] Institute of Logic and Computation, TU Wien, Vienna, Austria
anela@logic.at

Abstract. This paper focuses on a globally sound but possibly locally unsound analytic sequent calculus for the quantifier macro Q defined by $Q_{x,y}A(x,y) = \forall x \exists y A(x,y)$. It is demonstrated that no locally sound analytic representation exists.

Keywords: Sequent calculus · Cut-elimination · Quantifier macros

1 Introduction

The concept of an analytic proof introduced by Gottfried Wilhelm Leibniz is of fundamental importance for mathematics and logic. An analytic proof is a proof where all the information used in the proof is already contained in the end-sequent. This is of course an idealization, however sequent calculi which are cut-free complete can be considered since Gentzen 1934 [2] as a close approximation. In this paper we refine the concept of analyticity by considering macros of connectives and quantifiers[1].

In logic an analytic proof of a statement containing only macros of connectives and quantifiers would itself be based on these macros. The question is, whether it is possible to form inference rules for such macros that are compatible with cut-elimination. The answer is obviously "yes" for macros of connectives, "no" if macros of quantifiers are considered in the framework of usual eigenvariable conditions, which allow for a step-wise verification of the proof. In contrast an analytic framework can be constructed if globally sound but possibly locally unsound concepts of proof are introduced.

M. Baaz—This work is partially supported by FWF Project P 31063.
A. Lolic—Recipient of a DOC Fellowship of the Austrian Academy of Sciences at the Institute of Logic and Computation at TU Wien.

[1] Macros of connectives and quantifiers have a wide range of application in mathematics and are used to deal with explicit definitions, for example the handling of integrals as objects. In logic it is known that hierarchies of macros can be used to abbreviate proofs [3].

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 486–497, 2019.
https://doi.org/10.1007/978-3-662-59533-6_29

2 Macros for Connectives

We consider **LK** in a multiplicative version based on pairs of multisets as sequents.

Definition 1 (LK).

- *Axiom schema:* $A \rightarrow A$
- *The propositional rules:*
  $\wedge$*-introduction*

$$\frac{A, B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} \wedge_l \qquad \frac{\Gamma_1 \rightarrow \Delta_1, A \qquad \Gamma_2 \rightarrow \Delta_2, B}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, A \wedge B} \wedge_r$$

  $\vee$*-introduction*

$$\frac{A, \Gamma_1 \rightarrow \Delta_1 \qquad B, \Gamma_2 \rightarrow \Delta_2}{A \vee B, \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2} \vee_l \qquad \frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B} \vee_r$$

  $\rightarrow$*-introduction*

$$\frac{\Gamma_1 \rightarrow \Delta_1, A \qquad B, \Gamma_2 \rightarrow \Delta_2}{A \supset B, \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2} \supset_l \qquad \frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B} \supset_r$$

  $\neg$*-introduction*

$$\frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta} \neg_l \qquad \frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A} \neg_r$$

  weakening

$$\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} w_l \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A} w_r$$

  contraction

$$\frac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} c_l \qquad \frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A} c_r$$

  cut

$$\frac{\Gamma_1 \rightarrow \Delta_1, A \qquad A, \Gamma_2 \rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2} cut$$

- *The quantifier rules:*

$$\frac{A(a), \Gamma \rightarrow \Delta}{\exists x A(x) \Gamma \rightarrow \Delta} \exists_l \qquad \frac{\Gamma \rightarrow \Delta, A(a)}{\Gamma \rightarrow \Delta, \forall x A(x)} \forall_r$$

  where a is an eigenvariable.

$$\frac{A(t), \Gamma \rightarrow \Delta}{\forall x A(x) \Gamma \rightarrow \Delta} \forall_l \qquad \frac{\Gamma \rightarrow \Delta, A(t)}{\Gamma \rightarrow \Delta, \exists x A(x)} \exists_r$$

  where t is a term.

Definition 2. *A macro for connectives is a formula based on propositional variables which is considered as a connective in its own right.*

Example 1. We define the binary connective $\leftrightarrow$ as

$$A \leftrightarrow B = (A \supset B) \wedge (B \supset A).$$

The semantical meaning of such a macro is obvious.

Proposition 1. *To every macro $\Box$ defined by connectives a left inference rule, denoted by $\Box_l$, and right inference rule, denoted by $\Box_r$, are associated such that* **LK** *extended by $\Box_l$ and $\Box_r$ admits cut-elimination.*

Proof. Let $\mathcal{B}(A_1, \ldots, A_n)$ be the defining propositional formula.

$\Box_r$: Consider $\rightarrow \mathcal{B}(A_1, \ldots, A_n)$. Read the rules of **LK** backwards as long as this is possible. $\Phi, \Psi \subseteq A_1, \ldots, A_n$ for topmost sequents $\Phi \rightarrow \Psi$. Let

$$\Phi_1^R \rightarrow \Psi_1^R, \quad \ldots, \quad \Phi_{m_r}^R \rightarrow \Psi_{m_r}^R$$

be the topmost sequents with $\Phi_i^R \cap \Psi_i^R = \emptyset$. Then

$$\frac{\Gamma_1, \Phi_1^R \rightarrow \Delta_1, \Psi_1^R \qquad \Gamma_{m_r}, \Phi_{m_r}^R \rightarrow \Delta_{m_r}, \Psi_{m_r}^R}{\Gamma_1, \ldots, \Gamma_{m_r} \rightarrow \Delta_1, \ldots, \Delta_{m_r}, \Box(A_1, \ldots, A_n)}\ \Box_r$$

is the intended rule (the variables are instantiated by formulae).

$\Box_l$: analogous.

Obviously in **LK**

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Phi_1^L \rightarrow \Psi_1^L} \quad \dfrac{}{\Phi_{m_L}^L \rightarrow \Psi_{m_L}^L}}{\vdots \qquad\qquad \vdots}}{\rightarrow \mathcal{B}(A_1, \ldots, A_n)} \qquad \dfrac{\dfrac{\dfrac{}{\Phi_1^R \rightarrow \Psi_1^R} \quad \dfrac{}{\Phi_{m_R}^R \rightarrow \Psi_{m_R}^R}}{\vdots \qquad\qquad \vdots}}{\mathcal{B}(A_1, \ldots, A_n) \rightarrow}}{\rightarrow}$$

Consider the initial sequents as clauses. This clause set is unsatisfiable and therefore there is a refutation as propositional resolution is complete [4]. The critical step of Gentzen's cut-elimination is the grade reduction of a mix formula with outermost symbol $\Box$: any resolution refutation of the clauses above is suitable.

Example 2. $\Box_r$ is defined via

$$\dfrac{\dfrac{A \rightarrow B}{\rightarrow A \supset B}\ \supset_r \qquad \dfrac{B \rightarrow A}{\rightarrow B \supset A}\ \supset_r}{\rightarrow (A \supset B) \wedge (B \supset A)}\ \wedge_r$$

and $\Box_l$ is defined via

$$\dfrac{\dfrac{\dfrac{\rightarrow A, B \qquad A \rightarrow A}{B \supset A \rightarrow A}\ \supset_l \qquad \dfrac{B \rightarrow B \qquad A, B \rightarrow}{B, B \supset A \rightarrow}\ \supset_l}{(A \supset B), (B \supset A) \rightarrow}\ \supset_l}{(A \supset B) \wedge (B \supset A) \rightarrow}\ \wedge_l$$

Consequently, the inference rules for $\leftrightarrow$ are

$$\frac{A, \Gamma \to \Delta, B \qquad B, \Gamma \to \Delta, A}{\Gamma \to \Delta, A \leftrightarrow B} \leftrightarrow_r$$

$$\frac{A, B, \Gamma \to \Delta \qquad \Gamma \to \Delta, A, B}{A \leftrightarrow B, \Gamma \to \Delta} \leftrightarrow_l$$

The critical reduction step for $\leftrightarrow$ in the cut-elimination procedure of Gentzen is (contractions are hidden):

$$\frac{\dfrac{\Gamma_1 \to \Delta_1, A, B \qquad B, \Gamma_2 \to \Delta_2, A}{\Gamma_1, \Gamma_2' \to \Delta_1', \Delta_2, A} * \qquad \dfrac{A, \Gamma_3 \to \Delta_3, B \qquad A, B, \Gamma_4 \to \Delta_4}{A, \Gamma_3, \Gamma_4'' \to \Delta_3'', \Delta_4} *}{\Gamma_1, \Gamma_2', \Gamma_3''', \Gamma_4''' \to \Delta_1''', \Delta_2''', \Delta_3'', \Delta_4} *$$

where $*$ is the *mix* rule.

Therefore, a sequent calculus with $\leftrightarrow_l, \leftrightarrow_r$ with only logical rules admits cut-elimination.

In this paper we discuss the extension of this property to macros for quantifiers. We will concentrate on the macro for $\forall x \exists y$.

3 The Analytic Sequent Calculus LQ: A First Approach

Definition 3. *A macro for quantifiers M is a formula based on quantifiers $Q_i \in \{\forall, \exists\}, 1 \leq i \leq n$ which is considered as a quantifier in its own right:*

$$M_{x_1,\ldots,x_n} A(x_1, \ldots, x_n) = Q_1 x_1, \ldots, Q_n x_n A(x_1, \ldots, x_n).$$

We will concentrate on the quantifier macro Q defined as

$$Q_{x,y} A(x, y) = \forall x \exists y A(x, y).$$

The language $\mathcal{L}_Q$ of the calculus **LQ** is based on the usual language of first-order logic with exception that the quantifiers are replaced by the quantifier Q.

Definition 4 (LQ). *The calculus* **LQ** *is* **LK***, where the quantifier rules are exchanged by*

$$\frac{\Gamma \to \Delta, A(a, t)}{\Gamma \to \Delta, Q_{x,y} A(x, y)} Q_r$$

where a does not occur in the lower sequent and

$$\frac{A(t, a), \Gamma \to \Delta}{Q_{x,y} A(x, y), \Gamma \to \Delta} Q_l$$

where a *does not occur in the lower sequent or in* t *(the inference* Q_r *is derived from*

$$\cfrac{\cfrac{\Gamma \to \Delta, A(a,t)}{\Gamma \to \Delta, \exists y A(a,y)}\ \exists_r}{\Gamma \to \Delta, \forall x \exists y A(x,y)}\ \forall_r$$

where a *is an eigenvariable not allowed to occur in the lower sequent and the inference* Q_l *is derived from*

$$\cfrac{\cfrac{A(t,a), \Gamma \to \Delta}{\exists y A(t,y), \Gamma \to \Delta}\ \exists_l}{\forall x \exists y A(x,y), \Gamma \to \Delta}\ \forall_l$$

where a *is an eigenvariable not allowed to occur in the lower sequent or in* t*). The dual quantifier* Q^D *to* Q *can be defined in the usual dual way*

$$Q^D_{x,y} A(x,y) = \neg Q_{x,y} \neg A(x,y) = \exists x \forall y \neg A(x,y).$$

The quantifier introduction rules for Q^D *are*

$$\frac{\Gamma \to \Delta, A(t,a)}{\Gamma \to \Delta, Q^D_{x,y} A(x,y)}\ Q^D_r$$

where a *does not occur in the lower sequent and in* t *and*

$$\frac{A(a,t), \Gamma \to \Delta}{Q^D_{x,y} A(x,y), \Gamma \to \Delta}\ Q^D_l$$

where a *does not occur in the lower sequent.*

The usual quantifier rules of **LK** *can be obtained by partial dummy applications of* Q*. We denote these dummy quantifiers and their introduction rules by* $\forall$ *and* $\exists$ *(*$\forall_r, \forall_l, \exists_r, \exists_l$*).*

An **LQ***-proof is a tree formed according to the rules of* **LQ** *such that all leaves are axioms. The notion of context formulae, auxiliary formulae and principal formulae is as in* **LK***.*

Example 3. The sequent $Q_{x,y} A(x,y) \to \forall x \exists y A(x,y)$ is derivable in **LQ**:

$$\cfrac{\cfrac{\cfrac{A(a,b) \to A(a,b)}{A(a,b) \to \exists y A(a,y)}\ \exists_r}{Q_{x,y} A(x,y) \to \exists y A(a,y)}\ Q_l}{Q_{x,y} A(x,y) \to \forall x \exists y A(x,y)}\ \forall_r$$

Theorem 1. LQ *is sound.*

Proof. The macro Q can be replaced by $\forall\exists$ everywhere in the derivation. The resulting derivation is an **LK**-derivation.

Theorem 2. **LQ** *admits cut-elimination.*

Proof. We follow Gentzen's procedure, cf. [5]. The only difference to Gentzen's proof is the reduction of Q, which can be performed as follows:

$$\dfrac{\dfrac{\Gamma \to \Delta, A(a,t)}{\Gamma \to \Delta, Q_{x,y}A(x,y)}\,Q_r \qquad \dfrac{A(t',a'), \Pi \to \Lambda}{Q_{x,y}A(x,y), \Pi \to \Lambda}\,Q_l}{\Gamma, \Pi \to \Delta, \Lambda}\,mix$$

where a does not occur in the lower sequent, a' does not occur in the lower sequent and in t', all occurrences of a, a', t, t' are indicated (note that $Q_{x,y}A(x,y)$ does not occur in Δ or Π at this step). This can be reduced to

$$\dfrac{\Gamma \to \Delta, A(t',t) \qquad A(t',t), \Pi \to \Lambda}{\Gamma, \Pi' \to \Delta', \Lambda}\,mix$$

Corollary 1 (Midsequent-theorem). *For every proof of a prenex sequent in* **LQ** *there is a cut-free proof with a midsequent such that every inference above the midsequent is structural or propositional and every inference below the midsequent is structural or* Q_r*,* Q_l*.*

Proof. As in **LK** we can delay the quantifier inferences.

Proposition 2. **LQ** *is incomplete w.r.t. the sequents provable in* **LK***.*

Proof. Assume the sequent $Q_{x,y}A(x,y) \to Q_{x,y}(A(x,y) \vee C)$ was provable. Then it was provable without cuts. A cut-free derivation after deletion of weakenings and contractions has the initial form

$$\dfrac{\dfrac{A(a,b) \to A(a,b)}{A(a,b) \to A(a,b) \vee C}}{\vdots}$$

Due to the mixture of strong (eigenvariable dependent) and weak positions in Q none of the inference rules Q_r, Q_l can be applied.

Corollary 2. *Compound axioms* $A \to A$ *cannot be reduced to atomic ones.*

Corollary 3. *The sequent* $\forall x \exists y A(x,y) \to Q_{x,y}A(x,y)$ *is not derivable in* **LQ**.

Proof. We would obtain completeness in case $\forall x \exists y A(x,y) \to Q_{x,y}A(x,y)$ was derivable, because $Q_{x,y}A(x,y) \to \forall x \exists y A(x,y)$ is derivable by Example 3.

The usual quantifier shifts of classical logic are not derivable in **LQ**.

Definition 5 (quantifier shifts). *Let* $Q^* \in \{Q, Q^D\}$ *and* $\circ \in \{\wedge, \vee\}$*. Then the quantifier shifts for the operators* $\wedge, \vee$ *are:*

1. $Q^*_{x,y}(A \circ B(x,y)) \to A \circ Q^*_{x,y}B(x,y)$,
2. $Q^*_{x,y}(A(x,y) \circ B) \to Q^*_{x,y}A(x,y) \circ B$.

Let $(Q^*, Q^{D^*}) \subseteq \{(Q, Q^D), (Q^D, Q)\}$*, then the quantifier shifts for* $\supset$ *are:*

3. $Q^*_{x,y}(A \supset B(x,y)) \to A \supset Q^*_{x,y}B(x,y)$,
4. $Q^*_{x,y}(A(x,y) \supset B) \to Q^{D^*}_{x,y}A(x,y) \supset B$,
5. $A \supset Q^*_{x,y}B(x,y) \to Q^*_{x,y}(A \supset B(x,y))$,
6. $Q^{D^*}_{x,y}A(x,y) \supset B \to Q^*_{x,y}(A(x,y) \supset B)$.

The quantifier shifts for $\neg$ *are:*

7. $Q^*_{x,y}\neg A(x,y) \to \neg Q^{D^*}_{x,y}A(x,y)$,
8. $\neg Q^*_{x,y}A(x,y) \to Q^{D^*}_{x,y}\neg A(x,y)$.

Theorem 3. *None of the quantifier shifts from Definition 5 is derivable in* **LQ**.

4 LK, LK$^+$ and LK^{++}

The inherent incompleteness of **LQ** even for trivial statements is a consequence of the fact that Q represents a quantifier inference macro combining a strong and a weak occurrence of quantifiers. (**LQ**$'$ where Q' is a macro of only $\forall$ or only $\exists$ quantifiers is complete and admits cut-elimination.) The solution is to consider sequent calculi with concepts of proof which are globally but not necessarily locally sound. This means that all derived statements are true but that not every subderivation is meaningful.

Definition 6 (side variable relation $<_{\varphi,\mathbf{LK}}$). *Let* φ *be an* **LK**-*derivation. We say* b *is a side variable of* a *in* φ *(written* $a <_{\varphi,\mathbf{LK}} b$*) if* φ *contains a strong quantifier inference rule of the form*

$$\frac{\Gamma \to \Delta, A(a, b, \overline{c})}{\Gamma \to \Delta, \forall x A(x, b, \overline{c})} \forall_r$$

or of the form

$$\frac{A(a, b, \overline{c}), \Gamma \to \Delta}{\exists x A(x, b, \overline{c}), \Gamma \to \Delta} \exists_l$$

In addition to strong and weak quantifier inferences (in **LK** strong quantifier inferences are $\forall_r$ and $\exists_l$ and weak quantifier inferences are $\forall_l$ and $\exists_r$) we define **LK**-suitable quantifier inferences.

Definition 7 (LK-suitable quantifier inferences). *We say a quantifier inference is suitable for a proof* φ *if either it is a weak quantifier inference, or the following three conditions are satisfied:*

- *(substitutability) the eigenvariable does not appear in the conclusion of* φ.
- *(side variable condition) the relation* $<_{\varphi,\mathbf{LK}}$ *is acyclic.*
- *(weak regularity) the eigenvariables of an inference with eigenvariable conditions are not the eigenvariables of another inference with eigenvariable conditions in* φ.

Definition 8 ($\mathbf{LK}^+$). *We obtain* $\mathbf{LK}^+$ *from* $\mathbf{LK}$ *by replacing the usual eigenvariable conditions by* $\mathbf{LK}$*-suitable ones.*

A further weakening of the eigenvariable conditions gives rise to the notion of weak suitability.

Definition 9 (LK-weakly suitable quantifier inference). *A quantifier inference is weakly suitable for a proof* φ *if either it is a weak quantifier inference or it satisfies substitutability, the side-variable condition, and:*

- *(very weak regularity) the eigenvariable of an inference with principal formula* A *is different to the eigenvariable of an inference with principal formula* A' *whenever* $A \neq A'$.

Definition 10 ($\mathbf{LK}^{++}$). *We obtain* $\mathbf{LK}^{++}$ *from* $\mathbf{LK}$ *by replacing the usual eigenvariable conditions by* $\mathbf{LK}$*-weakly suitable ones.*

Remark 1. Note that eigenvariables may occur outside of the scope of the intended quantifier.

Theorem 4. *If a sequent is* $\mathbf{LK}^{++}$*-derivable, then it is already* $\mathbf{LK}$*-derivable.*

Proof. Let φ be an $\mathbf{LK}^{++}$-proof. Replace every universal quantifier inference unsound w.r.t. $\mathbf{LK}$ by an $\supset_l$ inference:

$$\frac{\Gamma \rightarrow \Delta, A(a) \qquad \forall x\, A(x) \rightarrow \forall x\, A(x)}{\Gamma, A(a) \supset \forall x\, A(x) \rightarrow \Delta, \forall x\, A(x)} \supset_l$$

Similarly replace every existential quantifier inference unsound w.r.t. $\mathbf{LK}$ by an $\supset_l$ inference

$$\frac{\exists x\, A(x) \rightarrow \exists x\, A(x) \qquad A(a), \Gamma \rightarrow \Delta}{\Gamma, \exists x\, A(x), \exists x\, A(x) \supset A(a) \rightarrow \Delta} \supset_l$$

By doing this, we obtain a proof of the desired sequent, together with formulae of the form $A(a) \supset \forall x\, A(x)$ or $\exists x\, A(x) \supset A(a)$ on the left-hand side. Note that the resulting derivation does not contain any inference based on eigenvariable conditions. We can eliminate each of $A(a) \supset \forall x\, A(x)$ or $\exists x\, A(x) \supset A(a)$ on the left-hand side by adding an existential quantifier inference and cutting with formulae of the form

$$\rightarrow \exists y \left(A(y) \supset \forall x\, A(x)\right)$$

or of the form

$$\rightarrow \exists y \left(\exists x\, A(x) \supset A(y)\right),$$

both of which are easily derivable. Note that the existential quantifier inferences can be carried out in a way that is permissible by $\mathbf{LK}$ because $<_{\varphi,\mathbf{LK}}$ does not loop.

Corollary 4. *If a sequent is derivable in* $\mathbf{LK}^+$*, then it is already derivable in* $\mathbf{LK}$.

Example 4. Consider the following locally unsound but globally sound derivation φ in $\mathbf{LK}^+$ (and $\mathbf{LK}^{++}$):

$$\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to \forall y A(y)}\,\forall_r}{\to A(a) \supset \forall y A(y)}\,\supset_r}{\to \exists x(A(x) \supset \forall y A(y))}\,\exists_r$$

As a is the only eigenvariable the side variable relation $<_{\varphi,\mathbf{LK}}$ is empty.

It is essential that the order defined by the side variable relation does not loop.

Example 5. Let φ be the following cut-free derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a,b) \to A(a,b)}{A(a,b) \to \forall y A(a,y)}\,\forall_r}{A(a,b) \to \exists x \forall y A(x,y)}\,\exists_r}{\exists y A(y,b) \to \exists x \forall y A(x,y)}\,\exists_l}{\forall x \exists y A(y,x) \to \exists x \forall y A(x,y)}\,\forall_l$$

This enforces the following side variable conditions, which loop:

$$a <_{\varphi,\mathbf{LK}} b \qquad b <_{\varphi,\mathbf{LK}} a.$$

Note that the other conditions for $\mathbf{LK}$-suitable quantifier inferences, substitutability and weak regularity, hold. However, the construction in the proof of Theorem 4 is impossible as neither

$$\exists x(\exists y A(y,b) \supset A(y,b)), A(a,b) \supset \forall y A(a,y), \Pi \to \Gamma$$

nor

$$\exists y A(y,b) \supset A(a,b), \exists x(A(a,x) \supset \forall y A(a,y), \Pi \to \Gamma$$

are derivable in $\mathbf{LK}$ from

$$\exists y A(y,b) \supset A(a,b), A(a,b) \supset \forall y A(a,y), \Pi \to \Gamma.$$

The concept of $\mathbf{LK}^{++}$-proofs can be used to handle cut-free complete quantifier macros as Q.

In [1] the focus has been on the strongly reduced complexity of cut-free $\mathbf{LK}^+$ and $\mathbf{LK}^{++}$ proofs (Theorem 2.6 and Corollary 2.7). The focus of this note is to achieve cut-free completeness for sequent calculi where logical rules introduce mixed blocks of strong and weak quantifiers.

5 The Analytic Sequent Calculus $\mathbf{LQ}^{++}$

From the example in the Sect. 3 it becomes obvious that there will be no analytic calculus with local rules to represent any reasonable fragment of the full logic with Q: the reason is that the inference rules for Q need eigenvariables in both polarities. The solution is to keep global soundness but to give up local soundness, as in [1]. To this aim, the eigenvariable conditions will be weakened.

Definition 11 (side variable relation $<_{\varphi,\mathbf{LQ}}$). *Let φ be an $\mathbf{LQ}$-derivation. We say b is a side variable of a in φ (written $a <_{\varphi,\mathbf{LQ}} b$) if φ contains a strong quantifier inference of the form*

$$\frac{A(t,a), \Gamma \rightarrow \Delta}{Q_{x,y}A(x,y), \Gamma \rightarrow \Delta}\, Q_l$$

and b occurs in t.

Definition 12 (LQ-weakly suitable quantifier inferences). *A quantifier inference is $\mathbf{LQ}$-suitable for a proof φ if the following three conditions are satisfied:*

- *(substitutability) the eigenvariable does not appear in the conclusion of φ.*
- *(side variable condition) the relation $<_{\varphi,\mathbf{LQ}}$ is acyclic.*
- *(very weak regularity) the eigenvariables of an inference with principal formula A are not the eigenvariables of an inference in φ with another principal formula A'.*

Definition 13 (analytic sequent calculus $\mathbf{LQ}^{++}$). *The analytic sequent calculus $\mathbf{LQ}^{++}$ is $\mathbf{LQ}$, except that we replace quantifier inferences with $\mathbf{LQ}$-suitable quantifier inferences.*

Example 6. The sequent $Q_{x,y}A(x,y) \rightarrow Q_{x,y}(A(x,y) \vee C)$ is $\mathbf{LQ}^{++}$-derivable. Consider the derivation $\varphi =$

$$\cfrac{\cfrac{\cfrac{A(a,b) \rightarrow A(a,b)}{A(a,b) \rightarrow A(a,b) \vee C}\, w_r + \vee_r}{A(a,b) \rightarrow Q_{x,y}(A(x,y) \vee C)}\, Q_r}{Q_{x,y}A(x,y) \rightarrow Q_{x,y}(A(x,y) \vee C)}\, Q_l$$

with $b <_{\varphi,\mathbf{LQ}} a$.

In contrast to $\mathbf{LQ}$ the usual quantifier shifts are derivable in $\mathbf{LQ}^{++}$.

Theorem 5. *The quantifier shifts from Definition 5 are $\mathbf{LQ}^{++}$-derivable.*

Proof. Due to space limitations we will only demonstrate the $\mathbf{LQ}^{++}$-derivations of two quantifier shifts. All other derivations of quantifier shifts can be carried out analogously.

The quantifier shift $Q^D_{x,y}(A(x,y) \supset B) \rightarrow Q_{x,y}A(x,y) \supset B$ is derivable in $\mathbf{LQ}^{++}$. Its derivation is $\varphi =$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a,b) \to A(a,b)}{A(a,b) \to B, A(a,b)}\,w_r \quad \dfrac{B \to B}{B, A(a,b) \to B}\,w_l}{A(a,b) \supset B, A(a,b) \to B}\,\supset_l}{A(a,b) \supset B, Q_{x,y}A(x,y) \to B}\,Q_l}{Q^D_{x,y}(A(x,y) \supset B), Q_{x,y}A(x,y) \to B}\,Q^D_l}{Q^D_{x,y}(A(x,y) \supset B) \to Q_{x,y}A(x,y) \supset B}\,\supset_r$$

with $b <_{\varphi,\mathbf{LQ}} a$.

The quantifier shift $Q_{x,y}A(x,y) \supset B \to Q^D_{x,y}(A(x,y) \supset B)$ is derivable in $\mathbf{LQ}^{++}$. Its derivation is $\varphi =$

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a,b) \to A(a,b)}{A(a,b) \to Q_{x,y}A(x,y)}\,Q_r \quad B \to B}{A(a,b), Q_{x,y}A(x,y) \supset B \to B}\,\supset_l}{Q_{x,y}A(x,y) \supset B \to A(a,b) \supset B}\,\supset_r}{Q_{x,y}A(x,y) \supset B \to Q^D_{x,y}(A(x,y) \supset B)}\,Q^D_r$$

with $b <_{\varphi,\mathbf{LQ}} a$.

6 Cut-Elimination for $\mathbf{LQ}^{++}$

To show cut-elimination for $\mathbf{LQ}^{++}$ we will translate $\mathbf{LQ}^{++}$-derivations into cut-free **LK**-derivations and vice versa.

Definition 14. *Let S_Q be an* **LQ***-sequent. Then $S_{\forall\exists}$ is the result by replacing the quantifier macro $Q_{x,y}$ everywhere in S_Q by $\forall x \exists y$.*

Let $S_{\forall\exists}$ be an **LK***-sequent containing quantifier occurrences only in blocks of the form $\forall x \exists y$. Then S_Q is the result by replacing $\forall x \exists y$ everywhere in $S_{\forall\exists}$ by the quantifier macro $Q_{x,y}$.*

Lemma 1. *An $\mathbf{LQ}^{++}$-derivation φ of S_Q can be effectively transformed into a cut-free* **LK***-derivation from atomic axioms of $S_{\forall\exists}$.*

Proof. By translating Q to $\forall\exists$ we obtain an $\mathbf{LK}^{++}$-derivation which can be transformed into an **LK**-derivation by Theorem 4. As **LK** admits cut-elimination we obtain a cut-free **LK**-derivation. Compound axioms in **LK** can be replaced by atomic ones.

Lemma 2. *A cut-free* **LK***-derivation from atomic axioms of a sequent $S_{\forall\exists}$ containing quantifiers only in the form of blocks $\forall x \exists y$ can be transformed into a cut-free $\mathbf{LQ}^{++}$-derivation of S_Q.*

Proof. First we transform the **LK**-proof in the following way: whenever there is an $\exists_l$ inference we immediately infer $\forall_l$ afterwards. This has no impact on the result nor on the proof being cut-free. For $\forall_r$ inferences with principal formula $\forall x \exists y A(x, y)$ and eigenvariable a we determine all existential inferences with principal formula $\exists y A(a, y)$ and introduce $\forall x \exists y A(x, y)$ immediately after these inferences (note that the original **LK**-derivation is regular). The resulting derivation is an $\mathbf{LQ}^{++}$-derivation because the weak regularity holds, the eigenvariables do not occur in the end-sequent and $<_{\varphi,\mathbf{LQ}}$ does not loop as the order on the inferences is respected.

Theorem 6. $\mathbf{LQ}^{++}$ *is sound and admits effective cut-elimination.*

Remark 2. In $\mathbf{LK}^{+}$ regularity is defined by using eigenvariables exclusively for one inference, see [1]. Let $\mathbf{LQ}^{+}$ be the corresponding variant of $\mathbf{LQ}^{++}$. $\mathbf{LQ}^{+}$ is however not a complete calculus, because the sequent

$$\forall x(A(x, c_1) \vee A(x, c_2)) \to Q_{x,y} A(x, y)$$

is not $\mathbf{LQ}^{+}$-derivable. However, it is $\mathbf{LQ}^{++}$-derivable:

$$\dfrac{\dfrac{\dfrac{A(a, c_1) \to A(a, c_1)}{A(a, c_1) \to Q_{x,y} A(x, y)}\, Q_r \qquad \dfrac{A(a, c_2) \to A(a, c_2)}{A(a, c_2) \to Q_{x,y} A(x, y)}\, Q_r}{A(a, c_1) \vee A(a, c_2) \to Q_{x,y} A(x, y)}\, \vee_l}{\forall x(A(x, c_1) \vee A(x, c_2)) \to Q_{x,y} A(x, y)}\, \forall_l$$

7 Conclusion

It is conjectured that all mixed macros for connectives and quantifiers can be analytically represented in the framework of globally sound but locally unsound calculi. The problem is that the inferences of the defining formulae might arise from a plethora of premises, contrary to the quantifier macro defining $\forall\exists$. The solution might be the use of elaborated eigenvariable orders which guarantee a specific derivation of the macro.

References

1. Aguilera, J.P., Baaz, M.: Unsound inferences make proofs shorter. J. Symb. Log. **84**(1), 102–122 (2019)
2. Gentzen., G.: Untersuchungen über das logische Schließen. Mathematische Zeitschrift **39**, 176–210, 405–431 (1934–1935)
3. Mac Lane, S.: Abgekürzte Beweise im Logikkalkül. Hubert, Columbus (1934)
4. Robinson, J.A.: A machine-oriented logic based on the resolution principle. J. ACM (JACM) **12**(1), 23–41 (1965)
5. Takeuti, G.: Proof Theory. Courier Dover Publications, Mineola (2013)

Closure Ordinals of the Two-Way Modal μ-Calculus

Gian Carlo Milanese[1](✉) and Yde Venema[2]

[1] University of Milano-Bicocca, Milan, Italy
giancarlo.milanese@gmail.com
[2] Institute for Logic, Language and Computation, Universiteit van Amsterdam, Amsterdam, The Netherlands
y.venema@uva.nl
https://staff.fnwi.uva.nl/y.venema/

Abstract. The closure ordinal of a μ-calculus formula $\varphi(x)$ is the least ordinal α, if it exists, such that, in any model, the least fixed point of $\varphi(x)$ can be computed in at most α many steps, by iteration of the meaning function associated with $\varphi(x)$, starting from the empty set. In this paper we focus on closure ordinals of the two-way modal μ-calculus. Our main technical contribution is the construction of a two-way formula φ_n with closure ordinal ω^n for an arbitrary $n \in \omega$. Building on this construction, as our main result we define a two-way formula φ_α with closure ordinal α for an arbitrary $\alpha < \omega^\omega$.

Keywords: Modal logic · Fixed points · Closure ordinals · Two-way μ-calculus

1 Introduction

The modal μ-calculus μML, introduced by Kozen [11] in the form known today, is an extension of basic modal logic with explicit least- and greatest fixed point operators. The addition of these operators significantly increases the expressive power of the formalism, enabling it to deal with various forms of *recursion*, as required by applications in for instance the area of program verification. In fact, the modal μ-calculus was shown to be expressively complete with respect to the bisimulation-invariant fragment of monadic second-order logic [10], and it embeds many other logics such as PDL, CTL, and CTL*. Despite this expressive power, the modal μ-calculus has remarkably fine computational properties, such as a quasi-polynomial model checking problem [3] and a satisfiability problem that can be solved in exponential time [6].

In addition, the system admits a nice logical meta-theory: it has the finite model property, uniform interpolation, and a decent model theory [5,8,12]. The set of all valid μ-calculus formulas admits an elegant axiomatisation, which was already introduced by Kozen in his original paper [11], and proved to be complete

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 498–515, 2019.
https://doi.org/10.1007/978-3-662-59533-6_30

some years later by Walukiewicz [15]. Recently, a cut-free proof system was introduced by Afshari and Leigh [2].

Over the years, the modal μ-calculus has developed into the 'canonical' or 'universal' modal fixed point logic. This status motivates the full development of the meta-logical theory of the logic $\mu\mathtt{ML}$, and of its variants such as the *two-way μ-calculus*, which features both forward- and backward modalities, to be interpreted by the corresponding directions of the model's accessibility relation.

A relatively recent line of research on the modal μ-calculus concerns its *closure ordinals*. For an introduction to this notion, consider a formula $\varphi(x)$ (with only positive occurrences of the variable x) and a Kripke model $\mathbb{S} = (S, R, V)$. We may define a monotone function $\varphi_x^{\mathbb{S}} : \wp(S) \to \wp(S)$, which intuitively expresses how in $\mathbb{S}$ the meaning of φ depends on the valuation of x. The formula $\mu x.\varphi$ is then interpreted in $\mathbb{S}$ as the least fixed point of this map $\varphi_x^{\mathbb{S}}$ – that is, as the least subset $L \subseteq S$ such that $\varphi_x^{\mathbb{S}}(L) = L$ – and the point is that this least fixed point can be 'computed' by performing an iterative process involving the function $\varphi_x^{\mathbb{S}}$. Starting from the empty set, we define the following ordinal-indexed sequence $(\varphi_{\mathbb{S}}^{\alpha})_{\alpha \in \mathrm{On}}$ of subsets of S:

$$\varphi_{\mathbb{S}}^{0} := \emptyset, \quad \varphi_{\mathbb{S}}^{\beta+1} := \varphi_x^{\mathbb{S}}(\varphi_{\mathbb{S}}^{\beta}), \quad \varphi_{\mathbb{S}}^{\lambda} := \bigcup_{\beta < \lambda} \varphi_x^{\mathbb{S}}(\varphi_{\mathbb{S}}^{\beta}),$$

where λ denotes an arbitrary limit ordinal. By monotonicity of the function $\varphi_x^{\mathbb{S}}$, the sequence $(\varphi_{\mathbb{S}}^{\alpha})_{\alpha \in \mathrm{On}}$ converges: there must be a least ordinal α such that $\varphi_{\mathbb{S}}^{\alpha} = \varphi_{\mathbb{S}}^{\alpha+1}$. The element $\varphi_{\mathbb{S}}^{\alpha}$ of the sequence then coincides with the least fixed point of $\varphi_x^{\mathbb{S}}$ so that we say that the function $\varphi_x^{\mathbb{S}}$ converges to its least fixed point in α many steps.

The *closure ordinal* of a formula $\varphi(x)$ is the least ordinal α such that the function $\varphi_x^{\mathbb{S}}$ converges to its least fixed point in at most α many steps across every model $\mathbb{S}$, if such an ordinal exists. In other words, we are interested in the least number of steps that a least fixed point formula needs to converge to its meaning in every model. Not every formula will have a closure ordinal; for instance, let $\mathbb{S}_\alpha = (S_\alpha, >)$ be the structure where S_α is the set of all ordinals smaller than α and the accessibility relation is the converse order relation on these ordinals. It is not hard to see that on this model, the formula $\Box x$ needs exactly α steps to converge to its least fixed point. Clearly then, this formula does not have a closure ordinal across all models.

Intuitively, the closure ordinal of a formula is some measure of its complexity. For instance, a (basic) modal logic formula $\varphi(x)$ has a finite closure ordinal if and only if $\mu x.\varphi$ is definable in (basic) modal logic [14]. Another interesting example is obtained if we involve the first infinite ordinal ω: call a formula $\varphi(x)$ *constructive in* x it has a closure ordinal $\alpha \leq \omega$. The name 'constructive' is taken loosely here, motivated by the observation that since $\varphi_{\mathbb{S}}^{\omega} = \bigcup_{n<\omega} \varphi_{\mathbb{S}}^{n}$ in every model, a formula $\varphi(x)$ is constructive iff for each model $\mathbb{S}$ and for each point s in $\mathbb{S}$ we only need *finitely* many iterations of the map $\varphi_x^{\mathbb{S}}$ in order to find out whether s satisfies the formula $\mu x.\varphi$ or not.

Generally, there are many interesting questions to ask about closure ordinals, and at this moment only few of these have been answered. In fact, it seems that we can summarize our knowledge in one paragraph. Otto [14] proved that it is decidable whether a modal μ-calculus formula can equivalently be expressed in (basic) modal logic. As a corollary, we can also decide whether a formula of modal logic has a finite closure ordinal. Czarnecki [4] showed how to construct a formula φ_α with closure ordinal α for an arbitrary $\alpha < \omega^2$. An interesting result by Afshari and Leigh [1] confirms the intuition that closure ordinals are an indication of the complexity of a formula: they proved that the closure ordinals reached by formulas in the alternation-free fragment of the μ-calculus are all smaller than the ordinal ω^2. Gouveia and Santocanale [9] presented a formula with closure ordinal ω_1 and proved that closure ordinals are closed under ordinal sum.

In this paper we contribute to the theory of closure ordinals by taking a look at the two-way modal μ-calculus. After recalling the syntax and semantics of the logic and providing some definitions concerning closure ordinals in Sect. 2, in the following section we show how to define a formula φ_n with closure ordinal ω^n for every $n \in \omega$ (some of the technical proofs of this section are delayed to the appendix of the paper). In Sect. 4 we build on this result by proving that every ordinal smaller than ω^ω is a closure ordinal in the two-way setting. One way to achieve this is via transferring a result by Gouveia and Santocanale [9] – stating that the class of closure ordinals is closed under taking ordinal sum – to the two-way setting. We also define, given a representation $\alpha = \omega^n \cdot k_1 + \omega^{n-1} \cdot k_2 + \ldots + \omega \cdot k_n + k_{n+1}$ of an arbitrary ordinal $\alpha < \omega^\omega$, an explicit formula φ_α with closure ordinal α. We finish the paper with mentioning some questions for further research.

Source. The results in this paper are taken from the MSc thesis [13], which was written by the first author under the supervision of the second.

2 Preliminaries

Definition 1. *The language μ*`TML` *of the two-way modal μ-*calculus *is given by the following grammar:*

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid F\varphi \mid P\varphi \mid \mu x.\varphi$$

where $p, x \in$ `PROP` *and the formation of the formula $\mu x.\varphi$ is subject to the constraint that the variable x is positive in φ, that is, every occurrence of x in φ is under the scope of an even number of negations.*

We can define $\top$, $\wedge$ and the box operators by letting $G\varphi := \neg F \neg\varphi$ and $H\varphi := \neg P \neg\varphi$, as well as the greatest fixed point operator as $\nu x.\varphi := \neg\mu x.\neg\varphi(\neg x)$. The intended interpretation of a formula $F\varphi$ is 'φ is true at some (one-step) *future* state', while that of $P\varphi$ is 'φ is true at some (one-step) *past* state'.

Formulas of this language will be interpreted in two-way models. These can be defined as Kripke models featuring a pair of accessibility relations that are each other's converse, where we recall that the *converse* of a relation R is the relation $R^{-1} := \{(s,t) \mid (t,s) \in R\}$. It will be more convenient to simply identify two-way models with standard Kripke models with one single relation, and make sure that the diamonds F and P access this relation in its two different directions.

Definition 2. *A Kripke model is a triple $\mathbb{S} = (S, R, V)$ where S, the domain or underlying set, is a set of points or states, R is a binary relation on S, and V is a valuation on S, that is, a function $V : \mathtt{PROP} \to \wp(S)$.*

Given a model $\mathbb{S} = (S, R, V)$, a propositional variable x and a subset $X \subseteq S$, we define $V[x \mapsto X]$ as the valuation given by $V[x \mapsto X](p) = X$ if $p = x$, and $V[x \mapsto X](p) = V(p)$ otherwise. We denote the model $(S, R, V[x \mapsto X])$ by $\mathbb{S}[x \mapsto X]$.

Given a subset $S' \subseteq S$, the submodel of $\mathbb{S}$ induced by S' *is the model $\mathbb{S}' = (S', R', V')$, where $R' = R \cap (S' \times S')$ and $V'(p) = V(p) \cap S'$ for all $p \in \mathtt{PROP}$.*

We now inductively define the meaning of a formula φ in a model $\mathbb{S}$ as the set of states where this formula is true, or satisfied. At the same time we define the function $\varphi_x^{\mathbb{S}}$, which intuitively expresses how in $\mathbb{S}$ the meaning of the formula φ varies depending on the meaning of the variable x.

Definition 3. *Given a $\mu\mathtt{TML}$-formula φ and a model $\mathbb{S} = (S, R, V)$, we define the meaning $[\![\varphi]\!]^{\mathbb{S}}$ of φ in $\mathbb{S}$, together with the function $\varphi_x^{\mathbb{S}} : \wp(S) \to \wp(S)$ mapping a subset $X \subseteq S$ to $[\![\varphi]\!]^{\mathbb{S}[x \mapsto X]}$, by the following simultaneous induction:*

$$
\begin{array}{llll}
[\![\bot]\!]^{\mathbb{S}} & = \emptyset, & [\![p]\!]^{\mathbb{S}} & = V(p), \\
[\![\varphi \vee \psi]\!]^{\mathbb{S}} & = [\![\varphi]\!]^{\mathbb{S}} \cup [\![\psi]\!]^{\mathbb{S}}, & [\![\neg\varphi]\!]^{\mathbb{S}} & = S \setminus [\![\varphi]\!]^{\mathbb{S}}, \\
[\![F\varphi]\!]^{\mathbb{S}} & = \{s \in S \mid R[s] \cap [\![\varphi]\!]^{\mathbb{S}} \neq \emptyset\}, & [\![P\varphi]\!]^{\mathbb{S}} & = \{s \in S \mid R^{-1}[s] \cap [\![\varphi]\!]^{\mathbb{S}} \neq \emptyset\}, \\
[\![\mu x.\varphi]\!]^{\mathbb{S}} & = \bigcap\{U \subseteq S \mid \varphi_x^{\mathbb{S}}(U) \subseteq U\}, & &
\end{array}
$$

where $R[s] := \{t \in S \mid (s,t) \in R\}$ and similarly for R^{-1}. For an element $s \in S$ we write $\mathbb{S}, s \Vdash \varphi$ if $s \in [\![\varphi]\!]^{\mathbb{S}}$.

Let $\varphi \in \mu\mathtt{TML}$ be a formula in which the variable x occurs only positively and let $\mathbb{S}$ be a model. By induction on φ one can prove that $\varphi_x^{\mathbb{S}} : \wp(S) \to \wp(S)$ is a monotone operation. Consequently, by the Knaster-Tarski theorem we obtain that $[\![\mu x.\varphi]\!]^{\mathbb{S}}$ is the least fixed point of $\varphi_x^{\mathbb{S}}$, denoted by $\mathrm{LFP}.\varphi_x^{\mathbb{S}}$. As we saw in the introduction, the meaning of $\mu x.\varphi$ in a model $\mathbb{S}$ can also be computed by performing an iteration of the function $\varphi_x^{\mathbb{S}}$ starting from the empty set, resulting in the ordinal-indexed sequence $(\varphi_{\mathbb{S}}^{\alpha})_{\alpha \in \mathrm{On}}$. When the model $\mathbb{S}$ is clear from context we will write φ^{α} instead of $\varphi_{\mathbb{S}}^{\alpha}$; we shall also exclusively take x as the fixed point variable of the formulas that we are looking at, so that we need not mention this explicitly in the sequel.

In this paper we are interested in the number of times we need to iterate the function $\varphi_{\mathbb{S}}^{\alpha}$ before we reach its least fixed point.

Definition 4. *Let $\varphi(x)$ be a formula which is positive in x. Then for a Kripke model $\mathbb{S}$, we let $\gamma_x(\varphi, \mathbb{S})$ denote the closure ordinal of φ in $\mathbb{S}$ with respect to x, that is, the least ordinal α such that $\varphi^\alpha_\mathbb{S} = \varphi^{\alpha+1}_\mathbb{S}$.*

The closure ordinal of φ with respect to the variable x is the least ordinal α such that $\gamma_x(\varphi, \mathbb{S}) \leq \alpha$ for every model $\mathbb{S}$, if it exists. If α is the closure ordinal of some (two-way) formula, we say that α is a (two-way) closure ordinal.

When proving results about closure ordinals an equivalent characterisation, given in Proposition 1, is often useful.

Proposition 1. *An ordinal α is the closure ordinal of $\varphi(x)$ if and only if (1) $\gamma_x(\varphi, \mathbb{S}) \leq \alpha$ for every model $\mathbb{S}$ and (2) $\gamma_x(\varphi, \mathbb{S}) = \alpha$ for some model $\mathbb{S}$.*

Proof. The only nontrivial observation in the proof concerns the case, in the direction from left to right, where the closure ordinal α of φ is a limit ordinal. In order to prove (2), let B be the set of ordinals $\beta < \alpha$ for which there is a model $\mathbb{S}_\beta$ with $\gamma_x(\varphi, \mathbb{S}_\beta) = \beta$. This set must be cofinal in α, and it is then easy to show that if we take $\mathbb{S}$ to be the disjoint union of the collection $\{\mathbb{S}_\beta \mid \beta \in B\}$, we find $\gamma_x(\varphi, \mathbb{S}) = \alpha$ as required.

Example 1. The closure ordinal of $\varphi := (G\bot \vee Fx)$ is ω. It is not hard to prove that $\gamma_x(\varphi, \mathbb{S}) \leq \omega$ for every model $\mathbb{S}$, and that φ converges to its least fixed point in exactly ω steps in the model $\mathbb{S}$ depicted in Fig. 1. Indeed, one can show that $\varphi^n = \{m \in \omega \mid m < n\}$ for all $n \in \omega$: the iteration of φ in $\mathbb{S}$ traverses the ordinal ω by adding each finite ordinal to the iteration, one by one. After ω many steps in the iteration we observe that $\varphi^\omega = \bigcup_{n<\omega} \varphi^n = \{0, 1, 2, \ldots\} = \omega$ and $\varphi^{\omega+1} = \varphi^\mathbb{S}_x(\varphi^\omega) = \varphi^\omega$, so that the iteration converges in exactly ω steps.

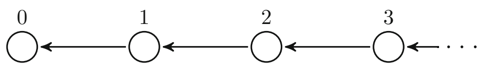


Fig. 1. Model where $G\bot \vee Fx$ converges in ω many steps

3 Two-Way Formulas: Closure Ordinal ω^n

In this section we define a two-way formula φ_n with closure ordinal ω^n for an arbitrary $n \in \omega$. We first need to define *colours*, which are essentially conjunctions of literals as specified in the next definition.

Definition 5. *Fix a subset $\{q_i \mid i \in \omega\}$ of propositional variables. For every $0 < n < \omega$ we define the colour c_n as the conjunction of literals $c_n := \bigwedge_{0<i<n} \neg q_i \wedge q_n$.*

For example, $c_1 = q_1$ and $c_2 = \neg q_1 \wedge q_2$. Clearly, $c_i \wedge c_j \equiv \bot$ for every $i \neq j$.

We now define, for all $0 < n < \omega$, a two-way formula φ_n.

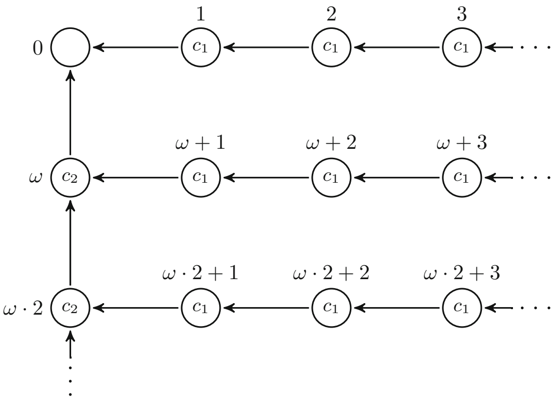


Fig. 2. Model corresponding to ω^2

Definition 6. *By induction on $i \in \omega$ we define the formulas π_i^∞ as follows:*

$$\begin{aligned} \pi_0^\infty &:= \top, \\ \pi_{i+1}^\infty &:= \nu y_{i+1}.(P(y_{i+1} \wedge c_{i+1}) \wedge \pi_i^\infty). \end{aligned}$$

For all $n \in \omega$ let φ_n be the formula

$$\varphi_n := G\bot \vee (c_1 \wedge Fx) \vee \bigvee_{i=2}^{n} (c_i \wedge \pi_{i-1}^\infty \wedge F(\nu y.P(y \wedge x \wedge c_{i-1}))).$$

Example 2. Consider the formula

$$\varphi_2 = G\bot \vee (c_1 \wedge Fx) \vee (c_2 \wedge \pi_1^\infty \wedge F(\nu y.P(y \wedge x \wedge c_1)))$$

and the model $\mathbb{S}$ depicted in Fig. 2, consisting of ω many copies of ω, thus intuitively corresponding to the ordinal ω^2.

The formula φ_2 crucially involves the formula $\nu y.P(y \wedge x \wedge c_1)$, which expresses the existence of an infinite R^{-1}-path of points where x and c_1 are always true starting from the R^{-1}-next state, and which allows the iteration to move from a copy of ω to the next, as we shall now see. The iteration of φ_2 in this model starts similarly as the one in Example 1, by including the state 0 through the disjunct $G\bot$ and then adding, one by one, each state labelled with a finite ordinal through the disjunct $(c_1 \wedge Fx)$. After ω many steps in the iteration we have $\varphi^\omega = \{0, 1, 2, \ldots\} = \omega$, so that every state labelled with a finite ordinal is inside the iteration. Now it holds that $\mathbb{S}[x \mapsto \varphi^\omega], \omega \Vdash (c_2 \wedge \pi_1^\infty \wedge F(\nu y.P(y \wedge x \wedge c_1)))$, so that $\varphi^{\omega+1} = [\![\varphi]\!]^{\mathbb{S}[x \mapsto \varphi^\omega]} = \omega \cup \{\omega\}$: the state ω is added to the iteration. The iteration continues through the disjunct $(c_1 \wedge Fx)$, with $\varphi^{\omega+n} = \omega \cup \{\omega, \ldots, \omega + (n-1)\}$, arriving at $\varphi^{\omega \cdot 2} = \omega \cup \{\omega, \omega+1, \omega+2, \ldots\}$, at which point the state $\omega \cdot 2$ will satisfy $(c_2 \wedge \pi_1^\infty \wedge F(\nu y.P(y \wedge x \wedge c_1)))$, and so on. The iteration will progress in a similar way, traversing all the copies of ω and converging in exactly ω^2 steps.

The following example concerns the formulas of shape π_i^∞ that appear as subformulas of φ_n. These formulas will make sure that the models of φ_n have a particular grid-like shape: we will need that whenever a state s in a model makes φ_n true and has colour c_i, then this state is the starting point of an infinite R^{-1}-path of points where c_{i-1} is always true, and from every point on this path an infinite R^{-1}-path starts of points where c_{i-2} is always true, and so on.

Example 3. Consider for instance

$$\pi_3^\infty = \nu y_3.(P(y_3 \wedge c_3) \wedge \nu y_2.(P(y_2 \wedge c_2) \wedge \nu y_1.(P(y_1 \wedge c_1) \wedge \top))).$$

This formula expresses the existence of an infinite R^{-1}-path $t_0 t_1 t_2 \ldots$ such that *(i)* c_3 is true at every t_i with $i > 0$; *(ii)* every t_i makes $\nu y_2.(P(y_2 \wedge c_2) \wedge \nu y_1.P(y_1 \wedge c_1))$ true, so from each t_i there is an infinite R^{-1}-path $u_0 u_1 u_2 \ldots$ where $u_0 = t_i$ and c_2 is true at every u_j with $j > 0$; *(iii)* every u_j makes $\nu y_1.P(y_1 \wedge c_1)$ true, so from each u_j there exists a R^{-1}-path $v_0 v_1 \ldots$, with $v_0 = u_j$, such that c_1 is true at v_k for every $k > 0$. For example, the point 0 in the model of Fig. 3 makes π_3^∞ true (as does every state of the form $\omega^2 \cdot n$ for $n \in \omega$).

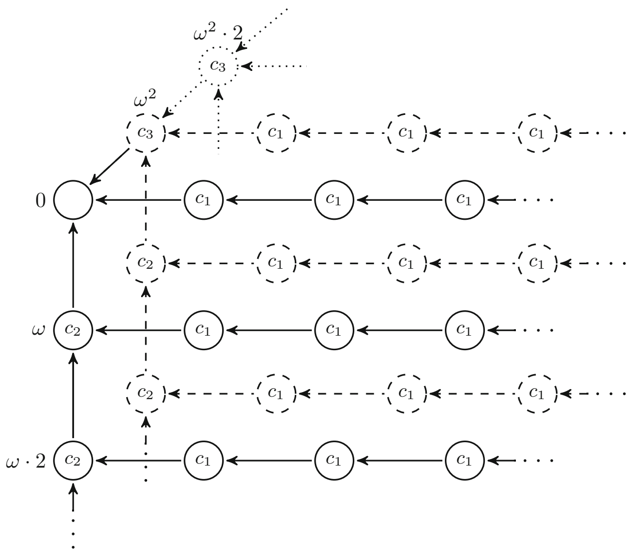


Fig. 3. Model corresponding to ω^3

Example 4. As a further example, consider the formula

$$\varphi_3 = \varphi_2 \vee (c_3 \wedge \pi_2^\infty \wedge F(\nu y.P(y \wedge x \wedge c_2)))$$

and the model pictured in Fig. 3, which consists of ω many copies of the model from Fig. 2, each attached to a state of the infinite path $0R^{-1}\omega^2 R^{-1}\omega^2 \cdot 2 \ldots$.

The iteration of φ_3 in this model starts similarly as the one in Example 2, but after ω^2 many steps, when the first copy of ω^2 is inside the approximating set φ^{ω^2}, the state ω^2 will satisfy the disjunct $(c_3 \wedge \pi_2^\infty \wedge F(\nu y.P(y \wedge x \wedge c_2)))$ of φ_3, so that the iteration will move to the second copy of ω^2 and continue in an analogous way, with convergence in exactly ω^3 steps.

The last example also suggests a recipe for constructing a model where the formula φ_n converges in exactly ω^n steps. For $n = 4$, we could take an infinite R^{-1}-chain of c_4-states, where to each such state is attached a copy of ω^3 (that is, a copy of the model of Fig. 3): the disjunct $(c_4 \wedge \pi_3^\infty \wedge F(\nu y.P(y \wedge x \wedge c_3)))$ of the formula φ_4 would allow the iteration to move between the copies of ω^3, exactly as the disjunct $(c_3 \wedge \pi_2^\infty \wedge F(\nu y.P(y \wedge x \wedge c_2)))$ of φ_3 allowed the iteration to move from a copy of ω^2 to the next. For $n = 5$ we could consider an infinite R^{-1}-chain of c_5-states, where to each such state is attached a copy of the model we have just described, and so on.

Lemma 1. *Let $0 < n < \omega$ be a finite ordinal. Then there is a model $\mathbb{S}$ where $\gamma_x(\varphi_n, \mathbb{S}) = \omega^n$.*

Up to this point we have only focused on one of the two conditions that the ordinal ω^n must satisfy in order to qualify as the closure ordinal of φ_n, namely the one concerning convergence in exactly ω^n steps in some model. It is less intuitive to see why ω^n should be an upper bound for the number of steps in the iteration of φ_n in an arbitrary model. Indeed, the previous models present a particular grid-like structure, which allows the iteration to progress in a very controlled way: if a state is added to the iteration at some step α, then the chain of c_1-states attached to it is included in the iteration in at most ω more steps, the chain of c_2-states attached to it is included in the iteration in at most ω^2 more steps, and the chain of c_3-states attached to it is included in the iteration in at most ω^3 more steps (in case these chains exist). This is formulated in a more general way in the next lemma, which states that if we have an infinite R^{-1}-path of c_i-states presenting the desired grid-like structure (that is, each satisfying π_{i-1}^∞) and the first state of this path belongs to the approximating set φ_n^α, then all the states of the path will be inside the iteration after at most ω^i more steps. Put differently, if a state t_0 in a model satisfies π_i^∞ and is in the approximating set φ_n^α, then all the states forming the R^{-1}-path that witnesses the truth of π_i^∞ at t_0 will belong to $\varphi_n^{\alpha+\omega^i}$.

Lemma 2. *Let $\mathbb{S} = (S, R, V)$ be a model and let $n \in \omega$. For $1 \leq i \leq n$, let $t_0 t_1 t_2 \ldots$ be an infinite R^{-1}-path such that*

$$\mathbb{S}, t_0 \Vdash \pi_{i-1}^\infty \textit{ and, for all } j > 0, \mathbb{S}, t_j \Vdash c_i \wedge \pi_{i-1}^\infty.$$

Then, for any ordinal α: if $t_0 \in \varphi_n^\alpha$ then $t_j \in \varphi_n^{\alpha+\omega^{i-1}\cdot j+1}$ for all $j \in \omega$.

In order to make sure that something similar also happens in an arbitrary model, the presence in φ_n of the subformulas π_{i+1}^∞'s from Definition 6 is necessary: by the definition of φ_n, if a state s in a model $\mathbb{S}$ satisfies $(\varphi_n \wedge c_i \wedge F\top)$, then it must also satisfy π_{i-1}^{∞}, so that the model $\mathbb{S}$ will present the desired grid-like structure. This fact and Lemma 2 are essential for proving that indeed φ_n converges to its least fixed point in at most ω^n steps in every model.

Lemma 3. *For an arbitrary model $\mathbb{S}$ and $0 < n < \omega$: $\gamma_x(\varphi_n, \mathbb{S}) \leq \omega^n$.*

By Proposition 1, and the Lemmas 1 and 3, the following is immediate.

Theorem 1. *For all $0 < n < \omega$, the two-way closure ordinal of $\varphi_n(x)$ is ω^n.*

The proofs of all the statements of this section can be found in the appendix.

4 Two-Way Formulas: Closure Ordinals Below ω^ω

This section is devoted to the main result of our paper, stating that every ordinal below ω^ω is a closure ordinal in the two-way setting. In the next subsection we transfer a result by Gouveia and Santocanale [9] to the two-way setting. That is, we show that for two-way formulas $\varphi_0(x)$ and $\varphi_1(x)$ with closure ordinals α_0 and α_1, respectively, we can define a two-way formula $\psi(x)$ with closure ordinal $\alpha_0 + \alpha_1$. From this observation and Theorem 1, our main result follows, since every ordinal α below ω^ω can be written as a finite sum of ordinals of the form ω^n. In the following subsection we improve on this result by defining, for an arbitrary ordinal $\alpha < \omega^\omega$, an explicit two-way formula φ_α with closure ordinal α.

4.1 Two-Way Formulas: Sum of Ordinals

In the introduction we already mentioned that Gouveia and Santocanale showed the class of closure ordinals to be closed under taking ordinal sums [9]. We will now see that their result also holds in the two-way setting.

Theorem 2. *There is an effective construction transforming a pair of two-way formulas $\varphi_0(x)$ and $\varphi_1(x)$ into a formula ψ such that, if $\varphi_0(x)$ and $\varphi_1(x)$ have closure ordinals α_0 and α_1, respectively, then $\psi(x)$ has closure ordinal $\alpha_0 + \alpha_1$.*

Our proof follows the approach from [9], but we provide some proof details here in order to keep our presentation self-contained, and because we can make some simplifications in the two-way setting. We confine ourselves to a proof sketch, focusing on intuitions rather than on technicalities. One concept we will need is that of a *strong* closure ordinal.

Definition 7. *An ordinal α is a* strong *closure ordinal for a (two-way) μ-calculus formula $\varphi(x)$ if $\gamma(\varphi, \mathbb{S}) \leq \alpha$ for all models $\mathbb{S}$, while there is a model $\mathbb{S} = (S, R, V)$ such that*

$$S = [\![\mu x.\varphi]\!]^{\mathbb{S}} = \varphi_{\mathbb{S}}^{\alpha} \text{ and } \varphi_{\mathbb{S}}^{\alpha} \neq \varphi_{\mathbb{S}}^{\beta} \text{ for every } \beta < \alpha.$$

Proposition 2. *If α is the two-way closure ordinal of some formula $\varphi(x)$, then it is a strong closure ordinal for the formula $\widehat{\varphi}(x) := (\mu x.\varphi) \to \varphi(x \wedge \mu x.\varphi)$.*

Proof. As in [9] the key observation is that in any model $\mathbb{S} = (S, R, V)$ we have

$$\widehat{\varphi}^{\gamma}_{\mathbb{S}} = (S \setminus \mathrm{LFP}.\varphi^{\mathbb{S}}_{x}) \cup \varphi^{\gamma}_{\mathbb{S}},$$

for any $\gamma \geq 1$—this claim can be proved by a straightforward transfinite induction. Consequently, for $\gamma = \alpha$ we obtain $\widehat{\varphi}^{\alpha}_{\mathbb{S}} = (S \setminus \mathrm{LFP}.\varphi^{\mathbb{S}}_{x}) \cup \varphi^{\alpha}_{\mathbb{S}} = (S \setminus \mathrm{LFP}.\varphi^{\mathbb{S}}_{x}) \cup \mathrm{LFP}.\varphi^{\mathbb{S}}_{x} = S$.

We now turn to the proof of Theorem 2. Throughout this subsection we let $\varphi_0(x)$ and $\varphi_1(x)$ be two-way formulas with closure ordinals α_0 and α_1, respectively. Our aim is to define a two-way formula $\psi(x)$ with closure ordinal $\alpha_0 + \alpha_1$. Because of Proposition 2 we may without loss of generality assume that α_0 is a *strong* closure ordinal for φ_0.

The idea underlying the definition of $\psi(x)$ is that in any model $\mathbb{S}$, in order to calculate the least fixed point of $\psi(x)$, one may first focus on φ_0 and then move on to φ_1. More precisely, with each model $\mathbb{S} = (S, R, V)$ we will associate two submodels $\mathbb{S}_0$ and $\mathbb{S}_1$ such that

$$\gamma(\psi, \mathbb{S}) \leq \gamma(\varphi_0, \mathbb{S}_0) + \gamma(\varphi_1, \mathbb{S}_1). \tag{1}$$

This implies that ψ has a closure ordinal β indeed, and that $\beta \leq \alpha_0 + \alpha_1$. To prove that $\beta \geq \alpha_0 + \alpha_1$ we will employ a special model $\mathbb{S}$ such that, for each i, $\mathbb{S}_i$ is a model witnessing that α_i is a strong closure ordinal for φ_i.

For the details of the construction of the submodels $\mathbb{S}_0$ and $\mathbb{S}_1$, note that the formula ψ will use one fresh variable p (so that in particular, p occurs neither in φ_0 nor in φ_1), and write $\mathtt{PROP}_p = \mathtt{PROP} \cup \{p\}$. Now, given a $\mathtt{PROP}_p$-model $\mathbb{S} = (S, R, V)$, we define $S_0 = S \setminus V(p)$ and $S_1 = V(p)$, and for $i = 0, 1$ let $\mathbb{S}_i$ be the submodel of $\mathbb{S}$ induced by the set S_i (and with V_i restricted to the set $\mathtt{PROP}$).

Syntactically, we need the following definition.

Definition 8. *Let $p \notin \mathtt{PROP}$ be a fresh variable and set $\mathtt{p}_0 := \neg p$ and $\mathtt{p}_1 := p$. For $i \in \{0, 1\}$ we define the* restriction *of φ to $\mathtt{p}_i$ as follows:*

$$\begin{array}{llll}
\mathtt{tr}_i(y) & := \mathtt{p}_i \wedge y & \mathtt{tr}_i(\psi_0 \wedge \psi_1) & := \mathtt{tr}_i(\psi_0) \wedge \mathtt{tr}_i(\psi_1) \\
\mathtt{tr}_i(\neg y) & := \mathtt{p}_i \wedge \neg y & \mathtt{tr}_i(\psi_0 \vee \psi_1) & := \mathtt{tr}_i(\psi_0) \vee \mathtt{tr}_i(\psi_1) \\
\mathtt{tr}_i(\bot) & := \bot & \mathtt{tr}_i(F\psi) & := \mathtt{p}_i \wedge F(\mathtt{p}_i \wedge \mathtt{tr}_i(\psi)) \\
\mathtt{tr}_i(\top) & := \mathtt{p}_i & \mathtt{tr}_i(G\psi) & := \mathtt{p}_i \wedge G(\mathtt{p}_i \to \mathtt{tr}_i(\psi)) \\
\mathtt{tr}_i(\mu z.\psi) & := \mu z.\mathtt{tr}_i(\psi) & \mathtt{tr}_i(P\psi) & := \mathtt{p}_i \wedge P(\mathtt{p}_i \wedge \mathtt{tr}_i(\psi)) \\
\mathtt{tr}_i(\nu z.\psi) & := \nu z.\mathtt{tr}_i(\psi) & \mathtt{tr}_i(H\psi) & := \mathtt{p}_i \wedge H(\mathtt{p}_i \to \mathtt{tr}_i(\psi))
\end{array}$$

We need the following properties of these restriction formulas.

Proposition 3. *Let $\varphi(x)$ be a formula in the two-way μ-calculus and let $\mathbb{S} = (S, R, V)$ be an arbitrary model. Then for $i = 0, 1$ we have*

1. $[\![\mathtt{tr}_i(\varphi)]\!]^{\mathbb{S}} = [\![\varphi]\!]^{\mathbb{S}_i}$
2. *with x free in φ, $(\mathtt{tr}_i(\varphi))^{\alpha}_{\mathbb{S}} = \varphi^{\alpha}_{\mathbb{S}_i}$, for every ordinal α.*

We are now ready for the definition of the formula $\psi(x)$. Consider the following formulas (which are somewhat simpler than the corresponding one-way formulas of [9]):

$\psi_0 \quad := \neg p \wedge \mathtt{tr}_0(\varphi_0)(x)$
$\psi_1 \quad := p \wedge \mathtt{tr}_1(\varphi_1)(x) \wedge G(\neg p \to x)$
$\psi(x) := \psi_0(x) \vee \psi_1(x)$.

To compute the least fixed point of the formula $\psi(x)$ on an arbitrary model $\mathbb{S}$, first consider its disjunct $\psi_0(x) = \neg p \wedge \mathtt{tr}_0(\varphi_0)(x)$. By Proposition 3 we may think of the computation of its least fixed point as taking place in the $\neg p$-part S_0 of $\mathbb{S}$, parallel to that of $\mu x.\varphi_0$ in $\mathbb{S}_0$, and so this computation finishes after $\gamma(\varphi_0, \mathbb{S}_0)$ steps. Similarly, the iterative process approximating the least fixed point of the formula $\psi_1' := p \wedge \mathtt{tr}_0(\varphi_1)(x)$ can be fully executed in the p-part S_1 of $\mathbb{S}$, and this computation would finish after $\gamma(\varphi_1, \mathbb{S}_1)$ steps. The formula $\psi_1(x)$, however, has an additional conjunct, viz., the formula $G(\neg p \to x)$; this ensures that a point in S_1 will only be included in an approximating set $\psi^{\alpha+1}$ if each of its successors in S_0 has been included in the set ψ^{α}. As a consequence, the computation of the S_1-part of the least fixed point of $\psi(x)$ need not be (fully) operational before the computation of the S_0-part is completed. Nevertheless, once the latter computation has terminated indeed, the conjunct $G(\neg p \to x)$ evaluates to true in every state in S_1, and so from that moment on at most $\gamma(\varphi_1, \mathbb{S}_1)$ steps are needed to finish the computation of $[\![\mu x.\psi]\!]^{\mathbb{S}}$. This finishes a proof sketch of the statement (1).

It remains to provide a model $\mathbb{S}$ where the closure ordinal of $\psi(x)$ is actually identical to $\alpha_0 + \alpha_1$. For this purpose, consider two models $\mathbb{S}_0$ and $\mathbb{S}_1$ such that $\gamma(\varphi_i, \mathbb{S}_i) = \alpha_i$ for $i = 0, 1$. Additionally, we require that $[\![\mu x.\varphi_0]\!]^{\mathbb{S}_0} = S_0$—such a model exists by our assumption that α_0 is a *strong* closure ordinal for φ_0. Now take the disjoint union of $\mathbb{S}_0$ and $\mathbb{S}_1$, add an arrow from every state of S_1 to every state of S_0, and set $V(p) := S_1$. Call the resulting model $\mathbb{S}$; it is easy to see that this definition does not cause notational confusion, since the models $\mathbb{S}_0$ and $\mathbb{S}_1$ are identical to the submodels of $\mathbb{S}$ induced by the sets $S_0 = [\![\neg p]\!]^{\mathbb{S}}$ and $S_1 = [\![p]\!]^{\mathbb{S}}$, respectively. The crux of this construction is that in the model $\mathbb{S}$, because every state s in S_1 has the *full* set S_0 among its successors, and we need *exactly* α_0 steps to get all S_0-points in the least fixed point of ψ, we can only start adding S_1-states to the least fixed point of $\psi(x)$ after we have added *all* S_0-states, that is, at stage $\alpha_0 + 1$. It is then not hard to see that another α_1 steps are needed to include all S_1-states, so that all in all we need exactly $\alpha_0 + \alpha_1$ steps for $\psi(x)$ to converge. This shows that the closure ordinal of the formula $\psi(x)$ is $\alpha_0 + \alpha_1$ indeed.

4.2 An Explicit Formula for Every Ordinal Below ω^ω

In this section we shall provide, for every ordinal $\alpha < \omega^\omega$, an explicit two-way formula φ_α with closure ordinal α.

In the case α is *finite*, it is not hard to see that n is the closure ordinal of $\varphi_n := (Gx \wedge G^n\bot)$ for every $n \in \omega$, so that in the sequel we confine attention to the infinite case. Recall that every ordinal α with $\omega \leq \alpha < \omega^\omega$ can be written in a unique normal form

$$\alpha = \omega^n \cdot k_1 + \omega^{n-1} \cdot k_2 + \ldots + \omega \cdot k_n + k_{n+1} \tag{2}$$

for some finite ordinals $n, k_1, \ldots, k_{n+1}$ with $n, k_1 > 0$. We may then use the Theorems 1 and 2 to construct, for every such ordinal α, an explicit two-way formula taking α as its closure ordinal.

Alternatively, in Definition 11 below we provide a different two-way formula φ_α with closure ordinal α; this definition is directly based on the normal form (2). In order to achieve this, we need to define a second set of colours.

Definition 9. *Fix a subset $\{p_i \mid i \in \omega\}$ of propositional variables that is disjoint from the set $\{q_i \mid i \in \omega\}$ from Definition 5. For every $0 < n < \omega$ we define the colour f_n as the conjunction of literals $f_n := \bigwedge_{0<i<n} \neg p_i \wedge p_n$.*

Definition 10. *For every $i, k \in \omega$ we define a formula $\pi^\infty_{i,k}$ inductively on i as follows:*

$$\begin{aligned} \pi^\infty_{0,k} &:= f_k \\ \pi^\infty_{i+1,k} &:= \nu y_{i+1}.(P(y_{i+1} \wedge c_{i+1} \wedge f_k \wedge G f_k) \wedge \pi^\infty_i). \end{aligned}$$

We finally state the definition of the formula φ_α.

Definition 11. *For $n, k \in \omega$ define the formulas*

$$\begin{aligned} \varphi_{(n,k)} &:= (Fx \wedge c_1 \wedge f_k \wedge G f_k) \vee \\ &\quad \bigvee_{i=2}^{n} (c_i \wedge f_k \wedge G f_k \wedge \pi^\infty_{i-1,k} \wedge F(\nu y. f_k \wedge P(y \wedge x \wedge G f_k \wedge c_{i-1}))), \\ \chi_k &:= (Gx \wedge f_{k+1} \wedge G f_k). \end{aligned}$$

Now let, for $n > 0$, $\alpha = \omega^n \cdot k_1 + \omega^{n-1} \cdot k_2 + \ldots + \omega \cdot k_n + k_{n+1}$. For all $0 \leq m \leq n$ define $k(\overrightarrow{m}) := \sum_{i=0}^{m} k_i$, where we let $k_0 := 0$. The formula φ_α is defined by letting

$$\begin{aligned} \psi &:= \bigvee_{i=0}^{k_{n+1}-1} (Gx \wedge \bigwedge_{j=0}^{i} G^j f_{k(\overrightarrow{n})+1} \wedge G^{i+1} f_{k(\overrightarrow{n})}), \\ \varphi_\alpha &:= G\bot \vee \bigvee_{k=1}^{k(\overrightarrow{n})-1} \chi_k \vee \bigvee_{m=0}^{n-1} \left(\bigvee_{k=k(\overrightarrow{m})+1}^{k(\overrightarrow{m+1})} \varphi_{(n-m,k)} \right) \vee \psi. \end{aligned}$$

Example 5. Consider the formulas

$$\begin{aligned} \varphi_{(2,i)} &:= (Fx \wedge c_1 \wedge f_i \wedge G f_i) \vee \\ &\qquad (c_2 \wedge f_i \wedge G f_i \wedge \pi^\infty_{1,i} \wedge F(\nu y. f_i \wedge P(y \wedge x \wedge G f_i \wedge c_1))), \\ \varphi_{\omega^2 \cdot 2} &:= G\bot \vee (Gx \wedge f_2 \wedge G f_1) \vee \varphi_{(2,1)} \vee \varphi_{(2,2)} \end{aligned}$$

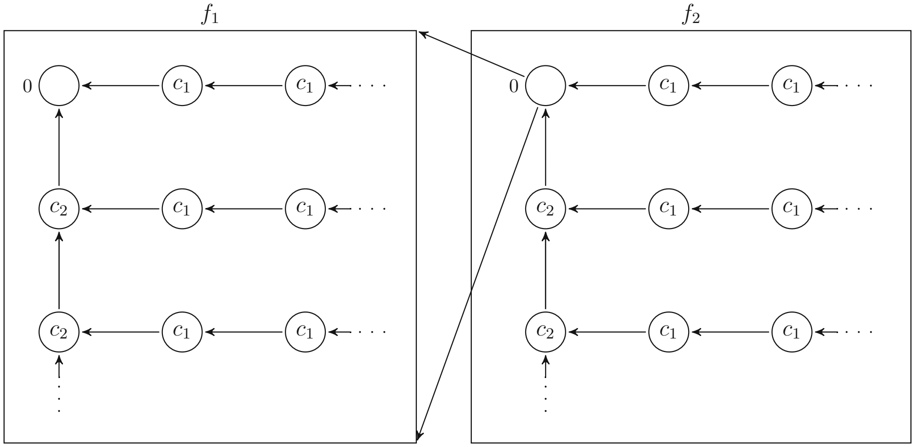


Fig. 4. Model corresponding to $\omega^2 \cdot 2$

and a model $\mathbb{S}$ consisting of two submodels $\mathbb{S}_1$ and $\mathbb{S}_2$, both copies of the model of Fig. 2, such that from the point corresponding to 0 in $\mathbb{S}_2$ there is an arrow to every state of $\mathbb{S}_1$, and moreover $S_1 = [\![f_1]\!]^{\mathbb{S}_1}$ and $S_2 = [\![f_2]\!]^{\mathbb{S}_2}$, as shown in Fig. 4. The colours f_1 and f_2 work similarly as the *fuses* used by Czarnecki in [4]: these force the iteration of $\varphi_{\omega^2 \cdot 2}$ to first traverse the f_1-copy $\mathbb{S}_1$ of ω^2 through the disjuncts $G\bot$ and $\varphi_{(2,1)}$, then move to the state 0 of the f_2-copy $\mathbb{S}_2$ of ω^2 through the disjunct $(Gx \wedge f_2 \wedge Gf_1)$, and finally traverse all $\mathbb{S}_2$ through the disjunct $\varphi_{(2,2)}$.

Theorem 3 below states that the two-way formula φ_α has closure ordinal α indeed. Due to space limitations, we have to omit the rather tedious proof; the interested reader can find the details in [13].

Theorem 3. *For every ordinal α with $\omega \leq \alpha < \omega^\omega$, the two-way closure ordinal of φ_α is α.*

5 Further Research

The research regarding closure ordinals of the μ-calculus has barely scratched the surface, and many questions remain open. We point out some possible future research lines.

Generally, we would like to understand better which ordinals feature as closure ordinals, and which ones don't. In particular, is there a two-way formula with closure ordinal at least ω^ω? Is there a standard (i.e., 'one-way') formula with a countable closure ordinal α at least ω^2? The approach taken here does not seem to work in the one-way setting—we refer to [13] for the details.

Another research direction involves decidability results. Given a formula $\varphi(x)$, is it decidable whether it has a closure ordinal, and can this be read off from its syntactic shape? Given an ordinal α, is it decidable whether a formula has closure ordinal α?

A more specific question concerns the *number of proposition letters* that is needed to characterize closure ordinals. In our approach we need an infinite set of atomic propositions to capture *all* ordinals below ω^ω. It is an interesting question to see whether this can be done with a *finite* set as well. We conjecture that this is indeed the case, by replacing the colors and fuses of Definition 11 by suitably chosen (basic) two-way formulas.[1]

Gouveia and Santocanale proved that closure ordinals are closed under ordinal sum [9] and we have transferred this result to the two-way setting. Is the class of closure ordinals closed under other ordinal operations as well, such as multiplication? Conversely, one may ask whether the formulas $\varphi \vee \psi, \varphi \wedge \psi, \varphi[\psi/x], \ldots$ have a closure ordinal whenever $\varphi(x)$ and $\psi(x)$ do.[2]

Finally, we mentioned the property of constructivity in the introduction. An interesting research direction involves the relationship between this property and that of continuity, where a formula $\varphi(x)$ is said to be *(Scott) continuous* in the variable x if, for an arbitrary model $\mathbb{S}$: $\mathbb{S}, s \Vdash \varphi$ iff $\mathbb{S}[x \mapsto V(p) \cap F], s \Vdash \varphi$, for some finite subset $F \subseteq S$. In particular, the second author [7,8] has formulated the question whether for every formula $\varphi(x)$ that is constructive in x one may find some formula $\psi(x)$ that is continuous in x, and equivalent to $\varphi(x)$ 'modulo an application of the least fixed point operator' (i.e., such that $\mu x.\varphi \equiv \mu x.\psi$). Some evidence supporting a positive answer can be found in [8,13].

A Proof of the Main Result in Section 3

The statement of Theorem 1 from Sect. 3 is a direct consequence of the following lemmas.

Lemma 1. *Let $0 < n < \omega$ be a finite ordinal. Then there is a model $\mathbb{S}$ where $\gamma_x(\varphi_n, \mathbb{S}) = \omega^n$.*

Proof. For the rest of the proof we adopt the following notation: since every ordinal $\alpha < \omega^n$ can be written as $\omega^{n-1} \cdot k_1 + \ldots + \omega \cdot k_{n-1} + k_n$, we also denote α as $(k_1, \ldots, k_n)$. From now on, if we write $\alpha = (k_1, \ldots, k_n)$ we mean that $\alpha = \omega^{n-1} \cdot k_1 + \ldots + \omega \cdot k_{n-1} + k_n$. Also, if a tuple $(k_1, \ldots, k_n)$ is of the form $(k_1, \ldots, k_i, 0, \ldots, 0)$, we mean that $k_j = 0$ for $i + 1 \leq j \leq n$.
Fix $n > 0$ and let $\varphi := \varphi_n$ as an abbreviation. We define $\mathbb{S} = (S, R, V)$ to be the model where:

- $S := \omega^n = \{(k_1, \ldots, k_n) \mid k_j \in \omega\}$;
- $R := \bigcup_{1 \leq i \leq n} \{((k_1, \ldots, k_i + 1, 0, \ldots, 0), (k_1, \ldots, k_i, 0, \ldots, 0)) \mid k_j \in \omega\}$;
- for $1 \leq i \leq n$, $V(q_i) := \{(k_1, \ldots, k_{n-i+1} + 1, 0, \ldots, 0) \mid k_j \in \omega\}$.

[1] This suggestion was raised by one of the referees.

[2] One of the reviewers pointed out that the formulas $p \wedge \Box(\neg p \wedge x)$ and $\neg p \wedge \Box(p \wedge x)$ both have closure ordinals, but their disjunction, behaving similarly to the formula $\Box x$, does not.

Note that $R[(0,\dots,0)] = \emptyset$ and that $(0,\dots,0)$ falsifies q_i for every $1 \le i \le n$.

Before proving the key claim we make an observation about notation. Note that an ordinal $\beta < \omega^n$ can both be seen as an *element* $\beta \in S = \omega^n$ of the model and as a *subset* $\beta = \{\gamma \mid \gamma < \beta\} \subseteq S = \omega^n$. To avoid confusion, until the end of the proof we write β when we consider it as an element of the domain, and S_β when we consider it as a subset of the domain ($S_\beta = \beta$ holds in any case).

Claim. For every $\alpha < \omega^n$, $\varphi^\alpha = S_\alpha$.

Proof of Claim. The proof goes by induction on α. The case for $\alpha = 0$ is immediate. If α is a limit ordinal, then $\varphi^\alpha = \bigcup_{\beta<\alpha} \varphi^\beta =_{IH} \bigcup_{\beta<\alpha} S_\beta = S_\alpha$.

Now suppose that $\alpha = \beta + 1$. We want to show that $\varphi^{\beta+1} = S_{\beta+1}$. We have that $\varphi^{\beta+1} = \varphi^{\mathbb{S}}_x(\varphi^\beta) =_{IH} \varphi^{\mathbb{S}}_x(S_\beta)$: we show

$$\varphi^{\mathbb{S}}_x(S_\beta) = S_{\beta+1}. \tag{3}$$

For the $\supseteq$ inclusion of (3) it suffices to show that $\mathbb{S}[x \mapsto S_\beta], \beta \Vdash \varphi$, since $S_{\beta+1} = S_\beta \cup \{\beta\}$ and $S_\beta = \varphi^\beta \subseteq \varphi^{\beta+1} = \varphi^{\mathbb{S}}_x(\varphi^\beta)$. If $\beta = 0 = (0,\dots,0)$ we are done. If $\beta = (k_1,\dots,k_n+1)$, then $\beta \in V(q_1)$ and $(k_1,\dots,k_n) \in S_\beta \cap R[\beta]$, so $\mathbb{S}[x \mapsto S_\beta], \beta \Vdash c_1 \wedge Fx$ and $\beta \in \varphi^{\mathbb{S}}_x(S_\beta)$.

Otherwise let $\beta = (k_1,\dots,k_i+1,0,\dots,0)$ for some $1 \le i < n$, so that $\beta \in V(q_{n-i+1})$. Note that

$$\begin{aligned}
&(k_1,\dots,k_i,k,0,\dots,0) \in S_\beta \text{ for all } k \in \omega,\\
&(k_1,\dots,k_i,0,0,\dots,0) \in R[\beta] \text{ and}\\
&(k_1,\dots,k_i,k,0,\dots,0) \in R[(k_1,\dots,k_i,k+1,0,\dots,0)] \cap V(q_{n-i}) \text{ for all } k > 0.
\end{aligned}$$

By construction of the model $\beta \Vdash \pi^\infty_{n-i}$ also holds: then $\mathbb{S}[x \mapsto S_\beta], \beta \Vdash c_{n-i+1} \wedge \pi^\infty_{n-i} \wedge F(\nu y.P(x \wedge y \wedge c_{n-i}))$, so $\beta \in \varphi^{\mathbb{S}}_x(S_\beta)$.

Now we move to the $\subseteq$ inclusion of (3). Let $\gamma \in \varphi^{\mathbb{S}}_x(S_\beta)$. We want to show that $\gamma \in S_{\beta+1}$. Since $\mathbb{S}[x \mapsto S_\beta], \gamma \Vdash \varphi$ holds, we proceed by case distinction as to which disjunct of φ is satisfied by γ. If $\gamma \Vdash G\bot$ then $\gamma = 0 \in S_{\beta+1}$. If $\gamma \Vdash c_1 \wedge Fx$, then $\gamma \in V(q_1)$, so that $\gamma = (k_1,\dots,k_n+1)$ and $\gamma' = (k_1,\dots,k_n) \in R[\gamma] \cap S_\beta$: as $\gamma' \in S_\beta$, then $\gamma = \gamma' + 1 \in S_{\beta+1}$.

Now suppose $\gamma \Vdash c_i \wedge \pi^\infty_{i-1} \wedge F(\nu y.P(y \wedge x \wedge c_{i-1}))$ for some $2 \le i \le n$. Then $\gamma \in V(q_i)$, so $\gamma = (k_1,\dots,k_{n-i+1}+1,0,\dots,0)$. For $j \in \omega$ let

$$\delta_j := (k_1,\dots,k_{n-i+1},j,0,\dots,0).$$

By construction $\delta_0 \in R[\gamma]$ and $\delta_j \in R[\delta_{j+1}]$ for all $j \ge 0$. Since $\mathbb{S}[x \mapsto S_\beta], \gamma \Vdash c_i \wedge \pi^\infty_{i-1} \wedge F(\nu y.P(y \wedge x \wedge c_{i-1}))$ then $\delta_j \in S_\beta$ for all $j > 0$. Hence

$$\beta > (k_1,\dots,k_{n-i+1},j,0,\dots,0) \text{ for all } j > 0,$$

implying $\beta \ge (k_1,\dots,k_{n-i+1}+1,0,\dots,0) = \gamma$, so $\gamma \in S_{\beta+1}$. $\lhd$

Now that we have the claim, it follows that there is a $\gamma \in \varphi^{\omega^n} \setminus \varphi^\beta$ for each $\beta < \omega^n$.

Proposition 4. *For all $m, n \in \omega$, if $m \geq n$, then $\pi_m^\infty \models \pi_n^\infty$. Moreover, if $\mathbb{S}$ is a model, $\mathbb{S}, s \Vdash \pi_m^\infty$ for some state s, and $t_0 t_1 \ldots$ is an R^{-1}-path witnessing the truth of π_m^∞ at s, then $t_j \Vdash \pi_m^\infty$ for all $j \in \omega$.*

Lemma 2. *Let $\mathbb{S} = (S, R, V)$ be a model and let $n \in \omega$. For $1 \leq i \leq n$, let $t_0 t_1 t_2 \ldots$ be an infinite R^{-1}-path such that*

$$\mathbb{S}, t_0 \Vdash \pi_{i-1}^\infty \textit{ and, for all } j > 0, \mathbb{S}, t_j \Vdash c_i \wedge \pi_{i-1}^\infty.$$

Then, for any ordinal α: if $t_0 \in \varphi_n^\alpha$ then $t_j \in \varphi_n^{\alpha + \omega^{i-1} \cdot j + 1}$ for all $j \in \omega$.

Proof. We prove the statement by induction on $1 \leq i \leq n$.

As the base case take $i = 1$, so that by assumption we have an infinite R^{-1}-path $t_0 t_1 t_2 \ldots$ such that $\mathbb{S}, t_j \Vdash c_1$ for all $j > 0$. Let $t_0 \in \varphi_n^\alpha$. We want to show that, for all $j \in \omega$, $t_j \in \varphi_n^{\alpha+j+1}$: we prove this by induction on $j \in \omega$. If $j = 0$, then $t_0 \in \varphi_n^\alpha \subseteq \varphi_n^{\alpha+1}$. Next, inductively assume that $t_j \in \varphi_n^{\alpha+j+1}$: then, since $t_j \in R[t_{j+1}]$, it follows that $\mathbb{S}[x \mapsto \varphi_n^{\alpha+j+1}], t_{j+1} \Vdash (c_1 \wedge Fx)$, so $t_{j+1} \in \varphi_n^{\alpha+(j+1)+1}$.

For the inductive step assume that the statement holds for i. We prove it for $i+1$, where $i < n$. Suppose then that $t_0 t_1 t_2 \ldots$ is an infinite R^{-1}-path such that $t_0 \Vdash \pi_i^\infty$ and for all $j > 0$, $t_j \Vdash c_{i+1} \wedge \pi_i^\infty$. Let $t_0 \in \varphi_n^\alpha$. We want to show that

$$\text{for every } j \in \omega, t_j \in \varphi_n^{\alpha + \omega^i \cdot j + 1}.$$

The proof of this last statement goes by induction on $j \in \omega$. The base case with $j = 0$ follows immediately, as by assumption $t_0 \in \varphi_n^\alpha$.

Now suppose that $t_j \in \varphi_n^{\alpha + \omega^i \cdot j + 1}$: we show that $t_{j+1} \in \varphi_n^{\alpha + \omega^i \cdot (j+1) + 1}$. By assumption $t_j \in R[t_{j+1}]$ and $t_j \Vdash \pi_i^\infty$, which in particular means that there is an infinite R^{-1}-path $u_0 u_1 \ldots$ (with $u_0 = t_j$) such that, for all $k > 0$, $u_k \Vdash c_i$. But then this path satisfies the conditions of the inductive hypothesis: by Proposition 4, since $u_0 \Vdash \pi_i^\infty$, then $u_0 \Vdash \pi_{i-1}^\infty$, and for every $k > 0$, $u_k \Vdash c_i \wedge \pi_{i-1}^\infty$. Then, by inductive hypothesis, since $u_0 = t_j \in \varphi_n^{\alpha + \omega^i \cdot j + 1}$ it follows that, for every $k \in \omega$, $u_k \in \varphi_n^{\alpha + \omega^i \cdot j + 1 + \omega^{i-1} \cdot k + 1}$. Since for all $k \in \omega$ it holds that

$$\begin{aligned}
\omega^i \cdot j + 1 + \omega^{i-1} \cdot k + 1 &< \omega^i \cdot j + 1 + \omega^i && (\text{as } \omega^{i-1} \cdot k + 1 < \omega^i \text{ for } i > 0) \\
&= \omega^i \cdot j + \omega^i && (1 + \omega^i = \omega^i \text{ for } i > 0) \\
&= \omega^i \cdot (j+1)
\end{aligned}$$

then also

$$\alpha + \omega^i \cdot j + 1 + \omega^{i-1} \cdot k + 1 < \alpha + \omega^i \cdot (j+1).$$

It follows that $u_k \in \varphi_n^{\alpha + \omega^i \cdot (j+1)}$ for all $k \in \omega$, so that

$$\mathbb{S}[x \mapsto \varphi_n^{\alpha + \omega^i \cdot (j+1)}], t_{j+1} \Vdash c_{i+1} \wedge \pi_i^\infty \wedge F(\nu y. P(x \wedge y \wedge c_i)).$$

We conclude that $t_{j+1} \in \varphi_n^{\alpha + \omega^i \cdot (j+1) + 1}$ as desired.

Lemma 3. *For an arbitrary model $\mathbb{S}$ and $0 < n < \omega$: $\gamma_x(\varphi_n, \mathbb{S}) \leq \omega^n$.*

Proof. It is sufficient to prove that $\varphi_n^{\omega^n+1} \subseteq \varphi_n^{\omega^n}$ for every model $\mathbb{S}$. Let $s \in \varphi_n^{\omega^n+1}$, that is, $\mathbb{S}[x \mapsto \varphi_n^{\omega^n}], s \Vdash \varphi_n$. We proceed by case distinction as to which disjunct of φ_n is satisfied by s to prove that $s \in \varphi_n^{\omega^n}$. If $s \Vdash G\bot$ then $s \in (\varphi_n)_x^{\mathbb{S}}(\emptyset) \subseteq \varphi_n^{\omega^n}$, while if $s \Vdash c_1 \wedge Fx$, then there is a $t \in R[s]$ such that $t \in \varphi_n^{\alpha}$ for some $\alpha < \omega^n$, so that $s \in \varphi_n^{\alpha+1} \subseteq \varphi_n^{\omega^n}$.

Now suppose $s \Vdash c_i \wedge \pi_{i-1}^{\infty} \wedge F(\nu y.P(y \wedge x \wedge c_{i-1}))$ for some $2 \leq i \leq n$. Then in particular there is a point $t \in R[s]$ and a R^{-1}-path $t_0 t_1 \ldots$ such that: *(i)* $t \in R[t_0]$, *(ii)* for all $j \in \omega$, $t_j \in \varphi_n^{\omega^n}$ and $t_j \Vdash c_{i-1}$. In particular, $t_0 \in \varphi_n^{\alpha}$ for some $\alpha < \omega^n$. Observe that $\varphi_n \wedge c_{i-1} \wedge F\top \models \pi_{i-2}^{\infty}$: this implies that $t_j \Vdash \pi_{i-2}^{\infty}$ for all $j \in \omega$, since $t_j \in \varphi_n^{\omega^n}$, $t_j \Vdash c_{i-1}$ and $R[t_j] \neq \emptyset$. This means that we can apply Lemma 2 and it follows that $t_j \in \varphi_n^{\alpha+\omega^{i-2}\cdot j+1} \subseteq \varphi_n^{\alpha+\omega^{i-1}}$ for all $j \in \omega$. Hence $\mathbb{S}[x \mapsto \varphi^{\alpha+\omega^{i-1}}], s \Vdash \varphi_n$ and $s \in \varphi_n^{\alpha+\omega^{i-1}+1} \subseteq \varphi_n^{\omega^n}$ (since $i \leq n$ and $\alpha < \omega^n$ imply $\alpha + \omega^{i-1} + 1 < \omega^n$).

References

1. Afshari, B., Leigh, G.E.: On closure ordinals for the modal μ-calculus. In: Computer Science Logic 2013 (CSL 2013). Leibniz International Proceedings in Informatics (LIPIcs), vol. 23, pp. 30–44. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik (2013). https://doi.org/10.4230/LIPIcs.CSL.2013.30. http://drops.dagstuhl.de/opus/volltexte/2013/4188
2. Afshari, B., Leigh, G.: Cut-free completeness for modal μ-calculus. In: Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2017), pp. 1–12 (2017). https://doi.org/10.1109/LICS.2017.8005088
3. Calude, C., Jain, S., Khoussainov, B., Li, W., Stephan, F.: Deciding parity games in quasipolynomial time. In: Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, pp. 252–263 (2017). https://doi.org/10.1145/3055399.3055409
4. Czarnecki, M.: How fast can the fixpoints in modal μ-calculus be reached? In: Fixed Points in Computer Science 2010 (FICS 2010), pp. 35–39, August 2010. https://hal.archives-ouvertes.fr/hal-00512377/document#page=36
5. D'Agostino, G., Hollenberg, M.: Logical questions concerning the μ-calculus. J. Symb. Log. **65**, 310–332 (2000)
6. Emerson, E.A., Jutla, C.S.: The complexity of tree automata and logics of programs (extended abstract). In: Proceedings of the 29th Annual Symposium on Foundations of Computer Science, pp. 328–337. IEEE Computer Society Press (1988). https://doi.org/10.1109/SFCS.1988.21949
7. Fontaine, G.: Continuous fragment of the μ-calculus. In: Kaminski, M., Martini, S. (eds.) CSL 2008. LNCS, vol. 5213, pp. 139–153. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-87531-4_12
8. Fontaine, G., Venema, Y.: Some model theory for the modal μ-calculus: syntactic characterisations of semantic properties. Log. Methods Comput. Sci. **14**(1) (2018)

9. Gouveia, M.J., Santocanale, L.: $\aleph_1$ and the modal μ-calculus. In: 26th EACSL Annual Conference on Computer Science Logic (CSL 2017). Leibniz International Proceedings in Informatics (LIPIcs), vol. 82, pp. 38:1–38:16. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik (2017). https://doi.org/10.4230/LIPIcs.CSL.2017.38. http://drops.dagstuhl.de/opus/volltexte/2017/7692. An updated version can be found at https://arxiv.org/abs/1704.03772v2
10. Janin, D., Walukiewicz, I.: On the expressive completeness of the propositional μ-calculus with respect to monadic second order logic. In: Montanari, U., Sassone, V. (eds.) CONCUR 1996. LNCS, vol. 1119, pp. 263–277. Springer, Heidelberg (1996). https://doi.org/10.1007/3-540-61604-7_60
11. Kozen, D.: Results on the propositional μ-calculus. Theor. Comput. Sci. **27**, 333–354 (1983)
12. Kozen, D.: A finite model theorem for the propositional μ-calculus. Stud. Log. **47**, 233–241 (1988)
13. Milanese, G.: An exploration of closure ordinals in the modal μ-calculus. Master's thesis, Institute for Logic, Language and Computation, University of Amsterdam (2018)
14. Otto, M.: Eliminating recursion in the μ-calculus. In: Meinel, C., Tison, S. (eds.) STACS 1999. LNCS, vol. 1563, pp. 531–540. Springer, Heidelberg (1999). https://doi.org/10.1007/3-540-49116-3_50
15. Walukiewicz, I.: Completeness of Kozen's axiomatisation of the propositional μ-calculus. Inf. Comput. **157**(1), 142–182 (2000). http://www.sciencedirect.com/science/article/pii/S0890540199928365

$\mathbf{SIXTEEN}_3$ in Light of Routley Stars

Hitoshi Omori(✉) and Daniel Skurt(✉)

Department of Philosophy I, Ruhr-University Bochum, Bochum, Germany
{Hitoshi.Omori,Daniel.Skurt}@rub.de

Abstract. For one of the most well-known many-valued logics **FDE**, there are several semantics, including the star semantics by Richard Routley and Valerie Routley, the two-valued relational semantics by Michael Dunn and the four-valued semantics by Nuel Belnap. The last semantics inspired Yaroslav Shramko and Heinrich Wansing to introduce the trilattice $\mathbf{SIXTEEN}_3$. In this article, we offer two alternative semantical presentations for $\mathbf{SIXTEEN}_3$, by applying the Routleys' semantics and the Dunn semantics. Based on our new semantics, we discuss related systems with less truth values, as well as the relation to **FDE**-based modal logics.

Keywords: **FDE** · $\mathbf{SIXTEEN}_3$ · Routley star · Dunn semantics

1 Introduction

1.1 Background (I): From Belnap to Shramko-Wansing

Ever since Jan Łukasiewicz and Emil Post started to explore more than two truth values independently in the 1920s, infinitely many kinds of many-valued logics have been introduced. The one that plays the crucial role in this paper is the four-valued logic of Belnap and Dunn, also known as **FDE**.

The four-valued truth tables for **FDE** were known since the 1950s, when Timothy Smiley pointed this out to Nuel Belnap, but the four values did not have an intuitive reading. It was Dunn who explicitly connected these four values to the classical truth values, true and false (see [6]). This then inspired Belnap to write the two influential papers [2,3]. In particular, the four values are now seen as the power set $\mathcal{P}(\{1,0\})$ of the set of the classical truth-values $\{1,0\}$, and receive the following intuitive reading:

The work reported in this paper started during DS's visit to Japan which was supported by JSPS KAKENHI Grant Number JP18K12183 granted to HO. HO was supported by a Sofja Kovalevskaja Award of the Alexander von Humboldt-Foundation, funded by the German Ministry for Education and Research. The work of DS has been carried out as part of the research project "**FDE**-based modal logics", supported by the Deutsche Forschungsgemeinschaft, DFG, grant WA 936/13-1. We would like to thank Sergei Odintsov, Yaroslav Shramko, Heinrich Wansing, Zach Weber and the referees for helpful discussions and/or comments on an earlier draft.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 516–532, 2019.
https://doi.org/10.1007/978-3-662-59533-6_31

$$\begin{aligned}\{0\} &= \text{told false}, & \{\} &= \text{told neither true nor false},\\ \{1\} &= \text{told true}, & \{1,0\} &= \text{told both true and false}.\end{aligned}$$

The above reading also inspired another perspective on the four-values, namely the bilattice of the power set of $\{1, 0\}$ (cf. [1,8]). In particular, two orders measure the degree of truth and the amount of information.

In [21–23] Shramko and Wansing then took this idea of Belnap even a step further. By arguing that the *computer* metaphor of Belnap can be transformed into considering a *computer network* communicating with each other about propositions, Shramko and Wansing developed the idea that such computers should be able to handle information that can be, for example, overcomplete and at the same time just true or false. In this way, they introduced **SIXTEEN**$_3$ which takes the power set of $\mathcal{P}(\{1, 0\})$ to generate a "useful sixteen-valued logic" which is meant to represent "how a computer network should think". This is thus a generalization of Belnap's "useful four-valued logic" which is meant to represent "how a computer should think". Moreover, **SIXTEEN**$_3$ is now a trilattice, rather than a bilattice, where an independent degree of falsity can be defined as an additional order.

Due to the interesting motivation, **SIXTEEN**$_3$ has now collected a lot of the attention it deserves. Just to mention some relevant work, Odintsov in [12], added some new algebraic insights and marked an important step on the problem of axiomatization. Heinrich Wansing considers sequent calculi related to **SIXTEEN**$_3$ in [25], and an analytic tableaux calculus is devised by Muskens and Wintein in [10]. Finally, the property of interpolation is studied again by Muskens and Wintein in [11].

1.2 Background (II): Routley and Dunn Semantics for FDE

As it is well-known, the four-valued interpretation of **FDE** is not the only semantics.[1] For the purpose of this paper, we focus on the following two: Routleys' star semantics and Dunn's relational semantics. Let us briefly highlight the key ideas of the two semantics which are both *two*-valued semantics.[2]

Routleys' star semantics, devised by Routley and Routley in [20], is a two-valued world semantics, as in the well-known Kripke semantics, but includes the so-called star operation which is an involutive operation on worlds. This star operation is used to interpret the negation. For conjunction and disjunction, it remains to be completely classical.

Dunn's relational semantics (or Dunn semantics in short) is yet another two-valued semantics which is also free of worlds. The crucial idea is to use a *relation* rather than a *function* in interpreting the language. In particular, formulas may be related to *both* true and false, or *neither* true nor false. As a consequence, truth and falsity conditions are both necessary, though in the case of **FDE**, those conditions remain completely classical.

[1] For a recent overview, see for example [17].

[2] The formal details will be given in the next section, so we are justified to be brief.

Both approaches have virtues of their own. On the one hand, Routleys' semantics is rather successful when applied to relevant logics. On the other hand, Dunn gives wonderful insights by giving an intuitive reading of truth values, as we already observed above through Belnap's semantics. In any case, the important thing here is that there are interesting two-valued semantics for **FDE**.

1.3 Aim

Based on these backgrounds, the motivation for this paper is rather simple: can we also devise two-valued semantics for logics related to $\mathbf{SIXTEEN}_3$? To the best of our knowledge, this seems to be not addressed yet in the literature. Therefore, we aim at marking the first step towards filling that gap.

On a broader scope, reducing the number of truth-values of a given system can be traced back to Suszko (cf. [24]), who believed that any multiplication of truth-values is a "mad idea". We do not wish to conflate our approach of reducing the number of truth-values with Suszko's critique about many-valued logics in general, but rather during the course of this article we will present an alternative strategy to obtain that goal.[3]

The paper is organized as follows. In Sects. 2 and 3 we will briefly recapitulate the basics of **FDE** and $\mathbf{SIXTEEN}_3$. These are followed by Sects. 4 and 5 in which we introduce the new two-valued semantics for $\mathbf{SIXTEEN}_3$. Based on the new semantics, we will reflect upon the implications in Sect. 6. Finally, we conclude the paper in Sect. 7 by summarizing our main observations and discuss some possible topics for further research.

2 Two-Valued Semantics for FDE

Our propositional languages consist of a finite set C of propositional connectives and a countable set Prop of propositional variables which we refer to as $\mathcal{L}_{\mathsf{C}}$. Furthermore, we denote by $\mathsf{Form}_{\mathsf{C}}$ the set of formulas defined as usual in $\mathcal{L}_{\mathsf{C}}$. In this paper, we always assume that $\{\sim, \wedge, \vee\} \subseteq \mathsf{C}$ and just include the propositional connective(s) not from $\{\sim, \wedge, \vee\}$ in the subscript of $\mathcal{L}_{\mathsf{C}}$. Moreover, we denote a formula of $\mathcal{L}_{\mathsf{C}}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}_{\mathsf{C}}$ by Γ, Δ, Σ, etc.

First, we review Routleys' star semantics.

Definition 1. *A* Routley interpretation *for $\mathcal{L}$ is a structure $\langle W, *, v\rangle$ where $W \neq \emptyset$ is a set of worlds, $* : W \longrightarrow W$ is a function with $w^{**} = w$, and $v : W \times \mathsf{Prop} \longrightarrow \{0, 1\}$. The function v is extended to $I : W \times \mathsf{Form} \longrightarrow \{0, 1\}$ as follows:*

$$I(w, p) = v(w, p), \qquad I(w, A \wedge B) = 1 \textit{ iff } I(w, A) = 1 \textit{ and } I(w, B) = 1,$$
$$I(w, \sim A) = 1 \textit{ iff } I(w^*, A) \neq 1, \quad I(w, A \vee B) = 1 \textit{ iff } I(w, A) = 1 \textit{ or } I(w, B) = 1.$$

[3] For a mechanical procedure to reduce the number of truth values in **FDE** and its expansions, see [16].

Definition 2. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_* A$ *iff for all Routley interpretations* $\langle W, *, v\rangle$ *and for all* $w \in W$, *if* $I(w, B) = 1$ *for all* $B \in \Gamma$ *then* $I(w, A) = 1$.

Second, we review Dunn's relational semantics.

Definition 3. *A* Dunn-interpretation *for* $\mathcal{L}$ *is a relation,* r, *between propositional variables and the values 1 and 0, namely* $r \subseteq \mathsf{Prop} \times \{1, 0\}$. *Given an interpretation,* r, *this is extended to a relation between all formulas and truth values by the following clauses:*

$$\begin{array}{ll} {\sim}Ar1 \textit{ iff } Ar0, & {\sim}Ar0 \textit{ iff } Ar1, \\ A \wedge Br1 \textit{ iff } Ar1 \textit{ and } Br1, & A \wedge Br0 \textit{ iff } Ar0 \textit{ or } Br0, \\ A \vee Br1 \textit{ iff } Ar1 \textit{ or } Br1, & A \vee Br0 \textit{ iff } Ar0 \textit{ and } Br0. \end{array}$$

Definition 4. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_r A$ *iff for all Dunn-interpretations* r, *if* $Br1$ *for all* $B \in \Gamma$ *then* $Ar1$.

Then, the following result is rather well-known.

Fact 5. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_r A$ *iff* $\Gamma \models_* A$.

A proof can be found, e.g., in [18, 8.7.17, 8.7.18]. In fact, something stronger can be established by a careful examination of Graham Priest's proof. To this end, we introduce another semantic consequence relation.

Definition 6. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{*,2} A$ *iff for all Routley interpretations* $\langle W, *, v\rangle$ *such that the number of worlds is 2 and for all* $w \in W$, *if* $I(w, B){=}1$ *for all* $B \in \Gamma$ *then* $I(w, A) = 1$.

Then, we obtain the following.

Lemma 1. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_r A$ *iff* $\Gamma \models_{*,2} A$.

Proof. For the proof of the left-to-right direction, Priest's construction works perfectly well with the two-world case. For the other direction, Priest's construction already establishes the desired result. □

As an immediate corollary, we obtain the following result, which can be regarded as logical folkore.

Theorem 1. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_* A$ *iff* $\Gamma \models_{*,2} A$. *That is, two worlds suffice for the extensional fragment.*

Remark 1. In view of the above result, we may conclude that there is a clear understanding of the star in the context of the above language. The star world is simply *the other* world. Of course, this only works with the simple language, not in the language with the intensional conditional. In the latter case, the star operation is elegantly characterized by Restall (cf. [19]).

3 Basics of $\mathbf{SIXTEEN_3}$

3.1 Language

There are several languages discussed in relation to the trilattice $\mathbf{SIXTEEN}_3$. Following the convention specified in the previous section, we will mainly deal with $\mathcal{L}_{\sim_f}$ and $\mathcal{L}_{\sim_f,\wedge_f,\vee_f}$. The latter is referred to as $\mathcal{L}_{tf}$ in the literature, but for the sake of presentation, we will use the above notation with the hope of being more accessible to wider audience.

Note too that we are omitting the subscript t for connectives. We fully understand that this goes very much against the spirit of the trilattice in general, but for the sake of presentation, and ease of comparison between **FDE** and $\mathbf{SIXTEEN}_3$, we keep the basic connectives free of subscripts.

3.2 Semantics

Let **16** be the set of generalized truth values which consists of the following 16 values:

1. $\emptyset = \{\ \}$
2. $\mathbf{N} = \{\{\ \}\}$
3. $\mathbf{F} = \{\{0\}\}$
4. $\mathbf{T} = \{\{1\}\}$
5. $\mathbf{B} = \{\{0,1\}\}$
6. $\mathbf{NF} = \{\{\ \},\{0\}\}$
7. $\mathbf{NT} = \{\{\ \},\{1\}\}$
8. $\mathbf{NB} = \{\{\ \},\{0,1\}\}$
9. $\mathbf{FT} = \{\{0\},\{1\}\}$
10. $\mathbf{FB} = \{\{0\},\{0,1\}\}$
11. $\mathbf{TB} = \{\{1\},\{0,1\}\}$
12. $\mathbf{NFT} = \{\{\ \},\{0\},\{1\}\}$
13. $\mathbf{NFB} = \{\{\ \},\{0\},\{0,1\}\}$
14. $\mathbf{NTB} = \{\{\ \},\{1\},\{0,1\}\}$
15. $\mathbf{FTB} = \{\{0\},\{1\},\{0,1\}\}$
16. $\mathbf{A} = \{\{\ \},\{0\},\{1\},\{0,1\}\}$

Note here that we changed the notation slightly from the original presentation. More specifically, we replaced T and F by 1 and 0. Moreover, the naming strategy for the truth values is very simple. Recall the following representation:

1. **n**=$\{\ \}$, for **n**either true nor false
2. **f**=$\{0\}$, for **f**alse only
3. **t**=$\{1\}$, for **t**rue only
4. **b**=$\{0,1\}$, for **b**oth true and false

Then, except for the value **A**, the inclusion of capital letters **N**, **F**, **T** and **B** corresponds to the fact that **n**, **f**, **t** and **b** are members of the generalized truth value. And, for **A**, it stands for **a**ll values **n**, **f**, **t** and **b** are members of the set.

Now we can define three different orderings on **16**.

Definition 7. *For every* $x, y \in \mathbf{16}$*:*

1. $x \leq_i y$ *iff* $x \subseteq y$*;*
2. $x \leq_t y$ *iff* $x^1 \subseteq y^1$ *and* $y^{-1} \subseteq x^{-1}$*,*
where $x^1 := \{z \in x : 1 \in z\}$ *and* $x^{-1} := \{z \in x : 1 \notin z\}$*;*
3. $x \leq_f y$ *iff* $x^0 \subseteq y^0$ *and* $y^{-0} \subseteq x^{-0}$*,*
where $x^0 := \{z \in x : 0 \in z\}$ *and* $x^{-0} := \{z \in x : 0 \notin z\}$*.*

We can then easily see that meets and joins exist in **16** for all three partial orders. Therefore, we use $\sqcap$ and $\sqcup$ with the appropriate subscripts for these operations under the corresponding orders. Then, the algebraic structure of **16** comes out as the trilattice $\mathbf{SIXTEEN}_3 = \langle \mathbf{16}, \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f \rangle$.

We can associate with each of the lattice orders of **SIXTEEN**$_3$ a unary operation which is an involution of order two with respect to this ordering and preserves the other orders. The unary operations $-_t$, $-_f$, and $-_i$ corresponding to the orders $\leq_t$, $\leq_f$ and $\leq_i$, respectively, are defined as follows.

x	$-_t x$	$-_f x$	$-_i x$	x	$-_t x$	$-_f x$	$-_i x$	x	$-_t x$	$-_f x$	$-_i x$	x	$-_t x$	$-_f x$	$-_i x$
$\emptyset$	$\emptyset$	$\emptyset$	**A**	**B**	**F**	**T**	**FTB**	**NB**	**FT**	**FT**	**FT**	**NFB**	**FTB**	**NFT**	**F**
N	**T**	**F**	**NFT**	**NF**	**TB**	**NF**	**NF**	**FB**	**FB**	**NT**	**FB**	**NTB**	**NFT**	**FTB**	**T**
F	**B**	**N**	**NFB**	**NT**	**NT**	**FB**	**NT**	**TB**	**NF**	**TB**	**TB**	**FTB**	**NFB**	**NTB**	**B**
T	**N**	**B**	**NTB**	**FT**	**NB**	**NB**	**NB**	**NFT**	**NTB**	**NFB**	**N**	**A**	**A**	**A**	$\emptyset$

We are now ready to assign generalized truth values of **16** to our language. More specifically, given a **16**-valuation $v : \mathsf{Prop} \to \mathbf{16}$, we extend the valuation to $\mathsf{Form}_{\sim_f, \wedge_f, \vee_f}$ as follows.

Definition 8. *For every* $A, B \in \mathsf{Form}_{\sim_f, \wedge_f, \vee_f}$*:*

1. $v(A \wedge B) = v(A) \sqcap_t v(B)$
2. $v(A \vee B) = v(A) \sqcup_t v(B)$
3. $v(\sim A) = -_t v(A)$
4. $v(A \wedge_f B) = v(A) \sqcap_f v(B)$
5. $v(A \vee_f B) = v(A) \sqcup_f v(B)$
6. $v(\sim_f A) = -_f v(A)$

Based on this, we can finally define the semantic consequence relations.

Definition 9. *For every* $A, B \in \mathsf{Form}_{\sim_f, \wedge_f, \vee_f}$*:*

- $A \models_t B$ *iff for all* **16***-valuations* v*:* $v(A) \leq_t v(B)$*;*
- $A \models_f B$ *iff for all* **16***-valuations* v*:* $v(A) \leq_f v(B)$*.*

Remark 2. We are *not* using the information order at all to interpret our language, but we introduced them above to emphasize that **16** is a trilattice. We will come back to the unary connective interpreted via $-_i$ towards the end of this paper, but only briefly, in the conclusion section. For discussions on the language including informational connectives, see e.g. [14].

3.3 Proof Systems

We now turn to the proof system. Note that we will only offer the proof system for the language $\mathcal{L}_{\sim_f}$, and just remark on the case of full language, namely the language $\mathcal{L}_{\sim_f, \wedge_f, \vee_f}$.

Definition 10. $\vdash$ *is a binary consequence relation on the language* $\mathcal{L}_{\sim_f}$ *satisfying the following axioms and rules.*

$A \wedge B \vdash A$	(a_t1)		
$A \wedge B \vdash B$	(a_t2)	$\dfrac{A \vdash B \quad B \vdash C}{A \vdash C}$	(r_t1)
$A \vdash A \vee B$	(a_t3)		
$B \vdash A \vee B$	(a_t4)	$\dfrac{A \vdash B \quad A \vdash C}{A \vdash B \wedge C}$	(r_t2)
$A \wedge (B \vee C) \vdash (A \wedge B) \vee C$	(a_t5)		
$A \vdash {\sim}{\sim}A$	(a_t6)	$\dfrac{A \vdash C \quad B \vdash C}{A \vee B \vdash C}$	(r_t3)
${\sim}{\sim}A \vdash A$	(a_t7)		
$A \vdash {\sim_f}{\sim_f}A$	(a_t8)	$\dfrac{A \vdash B}{{\sim}B \vdash {\sim}A}$	(r_t4)
${\sim_f}{\sim_f}A \vdash A$	(a_t9)		
${\sim_f}{\sim}A \vdash {\sim}{\sim_f}A$	(a_t10)	$\dfrac{A \vdash B}{{\sim_f}A \vdash {\sim_f}B}$	(r_t5)

Remark 3. Note that the binary consequence relation characterized in terms of the axioms from (a_t1) to (a_t7), as well as the rules from (r_1) to (r_t4) is sound and complete with respect to **FDE** for the language $\mathcal{L}$.

Finally, the following result was established by Shramko and Wansing in [22, Theorems 4.10, 4.13].

Theorem 2 (Shramko & Wansing). *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \vdash B$ iff $A \models_t B$.*

Remark 4. The problem of axiomatizing $\models_t$ for the language $\mathcal{L}_{\sim_f,\wedge_f,\vee_f}$ was left open in [22], but Odintsov in [12] marked the first step by showing that $\models_t$ is axiomatizable and that the consequence relation can be characterized by the intersection of two related consequence relations. Odintsov also introduced an expansion of $\mathcal{L}_{\sim_f,\wedge_f,\vee_f}$ by adding an implication, and presented an axiomatization of $\models_t$ in the expanded language. A definite solution to the original problem was given in [14] by Odintsov and Wansing by making use of algebraic results related to $\mathbf{SIXTEEN}_3$.

4 Alternative Semantics for $\mathbf{SIXTEEN_3}$ (I)

The first alternative semantics will have two star operations. More specifically, we take the star semantics for **FDE**, and add one more star to capture the additional connective ${\sim_f}$. Our strategy here is to prove the soundness and completeness with respect to the proof system given by Shramko and Wansing to establish the equivalence between the original semantics and the two-star semantics.

4.1 Semantics

Definition 11. *A* two-star interpretation *for $\mathcal{L}_{\sim_f}$ is at tuple $\mathcal{M} = \langle W, g, *_1, *_2, v\rangle$ where $W \neq \emptyset$ is a set of worlds, $g \in W$; $*_i : W \longrightarrow W$ is a function with $w^{*_i *_i} = w$ and $w^{*_i *_j} = w^{*_j *_i}$; $v : W \times \mathsf{Prop} \to \{0, 1\}$. The function v is extended to $I : W \times \mathsf{Form} \to \{0, 1\}$ by the following condition:*

$I(w, p) = v(w, p)$,
$I(w, {\sim}A){=}1$ *iff* $I(w^{*_1}, A){\neq}1$, $I(w, A \wedge B){=}1$ *iff* $I(w, A){=}1$ *and* $I(w, B){=}1$,
$I(w, {\sim}_f A){=}1$ *iff* $I(w^{*_2}, A){=}1$, $I(w, A \vee B){=}1$ *iff* $I(w, A){=}1$ *or* $I(w, B){=}1$.

Remark 5. It should be clear, from the definition, that the fragment with only the "truth connectives" will coincide with **FDE**. Note also that the truth condition for ${\sim}_f$ does *not* look like a truth condition for negation. We will reflect upon this connective in Sect. 6.

We then define two kinds of semantic consequence relation.

Definition 12. *Let $\Gamma \cup \{A\}$ be set of sentences in $\mathcal{L}_{\sim_f}$. Then,*

- $\Gamma \models_{*,\forall} A$ *iff for all two-star interpretations* $\langle W, g, *_1, *_2, v\rangle$ *and for all* $w \in W$, $I(w, A) = 1$ *if* $I(w, B) = 1$ *for all* $B \in \Gamma$.
- $\Gamma \models_{*,g} A$ *iff for all two-star interpretations* $\langle W, g, *_1, *_2, v\rangle$, $I(g, A) = 1$ *if* $I(g, B) = 1$ *for all* $B \in \Gamma$.

Remark 6. As we will establish below, these two consequence relations are equivalent as in some (not all!) modal logics (recall Kripke's seminal paper and the more recent text books). However, it will be useful to have both for our purposes.

4.2 Equivalence of Three Semantic Consequence Relations

We will now establish the equivalence of $\models_t$, $\models_{*,\forall}$ and $\models_{*,g}$ via the proof system. More specifically, in view of Theorem 2 of Shramko and Wansing, we prove the following three statements: for all $A, B \in \mathsf{Form}_{\sim_f}$,

if $A \vdash B$ then $A \models_{*,\forall} B$, if $A \models_{*,\forall} B$ then $A \models_{*,g} B$, if $A \models_{*,g} B$ then $A \vdash B$.

Note here that the second item is obvious. Therefore, we prove the first and the third item. The first item, which is soundness, is quite straightforward.

Proposition 1. *For all* $A, B \in \mathsf{Form}_{\sim_f}$, *if* $A \vdash B$ *then* $A \models_{*,\forall} B$.

Proof. We only note that we need $\models_{*,\forall}$, instead of $\models_{*,g}$, to establish the soundness, especially for the rules (r_t4) and (r_t5). □

For the purpose of establishing the third item, we construct a suitable canonical model. To this end, we introduce some standard notions.

Definition 13. *Let Γ be a set of sentences. Then, Γ is*

- *a* theory *iff Γ is closed under $\vdash$ and $\wedge$, i.e., for all A, B, if $A \in \Gamma$ and $A \vdash B$ then $B \in \Gamma$, and if $A \in \Gamma$ and $B \in \Gamma$, then $A \wedge B \in \Gamma$;*
- prime *iff for all A, B, if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$.*

The following fact is well known, due to Lindenbaum.

Lemma 2 (Lindenbaum). *For all A, B, if $A \nvdash B$ then there is a prime theory Γ such that $A \in \Gamma$ and $B \notin \Gamma$.*

We will also make use of the following lemma which is already established by Shramko and Wansing in [22, Lemma 4.11].

Lemma 3 (Shramko & Wansing). *Let Γ be a theory, and let Γ^* be defined as follows:*

$$\Gamma^* := \{A : {\sim_f} A \in \Gamma\}$$

Then Γ^* is a theory, ${\sim_f} A \in \Gamma^*$ iff $A \in \Gamma$, and Γ^* is prime iff Γ is prime.

We can then prove completeness as well.

Theorem 3. *For all $A, B \in \mathsf{Form}_{\sim_f}$, if $A \models_{*,g} B$ then $A \vdash B$.*

Proof. The details can be found in Appendix A. □

As a corollary, we obtain the following desired result:

Corollary 1. *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_t B$ iff $A \models_{*,g} B$ iff $A \models_{*,\forall} B$.*

We will now turn to two observations related to this result.

4.3 Two Basic Observations

First, we observe that we only need four worlds for two-star interpretations to characterize the syntactic consequence relation $\vdash$. To this end, we introduce one more semantic consequence relation.

Definition 14. *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_{*,g,4} B$ iff for all two-star interpretations $\langle W, g, *_1, *_2, v\rangle$ such that the number of worlds is 4, $I(g, B) = 1$ if $I(g, A) = 1$.*

Then, we obtain in analogy to Theorem 1 the following result:

Proposition 2. *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_{*,g} B$ iff $A \models_{*,g,4} B$.*

Proof. The left-to-right direction is obvious. For the other direction, it suffices to prove that $A \vdash B$ if $A \models_{*,g,4} B$ in view of Proposition 1. But this is already established by the proof for Theorem 3. □

Remark 7. We have a relatively clear formal understanding of star operations. However, as in the case for **FDE**, we do not know what they *mean*. Only that each star corresponds to a different "mate" relation, cf. [18, p. 151].

The second observation, which relies on the first observation, is that $\models_t$ is equivalent to yet another semantic consequence relation defined in terms of preservation of designated values. More precisely, we introduce the following consequence relation.

Definition 15. *For all $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_{16} B$ iff for all **16**-valuations v: $v(B) \in \mathcal{D}$ if $v(A) \in \mathcal{D}$, where $\mathcal{D} := \{x \in \mathbf{16} : \mathbf{T} \in x\}$.*

Then, by unpacking the definition of $\models_{*,g,4}$, we obtain the following result:

Proposition 3. *For all* $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_t B$ *iff* $A \models_{16} B$.

Remark 8. The reason of introducing $\models_{*,g}$ is to establish this connection to the 16-valued semantic consequence relation defined via designated values.

Note also that the result of the proposition above was already discussed in Lemma 4.3 in [22], for the language $\mathcal{L}$. In Lemma 4.9 of the same paper an additional restriction for the consequence relation is discussed for the language $\mathcal{L}_{\sim_f, \wedge_f, \vee_f}$ In the language $\mathcal{L}_{\sim_f}$, however, we do not need such additional restriction.

5 Alternative Semantics for SIXTEEN$_3$ (II)

The second alternative semantics will have only one star operation, but will be based on four-valued worlds, in analogy to the relational semantics of **FDE**. Therefore, the new semantics presented in this section can be seen as a hybrid of Routleys' semantics and Dunn semantics. The equivalence of the semantics will be established through the semantics given in the previous section.

5.1 Semantics

Definition 16. *A* one-star interpretation *for* $\mathcal{L}_{\sim_f}$ *is a tuple* $\mathcal{M} = \langle W, g, *, r \rangle$ *where* W *is a non-empty set of worlds,* $g \in W$*;* $* : W \longrightarrow W$ *is a function with* $w^{**} = w$*; and* $r_w \subseteq \mathsf{Prop} \times \{0, 1\}$ *for all* $w \in W$*. Given an interpretation,* r_w*, this is extended to a relation between all formulas and truth values by the following clauses:*

$$\begin{array}{rlrl}
\sim A r_w 1 & \textit{iff } A r_{w^*} 0, & \sim A r_w 0 & \textit{iff } A r_{w^*} 1, \\
A \wedge B r_w 1 & \textit{iff } A r_w 1 \textit{ and } B r_w 1, & A \wedge B r_w 0 & \textit{iff } A r_w 0 \textit{ or } B r_w 0, \\
A \vee B r_w 1 & \textit{iff } A r_w 1 \textit{ or } B r_w 1, & A \vee B r_w 0 & \textit{iff } A r_w 0 \textit{ and } B r_w 0, \\
\sim_f A r_w 1 & \textit{iff } A r_{w^*} 1, & \sim_f A r_w 0 & \textit{iff } A r_{w^*} 0.
\end{array}$$

Remark 9. As one can see from the above definition, the one-star interpretation is a hybrid of Routleys' semantics, for the use of the star operation, and Dunn semantics, for the use of the relation instead of the function.

Definition 17. *For all* $A, B \in \mathsf{Form}_{\sim_f}$, $A \models_r B$ *iff for all one-star interpretations* $\mathcal{M}$*,* $B r_g 1$ *if* $A r_g 1$.

5.2 Equivalence of Two Semantics

Proposition 4. *For all* $A, B \in \mathsf{Form}_{\sim_f}$*, if* $A \models_{*,g} B$ *then* $A \models_r B$.

Proof. The details are spelled out in Appendix B. □

Proposition 5. *For all* $A, B \in \mathsf{Form}_{\sim_f}$*, if* $A \models_r B$ *then* $A \models_{*,g} B$.

Proof. The details are spelled out in Appendix C. □

Remark 10. As in the case for **FDE** it is possible that the number of worlds for $\models_{*,g}$ can be reduce to 2. This can be seen by careful examination of the proofs of Lemma 1 and Proposition 5.

6 Reflections on $\sim_f$

The operator $\sim_f$ can be regarded as the negation with respect to the falsity order of the trilattice $\mathbf{SIXTEEN}_3$. However, in the context of this article, in which we focus solely on truth-order, it can be observed that $\sim_f$ is more than just a simple negation.

6.1 $\sim_f$ in Special Cases

The introduction of $\mathbf{SIXTEEN}_3$ inspired Dmitri Zaitsev to consider some variants with less truth values in [26]. In brief, Zaitsev suggests to apply the power set of a three-element set, rather than the four-element set used by Shramko and Wansing. Due to the limitation of space, we cannot discuss the details of how our two-valued semantics will capture one of Zaitsev's systems.

However, since it is rather natural to consider some variants with less truth values, we briefly consider three special cases of two-star interpretations, and connect the resulting system to those known in the literature.

First, as expected, if we require $w^{*_2} = w$ for all $w \in W$, then we simply obtain an expansion of **FDE** with $\sim_f A \vdash A$ and $A \vdash \sim_f A$. Second, if we require $w^{*_1} = w^{*_2}$ for all $w \in W$, then we obtain an expansion of **FDE** with $\sim_f$ as conflation.[4] Since classical negation is definable in terms of de Morgan negation and conflation, and conflation is definable in terms of de Morgan negation and classical negation, the resulting system is equivalent to the expansion of **FDE** by classical negation, called **BD+** in [5]. Finally, if we require $w^{*_1} = w$ for all $w \in W$, then $\sim$ is a classical negation, and $\sim_f$ is again conflation. Since de Morgan negation is definable in terms of classical negation and conflation, the resulting system is again equivalent to **BD+**.

6.2 $\sim_f$ as a Modal Operator

In $\mathbf{SIXTEEN}_3$, the operator $\sim_f$ serves as a negation over the falsity ordering. In what follows, we will, however, show that truth condition for $\sim_f$, understood as in Sect. 4, suffice to interpret $\sim_f$ as a modal operator satisfying the **K**-axiom, as well as the rule of necessitation. Since our language is rather weak, we add $\to$ which satisfies the following truth condition in a two-star interpretation.

$$I(w, A \to B) = 1 \text{ iff } I(w, A) \neq 1 \text{ or } I(w, B) = 1.$$

In fact, this connective is the implication introduced by Odintsov in [12] as $\to_t$.

It is now possible to prove the following proposition.

Proposition 6. *For all $A, B \in \mathsf{Form}_{\sim_f,\to}$,*

1. $\models_{*,\forall} \sim_f(A \to B) \to (\sim_f A \to \sim_f B)$,

[4] Given a Dunn interpretation, conflation, written as $-$, is characterized by the following truth and falsity conditions: $-Ar1$ iff not $Ar0$, and $-Ar0$ iff not $Ar1$.

2. $\dfrac{\models_{*,\forall} A}{\models_{*,\forall} \sim_f A}$ *and* $\dfrac{\models_{*,\forall} \sim_f A}{\models_{*,\forall} A}$.

Remark 11. The **T** and **S4** axiom are not valid in this semantics. Furthermore, the equivalence $\sim\sim_f\sim A \leftrightarrow \sim_f A$ shows that $\sim_f$ is self-dual and hence also contains properties of a possibility operator. The negative modality $\sim$ behaves in a similar way.[5]

Given that $\sim_f$ is not defined via an accessibility relation over worlds, but rather a function that maps worlds to worlds, one may doubt that $\sim_f$ counts as modal operator at all. However, as described by van Benthem in [4], it is possible to model propositional modal logic with a family of functions $\mathcal{F}$, rather than accessibility relations. A model $\mathcal{M} = \langle W, \mathcal{F}, V\rangle$ is then a tuple in the usual manner, with the following clause for the necessity operator: $I(w, \Box A) = 1$ iff $I(f(w), A) = 1$ for all $f \in \mathcal{F}$. For example, the modal logic **T** is complete with respect "for all frames whose function set $\mathcal{F}$ contains the identity function" [4].

In analogy to van Benthem's approach, we may regard our two-star interpretation as a model $\mathcal{M} = \langle W, g, *_1, \mathcal{F}, V\rangle$ where $\mathcal{F} = \{*_2\}$ (recall Definition 11). We would then have $I(w, \sim_f A) = 1$ iff $I(f(w), A) = 1$ for all $f \in \mathcal{F}$. Therefore, if van Benthem's approach is seen as an approach to modality, then $\sim_f$ will be also counted as a modality at least in that sense. Hence, the language $\mathcal{L}_{\sim_f}$ can be interpreted as an **FDE**-based modal language, where **FDE** is captured in terms of the star semantics (recall Definition 1), as, for example, in [7,9].[6]

7 Concluding Remarks

What we hope to have established in this paper is that it is possible to provide two-valued semantics for a logic based on **SIXTEEN**$_3$. In particular, we made essential use of Routleys' star operation for both two-valued semantics. However, our result here is just a first step, and there seem to be a number of problems to be explored in more details. We will mention two of them.

The first problem is related to the language. In this paper, we focused on the most simple language associated to **SIXTEEN**$_3$, namely $\mathcal{L}_{\sim_f}$. However, this is only one of the many possible choices. In particular, it seems more than natural to deal with $\wedge_f$ and $\vee_f$, but these connectives seem to be resistant. For example, if we consider the truth condition for $\wedge_f$ in a two-star interpretation, then a straightforward application of our method suggests to split truth condition depending on the number of stars applied at the state. We do not know, at the time of writing, if we can capture $\wedge_f$ in a two-star interpretation by a single truth condition. We should also note that some connectives discussed in the literature can be captured. For example, $\neg$ and $\sim_i$, in a two-star interpretation, will have the following truth conditions respectively:

[5] We thank Sergei Odintsov for pointing this out.

[6] For a different approach to **FDE**-based modal logic, where **FDE** is captured in terms of the Dunn semantics (recall Definition 3), see, for example [13,15]. Comparing the two approaches will be future work.

- $I(w, \neg A) = 1$ iff $I(w, A) \neq 1$
- $I(w, {\sim_i} A) = 1$ iff $I(w^{*_1 *_2}, A) = 1$

The second problem is to explore the relation between the two-valued semantics and the trilattice. Note that in our two-valued semantics, we are making essential use of the star operation, but this seems to give rise to some difficulties. Here is a reason: In the context of **FDE**, informational join and meet of the bilattice naturally inspire to introduce binary connectives, and these connectives can be captured easily in terms of Dunn semantics by giving truth and falsity conditions. However, it is far from obvious if we can capture the same connectives based on the star semantics by equally simple conditions. And a similar issue may carry over to the case with $\mathbf{SIXTEEN}_3$. In fact, this might also be related to the first problem related to $\wedge_f$ and $\vee_f$.

A Details of the Proof of Theorem 3

We prove the contrapositive. Assume $A \not\vdash B$. Then, by Lindenbaum's lemma, there is a prime theory Γ such that $A \in \Gamma$ and $B \notin \Gamma$. We then define a two-star interpretation $\langle W, g, *_1, *_2, v\rangle$ as follows:

- $W = \{a, b, c, d\}$, $g = a$;
- $a^{*_1} = b$, $b^{*_1} = a$, $c^{*_1} = d$, $d^{*_1} = c$, $a^{*_2} = c$, $b^{*_2} = d$, $c^{*_2} = a$, $d^{*_2} = b$;
- $v : W \times \mathsf{Prop} \to \{0, 1\}$ is defined as follows:

$$\begin{array}{ll} v(a,p)=1 \text{ iff } p \in \Gamma; & v(c,p)=1 \text{ iff } p \in \Gamma^*; \\ v(b,p)=1 \text{ iff } {\sim} p \notin \Gamma; & v(d,p)=1 \text{ iff } {\sim} p \notin \Gamma^*. \end{array}$$

If we can show that the above condition holds for all formulas, then the result follows since at $a \in W$, $I(a, A) = 1$ but $I(a, B) \neq 1$, i.e. $A \not\models_* B$. We prove this by induction on the complexity of A. We only prove the cases for $\sim$ and $\sim_f$, since the cases for $\wedge$ and $\vee$ are straightforward.

Case 1. If A is an element of Prop, the result holds by definition.

Case 2. If $A = {\sim} B$, then

$$\begin{array}{lll} v(a,{\sim}B){=}1 & \text{iff } v(a^{*_1},B){\neq}1 & \\ & \text{iff } v(b,B){\neq}1 & \text{Def. } *_1 \\ & \text{iff } {\sim}B \in \Gamma & \text{IH} \end{array} \qquad \begin{array}{lll} v(c,{\sim}B){=}1 & \text{iff } v(c^{*_1},B){\neq}1 & \\ & \text{iff } v(d,B){\neq}1 & \text{Def. } *_1 \\ & \text{iff } {\sim}B \in \Gamma^* & \text{IH} \end{array}$$

$$\begin{array}{lll} v(b,{\sim}B){=}1 & \text{iff } v(b^{*_1},B){\neq}1 & \\ & \text{iff } v(a,B){\neq}1 & \text{Def. } *_1 \\ & \text{iff } B \notin \Gamma & \text{IH} \\ & \text{iff } {\sim}{\sim}B \notin \Gamma & (\mathrm{a}_t6), (\mathrm{a}_t7) \end{array} \qquad \begin{array}{lll} v(d,{\sim}B){=}1 & \text{iff } v(d^{*_1},B){\neq}1 & \\ & \text{iff } v(c,B){\neq}1 & \text{Def. } *_1 \\ & \text{iff } B \notin \Gamma^* & \text{IH} \\ & \text{iff } {\sim}{\sim}B \notin \Gamma^* & (\mathrm{a}_t6), (\mathrm{a}_t7) \end{array}$$

Case 3. If $A = {\sim_f} B$, then

$$\begin{array}{llllll}
v(a, {\sim_f} B)=1 & \text{iff } v(a^{*_2}, B)=1 & & v(c, {\sim_f} B)=1 & \text{iff } v(c^{*_2}, B)=1 & \\
& \text{iff } v(c, B)=1 & \text{Def. } *_2 & & \text{iff } v(a, B)=1 & \text{Def. } *_2 \\
& \text{iff } B \in \Gamma^* & \text{IH} & & \text{iff } B \in \Gamma & \text{IH} \\
& \text{iff } {\sim_f}{\sim_f} B \in \Gamma^* & (\mathrm{a}_t8), (\mathrm{a}_t9) & & \text{iff } {\sim_f} B \in \Gamma^* & \text{Lem. 3} \\
& \text{iff } {\sim_f} B \in \Gamma & \text{Lem. 3} & & &
\end{array}$$

$$\begin{array}{llllll}
v(b, {\sim_f} B)=1 & \text{iff } v(b^{*_2}, B)=1 & & v(d, {\sim_f} B)=1 & \text{iff } v(d^{*_2}, B)=1 & \\
& \text{iff } v(d, B)=1 & \text{Def. } *_2 & & \text{iff } v(b, B)=1 & \text{Def. } *_2 \\
& \text{iff } {\sim} B \notin \Gamma^* & \text{IH} & & \text{iff } {\sim} B \notin \Gamma & \text{IH} \\
& \text{iff } {\sim_f}{\sim} B \notin \Gamma & \text{Lem. 3} & & \text{iff } {\sim_f}{\sim} B \notin \Gamma^* & \text{Lem. 3} \\
& \text{iff } {\sim}{\sim_f} B \notin \Gamma & (\mathrm{a}_t10) & & \text{iff } {\sim}{\sim_f} B \notin \Gamma^* & (\mathrm{a}_t10)
\end{array}$$

This completes the proof. □

B Details of the Proof of Proposition 4

We prove the contrapositive. Assume $A \not\models_r B$. Then, there is a one-star interpretation $\langle W, g, *, r\rangle$ such that Ar_g1, but not Br_g1. We then define a two-star interpretation $\langle W, g{*_1}, *_2, v\rangle$ as follows:

- $W = \{a, b, c, d\}$, $g = a$;
- $a^{*_1} = b$, $b^{*_1} = a$, $c^{*_1} = d$, $d^{*_1} = c$, $a^{*_2} = c$, $b^{*_2} = d$, $c^{*_2} = a$, $d^{*_2} = b$;
- $v : W \times \mathsf{Prop} \to \{0, 1\}$ is defined as follows:

$$\begin{array}{rr}
v(a, p) = 1 \text{ iff } pr_g1; & v(c, p) = 1 \text{ iff } pr_{g^*}1; \\
v(b, p) = 1 \text{ iff not } pr_{g^*}0 & v(d, p) = 1 \text{ iff not } pr_g0.
\end{array}$$

If we can show that the above condition holds for all formulas, then the result follows since at $a \in W$, $v(a, A) = 1$ but $v(a, B) \neq 1$, i.e. $A \not\models_{*,g} B$. We prove this by induction. We only prove the cases for $\sim$ and $\sim_f$, since the cases for $\wedge$ and $\vee$ are straightforward.

Case 1. If A is an element of Prop, the result holds by definition.

Case 2. If $A = {\sim} B$, then

$$\begin{array}{llllll}
v(a, {\sim} B) = 1 & \text{iff } v(a^{*_1}, B) \neq 1 & & v(c, {\sim} B) = 1 & \text{iff } v(c^{*_1}, B) \neq 1 & \\
& \text{iff } v(b, B) \neq 1 & \text{Def. } *_1 & & \text{iff } v(d, B) \neq 1 & \text{Def. } *_1 \\
& \text{iff } Br_{g^*}0 & \text{IH} & & \text{iff } Br_g0 & \text{IH} \\
& \text{iff } {\sim} Br_g1 & & & \text{iff } {\sim} Br_{g^*}1 &
\end{array}$$

$$\begin{array}{llllll}
v(b, {\sim} B) = 1 & \text{iff } v(b^{*_1}, B) \neq 1 & & v(d, {\sim} B) = 1 & \text{iff } v(d^{*_1}, B) \neq 1 & \\
& \text{iff } v(a, B) \neq 1 & \text{Def. } *_1 & & \text{iff } v(c, B) \neq 1 & \text{Def. } *_1 \\
& \text{iff not } Br_g1 & \text{IH} & & \text{iff not } Br_{g^*}1 & \text{IH} \\
& \text{iff not } {\sim} Br_{g^*}0 & & & \text{iff not } {\sim} Br_g0 &
\end{array}$$

Case 3. If $A = {\sim_f} B$, then

$$\begin{array}{llll} v(a, {\sim_f} B) = 1 & \text{iff } v(a^{*_2}, B) = 1 & & \\ & \text{iff } v(c, B) = 1 & \text{Def. } *_2 \\ & \text{iff } Br_{g^*}1 & \text{IH} \\ & \text{iff } {\sim_f} Br_g 1 & \end{array} \qquad \begin{array}{llll} v(c, {\sim_f} B) = 1 & \text{iff } v(c^{*_2}, B) = 1 & \\ & \text{iff } v(a, B) = 1 & \text{Def. } *_2 \\ & \text{iff } Br_g 1 & \text{IH} \\ & \text{iff } {\sim_f} Br_{g^*}1 & \text{Lem. 3} \end{array}$$

$$\begin{array}{llll} v(b, {\sim_f} B) = 1 & \text{iff } v(b^{*_2}, B) = 1 & \\ & \text{iff } v(d, B) = 1 & \text{Def. } *_2 \\ & \text{iff not } Br_g 0 & \text{IH} \\ & \text{iff not } {\sim_f} Br_{g^*}0 & \end{array} \qquad \begin{array}{llll} v(d, {\sim_f} B) = 1 & \text{iff } v(d^{*_2}, B) = 1 & \\ & \text{iff } v(b, B) = 1 & \text{Def. } *_2 \\ & \text{iff not } Br_{g^*}0 & \text{IH} \\ & \text{iff not } {\sim_f} Br_g 0 & \end{array}$$

This completes the proof. □

C Details of the Proof for Proposition 5

We prove the contrapositive. Assume $A \not\models_{*,g} B$. Then, there is a two-star interpretation $\langle W, g, *_1, *_2, v\rangle$ such that $I(g, A) = 1$ but $I(g, B) \neq 1$. We then define a one-star interpretation $\langle W, g, *, r\rangle$ as follows:

- $W = \{a, b\}$, $g = a$;
- $a^* = b$, $b^* = a$;
- $r_w \subseteq \mathsf{Prop} \times \{0, 1\}$ is defined as follows:

$$\begin{array}{ll} pr_a 1 \text{ iff } I(g, p) = 1; & pr_b 1 \text{ iff } I(g^{*_2}, p) = 1; \\ pr_a 0 \text{ iff } I(g^{*_1 *_2}, p) \neq 1; & pr_b 0 \text{ iff } I(g^{*_1}, p) \neq 1. \end{array}$$

If we can show that the above condition holds for all formulas, then the result follows since at $a \in W$, $Ar_a 1$ but not $Br_a 1$, i.e. $A \not\models_r B$. We prove this by induction. We only prove the cases for $\sim$ and ${\sim_f}$, since the cases for $\wedge$ and $\vee$ are straightforward.

Case 1. If A is an element of Prop, the result holds by definition.

Case 2. If $A = {\sim} B$, then

$$\begin{array}{lll} {\sim} Br_a 1 & \text{iff } Br_{a^*}0 & \\ & \text{iff } Br_b 0 & \text{Def. } * \\ & \text{iff } I(g^{*_1}, B) \neq 1 & \text{IH} \\ & \text{iff } I(g, {\sim} B) = 1 & \end{array} \qquad \begin{array}{lll} {\sim} Br_b 1 & \text{iff } Br_{b^*}0 & \\ & \text{iff } Br_a 0 & \text{Def. } * \\ & \text{iff } I(g^{*_1 *_2}, B) \neq 1 & \text{IH} \\ & \text{iff } I(g^{*_2}, {\sim} B) = 1 & \end{array}$$

$$\begin{array}{lll} {\sim} Br_a 0 & \text{iff } Br_{a^*}1 & \\ & \text{iff } Br_b 1 & \text{Def. } * \\ & \text{iff } I(g^{*_2}, B) = 1 & \text{IH} \\ & \text{iff } I(g^{*_2 *_1}, {\sim} B) \neq 1 & \end{array} \qquad \begin{array}{lll} {\sim} Br_b 0 & \text{iff } Br_{b^*}1 & \\ & \text{iff } Br_a 1 & \text{Def. } * \\ & \text{iff } I(g, B) = 1 & \text{IH} \\ & \text{iff } I(g^{*_1}, {\sim} B) \neq 1 & \end{array}$$

Case 3. If $A = {\sim_f} B$, then

$$\begin{array}{llll@{\qquad}llll}
{\sim_f}Br_a1 & \text{iff } Br_{a^*}1 & & & {\sim_f}Br_b1 & \text{iff } Br_{b^*}1 & \\
 & \text{iff } Br_b1 & \text{Def. } * & & & \text{iff } Br_a1 & \text{Def. } * \\
 & \text{iff } I(g^{*_2}, B) = 1 & \text{IH} & & & \text{iff } I(g, B) = 1 & \text{IH} \\
 & \text{iff } I(g, {\sim_f}B) = 1 & & & & \text{iff } I(g^{*_2}, {\sim_f}B) = 1 &
\end{array}$$

$$\begin{array}{llll@{\qquad}llll}
{\sim_f}Br_a0 & \text{iff } Br_{a^*}0 & & & {\sim_f}Br_b0 & \text{iff } Br_{b^*}0 & \\
 & \text{iff } Br_b0 & \text{Def. } * & & & \text{iff } Br_a0 & \text{Def. } * \\
 & \text{iff } I(g^{*_1}, B) \neq 1 & \text{IH} & & & \text{iff } I(g^{*_1 *_2}, B) \neq 1 & \text{IH} \\
 & \text{iff } I(g^{*_1 *_2}, {\sim_f}B) \neq 1 & & & & \text{iff } I(g^{*_1}, {\sim_f}B) \neq 1 &
\end{array}$$

This completes the proof. □

References

1. Arieli, O., Avron, A.: Reasoning with logical bilattices. J. Log. Lang. Inf. **5**(1), 25–63 (1996)
2. Belnap, N.: How a computer should think. In: Ryle, G. (ed.) Contemporary Aspects of Philosophy, pp. 30–55. Oriel Press (1976)
3. Belnap, N.: A useful four-valued logic. In: Dunn, J., Epstein, G. (eds.) Modern Uses of Multiple-Valued Logic, pp. 8–37. D. Reidel Publishing Co. (1977)
4. van Benthem, J.: Beyond accessibility. In: de Rijke, M. (ed.) Diamonds and Defaults: Studies in Pure and Applied Intensional Logic. SYLI, vol. 229, pp. 1–18. Springer, Dordrecht (1993). https://doi.org/10.1007/978-94-015-8242-1_1
5. De, M., Omori, H.: Classical negation and expansions of Belnap-Dunn logic. Stud. Log. **103**(4), 825–851 (2015)
6. Dunn, J.M.: Intuitive semantics for first-degree entailment and 'coupled trees'. Philos. Stud. **29**, 149–168 (1976)
7. Fuhrmann, A.: Models for relevant modal logics. Stud. Log. **49**(4), 501–514 (1990)
8. Ginsberg, M.: Multi-valued logics: a uniform approach to AI. Comput. Intell. **4**, 243–247 (1988)
9. Mares, E.D., Meyer, R.K.: The semantics of R4. J. Philos. Log. **22**(1), 95–110 (1993)
10. Muskens, R., Wintein, S.: Analytic tableaux for all of SIXTEEN$_3$. J. Philos. Log. **44**(5), 473–487 (2015)
11. Muskens, R., Wintein, S.: Interpolation in 16-valued trilattice logics. Stud. Log. **106**(2), 345–370 (2018)
12. Odintsov, S.P.: On axiomatizing Shramko-Wansing's logic. Stud. Log. **91**(3), 407–428 (2009)
13. Odintsov, S.P., Wansing, H.: Modal logics with Belnapian truth values. J. Appl. Non-Class. Log. **20**, 279–301 (2010)
14. Odintsov, S.P., Wansing, H.: The logic of generalized truth values and the logic of bilattices. Stud. Log. **103**(1), 91–112 (2015)
15. Odintsov, S.P., Wansing, H.: Disentangling FDE-based paraconsistent modal logics. Stud. Log. **105**(6), 1221–1254 (2017)
16. Omori, H., Sano, K.: Generalizing functional completeness in Belnap-Dunn logic. Stud. Log. **103**(5), 883–917 (2015)
17. Omori, H., Wansing, H.: 40 years of FDE: an introductory overview. Stud. Log. **105**(6), 1021–1049 (2017)

18. Priest, G.: An Introduction to Non-Classical Logic: From If to Is, 2nd edn. Cambridge University Press, Cambridge (2008)
19. Restall, G.: Negation in relevant logics (how i stopped worrying and learned to love the routley star). In: Gabbay, D.M., Wansing, H. (eds.) What is Negation?, pp. 53–76. Kluwer Academic Publishers (1999)
20. Routley, R., Routley, V.: Semantics for first degree entailment. Noûs **6**, 335–359 (1972)
21. Shramko, Y., Wansing, H.: Truth and Falsehood - An Inquiry into Generalized Logical Values, 1st edn. Springer, Dordrecht (2012). https://doi.org/10.1007/978-94-007-0907-2
22. Shramko, Y., Wansing, H.: Some useful 16-valued logics: how a computer network should think. J. Philos. Log. **34**(2), 121–153 (2005)
23. Shramko, Y., Wansing, H.: Hyper-contradictions, generalized truth values and logics of truth and falsehood. J. Log. Lang. Inf. **15**(4), 403–424 (2006)
24. Suszko, R.: Remarks on Łukasiewicz's three-valued logic. Bull. Sect. Log. **4**, 87–90 (1975)
25. Wansing, H.: The power of Belnap: sequent systems for $\mathrm{SIXTEEN}_3$. J. Philos. Log. **39**, 369–393 (2010)
26. Zaitsev, D.: A few more useful 8-valued logics for reasoning with tetralattice EIGHT_4. Stud. Log. **92**(2), 265–280 (2009)

An Algorithmic Approach to the Existence of Ideal Objects in Commutative Algebra

Thomas Powell[1(✉)], Peter Schuster[2], and Franziskus Wiesnet[3]

[1] Technische Universität Darmstadt, Darmstadt, Germany
powell@mathematik.tu-darmstadt.de
[2] University of Verona, Verona, Italy
[3] University of Trento, Trento, Italy

Abstract. The existence of ideal objects, such as maximal ideals in nonzero rings, plays a crucial role in commutative algebra. These are typically justified using Zorn's lemma, and thus pose a challenge from a computational point of view. Giving a constructive meaning to ideal objects is a problem which dates back to Hilbert's program, and today is still a central theme in the area of dynamical algebra, which focuses on the elimination of ideal objects via syntactic methods. In this paper, we take an alternative approach based on Kreisel's no counterexample interpretation and sequential algorithms. We first give a computational interpretation to an abstract maximality principle in the countable setting via an intuitive, state based algorithm. We then carry out a concrete case study, in which we give an algorithmic account of the result that in any commutative ring, the intersection of all prime ideals is contained in its nilradical.

Keywords: Proof theory · Program extraction · Commutative algebra · No-counterexample interpretation

1 Introduction

This paper is an application of proof theory in commutative algebra. To be more precise, we use proof theoretic methods to give a computational interpretation to a general maximality principle (Theorem 1), which in particular implies the existence of maximal ideals in commutative rings (Krull's lemma). In the context of second order arithmetic, the latter statement is equivalent to arithmetical comprehension [41, Chapter III.5], and thus Theorem 1 is a genuinely strong principle, and highly non-trivial from a computational perspective.

The first, second and third author were supported by the German Science Foundation (DFG Project KO 1737/6-1); by the John Templeton Foundation (ID 60842) and by a Marie Skłodowska-Curie fellowship of the Istituto Nazionale di Alta Matematica, respectively. The opinions expressed in this paper are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 533–549, 2019.
https://doi.org/10.1007/978-3-662-59533-6_32

The extraction of programs from proofs has a long and rich history, dating back to Kreisel's pioneering work on the 'unwinding' of proofs [17,18]. In the ensuing decades, the application of proof interpretations in particular has become a major topic in proof theory, and today encompasses both *proof mining* [12–14], which focuses on obtaining quantitative information primarily from proofs in areas of mathematical analysis, and the mechanized synthesis of programs from proofs, which has found many concrete applications in discrete mathematics and computer science [3,4,40].

Though as far back as the 1950s Kreisel already discusses the use of proof theoretic techniques to extract quantitative information from proofs in abstract algebra [19], specifically Hilbert's 17th problem together with his *Nullstellensatz*, to date there are comparatively few formal applications of proof interpretations in algebra, the computational analysis of which is done largely on a case by case basis. This typically involves replacing *semantic* conservation theorems with appropriate *syntactic* counterparts both sufficient for proofs of elementary statements and provable by elementary means. This method has proved possible in numerous different settings [5,6,21,22,34,44], and in the context of *commutative* algebra the so-called dynamical method is especially dominant [7,20,45,46].[1] In dynamical algebra one deals with a supposed ideal object (such as a maximal ideal) only by means of concrete, finitary approximations (such as finitely generated ideals, or rather the finite sets of generators), where the latter provide partial but sufficiently complete information about the former.

Interestingly, the idea of replacing ideal objects with suitable finitary approximations is already implicit in Kreisel's unwinding program, and is captured by his famous no-counterexample interpretation (n.c.i.). The n.c.i. plays an important role in proof mining, where in particular it corresponds to the notion of *metastability* [11,15,16], which has been made popular by Tao [43] and more recently has featured in higher order computability theory [35].

In this article, we take a new approach to eliminating ideal objects in abstract algebra, by solving an appropriate metastable reformulation of our general maximality principle. We then use this solution to extract *direct witnesses* from a variant of Krull's lemma.

The novelty of our approach lies not just in our use of the n.c.i., but in our description of its solution as a state based algorithm, inspired by recent work of the first author [23,24,26–28] which focuses on the algorithmic meaning of extracted programs. This form of presentation allows us to bridge the gap between the *rigorous* extraction of programs from proofs as terms in some formal calculus, and the more algorithmic style of dynamical algebra.

It also enables us to present our results in an entirely self-contained manner, without needing to introduce any heavy proof theoretic machinery. Though behind the scenes at least, aspects of our work are influenced by Gödel's functional interpretation [8] and Spector's bar recursion [42], neither of these make

[1] The second author has contributed to a universal conservation criterion [31–33] that includes many of the those cases [30,36,39].

an official appearance, and we have endeavoured to make everything as accessible to the non-specialist as possible.

Our first main contribution, given as Theorems 3 and 4, is a time sequential algorithm (in the sense of Gurevich [9]), whose states evolve step by step until they terminate in some final state s_j which represents a solution to the n.c.i. of Theorem 1. Each step in this process represents an *improvement* to our construction of an approximate ideal object, and so can also be viewed as a learning procedure in the style of [1].

We then present a concrete application of our abstract result, in which we analyse a classic maximality argument used to prove the well known fact that in any commutative ring, if some element r is contained in intersection of all prime ideals, then it must be nilpotent. We show that an instance of our sequential algorithm can be used to directly compute an exponent $e > 0$ such that $r^e = 0$, and thus our case study is another illustration of how the proof theoretic analysis of a highly nonconstructive proof can yield direct, computational information. We conclude by instantiating our algorithm in case of nonconstant coefficients of invertible polynomials. This is a well known example which has been widely studied from a computational perspective [25,29,37,38], thus facilitating a future analysis of our work with other approaches.

2 A General Maximality Argument

We begin by presenting our abstract maximality principle, which forms the main subject of the paper. Let X be some set (which for now is arbitrary but later will be countable), and denote by $\mathcal{P}_{fin}(X)$ the set of all finite subsets of X. Simple lemmas are stated without proof.

Definition 1. *Let $\rhd$ be some subset of $\mathcal{P}_{fin}(X) \times X$. We treat $\rhd$ as a binary relation and say that the element x is generated by the finite set A whenever $A \rhd x$. We extend $\rhd$ to arbitrary (not necessarily finite) $S \subseteq X$ by defining $S \rhd^* x$ whenever there exists some finite $A \subseteq S$ such that $A \rhd x$.*

Definition 2. *Given some $S \subseteq X$, define the sequence $(S_i)_{i \in \mathbb{N}}$ of sets by*

$$S_0 := S \quad \text{and} \quad S_{i+1} := \{x \mid \bigcup_{j \leq i} S_j \rhd^* x\}$$

and let $\langle S \rangle := \bigcup_{i \in \mathbb{N}} S_i$. We say that $\langle S \rangle$ is the closure of S w.r.t. $\rhd$, since whenever $\langle S \rangle \rhd^* x$ then $x \in \langle S \rangle$.

Definition 3. *For any $S \subseteq X$ and $x \in X$, $S \oplus x := \langle S \cup \{x\} \rangle$ denotes the closed extension of S with x.*

Lemma 1. *Suppose that $S \rhd^* x$. Then $S \oplus x = \langle S \rangle$.*

Definition 4. *Let $Q(x)$ be some predicate on X. For $S \subseteq X$ write $Q(S)$ for $(\forall x \in S) Q(x)$. Note in particular that $Q(S)$ and $S \supseteq T$ implies $Q(T)$.*

Definition 5. *We say that $M \subseteq X$ is maximal w.r.t. $\rhd$ and Q if*

(i) M *is closed w.r.t.* $\rhd^*$,
(ii) $Q(M)$,
(iii) $\neg Q(M \oplus x)$ *for any* $x \notin M$.

Theorem 1. *Suppose that $Q(\langle\emptyset\rangle)$. Then there exists some $M \subseteq X$ which is maximal w.r.t. $\rhd$ and Q.*

Proof. Define $\mathcal{S} := \{S \subseteq X \mid S \text{ is closed w.r.t } \rhd^* \text{ and } Q(S)\}$. We show that $\mathcal{S}$ is nonempty and chain complete w.r.t. set inclusion. Nonemptyness follows from the fact that $\langle\emptyset\rangle \in \mathcal{S}$, so it remains to prove chain completeness. Let γ be a chain in $\mathcal{S}$. Then $\hat{S} := \bigcup_{S\in\gamma} S$ is clearly closed, and moreover, if $x \in \hat{S}$ then $x \in S$ for some $S \in \gamma$, and therefore $Q(x)$. This establishes $\hat{S} \in \mathcal{S}$.

Thus by Zorn's lemma, $\mathcal{S}$ has some maximal element M, which by definition satisfies (i) and (ii). But for $x \notin M$ we have $M \subset M \oplus x$ and thus $M \oplus x \notin \mathcal{S}$. But since $M \oplus x$ is closed, it follows that $\neg Q(M \oplus x)$.

Corollary 1. *Any commutative ring X with $0 \neq 1$ has a maximal ideal.*

Proof. We follow the standard proof. Define $\rhd$ by $A \rhd x$ iff $x = x_1 \cdot a_1 + \ldots + x_k \cdot a_k$ for some $a_1, \ldots, a_k \in A$ and $x_1, \ldots, x_k \in X$. Note that $\emptyset \rhd 0$ by the convention that an empty sum is equal to zero. In addition, define $Q(x) :\equiv (x \neq 1)$. Then $S \subseteq X$ is closed iff it is an ideal, with $Q(S)$ iff S is proper. Now $\langle\emptyset\rangle = \{0\}$ (since $\emptyset \rhd 0$) and if $0 \neq 1$ then $Q(\{0\})$, thus by Theorem 1 there exists some maximal structure M. To see that M is a maximal ideal, if there were some $M \subset I \subseteq X$ then we would have $M \subset M \oplus x \subseteq I$ for some $x \notin M$, and by $\neg Q(M \oplus x)$ we would have $1 \in M \oplus x$ and thus $I = X$.

3 A Logical Analysis of Theorem 1

From now on, we assume that X is countable and comes equipped with some explicit enumeration $\{x_n \mid n \in \mathbb{N}\}$. Given some $S \subseteq X$, the initial segment of S of length n is defined by $[S](n) := S \cap \{x_m \mid m < n\}$. Note that $S = \bigcup_{n\in\mathbb{N}} [S](n)$. We define $\mathrm{dom}(S) \subseteq \mathbb{N}$ by $\mathrm{dom}(S) := \{n \in \mathbb{N} \mid x_n \in S\}$.

Theorem 2. *Suppose that $M \subseteq X$ satisfies*

$$x_n \in M \Leftrightarrow Q([M](n) \oplus x_n) \tag{1}$$

for all $n \in \mathbb{N}$. If $Q(\langle\emptyset\rangle)$ then M is maximal w.r.t. $\rhd$ and Q.

Proof. Let $M_n := \langle [M](n) \rangle$. We first observe that $Q(M_n)$ for all $n \in \mathbb{N}$, which follows by induction: For $n = 0$ we have $M_0 = \langle\emptyset\rangle$ and so $Q(M_0)$ is true by assumption. Now supposing that $Q(M_n)$ holds for some $n \in \mathbb{N}$ there are two possibilities: If $Q([M](n) \oplus x_n)$ then $x_n \in M$ and hence $M_{n+1} = \langle [M](n) \cup \{x_n\} \rangle = [M](n) \oplus x_n$, and if $\neg Q([M](n) \oplus x_n)$ then $x_n \notin M$ and hence $M_{n+1} = \langle [M](n) \rangle = M_n$. Either way we have $Q(M_{n+1})$.

We now establish each of the maximality conditions in turn. For closure, suppose that $M \rhd^* x_n$ but $x_n \notin M$, and so by definition $\neg Q([M](n) \oplus x_n)$. Since $M \rhd^* x_n$ we have $[M](k) \rhd^* x_n$ for some $k \in \mathbb{N}$. First, let $k \leq n$. Then $[M](k) \subseteq [M](n)$ and thus $[M](n) \rhd^* x_n$, which implies that $x_n \in M_n$ and thus by Lemma 1

$$[M](n) \oplus x_n = \langle [M](n) \rangle = M_n.$$

Since $Q(M_n)$ this contradicts $\neg Q([M](n) \oplus x_n)$. But if $n < k$ then $[M](n) \oplus x_n \subseteq [M](k) \oplus x_n$ and thus $\neg Q([M](n) \oplus x_n)$ implies $\neg Q([M](k) \oplus x_n)$. But $[M](k) \rhd^* x_n$ and thus by Lemma 1 again, $[M](k) \oplus x_n = M_k$, contradicting $Q(M_k)$.

That $Q(M)$ holds is straightforward: For if $x_n \in M$ then $x_n \in [M](n+1) \subseteq M_{n+1}$ and thus $Q(x_n)$ follows from $Q(M_{n+1})$. Finally, to show that $\neg Q(M \oplus x_n)$ for $x_n \notin M$, note that $x_n \notin M$ implies $\neg Q([M](n) \oplus x_n)$, and since $[M](n) \oplus x_n \subseteq M \oplus x_n$ the result follows.

The purpose of the above theorem was to give a more syntactic formulation of Theorem 1 in the countable setting: If $Q(\langle \emptyset \rangle)$ then the existence of a some maximal $M \subseteq X$ is implied by the existence of some M satisfying (1). In order to proceed, we will now take a closer look at the structure of (1) and make some restrictions on the logical complexity of certain parameters.

Lemma 2. *Suppose that the relation $A \rhd x$ can be encoded as a Σ^0_1-formula. Then the membership relation $x \in \langle A \rangle$ can also be encoded as a Σ^0_1-formula.*

Remark 1. The reader may assume that we are working in some reasonable base theory, and that formulas can be expressed in the language of Peano arithmetic: Thus a Σ^0_1-formula is a formula of the form $(\exists y)P(y)$ where $P(y)$ is primitive recursive.

Proof. We only sketch the proof, since explicit encodings will be given in the case studies that follow. We have $x \in \langle A \rangle$ iff there exists some finite derivation tree for x whose leaves are elements of A and whose nodes represent instances of $\rhd$. Given that $\rhd$ can be encoded as a Σ^0_1-formula, it is clear that the existence of a derivation trees can in turn be represented as Σ^0_1-formula via a suitable encoding.

Lemma 3. *Suppose that $Q(x)$ is a Π^0_1-formula and that $A \rhd x$ can be encoded as a Σ^0_1-formula. Then $Q(\langle A \rangle)$ is a Π^0_1-formula i.e. $Q(\langle A \rangle) \Leftrightarrow (\forall p)R_A(p)$ for some decidable predicate $R_A(p)$ on $\mathcal{P}_{fin}(A) \times \mathbb{N}$.*

Proof. We can write $Q(x) \Leftrightarrow (\forall e)Q_0(x, e)$ for some decidable $Q_0(x, e)$, and by Lemma 2, $x \in \langle A \rangle \Leftrightarrow (\exists t)G_A(x, t)$ for some decidable $G_A(x, t)$. Then

$$\begin{aligned} Q(\langle A \rangle) &\Leftrightarrow (\forall m)(x_m \in \langle A \rangle \Rightarrow Q(x_m)) \\ &\Leftrightarrow (\forall m)((\exists t)G_A(x_m, t) \Rightarrow (\forall e)Q_0(x_m, e)) \\ &\Leftrightarrow (\forall m, t, e)(G_A(x_m, t) \Rightarrow Q_0(x_m, e)) \end{aligned}$$

and the latter formula can be encoded as $(\forall p)R_A(p)$ for suitable $R_A(p)$ and using some pairing function for the tuple m, t, e.

Lemma 4. *Under the conditions of Lemma 3, (1) holds iff for all $n \in \mathbb{N}$:*

$$x_n \in M \Leftrightarrow (\forall p) R_{[M](n)\cup\{x_n\}}(p) \tag{2}$$

Proof. By Lemma 3 setting $A = [M](n) \cup \{x_n\}$, so that $\langle A \rangle = [M](n) \oplus x_n$.

Written out in full, the existence of some M satisfying (2) becomes

$$(\exists M)(\forall n)((x_n \in M \Rightarrow (\forall p) R_{[M](n)\cup\{x_n\}}(p)) \wedge (x_n \notin M \Rightarrow (\exists q) R_{[M](n)\cup\{x_n\}}(q)))$$

and so written out in Skolem normal form, this becomes

$$(\exists M, f)(\forall n, p)(x_n \in M \Rightarrow R_{[M](n)\cup\{x_n\}}(p) \wedge x_n \notin M \Rightarrow R_{[M](n)\cup\{x_n\}}(f(n))). \tag{3}$$

This motivates our final version of maximality, which is now in a form where we can directly apply the no-counterexample interpretation.

Definition 6. *An explicit maximal object w.r.t. $\rhd$ and Q is a set $M \subseteq X$ together with a function $f : \mathrm{dom}(X \backslash M) \to \mathbb{N}$ such that*

- $x_n \in M \Rightarrow R_{[M](n)\cup\{x_n\}}(p)$
- $x_n \notin M \Rightarrow \neg R_{[M](n)\cup\{x_n\}}(f(n))$

for all $n, p \in \mathbb{N}$.

The idea here is that the function f provides concrete evidence for why x_n is excluded from the maximal structure M: in other words, it encodes an element x_m together with some tree t and e such that $x_m \in [M](n) \oplus x_n$ with respect to t but $Q(x_m)$ fails relative to e.

4 An Approximating Algorithm for Maximal Objects

In general, it is impossible to effectively compute a set M together with an f satisfying Definition 6. However, we demonstrate how an *approximate*, or *metastable*, formulation of maximality in the spirit of Kreisel's no-counterexample interpretation, can be directly witnessed via an intuitive stateful procedure.

For a detailed and modern account of the n.c.i., the reader is encouraged to consult e.g. [10,13]. The rough idea is the following: Given some prenex formula of the form $A :\equiv (\exists x \in X)(\forall y \in Y) P_0(x, y)$, a functional $\Phi : (X \to Y) \to X$ is said to witness the n.c.i. of A if it witnesses $(\forall \omega : X \to Y)(\exists x) P_0(x, \omega(x))$ i.e. $(\forall \omega) P_0(\Phi\omega, \omega(\Phi\omega))$. This definition generalises in the obvious way to prenex formulas of arbitrary complexity. In this section, we give an algorithmic description of such an Φ for A being the statement that an explicit maximal object exists, as in Definition 6.

Definition 7. *Let (ω, ϕ) be functionals which take as input M and f and each return as output a natural number. An* approximate *explicit maximal object w.r.t $\rhd$, Q and (ω, ϕ) is a set $M \subseteq X$ together with a function f such that*

– $x_n \in M \Rightarrow R_{[M](n)\cup\{x_n\}}(p)$
– $x_n \notin M \Rightarrow \neg R_{[M](n)\cup\{x_n\}}(f(n))$

but now only for $n \leq \omega(M, f)$ *and* $p = \phi(M, f)$*.*

Note that Definition 7 is slightly stronger than the n.c.i. of (3), since it works for all $n \leq \omega(M, f)$ and not just $n = \omega(M, f)$.

Approximate maximal objects are useful because when a proof of a pure existential statement relies on the existence of some maximal M, we are typically able to find functionals (ω, ϕ) which calibrate exactly how this maximal object is used, and thereby construct a witness to the existential statement in terms of an approximate maximal object relative to (ω, ϕ). For a more detailed discussion of this phenomenon in the context of sequential algorithms the reader is directed to [28, Section 4.5]. We will see a concrete example in Sect. 5.

4.1 The Algorithm

We now present our algorithm, which computes approximate maximal objects given some input functionals (ω, ϕ). Our algorithm will be described as an evolving sequence of states

$$s_0 \mapsto s_1 \mapsto \cdots \mapsto s_k.$$

The basic idea is as follows: We start in some initial state s_0 which contains no information and gives rise to an 'empty' approximation. In each step of the computation we query our mathematical environment to asses whether or not our current approximation is good enough. If it is, the computation terminates in that state. If not, we use the information gained from this query to *improve* our approximation. The hope is that our algorithm always terminates on some reasonable set of inputs. In this section we describe how the states evolve, and in the next we deal with termination.

For us, states s_i are defined to be a functions of type $\mathbb{N} \to \{(*)\} + \mathbb{N}$ i.e. s_i is an array, whose nth entry $s_i(n)$ is either a natural number or some default value $(*)$. Any given state encodes a current approximation $(M[s_i], f[s_i])$ to an explicit maximal object by defining the set $M[s_i] \subseteq X$ as

$$M[s_i] := \{x_n \in \mathbb{N} \mid s_i(n) = (*)\}$$

and the function $f[s_i] : \mathrm{dom}(X \backslash M[s_i]) \to \mathbb{N}$ by

$$f[s_i](n) := s_i(n) \in \mathbb{N}$$

where $s_i(n) \in \mathbb{N}$ follows from the assumption that $n \notin M[s_i]$. Fixing some input functionals (ω, ϕ), we imagine for convenience that these now act directly on states, and write $\omega(s_i)$ as shorthand for $\omega(M[s_i], f[s_i])$.

We now need to explain how our state evolves. As an initial state, we set

$$s_0(m) := (*)$$

and so $M[s_0] = X$ and $f[s_0]$ has an empty domain. Now, supposing that we are in the ith state, we define

$$(n_i, p_i) := (\omega, \phi)(s_i).$$

and carry out the following steps:

– Search from 0 up to n_i until some $0 \leq n \leq n_i$ is found such that each of the following hold
  - $x_n \in M[s_i]$,
  - $\neg R_{[M[s_i]](n) \cup \{x_n\}}(p_i)$
– If no such n is found, the algorithm terminates in state s_i.
– Otherwise, define

$$s_{i+1}(m) := \begin{cases} s_i(m) & \text{if } m < n \\ p_i & \text{if } m = n \\ (*) & \text{if } m > n \end{cases}$$

and so in particular

$$M[s_{i+1}] = [M[s_i]](n) \cup \{x_k \in \mathbb{N} \mid k > n\}$$

and $x_n \notin M[s_{i+1}]$.

Lemma 5. *For all states $s_i \in \mathbb{N}$ and $n \in \mathbb{N}$ we have*

$$x_n \notin M[s_i] \Rightarrow \neg R_{[M[s_i]](n) \cup \{x_n\}}(f[s_i](n)).$$

Proof. Induction on i. For $i = 0$ the statement is trivially true, since $M[s_0] = X$. So suppose the statement is true for some i, and that $x_n \notin M[s_{i+1}]$. Since $M[s_{i+1}] = [M[s_i]](n') \cup \{x_k \in \mathbb{N} \mid k > n'\}$ for some $n' \leq n_i$ there are two possibilities. Either $n < n'$ and $x_n \notin M[s_i]$ and so the result follow by the induction hypothesis since $f[s_{i+1}](n) = s_{i+1}(n) = s_i(n) = f[s_i](n)$ and $[M[s_{i+1}]](n) = [M[s_i]](n)$. Or $n = n'$ and so $f[s_{i+1}(n)] = p_i$ which is defined to satisfy $\neg R_{[M[s_i]](n) \cup \{x_n\}}(p_i)$, and thus the result follows since $[M[s_{i+1}]](n) = [M[s_i]](n)$.

Theorem 3. *Suppose that the algorithm terminates in state s_j. Then s_j forms an approximate explicit maximal object w.r.t. $\rhd, Q$ and (ω, ϕ).*

Proof. If the algorithm terminates, then by definition it holds that for all $n \leq n_j = \omega(s_j)$, if $x_n \in M[s_j]$ then $R_{[M[s_j]](n) \cup \{x_n\}}(p_j)$ where $p_j = \phi(s_j)$. But if $x_n \notin M[s_j]$ then $\neg R_{[M[s_j]](n) \cup \{x_n\}}(f[s_j](n))$ by Lemma 5, and so we're done.

4.2 Termination

It remains, then, to show that our algorithm actually terminates on some reasonable set of parameters! Here, we make an additional standard assumption, namely that the functionals (ω, ϕ) are *continuous*.

Definition 8. *We say that* (ω, ϕ) *are continuous if for all states* $s : \mathbb{N} \to \{*\} + \mathbb{N}$ *(which encode* M, f*) there exists some natural number* L *such that for any other input state* s'*, if* $[s](L) = [s'](L)$ *then*

$$(\omega, \phi)(s) = (\omega, \phi)(s').$$

Note that whenever (ω, ϕ) are instantiated by computable functionals, they will automatically be continuous, so restricting ourselves to the continuous setting is entirely reasonable.

Theorem 4. *Whenever the algorithm runs on continuous parameters* (ω, ϕ)*, it terminates after a finite number of steps.*

Proof. Suppose that the algorithm does not terminate and thus results in an infinite run $\{s_i\}_{i \in \mathbb{N}}$. We first show that for each $n \in \mathbb{N}$, the value of $s_i(n)$ can only change finitely many times as $i \to \infty$. More precisely, we define a sequence $j_0 \leq j_1 \leq j_2 \leq \ldots$ satisfying

$$(\forall i \geq j_n)([s_i](n) = [s_{j_n}](n)). \tag{4}$$

The $(j_n)_{n \in \mathbb{N}}$ are defined inductively as follows: We let $j_0 := 0$, and if j_n has been defined, either there exists some $j \geq j_n$ such that $x_n \notin M[s_j]$, in which case we define $j_{n+1} = j$, or $x_n \in M[s_j]$ for all $j \geq j_n$ and we set $j_{n+1} := j_n$. To see that this construction satisfies (4) we use induction on n. The base case is trivial, so let's fix some n. By the induction hypothesis and the fact that $j_{n+1} \geq j_n$ we have $[s_i](n) = [s_{j_{n+1}}](n)$ for all $i \geq j_{n+1}$, and so we only need to check point n. Now, in the case $x_n \in M[s_i]$ for all $i \geq j_n = j_{n+1}$ we're done since this means that $s_i(n) = (*)$ for all $i \geq j_{n+1}$. In the other case, if $x_n \notin M[s_{j_{n+1}}]$ then $s_{j_{n+1}}(n) = p \in \mathbb{N}$ and observing the manner in which the states evolves at each step, the only way this can change is if x_m is removed from to s_i for some $i \geq j_{n+1}$ and $m < n$. But this contradicts the induction hypothesis.

The second part of the proof is where we make use of continuity. Define s_∞ to be the limit of the $[s_{j_n}](n)$, and let L be a point of continuity for (ω, ϕ) on this input. Define

$$j := j_N \quad \text{for} \quad N := \max\{L, \omega(s_\infty) + 1\}$$

Then in particular, since $[s_\infty](L) = [s_j](L)$ we must have

$$n_j := \omega(s_j) = \omega(s_\infty) < N.$$

But since the algorithm does not terminate, there is some $0 \leq n \leq n_j$ with $x_n \in M[s_j]$ but $x_n \notin M[s_{j+1}]$. But by definition of $j = j_N$, since $n < N$ then $x_n \in M[s_j]$ implies that $x_n \in M[s_i]$ for all $i \geq j$, a contradiction.

5 Case Study: The Nilradical as the Intersection of All Prime Ideals

We now use our algorithm to carry out a computational analysis of the following well known fact [2, Proposition 1.8], which is a frequently used form of Krull's lemma. Recall that a ring element r is nilpotent if $r^e = 0$ for some integer $e > 0$.

Theorem 5. *Let X be a countable commutative ring. Suppose that r lies in the intersection of all prime ideals of X. Then r is nilpotent.*

We first show how the standard proof follows from our general maximality principle Theorem 1. Now our countable set X comes equipped with a ring structure, which will be used to instantiate our parameters $\triangleright$ and Q.

Proof. Define $\triangleright$ as in Corollary 1, but now let $Q(x) := (\forall e)(e > 0 \Rightarrow x \neq r^e)$. Then $S \subseteq X$ is closed w.r.t $\triangleright$ and satisfies $Q(S)$ iff it is an ideal which does not contain r^e for any $e > 0$. Suppose for contradiction that r is not nilpotent, which would mean that $Q(\{0\})$ and thus $Q(\langle\emptyset\rangle)$ hold. By Theorem 1 there is some M which is maximal w.r.t. $\triangleright$ and Q, and in this case $M \oplus x = \langle M \cup \{x\}\rangle$ is just the ideal generated by M and x.

Take $x, y \notin M$. Then $\neg Q(M \oplus x)$ and hence there exists some $e_1 > 0$ such that $r^{e_1} \in M \oplus x$. Similarly, there exists some $e_2 > 0$ with $r^{e_2} \in M \oplus y$. But then $r^{e_1+e_2} \in M \oplus xy$ and thus $xy \notin M$. This would mean that M is prime, but then $Q(M)$ contradicts the assumption that $r \in M$.

Lemma 6. *For $\triangleright$ and Q defined as in the proof of Theorem 5, we have*

$$Q(\langle A\rangle) \Leftrightarrow (\forall b \in X^*, e)(\underbrace{|b| = k \wedge e > 0 \Rightarrow a_1 \cdot b_1 + \ldots + a_k \cdot b_k \neq r^e}_{R_A(b,e)})$$

where $A := \{a_1, \ldots, a_k\}$, X^* as usual denotes the set of lists over X and $|b|$ is the length of b.

Our aim will be to address the following computational challenge, given any fixed X and r,

- **Input.** Evidence that r lies in the intersection of all prime ideals
- **Output.** An exponent $e > 0$ such that $r^e = 0$

The first question is what we take to be evidence that r lies in all prime ideals. Note that this assumption is logically equivalent to the statement

$$(\forall S \subseteq X)(S \text{ is not a prime ideal} \vee r \in S),$$

so for a computational interpretation of the above it would be reasonable to ask for a procedure which takes some $S \subseteq X$ as input, and either confirms that $r \in S$ or demonstrates that S is not a prime ideal.

Let's now fix some enumeration of X, where we assume for convenience that $x_0 = 0_X$, $x_1 = 1_X$ and $x_2 = r$. This assumption is not essential, and is there merely to simplify some of the bureaucratic details which follow. From now on we assume that we have some function

$$\psi : \mathcal{P}(X) \to \{0, 1, 2\} + (\{3, 4, 5\} \times \mathbb{N}^3)$$

which for any $S \subseteq X$ satisfies

- $\psi(S) = 0 \Rightarrow 0_X \notin S$
- $\psi(S) = 1 \Rightarrow 1_X \in S$
- $\psi(S) = 2 \Rightarrow r \in S$
- $\psi(S) = (3, i, j, k) \Rightarrow (x_i + x_j = x_k) \wedge (x_i, x_j \in S) \wedge (x_k \notin S)$
- $\psi(S) = (4, i, j, k) \Rightarrow (x_i \cdot x_j = x_k) \wedge (x_i \in S) \wedge (x_k \notin S)$
- $\psi(S) = (5, i, j, k) \Rightarrow (x_i \cdot x_j = x_k) \wedge (x_i, x_j \notin S) \wedge (x_k \in S)$

The functional ψ witnesses the statement that $r \in S$ or S is not a prime ideal.

Lemma 7. *Suppose that $M \subseteq X$ and f satisfy*

$$x_n \notin M \Rightarrow \neg R_{[M](n) \cup \{x_n\}}(f_1(n), f_2(n)) \tag{5}$$

where $R_A(b, e)$ is as in Lemma 6 and if $f(n) = \langle b, e \rangle$ then $f_1(n) = b$ and $f_2(n) = e$. Whenever $\psi(M) \neq 0$ there exists some nonempty $A = \{a_1, \ldots, a_l\} \subseteq M$ together with a sequence $[b_1, \ldots, b_l]$ of elements of X and $e > 0$ such that

$$a_1 \cdot b_1 + \ldots + a_l \cdot b_l = r^e.$$

Moreover, e, A and b are computable in ψ, M and f.

Remark 2. Note that here $\langle b, e \rangle$ denotes the encoding of the pair b, e as a single natural number, so that the type of f matches that of Sect. 4.

Proof. This is a fairly routine case analysis. Since $\psi(M) \neq 0$ there are five remaining possibilities:

- $\psi(M) = 1$, i.e. $x_1 = 1_X \in M$ and so we set $e := 1$, $A := \{x_1\}$ and $b := [x_2]$ (recall that $x_2 = r$).
- $\psi(M) = 2$, i.e. $x_2 = r \in M$ and so $e := 1$, $A := \{x_2\}$ and $b := [x_1]$ work.
- $\psi(M) = (3, i, j, k)$. Since $x_k \notin M$, by (5) for $b' = f_1(k)$ we have

  $$x_{\alpha_1} \cdot b'_1 + \ldots + x_{\alpha_p} \cdot b'_p + x_k \cdot b'_{p+1} = r^{f_2(k)}$$

  for $\{x_{\alpha_1}, \ldots, x_{\alpha_p}\} = [M](k)$. But then

  $$x_{\alpha_1} \cdot b'_1 + \ldots + x_{\alpha_p} \cdot b'_p + (x_i + x_j) \cdot b'_{p+1} = r^{f_2(k)}$$

  and so $e := f_2(k)$, together with $A := \{x_{\alpha_1}, \ldots, x_{\alpha_p}, x_i, x_j\} \subseteq M$ and $b := [b'_1, \ldots, b'_p, b'_{p+1}, b'_{p+1}]$ work.
- $\psi(M) = (4, i, j, k)$. Entirely analogously, but this time we have

  $$x_{\alpha_1} \cdot b'_1 + \ldots + x_{\alpha_p} \cdot b'_p + x_i \cdot (x_j \cdot b'_{p+1}) = r^{f_2(k)}$$

  and so $e := f_2(k)$, $A := \{x_{\alpha_1}, \ldots, x_{\alpha_p}, x_i\}$ and $b := [b'_1, \ldots, b'_p, x_j \cdot b'_{p+1}]$ work.
- $\psi(M) = (5, i, j, k)$. For $b' = f_1(i)$ and $b'' = f_1(j)$ we have $x_{\alpha_1} \cdot b'_1 + \ldots + x_{\alpha_p} \cdot b'_p + x_i \cdot b'_{p+1} = r^{f_2(i)}$ and $x_{\beta_1} \cdot b''_1 + \ldots + x_{\beta_q} \cdot b''_q + x_j \cdot b''_{q+1} = r^{f_2(j)}$ where $\{x_{\alpha_1}, \ldots, x_{\alpha_p}\} = [M](i)$ and $\{x_{\beta_1}, \ldots, x_{\beta_q}\} = [M](j)$, and therefore

  $$\begin{aligned}(x_{\alpha_1} \cdot b'_1 + \ldots + x_{\alpha_p} \cdot b'_p) \cdot r^{f_2(j)} + x_i \cdot b'_{p+1} \cdot (x_{\beta_1} \cdot b''_1 + \ldots + x_{\beta_q} \cdot b''_q) \\ + x_i \cdot x_j \cdot b'_{p+1} \cdot b''_{q+1} = r^{f_2(i) + f_2(j)}\end{aligned}$$

  and so $e := f_1(i) + f_2(j)$, $A := \{x_{\alpha_1}, \ldots, x_{\alpha_p}, x_{\beta_1}, \ldots, x_{\beta_q}, x_i \cdot x_j\}$ and the corresponding b from the above equation work.

Lemma 8. *Suppose that M and f satisfy (5) as in Lemma 7 and that $\psi(M) \neq 0$. Then there exists some $n \in \mathbb{N}$, sequence b and $e > 0$ such that*

- $x_n \in M$,
- $\neg R_{[M](n)\cup\{x_n\}}(b, e)$

and moreover, n, b and e are computable in ψ, M and f.

Proof. By Lemma 7 there exist, computable in ψ, M and f, a nonempty $A = \{a_1, \ldots, a_l\} \subseteq M$ together with $b = [b_1, \ldots, b_l]$ and $e > 0$ satisfying $a_1 \cdot b_1 + \ldots + a_l \cdot b_l = r^e$. In particular, we can find some $n \in \mathbb{N}$ which is the maximal with $x_n \in A \subseteq M$, and thus $A \subseteq [M](n) \cup \{x_n\}$. But by expanding b to some sequence b' with zeroes added wherever needed, we have

$$x_{\alpha_1} \cdot b'_1 + \ldots + x_{\alpha_p} \cdot b'_p + x_n \cdot b'_{p+1} = r^e$$

where $\{x_{\alpha_1}, \ldots, x_{\alpha_p}\} = [M](n)$, and thus $\neg R_{[M](n)\cup\{x_n\}}(b', e)$ holds.

Theorem 6. *Given an input functional ψ which for any S witnesses that $r \in S$ or S is not a prime ideal, define the functionals ω, ϕ by*

$$(\omega, \phi)(M, f) := \begin{cases} n, \langle b, e\rangle & \text{if } \psi(M) \neq 0, \text{ where } n, b \text{ and } e \text{ satisfy Lemma 8} \\ 0, \langle [], 0\rangle & \text{otherwise} \end{cases}$$

Suppose that the algorithm $\{s_i\}_{i\in\mathbb{N}}$ described in Sect. 4.1 is run on (ω, ϕ), and for $R_A(b, e)$ as defined in Lemma 6. Then the algorithm terminates in some final state s_j satisfying

$$s_j(0)_2 > 0 \wedge r^{s_j(0)_2} = 0_X.$$

Proof. First of all, we note that (ω, ϕ) are computable, and so in particular must be continuous in the sense of Definition 8. Therefore the algorithm terminates in some final state s_j. By Lemma 5 we have

$$x_n \notin M[s_j] \Rightarrow \neg R_{[M[s_j]](n)\cup\{x_n\}}(f_1[s_j](n), f_2[s_j](n)). \tag{6}$$

We claim that $\psi(M[s_j]) = 0$. If this were not the case, then by Lemma 8 and the definition of (ω, ϕ) we would have $x_{n_j} \in M[s_j]$ and $\neg R_{[M[s_j]](n_j)\cup\{x_{n_j}\}}(b_j, e_j)$ for

$$(n_j, \langle b_j, e_j\rangle) = (\omega, \phi)s_j$$

and so by definition the algorithm cannot be in a final state. This proves the claim. But $\psi(M[s_j]) = 0$ implies that $x_0 = 0_X \notin M[s_j]$, and therefore by (6) we have $\neg R_{\{x_0\}}(b, e)$ where $\langle b, e\rangle = f[s_j](0) = s_j(0)$, which is just

$$|b| = 1 \wedge e > 0 \wedge x_0 \cdot b_0 = r^e.$$

But since $x_0 \cdot b_0 = 0_X \cdot b_0 = 0$ we have $r^e = 0$ i.e. $r^{s_j(0)_2} = 0_X$.

5.1 Informal Description of the Algorithm

The basic idea behind the algorithm in this section is the following.

- Each state s_i encodes some $M[s_i] \subseteq X$, where $x_n \notin M[s_i]$ only if we have found evidence that $[M[s_i]](n) \cup \{x_n\}$ generates r^e for some $e > 0$, in which case this evidence is encoded as $s_i(n) \in \mathbb{N}$.
- We start off at s_0 with the full set $M[s_0] = X$.
- At state s_i we interact with our functional ψ, which provides us with evidence that either $M[s_i]$ is not a prime ideal, or $r \in M[s_i]$.
- If this evidence takes the form of anything other than $0_X \notin S$, then we are able to use this to find some $x_n \in M$ and evidence that $[M](n) \cup \{x_n\}$ generates r^e for some $e > 0$. We exclude x_n from $M[s_i]$ but add all x_k for all $k > n$ (since now the evidence that $[M[s_i]](k) \cup \{x_k\}$ generates $r^{e'}$ could be falsified by the removal of x_n).

Eventually, using a continuity argument as in Theorem 4, the algorithm terminates in some state s_j. But the only way this can be is if $\psi(M[s_j]) = 0$, which indicates that $0_X \notin M[s_j]$. Thus $\{0_X\}$ generates r^e for some $e > 0$ encoded in the state.

5.2 Example: Nilpotent Coefficients of Invertible Polynomials

We conclude by outlining a simple and very concrete application [2, pp. 10–11] of Theorem 5, and sketching how our algorithm would be implemented in this case. Fixing our countable commutative ring X, let $f = \sum_{i=0}^{n} a_i T_i$ be a unit in the polynomial ring $X[T]$. Then each a_i for $i > 0$ is nilpotent.

To prove this, by Theorem 5 it suffices to show that $a_i \in P$ for all prime ideals P of X. Let $g \in X[T]$ be such that $fg = 1$, and let P be some arbitrary prime ideal. Then we also have $fg = 1$ in $(X/P)[T]$, but since P is prime, X/P is an integral domain, and thus $0 = \deg(fg) = \deg(f) + \deg(g)$. This implies that $\deg(f) = 0$ in $(X/P)[T]$ and thus $a_i \in P$ for all $i > 0$.

In order to obtain a concrete algorithm, which for any a_i for $i > 0$, produces some $e > 0$ such that $r^e = 0$, we need to analyse the above argument to produce a specific functional ψ which for any $S \subseteq X$, witnesses the statement that either $a_i \in S$ or S is not a prime ideal. Fixing $i > 0$ and S, we define $\psi(S)$ via the following algorithm:

- Check in turn whether any of $0 \notin S$, $1 \in S$ or $a_i \in S$ are true. In the first case, return $\psi(S) = 0$, and in the others, $\psi(S) = 1$ and $\psi(S) = 2$ respectively.
- Otherwise, let $g = \sum_{j=0}^{m} b_j T^j \in X[T]$ be such that

$$1 = fg = \sum_{k=0}^{n+m} c_k T^k$$

for $c_k = \sum_{j=0}^{k} a_j b_{k-j}$. Then in particular, for $i > 0$ we have $0 = c_i = \sum_{j=0}^{i-1} a_j b_{i-j} + a_i b_0$ and so (using that $a_0 b_0 = c_0 = 1$):

$$a_i = -a_0 \sum_{j=0}^{i-1} a_j b_{i-j}. \tag{7}$$

There are now two subcases to consider.

- If all of $b_1, \ldots, b_i \in S$, then because $a_i \notin S$, an analysis of the r.h.s. of (7) allows us to find, in a finite number of steps, either some $x_u, x_v \in S$ and $x_w \notin S$ such that $x_w = x_u + x_v$, in which case we return $\psi(S) = (3, u, v, w)$, or some $x_u \in S, x_v$ and $x_w \notin S$ such that $x_w = x_u x_v$, in which case we return $\psi(S) = (4, u, v, w)$.
- Otherwise we have $b_j \notin S$ for some $1 \leq j \leq i$. Take $1 \leq k \leq n$ and $1 \leq l \leq m$ to be the maximal such that $a_k, b_l \notin S$ (note that because $a_i \notin S$ then this maximal a_k also exists) and consider

$$0 = c_{k+l} = a_k b_l + \sum_{p+q=k+l \wedge (p>k \vee q>l)} a_p b_q.$$

  Then, splitting into two further subcases: Either $a_k b_l \in S$, in which case we return $\psi(S) = (5, u, v, w)$ for $x_u = a_k$, $x_v = b_l$ and $x_w = a_k b_l$, or

$$-\sum a_p b_q = a_k b_l \notin S$$

  and since for each summand $a_p b_q$ either $a_p \in S$ or $b_q \in S$, an analysis analogous to the previous case returns $\psi(S) = (3, u, v, w) or (4, u, v, w)$ for suitable u, v, w.

Therefore, running our algorithm for ψ as defined above results in a sequential algorithm which, by Theorem 6 terminates in some final state s_j with $f[s_j] = \langle b, e \rangle$ for $e > 0$ and $a_i^e = 0$.

Example 1. In the very simple case where $X = \mathbb{Z}_4$ and $f = a_0 + a_1 T = 1 + 2T$, the corresponding run our algorithm for $a_1 = 2$ would be as follows;

- $M[s_0] = \mathbb{Z}_4$, and since $1 \in \mathbb{Z}_4$ we are in the first main case of the definition of ψ above, and we have $\psi(\mathbb{Z}_4) = 1$. Therefore we remove 1 from $\mathbb{Z}_4$, citing $1 \cdot 2 = 2^1$ as evidence.
- $M[s_1] = \mathbb{Z}_4 \backslash \{1\}$, and since $a_1 = 2 \in \mathbb{Z}_4 \backslash \{1\}$ we are again in the first main case. Therefore we set $\psi(\mathbb{Z}_4 \backslash \{1\}) = 2$ and remove 2 with evidence $2 \cdot 1 = 2^1$.
- $M[s_2] = \mathbb{Z}_4 \backslash \{1, 2\}$. We now fall into the second main case. Picking $g = b_0 + b_1 T = 1 + 2T$ as our inverse for f, since in $\mathbb{Z}_4$:

$$(1 + 2T)(1 + 2T) = 1,$$

  we have $b_1 = 2 \notin \mathbb{Z}_4 \backslash \{1, 2\}$. This puts us in the second subcase, where we observe that $a_1 = b_1 = 2$ are the maximal coefficients with $a_1, b_1 \notin \mathbb{Z}_4 \backslash \{1, 2\}$. Then $0 = c_2 = a_1 \cdot b_1 \in \mathbb{Z}_4 \backslash \{1, 2\}$, and thus $\psi(\mathbb{Z}_4 \backslash \{1, 2\}) = (5, 2, 2, 0)$, and so we remove 0 with evidence $0 = 2^2$.

– Finally, $M[s_3] = \mathbb{Z}_4 \backslash \{0\}$ (since we now re-add both 1 and 2 to the approximation) and $\psi(\mathbb{Z}_4 \backslash \{0\}) = 0$, and since we have already stored the evidence that $0 = 2^2$, the algorithm terminates with $e = 2$.

Acknowledgements. The authors are grateful to the anonymous referees for their detailed comments, which led to a much improved version of the paper.

References

1. Aschieri, F., Berardi, S.: Interactive learning-based realizability for Heyting arithmetic with EM1. Log. Methods Comput. Sci. **6**(3), 1–22 (2010)
2. Atiyah, M., Macdonald, I.: Introduction to Commutative Algebra. Addison-Wesley Publishing Co., Boston (1969)
3. Berger, U., Lawrence, A., Forsberg, F., Seisenberger, M.: Extracting verified decision procedures: DPLL and resolution. Log. Methods Comput. Sci. **11**(1:6), 1–18 (2015)
4. Berger, U., Miyamoto, K., Schwichtenberg, H., Seisenberger, M.: Minlog - a tool for program extraction supporting algebras and coalgebras. In: Corradini, A., Klin, B., Cîrstea, C. (eds.) CALCO 2011. LNCS, vol. 6859, pp. 393–399. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-22944-2_29
5. Cederquist, J., Coquand, T.: Entailment relations and distributive lattices. In: Buss, S.R., Hájek, P., Pudlák, P. (eds.) Logic Colloquium 1998, Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic. Lect. Notes Logic, Prague, Czech Republic, 9–15 August 1998, vol. 13, pp. 127–139. A. K. Peters, Natick (2000)
6. Cederquist, J., Negri, S.: A constructive proof of the Heine-Borel covering theorem for formal reals. In: Berardi, S., Coppo, M. (eds.) TYPES 1995. LNCS, vol. 1158, pp. 62–75. Springer, Heidelberg (1996). https://doi.org/10.1007/3-540-61780-9_62
7. Coste, M., Lombardi, H., Roy, M.F.: Dynamical method in algebra: Effective Nullstellensätze. Ann. Pure Appl. Logic **111**(3), 203–256 (2001)
8. Gödel, K.: Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica **12**, 280–287 (1958)
9. Gurevich, Y.: Sequential abstract-state machines capture sequential algorithms. ACM Trans. Comput. Log. (TOCL) **1**, 77–111 (2000)
10. Kohlenbach, U.: On the no-counterexample interpretation. J. Symb. Log. **64**, 1491–1511 (1999)
11. Kohlenbach, U.: Some computational aspects of metric fixed point theory. Nonlinear Anal. **61**(5), 823–837 (2005)
12. Kohlenbach, U.: Some logical metatheorems with applications in functional analysis. Trans. Amer. Math. Soc. **357**, 89–128 (2005)
13. Kohlenbach, U.: Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Monographs in Mathematics. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-77533-1
14. Kohlenbach, U.: Proof-theoretic methods in nonlinear analysis. In: Proceedings of the ICM 2018, vol. 2, pp. 79–100. World Scientific (2019)
15. Kohlenbach, U., Koutsoukou-Argyraki, A.: Rates of convergence and metastability for abstract Cauchy problems generated by accretive operators. J. Math. Anal. Appl. **423**, 1089–1112 (2015)

16. Kohlenbach, U., Leuştean, L.: Effective metastability of Halpern iterates in CAT(0) spaces. Adv. Math. **321**, 2526–2556 (2012)
17. Kreisel, G.: On the interpretation of non-finitist proofs. Part I. J. Symb. Log. **16**, 241–267 (1951)
18. Kreisel, G.: On the interpretation of non-finitist proofs, Part II: interpretation of number theory. J. Symb. Log. **17**, 43–58 (1952)
19. Kreisel, G.: Mathematical significance of consistency proofs. J. Symb. Log. **23**(2), 155–182 (1958)
20. Lombardi, H., Quitté, C.: Commutative Algebra: Constructive Methods: Finite Projective Modules. Springer, Dordrecht (2015). https://doi.org/10.1007/978-94-017-9944-7
21. Mulvey, C., Wick-Pelletier, J.: A globalization of the Hahn-Banach theorem. Adv. Math. **89**, 1–59 (1991)
22. Negri, S., von Plato, J., Coquand, T.: Proof-theoretical analysis of order relations. Arch. Math. Logic **43**, 297–309 (2004)
23. Oliva, P., Powell, T.: A game-theoretic computational interpretation of proofs in classical analysis. In: Kahle, R., Rathjen, M. (eds.) Gentzen's Centenary, pp. 501–531. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-10103-3_18
24. Oliva, P., Powell, T.: Spector bar recursion over finite partial functions. Ann. Pure Appl. Log. **168**(5), 887–921 (2017)
25. Persson, H.: An application of the constructive spectrum of a ring. In: Type Theory and the Integrated Logic of Programs. Chalmers University and University of Göteborg (1999). Ph.D. thesis
26. Powell, T.: On bar recursive interpretations of analysis. Ph.D. thesis, Queen Mary University of London (2013)
27. Powell, T.: Gödel's functional interpretation and the concept of learning. In: Proceedings of Logic in Computer Science (LICS 2016), pp. 136–145. ACM (2016)
28. Powell, T.: Sequential algorithms and the computational content of classical proofs (2018). https://arxiv.org/abs/1812.11003
29. Richman, F.: Nontrivial uses of trivial rings. Proc. Am. Math. Soc. **103**(4), 1012–1014 (1988)
30. Rinaldi, D., Schuster, P.: A universal Krull-Lindenbaum theorem. J. Pure Appl. Algebra **220**, 3207–3232 (2016)
31. Rinaldi, D., Schuster, P., Wessel, D.: Eliminating disjunctions by disjunction elimination. Bull. Symb. Logic **23**(2), 181–200 (2017)
32. Rinaldi, D., Schuster, P., Wessel, D.: Eliminating disjunctions by disjunction elimination. Indag. Math. (N.S.) **29**(1), 226–259 (2018)
33. Rinaldi, D., Wessel, D.: Cut elimination for entailment relations. Arch. Math. Log. (2018). https://doi.org/10.1007/s00153-018-0653-0
34. Rinaldi, D., Wessel, D.: Extension by conservation. Sikorski's theorem. Log. Methods Comput. Sci. **14**(4:8), 1–17 (2018)
35. Sanders, S.: Metastability and higher-order computability. In: Artemov, S., Nerode, A. (eds.) LFCS 2018. LNCS, vol. 10703, pp. 309–330. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-72056-2_19
36. Schlagbauer, K., Schuster, P., Wessel, D.: Der Satz von Hahn-Banach im Rahmen einer allgemeinen Idealtheorie. Confluentes Math, forthcoming
37. Schuster, P.: Induction in algebra: a first case study. In: 2012 27th Annual ACM/IEEE Symposium on Logic in Computer Science, pp. 581–585. IEEE Computer Society Publications (2012). Proceedings, LICS 2012, Dubrovnik, Croatia
38. Schuster, P.: Induction in algebra: a first case study. Log. Methods Comput. Sci. **9**(3), 20 (2013)

39. Schuster, P., Wessel, D.: A general extension theorem for directed-complete partial orders. Rep. Math. Logic **53**, 79–96 (2018)
40. Schwichtenberg, H., Seisenberger, M., Wiesnet, F.: Higman's lemma and its computational content. In: Kahle, R., Strahm, T., Studer, T. (eds.) Advances in Proof Theory. PCSAL, vol. 28, pp. 353–375. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-29198-7_11
41. Simpson, S.G.: Subsystems of Second Order Arithmetic. Perspectives in Mathematical Logic. Springer, Berlin (1999)
42. Spector, C.: Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles in current intuitionistic mathematics. In: Dekker, F.D.E. (ed.) Recursive Function Theory: Proceedings of the Symposia in Pure Mathematics, vol. 5, pp. 1–27. American Mathematical Society, Providence, Rhode Island (1962)
43. Tao, T.: Soft analysis, hard analysis, and the finite convergence principle. Essay, published as Chap. 1.3 of T. Tao, Structure and Randomness: Pages from Year 1 of a Mathematical Blog, Amer. Math. Soc (2008). Original version available online at http://terrytao.wordpress.com/2007/05/23/soft-analysis-hard-analysis-and-the-finite-convergence-principle/
44. Wessel, D.: Ordering groups constructively. Commun. Algebra, forthcoming
45. Yengui, I.: Making the use of maximal ideals constructive. Theor. Comput. Sci. **392**, 174–178 (2008)
46. Yengui, I.: Constructive Commutative Algebra. LNM, vol. 2138. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-19494-3

Reverse Mathematics and Computability Theory of Domain Theory

Sam Sanders(✉)

School of Mathematics, Leeds University, Leeds, UK
sasander@me.com

Abstract. This paper deals with the foundations of mathematics and computer science, domain theory in particular; the latter studies certain ordered sets, called domains, with close relations to topology. Conceptually speaking, domain theory provides a highly abstract and general formalisation of the intuitive notions 'approximation' and 'convergence'. Thus, a major application in computer science is the semantics of programming languages. We study the following foundational questions:
(Q1) Which axioms are needed to prove basic results in domain theory?
(Q2) How hard it is to compute the objects in these basic results?
Clearly, (Q1) is part of the program *Reverse Mathematics*, while (Q2) is part of *computability theory* in the sense of Kleene. Our main result is that even very basic theorems in domain theory are *extremely* hard to prove, while the objects in these theorems are similarly *extremely* hard to compute; this hardness is measured relative to the usual hierarchy of comprehension axioms, namely one requires full *second-order arithmetic* in each case. By contrast, we show that the formalism of domain theory obviates the need for the Axiom of Choice, a foundational concern.

1 Introduction

In a nutshell, our main result is that even the most basic theorems in *domain theory* are *extremely* hard to prove, while the objects in these theorems are similarly *extremely* hard to compute; this hardness is measured relative to the usual hierarchy of comprehension axioms, namely one requires full *second-order arithmetic* in each case. This observed hardness has nothing to do with the Axiom of Choice, and we even show that the formalism of domain theory can obviate the need for this axiom. In this light, our paper deals with the study of domain theory from the point of view of *Reverse Mathematics* and *computability theory*. Let us first introduce some of the aforementioned italicised notions.

First of all, *Reverse Mathematics* (RM hereafter) is a program in the foundations of mathematics where the aim is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics. We provide an

This research was supported by the John Templeton Foundation grant *a new dawn of intuitionism* with ID 60842. Opinions expressed in this paper do not necessarily reflect those of the John Templeton Foundation. See also Remark 4.7 below.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 550–568, 2019.
https://doi.org/10.1007/978-3-662-59533-6_33

introduction to RM in Sect. 2.1. We note that the RM of domain theory (resp. topology) has already been initiated in [29,40] (resp. [37–39]).

Secondly, *domain theory* is a field in the intersection of mathematics and computer science [1,19–21]. At a conceptual level, domain theory provides a highly abstract and general formalisation for the intuitive notions of 'approximation' and 'convergence', which can be used to provide semantics for programming languages. In particular, domain theory deals with the study of certain ordered sets, called domains, with close relations to topology. The central notions are *directed sets* and *nets*; nets generalise the concept of sequence to uncountable index sets, given by directed sets. For instance, nets and directed sets yield intuitive definitions of the Scott and Lawson topologies (see [20, II and III]). Moreover, there are rather strong opinions about the central role of nets and directed sets in domain theory, even pertaining to the Axiom of Choice.

> clinging to ascending sequences would produce a mathematical theory that becomes rather bizarre, whence our move to directed [21, p. 59]

> Turning to foundations, we feel that the necessity to choose chains where directed subsets are naturally available (such as in function spaces) and thus to rely on the Axiom of Choice without need, is a serious stain on this approach. [1, §2.2.4].

In light of these quotes, directed sets are first-class citizens in domain theory while sequences do not provide an acceptable substitute. We have therefore opted to work in Kohlenbach's *higher-order* RM (see Sect. 2.1) where (uncountable) directed sets are directly available. By contrast, 'classical' RM is developed in the language of second-order arithmetic, i.e. directed sets and nets are represented by *countable* objects, which runs counter to the opinions in the above quotes. Similarly, we work with Kleene's *higher-order* notion of computability, given by his S1–S9 schemes (see Sect. 2.2) and which extends Turing's framework [55]. Based on the above, we answer the following foundational questions in this paper.

(Q1) As part of the program Reverse Mathematics, which axioms are needed to prove basic results in domain theory?
(Q2) How hard it is to compute, in the sense of Kleene's S1-S9, the objects in basic results in domain theory?

Regarding (Q1), we study the *monotone convergence theorem for nets*, which is implicit in the central object of study from domain theory, namely directed-complete posets. Even for the basic case of nets in the unit interval indexed by Baire space, this theorem is *extremely* hard to prove, relative to the usual hierarchy of comprehension axioms; a proof namely requires full second-order arithmetic which is also sufficient (see Sect. 3.1). Thus, the observed hardness has nothing to do with the Axiom of Choice, and we show in Sect. 3.2 that using nets rather than sequences obviates the need for this axiom, a foundational concern expressed in the above quotes.

Regarding (Q2), it turns out that the limit in the monotone convergence theorem (for nets in the unit interval indexed by Baire space) is similarly *extremely* hard to compute, but we moreover show that more general index sets (definable in the finite type hiearchy) yield a hierarchy including n-th order arithmetic for any $n \geq 2$ (see Sect. 4). We have relegated certain technical definitions (Sect. A.1) and certain similar proofs (Sect. A.2) to the appendix.

Finally, as to prior art, it is shown in [48] that the Bolzano-Weierstrass theorem for nets in the unit interval indexed by subsets of Baire space, implies the Heine-Borel theorem for uncountable covers. Dini's theorem for nets is shown to be *equivalent* to this covering theorem even. We discuss the general context and foundational implications of these results in Sect. 5.

2 Preliminaries

We introduce *Reverse Mathematics* in Sect. 2.1, as well as its generalisation to *higher-order arithmetic*, and the associated base theory RCA_0^ω. We introduce some essential axioms in Sect. 2.2. To obtain our main RM-results it suffices to study nets *indexed by subsets of Baire space*; the latter bit of set theory shall be represented in RCA_0^ω as in Definition 2.2 in Sect. 2.1.

2.1 Reverse Mathematics

Reverse Mathematics is a program in the foundations of mathematics initiated around 1975 by Friedman [13,14] and developed extensively by Simpson [51]. The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics.

We refer to [53] for a basic introduction to RM and to [50,51] for an overview of RM. We expect basic familiarity with RM, but do sketch some aspects of Kohlenbach's *higher-order* RM [27] essential to this paper, including the base theory RCA_0^ω (Definition A.1). As will become clear, the latter is officially a type theory but can accommodate (enough) set theory via Definition 2.2.

First of all, in contrast to 'classical' RM based on *second-order arithmetic* Z_2, higher-order RM uses the richer language L_ω of *higher-order arithmetic*. Indeed, while second-order RM is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of *all finite types* $\mathbf{T}$, defined by the two clauses:

$$\text{(i) } 0 \in \mathbf{T} \text{ and (ii) If } \sigma, \tau \in \mathbf{T} \text{ then } (\sigma \to \tau) \in \mathbf{T},$$

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type σ to objects of type τ. In this way, $1 \equiv 0 \to 0$ is the type of functions from numbers to numbers, and where $n + 1 \equiv n \to 0$. Viewing sets as given by characteristic functions, Z_2 only includes objects of type 0 and 1.

Secondly, the language L_ω includes variables $x^\rho, y^\rho, z^\rho, \ldots$ of any finite type $\rho \in \mathbf{T}$. Types may be omitted when they can be inferred from context. The constants of L_ω includes the type 0 objects $0, 1$ and $<_0, +_0, \times_0, =_0$ which are intended to have their usual meaning pertaining to $\mathbb{N}$. Equality at higher types is defined in terms of '$=_0$' as follows: for any objects x^τ, y^τ, we have

$$[x =_\tau y] \equiv (\forall z_1^{\tau_1} \dots z_k^{\tau_k})[x z_1 \dots z_k =_0 y z_1 \dots z_k], \tag{2.1}$$

if the type τ is composed as $\tau \equiv (\tau_1 \to \dots \to \tau_k \to 0)$. Furthermore, L_ω also includes the *recursor constant* $\mathbf{R}_\sigma$ for any $\sigma \in \mathbf{T}$, which allows for iteration on type σ-objects as in the special case (A.1). Formulas and terms are defined as usual. One obtains the sub-language L_{n+2} by restricting the above type formation rule to produce only type $n+1$ objects (and related types of similar complexity).

As discussed in [27, §2], RCA_0^ω and RCA_0 prove the same sentences 'up to language' as the latter is set-based and the former function-based. Recursion as in (A.1) is called *primitive recursion*; the class of functionals obtained from $\mathbf{R}_\rho$ for all $\rho \in \mathbf{T}$ is *Gödel's system T* of all (higher-order) primitive recursive functionals.

We use the usual notations for natural, rational, and real numbers, and the associated functions, as introduced in [27, pp. 288–289].

Definition 2.1 (Real numbers and related notions in RCA_0^ω)

(a) Natural numbers correspond to type zero objects, and we use 'n^0' and '$n \in \mathbb{N}$' interchangeably. Rational numbers are defined as signed quotients of natural numbers, and '$q \in \mathbb{Q}$' and '$<_\mathbb{Q}$' have their usual meaning.
(b) Real numbers are coded by fast-converging Cauchy sequences $q_{(\cdot)} : \mathbb{N} \to \mathbb{Q}$, i.e. such that $(\forall n^0, i^0)(|q_n - q_{n+i})| <_\mathbb{Q} \frac{1}{2^n})$. We use Kohlenbach's 'hat function' from [27, p. 289] to guarantee that every q^1 defines a real number.
(c) We write '$x \in \mathbb{R}$' to express that $x^1 := (q^1_{(\cdot)})$ represents a real as in the previous item and write $[x](k) := q_k$ for the k-th approximation of x.
(d) Two reals x, y represented by $q_{(\cdot)}$ and $r_{(\cdot)}$ are *equal*, denoted $x =_\mathbb{R} y$, if $(\forall n^0)(|q_n - r_n| \leq 2^{-n+1})$. Inequality '$<_\mathbb{R}$' is defined similarly. We sometimes omit the subscript '$\mathbb{R}$' if it is clear from context.
(e) Functions $F : \mathbb{R} \to \mathbb{R}$ are represented by $\Phi^{1\to 1}$ mapping equal reals to equal reals, i.e. $(\forall x, y \in \mathbb{R})(x =_\mathbb{R} y \to \Phi(x) =_\mathbb{R} \Phi(y))$.
(f) The relation '$x \leq_\tau y$' is defined as in (2.1) but with '$\leq_0$' instead of '$=_0$'. Binary sequences are denoted '$f^1, g^1 \leq_1 1$', but also '$f, g \in C$' or '$f, g \in 2^\mathbb{N}$'. Elements of Baire space are given by f^1, g^1, but also denoted '$f, g \in \mathbb{N}^\mathbb{N}$'.
(g) Sets of type ρ objects $X^{\rho\to 0}, Y^{\rho\to 0}, \dots$ are given by their characteristic functions $F_X^{\rho\to 0} \leq_{\rho\to 0} 1$, i.e. we write '$x \in X$' for $F_X(x) =_0 1$.

The following special case of item (g) is singled out, as it will be used below.

Definition 2.2. [RCA_0^ω] A 'subset D of $\mathbb{N}^\mathbb{N}$' is given by its characteristic function $F_D^2 \leq_2 1$, i.e. we write '$f \in D$' for $F_D(f) = 1$ for any $f \in \mathbb{N}^\mathbb{N}$. A 'binary relation $\preceq$ on a subset D of $\mathbb{N}^\mathbb{N}$' is given by the associated characteristic function $G_\preceq^{(1\times 1)\to 0}$, i.e. we write '$f \preceq g$' for $G_\preceq(f, g) = 1$ and any $f, g \in D$. Assuming extensionality on the reals as in item (e), we obtain characteristic functions that represent subsets of $\mathbb{R}$ and relations thereon. Using pairing functions, it is clear we can also represent sets of finite sequences (of reals), and relations thereon. To improve readability, the variables $v, w, u, \ldots$ are reserved for finite sequence of reals, which have the dedicated type '1^*', as detailed in Notation A.3. A finite sequence $w^{1^*} = \langle y_0, \ldots, y_k \rangle$ has 'length' $|w| = k+1$, and $|\langle\rangle| = 0$.

2.2 Higher-Order Computability Theory

As noted above, some of our main results are part of computability theory. Thus, we first make our notion of 'computability' precise as follows.

(I) We adopt ZFC, i.e. Zermelo-Fraenkel set theory with the Axiom of Choice, as the official metatheory for all results, unless explicitly stated otherwise.
(II) We adopt Kleene's notion of *higher-order computation* as given by his nine clauses S1-S9 (see [30,45]) as our official notion of 'computable'.

We only need Kleene's S1-S9 as a 'background theory', as we will always obtain *primitive recursive* computability, in the sense of Gödel's T (see [30]).

For the rest of this section, we introduce some existing functionals which will be used below. These also constitute the higher-order counterparts of second-order arithmetic Z_2, and some of the Big Five systems, in higher-order RM. We use the formulation from [27,41]. First of all, ACA_0 is readily derived from:

$$(\exists \mu^2)(\forall f^1)\big[(\exists n)(f(n)=0) \to [(f(\mu(f))=0) \wedge (\forall i < \mu(f)) f(i) \neq 0] \wedge [(\forall n)(f(n) \neq 0) \to \mu(f) = 0]\big], \tag{μ^2}$$

and $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mu^2)$ proves the same sentences as ACA_0 by [24, Theorem 2.5]. The (unique) functional μ^2 in (μ^2) is also called *Feferman's* μ [2], and is *discontinuous* at $f =_1 11\ldots$; in fact, (μ^2) is equivalent to the existence of $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = 1$ if $x >_{\mathbb{R}} 0$, and 0 otherwise [27, §3], and to

$$(\exists \varphi^2 \leq_2 1)(\forall f^1)\big[(\exists n)(f(n)=0) \leftrightarrow \varphi(f) = 0\big]. \tag{$\exists^2$}$$

Secondly, $\Pi_1^1\text{-}\mathsf{CA}_0$ is readily derived from the following sentence:

$$(\exists \mathsf{S}^2 \leq_2 1)(\forall f^1)\big[(\exists g^1)(\forall n^0)(f(\overline{g}n)=0) \leftrightarrow \mathsf{S}(f) = 0\big], \tag{S^2}$$

and $\Pi_1^1\text{-}\mathsf{CA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mathsf{S}^2)$ proves the same Π_3^1-sentences as $\Pi_1^1\text{-}\mathsf{CA}_0$ by [46, Theorem 2.2]. The (unique) functional S^2 in (S^2) is also called *the Suslin functional* [27]. By definition, the Suslin functional S^2 can decide whether a Σ_1^1-formula (as in the left-hand side of (S^2)) is true or false. We similarly define the functional S_k^2 which decides the truth or falsity of Σ_k^1-formulas; we also define the system $\Pi_k^1\text{-}\mathsf{CA}_0^\omega$ as $\mathsf{RCA}_0^\omega + (\mathsf{S}_k^2)$, where (S_k^2) expresses that S_k^2 exists. Note that we allow formulas with *function* parameters, but **not** *functionals* here. In fact, Gandy's *Superjump* $\mathbb{S}^3$ [18] constitutes a way of extending $\Pi_1^1\text{-}\mathsf{CA}_0^\omega$ to parameters of type two, as follows:

$$\mathbb{S}(F^2, e^0) := \begin{cases} 0 & \text{if } \{e\}(F) \text{ terminates} \\ 1 & \text{otherwise} \end{cases}, \tag{$\mathbb{S}^3$}$$

where the formula '$\{e\}(F)$ terminates' is a Π^1_1-formula, defined by Kleene's S1–S9 and (obviously) involving type two parameters.
Thirdly, full second-order arithmetic Z_2 is derived from $\cup_k \Pi^1_k\text{-}\mathsf{CA}^\omega_0$, or from:

$$(\exists E^3 \leq_3 1)(\forall Y^2)\big[(\exists f^1)Y(f) = 0 \leftrightarrow E(Y) = 0\big], \qquad (\exists^3)$$

and we therefore define $\mathsf{Z}^\Omega_2 \equiv \mathsf{RCA}^\omega_0 + (\exists^3)$ and $\mathsf{Z}^\omega_2 \equiv \cup_k \Pi^1_k\text{-}\mathsf{CA}^\omega_0$, which are conservative over Z_2 by [24, Cor. 2.6]. Despite this close connection, Z^ω_2 and Z^Ω_2 can behave quite differently, as discussed in e.g. [41, §2.2]. The functional from $(\exists^3)$ is also called '$\exists^3$', and we use the same convention for other functionals.

Finally, the Heine-Borel theorem states the existence of a finite sub-cover for an open cover of certain spaces. Now, a functional $\Psi : \mathbb{R} \to \mathbb{R}^+$ gives rise to the *canonical* cover $\cup_{x\in I} I^\Psi_x$ for $I \equiv [0, 1]$, where I^Ψ_x is the open interval $(x - \Psi(x), x + \Psi(x))$. Hence, the uncountable cover $\cup_{x\in I} I^\Psi_x$ has a finite sub-cover by the Heine-Borel theorem; in symbols:

$$(\forall \Psi : \mathbb{R} \to \mathbb{R}^+)(\exists y_1, \ldots, y_k \in I)(\forall x \in I)(\exists i \leq k)(x \in I^\Psi_{y_i}). \qquad (\mathsf{HBU})$$

Note that HBU is almost verbatim *Cousin's lemma* (see [9, p. 22]), i.e. the Heine-Borel theorem restricted to canonical covers. The latter restriction does not make much of a big difference, as studied in [47]. By [41,42], Z^Ω_2 proves HBU but $\mathsf{Z}^\omega_2 + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ cannot, and many basic properties of the *gauge integral* [36,54] are equivalent to HBU. Although strictly speaking incorrect, we sometimes use set-theoretic notation, like reference to the cover $\cup_{x\in I} I^\Psi_x$ inside RCA^ω_0, to make proofs more understandable. Such reference can in principle be removed in favour of formulas of higher-order arithmetic.

2.3 Introducing Nets

We introduce the notion of directed set and nets, and associated concepts. On a historical note, Moore-Smith and Vietoris independently introduced these notions about a century ago in [34,56]. We first consider the following standard definition (see e.g. [25, Ch. 2]).

Definition 2.3. [Nets] A set $D \neq \emptyset$ with a binary relation '$\preceq$' is *directed* if

(a) The relation $\preceq$ is transitive, i.e. $(\forall x, y, z \in D)([x \preceq y \wedge y \preceq z] \to x \preceq z)$.
(b) For $x, y \in D$, there is $z \in D$ such that $x \preceq z \wedge y \preceq z$.
(c) The relation $\preceq$ is reflexive, i.e. $(\forall x \in D)(x \preceq x)$.

For such $(D, \preceq)$ and topological space X, any map $x : D \to X$ is a *net* in X.

We denote $x(d)$ as x_d to suggest the connection to sequences. Note that a net is officially a triple $(D, \preceq, x_d)$, but the first two are often not mentioned explicitly.

In Sect. 3, we only study directed sets that are subsets of Baire space, i.e. as given by Definition 2.2. Similarly, we only study nets $x_d : D \to \mathbb{R}$ where D is a subset of Baire space. Thus, a net x_d in $\mathbb{R}$ is just a type $1 \to 1$ functional with extra structure on its domain D provided by '$\preceq$' as in Definition 2.2.

The definitions of convergence and increasing net are of course familiar.

Definition 2.4. [Convergence of nets] If x_d is a net in X, we say that x_d *converges* to the limit $\lim_d x_d = y \in X$ if for every neighbourhood U of y, there is $d_0 \in D$ such that for all $e \succeq d_0$, $x_e \in U$.

Definition 2.5. [Increasing nets] A net x_d in $\mathbb{R}$ is *increasing* if $a \preceq b$ implies $x_a \leq_{\mathbb{R}} x_b$ for all $a, b \in D$.

The notion of 'sub-net' was first given by Moore in [35] and is used in [25], but is not needed in this paper. Finally, we point out that $\mathbb{N}$ with its usual ordering yields a directed set, i.e. convergence results about nets do apply to sequences.

3 Reverse Mathematics

3.1 Monotone Convergence for Nets

We show that the monotone convergence theorem for nets in $[0,1]$ is extremely hard to prove, while the associated limit is similarly hard to compute. Indeed, one needs full second-order arithmetic as in $(\exists^3)$ in each case.

Let $\mathsf{MCT}^0_{\mathsf{net}}$ state that every increasing net in $[0,1]$ converges. As discussed in Sect. 2.3, $\mathsf{MCT}^0_{\mathsf{net}}$ is restricted to nets that are indexed by subsets of $\mathbb{N}^{\mathbb{N}}$. We show $\mathsf{MCT}^0_{\mathsf{net}} \rightarrow \mathsf{HBU}$ in Theorem 3.1, but Theorem 3.2 is of more importance: $\mathsf{MCT}^0_{\mathsf{net}}$ is provable without the Axiom of Choice, i.e. the 'hardness' of the former theorem has nothing to do with the latter. We obtain a relative computability result in Theorem 3.3, the foundation for Sect. 4.

As to the provenance of $\mathsf{MCT}^0_{\mathsf{net}}$, this theorem can be found in e.g. [7, p. 103], but is also implicit in domain theory. Indeed, the main objects of study of domain theory are *dcpos*, i.e. directed-complete posets, and every monotone net converges to its supremum in any dcpo.

Theorem 3.1. *The system* $\mathsf{RCA}_0^\omega + \mathsf{MCT}^0_{\mathsf{net}}$ *proves* HBU.

Proof. Note that $\mathsf{MCT}^0_{\mathsf{net}}$ implies the monotone convergence theorem *for sequences*, as the latter are nets. Hence, we have access to ACA_0 by [51, III.2.2]. We shall prove the theorem twice: once in case $(\exists^2)$ and once in case $\neg(\exists^2)$; the law of excluded middle as in $(\exists^2) \vee \neg(\exists^2)$ then finishes the proof.

Now, in case $\neg(\exists^2)$, all functions on $\mathbb{R}$ are continuous by [27, Prop. 3.12]. Hence, $\cup_{q\in\mathbb{Q}\cap[0,1]} I^\Psi_q$ is a countable sub-cover of $\cup_{x\in[0,1]} I^\Psi_x$ for any $\Psi : I \rightarrow \mathbb{R}^+$. By [51, VI.1], WKL already implies that the former has a finite sub-cover, i.e. this case is finished. For the case $(\exists^2)$, fix some $\Psi : I \rightarrow \mathbb{R}^+$ and use $(\exists^2)$ to define D as the set of finite sequences of reals w^{1^*} such that $0 \in w$ and the cover $\cup_{i<|w|} I^\Psi_{w(i)}$ has 'no holes', i.e. any point between two intervals of this cover, is also in the cover. We define '$v \preceq w$' as $(\forall i < |v|)(\exists j < |w|)(v(i) =_{\mathbb{R}} w(j))$. Clearly, $(D, \preceq)$ is a directed set and we define the net $x_w : D \rightarrow [0,1]$ as the right end-point of the right-most interval in $\cup_{i<|w|} I^\Psi_{w(i)}$, capped by 1 if necessary.

Since x_w is increasing by definition, let $x \in [0,1]$ be the limit provided by $\mathsf{MCT}^0_{\mathsf{net}}$. If $x =_{\mathbb{R}} 1$, then apply $\lim x_d = x$ for $\varepsilon = \Psi(1)$ to find a finite sub-cover for the canonical cover associated to Ψ. In case $x <_{\mathbb{R}} 1$, apply $\lim x_d = x$ for $\varepsilon_0 = \min(\Psi(x), |x-1|/2)$, i.e. there is $w_0 \in D$ such that for all $v \succeq w_0$, we have $|x_v - x| < \varepsilon_0$, implying $x_v \in I_x^{\Psi}$. Fix such w_0 and consider $v_0 := w_0 * \langle x \rangle$. The latter is in D and satisfies $v_0 \succeq w_0$. Hence, x_{v_0} must be in I_x^{Ψ} by the aforementioned convergence, but $x_{v_0} \notin I_x^{\Psi}$ by definition of the net x_w. Hence, we have obtained a contradiction in case $x <_{\mathbb{R}} 1$. □

On one hand, the previous implies that nets indexed by subsets of Baire space already give rise to HBU. On the other hand, the proof of the following corollary suggests that such nets are 'all we can handle' in Z_2^{Ω}. Moreover, the proof suggests that $\mathsf{Z}_2^{\omega} + \mathsf{HBU}$ cannot prove $\mathsf{MCT}_{\mathsf{net}}^0$.

Theorem 3.2. *The system* Z_2^{Ω} *proves* $\mathsf{MCT}_{\mathsf{net}}^0$*, while* $\mathsf{Z}_2^{\omega} + \mathsf{QF\text{-}AC}^{0,1}$ *does not.*

Proof. The negative result follows from [42, Theorem 4.3]. For the remaining result, note that HBU is available thanks to [42, Theorem 4.2]. Suppose $\neg\mathsf{MCT}_{\mathsf{net}}^0$, i.e. there is some increasing net x_d in I that does not converge to any point in I. Hence, for every $x \in I$ there is $n \in \mathbb{N}$ such that for all $d \in D$ there is $e \succeq d$ such that $|x - x_e| \geq \frac{1}{2^n}$. Since $\exists^3$ is given, we may use $\mathsf{QF\text{-}AC}^{1,0}$ to obtain $\Phi : I \to \mathbb{R}$ such that $\Phi(x)$ is the least such $n \in \mathbb{N}$. Define $\Psi(x) := \frac{1}{2^{\Phi(x)}}$ and use HBU to find $y_1, \ldots, y_k \in I$ such that $\cup_{i \leq k} I_{y_i}^{\Psi}$ covers I. By definition, for any $i \leq k$, either x_d is 'below' $I_{y_i}^{\Psi}$ for all $d \in D$ or there is $d_i \in D$ such that x_e is 'above' $I_{y_i}^{\Psi}$ for all $e \succeq d_i$. Let $d_{i_1}, \ldots, d_{i_m} \in D$ be all such numbers from the second case. There is $e_0 \succeq d_{i_j}$ for $j \leq m$ by Definition 2.3, but x_{e_0} cannot be in I, a contradiction. □

The previous theorem also implies that $\mathsf{MCT}_{\mathsf{net}}^0$ has the same first-order strength as ACA_0 using the above 'excluded middle trick' and the 'splitting' of $(\exists^3)$ as $[(\kappa_0^3) + (\exists^2)] \leftrightarrow (\exists^3)$, where (κ_0^3) may be found in [49, §3.1].

Next, it is well-known that $\exists^2$ computes a realiser for the monotone convergence theorem for *sequences* via a term of Gödel's T, and vice versa (see [46, §4]). Inspired by this observation, we obtain an elegant 'one type up' generalisation in Theorem 3.3. A realiser for $\mathsf{MCT}_{\mathsf{net}}^0$ is a functional taking as input $(D, \preceq_D, x_d)$ and outputting the real $x = \lim_d x_d$ if the inputs satisfy the conditions of $\mathsf{MCT}_{\mathsf{net}}^0$.

Theorem 3.3. *A realiser for* $\mathsf{MCT}_{\mathsf{net}}^0$ *computes* $\exists^3$ *via a term of Gödel's* T*, and vice versa.*

Proof. For the 'vice versa' direction, one uses the usual 'interval halving technique' where $\exists^3$ is used to decide whether there is $d \in D$ such that x_d is in the relevant interval. Indeed, define $\mathbb{r} : C \to [0,1]$ as $\mathbb{r}(f) := \sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}}$ and define $f_0 \in C$ as follows: $f_0(0) = 1$ if and only if $(\exists d \in D)(x_d \geq \frac{1}{2})$ and $f_0(n+1) = 1$ if and only if $(\exists d \in D)(x_d \geq \mathbb{r}(\overline{f_0}n * 00\ldots))$. Then $\lim_d x_d = \mathbb{r}(f_0)$, as required.

For the other direction, fix Y^2, let D be Baire space, and define '$f \preceq g$' by $Y(f) \geq_0 Y(g)$ for any $f, g \in D$. It is straightforward to show that $(D, \preceq)$ is a directed set. Define the net $x_d : D \to I$ by 0 if $Y(d) > 0$, and 1 if $Y(d) = 0$, which is increasing by definition. Hence, x_d converges, say to $y_0 \in I$, and if $y_0 >_{\mathbb{R}} 1/3$, then there is f^1 such that $Y(f) = 0$, while if $y_0 <_{\mathbb{R}} 2/3$, then $(\forall f^1)(Y(f) > 0)$. Clearly, this yields a term of Gödel's T that computes $\exists^3$. □

Corollary 3.4. *The system* Z_2^Ω *proves* $\mathsf{MCT}^0_{\mathsf{net}}$.

The previous results show that $\mathsf{MCT}^0_{\mathsf{net}}$ is extremely hard to prove, the limit therein similarly hard to compute. We establish in Sect. 4 that generalisations of $\mathsf{MCT}^0_{\mathsf{net}}$ to 'larger' index sets have yet more extreme properties, even compared to e.g. $\exists^3$. We also discuss special cases for e.g. the superjump $\mathbb{S}$.

3.2 The Axiom of Choice and Nets

As is clear from Sect. 1, a certain value is placed in the development of domain theory on avoiding 'stains' caused by the unnecessary use of the Axiom of Choice. In this section, we provide two important results pertaining to nets and the Axiom of Choice. In particular, we show that replacing limits involving nets by limits involving sequences implies $\mathsf{QF\text{-}AC}^{0,1}$, and we even obtain an equivalence (Corollary 3.7). Moreover, we show that the definition of continuity based on nets, as can be found in [20, p. 45], is equivalent to the epsilon-delta definition of continuity in RCA_0^ω (Theorem 3.9). By contrast, the equivalence involving *sequential* continuity is not provable in ZF as it requires $\mathsf{QF\text{-}AC}^{0,1}$. In conclusion, replacing sequences by nets can obviate the use of the Axiom of Choice, and the latter is essential when replacing limits of nets by limits of sequences.

Nets and Sequentialisation. By the above, basic theorems regarding nets imply HBU and are therefore extremely hard to prove. In line with the coding practise of RM, one may therefore want to replace limits involving nets by 'countable' limits, i.e. if a net converges to some limit, then there should be a *sequence* in the net that also converges to the same limit. In this section, we show that even elementary versions of such 'sequentialisation' theorems imply $\mathsf{QF\text{-}AC}^{0,1}$. In general, it should be noted that such sequentialisation theorems are only valid/possible for first-countable spaces.

We study the following basic sequentialisation theorem, which can be found in Bourbaki's general topology in general form; see [6, p. 337]. Recall $I \equiv [0,1]$.

Definition 3.5. [SUB_0] For $x_d : D \to I$ an increasing net converging to $x \in I$, there is $\Phi : \mathbb{N} \to D$ such that $\lambda n.x_{\Phi(n)}$ is increasing and $\lim_{n\to\infty} x_{\Phi(n)} =_{\mathbb{R}} x$.

Let IND be the induction schema for all formulas in L_ω. The system $\mathsf{RCA}_0^\omega + \mathsf{IND}$ has the same first-order strength as ACA_0 (see [2]).

Theorem 3.6. *The system* $\mathsf{RCA}_0^\omega + \mathsf{IND}$ *proves* $\mathsf{SUB}_0 \to \mathsf{QF\text{-}AC}^{0,1}$.

Proof. In case $\neg(\exists^2)$, all functions on Baire space are continuous by [27, Prop. 3.7], and $\mathsf{QF\text{-}AC}^{0,1}$ reduces to $\mathsf{QF\text{-}AC}^{0,0}$, included in RCA_0^ω. To observe the latter reduction, note that the antecedent of $\mathsf{QF\text{-}AC}^{0,1}$ can be brought into the following form in RCA_0^ω: $(\forall n^0)(\exists f^1)(Y(f,n)=0)$, for some Y^2. Since $\lambda f.Y$ is continuous by assumption, this formula is equivalent to $(\forall n^0)(\exists \sigma^{0^*})(Y(\sigma * 00\ldots, n) = 0)$.

For the case $(\exists^2)$, note that we also have (μ^2) by [27, §3]. Define $\mathbb{b}^{1\to 1^*}$ as follows: $|\mathbb{b}(f)| = f(0)+1$ and $\mathbb{b}(f)(i)$ for $i < |\mathbb{b}(f)|$ is the sequence $f(1+$

$i), f(1+i+|\mathbb{b}(f)|), f(1+i+2|\mathbb{b}(f)|), \ldots$. Note that $\mathbb{b}^{1\to 1^*}$ provides the inverse of a pairing function. Fix some $F^{(0\times 1)\to 0}$ satisfying the antecedent of $\mathsf{QF\text{-}AC}^{0,1}$, i.e. $(\forall n^0)(\exists f^1)(F(n,f)=0)$, and use IND to prove the following:

$$(\forall n^0)(\exists f^1)\underline{(\forall i \le n)(F(i, \mathbb{b}(\langle n\rangle * f)(i)) = 0)}. \tag{3.1}$$

The underlined formula in (3.1) is also written '$G(n,f)=0$' and if there is f_0^1 such that $(\forall n^0)(G(n,f_0)=0)$, then $Y(n) := \mathbb{b}(\langle n\rangle * f_0)(n)$ is as required for the consequent of $\mathsf{QF\text{-}AC}^{0,1}$. Otherwise, i.e. in case $(\forall f^1)(\exists n^0)(G(n,f)\neq 0)$, define $D := \{f^1 : (\exists n^0)G(n,f)=0\}$ and define '$\preceq$' as: $f \preceq g$ if and only if

$$(\mu n)(G(n,f)\neq 0) \le (\mu m)(G(m,g)\neq 0), \tag{3.2}$$

which is well-defined by assumption. Note that $(D,\preceq)$ is a directed set by assumption. Define the increasing net $x_d := 1 - 2^{-(\mu n)(G(n,d)\neq 0)}$ and note that $\lim_d x_d = 1$ by assumption and (3.2). By SUB_0, there is some $\Phi^{0\to 1}$ such that $\lim_{n\to\infty} x_{\Phi(n)} = 1$, i.e. $(\forall \varepsilon > 0)(\exists m^0)(\forall k^0 \ge m)(|x_{\Phi(k)} - 1| < \varepsilon)$, and use μ^2 to find Ψ^2 computing such m^0 from ε. Then the functional $Y(n) := \Phi(\Psi(\frac{1}{2^{n+1}}))$ provides the witness as required for the conclusion of $\mathsf{QF\text{-}AC}^{0,1}$. □

Let ADS be the L_2-sentence from the RM zoo (see [22, Def. 9.1]) that every infinite linear order has an infinite ascending or descending sequence.

Corollary 3.7. *The system* $\mathsf{RCA}_0^\omega + \mathsf{IND} + \mathsf{ADS}$ *proves* $\mathsf{SUB}_0 \leftrightarrow \mathsf{QF\text{-}AC}^{0,1}$.

Proof. See Sect. A.2 □

Nets and Continuity. We establish that 'net-continuity' as in Definition 3.8 and 'epsilon-delta' continuity are *locally* equivalent over RCA_0^ω. As discussed in [27, Rem. 3.13], ZF cannot prove the local[1] equivalence of *sequential* and epsilon-delta continuity [12], while $\mathsf{QF\text{-}AC}^{0,1}$ suffices to establish the general case.

Definition 3.8. [Net-continuity] A function $f : \mathbb{R} \to \mathbb{R}$ is *net-continuous* at $x \in \mathbb{R}$ if for any net $x_d : D \to \mathbb{R}$, $\lim_d x_d = x$ implies $\lim_d f(x_d) = f(x)$.

Note that net-continuity is equivalent to the topological definition of continuity by [3, Example 2.7]. As it happens, the definition of continuity in [20, p. 45] *is* the definition of net-continuity. Nonetheless, *Scott continuity* is much more important in domain theory than (plain) net continuity.

Theorem 3.9 (RCA_0^ω)**.** *For* $f : \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$, *the following are equivalent:*

(a) *the function* $f : \mathbb{R} \to \mathbb{R}$ *is net-continuous at* x,
(b) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in \mathbb{R})(|x-y| < \delta \to |f(x) - f(y)| < \varepsilon)$.

[1] By [27, Prop. 3.6], RCA_0^ω can prove the *global* equivalence of sequential continuity and epsilon-delta continuity on Baire space, i.e. when those continuity properties hold everywhere on the latter.

Proof. The implication (b) $\rightarrow$ (a) is immediate. For the remaining implication, note that in case of $\neg(\exists^2)$, all $f:\mathbb{R}\rightarrow\mathbb{R}$ are continuous by [27, Prop. 3.12]. In case $(\exists^2)$, fix $x\in\mathbb{R}, f:\mathbb{R}\rightarrow\mathbb{R}$ and suppose f is net-continuous at x, but not epsilon-delta continuous at x, i.e. there is $\varepsilon_0>0$ such that

$$(\forall k\in\mathbb{N})(\exists y\in\mathbb{R})(|x-y|<_{\mathbb{R}}\tfrac{1}{2^k}\wedge|f(x)-f(y)|\geq_{\mathbb{R}}\varepsilon_0). \tag{3.3}$$

Using $(\exists^2)$, let D be the set of all $y\in\mathbb{R}$ such that $|f(x)-f(y)|\geq_{\mathbb{R}}\varepsilon_0$ and define '$y_1\preceq y_2$' for $y_1,y_2\in D$ by $|x-y_1|\geq_{\mathbb{R}}|x-y_2|$. Clearly, the relation $\preceq$ yields a directed set. Now define a net $x_d:D\rightarrow\mathbb{R}$ by $x_d:=d$ and note that x_d converges to x by (3.3). By the net-continuity of f, $f(x_d)$ then converges to $f(x)$, which yields a clear contradiction. $\square$

The previous proof highlights a conceptual advantage of nets compared to sequences: to define a sequence $\lambda n^0.x_n$, one has to list the members one by one. In this light, to get a sequence from (3.3), $\mathsf{QF\text{-}AC}^{0,1}$ seems unavoidable. By contrast, to define a net x_d, one only needs to satisfy Definition 2.3, i.e. show that there always *exist* 'bigger' (in the sense of $\preceq$) elements *without* listing them.

Corollary 3.10. *The system* ZF *cannot prove the local equivalence between net-continuity and sequential continuity on* $\mathbb{R}$.

In conclusion, nets have the advantage that the associated notion of net-continuity is locally equivalent to the usual epsilon-delta definition *without* the Axiom of Choice as in $\mathsf{QF\text{-}AC}^{0,1}$. This confirms the opinion expressed in Sect. 1.

4 Computability Theory

The previous section is devoted to the RM-study of nets indexed by subsets of Baire space. Our principal motivation for this restriction was simplicity: we already obtain HBU from basic theorems pertaining to such nets. In this section, we show that nets become more powerful when the index set is more general. In particular, we show that for index sets expressible in L_n $(n\geq 2)$, the language of n-th order arithmetic, we obtain full n-th order arithmetic from a realiser for the associated monotone convergence theorem for nets. Thus, the 'size' of a net is directly proportional to the power of the associated convergence theorem. We have relegated the proofs of the below theorems to Sect. A.2, in light of the similarities with the proof of Theorem 3.3.

We stress that the results in this section are included by way of illustration: the general study of nets is perhaps best undertaken in a suitable set theoretic framework. That is not to say this section should be dismissed as *spielerei*: index sets beyond Baire space do occur 'in the wild', namely in *fuzzy mathematics* or the *iterated limit theorem*, as discussed in Remark 4.6 below.
First of all, we introduce the following hierarchy of comprehension functionals:

$$(\exists E^{(\sigma\rightarrow 0)\rightarrow 0})(\forall Y^{\sigma\rightarrow 0})\big[E(Y)=_0 0\leftrightarrow(\exists f^\sigma)(Y(f)=0)\big]. \tag{$\exists^{\sigma+2}$}$$

where σ is any finite type. Similar to Definition 2.2, we introduce the following.

Definition 4.1. [RCA_0^ω] A 'subset E of $\mathbb{N}^\mathbb{N} \to \mathbb{N}$' is given by its characteristic function $F_E^3 \leq_3 1$, i.e. we write '$Y \in E$' for $F_E(Y) = 1$ for any Y^2. A 'binary relation $\preceq$ on the subset E of $\mathbb{N}^\mathbb{N} \to \mathbb{N}$' is given by the associated characteristic function $G_\preceq^{(2\times 2)\to 0}$, i.e. we write '$Y \preceq Z$' for $G_\preceq(Y,Z) = 1$ and any $Y, Z \in E$.

Secondly, let $\mathsf{MCT}^1_{\mathsf{net}}$ be the statement that any increasing net $x_e : E \to [0,1]$, i.e. indexed by subsets of $\mathbb{N}^\mathbb{N} \to \mathbb{N}$, converges to a limit in $[0,1]$. A realiser for $\mathsf{MCT}^1_{\mathsf{net}}$ is a fifth-order object that takes as input $(E, \preceq_E, x_e)$ and outputs the real $x =_\mathbb{R} \lim_e x_e$ if the inputs satisfy the conditions of the theorem.

Theorem 4.2. *A realiser for* $\mathsf{MCT}^1_{\mathsf{net}}$ *computes* $\exists^4$ *via a term of Gödel's* T*, and vice versa.*

Let $\mathsf{MCT}^\sigma_{\mathsf{net}}$ be the obvious generalisation of $\mathsf{MCT}^1_{\mathsf{net}}$ to sets of type $\sigma+1$ objects. A realiser for the former computes $\exists^{\sigma+3}$, and vice versa, via a straightforward modification of Theorem 4.2. Hence, the general monotone convergence theorem for nets is extremely hard to prove, even compared to e.g. $\exists^3$.

Thirdly, we also study a special case of $\mathsf{MCT}^0_{\mathsf{net}}$ as follows. Let $\mathsf{MCT}^{\mathsf{S}}_{\mathsf{net}}$ be $\mathsf{MCT}^0_{\mathsf{net}}$ restricted to directed sets $(D, \preceq)$ and nets $x_d : D \to I$ *defined via arithmetical formulas*. To be absolutely clear, we assume that 'arithmetical formulas' are part of L_2, i.e. *only type zero and one parameters* are allowed.

Theorem 4.3. *A realiser for* $\mathsf{MCT}^{\mathsf{S}}_{\mathsf{net}}$ *computes* S^2 *via a term of Gödel's* T*, and vice versa.*

The restriction on parameters in $\mathsf{MCT}^{\mathsf{S}}_{\mathsf{net}}$ turns out to be essential: we show that allowing type two parameters yields Gandy's superjump $\mathbb{S}$. Let $\mathsf{MCT}^{\mathbb{S}}_{\mathsf{net}}$ be $\mathsf{MCT}^0_{\mathsf{net}}$ restricted to directed sets $(D, \prec)$ and nets $x_d : D \to I$ defined via arithmetical formulas, possibly involving type two parameters.

Corollary 4.4. *A realiser for* $\mathsf{MCT}^{\mathbb{S}}_{\mathsf{net}}$ *computes* $\mathbb{S}^3$ *via a term of Gödel's* T*.*

To obtain a realiser for ATR_0 (only), one could formulate a version of $\mathsf{MCT}^0_{\mathsf{net}}$ restricted to directed sets $(D, \preceq)$ and nets $x_d : D \to I$ *defined via a quantifier-free formula with* ***continuous*** *type two parameters*. The technical details are however somewhat involved, and we omit the proof.

Finally, let $\mathsf{MCT}^{\mathsf{seq}}_{\mathsf{net}}$ be the generalisation of $\mathsf{MCT}^{\mathsf{S}}_{\mathsf{net}}$ that states that for a *sequence* of nets $x_{d,n} : (D \times \mathbb{N}) \to [0,1]$ increasing in D, there is a sequence y_n in $[0,1]$ such that $\lim_d x_{d,n} = y_n$. Note that such 'sequential' theorems are well-studied in RM, starting with [51, IV.2.12], and can also be found in e.g. [10,11,16,17,23]. Moreover, 'double' nets $x_{e,d} : (D \times E) \to X$ that depend on two index sets D, E are studied (see [25, p. 69]) for their unique convergence properties, i.e. sequences of nets are not that exotic. Now, $\mathsf{MCT}^{\mathsf{seq}}_{\mathsf{net}}$ is part of third-order arithmetic, and we obtain the following in the same way as for Theorem 4.3.

Corollary 4.5. *The system* $\mathsf{RCA}_0^\omega + \mathsf{MCT}^{\mathsf{seq}}_{\mathsf{net}}$ *proves* Π^1_1-CA_0.

Next, we consider a conceptual remark on 'large' index sets and their (rather plentiful) occurrence in mathematics.

Remark 4.6 (Nets in fuzzy mathematics). Zadeh founded the field of *fuzzy mathematics* in [58]. The core notion of *fuzzy set* is a mapping that assigns values in $[0, 1]$, i.e. a 'level' of membership, rather than the binary relation from usual set theory. The first two chapters of Kelley's *General Topology* [25] are generalised to the setting of fuzzy mathematics in [44]. As an example, [44, Theorem 11.1] is the fuzzy generalisation of the classical statement that a point is in the closure of a set if and only if there is a net that converges to this point. However, as is clear from the proof of this theorem, to accommodate fuzzy points in X, the net is indexed by the space $X \to [0, 1]$. Moreover, the *iterated limit theorem* for nets, be it the standard version ([25, p. 69]; [34, §7]) or the fuzzy one [44, Theorem 12.2], involves an index set E_m *indexed by* $m \in D$, where D is another index set. Thus, 'large' index sets can be found in the wild.

Remark 4.7. This paper constitutes a spin-off from the joint project with Dag Normann on the logical and computational properties of the uncountable. A good starting point for those interested in this project is [41]. We thank Dag Normann, Thomas Streicher, and Anil Nerode for their valuable advice. We also thank the referees for all their useful comments and suggestions.

5 Nets and the Gödel Hierarchy

We discuss the foundational implications of our results, esp. as they pertain to the *Gödel hierarchy*. Now, the latter is a collection of logical systems ordered via consistency strength. This hierarchy is claimed to capture most systems that are natural or have foundational import, as follows.

> *It is striking that a great many foundational theories are linearly ordered by $<$. Of course it is possible to construct pairs of artificial theories which are incomparable under $<$. However, this is not the case for the "natural" or non-artificial theories which are usually regarded as significant in the foundations of mathematics* [52].

Burgess and Koellner corroborate this claim in [8, §1.5] and [26, §1.1]. The Gödel hierarchy is a central object of study in mathematical logic, as e.g. argued by Simpson in [52, p. 112] or Burgess in [8, p. 40]. Precursors to the Gödel hierarchy may be found in the work of Wang [57] and Bernays (see [4,5]). Friedman [15] studies the linear nature of the Gödel hierarchy in detail. Moreover, the Gödel hierarchy exhibits some remarkable *robustness*: we can perform the following modifications and the hierarchy remains largely unchanged:

1. Instead of the ordering via consistency strength, we can order via inclusion: Simpson claims that inclusion and consistency strength yield the same[2] Gödel

[2] Simpson mentions in [52] the caveat that e.g. PRA and WKL_0 have the same first-order strength, but the latter is strictly stronger than the former.

hierarchy as depicted in [52, Table 1]. Some exceptional (semi-natural) statements[3] do fall outside of the Gödel hierarchy based on inclusion.

2. We can replace the systems with their higher-order (eponymous but for the 'ω') counterparts. The higher-order systems are generally conservative over their second-order counterpart for (large parts of) the second-order language. Hunter's dissertation contains a number of such general results [24, Ch. 2]

Now, *if* one accepts the modifications (inclusion ordering and higher types) described in the previous two items, *then* an obvious question is where e.g. HBU fits into the (inclusion-based) Gödel hierarchy. Indeed, the Heine-Borel theorem has a central place in analysis and a rich history predating set theory (see [31]).

The answer to this question may come as a surprise: starting with the results in [41–43], Dag Normann and the author have identified a *large* number of *natural* theorems of third-order arithmetic, including HBU, forming a branch *independent* of the medium range of the Gödel hierarchy based on inclusion. Indeed, none of the systems $\Pi^1_k\text{-}\mathsf{CA}^\omega_0 + \mathsf{QF\text{-}AC}^{0,1}$ can prove HBU, while Z_2^Ω can. We stress that both $\Pi^1_k\text{-}\mathsf{CA}^\omega_0 + \mathsf{QF\text{-}AC}^{0,1}$ and HBU are part of the language of *third-order arithmetic*, i.e. expressible in the same language.

In more detail, results pertaining to 'local-global' theorems are obtained in [42]. Measure theory is studied in [43] while results pertaining to HBU and the gauge integral may be found in [41]. In this paper and [43,48], we have shown that a number of basic theorems about nets similarly fall outside of the Gödel hierarchy, including the Bolzano-Weierstrass theorem for nets ($\mathsf{BW}_{\mathsf{net}}$; see [48]) and the monotone convergence theorem for nets of continuous functions ($\mathsf{MCT}_{\mathsf{net}}$; see [43]) We recall that convergence theorems concerning nets are old and well-established, starting with Moore-Smith more than a centure ago [33,34]. Our results are summarised in Fig. 1 below.

Our results highlight a fundamental difference between second-order and higher-order arithmetic. Such differences are discussed in detail in [49, §4], based on helpful discussion with Steve Simpson, Denis Hirschfeldt, and Anil Nerode. We now discuss some the technical details concerning Fig. 1 as follows.

Remark 5.1. First of all, Z_2^Ω is placed *between* the medium and strong range, as the combination of the recursor R_2 from Gödel's T and $\exists^3$ yields a system stronger than Z_2^Ω. The system $\Pi^1_k\text{-}\mathsf{CA}^\omega_0$ does not change in the same way.

Secondly, while HBU clearly implies WKL, $\mathsf{MCT}_{\mathsf{net}}$ from [43] only implies WWKL as far as we know, and this is symbolised by the dashed line.

In conclusion, in light of the results in this paper and [41–43,48], we observe a serious challenge to the linear nature of the Gödel hierarchy, as well as Feferman's claim that the mathematics necessary for the development of physics can be formalised in relatively weak logical systems (see e.g. [41, p. 24]).

[3] There are some examples of theorems (predating HBU and [41]) that fall outside of the Gödel hierarchy *based on inclusion*, like *special cases* of Ramsey's theorem and the axiom of determinacy from set theory [22,32]. These are far less natural than e.g. Heine-Borel compactness, in our opinion.

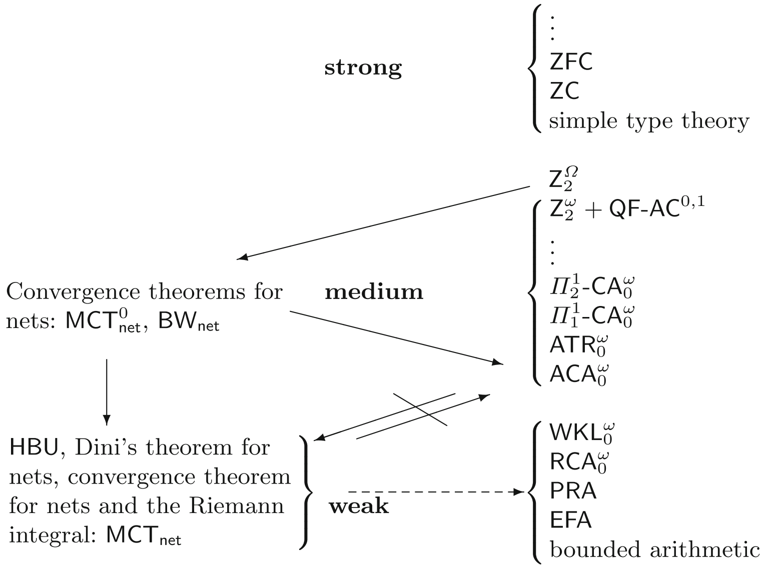


Fig. 1. The Gödel hierarchy with a side-branch for the medium range

A Technical Appendix

We provide the full definition of the system RCA_0^ω in Sect. A.1, while we list some proofs in Sect. A.2.

A.1 The Base Theory of Higher-Order Reverse Mathematics

We list all the axioms of the base theory RCA_0^ω, first introduced in [27, §2].

Definition A.1. The base theory RCA_0^ω consists of the following axioms.

(a) Basic axioms expressing that $0, 1, <_0, +_0, \times_0$ form an ordered semi-ring with equality $=_0$.
(b) Basic axioms defining the well-known Π and Σ combinators (aka K and S in [2]), which allow for the definition of λ-*abstraction*.
(c) The defining axiom of the recursor constant $\mathbf{R}_0$: For m^0 and f^1:

$$\mathbf{R}_0(f, m, 0) := m \text{ and } \mathbf{R}_0(f, m, n+1) := f(n, \mathbf{R}_0(f, m, n)). \tag{A.1}$$

(d) The *axiom of extensionality*: for all $\rho, \tau \in \mathbf{T}$, we have:

$$(\forall x^\rho, y^\rho, \varphi^{\rho \to \tau})\big[x =_\rho y \to \varphi(x) =_\tau \varphi(y)\big]. \tag{$\mathsf{E}_{\rho,\tau}$}$$

(e) The induction axiom for quantifier-free[4] formulas of L_ω.

[4] To be absolutely clear, variables (of any finite type) are allowed in quantifier-free formulas of the language L_ω: only quantifiers are banned.

(f) $\mathsf{QF\text{-}AC}^{1,0}$: The quantifier-free Axiom of Choice as in Definition A.2.

Definition A.2. The axiom $\mathsf{QF\text{-}AC}$ consists of the following for all $\sigma, \tau \in \mathbf{T}$:

$$(\forall x^\sigma)(\exists y^\tau)A(x,y) \to (\exists Y^{\sigma\to\tau})(\forall x^\sigma)A(x,Y(x)), \qquad (\mathsf{QF\text{-}AC}^{\sigma,\tau})$$

for any quantifier-free formula A in the language of L_ω.

For completeness, we list the following notational convention on finite sequences.

Notation A.3 (Finite sequences). We assume a dedicated type for 'finite sequences of objects of type ρ', namely ρ^*. Since the usual coding of pairs of numbers goes through in RCA_0^ω, we shall not always distinguish between 0 and 0^*. Similarly, we do not always distinguish between 's^ρ' and '$\langle s^\rho\rangle$', where the former is 'the object s of type ρ', and the latter is 'the sequence of type ρ^* with only element s^ρ'. The empty sequence for the type ρ^* is denoted by '$\langle\rangle_\rho$', usually with the typing omitted.

Furthermore, we denote by '$|s| = n$' the length of the finite sequence $s^{\rho^*} = \langle s_0^\rho, s_1^\rho, \dots, s_{n-1}^\rho\rangle$, where $|\langle\rangle| = 0$, i.e. the empty sequence has length zero. For sequences s^{ρ^*}, t^{ρ^*}, we denote by '$s*t$' the concatenation of s and t, i.e. $(s*t)(i) = s(i)$ for $i < |s|$ and $(s*t)(j) = t(|s|-j)$ for $|s| \le j < |s|+|t|$. For a sequence s^{ρ^*}, we define $\overline{s}N := \langle s(0), s(1), \dots, s(N-1)\rangle$ for $N^0 < |s|$. For a sequence $\alpha^{0\to\rho}$, we also write $\overline{\alpha}N = \langle \alpha(0), \alpha(1), \dots, \alpha(N-1)\rangle$ for *any* N^0. By way of shorthand, $(\forall q^\rho \in Q^{\rho^*})A(q)$ abbreviates $(\forall i^0 < |Q|)A(Q(i))$, which is (equivalent to) quantifier-free if A is.

A.2 Some Proofs

We provide the proofs of some of the above theorems. First of all, the proof of Corollary 3.7 is as follows.

Proof. We only need to prove the reverse implication. To this end, let $x_d : D \to I$ be an increasing net converging to some $x \in I$. This convergence trivially implies:

$$(\forall k \in \mathbb{N})(\exists d \in D)(|x - x_d| < \tfrac{1}{2^k}), \tag{A.2}$$

and applying $\mathsf{QF\text{-}AC}^{0,1}$ to (A.2) yields $\Phi : \mathbb{N} \to D$ such that *the sequence* $\lambda k^0.x_{\Phi(k)}$ also converges to x as $k \to \infty$. Since ADS is equivalent to the statement that a sequence in $\mathbb{R}$ has a monotone sub-sequence [28, §3], SUB_0 follows. □

Secondly, the proof of Theorem 4.2 is as follows.

Proof. For the 'vice versa' direction, one uses the usual 'interval halving technique' where $\exists^4$ is used to decide whether there is $e \in E$ such that x_e is in the relevant interval. For the other direction, fix F^3, let E be $\mathbb{N}^\mathbb{N} \to \mathbb{N}$ itself, and define '$X \preceq Y$' by $F(X) \ge_0 F(Y)$ for any X^2, Y^2. It is easy to show that $(E, \preceq)$ is a directed set. Define the net $x_e : E \to I$ by 0 if $F(e) > 0$, and 1 if $F(e) = 0$, which is increasing by definition. Hence, x_e converges, say to $y_0 \in I$, and if $y_0 > 2/3$, then there must be Y^2 such that $F(Y) = 0$, while if $y_0 < 1/3$, then $(\forall Y^2)(F(Y) > 0)$. Clearly, this yields a term of Gödel's T computing $\exists^3$. □

Thirdly, we provide the proof of Theorem 4.3.

Proof. For the 'vice versa' direction, one uses the usual 'interval halving technique' where S^2 is used to decide whether there is $d \in D$ such that x_d is in the relevant interval. For the other direction, fix f^1, let D be Baire space, and define '$h \preceq g$' by the following arithmetical formula

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\big[f(\overline{g}n) > 0 \to f(\overline{h}m) \geq f(\overline{g}n)\big],$$

for any $h, g \in D$. It is easy to show that $(D, \preceq)$ is a directed set. Define the net $x_g : D \to I$ by 0 if $(\exists n^0)(f(\overline{g}n) > 0)$, and 1 if otherwise, which is arithmetical and increasing. Hence, x_d converges, say to $y_0 \in I$, and if $y_0 > 2/3$, then there is g^1 such that $(\forall n^0)(f(\overline{g}n) = 0)$, while if $y_0 < 1/3$, then $(\forall g^1)(\exists n^0)(f(\overline{g}n) > 0)$. Clearly, this provides a term of Gödel's T computing S^2. □

Fourth, we provide the proof of Corollary 4.4

Proof. Let $(\forall f^1)\varphi(f, F^2, e^0)$ be the formula expressing that the e-th algorithm with input F^2 terminates, i.e. $\varphi(f, F, e)$ is arithmetical with type two parameters. Let D be Baire space and define '$f \preceq_D g$' by $\varphi(f, F, e) \to \varphi(g, F, e)$, which readily yields a directed set. The net $x_d : D \to \mathbb{R}$ is defined as follows: x_f is 0 if $\varphi(f, F, e)$, and 1 otherwise. This net is increasing and $\mathsf{MCT}^{\mathbb{S}}_{\mathsf{net}}$ yields a limit $y_0 \in I$; if $y_0 > 1/3$, then $\{e\}(F)$ does not terminate, and if $y < 2/3$, then $\{e\}(F)$ terminates. □

References

1. Abramsky, S., Jung, A.: Domain Theory. Handbook of Logic in Computer Science, vol. 3, pp. 1–168. Oxford University Press (1994)
2. Avigad, J., Feferman, S.: Gödel's Functional ("Dialectica") Interpretation. Handbook of Proof Theory. Studies in Logic and the Foundations of Mathematics, vol. 137, pp. 337–405 (1998)
3. Bartle, R.G.: Nets and filters in topology. Am. Math. Monthly **62**, 551–557 (1955)
4. Benacerraf, P., Putnam, H.: Philosophy of Mathematics: Selected Readings, 2nd edn. Cambridge University Press, Cambridge (1984)
5. Bernays, P.: Sur le Platonisme Dans les Mathé matiques. L'Enseignement Mathématique **34**, 52–69 (1935)
6. Bourbaki, N.: Elements of Mathematics. General Topology. Part 2. Addison-Wesley, Boston (1966)
7. Brown, A., Pearcy, C.: An Introduction to Analysis. Graduate Texts in Mathematics, vol. 154. Springer, Heidelberg (1995). https://doi.org/10.1007/978-1-4612-0787-0
8. Burgess, J.P.: Fixing Frege. Princeton Monographs in Philosophy. Princeton University Press, Princeton (2005)
9. Cousin, P.: Sur les fonctions de n variables complexes. Acta Math. **19**, 1–61 (1895)
10. Dorais, F.G.: Classical consequences of continuous choice principles from intuitionistic analysis. Notre Dame J. Form. Log. **55**(1), 25–39 (2014)
11. Dorais, F.G., Dzhafarov, D.D., Hirst, J.L., Mileti, J.R., Shafer, P.: On uniform relationships between combinatorial problems. Trans. Am. Math. Soc. **368**(2), 1321–1359 (2016)

12. Felgner, U.: Models of ZF-Set Theory. LNM, vol. 223. Springer, Heidelberg (1971). https://doi.org/10.1007/BFb0061160
13. Friedman, H.: Some systems of second order arithmetic and their use. In: Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), vol. 1, pp. 235–242 (1975)
14. Friedman, H.: Systems of second order arithmetic with restricted induction, I & II (Abstracts). J. Symb. Log. **41**, 557–559 (1976)
15. Friedman, H.: Interpretations, According to Tarski. Interpretations of Set Theory in Discrete Mathematics and Informal Thinking, The Nineteenth Annual Tarski Lectures, vol. 1, p. 42 (2007). http://u.osu.edu/friedman.8/files/2014/01/Tarski1052407-13do0b2.pdf
16. Fujiwara, M., Yokoyama, K.: A note on the sequential version of Π^1_2 statements. In: Bonizzoni, P., Brattka, V., Löwe, B. (eds.) CiE 2013. LNCS, vol. 7921, pp. 171–180. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-39053-1_20
17. Fujiwara, M., Higuchi, K., Kihara, T.: On the strength of marriage theorems and uniformity. MLQ Math. Log. Q. **60**(3), 136–153 (2014)
18. Gandy, R.: General recursive functionals of finite type and hierarchies of functions. Ann. Fac. Sci. Univ. Clermont-Ferrand **35**, 5–24 (1967)
19. Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S.: A Compendium of Continuous Lattices. Springer, Heidelberg (1980). https://doi.org/10.1007/978-3-642-67678-9
20. Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S.: Continuous Lattices and Domains. Encyclopedia of Mathematics and its Applications, vol. 93. Cambridge University Press (2003)
21. Goubault-Larrecq, J.: Non-Hausdorff Topology and Domain Theory. New Mathematical Monographs, vol. 22. Cambridge University Press, Cambridge (2013)
22. Hirschfeldt, D.R.: Slicing the Truth. Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, vol. 28. World Scientific Publishing (2015)
23. Hirst, J.L., Mummert, C.: Reverse mathematics and uniformity in proofs without excluded middle. Notre Dame J. Form. Log. **52**(2), 149–162 (2011)
24. Hunter, J.: Higher-Order Reverse Topology. ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–The University of Wisconsin - Madison (2008)
25. Kelley, J.L.: General Topology. Springer, Heidelberg (1975). Reprint of the 1955 edition, Graduate Texts in Mathematics, No. 27
26. Koellner, P.: Large Cardinals and Determinacy. The Stanford Encyclopedia of Philosophy (2014). https://plato.stanford.edu/archives/spr2014/entries/large-cardinals-determinacy/
27. Kohlenbach, U.: Higher order reverse mathematics. Reverse Mathematics (2001). Lect. Notes Log., vol. 21, ASL, 2005, pp. 281–295
28. Kreuzer, A.P.: Primitive recursion and the chain antichain principle. Notre Dame J. Form. Log. **53**(2), 245–265 (2012)
29. Li, G., Ru, J., Wu, G.: Rudin's lemma and reverse mathematics. Ann. Jpn. Assoc. Philos. Sci. **25**, 57–66 (2017)
30. Longley, J., Normann, D.: Higher-Order Computability. Theory and Applications of Computability. Springer, Heidelberg (2015). https://doi.org/10.1007/978-3-662-47992-6
31. Medvedev, F.A.: Scenes from the History of Real Functions. Science Networks. Historical Studies, vol. 7. Birkhœuser Verlag, Basel (1991)
32. Montalbán, A., Shore, R.A.: The limits of determinacy in second order arithmetic. Proc. Lond. Math. Soc. (3) **104**(2), 223–252 (2012)

33. Moore, E.H.: Definition of limit in general integral analysis. Proc. Natl. Acad. Sci. U.S.A **1**(12), 628–632 (1915)
34. Moore, E., Smith, H.: A general theory of limits. Am. J. Math. **44**, 102–121 (1922)
35. Moore, E.: General Analysis. Part I. The Algebra of Matrices, vol. 1. Memoirs of the American Philosophical Society, Philadelphia (1935)
36. Muldowney, P.: A General Theory of Integration in Function Spaces, Including Wiener and Feynman Integration, vol. 153. Longman Scientific & Technical, Harlow, Wiley (1987)
37. Mummert, C., Simpson, S.G.: Reverse mathematics and Π_2^1 comprehension. Bull. Symb. Log. **11**(4), 526–533 (2005)
38. Mummert, C.: On the Reverse Mathematics of General Topology. ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–The Pennsylvania State University (2005)
39. Mummert, C.: Reverse mathematics of MF spaces. J. Math. Log. **6**(2), 203–232 (2006)
40. Mummert, C., Stephan, F.: Topological aspects of poset spaces. Michigan Math. J. **59**(1), 3–24 (2010)
41. Normann, D., Sanders, S.: On the mathematical and foundational significance of the uncountable. J. Math. Log. (2018). https://doi.org/10.1142/S0219061319500016
42. Normann, D., Sanders, S.: Pincherle's theorem in Reverse Mathematics and computability theory (2018, submitted). arXiv: https://arxiv.org/abs/1808.09783
43. Normann, D., Sanders, S.: Representations in measure theory (2019). arXiv: https://arxiv.org/abs/1902.02756
44. Pu, P.M., Liu, Y.M.: Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. J. Math. Anal. Appl. **76**(2), 571–599 (1980)
45. Sacks, G.E.: Higher Recursion Theory. Perspectives in Mathematical Logic. Springer, Heidelberg (1990)
46. Sakamoto, N., Yamazaki, T.: Uniform versions of some axioms of second order arithmetic. MLQ Math. Log. Q. **50**(6), 587–593 (2004)
47. Sanders, S.: Reverse Mathematics of topology: dimension, paracompactness, and splittings, p. 17 (2018). arXiv: https://arxiv.org/abs/1808.08785
48. Sanders, S.: Nets and Reverse Mathematics: Initial Results. Proceedings of CiE19. LNCS, p. 12. Springer, Heidelberg (2019, to appear)
49. Sanders, S.: Splittings and disjunctions in Reverse Mathematics. Notre Dame J. Formal Log. 18 (2019, to appear). arXiv: https://arxiv.org/abs/1805.11342
50. Simpson, S.G. (ed.): Reverse Mathematics (2001). Lecture Notes in Logic, vol. 21, ASL, La Jolla, CA (2005)
51. Simpson, S.G. (ed.): Subsystems of Second Order Arithmetic, 2nd edn. Perspectives in Logic, CUP (2009)
52. Simpson, S.G. (ed.): The Gödel hierarchy and reverse mathematics., Kurt Gödel. Essays for his centennial, pp. 109–127 (2010)
53. Stillwell, J.: Reverse Mathematics, Proofs from the Inside Out. Princeton University Press, Princeton (2018)
54. Swartz, C.: Introduction to Gauge Integrals. World Scientific (2001)
55. Turing, A.: On computable numbers, with an application to the Entscheidungsproblem. Proc. Lond. Mat. Soc. **42**, 230–265 (1936)
56. Vietoris, L.: Stetige mengen. Monatsh. Math. Phys. **31**(1), 173–204 (1921). (German)
57. Wang, H.: Eighty years of foundational studies. Dialectica **12**, 466–497 (1958)
58. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)

Cut Elimination for the Weak Modal Grzegorczyk Logic via Non-well-Founded Proofs

Yury Savateev[1(✉)] and Daniyar Shamkanov[1,2]

[1] National Research University Higher School of Economics, Moscow, Russia
estlladon@gmail.com
[2] Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow, Russia

Abstract. We present a sequent calculus for the weak Grzegorczyk logic Go allowing non-well-founded proofs and obtain the cut-elimination theorem for it by constructing a continuous cut-elimination mapping acting on these proofs.

Keywords: Non-well-founded proofs · Weak Grzegorczyk logic · Logic Go · Cut-elimination · Cyclic proofs

1 Introduction

The logic Go, also known as the weak Grzegorzyk logic, is the smallest normal modal logic containing the axiom K and the axioms $\Box A \to \Box\Box A$ and $\Box(\Box(A \to \Box A) \to A) \to \Box A$. A survey of results on Go can be found in [5]. The logic is sound and complete with respect to the class of transitive frames with no proper clusters and infinite ascending chains [2], and it is a proper sublogic of both Gödel-Löb logic GL (also known as provability logic) and Grzegorzyk logic Grz.

Recently a new proof-theoretic presentation for the logic GL in the form of a sequent calculus allowing non-well-founded proofs was given in [4,10]. Later, the same ideas were applied to the modal Grzegorczyk logic Grz in [8,9], where it allowed to prove several proof-theoretic properties of this logic syntactically.

In this paper we use the same approach for the logic Go. We consider a sequent calculus allowing non-well-founded proofs Go_∞ and present the cut-elimination theorem for it. We consider the set of non-well-founded proofs of Go_∞ and various sets of operations acting on theses proofs as ultrametric spaces and define our cut-elimination operator using the Prieß-Crampe fixed-point theorem (see [7]), which is a strengthening of the Banach's theorem.

In [3] Goré and Ramanayake remark that their method for cut elimination for the logic Go is more complex than the similar methods for the logics GL and Grz. This difference in complexity seems to be present in our approach as well. The proofs of cut-elimination for Go_∞ and Grz_∞ turn out to be almost the same, but the system Go_∞ itself seems to be more complex (it includes rules of

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 569–583, 2019.
https://doi.org/10.1007/978-3-662-59533-6_34

arbitrary arity, where Grz_∞ has at most binary) and the translation from Go_∞ to the standard system seems to require bigger induction measure.

2 Preliminaries

In this section we recall the weak Grzegorczyk logic Go and define an ordinary sequent calculus for it.

Formulas of Go, denoted by A, B, C, are built up as follows:

$$A ::= \bot \mid p \mid (A \to A) \mid \Box A\ ,$$

where p stands for atomic propositions.

The Hilbert-style axiomatization of Go is given by the following axioms and inference rules:

Axioms:

(i) Boolean tautologies;
(ii) $\Box(A \to B) \to (\Box A \to \Box B)$;
(iii) $\Box A \to \Box\Box A$;
(iv) $\Box(\Box(A \to \Box A) \to A) \to \Box A$.

Rules: modus ponens, $A / \Box A$.

Now we define an ordinary sequent calculus for Go. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. For a multiset of formulas $\Gamma = A_1, \dots, A_n$, we set $\Box\Gamma := \Box A_1, \dots, \Box A_n$.

The system $\mathsf{Go}_{\mathsf{Seq}}$, is defined by the following initial sequents and inference rules:

$$\Gamma, A \Rightarrow A, \Delta\ , \qquad \Gamma, \bot \Rightarrow \Delta\ ,$$

$$\to_{\mathsf{L}}\ \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}\ , \qquad \to_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\ ,$$

$$\Box_{\mathsf{Go}}\ \frac{\Box\Pi, \Pi, \Box(A \to \Box A) \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\ .$$

The cut rule has the form

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ ,$$

where A is called the *cut formula* of the given inference.

Lemma 21. *$\mathsf{Go}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$ if and only if $\mathsf{Go} \vdash \bigwedge \Gamma \to \bigvee \Delta$.*

Proof. Standard transformations of proofs.

Theorem 22. *If $\mathsf{Go}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{Go}_{\mathsf{Seq}} \vdash \Gamma \Rightarrow \Delta$.*

A syntactic cut-elimination for Go was obtained by Goré and Ramanayake in [3]. In this paper, we will give another proof of this cut-elimination theorem.

3 Non-well-Founded Proofs

Now we define a sequent calculus for Go allowing non-well-founded proofs.

Inference rules and initial sequents of the sequent calculus Go_∞ have the following form:

$$\Gamma, p \Rightarrow p, \Delta\ , \qquad \Gamma, \bot \Rightarrow \Delta\ ,$$

$$\rightarrow_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}\ , \qquad \rightarrow_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}\ ,$$

$$\Box \frac{\Box\Pi, \Pi \Rightarrow A_1, \dots, A_n, \Box A_1, \dots \Box A_n \qquad \Box\Pi, \Pi \Rightarrow A_1 \qquad \dots \qquad \Box\Pi, \Pi \Rightarrow A_n}{\Gamma, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Delta}\ .$$

The system $\mathsf{Go}_\infty + \mathsf{cut}$ is defined by adding the rule (cut) to the system Go_∞.

We will refer to all but the leftmost premises of the rule $(\Box)$ as "right".

An ∞*–proof* in Go_∞ ($\mathsf{Go}_\infty + \mathsf{cut}$) is a (possibly infinite) tree whose nodes are marked by sequents and whose leaves are marked by initial sequents and that is constructed according to the rules of the sequent calculus. In addition, every infinite branch in an ∞–proof must pass through a right premise of the rule $(\Box)$ infinitely many times. A sequent $\Gamma \Rightarrow \Delta$ is *provable* in Go_∞ ($\mathsf{Go}_\infty + \mathsf{cut}$) if there is an ∞–proof in Go_∞ ($\mathsf{Go}_\infty + \mathsf{cut}$) with the root marked by $\Gamma \Rightarrow \Delta$.

For a multiset of formulas $\Gamma = A_1, \dots, A_n$, we set

$$\boxtimes\Gamma := A_1, \dots, A_n, \Box A_1, \dots, \Box A_n.$$

Then the rule $(\Box)$ can be written as

$$\Box \frac{\boxtimes\Pi \Rightarrow \boxtimes(A_1, \dots, A_n) \qquad \boxtimes\Pi \Rightarrow A_1 \qquad \dots \qquad \boxtimes\Pi \Rightarrow A_n}{\Gamma, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Delta}\ .$$

Let us construct an ∞–proof of the sequent $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \Rightarrow \Box p$. We will do it by defining several finite proof parts and show how they connect together to form the full ∞-proof. A proof part is a tree with non-axiom leaves marked by numbered labels that indicate that another part must be attached there to form a correct proof. For proofs π, ρ and τ, and a proof part α with three non-axiom leaves by

$$\frac{\pi \qquad \rho \qquad \tau}{\alpha}$$

we will denote the proof obtained by attaching the proofs π, ρ and τ to the leaves with labels (1), (2), and (3) respectively.

Let $F = \Box(p \rightarrow \Box p) \rightarrow p$ and let ψ be the following proof:

$$\rightarrow_{\mathsf{R}} \frac{\dfrac{\mathsf{Ax}}{\boxtimes F, p \Rightarrow p, \Box p, \Box p, \Box(p \rightarrow \Box p)}}{\boxtimes F \Rightarrow \boxtimes(p, p \rightarrow \Box p)}\ .$$

Let ϕ be the following proof part:

$$\to_{\mathsf{L}}\dfrac{\overset{\mathsf{Ax}}{\Box F, p \Rightarrow p, \Box p} \qquad \Box\dfrac{\psi \qquad \to_{\mathsf{R}}\dfrac{\overset{(1)}{F, p, \Box F \Rightarrow \Box p}}{\boxtimes F \Rightarrow p \to \Box p} \qquad \overset{(2)}{\boxtimes F \Rightarrow p}}{\Box F \Rightarrow \Box(p \to \Box p), \Box p, p}}{\Box F, \Box(p \to \Box p) \to p \Rightarrow p, \Box p}$$

Let θ be the following proof part:

$$\to_{\mathsf{L}}\dfrac{\overset{\mathsf{Ax}}{\Box F, p \Rightarrow p} \qquad \Box\dfrac{\to_{\mathsf{R}}\dfrac{\Box\dfrac{\psi \qquad \to_{\mathsf{R}}\dfrac{\overset{(1)}{F, p, \Box F \Rightarrow \Box p}}{\boxtimes F \Rightarrow p \to \Box p} \qquad \overset{(2)}{\boxtimes F \Rightarrow p}}{p, F, \Box F \Rightarrow \Box p, \Box(p \to \Box p)}}{\boxtimes F \Rightarrow \boxtimes(p \to \Box p)} \qquad \to_{\mathsf{R}}\dfrac{\overset{(3)}{F, p, \Box F \Rightarrow \Box p}}{\boxtimes F \Rightarrow p \to \Box p}}{\Box F \Rightarrow \Box(p \to \Box p), p}}{\Box F, \Box(p \to \Box p) \to p \Rightarrow p}$$

An ∞–proof of the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow \Box p$ can be constructed as follows:

$$\Box\dfrac{\dfrac{\Box\dfrac{\overset{\vdots}{\phi} \qquad \overset{\vdots}{\theta}}{F, p, \Box F \Rightarrow \Box p} \qquad \overset{\vdots}{\theta}}{\phi} \qquad \dfrac{\Box\dfrac{\overset{\vdots}{\phi} \qquad \overset{\vdots}{\theta}}{F, p, \Box F \Rightarrow \Box p} \qquad \overset{\vdots}{\theta} \qquad \Box\dfrac{\overset{\vdots}{\phi} \qquad \overset{\vdots}{\theta}}{F, p, \Box F \Rightarrow \Box p}}{\theta}}{\Box(\Box(p \to \Box p) \to p) \Rightarrow \Box p}\ .$$

The *n-fragment* of an ∞–proof is a finite tree obtained from the ∞–proof by cutting every branch at the nth from the root right premise of a $\Box$-rule. The 1-fragment of an ∞–proof is also called its *main fragment*. The *local height of an ∞–proof* π denoted by $|\pi|$ is the length of the longest branch in its main fragment. An ∞–proof only consisting of an initial sequent has height 0.

We denote the set of all ∞-proofs in the system $\mathsf{Go}_\infty + \mathsf{cut}$ by $\mathcal{P}$.

For $\pi, \tau \in \mathcal{P}$, we write $\pi \sim_n \tau$ if n-fragments of these ∞-proofs coincide. For any $\pi, \tau \in \mathcal{P}$, we also set $\pi \sim_0 \tau$.

Now we define two translations that connect ordinary and non-well-founded sequent calculi for the logic Go.

Lemma 31. *We have $\mathsf{Go}_\infty \vdash \Gamma, A \Rightarrow A, \Delta$ for any sequent $\Gamma \Rightarrow \Delta$ and any formula A.*

Proof. Standard induction on the structure of A.

Lemma 32. *We have $\mathsf{Go}_\infty \vdash \Box(\Box(A \to \Box A) \to A) \Rightarrow \Box A$ for any formula A.*

Proof. Consider an example of ∞–proof for the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow \Box p$ given above. We transform this example into an ∞–proof for $\Box(\Box(A \to \Box A) \to A) \Rightarrow A$ by replacing p with A and adding required ∞–proofs instead of initial sequents using Lemma 31.

Recall that an inference rule is called admissible (in a given proof system) if, for any instance of the rule, the conclusion is provable whenever all premises are provable.

Lemma 33. *The rule*

$$\mathsf{weak}\ \frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}$$

is admissible in the systems $\mathsf{Go_{Seq}}$ *and* $\mathsf{Go}_\infty + \mathsf{cut}$*.*

The rule

$$\mathsf{ctr}\ \frac{\Gamma, \Pi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}$$

is admissible in the system $\mathsf{Go_{Seq}}$*.*

Proof. Standard induction on the structure (local height) of a proof of $\Gamma \Rightarrow \Delta$.

Theorem 34. *If* $\mathsf{Go_{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathsf{Go}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$*.*

Proof. Assume π is a proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{Go_{Seq}} + \mathsf{cut}$. By induction on the size of π we prove $\mathsf{Go}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$.

If $\Gamma \Rightarrow \Delta$ is an initial sequent of $\mathsf{Go_{Seq}} + \mathsf{cut}$, then it is provable in $\mathsf{Go}_\infty + \mathsf{cut}$ by Lemma 31. Otherwise, consider the last application of an inference rule in π.

The only non-trivial case is when the proof π has the form

$$\Box_{\mathsf{Go}}\ \frac{\begin{array}{c}\pi' \\ \Box\Pi, \Pi, \Box(A \to \Box A) \Rightarrow A\end{array}}{\Sigma, \Box\Pi \Rightarrow \Box A, \Lambda}\ ,$$

where $\Sigma, \Box\Pi = \Gamma$ and $\Box A, \Lambda = \Delta$. By the induction hypothesis there is an ∞–proof ξ of $\boxtimes\Pi, \Box(A \to \Box A) \Rightarrow A$ in $\mathsf{Go}_\infty + \mathsf{cut}$.

The required ∞–proof for $\Sigma, \Box\Pi \Rightarrow \Box A, \Delta$ has the form:

$$
\begin{prooftree}
\AxiomC{ξ}
\noLine
\UnaryInfC{$\boxtimes\Pi, \Box(A \to \Box A) \Rightarrow A$}
\LeftLabel{$\to_{\mathsf{R}}$}
\UnaryInfC{$\boxtimes\Pi \Rightarrow F$}
\LeftLabel{weak}
\UnaryInfC{$\boxtimes\Pi \Rightarrow \boxtimes F$}
\AxiomC{ξ}
\noLine
\UnaryInfC{$\boxtimes\Pi, \Box(A \to \Box A) \Rightarrow A$}
\LeftLabel{$\to_{\mathsf{R}}$}
\UnaryInfC{$\boxtimes\Pi \Rightarrow F$}
\LeftLabel{$\Box$}
\BinaryInfC{$\Sigma, \Box\Pi \Rightarrow \Box F, \Box A, \Lambda$}
\AxiomC{χ}
\noLine
\UnaryInfC{$\Box F \Rightarrow \Box A$}
\LeftLabel{weak}
\UnaryInfC{$\Sigma, \Box\Pi, \Box F \Rightarrow \Box A, \Lambda$}
\LeftLabel{cut}
\BinaryInfC{$\Sigma, \Box\Pi \Rightarrow \Box A, \Lambda$}
\end{prooftree}
$$

where $F = \Box(A \to \Box A) \to A$ and χ is an ∞–proof of $\Box F \Rightarrow \Box A$, which exists by Lemma 32.

The cases of other inference rules being last in π are straightforward, so we omit them.

For a sequent $\Gamma \Rightarrow \Delta$, let $Sub(\Gamma \Rightarrow \Delta)$ be the set of all subformulas of the formulas from $\Gamma \cup \Delta$. For a finite set of formulas Λ, let Λ^* be the set $\{A \to \Box A \mid A \in \Lambda\}$.

Theorem 35. *If* $\mathsf{Go}_\infty \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Go}_{\mathsf{Seq}} \vdash \Gamma \Rightarrow \Delta$.

Proof. First we will prove a more general statement: if $\mathsf{Go}_\infty \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{Go}_{\mathsf{Seq}} \vdash \Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \Gamma \Rightarrow \Delta$ for any finite sets of formulas Λ_1, Λ_2, and Ω such that $\Lambda_2 \subset \Lambda_1$.

Assume π is an ∞–proof of the sequent $\Gamma \Rightarrow \Delta$ in Go_∞ and Λ_1, Λ_2, and Ω are finite sets of formulas, such that $\Lambda_2 \subset \Lambda_1$.

We prove that $\mathsf{Go}_{\mathsf{Seq}} \vdash \Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \Gamma \Rightarrow \Delta$ by quadruple induction: by induction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta)\backslash\Lambda_1$ with a subinduction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta)\backslash\Lambda_2$, subinduction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta)\backslash\Omega$, and with subinduction on $|\pi|$.

If $|\pi| = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. We see that the sequent $\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \Gamma \Rightarrow \Delta$ is an initial sequent and it is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Otherwise, consider the last application of an inference rule in π.

The cases when this application is of rules $\to_{\mathsf{R}}$ or $\to_{\mathsf{L}}$ are trivial, since these rules are common to both systems and the weakening rule is admissible in $\mathsf{Go}_{\mathsf{Seq}}$.

Suppose that π has the form

$$\frac{\begin{matrix}\pi_0 \\ \boxtimes\Pi \Rightarrow \boxtimes(A_1, \ldots, A_n)\end{matrix} \qquad \begin{matrix}\pi_1 \\ \boxtimes\Pi \Rightarrow A_1\end{matrix} \qquad \ldots \qquad \begin{matrix}\pi_n \\ \boxtimes\Pi \Rightarrow A_n\end{matrix}}{\Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma}$$

where $\Phi, \Box\Pi = \Gamma$ and $\Box A_1, \ldots, \Box A_n, \Sigma = \Delta$.

Subcase 1: For some i, we have $A_i \notin \Lambda_1$. We have that the number of elements in $Sub(\Box\Pi, \Pi \Rightarrow A)\backslash(\Lambda_1 \cup \{A_i\})$ is strictly less than the number of elements in $Sub(\Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma)\backslash\Lambda_1$. By the induction hypothesis for $\Lambda_1 \cup \{A_i\}$, Λ_1, $\varnothing$ and π_i, the sequent $\Box(\Lambda_1^*), \Box(A_i \to \Box A_i), \Lambda_1^*, \Box\Pi, \Pi \Rightarrow A$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Then we have

$$\Box_{\mathsf{Go}}\,\frac{\Box(\Lambda_1^*), \Box(A_i \to \Box A_i), \Lambda_1^*, \Box\Pi, \Pi \Rightarrow A_i}{\Lambda_2^*, \Box(\Lambda_1^*), \boxtimes\Omega, \Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_i, \ldots, \Box A_n, \Sigma}\,.$$

Subcase 2: For all i, we have $A_i \in \Lambda_1$, but there is i, such that $A_i \notin \Lambda_2$. We have that the number of elements in $Sub(\Box\Pi, \Pi \Rightarrow A_i)\backslash\Lambda_1$ is strictly less than the number of elements in $Sub(\Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma)\backslash\Lambda_2$. By the induction hypothesis for Λ_1, Λ_1, $\varnothing$ and π_i, the sequent $\Box(\Lambda_1^*), \Lambda_1^*, \Box\Pi, \Pi \Rightarrow A_i$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Then we have

$$\Box_{\mathsf{Go}}\,\frac{\mathsf{weak}\,\dfrac{\Box(\Lambda_1^*), \Lambda_1^*, \Box\Pi, \Pi \Rightarrow A_i}{\Box(\Lambda_1^*), \Lambda_1^*, \Box\Pi, \Pi, \Box(A_i \to \Box A_i) \Rightarrow A_i}}{\Lambda_2^*, \Box(\Lambda_1^*), \boxtimes\Omega, \Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_i, \ldots, \Box A_n, \Sigma}\,.$$

Subcase 3: For all i, we have $A_i \in \Lambda_2 \subset \Lambda_1$, but there is a formula F in Π, such that $F \notin \Omega$. We have that the number of elements in $Sub(\Box\Pi, \Pi \Rightarrow A_1)\backslash(\Omega \cup \Pi)$ is strictly less than the number of elements in $Sub(\Phi, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma)\backslash\Omega$. By the induction hypothesis for Λ_1, Λ_1, $\Omega \cup \Pi$ and π_1, the sequent $\Box(\Lambda_1^*), \Lambda_1^*, \boxtimes\Omega, \boxtimes(\Pi\backslash\Omega), \boxtimes\Pi \Rightarrow A_1$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Then we have

$$\dfrac{\dfrac{\Box(\Lambda_1^*), \Lambda_1^*, \boxtimes\Omega, \boxtimes(\Pi\backslash\Omega), \boxtimes\Pi \Rightarrow A_1}{\Box(\Lambda_1^*), \Lambda_1^*, \boxtimes\Omega, \boxtimes(\Pi\backslash\Omega), \boxtimes\Pi, \Box(A \to \Box A) \Rightarrow A_1}\,\mathsf{weak}}{\dfrac{\Lambda_2^*, \Box(\Lambda_1^*), \Box\Omega, \Omega, \Phi, \Box(\Pi\backslash\Omega), \Box(\Pi\backslash\Omega), \Box(\Pi \cap \Omega) \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma}{\Lambda_2^*, \Box(\Lambda_1^*), \boxtimes\Omega, \Phi, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma}\,\mathsf{ctr}}\,\Box_{\mathsf{Go}} .$$

Subcase 4: For all i, we have $A_i \in \Lambda_2 \subset \Lambda_1$ and $\Pi \subset \Omega$. We see that $|\pi_0| < |\pi|$. By the induction hypothesis for Λ_1, Λ_2, Ω and π_0 the sequent $\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \boxtimes\Pi \Rightarrow \boxtimes(A_1, \dots, A_n)$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Then we have

$$\dfrac{\dfrac{\mathsf{Ax}}{\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \boxtimes\Pi, \Box A_1 \Rightarrow \boxtimes(A_1, \dots, A_n)} \qquad \Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \boxtimes\Pi \Rightarrow \boxtimes(A_1, \dots, A_n)}{\dfrac{\Box(\Lambda_1^*), \Lambda_2^*, A_1 \to \Box A_1, \boxtimes\Omega, \boxtimes\Pi \Rightarrow \Box A_1, \boxtimes(A_2, \dots, A_n)}{\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \boxtimes\Pi \Rightarrow \Box A_1, \boxtimes(A_2, \dots, A_n)}\,\mathsf{ctr}}\,\to_L$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\Box(\Lambda_1^*), \Lambda_2^*, A_n \to \Box A_n, \boxtimes\Omega, \boxtimes\Pi \Rightarrow \Box A_1, \dots, \Box A_n}{\Box(\Lambda_1^*), \Lambda_2^*, \Box\Omega, \Omega\backslash\Pi, \Pi, \Box\Pi, \Pi \Rightarrow \Box A_1, \dots, \Box A_n}\,\mathsf{ctr}}{\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n}\,\mathsf{ctr}}{\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes\Omega, \Phi, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma}\,\mathsf{weak} .$$

4 Ultrametric Spaces

In this section we recall basic notions of the theory of ultrametric spaces (cf. [11]) and consider several examples concerning ∞-proofs.

An *ultrametric space* (M, d) is a metric space that satisfies a stronger version of the triangle inequality: for any $x, y, z \in M$, $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$.

For $x \in M$ and $r \in (0, +\infty)$, the set $B_r(x) = \{y \in M \mid d(x, y) \leqslant r\}$ is called the *closed ball* with *center* x and *radius* r. An ultrametric space (M, d) is called *spherically complete* if each descending sequence of closed balls

$$B_{r_0}(x_0) \supset B_{r_1}(x_1) \supset B_{r_2}(x_2) \supset \dots$$

has a common point. We recall that a metric space (M, d) is *complete* if any descending sequence of closed balls, with radii tending to 0, has a common point.

In an ultrametric space (M, d), a function $f\colon M \to M$ is called *(strictly) contractive* if $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$.

Theorem 41 (Prieß-Crampe [7], Petalas and Vidalis [6]). *Let (M, d) be a non-empty spherically complete ultrametric space. Then every strictly contractive mapping $f\colon M \to M$ has a unique fixed-point.*

Now consider the set $\mathcal{P}$ of all ∞-proofs of the system $\mathsf{Go}_\infty + \mathsf{cut}$. We can define an ultrametric $d_\mathcal{P}: \mathcal{P} \times \mathcal{P} \to [0,1]$ on $\mathcal{P}$ by putting

$$d_\mathcal{P}(\pi,\tau) = \inf\{\frac{1}{2^n} \mid \pi \sim_n \tau\}.$$

We see that $d_\mathcal{P}(\pi,\tau) \leqslant 2^{-n}$ if and only if $\pi \sim_n \tau$. Thus, the ultrametric $d_\mathcal{P}$ can be considered as a measure of similarity between ∞-proofs.

Proposition 42. *The ultrametric space* $(\mathcal{P}, d_\mathcal{P})$ *is complete.*

In an ultrametric space (M,d), a function $f: M \to M$ is called *non-expansive* if $d(f(x), f(y)) \leqslant d(x,y)$ for all $x,y \in M$. For ultrametric spaces (M, d_M) and (N, d_N), the Cartesian product $M \times N$ can be also considered as an ultrametric space with the metric $d_{M\times N}((x_1,y_1),(x_2,y_2)) = \max\{d_M(x_1,x_2), d_N(y_1,y_2)\}$.

Let us consider another example. For $m \in \mathbb{N}$, let $\mathcal{F}_m$ denote the set of all non-expansive functions from $\mathcal{P}^m$ to $\mathcal{P}$. Note that any function $\mathsf{u}: \mathcal{P}^m \to \mathcal{P}$ is non-expansive if and only if for any tuples $\vec{\pi}$ and $\vec{\pi}'$, and any $n \in \mathbb{N}$ we have

$$\pi_1 \sim_n \pi'_1, \ldots, \pi_m \sim_n \pi'_m \Rightarrow \mathsf{u}(\vec{\pi}) \sim_n \mathsf{u}(\vec{\pi}').$$

Now we introduce an ultrametric for $\mathcal{F}_m$. For $\mathsf{a}, \mathsf{b} \in \mathcal{F}_m$, we write $\mathsf{a} \sim_{n,k} \mathsf{b}$ if $\mathsf{a}(\vec{\pi}) \sim_n \mathsf{b}(\vec{\pi})$ for any $\vec{\pi} \in \mathcal{P}^m$ and, in addition, $\mathsf{a}(\vec{\pi}) \sim_{n+1} \mathsf{b}(\vec{\pi})$ whenever $\sum_{i=1}^{m} |\pi_i| < k$.[1] An ultrametric l_m on $\mathcal{F}_m$ is defined by

$$l_m(\mathsf{a},\mathsf{b}) = \frac{1}{2}\inf\{\frac{1}{2^n} + \frac{1}{2^{n+k}} \mid \mathsf{a} \sim_{n,k} \mathsf{b}\}.$$

We see that $l_m(\mathsf{a},\mathsf{b}) \leqslant 2^{-n-1} + 2^{-n-k-1}$ if and only if $\mathsf{a} \sim_{n,k} \mathsf{b}$.

Notice that any operator $\mathsf{U}: \mathcal{F}_m \to \mathcal{F}_m$ is strictly contractive if and only if for any $\mathsf{a}, \mathsf{b} \in \mathcal{F}_m$, and any $n,k \in \mathbb{N}$ we have

$$\mathsf{a} \sim_{n,k} \mathsf{b} \Rightarrow \mathsf{U}(\mathsf{a}) \sim_{n,k+1} \mathsf{U}(\mathsf{b}).$$

Proposition 43. *Every strictly contractive mapping* $\mathsf{U}: \mathcal{F}_m \to \mathcal{F}_m$ *has a unique fixed-point.*

This proposition can be derived from Theorem 41 by proving that the space (F_m, l_m) is spherically complete. However there is a direct proof of this result in order to show an explicit construction of the required fixed-point (See [9]).

[1] This definition is inspired by [1, Subsect. 2.1]. It reveals our intention to construct mappings on the set of ∞-proofs by co-induction with subinduction on the sum of local heights of the arguments.

5 Admissible Rules and Mappings

In this section, for the system $\mathsf{Go}_\infty + \mathsf{cut}$, we state admissibility of auxiliary inference rules, which will be used in the proof of the cut-elimination theorem.

Recall that the set $\mathcal{P}$ of all ∞-proofs of the system $\mathsf{Go}_\infty + \mathsf{cut}$ can be considered as an ultrametric space with the metric $d_\mathcal{P}$.

By $\mathcal{P}_n$ we denote the set of all ∞-proofs that do not contain applications of the cut rule in their n-fragments. We also set $\mathcal{P}_0 = \mathcal{P}$.

A mapping $\mathsf{u} : \mathcal{P}^m \to \mathcal{P}$ is called *adequate* if for any $n \in \mathbb{N}$ we have $\mathsf{u}(\pi_1, \dots, \pi_m) \in \mathcal{P}_n$, whenever $\pi_i \in \mathcal{P}_n$ for all $i \leqslant n$.

In $\mathsf{Go}_\infty + \mathsf{cut}$, we call a single-premise inference rule *strongly admissible* if there is a non-expansive adequate mapping $\mathsf{u} \colon \mathcal{P} \to \mathcal{P}$ that maps any ∞-proof of the premise of the rule to an ∞-proof of the conclusion. The mapping u must also satisfy one additional condition: $|\mathsf{u}(\pi)| \leqslant |\pi|$ for any $\pi \in \mathcal{P}$.

In the following lemma, non-expansive mappings are defined in a standard way by induction on the local heights of ∞-proofs for the premises. So we omit further details.

Lemma 51. *For any finite multisets of formulas Π and Σ, formulas A and B, and atomic proposition p the inference rules*

$$\mathsf{wk}_{\Pi;\Sigma}\ \frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}$$

$$\mathsf{li}_{A\to B}\ \frac{\Gamma, A \to B \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta} \qquad \mathsf{ri}_{A\to B}\ \frac{\Gamma, A \to B \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\mathsf{i}_{A\to B}\ \frac{\Gamma \Rightarrow A \to B, \Delta}{\Gamma, A \Rightarrow B, \Delta} \qquad \mathsf{i}_{\bot}\ \frac{\Gamma \Rightarrow \bot, \Delta}{\Gamma \Rightarrow \Delta}$$

$$\mathsf{acl}_p\ \frac{\Gamma, p, p \Rightarrow \Delta}{\Gamma, p \Rightarrow \Delta} \qquad \mathsf{acr}_p\ \frac{\Gamma \Rightarrow p, p, \Delta}{\Gamma \Rightarrow p, \Delta}$$

are strongly admissible in $\mathsf{Go}_\infty + \mathsf{cut}$.

Let us also define the mapping $\mathsf{clip} \colon \mathcal{P} \to \mathcal{P}$. Consider an ∞-proof π. If the last rule application in π is not of the rule $(\Box)$ then we put $\mathsf{clip}(\pi) = \pi$. If the ∞-proof π has the form

$$\frac{\begin{matrix}\pi_0 \\ \boxtimes\Pi \Rightarrow \boxtimes(A_1, \dots, A_n)\end{matrix} \qquad \begin{matrix}\pi_1 \\ \boxtimes\Pi \Rightarrow A_1\end{matrix} \qquad \dots \qquad \begin{matrix}\pi_n \\ \boxtimes\Pi \Rightarrow A_n\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Delta}$$

we define $\mathsf{clip}(\pi)$ to be

$$\frac{\begin{matrix}\pi_0 \\ \boxtimes\Pi \Rightarrow \boxtimes(A_1, \dots, A_n)\end{matrix} \qquad \begin{matrix}\pi_1 \\ \boxtimes\Pi \Rightarrow A_1\end{matrix} \qquad \dots \qquad \begin{matrix}\pi_n \\ \boxtimes\Pi \Rightarrow A_n\end{matrix}}{\Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n}\ .$$

Clearly this mapping is non-expansive, adequate, and $|\mathsf{clip}(\pi)| \leqslant |\pi|$ for any $\pi \in \mathcal{P}$.

6 Cut Elimination

In this section, we construct a continuous function from $\mathcal{P}$ to $\mathcal{P}$, which maps any ∞-proof of the system $\mathsf{Go}_\infty + \mathsf{cut}$ to a cut-free ∞-proof of the same sequent.

Let us call a pair of ∞-proofs (π, τ) a *cut pair* if π is an ∞-proof of the sequent $\Gamma \Rightarrow \Delta, A$ and τ is an ∞-proof of the sequent $A, \Gamma \Rightarrow \Delta$ for some Γ, Δ, and A. For a cut pair (π, τ), we call the sequent $\Gamma \Rightarrow \Delta$ its *cut result* and the formula A its cut formula.

For a modal formula A, a non-expansive mapping u from $\mathcal{P} \times \mathcal{P}$ to $\mathcal{P}$ is called *A-removing* if it maps every cut pair (π, τ) with the cut formula A to an ∞-proof of its cut result. By $\mathcal{R}_A$, let us denote the set of all A-removing mappings.

Lemma 61. *In an ultrametric space $(\mathcal{R}_A, l_2)$, any contractive operator $\mathsf{U} : \mathcal{R}_A \to \mathcal{R}_A$ has a unique fixed-point.*

Proof. Let us check that the set $\mathcal{R}_A$ is non-empty. Consider the mapping $\mathsf{u}_{cut} : \mathcal{P}^2 \to \mathcal{P}$ that is defined as follows. For a cut pair (π, τ) with the cut formula A, it joins the ∞-proofs π and τ with an appropriate instance of the rule (cut). For all other pairs, the mapping u_{cut} returns the first argument. Clearly, u_{cut} is non-expansive and therefore lies in $\mathcal{R}_A$.

The rest of the proof is completely analogous to the proof of Proposition 43.

In what follows, we use nonexpansive adequate mappings $\mathsf{wk}_{\Pi;\Sigma}$, $\mathsf{li}_{A\to B}$, $\mathsf{ri}_{A\to B}$, $\mathsf{i}_{A\to B}$, $\mathsf{i}_\bot$, acl_p, acr_p from Lemma 51.

Lemma 62. *For any atomic proposition p, there exists an adequate p-removing mapping re_p.*

Proof. Assume we have two ∞-proofs π and τ. If the pair (π, τ) is not a cut pair or is a cut pair with the cut formula being not p, then we put $\mathsf{re}_p(\pi, \tau) = \pi$. Otherwise, we define $\mathsf{re}_p(\pi, \tau)$ by induction on $|\pi|$. Let the cut result of the pair (π, τ) be $\Gamma \Rightarrow \Delta$.

If $|\pi| = 0$, then $\Gamma \Rightarrow \Delta, p$ is an initial sequent. Suppose that $\Gamma \Rightarrow \Delta$ is also an initial sequent. Then $\mathsf{re}_p(\pi, \tau)$ is defined as the ∞-proof consisting only of the sequent $\Gamma \Rightarrow \Delta$. If $\Gamma \Rightarrow \Delta$ is not an initial sequent, then Γ has the form p, Φ, and τ is an ∞-proof of the sequent $p, p, \Phi \Rightarrow \Delta$. Applying the non-expansive adequate mapping acl_p from Lemma 51, we put $\mathsf{re}_p(\pi, \tau) := \mathsf{acl}_p(\tau)$.

Now suppose that $|\pi| > 0$. We define $\mathsf{re}_p(\pi, \tau)$ according to the last application of an inference rule in π as shown in Fig. 1.

The mapping re_p is well defined, adequate and non-expansive.

Lemma 63. *Given an adequate B-removing mapping re_B, there exists an adequate $\Box B$-removing mapping $\mathsf{re}_{\Box B}$.*

Proof. Assume we have an adequate B-removing mapping re_B. The required $\Box B$-removing mapping $\mathsf{re}_{\Box B}$ is obtained as the fixed-point of a contractive operator $\mathsf{G}_{\Box B}: \mathcal{R}_{\Box B} \to \mathcal{R}_{\Box B}$.

$$\left(\rightarrow_{\mathsf{R}}\ \dfrac{\overset{\pi_0}{\Gamma, B \Rightarrow C, \Sigma, p}}{\Gamma \Rightarrow B \rightarrow C, \Sigma, p}, \tau\right) \longmapsto \rightarrow_{\mathsf{R}}\ \dfrac{\overset{\mathsf{re}_p(\pi_0, \mathsf{i}_{B\rightarrow C}(\tau))}{\Gamma, B \Rightarrow C, \Sigma}}{\Gamma \Rightarrow B \rightarrow C, \Sigma},$$

$$\left(\rightarrow_{\mathsf{L}}\ \dfrac{\overset{\pi_0}{\Sigma, C \Rightarrow \Delta, p} \qquad \overset{\pi_1}{\Sigma \Rightarrow B, \Delta, p}}{\Sigma, B \rightarrow C \Rightarrow \Delta, p}, \tau\right) \longmapsto \rightarrow_{\mathsf{L}}\ \dfrac{\overset{\mathsf{re}_p(\pi_0, \mathsf{li}_{B\rightarrow C}(\tau))}{\Sigma, C \Rightarrow \Delta} \qquad \overset{\mathsf{re}_p(\pi_1, \mathsf{ri}_{B\rightarrow C}(\tau))}{\Sigma \Rightarrow B, \Delta}}{\Sigma, B \rightarrow C \Rightarrow \Delta},$$

$$\left(\mathsf{cut}\ \dfrac{\overset{\pi_0}{\Gamma \Rightarrow B, \Delta, p} \qquad \overset{\pi_1}{\Gamma, B \Rightarrow \Delta, p}}{\Gamma \Rightarrow \Delta, p}, \tau\right) \longmapsto \mathsf{cut}\ \dfrac{\overset{\mathsf{re}_p(\pi_0, \mathsf{wk}_{\varnothing;B}(\tau))}{\Gamma \Rightarrow B, \Delta} \qquad \overset{\mathsf{re}_p(\pi_1, \mathsf{wk}_{B;\varnothing}(\tau))}{\Gamma, B \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta},$$

$$\left(\Box\ \dfrac{\overset{\pi_0}{\boxtimes\Pi \Rightarrow \boxtimes(A_1, \ldots, A_n)} \qquad \ldots}{\Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma, p}, \tau\right) \longmapsto \Box\ \dfrac{\overset{\pi_0}{\boxtimes\Pi \Rightarrow \boxtimes(A_1, \ldots, A_n)} \qquad \ldots}{\Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma}$$

Fig. 1. Definition of re_p.

For a mapping $\mathsf{u} \in \mathcal{R}_{\Box B}$ and a pair of ∞-proofs (π, τ), the ∞-proof $\mathsf{G}_{\Box B}(\mathsf{u})(\pi, \tau)$ is defined as follows. If (π, τ) is not a cut pair or a cut pair with the cut formula being not $\Box B$, then we put $\mathsf{G}_{\Box B}(\mathsf{u})(\pi, \tau) = \pi$.

Now let (π, τ) be a cut pair with the cut formula $\Box B$ and the cut result $\Gamma \Rightarrow \Delta$. If $|\pi| = 0$ or $|\tau| = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. In this case, we define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi, \tau)$ as the ∞-proof consisting only of the sequent $\Gamma \Rightarrow \Delta$.

Suppose that $|\pi| > 0$ and $|\tau| > 0$. We define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi, \tau)$ according to the last application of an inference rule in π as shown in Fig. 2.

Now we need to consider only the cases when π has the form

$$\Box\ \dfrac{\overset{\pi_0}{\boxtimes\Pi \Rightarrow \boxtimes(B, A_1, \ldots, A_n)} \qquad \overset{\pi_B}{\boxtimes\Pi \Rightarrow B} \qquad \overset{\pi_1}{\boxtimes\Pi \Rightarrow A_1} \qquad \ldots \qquad \boxtimes\Pi \Rightarrow \boxtimes A_n}{\Phi, \Box\Pi \Rightarrow \Box B, \Box A_1, \ldots, \Box A_n, \Sigma}.$$

We define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi, \tau)$ according to the last application of an inference rule in τ as shown in Fig. 3.

It remains to consider the case when τ has the form

$$\Box\ \dfrac{\overset{\tau_0}{\boxtimes\Lambda, \boxtimes B \Rightarrow \boxtimes(C_1, \ldots, C_k)} \qquad \overset{\tau_1}{\boxtimes\Lambda, \boxtimes B \Rightarrow C_1} \qquad \ldots \qquad \boxtimes\Lambda, \boxtimes B \Rightarrow \boxtimes C_k}{\Phi', \Box\Lambda, \Box B \Rightarrow \Box C_1, \ldots, \Box C_k, \Sigma'}.$$

Notice that the sequent $\Gamma \Rightarrow \Delta$, the sequent $\Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma$, and the sequent $\Phi', \Box\Lambda \Rightarrow \Box C_1, \ldots, \Box C_k, \Sigma'$ are the same.

Let $I := \{i \mid A_i \notin C_1, \ldots, C_k\}$ and $J := \{i \mid C_i \notin A_1, \ldots, A_n\}$, let π'_B be the following ∞-proof:

$$\Box\ \dfrac{\overset{\mathsf{wk}_{\varnothing,\Box\mathsf{B}}(\pi_B)}{\boxtimes\Pi \Rightarrow \boxtimes B} \qquad \overset{\pi_B}{\boxtimes\Pi \Rightarrow B}}{\boxtimes\Pi \Rightarrow \Box B},$$

$$\left({\to_{\mathsf{R}}}\,\dfrac{\overset{\pi_0}{\Gamma, C \Rightarrow D, \Sigma, \Box B}}{\Gamma \Rightarrow C \to D, \Sigma, \Box B}, \tau\right) \longmapsto {\to_{\mathsf{R}}}\,\dfrac{\overset{\mathsf{u}(\pi_0, \mathsf{i}_{C\to D}(\tau))}{\Gamma, C \Rightarrow D, \Sigma}}{\Gamma \Rightarrow C \to D, \Sigma}\ ,$$

$$\left({\to_{\mathsf{L}}}\,\dfrac{\overset{\pi_0}{\Sigma, D \Rightarrow \Delta, \Box B} \qquad \overset{\pi_1}{\Sigma \Rightarrow C, \Delta, \Box B}}{\Sigma, C \to D \Rightarrow \Delta, \Box B}, \tau\right) \longmapsto {\to_{\mathsf{L}}}\,\dfrac{\overset{\mathsf{u}(\pi_0, \mathsf{li}_{C\to D}(\tau))}{\Sigma, D \Rightarrow \Delta} \qquad \overset{\mathsf{u}(\pi_1, \mathsf{ri}_{C\to D}(\tau))}{\Sigma \Rightarrow C, \Delta}}{\Sigma, C \to D \Rightarrow \Delta}\ ,$$

$$\left(\mathsf{cut}\,\dfrac{\overset{\pi_0}{\Gamma \Rightarrow C, \Delta, \Box B} \qquad \overset{\pi_1}{\Gamma, C \Rightarrow \Delta, \Box B}}{\Gamma \Rightarrow \Delta, \Box B}, \tau\right) \longmapsto \mathsf{cut}\,\dfrac{\overset{\mathsf{u}(\pi_0, \mathsf{wk}_{\varnothing;C}(\tau))}{\Gamma \Rightarrow C, \Delta} \qquad \overset{\mathsf{u}(\pi_1, \mathsf{wk}_{C;\varnothing}(\tau))}{\Gamma, C \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\ ,$$

$$\left(\Box\,\dfrac{\overset{\pi_0}{\boxtimes\Pi \Rightarrow \boxtimes(A_1, \dots, A_n)} \qquad \dots}{\Phi, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma, \Box B}, \tau\right) \longmapsto \Box\,\dfrac{\overset{\pi_0}{\boxtimes\Pi \Rightarrow \boxtimes(A_1, \dots, A_n)} \qquad \dots}{\Phi, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma}$$

Fig. 2. Definition of $\mathsf{G}_{\Box B}$ part 1.

$$\left(\pi, {\to_{\mathsf{R}}}\,\dfrac{\overset{\tau_0}{\Gamma, \Box B, C \Rightarrow D, \Sigma}}{\Gamma, \Box B \Rightarrow C \to D, \Sigma}\right) \longmapsto {\to_{\mathsf{R}}}\,\dfrac{\overset{\mathsf{u}(\mathsf{i}_{C\to D}(\pi), \tau_0)}{\Gamma, C \Rightarrow D, \Sigma}}{\Gamma \Rightarrow C \to D, \Sigma}\ ,$$

$$\left(\pi, {\to_{\mathsf{L}}}\,\dfrac{\overset{\tau_0}{\Sigma, \Box B, D \Rightarrow \Delta} \qquad \overset{\tau_1}{\Sigma, \Box B \Rightarrow C, \Delta}}{\Sigma, \Box B, C \to D \Rightarrow \Delta}\right) \longmapsto {\to_{\mathsf{L}}}\,\dfrac{\overset{\mathsf{u}(\mathsf{li}_{C\to D}(\pi), \tau_0)}{\Sigma, D \Rightarrow \Delta} \qquad \overset{\mathsf{u}(\mathsf{ri}_{C\to D}(\pi_1), \tau_1)}{\Sigma \Rightarrow C, \Delta}}{\Sigma, C \to D \Rightarrow \Delta}\ ,$$

$$\left(\pi, \mathsf{cut}\,\dfrac{\overset{\tau_0}{\Gamma, \Box B \Rightarrow C, \Delta} \qquad \overset{\tau_1}{\Gamma, \Box B, C \Rightarrow \Delta}}{\Gamma, \Box B \Rightarrow \Delta}\right) \longmapsto \mathsf{cut}\,\dfrac{\overset{\mathsf{u}(\mathsf{wk}_{\varnothing;C}(\pi), \tau_0)}{\Gamma \Rightarrow C, \Delta} \qquad \overset{\mathsf{u}(\mathsf{wk}_{C;\varnothing}(\pi), \tau_1)}{\Gamma, C \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\ ,$$

$$\left(\pi, \Box\,\dfrac{\overset{\tau_0}{\boxtimes\Lambda \Rightarrow \boxtimes(C_1, \dots, C_k)} \qquad \dots}{\Phi, \Box B, \Box\Lambda \Rightarrow \Box C_1, \dots, \Box C_k, \Sigma}\right) \longmapsto \Box\,\dfrac{\overset{\tau_0}{\boxtimes\Lambda \Rightarrow \boxtimes(C_1, \dots, C_k)} \qquad \dots}{\Phi, \Box\Lambda \Rightarrow \Box C_1, \dots, \Box C_k, \Sigma}$$

Fig. 3. Definition of $\mathsf{G}_{\Box B}$ part 2.

and consider the following ∞-proofs:

$$\psi_1 := \mathsf{u}(\mathsf{wk}_{\Lambda\setminus\Pi, \Box\Lambda\setminus\Pi; \{\boxtimes C_i\}_{i\in J}}(\pi_0), \mathsf{wk}_{\Pi\cup\Lambda, \Box(\Pi\setminus\Lambda); \{\boxtimes A_i\}_{i\in I}, \{C_i\}}(\mathsf{clip}(\tau))),$$

$$\psi_2 := \mathsf{u}(\mathsf{wk}_{\Pi\cup\Lambda, \Box\Lambda\setminus\Pi; \{\boxtimes C_i\}_{i\in J}, \{A_i\}}(\mathsf{clip}(\pi)), \mathsf{wk}_{\Pi\setminus\Lambda, \Box(\Pi\setminus\Lambda); \{\boxtimes A_i\}_{i\in I}}(\tau_0)),$$

$$\phi_j := \mathsf{u}(\mathsf{wk}_{\boxtimes(\Lambda\setminus\Pi); C_j}(\pi'_B), \mathsf{re}_{\mathsf{B}}(\mathsf{wk}_{\boxtimes(\Lambda\setminus\Pi, \Box B; \varnothing)}(\pi_B)), \mathsf{wk}_{\boxtimes(\Pi\setminus\Lambda; \varnothing)}(\tau_j))).$$

We define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau)$ as

$$\Box\ \dfrac{\overset{\mathsf{re_B}(\psi_1,\psi_2)}{\boxtimes(\Pi\cup\Lambda)\Rightarrow\boxtimes A_i,\{\boxtimes C_j\}_{j\in J}} \qquad \overset{(\mathsf{wk}_{\boxtimes(\Lambda\setminus\Pi)}(\pi_i))}{(\boxtimes(\Pi\cup\Lambda)\Rightarrow A_i)} \qquad \overset{\phi_j}{(\boxtimes(\Pi\cup\Lambda)\Rightarrow C_j))_{j\in J}}}{\Gamma\Rightarrow\Delta}\ .$$

Now the operator $\mathsf{G}_{\Box\mathsf{B}}$ is well-defined. By the case analysis according to the definition of $\mathsf{G}_{\Box\mathsf{B}}$, we see that $\mathsf{G}_{\Box\mathsf{B}}(\mathsf{u})$ is non-expansive and belongs to $\mathcal{R}_{\Box B}$ whenever $\mathsf{u}\in\mathcal{R}_{\Box B}$.

We claim that $\mathsf{G}_{\Box\mathsf{B}}$ is contractive. It sufficient to check that for any $\mathsf{u},\mathsf{v}\in\mathcal{R}_{\Box B}$ and any $n,k\in\mathbb{N}$ we have $\mathsf{u}\sim_{n,k}\mathsf{v}\Rightarrow\mathsf{G}_{\Box\mathsf{B}}(\mathsf{u})\sim_{n,k+1}\mathsf{G}_{\Box\mathsf{B}}(\mathsf{v})$, which we prove by case analysis.

Now we define the required $\Box B$-removing mapping $\mathsf{re}_{\Box B}$ as the fixed-point of the the operator $\mathsf{G}_{\Box B}{:}\mathcal{R}_{\Box B}\to\mathcal{R}_{\Box B}$, which exists by Lemma 61.

Lemma 64. *For any formula A, there exists an adequate A-removing mapping* re_A.

Proof. We define re_A by induction on the structure of the formula A.

Case 1: A has the form p. In this case, re_p is defined by Lemma 62.

Case 2: A has the form $\bot$. Then we put $\mathsf{re}_\bot(\pi,\tau):=\mathsf{i}_\bot(\pi)$.

Case 3: A has the form $B\to C$. Then we put

$$\mathsf{re}_{B\to C}(\pi,\tau):=\mathsf{re}_C(\mathsf{re}_B(\mathsf{wk}_{\varnothing,C}(\mathsf{ri}_{B\to C}(\tau)),\mathsf{i}_{B\to C}(\pi)),\mathsf{li}_{B\to C}(\tau)).$$

Case 4: A has the form $\Box B$. By the induction hypothesis, there is an adequate B-removing mapping re_B. The required $\Box B$-removing mapping $\mathsf{re}_{\Box B}$ exists by Lemma 63.

A mapping $\mathsf{u}{:}\mathcal{P}\to\mathcal{P}$ is called *root-preserving* if it maps ∞-proofs to ∞-proofs of the same sequents. Let $\mathcal{T}$ denote the set of all root-preserving non-expansive mappings from $\mathcal{P}$ to $\mathcal{P}$.

Lemma 65. *In an ultrametric space $(\mathcal{T},l_1)$, any contractive operator $\mathsf{U}:\mathcal{T}\to\mathcal{T}$ has a unique fixed-point.*

Proof. The space is obviously non-empty, since the identity function lies in $\mathcal{T}$. The proof is analogous to the proof of Proposition 43.

Theorem 66 (cut-elimination). *If $\mathsf{Go}_\infty+\mathsf{cut}\vdash\Gamma\Rightarrow\Delta$, then $\mathsf{Go}_\infty\vdash\Gamma\Rightarrow\Delta$.*

Proof We construct the required cut-elimination mapping ce so it commutes with every application of inference rules except (cut) and satisfies the following condition:

$$\mathsf{ce}\left(\mathsf{cut}\ \dfrac{\overset{\pi_0}{\Gamma\Rightarrow A,\Delta} \qquad \overset{\pi_1}{\Gamma,A\Rightarrow\Delta}}{\Gamma\Rightarrow\Delta}\right)=\mathsf{re}_A(\mathsf{ce}(\pi_0),\mathsf{ce}(\pi_1)).$$

In order to do this, we define the contractive operator $\mathsf{F}\colon \mathcal{T} \to \mathcal{T}$ and obtain the mapping ce as the fixed-point of F.

For a mapping $\mathsf{u} \in \mathcal{T}$ and an ∞-proof π, the ∞-proof $\mathsf{F}(\mathsf{u})(\pi)$ is defined as follows. If $|\pi| = 0$, then we define $\mathsf{F}(\mathsf{u})(\pi)$ to be π.

Otherwise, we define $\mathsf{F}(\mathsf{u})(\pi)$ according to the last application of an inference rule in π as shown in Fig. 4.

$$\to_{\mathsf{L}}\ \dfrac{\overset{\pi_1}{\Gamma, B \Rightarrow \Delta} \qquad \overset{\pi_2}{\Gamma \Rightarrow A, \Delta}}{\Gamma, A \to B \Rightarrow \Delta} \longmapsto \to_{\mathsf{L}}\ \dfrac{\overset{\mathsf{u}(\pi_1)}{\Gamma, B \Rightarrow \Delta} \qquad \overset{\mathsf{u}(\pi_2)}{\Gamma \Rightarrow A, \Delta}}{\Gamma, A \to B \Rightarrow \Delta}\ ,$$

$$\to_{\mathsf{R}}\ \dfrac{\overset{\pi_0}{\Gamma, A \Rightarrow B, \Delta}}{\Gamma \Rightarrow A \to B, \Delta} \longmapsto \to_{\mathsf{R}}\ \dfrac{\overset{\mathsf{u}(\pi_0)}{\Gamma, A \Rightarrow B, \Delta}}{\Gamma \Rightarrow A \to B, \Delta}\ ,$$

$$\Box\ \dfrac{\overset{\pi_0}{\boxtimes \Pi \Rightarrow \boxtimes(A_1, \dots, A_n)} \qquad \overset{\pi_1}{\boxtimes \Pi \Rightarrow A_1} \qquad \dots \qquad \overset{\pi_n}{\boxtimes \Pi \Rightarrow \boxtimes A_n}}{\Phi, \Box \Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma} \longmapsto$$

$$\longmapsto\ \Box\ \dfrac{\overset{\mathsf{u}(\pi_0)}{\boxtimes \Pi \Rightarrow \boxtimes(A_1, \dots, A_n)} \qquad \overset{\mathsf{u}(\pi_1)}{\boxtimes \Pi \Rightarrow A_1} \qquad \dots \qquad \overset{\mathsf{u}(\pi_n)}{\boxtimes \Pi \Rightarrow \boxtimes A_n}}{\Phi, \Box \Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma}\ ,$$

$$\mathsf{cut}\ \dfrac{\overset{\pi_1}{\Gamma \Rightarrow A, \Delta} \qquad \overset{\pi_2}{\Gamma, A \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta} \longmapsto \mathsf{re}_A(\mathsf{u}(\pi_0), \mathsf{u}(\pi_1)).$$

Fig. 4. Definition of ce.

Now assume $\mathsf{Go}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$. Take an ∞-proof of the sequent $\Gamma \Rightarrow \Delta$ in the system $\mathsf{Go}_\infty + \mathsf{cut}$ and apply the mapping ce to it. We obtain an ∞-proof of the same sequent in the system Go_∞.

Theorem 22 is now established as a direct consequence of Theorems 34, 66 and 35.

7 Conclusion

We have proven the cut elimination theorem for the logic Go syntactically. The approach from [8,9] seems to be easily adaptable to different logics and to provide convenient tools for proving various proof-theoretic properties.

References

1. Di Gianantonio, P., Miculan, M.: A unifying approach to recursive and co-recursive definitions. In: Geuvers, H., Wiedijk, F. (eds.) TYPES 2002. LNCS, vol. 2646, pp. 148–161. Springer, Heidelberg (2003). https://doi.org/10.1007/3-540-39185-1_9
2. Goré, R.: Tableau methods for modal and temporal logics. In: D'Agostino, M., Gabbay, D.M., Hähnle, R., Posegga, J. (eds.) Handbook of Tableau Methods, pp. 297–396. Springer, Dordrecht (1999). https://doi.org/10.1007/978-94-017-1754-0_6
3. Goré, R., Ramanayake, R.: Cut-elimination for weak Grzegorczyk logic Go. Studia Logica **102**(1), 1–27 (2014)
4. Iemhoff, R.: Reasoning in circles. In: Liber Amicorum Alberti. A tribute to Albert Visser. Tributes, vol. 30, pp. 165–176 (2016)
5. Litak, T.: The non-reflexive counterpart of Grz. Bull. Sect. Logic **36**, 195–208 (2007)
6. Petalas, C., Vidalis, T.: A fixed point theorem in non-Archimedean vector spaces. Proc. Am. Math. Soc. **118**(3), 819–821 (1993)
7. Prieß-Crampe, S.: Der Banachsche Fixpunktsatz für ultrametrische Räume. Results Math. **18**(1–2), 178–186 (1990)
8. Savateev, Y., Shamkanov, D.: Cut-elimination for the modal Grzegorczyk logic via non-well-founded proofs. In: Kennedy, J., de Queiroz, R.J.G.B. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 321–335. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_23
9. Savateev, Y., Shamkanov, D.: Non-Well-Founded Proofs for the Grzegorczyk Modal Logic. The Review of Symbolic Logic (to submitted)
10. Shamkanov, D.S.: Circular proofs for the Gödel-Löb provability logic. Math. Notes **96**(3), 575–585 (2014)
11. Schörner, E.: Ultrametric fixed point theorems and applications. Valuat. Theory Appl. **II**(33), 353–359 (2003)

On First-Order Expressibility of Satisfiability in Submodels

Denis I. Saveliev[1,2](✉)

[1] Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia
[2] Institute for Information Transmission Problems of Russian Academy of Sciences, Moscow, Russia
d.i.saveliev@iitp.ru

Abstract. Let κ, λ be regular cardinals, $\lambda \leqslant \kappa$, let φ be a sentence of the language $\mathcal{L}_{\kappa,\lambda}$ in a given signature, and let $\vartheta(\varphi)$ express the fact that φ holds in a submodel, i.e., any model $\mathfrak{A}$ in the signature satisfies $\vartheta(\varphi)$ if and only if some submodel $\mathfrak{B}$ of $\mathfrak{A}$ satisfies φ. It was shown in [1] that, whenever φ is in $\mathcal{L}_{\kappa,\omega}$ in the signature having less than κ functional symbols (and arbitrarily many predicate symbols), then $\vartheta(\varphi)$ is equivalent to a monadic existential sentence in the second-order language $\mathcal{L}^2_{\kappa,\omega}$, and that for any signature having at least one binary predicate symbol there exists φ in $\mathcal{L}_{\omega,\omega}$ such that $\vartheta(\varphi)$ is not equivalent to any (first-order) sentence in $\mathcal{L}_{\infty,\omega}$. Nevertheless, in certain cases $\vartheta(\varphi)$ are first-order expressible. In this note, we provide several (syntactical and semantical) characterizations of the case when $\vartheta(\varphi)$ is in $\mathcal{L}_{\kappa,\kappa}$ and κ is ω or a certain large cardinal.

Keywords: Satisfiability in submodels · Infinitary language · Large cardinal · Ultraproduct · Model-theoretic language · Logic of submodels

Given a model-theoretic language $\mathcal{L}$ (in sense of [2]) and a sentence φ in $\mathcal{L}$, let $\vartheta(\varphi)$ express the fact that φ is satisfied in a submodel. Thus for any model $\mathfrak{A}$,

$$\mathfrak{A} \vDash \vartheta(\varphi) \text{ iff } \mathfrak{B} \vDash \varphi \text{ for some submodel } \mathfrak{B} \text{ of } \mathfrak{A}.$$

We study when $\vartheta(\varphi)$, considered a priori as a meta-expression, is equivalent to a sentence in another (perhaps, the same) given model-theoretic language $\mathcal{L}'$. Such questions naturally arise in studies of modal logics of submodels; if $\mathcal{L}$ is closed under ϑ, then ϑ induces a modal operator on sentences (where a possibility of φ means the satisfiability of φ in a submodel), and the resulting modal logic can be regarded as a fragment of $\mathcal{L}$ with the submodel relation on a given class of models. These logics are an instance of modal logics of various model-theoretic relations,

The work was supported by grant 16-11-10252 of Russian Science Foundation and was carried out at Steklov Mathematical Institute of Russian Academy of Sciences.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 584–593, 2019.
https://doi.org/10.1007/978-3-662-59533-6_35

which were introduced and studied in [1]; another instance is modal logic of forcing (see, e.g., [3]). As another source of motivation for studies undertaken in this note, let us point out the paper [4] discussing reduction of higher-order logics to second-order one.

Here we concentrate on first-order languages $\mathcal{L}_{\kappa,\lambda}$. Recall that $\mathcal{L}_{\omega,\omega}$ is the usual first-order finitary language; $\mathcal{L}_{\kappa,\lambda}$ expands it by involving Boolean connectives of any arities $<\kappa$ and quantifiers over $<\lambda$ first-order variables, where $\lambda \leqslant \kappa$ are given regular cardinals; and $\mathcal{L}_{\infty,\lambda}$ is the union of $\mathcal{L}_{\kappa,\lambda}$ for all κ; see [2,5,6]. It was shown in [1] that, even for φ in $\mathcal{L}_{\omega,\omega}$, it is possible that $\vartheta(\varphi)$ is not in $\mathcal{L}_{\infty,\omega}$; on the other hand, $\vartheta(\varphi)$ is equivalent to a second-order (in general, infinitary) sentence; these results are reproduced as Theorems 1 and 2 below. Nevertheless, in certain cases $\vartheta(\varphi)$ are first-order expressible. In this note, we provide several (syntactical as well as semantical) characterizations of the case when $\vartheta(\varphi)$ is equivalent to a sentence in $\mathcal{L}_{\kappa,\kappa}$ and κ is either ω or a certain large (e.g., compact) cardinal.

We start with some obvious observations, which confirm, in particular, that ϑ behaves like an S4 possibility operator.

Proposition 1. *For any sentences φ, ψ, and φ_i, $i \in I$, in a model-theoretic language $\mathcal{L}$ involving the syntactic operations under consideration, we have:*

(i) $\vartheta(\top)$ *is equivalent to* $\top$, *and* $\vartheta(\bot)$ *is equivalent to* $\bot$;
(ii) φ *implies* $\vartheta(\varphi)$;
(iii) $\vartheta(\varphi)$ *is equivalent to* $\vartheta(\vartheta(\varphi))$;
(iv) $\varphi \to \psi$ *implies* $\vartheta(\varphi) \to \vartheta(\psi)$;
(v) $\vartheta(\bigwedge_{i\in I} \varphi_i)$ *implies* $\bigwedge_{i\in I} \vartheta(\varphi_i)$, *and* $\vartheta(\bigvee_{i\in I} \varphi_i)$ *is equivalent to* $\bigvee_{i\in I} \vartheta(\varphi_i)$;
(vi) $\neg\,\vartheta(\varphi)$ *implies* $\neg\,\varphi$, *and* $\neg\,\varphi$ *implies* $\vartheta(\neg\,\varphi)$;
(vii) $\vartheta(\neg\,\vartheta(\varphi))$ *implies* $\vartheta(\neg\,\varphi)$.

Proof. Items (i)–(v) are immediate; (vi) follows from (ii); and (vii) from (iv) and (vi).

Corollary 1. *For every sentence φ, the sentence $\vartheta(\varphi)$ is preserved under extensions of models. A fortiori, it is preserved under elementary extensions, unions of increasing chains of models; in purely predicate signatures: under direct unions, direct products and powers; etc.*

Proof. This follows from item (iii) of Proposition 1.

Given cardinals κ, μ with $\mu \geqslant \kappa$, recall that κ is *μ-compact* iff the language $\mathcal{L}_{\kappa,\kappa}$ satisfies the (μ, κ)-compactness, i.e., any theory in $\mathcal{L}_{\kappa,\kappa}$ of cardinality $\leqslant\mu$ has a model whenever each its subtheory of cardinality $<\kappa$ has a model. A cardinal κ is *weakly compact* iff it is κ-compact, and *strongly compact* iff it is μ-compact for all $\mu \geqslant \kappa$ (e.g., ω is strongly compact). By a compact cardinal κ we shall mean strongly compact κ, and by an inaccessible, a strongly inaccessible, i.e., a regular κ such that $2^{\lambda} < \kappa$ for all $\lambda < \kappa$. For more on these and other large cardinals and their connections with infinitary languages, we refer the reader to [5,6].

Corollary 2. *Let κ be ω or, more generally, a compact cardinal, and φ a sentence in $\mathcal{L}_{\kappa,\kappa}$. The following are equivalent:*

(i) *$\vartheta(\varphi)$ is equivalent to a sentence in $\mathcal{L}_{\kappa,\kappa}$;*
(ii) *$\vartheta(\varphi)$ is equivalent to a sentence in $\Sigma^0_1(\mathcal{L}_{\kappa,\kappa})$, i.e., an existential sentence in $\mathcal{L}_{\kappa,\kappa}$.*

Proof. (i)→(ii). By Corollary 1 since sentences in $\mathcal{L}_{\kappa,\kappa}$ that are preserved under extensions are exactly existential ones (for $\kappa = \omega$ see [7]; for compact $\kappa > \omega$, modify the same argument).

Two following results on expressibility of $\vartheta(\varphi)$, one negative and one positive, were essentially obtained in [1].

Theorem 1. *For any signature having at least one binary predicate symbol R, there exists φ in $\mathcal{L}_{\omega,\omega}$ such that $\vartheta(\varphi)$ is not equivalent to any (first-order) sentence in $\mathcal{L}_{\infty,\omega}$.*

Proof. If φ states that there is no R-minimal element, then $\vartheta(\varphi)$ states that R is non-well-founded; as well known, the latter property is not expressible in $\mathcal{L}_{\infty,\omega}$ (moreover, it is not RPC in $\mathcal{L}_{\infty,\omega}$, see, e.g., [2], Theorem 3.2.20).

Given a model-theoretic language $\mathcal{L}$, let $\mathcal{L}^\alpha$ denote the αth-order extension of $\mathcal{L}$. A formula of $\mathcal{L}^2$ is *monadic* iff it involves only unary predicate variables, and *existential second-order*, respectively, *universal second-order* iff it involves only existential, respectively, universal quantifiers over second-order variables preceding a first-order formula (with arbitrary quantifiers). The monadic fragment of $\mathcal{L}^2$ consists of its monadic formulas; similarly for the existential and universal fragments of the language, which will be denoted by $\Sigma^1_1(\mathcal{L}^2)$ and $\Pi^1_1(\mathcal{L}^2)$, respectively.

Theorem 2. *Let κ be a regular cardinal and φ a (first-order) sentence of $\mathcal{L}_{\kappa,\omega}$ in a signature τ with $<\kappa$ functional (including constant) symbols and arbitrarily many predicate symbols. Then $\vartheta(\varphi)$ is equivalent to a monadic existential formula in $\mathcal{L}^2_{\kappa,\omega}$. Moreover, the following languages are closed under ϑ:*

(i) *the monadic fragment of $\mathcal{L}^2_{\kappa,\lambda}$ for any $\lambda \leqslant \kappa$;*
(ii) *the existential fragment of $\mathcal{L}^2_{\kappa,\lambda}$ for any $\lambda \leqslant \kappa$;*
(iii) *$\mathcal{L}^\alpha_{\kappa,\lambda}$ for any $\lambda \leqslant \kappa$ and $\alpha \geqslant 2$.*

Proof. If X is a second-order unary predicate variable, for each functional symbol F in τ let ψ_F be an $\mathcal{L}^2_{\omega,\omega}$-formula stating that X is closed under F, and let $\psi(X)$ be the $\mathcal{L}^2_{\kappa,\omega}$-formula

$$\exists x\big(X(x) \wedge \bigwedge\{\psi_F(X) : F \text{ is a functional symbol in } \tau\}\big)$$

stating that X forms a submodel. Then $\vartheta(\varphi)$ is equivalent to the sentence

$$\exists X\big(\psi(X) \wedge \varphi^X\big)$$

where φ^X is the relativization of φ to X.

As usual, a filter D is *κ-complete* iff $\bigcap E \in D$ for all $E \in P_\kappa(D)$, where $P_\kappa(A)$ denotes the set of all subsets of A which have cardinality $<\kappa$.

Corollary 3. *Let κ be ω or, more generally, a compact cardinal, and φ a sentence in $\mathcal{L}_{\kappa,\kappa}$ in a signature with $<\kappa$ functional (including constant) symbols. Then $\vartheta(\varphi)$ is preserved under ultraproducts by κ-complete ultrafilters. Moreover, this remains true for sentences φ in $\Sigma^1_1(\mathcal{L}^2_{\kappa,\kappa})$.*

Proof. By Theorem 2, for such a φ the statement $\vartheta(\varphi)$ is equivalent to some $\Sigma^1_1(\mathcal{L}^2_{\kappa,\kappa})$-sentence, therefore, it is preserved under ultraproducts (for $\kappa = \omega$ see, e.g., [7], Corollary 4.1.14; for compact κ modify the same argument).

The next result on non-expressibility of $\vartheta(\varphi)$ shows that the restriction on the number of functional symbols in Theorem 2 is optimal.

Theorem 3. *For any signature τ having $\geqslant\kappa$ functional (e.g., constant) symbols, there exists a sentence φ of $\mathcal{L}_{\omega,\omega}$ (in fact, in the empty signature) such that $\vartheta(\varphi)$ is not equivalent to any sentence in $\mathcal{L}^\alpha_{\kappa,\kappa}$ for all α, and moreover, in every language $\mathcal{L}$ whose formulas ψ have cardinality $|\psi| < \kappa$.*

Proof. Clearly, it suffices to consider only a signature τ consisting of κ constant symbols, say, c_α, $\alpha < \kappa$. Let φ be the sentence $\forall x \forall y\, x = y$; then $\vartheta(\varphi)$ states the existence of a single-point submodel. Toward a contradiction, assume that such an $\mathcal{L}$ has some ψ equivalent to $\vartheta(\varphi)$. Let two models $\mathfrak{A}$ and $\mathfrak{B}$ in τ have the same two-point universe $\{a, b\}$, and let for all $\alpha < \kappa$,

$$c^{\mathfrak{A}}_\alpha := a, \quad \text{and} \quad c^{\mathfrak{B}}_\alpha := \begin{cases} a & \text{if } c_\alpha \text{ occurs in } \psi, \\ b & \text{otherwise.} \end{cases}$$

Since $|\psi| < \kappa$, there exists $\alpha < \kappa$ such that $c^{\mathfrak{B}}_\alpha = b$. So we have: $\mathfrak{A} \vDash \psi$ iff $\mathfrak{B} \vDash \psi$ (as $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same formulas involving only symbols from ψ), however, $\mathfrak{A} \vDash \vartheta(\varphi)$ and $\mathfrak{B} \vDash \neg\,\vartheta(\varphi)$ (as the singleton $\{a\}$ forms a submodel of $\mathfrak{A}$ while $\mathfrak{B}$ has no single-point submodels).

Let $\vartheta_{\leqslant\lambda}(\varphi)$ denote that φ is satisfied in a submodel generated by a set of cardinality $\leqslant\lambda$. Obviously, $\vartheta_{\leqslant\lambda}(\varphi)$ implies $\vartheta(\varphi)$. We are going to show that $\vartheta_{\leqslant\lambda}(\varphi)$ is an existential sentence in an appropriate first-order language. To simplify some formulations, we shall consider *partial models* in which their operations can be only partial. An *atomic diagram* of a partial model $\mathfrak{A}$ is defined in the same way as for usual models with total operations, i.e., it consists of all true in $\mathfrak{A}$ atomic and negated atomic sentences of the language expanded by constant symbols for all elements of $\mathfrak{A}$.

Lemma 1. *Let κ be ω or, more generally, an inaccessible cardinal, $\lambda < \kappa$, and φ a sentence in $\mathcal{L}_{\kappa,\kappa}$ in a signature τ with $<\kappa$ functional (including constant) symbols. Then $\vartheta_{\leqslant\lambda}(\varphi)$ is equivalent to an existential sentence in $\mathcal{L}_{\kappa,\kappa}$.*

Proof. Let us first consider τ without functional symbols. Then $\vartheta_{\leqslant\lambda}(\varphi)$ is clearly equivalent to the first-order sentence

$$\exists_{\alpha<\lambda} x_\alpha\, \varphi^{\{x_\alpha :\, \alpha<\lambda\}}$$

where $\varphi^{\{x_\alpha:\,\alpha<\lambda\}}$ is the relativization of φ to the set of (first-order) variables x_α, $\alpha<\lambda$, which do not occur in φ. Let us verify that the relativization is equivalent to an open formula in $\mathcal{L}_{\kappa,\kappa}(\tau)$ (with parameters x_α, $\alpha<\lambda$); it will clearly follow that $\exists_{\alpha<\lambda} x_\alpha\, \varphi^{\{x_\alpha:\,\alpha<\lambda\}}$ is equivalent to a $\Sigma^0_1(\mathcal{L}_{\kappa,\kappa})$-sentence.

Indeed, the relativization is obtained from φ by successively replacing each subformula $\exists_{\beta<\gamma} y_\beta\, \psi$ with the $\mathcal{L}_{\kappa,\kappa}$-formula

$$\exists_{\beta<\gamma} y_\beta \big(\psi \wedge \bigwedge_{\beta<\gamma} \bigvee_{\alpha<\lambda} y_\beta = x_\alpha\big).$$

The latter formula is equivalent to the formula

$$\exists_{\beta<\gamma} y_\beta \big(\psi \wedge \bigvee_{f\in\lambda^\gamma} \bigwedge_{\beta<\gamma} y_\beta = x_{f(\beta)}\big),$$

which is still in $\mathcal{L}_{\kappa,\kappa}$ since $|\lambda^\gamma|<\kappa$ due to the condition that κ is inaccessible, and furthermore, to the open formula

$$\bigvee_{f\in\lambda^\gamma} \psi(y_\beta/x_{f(\beta)})_{\beta<\gamma}$$

where $\psi(y_\beta/x_{f(\beta)})_{\beta<\gamma}$ is obtained from ψ by substituting each variable y_β with the variable $x_{f(\beta)}$. This eliminates all quantifiers in all subformulas of $\varphi^{\{x_\alpha:\,\alpha<\lambda\}}$, as required.

In the general case, the construction is slightly more complex. Let τ' expand τ by λ new constant symbols c_α, $\alpha<\lambda$. We still have $|\tau'|<\kappa$. Hence, since κ is inaccessible, there exist only $<\kappa$ pairwise non-isomorphic partial models in τ' satisfying φ with the universe consisting of an interpretation of all closed terms; say, $\mathfrak{B}_\beta$, $\beta<\mu$, for some $\mu<\kappa$. Note that, though such partial models may have size $>\lambda$ (they interpret not only the c_α but all terms constructed from them), all they have size $\leqslant\nu$ for some fixed ν with $\lambda\leqslant\nu<\kappa$. For any $\beta<\mu$, let Δ_β be the atomic diagram of $\mathfrak{B}_\beta$, and ψ_β its conjunction $\bigwedge\Delta_\beta$, which is still in $\mathcal{L}_{\kappa,\kappa}$ as $|\Delta_\beta|<\kappa$. Let x_α, $\alpha<\lambda$, be variables not occurring in Δ_β, and let φ_β be the formula $\psi_\beta(c_\alpha/x_\alpha)$ obtained from ψ_β by replacing each constant symbol c_α with the variable x_α. Then φ_β is an open formula in τ, and the $\Sigma^0_1(\mathcal{L}_{\kappa,\kappa})$-sentence

$$\exists_{\alpha<\lambda} x_\alpha\ \varphi_\beta(x_\alpha)_{\alpha<\lambda}$$

characterizes the partial model $\mathfrak{B}_\beta$ up to isomorphism. It follows that $\vartheta_{\leqslant\lambda}(\varphi)$ is equivalent to the $\Sigma^0_1(\mathcal{L}_{\kappa,\kappa})$-sentence

$$\exists_{\alpha<\lambda} x_\alpha \bigvee_{\beta<\mu} \varphi_\beta(x_\alpha)_{\alpha<\lambda}.$$

This completes the proof.

Remark 1. The argument shows that, whenever κ is an inaccessible cardinal $>\omega$, then moreover, $\vartheta_{\leqslant\lambda}(\varphi)$ is equivalent to an existential sentence in $\mathcal{L}_{\kappa,\lambda^+}$. Also we can see that Lemma 1 remains true for signatures with $<\kappa$-ary symbols. For τ with $\geqslant\kappa$ functional symbols, even $\vartheta_{\leqslant 1}(\varphi)$ is non-expressible in any language with formulas of size $<\kappa$, by the proof of Theorem 3.

A *fragment* of a model $\mathfrak{A}$ is an its partial submodel, i.e., a subset of the universe of $\mathfrak{A}$ together with the inherited structure. Thus for models in signatures without functional symbols, fragments are just submodels; while for models in signatures with functional symbols, operations on fragments can be partial. A fragment can be considered as a submodel of the corresponding model in the purely predicate language obtained from the original language by replacing each functional symbol of arity $\geqslant 1$ with a predicate symbol having the same interpretation. Clearly, for any fragment there exists the smallest submodel including it, the submodel *generated* by the fragment.

Let $\mathfrak{A}$ be a model and I an ideal over A (the universe of $\mathfrak{A}$) with $\bigcup I = A$. We shall say that the system $(\mathfrak{B}_i)_{i\in I}$ of models (in the same signature) is *coherent in* $\mathfrak{A}$ iff for every $i \in I$, the set i is included into B_i (the universe of $\mathfrak{B}_i$) and the fragments of $\mathfrak{A}$ and of $\mathfrak{B}_i$ given by i coincide.

As usual, an *upper cone* of a partially ordered set $(P, \leqslant)$ is an $C \subseteq P$ which is upward closed, i.e., such that $b \in C$ whenever $a \leqslant b$ for some $a \in C$. Clearly, the set of upper cones of P generates a filter over P whenever P is directed.

Lemma 2. *Let $(\mathfrak{B}_i)_{i\in I}$ be coherent in $\mathfrak{A}$ and D a filter over I extending the filter generated by upper cones of $(I, \subseteq)$. Then $\mathfrak{A}$ isomorphically embeds into $\mathfrak{B} := \prod_D \mathfrak{B}_i$, the product of the models $\mathfrak{B}_i$ reduced by D.*

Proof. For each $i \in I$ we fix some $b_i \in B_i$, and for each $a \in A$, let $c_a \in \prod_{i\in I} B_i$ be the function defined by letting for all $i \in I$,

$$c_a(i) := \begin{cases} a & \text{if } a \in i, \\ b_i & \text{otherwise.} \end{cases}$$

Now define $f : A \to B$ by letting for all $a \in A$,

$$f(a) := [c_a]_D,$$

and check that f is an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

Let R be an n-ary predicate symbol in our signature. We must check that for all $a_0, \ldots, a_{n-1}$ in A,

$$R^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \text{ iff } R^{\mathfrak{B}}(f(a_0), \ldots, f(a_{n-1})).$$

Since $\bigcup I = A$, for any $k < n$ there is $i_k \in I$ with $a_k \in B_{i_k}$, and since I is an ideal, $\bigcup_{k<n} i_k \in I$. Moreover, since $i \subseteq \mathfrak{B}_i$ for all $i \in I$, whenever $\bigcup_{k<n} i_k \subseteq i$ then $\{a_k\}_{k<n} \subseteq B_i$ and $a_k = c_{a_k}(i)$, and so, since the fragments of $\mathfrak{A}$ and $\mathfrak{B}_i$ given by i coincide, $R^{\mathfrak{A}}(a_0, \ldots, a_{n-1})$ is equivalent to $R^{\mathfrak{B}_i}(c_{a_0}(i), \ldots, c_{a_{n-1}}(i))$. Thus we have:

$$\begin{aligned} R^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) &\text{ iff } \big\{i \in I : R^{\mathfrak{B}_i}(c_{a_0}(i), \ldots, c_{a_{n-1}}(i))\big\} \text{ is an upper cone of } I \\ &\text{ iff } \big\{i \in I : R^{\mathfrak{B}_i}(c_{a_0}(i), \ldots, c_{a_{n-1}}(i))\big\} \in D \end{aligned}$$

where one implication in the second equivalence holds since D extends the filter generated by upper cones of I while the converse implication holds since the property inherits upward. But by definition of reduced products, the latter assertion is equivalent to $R^{\mathfrak{B}}([c_{a_0}]_D, \ldots, [c_{a_{n-1}}]_D)$ and thus to $R^{\mathfrak{B}}(f(a_0), \ldots, f(a_{n-1}))$, as required.

Let now F be an n-ary functional symbol in the signature. We must check that for all $a_0, \ldots, a_{n-1}$ in A,

$$f(F^{\mathfrak{A}}(a_0, \ldots, a_{n-1})) = F^{\mathfrak{B}}(f(a_0), \ldots, f(a_{n-1})).$$

Indeed,

$$f(F^{\mathfrak{A}}(a_0, \ldots, a_{n-1})) = [c_{F^{\mathfrak{A}}(a_0,\ldots,a_{n-1})}]_D$$

while

$$F^{\mathfrak{B}}(f(a_0), \ldots, f(a_{n-1})) = F^{\mathfrak{B}}([c_{a_0}]_D, \ldots, [c_{a_{n-1}}]_D) = [b]_D$$
$$\text{where } b(i) = F^{\mathfrak{B}_i}(c_{a_0}(i), \ldots, c_{a_{n-1}}(i)).$$

Again, if for $k < n$, $i_k \in I$ is such that $a_k \in B_{i_k}$, and also $i_n \in I$ is such that $F^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \in B_{i_n}$, whenever $\bigcup_{k \leqslant n} i_k \subseteq i$ then $c_{F^{\mathfrak{A}}(a_0,\ldots,a_{n-1})}(i) = F^{\mathfrak{A}}(a_0, \ldots, a_{n-1}) = F^{\mathfrak{B}_i}(a_0, \ldots, a_{n-1})$ and also

$$F^{\mathfrak{B}_i}(c_{a_0}(i), \ldots, c_{a_{n-1}}(i)) = F^{\mathfrak{B}_i}(a_0, \ldots, a_{n-1}).$$

It follows $[c_{F^{\mathfrak{A}}(a_0,\ldots,a_{n-1})}]_D = [b]_D$, as required.

The proof is complete.

Remark 2. In general, even if D is an ultrafilter and all the $\mathfrak{B}_i$ are submodels of $\mathfrak{A}$, the embedding is not elementary, and moreover, $\mathfrak{A}$ and $\mathfrak{B}$ are not elementarily equivalent, even in the sense of $\mathcal{L}_{\omega,\omega}$. E.g., let $\mathfrak{A} = (\omega, <)$ and $\mathfrak{B}_i = (i, <)$ for all finite $i \subseteq \omega$. Then if φ is $\exists x\, \forall y\, \neg\, (x < y)$, we have: $\mathfrak{A} \vDash \varphi$, but for all i, $\mathfrak{B}_i \vDash \neg\, \varphi$, and hence, $\mathfrak{B} \vDash \neg\, \varphi$.

Remark 3. If the ideal I is κ-complete, i.e., $\bigcup E \in I$ for all $E \in P_\kappa(I)$, then Lemma 2 remains true even for signatures involving $<\!\kappa$-ary symbols. Let us point out also that whenever I is κ-complete then so is the filter D_I over I generated by upper cones in $(I, \subseteq)$ (but of course not any filter D extending D_I), and that in the case $I = P_\kappa(A)$, D_I is the least κ-complete *fine* filter over $P_\kappa(A)$.

The theorem below is the main result of this note; it extends Corollary 2 by providing new characterizations – syntactical in item (iii) and semantical in items (iv) and (v) – of the case when $\vartheta(\varphi)$ is equivalent to a first-order formula.

Theorem 4. *Let κ be ω or, more generally, a compact cardinal and φ a sentence in the language $\mathcal{L}_{\kappa,\kappa}$ in a signature τ with $<\!\kappa$ functional (including constant) symbols. The following are equivalent:*

(i) $\vartheta(\varphi)$ is equivalent to a $\Sigma^0_1(\mathcal{L}_{\kappa,\kappa})$-sentence;
(ii) $\vartheta(\varphi)$ is equivalent to an $\mathcal{L}_{\kappa,\kappa}$-sentence;
(iii) $\vartheta(\varphi)$ is equivalent to a $\Pi^1_1(\mathcal{L}^2_{\kappa,\kappa})$-sentence;

(iv) any model satisfying φ has a fragment of cardinality $<\kappa$ such that each model having the fragment satisfies $\vartheta(\varphi)$;
(v) there exists $\lambda < \kappa$ such that any model satisfying φ has a fragment of cardinality $\leqslant\lambda$ and such that each model having the fragment satisfies $\vartheta(\varphi)$.

Proof. (i)→(ii) and (ii)→(iii). Trivial.

(iii)→(iv). Assume that (iv) does not hold. Then there is $\mathfrak{A}$ such that $\mathfrak{A} \vDash \varphi$, and for every set $i \subseteq A$ of size $< \kappa$, there exists a model $\mathfrak{B}_i$ such that $i \subseteq B_i$, the fragments of $\mathfrak{A}$ and of $\mathfrak{B}_i$ given by i coincide, and $\mathfrak{B}_i$ has no submodels satisfying φ, thus $\mathfrak{B}_i \vDash \neg\vartheta(\varphi)$. Let $\mathfrak{B} := \prod_D \mathfrak{B}_i$ where D is a κ-complete ultrafilter over $P_\kappa(A)$ which is fine, i.e., extends the filter generated by the sets $\{i \in P_\kappa(A) : a \in i\}$ for all $a \in A$ (recall that the existence of such an ultrafilter follows from the compactness of κ; see, e.g., [6], Corollary 22.18). By Lemma 2, $\mathfrak{A}$ isomorphically embeds into $\mathfrak{B}$; therefore, $\mathfrak{B} \vDash \vartheta(\varphi)$. Let us show that (iii) fails.

Indeed, if $\vartheta(\varphi)$ is equivalent to a $\Pi^1_1(\mathcal{L}^2_{\kappa,\kappa})$-formula, then $\neg\vartheta(\varphi)$ is equivalent to a $\Sigma^1_1(\mathcal{L}^2_{\kappa,\kappa})$-formula, and hence, is preserved under ultraproducts by κ-complete ultrafilters (as was pointed out in the proof of Corollary 3), whence we get also $\mathfrak{B} \vDash \neg\vartheta(\varphi)$; a contradiction.

(iv)→(v). Assume that (v) does not hold. Let us again use an ultraproduct argument: for any $\alpha < \kappa$ pick a model $\mathfrak{A}_\alpha$ which satisfies φ and does not have fragments of size $\leqslant\lambda$ generating submodels satisfying φ, pick any κ-complete ultrafilter D over κ, and consider $\mathfrak{A} := \prod_D \mathfrak{A}_\alpha$. Clearly, $\mathfrak{A}$ satisfies φ. Let us show that $\mathfrak{A}$ does not have fragments of size $<\kappa$ generating submodels satisfying φ, thus proving that (iv) fails.

Indeed, if there is $\lambda < \kappa$ such that $\mathfrak{A}$ has some λ-generated submodel satisfying φ, i.e., $\mathfrak{A} \vDash \vartheta_{\leqslant\lambda}(\varphi)$, then this fact is expressed by a first-order (and even existential) sentence by Lemma 1, and hence, should hold in $\mathfrak{A}_\alpha$ for D-almost all α, which is, however, not true.

(v)→(i). Assume (v). Then $\vartheta(\varphi)$ is equivalent to $\vartheta_{\leqslant\lambda}(\varphi)$, which is equivalent to a $\Sigma^0_1(\mathcal{L}_{\kappa,\kappa})$-sentence by Lemma 1, thus proving (i).

The theorem is proved.

Corollary 4. *Let κ be ω or, more generally, a compact cardinal and φ a sentence in $\mathcal{L}_{\kappa,\kappa}$ in a signature without functional symbols. The following are equivalent:*

(i) $\vartheta(\varphi)$ is equivalent to a sentence in $\mathcal{L}_{\kappa,\kappa}$ (or $\Sigma^0_1(\mathcal{L}_{\kappa,\kappa})$, or $\Pi^1_1(\mathcal{L}^2_{\kappa,\kappa})$);
(ii) any model satisfying φ has a submodel of cardinality $<\kappa$ satisfying φ (or $\vartheta(\varphi)$);
(iii) there exists $\lambda < \kappa$ such that any model satisfying φ has a submodel of cardinality $\leqslant\lambda$ satisfying φ (or $\vartheta(\varphi)$).

Proof. As in such signatures the notions of fragments and submodels coincide, this follows from Theorem 4.

Example 1. Recall that models in signatures having only unary functional symbols are called *unoids*, and if such a symbol is unique, *unars*.

Let a signature τ consist of a single unary functional symbol F. Let φ be the $\mathcal{L}_{\omega,\omega}$-sentence $\exists x\, F(x) \neq x$ in τ. Then $\vartheta(\varphi)$ is equivalent to φ itself. Clearly, the cardinality $\lambda < \omega$ of a finite fragment determining satisfiability of $\vartheta(\varphi)$ in submodels extending it, stated in Theorem 4 (v), is 1; note that there are models of φ without finite submodels at all (e.g., so is the free 1-generated unar (ω, S) where S is the successor operation).

More generally, let τ consist of κ unary functional symbols F_α, $\alpha < \kappa$, and let φ be the $\mathcal{L}_{\kappa^+,\omega}$-sentence $\exists x \bigvee_{\alpha<\kappa} F_\alpha(x) \neq x$ in τ. Then again, $\vartheta(\varphi)$ is equivalent to φ, λ is 1, and there are models of φ without submodels of size $<\kappa$ (e.g., any free τ-unoid).

Example 2. Let a signature τ consist of a single binary predicate symbol R.

Let φ be the $\mathcal{L}_{\omega,\omega}$-sentence $\exists x\, \forall y\; R(x, y)$ in τ. Then $\vartheta(\varphi)$ is equivalent to $\exists x\, R(x, x)$, and λ from Corollary 4 (iii) is 1. Observe that φ is in $\Sigma^0_2 \setminus \Pi^0_2$. Indeed, assume that φ is in Π^0_2. Recall that Π^0_2-formulas are characterized as those that are preserved under chain unions (see [7], Theorem 3.2.2). Let $\mathfrak{A}_n$ be $(n+1, \geqslant)$, and let $\mathfrak{A}$ be the union of the chain of the $\mathfrak{A}_n$, $n \in \omega$. Then $\mathfrak{A}_n \vDash \varphi$ for all $n \in \omega$ but $\mathfrak{A} \vDash \neg\,\varphi$; a contradiction.

Let also ψ be the $\mathcal{L}_{\omega,\omega}$-sentence $\forall x\, \exists y R(x, y)$ in τ. Then $\vartheta(\psi)$ is not equivalent to any $\mathcal{L}_{\omega,\omega}$-sentence. Indeed, the model $(\omega, <)$ satisfies ψ but includes no finite submodels satisfying ψ; apply Corollary 4 (iii). Similar arguments show that ψ is in $\Pi^0_2 \setminus \Sigma^0_2$ and that $\vartheta(\neg\,\psi)$ is $\exists x \neg\, R(x, x)$ while $\vartheta(\neg\,\varphi)$ is not equivalent to any $\mathcal{L}_{\omega,\omega}$-sentence.

Remark 4. Although any φ with the first-order expressible $\vartheta(\varphi)$ determines the least size of a small fragment from Theorem 4, which is thus independent of a particular model, it does not determine, even up to isomorphism, the fragment itself. E.g., if φ is any true formula then $\vartheta(\varphi)$ is equivalent to φ; hence in a purely predicate signature τ the discussed size is 1 but there can be many non-isomorphic single-point models $-$arbitrarily many in an appropriate τ.

Remark 5. Despite the low complexity of the first-order expressible $\vartheta(\varphi)$, the complexity of φ itself can be much higher: there exist non-first-order expressible φ with first-order expressible $\vartheta(\varphi)$. Let φ state the finiteness, i.e., let any model satisfy φ iff it is finite. As well-known, φ is not $\mathcal{L}_{\omega,\omega}$-expressible (though is expressible in $\mathcal{L}_{\omega_1,\omega}$ or else in the weak second-order language; see, e.g., [2]). However, in any signature without functional symbols, $\vartheta(\varphi)$ is equivalent to any true formula. This example is generalized to the languages $\mathcal{L}_{\kappa,\kappa}$ with arbitrarily large κ.

Remark 6. Theorem 4 (and Corollary 4) remains true even for signatures with $<\kappa$-ary symbols; for Lemmas 1 and 2 this has been already noticed, other arguments in the proof do not require any modifying.

Remark 7. The results of this note admit further improvements and generalizations.

First, in Theorem 4 (and Corollary 4) it suffices to assume that κ is inaccessible. Moreover, for $\kappa > \omega$, if a sentence φ in $\mathcal{L}_{\kappa,\kappa}$ holds in a model then it holds in its submodel of size $<\kappa$ (see, e.g., [2], Corollary 3.1.3), hence $\vartheta(\varphi)$ is equivalent to $\vartheta_{\leqslant\lambda}(\varphi)$ for some $\lambda < \kappa$; then items (i)–(v) of Theorem 4 follow due to Lemma 1 (even in a nicer form with submodels instead of fragments, like as in Corollary 4).

Further, the obtained results can be relativized to theories T in a given language. Another natural generalization concerns the question when $\vartheta(T)$ is equivalent to some theory in the same language.

Question 1. Characterize first-order sentences φ for which $\vartheta(\varphi)$ are first-order sentences in $\mathcal{L}_{\omega,\omega}$.

Acknowledgement. I am grateful to N. L. Poliakov for discussions on the subject of this note and especially for his valuable help in handling the case of functional signatures in Lemma 1. I am indebted to F. N. Pakhomov for his remark about the number of functional symbols in that lemma, which leaded me to Theorem 3, and for his proposal to weaken the large cardinal property of κ to inaccessibility by using the downward Löwenheim–Skolem theorem for $\mathcal{L}_{\kappa,\kappa}$. I also express my appreciation to I. B. Shapirovsky who read this note and made several useful comments. Finally, I thank two (unknown to me) referees for some suggestions improving the text.

References

1. Saveliev, D.I., Shapirovsky, I.B.: On modal logics of model-theoretic relations. Studia Logica (2019, accepted). arXiv: 1804.09810
2. Barwise, J., Feferman, S. (eds.): Model-Theoretic Logics. Perspectives in Mathematical Logic, vol. 8. Springer, New York (1985)
3. Hamkins, J.D., Löwe, B.: The modal logic of forcing. Trans. Am. Math. Soc. **360**(4), 1793–1817 (2008)
4. Montague, R.: Reduction of higher-order logic. In: Symposium on the Theory of Models, pp. 251–264. North-Holland, Amsterdam (1965)
5. Drake, F.R.: Set theory: an introduction to large cardinals. In: Studies in Logic and the Foundations of Mathematics, vol. 76. North-Holland, Amsterdam, Oxford, New York (1974)
6. Kanamori, A.: The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings, 2nd edn. Springer, Heidelberg (2003)
7. Chang, C.C., Keisler, H.J.: Model Theory, 3rd edn. North-Holland, Amsterdam (1990)

Substructural Propositional Dynamic Logics

Igor Sedlár(✉)

The Czech Academy of Sciences, Institute of Computer Science,
Pod Vodárenskou věží 271/2, 182 07 Prague 8, Czech Republic
sedlar@cs.cas.cz
http://www.cs.cas.cz/sedlar/

Abstract. We prove completeness and decidability of a version of Propositional Dynamic Logic where the underlying non-modal propositional logic is a substructural logic in the vicinity of the Full Distributive Non-associative Lambek Calculus. Extensions of the result to stronger substructural logics are briefly discussed.

Keywords: Lambek calculus · Modal logic · Propositional dynamic logic · Relevant logic · Substructural logic

1 Introduction

Propositional Dynamic Logic, introduced in [8] following the ideas of [24], is a multi-modal logic for reasoning about structured actions with applications in formal verification of programs [11], automated planning [26,33], dynamic epistemic logic [1] and deontic logic [19], for example.

In its standard formulation, PDL is a normal modal logic extending classical logic. Nevertheless, many non-classical versions of PDL—non-classical PDLs—have been explored as well, ranging from intuitionistic versions [6,21,36], to many-valued [4,5,12,15,16,34] and paraconsistent ones [29,30]. In [32], the landscape is extended with a study of propositional dynamic logic based on weak substructural logics in the vicinity of the Non-associative Lambek calculus. In that paper a formula-formula sequent system is used on the proof-theoretic side to complement a simple relational semantics extending frames for the Lambek calculus [7,20]. This approach, however, is not typical in all areas of substructural logic; especially in relevant logic a Hilbert-style proof theory combined with models based on partially ordered sets is preferred [25,28]. One naturally wonders if PDLs can be easily formulated in this setting as well.

In this paper we explore completeness of Hilbert-style formulations of substructural PDLs with respect to partially ordered models. We employ the technique of [31] (itself based on [3]), where a fragment of the present setting was studied, in combination with Nishimura's approach to intuitionistic PDL [21]. We show that the approach works for PDLs based on some weak substructural

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 594–609, 2019.
https://doi.org/10.1007/978-3-662-59533-6_36

logics but it fails for some stronger logics (for instance, some undecidable relevant logics); these ramifications are similar to those pointed out in [32]. In addition, extensions of Hilbert-style PDLs with primitive existential modalities ("diamond" versions of the action-indexed modalities) are shown to be problematic. These observations suggest that the study of substructural propositional dynamic logic abounds with interesting challenges and, most probably, requires the development of novel techniques.

The paper is structured as follows. In Sect. 2 we give the necessary background on PDL and on substructural logics. Section 3 discusses the motivation for studying substructural PDLs (in addition to technical curiosity). Completeness and decidability of PDL based on a weak substructural logic close to the Non-associative Lambek calculus is established in Sect. 4. The ramifications of the technique used to obtain the result, along with a number of related open problems, are discussed in Sect. 5.

2 Preliminaries

In this section we give an outline of propositional dynamic logic based on classical logic (Sect. 2.1, where we build on [11]) and of substructural logics (Sect. 2.2, where we build mainly on [25]).

2.1 Classical PDL

Fix countable sets At of atomic formulas and Ac of atomic action expressions. *Formulas* and *action expressions* are defined by mutual induction as follows: 1. Each $p \in At$ is a formula, the truth constant $\bar{1}$ and the falsity constant $\bar{0}$ are formulas and each combination of formulas using Boolean connectives $\wedge, \vee, \rightarrow$ is a formula; moreover, if α is an action expression, then $[\alpha]\varphi$ is a formula. 2. Each $a \in Ac$ is an action expression; if α and β are action expressions, then so are $\alpha;\beta$ (expressing composition of actions, "doing α and then β"), $\alpha \cup \beta$ (expressing non-deterministic choice, "doing α or β"), α^* (representing iteration, "doing α some finite number of times"). Moreover, if φ is a formula, then $\varphi?$ is an action expression (expressing test, "testing whether φ holds"). This language will be called the *dynamic language*. We define $\top$ as $\bar{0} \rightarrow \bar{0}$, $\neg\varphi$ and $\varphi \leftrightarrow \psi$ are defined as usual. Conjunctions and disjunctions of finite sets of formulas are defined as usual, with $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \bar{0}$.[1]

Formulas and action expressions are referred to jointly as *expressions*. The notions of subformula and action-subexpression are defined as expected. The relation of *subexpression* is defined as the least relation satisfying the following: 1. Each subformula of φ is a subexpression of φ; if φ is of the form $[\alpha]\psi$, then each subexpression of α is a subexpression of φ. 2. Each action-subexpression of α is a subexpression of α; if α is of the form $\varphi?$, then each subexpression of φ

[1] We distinguish between $\bar{1}$ and $\top$ for the sake of presentation; these will not be equivalent in substructural logics. See Sect. 2.2. We need $\bar{1}$ in our language for a technical reason, see the proof of Lemma 6.

is a subexpression of α. Proofs by induction on subexpressions are often used in propositional dynamic logic.

Fix an axiomatization CPC of the classical propositional calculus in the language $\{\wedge, \vee, \rightarrow, \bar{1}, \bar{0}\}$ using axiom schemata and Modus Ponens as the only rule of inference. The axiom system PDL is obtained by adding to CPC the axiom schemata

$$
\begin{aligned}
&([\alpha]\varphi \wedge [\alpha]\psi) \rightarrow [\alpha](\varphi \wedge \psi),\\
&[\alpha \cup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi),\\
&[\alpha ; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi,\\
&[\alpha^*]\varphi \rightarrow (\varphi \wedge [\alpha][\alpha^*]\varphi),\\
&[\psi?]\varphi \wedge \psi \rightarrow \varphi;
\end{aligned}
$$

and the inference rules

$$
\frac{\varphi \rightarrow \psi}{[\alpha]\varphi \rightarrow [\alpha]\psi}, \ \frac{\varphi \rightarrow [\alpha]\varphi}{\varphi \rightarrow [\alpha^*]\varphi} \text{ and } \frac{(\varphi \wedge \psi) \rightarrow \chi}{\varphi \rightarrow [\psi?]\chi}.
$$

Let us refer to these additional modal axioms and rules as MAX. Theorems and derivability in PDL are defined in the usual way. (Hence, derivability is finitary: φ is derivable from Γ iff it is derivable from a finite subset of Γ.)

A *standard frame* is a couple $F = \langle W, R\rangle$, where W is a non-empty set ("worlds" or "states") and R is a function from Ac to binary relations on W. The "accessibility" relation $R(a)$ represents actions of type a—$R(a)(x, y)$ can be read as "state y is accessible from x by performing an action of type a". A *standard model* is a triple $M = \langle W, R, V\rangle$, where V is a function from At to subsets of W. We also say that $\langle W, R, V\rangle$ is a model *based on* the frame $\langle W, R\rangle$.

For each M, we define the *evaluation function* $[\![\ ]\!]_M$ that assigns subsets of W to formulas and binary relations on W to action expressions in the following way (again, the definition is by mutual induction):

– $[\![p]\!]_M = V(p)$, $[\![\bar{1}]\!]_M = W$ and $[\![\bar{0}]\!]_M = \emptyset$; the usual set-theoretic clauses are used for Boolean combinations of formulas. Moreover, $[\![[\alpha]\varphi]\!]_M$ is the set of x such that, for all y, if $x[\![\alpha]\!]_M y$, then $y \in [\![\varphi]\!]_M$.
– $[\![a]\!]_M = R(a)$, $[\![\alpha \cup \beta]\!]_M$ is the union of $[\![\alpha]\!]_M$ and $[\![\beta]\!]_M$, $[\![\alpha ; \beta]\!]_M$ is the composition of $[\![\alpha]\!]_M$ and $[\![\beta]\!]_M$, $[\![\alpha^*]\!]_M$ is the reflexive transitive closure of $[\![\alpha]\!]_M$, and $[\![\varphi?]\!]_M$ is the identity relation on $[\![\varphi]\!]_M$.

Infix notation $x[\![\alpha]\!]_M y$ is used for the fact that $\langle x, y\rangle$ is in the relation $[\![\alpha]\!]_M$. The subscript is often omitted.

Formula φ is *valid* iff $[\![\varphi]\!]_M$ is the set of worlds in M, for all models M. More generally, φ follows from a set of assumptions Γ iff $\left(\bigcap_{\psi \in \Gamma} [\![\psi]\!]_M\right) \subseteq [\![\varphi]\!]_M$, for all M.

Theorem 1. *φ is a theorem of PDL iff it is valid. The set of theorems of PDL is decidable.*

Decidability of the set of valid formulas was shown in [8] using a finite model construction. Completeness of a system equivalent to PDL without test and with a "converse" modality was shown in [13,23], using a finite model construction similar to the one used in [8]; it is noted in the papers that the proof strategy is compatible with adding test and removing converse. For the full proof of this fact consult [11].

A noteworthy feature of propositional dynamic logic is that it is not compact. To see this, note that $[a^*]p$ follows from $\{p\} \cup \{[a^n]p \mid n \in \omega\}$, where a^1 is a and a^{n+1} is $a; a^n$. However, $[a^*]p$ does not follow from any finite subset of that set of assumptions. Hence, $[a^*]p$ is not derivable from that set of assumptions in PDL. As a result, one cannot hope for a *strong* completeness theorem for PDL. In [14] an *infinitary* proof system PDL_ω is shown to be strongly complete with respect to the standard semantics.

2.2 Some Substructural Logics

For the sake of simplicity, we diverge somewhat from the usual presentation (e.g. [10,25]) and we discuss substructural logics in the language of CPC, that is, in $\{\wedge, \vee, \rightarrow, \bar{1}, \bar{0}\}$.[2] Substructural logics in this language can be seen as logics where $\rightarrow$ lacks some properties that implication has in classical logic. Some such properties are given in Fig. 1.

B	$(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$	Associativity
C	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$	Commutativity
CI	$\varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$	Weak commutativity
W	$(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$	Contraction
WI	$(\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \psi$	Weak contraction
K	$\varphi \rightarrow (\psi \rightarrow \varphi)$	Weakening

Fig. 1. "Structural schemata" that fail in some substructural logics.

The reasons to avoid the respective properties of implication are related to various possible informal readings of $\rightarrow$. The Weakening axiom is usually avoided based on the assumption that ψ has to be *relevant* to φ in order for $\psi \rightarrow \varphi$ to be true. Note that the Weakening axiom entails that $\psi \rightarrow \varphi$ is derivable from the mere assumption that φ is the case, without assuming anything about ψ at all. These considerations led to the study of *relevant* logics; the main examples of such logics—for example the logic R—include all the other schemata. The contraction axiom is usually omitted when implication $\varphi \rightarrow \psi$ is read in terms of *resource use*, for instance as "by using a resource of type φ, outcome of type ψ may be produced". It is clear that some outputs require *several* pieces of resource of some type to be used. These considerations are central to *linear*

[2] This means that we do not include the fusion $\circ$ and the dual implication $\leftarrow$.

logic, for instance. In addition, contraction is also avoided in some *fuzzy logics* (logics of graded truth). Note that Contraction is also not plausible when formulas are seen as expressing types of linguistic items (expressions) and $\varphi \to \psi$ represents the type of expression that, when concatenated with expression of type φ, results in an expression of type ψ, such as in the various versions of the *Lambek calculus*. This interpretation is also inconsistent with Commutativity—the order of expression concatenation usually matters. Finally, Associativity is omitted in some versions of the Lambek calculus (not dealing with strings, but with some more general class of linguistic items). See the introductory chapter of [22] for more details on these motivations, for example. We note, in addition, that the Explosion principle, $(\varphi \wedge \neg\varphi) \to \psi$, follows from Weak contraction and $\bar{0} \to \psi$. *Paraconsistent* logics avoid the Explosion principle since it trivializes inconsistent sets of assumptions.

Let us turn to axiomatic presentations of substructural logics. In what follows, a *logic* will be any set of formulas in the language $\{\wedge, \vee, \to, \bar{1}, \bar{0}\}$ containing all the formulas of the form

$$\begin{array}{l}\bar{1},\ \varphi \to \varphi,\ (\varphi \wedge \psi) \to \varphi,\ (\varphi \wedge \psi) \to \psi,\ \varphi \to (\varphi \vee \psi),\ \psi \to (\varphi \vee \psi),\\ \varphi \wedge (\psi \vee \chi) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)),\ \bar{0} \to \varphi,\ \varphi \to (\bar{0} \to \bar{0}),\\ ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)),\\ ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi);\end{array}$$

and closed under

$$\frac{\varphi}{\bar{1} \to \varphi},\ \frac{\varphi \to \psi \quad \varphi}{\psi},\ \frac{\varphi \quad \psi}{\varphi \wedge \psi},\quad \frac{\varphi \to \psi \quad \chi \to \theta}{(\psi \to \chi) \to (\varphi \to \theta)}.$$

We sometimes write $\vdash_{\Lambda} \varphi$ instead of $\varphi \in \Lambda$.

Let Λ_0 be the smallest logic; it can be seen as a Hilbert-style axiomatization of a fragment of the Distributive Full Non-associative Lambek calculus extended with the falsity constant $\bar{0}$ (or, as we may also say in the terminology of [10], the $\{\wedge, \vee, \to, \bar{1}, \bar{0}\}$-fragment of the "zero-bounded" DFNL).

A *Routley–Meyer frame* is a structure $\mathcal{F} = \langle S, \leq, L, T\rangle$ where $\langle S, \leq\rangle$ is a partially ordered set, L is an upwards closed subset of $\langle S, \leq\rangle$ and T is a ternary relation on S such that

$$Txyz, x' \leq x, y' \leq y, z \leq z' \implies Tx'y'z' \tag{1}$$

$$x \leq y \iff (\exists z)(z \in L \ \&\ Tzxy) \tag{2}$$

A *Routley–Meyer model* based on $\mathcal{F}$ is $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where V is a function from At to upwards closed subsets of the frame $\mathcal{F}$. For each $\mathcal{M}$, we define the *evaluation function* $[\![]\!]_{\mathcal{M}}$ that assigns subsets of the frame on which $\mathcal{M}$ is based to formulas (states that "satisfy" the formulas) in the following way: $[\![p]\!]_{\mathcal{M}} = V(p)$, $[\![\bar{0}]\!]_{\mathcal{M}} = \emptyset$ and

$$[\![\bar{1}]\!]_{\mathcal{M}} = L; \tag{3}$$

the usual set-theoretic clauses are used for $\wedge, \vee$ and

$$[\![\varphi \to \psi]\!]_{\mathcal{M}} = \{x \mid (\forall yz)((Txyz \ \& \ y \in [\![\varphi]\!]_{\mathcal{M}}) \implies z \in [\![\psi]\!]_{\mathcal{M}})\} \tag{4}$$

Note that $[\![\top]\!]_{\mathcal{M}} = S$ and so $[\![\top]\!]_{\mathcal{M}} \neq [\![\bar{1}]\!]_{\mathcal{M}}$ if $L \neq S$.

The following well-known facts outline the reasons why Routley–Meyer frames contain L and $\leq$ and why (1–2) are assumed.

Lemma 1. *For all $\mathcal{M}$ and φ, $[\![\varphi]\!]_{\mathcal{M}}$ is an upwards closed set.*

Proof. Use (1) for $\varphi = \psi \to \chi$; other cases are trivial.

A formula φ is *valid in $\mathcal{M}$* iff $L \subseteq [\![\varphi]\!]_{\mathcal{M}}$.

Lemma 2. *$\varphi \to \psi$ is valid in $\mathcal{M}$ iff $[\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$.*

Proof. "If": Take $x \in L$ and assume that $Txyz$ and $y \in [\![\varphi]\!]_{\mathcal{M}}$. By (2), $y \leq z$ and by Lemma 1 $z \in [\![\varphi]\!]_{\mathcal{M}}$. Hence $z \in [\![\psi]\!]_{\mathcal{M}}$ by the assumption.

"Only if": Take $x \in [\![\varphi]\!]_{\mathcal{M}}$. By (2), we have $Tyxx$ for some $y \in L$ and so $x \in [\![\psi]\!]_{\mathcal{M}}$ by the assumption.

We note that formulas in Λ_0 are typically *not* satisfied in *all* states in a model. For instance, $p \to p$ may fail in x if there are y, z such that $Rxyz$ and $y \not\leq z$. However, thanks to Lemma 2, $p \to p$ is clearly satisfied in all $x \in L$.

Let Λ be a logic. A set of formulas Γ is a *non-trivial prime Λ-theory* iff 1. Γ is non-empty, 2. $\varphi \to \psi \in \Lambda$ and $\varphi \in \Gamma$ implies $\psi \in \Gamma$, 3. $\varphi, \psi \in \Gamma$ only if $\varphi \wedge \psi \in \Gamma$, 4. $\varphi \vee \psi \in \Gamma$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Lemma 3 (Pair Extension). *Let Λ be any logic extending Λ_0. Assume that there is no conjunction γ of elements of Γ and a disjunction δ of elements of Δ such that $\gamma \to \delta \in \Lambda$. Then there is a non-trivial prime Λ-theory Σ extending Γ and disjoint from Δ.*

Proof. See [25, 92–94].

A formula is *valid in $\mathcal{F}$* iff it is valid in all models based on $\mathcal{F}$.

Theorem 2. *$\varphi \in \Lambda_0$ iff φ is valid in all Routley–Meyer frames.*

Proof. Canonical model construction, the argument uses the Pair Extension Lemma; see [25] for details.

We note that a simpler semantics (without L and $\leq$) is sufficient for some formula-formula sequent presentations of some substructural logics, i.e. where a logic is defined as a set of ordered pairs of formulas, not as a set of formulas.

3 Motivation

The previous section suggests an obvious way to produce proof systems for substructural propositional dynamic logics—take a substructural logic and add MAX. Semantics for these proof systems do not seem hard to come by as well. Following the lead of the literature on modal relevant logics [9,17,18,27], the idea is to add to Routley–Meyer frames a function R from atomic action expressions Ac to binary relations on S satisfying a tonicity condition in the style of (1) and then define the evaluation function on complex action expressions in the style of classical PDL. It is to be expected that if Λ is sound and complete with respect to a class of Routley–Meyer frames, then PDL_{Λ}, an extension of Λ with MAX, is sound and complete with respect to suitable "modal extensions" of frames in the class. We will show in Sect. 4 that this is indeed the case for PDL_{Λ_0} and we will point out some problems that pop up when stronger logics are considered in Sect. 5.

But first, we need to address another question, namely, *why* is it interesting to consider such substructural PDLs. We will not go into a detailed discussion of this important question here. We just point out some relations of the present question to the original motivations for omitting some of the structural schemata of Fig. 1.

One of the crucial properties of actions expressed in the language of PDL are *partial correctness assertions* of the type

$$\varphi \to [\alpha]\psi,$$

read "if φ is the case, then each (terminating) execution of action α leads to a state where ψ holds"; see [11]. One may insist that such assertions express meaningful properties of actions only if φ is *relevant*, in some sense close to the motivations of relevant logic, to $[\alpha]\psi$ (or to ψ).[3] This motivates the study of PDL without the Weakening axiom. We note that most non-classical PDLs studied in the literature so far (intuitionistic and fuzzy PDLs) assume Weakening.

In general, omissions of the structural schemata from PDL can be motivated by the goal of formulating logics of structured actions that modify the types of objects related to non-modal substructural logics without the respective structural schemata. For instance, assume that we want to study a logic for reasoning about structured actions modifying linguistic items (expressions) of some kind. It is reasonable to take a PDL based on some version of the Lambek calculus. Similarly, reasoning about actions in a setting where graded truth values are admitted (e.g. situations where graded predicates play an important role), requires a fuzzy version of PDL without Contraction.

4 The Basic Substructural PDL

In this section we prove completeness and decidability of the basic substructural propositional dynamic logic PDL_{Λ_0}, which we denote simply as PDL_0. To be

[3] For instance, on may wonder if $p \to [\alpha]\top$ express a meaningful specification of α.

more precise, PDL_0 is the least set of formulas of the dynamic language 1. containing all the formulas of the forms used in the definition of Λ_0 and closed under all the Λ_0-inference rules; and 2. containing (or closed under) all elements of MAX.

A *dynamic Routley–Meyer frame* is a structure $\mathfrak{F} = \langle S, \leq, L, T, R \rangle$ where $\langle S, \leq, L, T \rangle$ is a Routley–Meyer frame and R is a function from Ac to binary relations on S such that

$$R(a)xy, x' \leq x, y \leq y' \implies R(a)x'y' \tag{5}$$

A *dynamic Routley–Meyer model based on* $\mathfrak{F}$ is $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where V is as in Routley–Meyer models. The evaluation function $[\![\ ]\!]_{\mathfrak{M}}$ assigning subsets of S to formulas and binary relations on S to action expressions is defined as in dynamic and Routley–Meyer models, respectively (the clause for $\rightarrow$ is the one used in Routley–Meyer models), with one exception:

$$[\![\varphi?]\!]_{\mathfrak{M}} = \{\langle x, y \rangle \mid x \leq y \ \ \& \ \ y \in [\![\varphi]\!]_{\mathfrak{M}}\} \tag{6}$$

Lemma 4. *1. Each $[\![\varphi]\!]_{\mathfrak{M}}$ is upwards closed.*
2. For all α, $x[\![\alpha]\!]_{\mathfrak{M}}y$, $x' \leq x$ and $y \leq y'$ imply $x'[\![\alpha]\!]_{\mathfrak{M}}y'$.

Proof. The claims are established simultaneously by induction on subexpressions (in each case, the induction hypothesis is that *both* claims hold for all proper subexpressions of the expression at hand). 1. The only new claim is the one concerning formulas of the form $[\alpha]\psi$—and that claim is easily seen to follow from 2. for α (a subexpression of $[\alpha]\psi$). 2. We give details of the case $\psi?$ as it hinges on the non-standard evaluation condition (6). If $x \leq y$ and $y \in [\![\psi]\!]_{\mathfrak{M}}$, then $x' \leq x$ and $y \leq y'$ imply $x' \leq y'$ and $y' \in [\![\psi]\!]_{\mathfrak{M}}$ by transitivity of $\leq$ and claim 1. for ψ (a subexpression of $\psi?$).

The proof of Lemma 4 provides the justification for our choice of the non-standard evaluation condition (6)—note that the second claim of the lemma would fail if $[\![\varphi?]\!]_{\mathfrak{M}}$ was defined, classically, as the identity relation on $[\![\varphi]\!]_{\mathfrak{M}}$. Nevertheless, this definition seems to bring about a significant shift in "meaning" of the test action when compared to the classical case. Instead of "Test whether φ is satisfied; do not change state", we now have "Move to an arbitrary bigger state that supports φ", i.e. something along the lines of "Assume, *ceteris paribus*, that φ is satisfied". Should we even call this action *test*?

We note only that the classical definition of test is a special case, obtained under particular assumptions concerning the notion of a state, of the new definition. Note that if only the maximal elements in the partial ordering are considered (assume, for the sake of discussion, that we have a model where such maximal elements exist), then the newly defined $[\![\varphi?]\!]_{\mathfrak{M}}$ is in fact the identity relation on $[\![\varphi]\!]_{\mathfrak{M}}$—moving to an arbitrary bigger state supporting φ amounts to staying in the present state if φ is satisfied there and "aborting" otherwise, just as in the classical case. Hence, the shift here is not in the meaning of test, but in the kind of state allowed.[4]

Validity in dynamic Routley–Meyer models and frames, respectively, is defined in the same way as in Routley–Meyer models.

Lemma 5. *$\varphi \to \psi$ is valid in $\mathfrak{M}$ iff $[\![\varphi]\!]_{\mathfrak{M}} \subseteq [\![\psi]\!]_{\mathfrak{M}}$.*

Proof. Similar to the proof of Lemma 2, using Lemma 4.

Theorem 3. *Each element of PDL_0 is valid in all dynamic Routley–Meyer frames.*

Proof. Induction on the length of the proof; Lemma 5 provides a useful shortcut.

Completeness is established by a finitary method related to the standard proofs for PDL. Out of convenience we chose a combination of the method outlined in [31] with Nishimura's approach given in [21]. (It is also possible to obtain the result by combining the technique of [31] with the standard approach of [11,23] that uses non-standard models, but we have opted for a more direct approach that we deem more elegant.)

Definition 1 (Closure). *Let Σ be a set of formulas of the dynamic language. The* closure *of Σ is the least set $\Sigma^c \supseteq \Sigma$ closed under subformulas such that:*

- $\bar{0} \to \bar{0} \in \Sigma^c$ *and* $\bar{1} \in \Sigma^c$
- $[\alpha \cup \beta]\varphi \in \Sigma^c$ *implies* $[\alpha]\varphi \in \Sigma^c$ *and* $[\beta]\varphi \in \Sigma^c$
- $[\alpha;\beta]\varphi \in \Sigma^c$ *implies* $[\alpha][\beta]\varphi \in \Sigma^c$
- $[\alpha^*]\varphi \in \Sigma^c$ *implies* $[\alpha][\alpha^*]\varphi \in \Sigma^c$
- $[\psi?]\varphi \in \Sigma^c$ *implies* $\psi \in \Sigma^c$

Σ is closed *iff $\Sigma = \Sigma^c$.*

We say that a pair of sets of formulas $\underline{\Gamma} = \langle \underline{\Gamma}^+, \underline{\Gamma}^- \rangle$ is an *independent Λ-pair* (member of IP_Λ) iff there is no conjunction γ^+ of elements of $\underline{\Gamma}^+$ and disjunction γ^- of element of $\underline{\Gamma}^-$ such that $\gamma^+ \to \gamma^-$ is in Λ. (We note that both $\underline{\Gamma}^+$ and $\underline{\Gamma}^-$ may be empty or infinite.) Recall the Pair Extension Lemma 3 saying that for each $\underline{\Gamma} \in IP_\Lambda$ there is a non-trivial prime Λ-theory $\Delta \supseteq \underline{\Gamma}^+$ disjoint from $\underline{\Gamma}^-$.

Definition 2 (Canonical model). *Let Φ be a finite closed set. The Λ-*canonical model *for Φ is defined as consisting of the following elements:*

- *S^Φ is the set of all $\underline{\Gamma} \in IP_\Lambda$ such that $\underline{\Gamma}^+ \cup \underline{\Gamma}^- = \Phi$*
- *$\underline{\Gamma} \leq^\Phi \underline{\Delta}$ iff $\underline{\Gamma}^+ \subseteq \underline{\Delta}^+$*
- *L^Φ is the set of $\underline{\Gamma}$ such that $\bar{1} \in \underline{\Gamma}^+$*

[4] Another interesting observation is that, on the present evaluation condition, $[\varphi?]\psi$ is equivalent to intuitionistic implication $\varphi \to_{IL} \psi$. Hence, in a sense, our substructural PDLs contain intuitionistic PDL.

- $T^{\Phi}\underline{\Gamma}\,\underline{\Delta}\,\underline{\Sigma}$ *iff there are non-trivial prime* Λ*-theories* Γ', Δ' *and* Σ' *such that* $\underline{\Gamma}^{+} \subseteq \Gamma'$, $\underline{\Delta}^{+} \subseteq \Delta'$, $(\Sigma' \cap \Phi) \subseteq \underline{\Sigma}^{+}$ *such that*

$$(\forall \varphi, \psi)(\varphi \to \psi \in \Gamma' \ \& \ \varphi \in \Delta' \implies \psi \in \Sigma') \tag{7}$$

- $R^{\Phi}(a)\underline{\Gamma}\,\underline{\Delta}$ *iff, for all* $[a]\varphi \in \Phi$, *if* $[a]\varphi \in \underline{\Gamma}^{+}$, *then* $\varphi \in \underline{\Delta}^{+}$
- *If* $p \in \Phi$, *then* $V^{\Phi}(p) = \{\underline{\Gamma} \mid p \in \underline{\Gamma}^{+}\}$; $V^{\Phi}(p) = \emptyset$ *otherwise.*

The canonical evaluation function $[\![\ ]\!]^{\Phi}$ *is defined as in dynamic Routley–Meyer models.*

Lemma 6. *For each* Λ *and finite* Φ, $\mathfrak{M}^{\Phi}_{\Lambda}$ *is a dynamic Routley–Meyer model.*

Proof. We show that the "if" implication of (2) holds. Let $\varphi \in \underline{\Gamma}^{+}$ (recall that $\varphi \in \Phi$ as a result) and $T^{\Phi}\underline{\Sigma}\,\underline{\Gamma}\,\underline{\Delta}$ for some $\underline{\Sigma} \in L^{\Phi}$. The latter means that $\bar{1} \in \underline{\Sigma}^{+} \subseteq \Sigma'$ for some non-trivial prime theory Σ' such that there are non-trivial prime theories Γ' and Δ', where $\underline{\Gamma}^{+} \subseteq \Gamma'$ and $(\Delta' \cap \Phi) \subseteq \underline{\Delta}^{+}$, for which it holds that if $\varphi \to \psi \in \Sigma'$, then $\psi \in \Delta'$. But $\vdash_{\Lambda} \bar{1} \to (\varphi \to \varphi)$, so $\varphi \to \varphi \in \Sigma'$, so $\varphi \in \Delta'$, so $\varphi \in \underline{\Delta}^{+}$.

It is noteworthy that this argument could not have been simulated without $\bar{1}$ in the language. Then the only plausible definition of $\underline{\Sigma} \in L^{\Phi}$ is that $\underline{\Sigma}^{+} \subseteq \Sigma'$ for some $\Sigma' \supseteq \Lambda$. However, the fact that $T^{\Phi}\underline{\Sigma}\,\underline{\Gamma}\,\underline{\Delta}$ allows us to infer that there is *some* $\Sigma'' \supseteq \underline{\Sigma}^{+}$, possibly different from $\Sigma' \supseteq \Lambda$, such that a version of (7) holds for some $\Gamma' \supseteq \underline{\Gamma}^{+}$ and $(\Delta' \cap \Phi) \subseteq \underline{\Delta}^{+}$. Hence, we cannot infer that $\varphi \to \varphi \in \Sigma''$.

Clearly if $\varphi \notin \Lambda$, then $\bar{1} \to \varphi \notin \Lambda$ and so $\langle\{\bar{1}\}, \{\varphi\}\rangle \in IP_{\Lambda}$. We want to show now that there is a state in $\mathfrak{M}^{\{\bar{1}\to\varphi\}^{c}}_{\Lambda}$ that satisfies $\bar{1}$ (i.e. it is a logical state), but not φ. Then we will have shown that φ is not valid in all dynamic Routley–Meyer frames. This yields a completeness result for PDL_0 right away as, in this case, it is not necessary to show that the frame underlying $\mathfrak{M}^{\{\bar{1}\to\varphi\}^{c}}_{PDL_0}$ satisfies any additional frame conditions.

Lemma 7. *Let* $\underline{\Gamma} \in IP_{\Lambda}$ *such that* $\underline{\Gamma}^{+} \cup \underline{\Gamma}^{-} \subseteq \Phi$. *Then there is* $\underline{\Delta} \in IP_{\Lambda}$ *such that* $\underline{\Gamma}^{+} \subseteq \underline{\Delta}^{+}$, $\underline{\Gamma}^{-} \subseteq \underline{\Delta}^{-}$ *and* $\underline{\Delta}^{+} \cup \underline{\Delta}^{-} = \Phi$.

Proof. Similar to the proof of the Pair Extension Lemma, see [25, 92–94].

Let X, Y be subsets of S^{Φ}. We define αX as the set of such $\underline{\Gamma}$ where $\underline{\Gamma}[\![\alpha]\!]^{\Phi}\underline{\Delta}$ implies $\underline{\Delta} \in X$. Let us also define $f(\underline{\Gamma}) = \bigwedge \underline{\Gamma}^{+}$ and $f(\{\underline{\Gamma_1}, \ldots, \underline{\Gamma_n}\}) = \bigvee_{i \leq n} f(\Gamma_i)$.

Lemma 8. *1. For all* $\varphi \in \Phi$, $\varphi \in [\![\Sigma]\!]^{\Phi}$ *iff* $\varphi \in \underline{\Sigma}^{+}$.
2. If $[\alpha]\varphi \in \Phi$ *for some* φ, *then* $X \subseteq \alpha Y$ *implies that* $\vdash_{\Lambda} f(X) \to [\alpha]f(Y)$.

Proof. See the Technical appendix.

Theorem 4. *Each formula valid in all dynamic Routley–Meyer frames belongs to* PDL_0.

Proof. If $\nvdash_{PDL_0} \varphi$, then $\nvdash_{PDL_0} \bar{1} \to \varphi$, so $\langle \{\bar{1}\}, \{\varphi\} \rangle$ in IP_{PDL_0}. By Lemma 6, $\mathfrak{M}_{PDL_0}^{\{\bar{1}\to\varphi\}^c}$ is a dynamic Routley–Meyer model. By Lemma 8, φ is not valid in the model. Hence, φ is not valid in all dynamic Routley–Meyer frames.

Theorem 5. *PDL_0 is a decidable set.*

Proof. Note that if Γ is finite, then so is Γ^c. The number of models in $\mathfrak{M}_{PDL_0}^{\Gamma^c}$ is at most $2^{|\Gamma^c|}$.

5 Beyond the Minimal Substructural PDL

In this section we discuss the applicability of our technique to some extensions of PDL_0.

5.1 Axiomatic Extensions

Let us refer to the schemata shown in Fig. 1 as "structural schemata". Similarly to modal logic, structural schemata *define* various properties of Routley–Meyer frames in the sense of correspondence theory—the defining schema holds in a frame iff the frame has the defined property. Figure 2 shows the frame properties defined by the structural schemata, see [25, ch. 11] for proofs and details ($T(xy)zw$ means $(\exists u)(Txyu \ \& \ Tuzw)$, $Tx(yz)w$ means $(\exists u)(Tyzu \ \& \ Txuw)$.)

B	$T(xy)zw \to Tx(yz)w$
C	$T(yx)zw \to Tx(yz)w$
CI	$Txyz \to Tyxz$
W	$Txyz \to T(xy)yz$
WI	$Txxx$
K	$Txyz \to x \leq z$

Fig. 2. Frame properties defined by the structural schemata shown in Fig. 1.

Let us denote as $\Lambda_{\mathsf{S}_1 \dots \mathsf{S}_n}$ the extension of Λ_0 with $\mathsf{S}_1 \dots \mathsf{S}_n$ as extra axiom schemata (in the obvious sense); $PDL_{\mathsf{S}_1 \dots \mathsf{S}_n}$ is the extension of $\Lambda_{\mathsf{S}_1 \dots \mathsf{S}_n}$ with MAX. A plausible conjecture is the following:

Conjecture 1. $PDL_{\mathsf{S}_1 \dots \mathsf{S}_n}$ is sound and complete with respect to the class of dynamic Routley–Meyer models with the properties defined by $\mathsf{S}_1 \dots \mathsf{S}_n$.

As it happens, the present technique can be used to establish only some special cases of Conjecture 1.

Theorem 6. *Let $\mathsf{S}_1 \dots \mathsf{S}_n$ be any combination of CI, WI and K. Then $PDL_{\mathsf{S}_1 \dots \mathsf{S}_n}$ is sound and complete with respect to the corresponding class of dynamic Routley–Meyer models.*

Proof. It is sufficient to show that the frame underlying $\mathfrak{M}^{\Phi}_{PDL_{\mathsf{S}_1\ldots\mathsf{S}_n}}$ has the corresponding frame properties. Let $\mathsf{S}_1\ldots\mathsf{S}_n = \mathsf{CI}$. Assume that $T^{\Phi}\underline{\Gamma\Delta\Sigma}$; hence, (7) holds for some appropriate Γ', Δ' and Σ'. Now assume that $\varphi \to \psi \in \Delta' \supseteq \underline{\Delta}^+$ and $\varphi \in \Gamma' \supseteq \underline{\Gamma}^+$. Using the axiom schema CI and the fact that Γ' is a prime theory, we have $(\varphi \to \psi) \to \psi \in \Gamma'$. Hence, $\psi \in \Sigma'$ by (7). The argument is similar in the remaining cases.

It is easy to see that our "finitary" technique *cannot* be used for some combinations of structural schemata. For instance, Λ_{BC} and Λ_{BCW} are fragments of the *undecidable* relevant logics $R{-}W$ and R [35], [25, ch. 15], so we cannot hope for a finite model property for these logics. However, our proof always produces a *finite* countermodel for a unprovable formula.

It is not straightforward to imagine a modification of our technique that would yield a proof of Conjecture 1 in these undecidable cases. We leave this as a curious open problem.

A surrogate strategy that might look promising at first is to work at least with an *infinitary* proof system $PDL^{\omega}_{\mathsf{S}_1\ldots\mathsf{S}_n}$ in the problematic cases (infinitary in the sense of containing an inference rule for α^* with a countable set of assumptions). It is shown in [14] that, in the case based on classical logic, using an infinitary proof system allows to construct a well-behaved *infinite* canonical model. This sounds promising for logics without the finite model property. However, as shown in [2], the Pair Extension Lemma does not hold for infinitary logics. We do not see how our proof could be rephrased without using the Pair Extension Lemma.

5.2 Adding Diamonds

Another way to extend PDL_0 is to add to the language *primitive* existential modalities $\langle\alpha\rangle$ with evaluation defined as follows:

- $[\![\langle\alpha\rangle\varphi]\!]_{\mathfrak{M}} = \{x \mid (\exists y)(x[\![\alpha]\!]_{\mathfrak{M}}y \;\;\&\;\; y \in [\![\varphi]\!]_{\mathfrak{M}})\}$[5]

Without going into details we note two problems that arise from such an addition; both are related to the canonical model construction. Firstly, the presence of primitive diamonds requires to modify the definition of $R^{\Phi}(a)$ by adding the requirement that $R^{\Phi}(a)\underline{\Gamma\Delta}$ only if, for all $\langle a\rangle\varphi \in \Phi$, if $\varphi \in \underline{\Delta}^+$, then $\langle a\rangle\varphi \in \underline{\Gamma}^+$. This modification makes it problematic to prove a version of Lemma 8; we have failed to provide a proof without the extra assumption that $R(a)$ is a serial relation.

Secondly, neither (5) nor any other condition presently assumed entail that $[\![\langle\alpha\rangle\varphi]\!]_{\mathfrak{M}}$ be an upset. Some additional frame condition is required, for example:

$$R(a)xy \;\;\&\;\; x \le x' \implies (\exists y')(y \le y' \;\;\&\;\; R(a)x'y') \tag{8}$$

or the stronger

$$R(a)xy \;\;\&\;\; x \le x' \implies R(a)x'y \tag{9}$$

[5] Note that $\langle\alpha\rangle\varphi$ can be defined as $\neg[\alpha]\neg\varphi$ in the present setting, but the defined modality does not yield the same truth condition—recall that $\neg\psi$ is defined as $\psi \to \bar{0}$.

On the assumption of either one of these conditions, however, the proof of Lemma 6 seems to fail.

Acknowledgements. This work was carried out within the project *Enhancing human resources for research in theoretical computer science* (no. CZ.02.2.69/0.0/0.0/17_050/0008361), funded by the Operational Programme *Research, Development and Education* of the Ministry of Education, Youth and Sports of the Czech Republic. The project is co-funded by the EU. The author is grateful to two anonymous WoLLIC reviewers for useful feedback and to Vít Punčochář and Andrew Tedder for fruitful collaboration on the topic.

A Technical Appendix

Lemma 8.

1. For all $\varphi \in \Phi$, $\varphi \in [\![\Sigma]\!]^{\Phi}$ iff $\varphi \in \underline{\Sigma}^{+}$.
2. If $[\alpha]\varphi \in \Phi$ for some φ, then $X \subseteq \alpha Y$ implies that $\vdash_{\Lambda} f(X) \to [\alpha]f(Y)$.

Proof. Induction on subexpressions. 1. holds for $p \in At$ by definition and the inductive steps for the rest of the Boolean connectives are easy. The case for $[\alpha]\varphi$ is more complicated. Note that we have to prove that $[\alpha]\varphi \in \underline{\Gamma}^{+}$ iff $\underline{\Gamma} \in \alpha[\![\varphi]\!]^{\Phi}$. The "if" part is established using claim 2. for α (a subexpression of $[\alpha]\varphi$) as follows. If $\underline{\Gamma} \in \alpha[\![\varphi]\!]^{\Phi}$, then $\vdash_{\Lambda} f(\underline{\Gamma}) \to [\alpha]f([\![\varphi]\!]^{\Phi})$ by 2. Note that $\vdash_{\Lambda} f([\![\varphi]\!]^{\Phi}) \to \varphi$ by the induction hypothesis ($\varphi \in \underline{\Delta}^{+}$ for all $\underline{\Delta} \in [\![\varphi]\!]^{\Phi}$) and so $\vdash_{\Lambda} [\alpha]f([\![\varphi]\!]^{\Phi}) \to [\alpha]\varphi$ using the fact that Λ contains MAX. Hence, $\vdash_{\Lambda} f(\underline{\Gamma}) \to [\alpha]\varphi$. Now $[\alpha]\varphi$ is assumed to be in Φ, so if it were the case that $[\alpha]\varphi \notin \underline{\Gamma}^{+}$, then $\underline{\Gamma} \notin IP_{\Lambda}$ contrary to our assumption. Hence, $[\alpha]\varphi \in \underline{\Gamma}^{+}$. The "only if" part is established by induction on the complexity of α using the MAX axioms for the action operators; we skip the details.

2. Assume that $X \subseteq aY$. Take an arbitrary $\underline{\Gamma} \in X$. Suppose, for the sake of contradiction, that

$$\nvdash_{\Lambda} f(\underline{\Gamma}) \to [a]f(Y).$$

Let $Z = \{\psi \mid [a]\psi \in \underline{\Gamma}^{+}\}$. It follows that

$$\nvdash_{\Lambda} \bigwedge Z \to f(Y).$$

Hence, by the Pair Extension Lemma, there is a non-trivial prime Λ-theory Δ such that $Z \subseteq \Delta$ and $f(Y) \notin \Delta$. Now consider $\underline{\Sigma} = \langle \Delta \cap \Phi, \bar{\Delta} \cap \Phi \rangle$ (where $\bar{\Delta}$ is the complement of Δ). Obviously $\underline{\Sigma} \in IP_{\Lambda}$ and $R^{\Phi}(a)\underline{\Gamma}\underline{\Sigma}$. Hence, by our assumption, $\underline{\Sigma} \in Y$, so

$$\vdash_{\Lambda} f(\underline{\Sigma}) \to f(Y),$$

but also

$$\vdash_{\Lambda} \bigwedge Z \to f(\underline{\Sigma})$$

(by the construction of $\underline{\Sigma}$), so

$$\vdash_{\Lambda} \bigwedge Z \to f(Y),$$

contrary to our assumption. Consequently, it has to be the case that $\vdash_\Lambda f(\underline{\Gamma}) \to [a]f(Y)$. The same argument can be repeated for all $\underline{\Gamma}_i \in X$. Hence, $\vdash_\Lambda f(X) \to [a]f(Y)$.

The inductive steps for concatenation and choice are easily established using the MAX axioms characterising these action operators. It is worthwhile to go through the cases for α^* and $\varphi?$. Assume first that $X \subseteq \alpha^* Y$. Hence

$$\vdash_\Lambda f(X) \to f(\alpha^* Y)$$

It is easily seen that $\alpha^* Y \subseteq \alpha(\alpha^* Y)$. Hence, using induction hypothesis for α,

$$\vdash_\Lambda f(\alpha^* Y) \to [\alpha]f(\alpha^* Y).$$

So, by the MAX rule characterizing α^*,

$$\vdash_\Lambda f(\alpha^* Y) \to [\alpha^*]f(\alpha^* Y).$$

Yet, we have

$$\vdash_\Lambda [\alpha^*]f(\alpha^* Y) \to [\alpha^*]f(Y)$$

(note that $\alpha^* Y \subseteq Y$ and use the monotonicity MAX rule). Therefore,

$$\vdash_\Lambda f(X) \to [\alpha^*]f(Y).$$

The case for $\varphi?$ is established as follows. Assume that $X \subseteq (\varphi?)Y$. Take an arbitrary $\underline{\Gamma} \in X$ and assume, for the sake of contradiction, that

$$\nvdash_\Lambda f(\underline{\Gamma}) \to [\varphi?]f(Y).$$

Hence, using the MAX rule for $\varphi?$,

$$\nvdash_\Lambda (f(\underline{\Gamma}) \wedge \varphi) \to f(Y).$$

Using the Pair Extension Lemma, there is a non-trivial prime theory Δ containing $f(\underline{\Gamma}) \wedge \varphi$ but not containing $f(Y)$. Now take $\underline{\Delta} = \langle \Delta \cap \Phi, \bar{\Delta} \cap \Phi \rangle$. It is clear that $\underline{\Gamma} \leq^\Phi \underline{\Delta}$ and that $\underline{\Delta} \in [\![\varphi]\!]^\Phi$ (by induction hypothesis 1. applied to φ, the subexpression of $\varphi?$). By the definition of $[\![\varphi?]\!]^\Phi$, it follows that $\underline{\Gamma}[\![\varphi?]\!]^\Phi \underline{\Delta}$ and, hence, $\underline{\Delta} \in Y$. Consequently,

$$\vdash_\Lambda f(\underline{\Delta}) \to f(Y).$$

But this contradicts the observation that the prime theory Δ does not contain $f(Y)$. Hence, it must be the case that $\vdash_\Lambda (f(\underline{\Gamma}) \wedge \varphi) \to f(Y)$. Similar reasoning can be applied to each element of X, so

$$\vdash_\Lambda f(X) \to [\varphi?]f(Y).$$

References

1. Baltag, A., Moss, L.S.: Logics for epistemic programs. Synthese **139**(2), 165–224 (2004). https://doi.org/10.1023/B:SYNT.0000024912.56773.5e
2. Bílková, M., Cintula, P., Lávička, T.: Lindenbaum and pair extension lemma in infinitary logics. In: Moss, L.S., de Queiroz, R., Martinez, M. (eds.) WoLLIC 2018. LNCS, vol. 10944, pp. 130–144. Springer, Heidelberg (2018). https://doi.org/10.1007/978-3-662-57669-4_7
3. Bílková, M., Majer, O., Peliš, M.: Epistemic logics for sceptical agents. J. Log. Comput. **26**(6), 1815–1841 (2016)
4. Boutilier, C.: Toward a logic for qualitative decision theory. In: Doyle, J., Sandewall, E., Torasso, P. (eds.) Principles of Knowledge Representation and Reasoning, pp. 75–86. Morgan Kaufmann, Burlington (1994)
5. Běhounek, L.: Modeling costs of program runs in fuzzified propositional dynamic logic. In: Hakl, F. (ed.) Doktorandské dny '08, pp. 6–14. ICS AS CR and Matfyzpress, Prague (2008)
6. Degen, J., Werner, J.: Towards intuitionistic dynamic logic. Log. Log. Philos. 15(4), 305–324 (2006). http://apcz.umk.pl/czasopisma/index.php/LLP/article/view/LLP.2006.018
7. Došen, K.: A brief survey of frames for the Lambek calculus. Math. Logic Q. **38**(1), 179–187 (1992). https://doi.org/10.1002/malq.19920380113
8. Fischer, M.J., Ladner, R.E.: Propositional dynamic logic of regular programs. J. Comput. Syst. Sci. **18**, 194–211 (1979)
9. Fuhrmann, A.: Models for relevant modal logics. Studia Logica **49**(4), 501–514 (1990). https://doi.org/10.1007/BF00370161
10. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Elsevier, Amsterdam (2007)
11. Harel, D., Kozen, D., Tiuryn, J.: Dynamic Logic. MIT Press, Cambridge (2000)
12. Hughes, J., Esterline, A., Kimiaghalam, B.: Means-end relations and a measure of efficacy. J. Log. Lang. Inf. **15**(1), 83–108 (2006). https://doi.org/10.1007/s10849-005-9008-4
13. Kozen, D., Parikh, R.: An elementary proof of the completeness of PDL. Theor. Comput. Sci. **14**, 113–118 (1981)
14. de Lavalette, G.R., Kooi, B., Verbrugge, R.: Strong completeness and limited canonicity for PDL. J. Log. Lang. Inf. **17**(1), 69–87 (2008). https://doi.org/10.1007/s10849-007-9051-4
15. Liau, C.-J.: Many-valued dynamic logic for qualitative decision theory. In: Zhong, N., Skowron, A., Ohsuga, S. (eds.) RSFDGrC 1999. LNCS (LNAI), vol. 1711, pp. 294–303. Springer, Heidelberg (1999). https://doi.org/10.1007/978-3-540-48061-7_36
16. Madeira, A., Neves, R., Martins, M.A.: An exercise on the generation of many-valued dynamic logics. J. Log. Algebr. Methods Program. **85**(5, Part 2), 1011–1037 (2016). https://doi.org/10.1016/j.jlamp.2016.03.004. http://www.sciencedirect.com/science/article/pii/S2352220816300256. Articles dedicated to Prof. J. N. Oliveira on the occasion of his 60th birthday
17. Mares, E.D.: The semantic completeness of RK. Rep. Math. Log. **26**, 3–10 (1992)
18. Mares, E.D., Meyer, R.K.: The semantics of R4. J. Philos. Log. **22**(1), 95–110 (1993). https://doi.org/10.1007/BF01049182
19. Meyer, J.J.C.: A different approach to deontic logic: deontic logic viewed as a variant of dynamic logic. Notre Dame J. Formal Log. **29**(1), 109–136 (1987)

20. Moot, R., Retoré, C.: The Logic of Categorial Grammars. LNCS. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-31555-8
21. Nishimura, H.: Semantical analysis of constructive PDL. Publ. Res. Inst. Math. Sci. **18**(2), 847–858 (1982). https://doi.org/10.2977/prims/1195183579
22. Paoli, F.: Substructural Logics: A Primer. Kluwer, Dordrecht (2002)
23. Parikh, R.: The completeness of propositional dynamic logic. In: Winkowski, J. (ed.) MFCS 1978. LNCS, vol. 64, pp. 403–415. Springer, Heidelberg (1978). https://doi.org/10.1007/3-540-08921-7_88
24. Pratt, V.: Semantical considerations on Floyd-Hoare logic. In: 7th Annual Symposium on Foundations of Computer Science, pp. 109–121. IEEE Computing Society (1976)
25. Restall, G.: An Introduction to Substrucutral Logics. Routledge, London (2000)
26. Rosenschein, S.: Plan synthesis: a logical perspective. In: Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI) (1981)
27. Routley, R., Meyer, R.K.: The semantics of entailment-II. J. Philos. Logic **1**(1), 53–73 (1972). https://doi.org/10.1007/BF00649991
28. Routley, R., Meyer, R.K.: Semantics of entailment. In: Leblanc, H. (ed.) Truth Syntax and Modality, pp. 194–243. North Holland, Amsterdam (1973)
29. Sedlár, I.: Propositional dynamic logic with Belnapian truth values. In: Advances in Modal Logic, vol. 11. College Publications, London (2016)
30. Sedlár, I.: Non-classical PDL on the cheap. In: Arazim, P., Lávička, T. (eds.) The Logica Yearbook 2016, pp. 239–256. College Publications, London (2017)
31. Sedlár, I.: Substructural logics with a reflexive transitive closure modality. In: Kennedy, J., de Queiroz, R.J.G.B. (eds.) WoLLIC 2017. LNCS, vol. 10388, pp. 349–357. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-662-55386-2_25
32. Sedlár, I., Punčochář, V.: From positive PDL to its non-classical extensions. Log. J. IGPL (2019, forthcoming)
33. Spalazzi, L., Traverso, P.: A dynamic logic for acting, sensing, and planning. J. Log. Comput. **10**(6), 787–821 (2000). https://doi.org/10.1093/logcom/10.6.787
34. Teheux, B.: Propositional dynamic logic for searching games with errors. J. Appl. Log. **12**(4), 377–394 (2014)
35. Urquhart, A.: The undecidability of entailment and relevant implication. J. Symb. Log. **49**(4), 1059–1073 (1984). http://www.jstor.org/stable/2274261
36. Wijesekera, D., Nerode, A.: Tableaux for constructive concurrent dynamic logic. Ann. Pure Appl. Log. **135**(1), 1–72 (2005). https://doi.org/10.1016/j.apal.2004.12.001. http://www.sciencedirect.com/science/article/pii/S0168007204001794

Modal Logics of Finite Direct Powers of ω Have the Finite Model Property

Ilya Shapirovsky[1,2(✉)]

[1] Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia
[2] Institute for Information Transmission Problems of Russian Academy of Sciences, Moscow, Russia
shapir@iitp.ru

Abstract. Let $(\omega^n, \preceq)$ be the n-th direct power of $(\omega, \leq)$, natural numbers with the standard ordering, and let $(\omega^n, \prec)$ be the n-th direct power of $(\omega, <)$. We show that for all finite n, the modal algebras of $(\omega^n, \preceq)$ and of $(\omega^n, \prec)$ are locally finite. In particular, it follows that the modal logics of these frames have the finite model property.

Keywords: Modal logic · Modal algebra · Finite model property · Local finiteness · Tuned partition · Direct product of frames

1 Introduction

We consider modal logics of direct products of linear orders. It is known that the logics of finite direct powers of real numbers and of rational numbers with the standard non-strict ordering have the finite model property, are finitely axiomatizable, and consequently are decidable. These non-trivial results were obtained in [5], and independently in [16]. Later, analogous results were obtained for the logics of finite direct powers of $(\mathbb{R}, <)$ [14]. Recently, it was shown that the direct squares $(\mathbb{R}, \leq, \geq)^2$ and $(\mathbb{R}, <, >)^2$ have decidable bimodal logics [6,7]. Direct products of well-founded orders have never been investigated before in the context of modal logic.

Let $(\omega^n, \preceq)$ be the n-th direct power of $(\omega, \leq)$, natural numbers with the standard ordering: for $x, y \in \omega^n$, $x \preceq y$ iff $x(i) \leq y(i)$ for all $i < n$. Likewise, let $(\omega^n, \prec)$ be the the direct power $(\omega, <)^n$: $x \prec y$ iff $x(i) < y(i)$ for all $i < n$.

The main result of this paper (Theorem 1) shows that for all finite $n > 0$, the modal algebras of the frames $(\omega^n, \preceq)$ and $(\omega^n, \prec)$ are locally finite. It particular, it follows that the modal logics of these frames have the finite model property.

The work on this paper was supported by the Russian Science Foundation under grant 16-11-10252 and performed at Steklov Mathematical Institute of Russian Academy of Sciences.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 610–618, 2019.
https://doi.org/10.1007/978-3-662-59533-6_37

2 Partitions of Frames, Local Finiteness, and the Finite Model Property

We assume the reader is familiar with the basic notions of modal logics [1,4]. By a *logic* we mean a normal propositional modal logic. For a (Kripke) frame F, $Log(\mathsf{F})$ denotes its modal logic, i.e., the set of all modal formulas that are valid in F. For a set W, $\mathcal{P}(W)$ denotes the powerset of W. The *(complex) algebra of a frame* (W, R) is the modal algebra $(\mathcal{P}(W), R^{-1})$. The algebra of F is denoted by $\mathsf{A}(\mathsf{F})$. A logic has the *finite model property* if it is complete with respect to a class of finite frames (equivalently, finite algebras).

A partition $\mathcal{A}$ of a set W is a set of non-empty pairwise disjoint sets such that $W = \bigcup \mathcal{A}$. A partition $\mathcal{B}$ *refines* $\mathcal{A}$, if each element of $\mathcal{A}$ is the union of some elements of $\mathcal{B}$.

Definition 1. Let $\mathsf{F} = (W, R)$ be a Kripke frame. A partition $\mathcal{A}$ of W is *tuned (in* F) if for every $U, V \in \mathcal{A}$,

$$\exists u \in U\, \exists v \in V\, uRv \;\Rightarrow\; \forall u \in U\, \exists v \in V\, uRv.$$

F is *tunable* if for every finite partition $\mathcal{A}$ of F there exists a finite tuned refinement $\mathcal{B}$ of $\mathcal{A}$.

Proposition 1. *If* F *is tunable, then* $Log(\mathsf{F})$ *has the finite model property.*

Apparently, this fact was first observed by H. Franzén (see [13]). This proposition can be explained as follows. Let L be the logic of a frame F, or in other words, the logic of the modal algebra $\mathsf{A}(\mathsf{F})$. Equivalently, L is the logic of finitely generated subalgebras of $\mathsf{A}(\mathsf{F})$. Recall that an algebra A is *locally finite* if every finitely generated subalgebra of A is finite. It follows that if $\mathsf{A}(\mathsf{F})$ is locally finite, then L has the finite model property. Hence, Proposition 1 is a corollary of the following observation.

Proposition 2. *The algebra of a frame* F *is locally finite iff* F *is tunable.*

Proof. From Definition 1 we have: a finite partition $\mathcal{B}$ is tuned in $\mathsf{F} = (W, R)$ iff the family $\{\cup x \mid x \subseteq \mathcal{B}\}$ of subsets of W forms a subalgebra of the modal algebra $\mathsf{A}(\mathsf{F}) = (\mathcal{P}(W), R^{-1})$.

Assume that $\mathsf{A}(\mathsf{F})$ is locally finite and $\mathcal{A}$ is a finite partition of W. Consider the subalgebra B of $\mathsf{A}(\mathsf{F})$ generated by the elements of $\mathcal{A}$. Then the set $\mathcal{B}$ of the atoms of B is a finite tuned refinement of $\mathcal{A}$.

Now assume that F is tunable and B is the subalgebra of $\mathsf{A}(\mathsf{F})$ generated by a finite family $\mathcal{V}$ of subsets of W. Let $\mathcal{A}$ be the quotient set $W/{\sim}$, where

$$u \sim v \text{ iff } \forall A \in \mathcal{V}\, (u \in A \Leftrightarrow v \in A).$$

Since $\mathcal{A}$ is a finite partition of W, there exists its finite tuned refinement $\mathcal{B}$. The finite family $\{\cup x \mid x \subseteq \mathcal{B}\}$ is the carrier of a subalgebra of $\mathsf{A}(\mathsf{F})$ and contains $\mathcal{V}$. Hence the algebra B is finite. $\square$

Thus, logics of tunable frames have the finite model property, and moreover, algebras of tunable frames are locally finite.

Example 1. Consider the frame $(\omega, \leq)$, natural numbers with the standard ordering. Suppose that $\mathcal{A}$ is a finite partition of ω. If every $A \in \mathcal{A}$ is infinite, then $\mathcal{A}$ is tuned in $(\omega, \leq)$ and in $(\omega, <)$. Otherwise, let k_0 be the greatest element of the finite set $\bigcup\{A \in \mathcal{A} \mid A \text{ is finite}\}$, and $U = \{k \mid k_0 < k < \omega\}$. Consider the following finite partition $\mathcal{B}$ of ω:

$$\mathcal{B} = \{\{k\} \mid k \leq k_0\} \cup \{A \cap U \mid A \text{ is an infinite element of } \mathcal{A}\}.$$

Each element of $\mathcal{B}$ is either infinite, or a singleton, and singletons in $\mathcal{B}$ cover an initial segment of ω. Thus, $\mathcal{B}$ is a finite refinement of $\mathcal{A}$ which is tuned in $(\omega, \leq)$ and in $(\omega, <)$.

It follows that the algebras of the frames $(\omega, \leq)$ and $(\omega, <)$ are locally finite.

Remark 1. Recall that a logic L is *locally finite* (in another terminology, *locally tabular*) if the Lindenbaum algebra of L is locally finite [4]. Equivalently, a logic L is locally finite if the variety of its algebras is locally finite, i.e., every finitely generated algebra validating L is finite.

A logic of a transitive frame is locally finite iff the frame is of finite height [8,11]. Thus, although the algebras of the frames $(\omega, \leq)$ and $(\omega, <)$ are locally finite, the logics of these frames are not. Hence, local finiteness of the algebra $\mathsf{A}(\mathsf{F})$ does not imply local finiteness of the logic $\mathit{Log}(\mathsf{F})$.

Local finiteness of the variety generated by an algebra A of a finite signature is equivalent to uniform local finiteness of A: an algebra A is *uniformly locally finite* if there exists a function $f : \omega \to \omega$ such that the cardinality of a subalgebra of A generated by $m < \omega$ elements does not exceed $f(m)$; see [9, Sect. 14, Theorem 3].

Local finiteness of modal logics is formulated in terms of tuned partitions as follows [15]: the logic of a frame F is locally finite iff there exists a function $f : \omega \to \omega$ such that for every finite partition $\mathcal{A}$ of W there exists a refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$ and $\mathcal{B}$ is tuned in F.

3 Main Result

Theorem 1. *For all finite $n > 0$, the algebras $\mathsf{A}(\omega^n, \preceq)$ and $\mathsf{A}(\omega^n, \prec)$ are locally finite.*

The simple case $n = 1$ was considered in Example 1. To prove the theorem for the case of arbitrary finite n, we need some auxiliary constructions.

Definition 2. Consider a non-empty $V \subseteq \omega^n$. Put

$$J(V) = \{i < n \mid \exists x \in V\, \exists y \in V\ x(i) \neq y(i)\},$$
$$I(V) = \{i < n \mid \forall x \in V\, \forall y \in V\ x(i) = y(i)\} = n \backslash J(V).$$

The *hull of* V is the set

$$\overline{V} = \{y \in \omega^n \mid \forall i \in I(V)\,(y(i) = x(i) \text{ for some (for all) } x \in V)\}.$$

V is *pre-cofinal* if it is cofinal in its hull, i.e.,

$$\forall x \in \overline{V}\, \exists y \in V\, x \preceq y.$$

A partition $\mathcal{A}$ of $V \subseteq \omega^n$ is *monotone* if

– all of its elements are pre-cofinal, and
– for all $x, y \in V$ such that $x \preceq y$ we have $J([x]_{\mathcal{A}}) \subseteq J([y]_{\mathcal{A}})$,

where $[x]_{\mathcal{A}}$ is the element of $\mathcal{A}$ containing x.

Lemma 1. *If $\mathcal{A}$ is a monotone partition of ω^n, then $\mathcal{A}$ is tuned in $(\omega^n, \preceq)$ and in $(\omega^n, \prec)$.*

Proof. Let $A, B \in \mathcal{A}$, $x, y \in A$, $x \preceq z \in B$. Let u be the following point in ω^n:

$$u(i) = y(i) + 1 \text{ for } i \in J(A), \text{ and } u(i) = z(i) \text{ for } i \in I(A). \tag{1}$$

We have

$$\{i < n \mid u(i) \neq z(i)\} \;\subseteq\; n\backslash I(A) \;=\; J(A) \;\subseteq\; J(B);$$

the first inclusion follows from (1), the second follows from the monotonicity of $\mathcal{A}$. Hence, we have $u(i) = z(i)$ for all $i \in I(B)$. By the definition of $\overline{B}$, we have $u \in \overline{B}$. Since B is cofinal in $\overline{B}$ (we use monotonicity again), for some $u' \in B$ we have $u \preceq u'$.

By (1), we have $y(i) \leq u(i)$ for all $i < n$: indeed, $y(i) = x(i) \leq z(i) = u(i)$ for $i \in I(A)$, and $u(i) = y(i) + 1$ otherwise. Thus, $y \preceq u$, and so $y \preceq u'$. It follows that $\mathcal{A}$ is tuned in $(\omega^n, \preceq)$.

In order to show that $\mathcal{A}$ is tuned in $(\omega^n, \prec)$, we now assume that $x \prec z$. Then we have $y(i) < u(i)$ for all $i < n$, since $y(i) = x(i) < z(i) = u(i)$ for $i \in I(A)$, and $u(i) = y(i) + 1$ otherwise. Hence $y \prec u$. Since $u \preceq u'$, we have $y \prec u'$, as required. □

Let $\mathcal{A}$ be a partition of a set W. For $V \subseteq W$, the partition

$$\mathcal{A}{\restriction}V = \{A \cap V \mid A \in \mathcal{A} \;\&\; A \cap V \neq \varnothing\}$$

of V is called the *restriction of $\mathcal{A}$ to V*.

For a family $\mathcal{B}$ of subsets of W, the *partition induced by $\mathcal{B}$ on $V \subseteq W$* is the quotient set $V/{\sim}$, where

$$x \sim y \text{ iff } \forall A \in \mathcal{B}\,(x \in A \Leftrightarrow y \in A).$$

Lemma 2. *Any finite partition of ω^n has a finite monotone refinement.*

Proof. By induction on n. Let $\mathcal{A}$ be a finite partition of ω^n.

Suppose $n = 1$. Let k_0 be the greatest element of the finite set

$$\bigcup\{A \in \mathcal{A} \mid A \text{ is finite}\}.$$

Put $\mathcal{B} = \{\{k\} \mid k \leq k_0\}$. Let $\mathcal{C}$ be the partition induced by $\mathcal{A} \cup \mathcal{B}$ on ω. Consider $x \in \omega$ and put $A = [x]_\mathcal{C}$. If $x \leq k_0$, then $A = \overline{A} = \{x\}$ and $J(A) = \varnothing$. If $x > k_0$, then A is cofinal in $\omega = \overline{A}$, $J(A) = \{0\}$. It follows that $\mathcal{C}$ is the required monotone refinement of $\mathcal{A}$.

Suppose $n > 1$. For $k \in \omega$ let $U_k = \{y \in \omega^n \mid y(i) \geq k \text{ for all } i < n\}$. Since $\mathcal{A}$ is finite, we can choose a natural number k_0 such that

$$\text{if } y \in U_{k_0}, \text{ then } [y]_\mathcal{A} \text{ is cofinal in } \omega^n. \tag{2}$$

Indeed, if $A \in \mathcal{A}$ is not cofinal in ω^n, then $U_{k_A} \cap A = \varnothing$ for some $k_A < \omega$; hence, (2) holds whenever k_0 is greater than every such k_A.

It follows that the partition $\mathcal{A}{\restriction}U_{k_0}$ is monotone: it consists of sets that are cofinal in ω^n (and so, they are obviously pre-cofinal), and $J(A) = n$ for all $A \in \mathcal{A}{\restriction}U_{k_0}$.

We are going to extend this partition step by step in order to obtain a sequence of finite monotone partitions of $U_{k_0-1}, \ldots, U_0 = \omega^n$, respectively refining $\mathcal{A}{\restriction}U_{k_0-1}, \ldots, \mathcal{A}{\restriction}U_0 = \mathcal{A}$.

First, let us describe the construction for the case $k_0 = 1$, the crucial technical step of the proof.

Claim A. Suppose that $\mathcal{B}$ is a finite monotone partition of U_1 refining $\mathcal{A}{\restriction}U_1$. Then there exists a finite monotone partition $\mathcal{C}$ of ω^n refining $\mathcal{A}$ such that $\mathcal{B} \subseteq \mathcal{C}$.

Proof. $\mathcal{C}$ will be the union of $\mathcal{B}$ and a partition of the set

$$V = \{x \in \omega^n \mid x(i) = 0 \text{ for some } i < n\} = \omega^n \backslash U_1.$$

To construct the required partition of V, for $I \subseteq n$ put

$$V_I = \{x \mid \forall i < n\, (i \in I \Leftrightarrow x(i) = 0)\}.$$

Then $\{V_I \mid \varnothing \neq I \subseteq n\}$ is a partition of V, $V_\varnothing = U_1$.

Each V_I considered with the order $\preceq$ on it is isomorphic to $(\omega^{n-|I|}, \preceq)$. Thus, by the induction hypothesis, for a non-empty $I \subseteq n$ we have:

$$\text{Each finite partition of } V_I \text{ admits a finite monotone refinement.} \tag{3}$$

For $I \subseteq n$, by induction on the cardinality of I we define a finite partition $\mathcal{C}_I$ of V_I.

We put $\mathcal{C}_\varnothing = \mathcal{B}$.

Assume that I is non-empty. Consider the projection $\mathrm{Pr}_I : x \mapsto y$ such that $y(i) = 0$ whenever $i \in I$, and $y(i) = x(i)$ otherwise. Note that for all $K \subset I$, $x \in V_K$ implies $\mathrm{Pr}_I(x) \in V_I$. Let $\mathcal{D}$ be the partition induced on V_I by the family

$$\mathcal{A} \cup \bigcup_{K \subset I} \{\mathrm{Pr}_I(A) \mid A \in V_K\}. \tag{4}$$

By an immediate induction argument, $\mathcal{D}$ is finite. Let $\mathcal{C}_I$ be a finite monotone refinement of $\mathcal{D}$, which exists according to (3).

We put

$$\mathcal{C} = \bigcup_{I \subseteq n} \mathcal{C}_I.$$

Then $\mathcal{C}$ is a finite refinement of $\mathcal{A}$. We have to check monotonicity.

Every element A of $\mathcal{C}$ is pre-cofinal, because A is an element of a monotone partition $\mathcal{C}_I$ for some I. In order to check the second condition of monotonicity, we consider x, y in ω^n with $x \preceq y$ and show that

$$J([x]_{\mathcal{C}}) \subseteq J([y]_{\mathcal{C}}). \tag{5}$$

Let $x \in V_I$, $y \in V_K$ for some $I, K \subseteq n$. Since $x \preceq y$, we have $K \subseteq I$. If $K = I$, then (5) holds, since in this case $[x]_{\mathcal{C}}$ and $[y]_{\mathcal{C}}$ belong to the same monotone partition $\mathcal{C}_I$. Assume that $K \subset I$. In this case we have:

$$J([x]_{\mathcal{C}}) \subseteq J([\mathrm{Pr}_I(y)]_{\mathcal{C}}) \subseteq J(\mathrm{Pr}_I([y]_{\mathcal{C}})) \subseteq J([y]_{\mathcal{C}}).$$

To check the first inclusion, we observe that $\mathrm{Pr}_I(y)$ belongs to V_I (since $K \subset I$). This means that $[x]_{\mathcal{C}}$ and $[\mathrm{Pr}_I(y)]_{\mathcal{C}}$ are elements of the same partition $\mathcal{C}_I$. We have $x \preceq \mathrm{Pr}_I(y)$, since $x \in V_I$ and $x \preceq y$. Now the first inclusion follows from monotonicity of $\mathcal{C}_I$. By (4), $\mathrm{Pr}_I([y]_{\mathcal{C}})$ is the union of some elements of $\mathcal{C}_I$ (since $K \subset I$ and $[y]_{\mathcal{C}} \in \mathcal{C}_K$); trivially, $\mathrm{Pr}_I(y) \in \mathrm{Pr}_I([y]_{\mathcal{C}})$, hence $[\mathrm{Pr}_I(y)]_{\mathcal{C}}$ is a subset of $\mathrm{Pr}_I([y]_{\mathcal{C}})$. This yields the second inclusion. The third inclusion is immediate from Definition 2. Thus, we have (5), which proves the claim. □

From Claim A it is not difficult to obtain the following:

Claim B. Let $0 < k < \omega$. If $\mathcal{B}$ is a finite monotone partition of U_k refining $\mathcal{A}{\restriction}U_k$, then there exists a finite monotone partition $\mathcal{C}$ of U_{k-1} refining $\mathcal{A}{\restriction}U_{k-1}$ such that $\mathcal{B} \subseteq \mathcal{C}$.

Proof. Consider the translation $\mathrm{Tr} : U_{k-1} \to \omega^n$ taking $(x_i)_{i<n}$ to $(x_i - k + 1)_{i<n}$. Let $\mathcal{B}'$ be the set $\{\mathrm{Tr}(A) \mid A \in \mathcal{B}\}$ of images of elements of $\mathcal{B}$ by Tr, and $\mathcal{A}'$ be the set $\{\mathrm{Tr}(A) \mid A \in \mathcal{A}{\restriction}U_{k-1}\}$. Then $\mathcal{A}'$ is a partition of ω^n, $\mathcal{B}'$ is a finite monotone partition of U_1 refining $\mathcal{A}'{\restriction}U_1$. By Claim A, there exists a finite monotone partition $\mathcal{C}'$ of ω^n refining $\mathcal{A}'$ such that $\mathcal{B}' \subseteq \mathcal{C}'$. The family $\mathcal{C} = \{\mathrm{Tr}^{-1}(A) \mid A \in \mathcal{C}'\}$ is the required partition of U_{k-1}. □

Applying Claim B k_0 times, we obtain the required monotone refinement of $\mathcal{A}$. This proves Lemma 2. □

From the above two lemmas we obtain that the frames $(\omega^n, \preceq)$ and $(\omega^n, \prec)$, $0 < n < \omega$, are tunable. Now the proof of Theorem 1 immediately follows from Proposition 2.

Corollary 1. *For all finite n, the logics $Log(\omega^n, \preceq)$ and $Log(\omega^n, \prec)$ have the finite model property.*

4 Questions and Conjectures

It is well-known that every extension of $Log(\omega, \leq)$ has the finite model property [3].

Question 1. Let L be an extension of $Log(\omega^n, \preceq)$ for some finite $n > 1$. Does L have the finite model property?

Every extension of a locally finite logic is locally finite, and so has the finite model property. Although the algebras of the frames $(\omega^n, \preceq)$ and $(\omega^n, \prec)$ are locally finite, the logics of these frames are not (recall that a logic of a transitive frame is locally finite iff the frame is of finite height [8,11]). Thus, Theorem 1 does not answer Question 1.

At the same time, Theorem 1 yields another corollary. A *subframe* of a frame (W, R) is the restriction $(V, R \cap (V \times V))$, where V is a non-empty subset of W. It follows from Definition 1 that if a frame is tunable then all its subframes are (details can be found in the proof of Lemma 5.9 in [15]). From Proposition 2, we have:

Proposition 3. *If the algebra of a frame* F *is locally finite, then the algebra of any subframe of* F *is also locally finite.*

Corollary 2. *For all finite n, if* F *is a subframe of* $(\omega^n, \preceq)$ *or of* $(\omega^n, \prec)$*, then* $\mathsf{A}(\mathsf{F})$ *is locally finite, and* $Log(\mathsf{F})$ *has the finite model property.*

While $Log(\omega, \leq)$ is not locally finite, the intermediate logic $ILog(\omega, \leq)$ is (see, e.g., [17, Sect. 3.4]).

Conjecture 1. For all finite n, $ILog(\omega^n, \preceq)$ is locally finite.

The logics of finite direct powers of $(\mathbb{R}, \leq)$ and of $(\mathbb{R}, <)$ have the finite model property, are finitely axiomatizable, and consequently are decidable [5,14,16].

Question 2. Let $n > 1$. Are logics $Log(\omega^n, \preceq)$ and $Log(\omega^n, \prec)$ decidable or at least recursively axiomatizable?

In the one-dimensional case, decidability is a classical result: apparently, the first published proof of finite axiomatizability and the finite model property of the logic $Log(\omega, \leq)$ is given in [2]; for the logic $Log(\omega, <)$, these properties were established in [10] and [12].

Finally, let us address the following question: does the direct product operation on frames preserve local finiteness of their modal algebras?

Proposition 4. *If a frame* F *is tunable and a frame* G *is finite, then the direct product* $\mathsf{F} \times \mathsf{G}$ *is tunable.*

Proof. Let $\mathsf{F} = (F, R)$, $\mathsf{G} = (G, S)$, and $\mathcal{A}$ be a finite partition of $F \times G$. For A in $\mathcal{A}$ and y in G, we put $\mathrm{Pr}_y(A) = \{x \in F \mid (x, y) \in A\}$, $\mathcal{A}_y = \{\mathrm{Pr}_y(A) \mid A \in \mathcal{A}\}$.

Let $\mathcal{B}$ be the partition induced on F by the family $\bigcup_{y \in G} \mathcal{A}_y$. Since $\mathcal{B}$ is finite, there exists its finite refinement $\mathcal{C}$ that is tuned in F. Consider the partition

$$\mathcal{D} = \{A \times \{y\} \mid A \in \mathcal{C} \,\&\, y \in G\}$$

of $F \times G$. Then $\mathcal{D}$ is a finite refinement of $\mathcal{A}$. It is not difficult to check that $\mathcal{D}$ is tuned in $\mathsf{F} \times \mathsf{G}$. $\square$

If follows that if the algebra of F is locally finite and G is finite, then the algebra of $\mathsf{F} \times \mathsf{G}$ is locally finite.

Question 3. Consider tunable frames F_1 and F_2. Is the direct product $\mathsf{F}_1 \times \mathsf{F}_2$ tunable?

If this is true, then Theorem 1 immediately follows from the simple one-dimensional case. And, moreover, in this case Theorem 1 can be generalized to arbitrary ordinals in view of the following observation.

Proposition 5. *For every ordinal $\alpha > 0$, the modal algebras $\mathsf{A}(\alpha, \leq)$, $\mathsf{A}(\alpha, <)$ are locally finite.*

Proof. By induction on α we show that the frames $(\alpha, \leq)$, $(\alpha, <)$ are tunable.

For a finite α, the statement is trivial.

Suppose that $\mathcal{A}$ is a finite partition of an infinite α. If every element of $\mathcal{A}$ is cofinal in α, then $\mathcal{A}$ is tuned in $(\alpha, \leq)$ and in $(\alpha, <)$. Otherwise, we put

$$\beta = \sup \bigcup \{A \in \mathcal{A} \mid A \text{ is bounded in } \alpha\}.$$

Since $\mathcal{A}$ is finite, we have $\beta < \alpha$. Put $\mathcal{B} = \mathcal{A}{\restriction}\beta$. By the induction hypothesis, there exists a finite tuned refinement $\mathcal{C}$ of $\mathcal{B}$. Then the partition of α induced by $\mathcal{A} \cup \mathcal{C}$ is the required refinement of $\mathcal{A}$. $\square$

Conjecture 2. If $(\alpha_i)_{i<n}$ is a finite family of ordinals, then the algebras of the direct products $\prod_{i<n}(\alpha_i, \leq)$, $\prod_{i<n}(\alpha_i, <)$ are locally finite.

Acknowledgements. I am grateful to the reviewers for their suggestions and questions on an earlier version of the paper. I am also grateful to Alex Citkin and Denis Saveliev for useful comments and discussions.

The work on this paper was supported by the Russian Science Foundation under grant 16-11-10252 and performed at Steklov Mathematical Institute of Russian Academy of Sciences.

References

1. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic, Cambridge Tracts in Theoretical Computer Science, vol. 53. Cambridge University Press, Cambridge (2002)
2. Bull, R.A.: An algebraic study of Diodorean modal systems. J. Symbolic Logic **30**(1), 58–64 (1965)
3. Bull, R.A.: That all normal extensions of S4.3 have the finite model property. Math. Logic Q. **12**(1), 341–344 (1966)
4. Chagrov, A., Zakharyaschev, M.: Modal Logic, Oxford Logic Guides, vol. 35. Oxford University Press, Oxford (1997)
5. Goldblatt, R.: Diodorean modality in Minkowski spacetime. Stud. Logica: Int. J. Symbolic Logic **39**(2/3), 219–236 (1980)
6. Hirsch, R., McLean, B.: The temporal logic of two-dimensional Minkowski spacetime with slower-than-light accessibility is decidable. In: Advances in Modal Logic, vol. 12, pp. 347–366. College Publications (2018)
7. Hirsch, R., Reynolds, M.: The temporal logic of two-dimensional Minkowski spacetime is decidable. J. Symbolic Logic **83**(3), 829–867 (2018)
8. Maksimova, L.: Modal logics of finite slices. Algebra Logic **14**(3), 304–319 (1975)
9. Malcev, A.I.: Algebraic Systems. Die Grundlehren der mathematischen Wissenschaften, vol. 192. Springer, Berlin (1973)
10. Schindler, P.: Tense logic for discrete future time. J. Symbolic Logic **35**(1), 105–118 (1970)
11. Segerberg, K.: An Essay in Classical Modal Logic. Filosofska Studier, vol. 13. Uppsala Universitet, Uppsala (1971)
12. Segerberg, K.: Modal logics with linear alternative relations. Theoria **36**(3), 301–322 (1970)
13. Segerberg, K.: Franzen's proof of Bull's theorem. Ajatus **35**, 216–221 (1973)
14. Shapirovsky, I., Shehtman, V.: Chronological future modality in Minkowski spacetime. In: Advances in Modal Logic, vol. 4, pp. 437–459. College Publications, London (2003)
15. Shapirovsky, I., Shehtman, V.: Local tabularity without transitivity. In: Advances in Modal Logic, vol. 11, pp. 520–534. College Publications (2016)
16. Shehtman, V.: Modal logics of domains on the real plane. Stud. Logica **42**, 63–80 (1983)
17. Zakharyaschev, M., Wolter, F., Chagrov, A.: Advanced modal logic. In: Gabbay, D., Guenther, F. (eds.) Handbook of Philosophical Logic, vol. 3, pp. 83–266. Kluwer Academic, Dordrecht (2001)

Knowledge Without Complete Certainty

Hans van Ditmarsch[1] and Louwe B. Kuijer[2(✉)]

[1] CNRS, LORIA, Vandœuvre-lès-Nancy, France
hans.van-ditmarsch@loria.fr
[2] University of Liverpool, Liverpool, UK
Louwe.Kuijer@liverpool.ac.uk

Abstract. We present an epistemic logic ELF (Epistemic Logic with Filters) where knowledge does not require complete certainty. In this logic, instead of saying that an agent knows a particular fact if it is true in every accessible world, we say that it knows the fact if it is true in a sufficiently large set accessible worlds. On a technical level, we do this by enriching the standard Kripke models of epistemic logic with a set of filters: a sufficiently large set of worlds is one that is in the filter. We introduce semantics for ELF, and give a sound and complete proof system.

1 Introduction

In the standard Kripke semantics for epistemic logic, we say that an agent a *knows* a proposition φ if and only if φ is true in every world that is *epistemically accessible* for a [12,15]. In other words, according to such semantics a knows that φ if and only if φ is true in every world that is consistent with a's observations.

Unfortunately, it is generally not very hard to invent worlds that are consistent with a's observations where φ is false. Along a general line of skepticism going back to (at least) Descartes' evil demon (*le mauvais genie*) and that continues to flourish in logical and epistemological circles [10,25], consider the following example, adapted from Harman [14].

> Alice is sitting at her desk, writing a logic paper. Strictly speaking, the skeptical scenario where she is merely a "brain in a vat" that wrongly believes itself to be sitting at a desk is consistent with Alice's observations. In theory, this means that the possible world where Alice is a brain in a vat is epistemically accessible for her, so according to the standard Kripke semantics she does not *know* that she is sitting at her desk. Still, we would typically like to say that Alice does know this fact.

There are several solutions to this problem. Firstly, we could bite the bullet and conclude that Alice does not, and cannot, know that she is sitting at her desk. This skeptic's choice is internally consistent, but results in a rather trivial notion of (unobtainable) knowledge. So while we acknowledge that the skeptics may be correct, that is not the kind of knowledge that we are interested in here.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 619–632, 2019.
https://doi.org/10.1007/978-3-662-59533-6_38

Secondly, we could say that the scenario where Alice is a brain in a vat should not be considered a proper possible world, and therefore should not be among Alice's epistemic alternatives. Doing so can be justified from a *contextualist* point of view [11,21,28], which states that the conditions for knowledge depend on context. Whenever skeptical scenarios are irrelevant, they are excluded by the context. As long as we are modeling a context where skeptical scenarios are excluded, we may (and must) omit the worlds where Alice is a brain in a vat, allowing us to conclude that Alice knows that she is sitting at her desk.

This second solution is the most practical one, and commonly used in epistemic logic. Unfortunately, this solution is not always available. If we want to reason about whether Alice knows that she is not a brain in a vat, then clearly the world where she is in fact a brain in a vat is relevant to our context. So it cannot be omitted. What are the consequences? In the epistemic logical setting, and in particular in the modal logical propositional modeling of it, the typical notion to fall short of knowledge within a given context is called *belief*, and the minimal difference between belief and knowledge is that belief unlike knowledge may be *false* (incorrect). Even within that restriction there is a wide gap between *defeasible belief* [19] and so-called *conviction* [26]. Defeasible beliefs may be defeated, i.e., the agents may be willing to change their beliefs after further evidence or consideration. But false convictions remain false forever. When modeling *certain* knowledge this rather Platonic focus on *true* knowledge is peculiar. Why should one particular exception of the rule matter more than any other exception? Many works have been dedicated to the difference between (modal) knowledge and belief [16,18], and in particular on notions of knowledge closer to belief [15,27], *fallible knowledge* [2] and (the more dynamically motivated) *safe belief* [3]. They all fall short of modeling certain knowledge, because the set of accessible worlds where φ is false is always *too big*, even when there is only one.

In this paper we therefore choose a third solution: we *do* include the world where Alice is a brain in a vat, as well as other skeptical worlds. But we say that we *know* φ even if there are accessible worlds where φ is false, as long as the set of accessible φ worlds is *sufficiently large*.

Much like the second one, the third solution is justified by contextualism. The key observation is that our context as modelers may be different from the context of the agents being modelled. We are interested in whether Alice knows she is not a brain in a vat, so our context does not allow us to omit the worlds where she is a brain in a vat. But as long as Alice's context allows her to ignore such skeptical worlds, she can know she is not a brain in a vat even though the worlds are accessible.

The remaining question, then, is to decide on what we mean by the set of counterexamples being "small". A simple numerical ("up to n counterexamples") or finite fraction ("up to $\frac{n}{m}$ of the possible worlds may be counterexamples") rule would not solve Alice's problem: we can create infinitely many skeptical scenarios, so for every $n \in \mathbb{N}$ there are more than n counterexamples, and the ratio of worlds where she is a brain in a vat divided by those where she is not is $\frac{\infty}{\infty}$ and therefore not a finite fraction. Note also that a numerical or finite fraction threshold is vulnerable to the lottery paradox [20], while Alice's example above and further examples below are immune to finite lottery paradoxes.

A more promising approach would be to say that the number of counterexamples is small if the set of counterexamples has measure zero. Such a notion of knowledge, employing Keisler's infinitesimals for measure zero sets [17], has been proposed in [1] for modeling knowledge revision. Measure theory is unnecessarily heavy machinery for our current purpose, however: we do not need an exact measure of all sets of worlds, we only need to know which sets are small. We therefore prefer a very similar but somewhat more lightweight approach: we use *filters*. The notion goes back to [7], and it is frequently used within modal logic [6], also for default reasoning [4]. We say that the set of counterexamples is small if its complement is a member of the filter.

The structure of the rest of this paper is as follows. In Sect. 2 we formally define the syntax and semantics of our logic Epistemic Logic with Filters (ELF). Then, in Sect. 3 we present detailed examples. In Sect. 4 we provide a sound and complete axiomatization for ELF. Section 5 compares our framework to the class of non-normal modal logics known as regular modal logics.

2 Syntax and Semantics

Before defining the language, models and semantics, let us first define filters.

Definition 1. *Let S be a set. Then $F \subseteq 2^S$ is a* filter *if*

- $F \neq \emptyset$,
- *for every* $X_1, X_2 \in F$, *we have* $X_1 \cap X_2 \in F$,
- *for every* $X_1 \in F$ *and every* $X_2 \subseteq S$, *if* $X_1 \subseteq X_2$ *then* $X_2 \in F$.

A filter F is proper *if* $\emptyset \notin F$.

We have no use for improper filters, so for the remainder of this paper we assume all filters to be proper.

A filter serves to identify which subsets of S are small or large, with $X \subseteq S$ being *large* if $X \in F$ and $X \subseteq S$ being *small* if $S \setminus X \in F$. Note that, by the fact that for $X_1, X_2 \in F$ we have $X_1 \cap X_2 \in F$, the intersection of two large sets is itself large. Typical examples of filters include (i) the co-finite subsets of S (if S is infinite),[1] (ii) the sets of full measure in a measure space and (iii) for a fixed $C \subseteq S$, all sets that contain C.

The latter kind of filter, where $F = \{X \subseteq S \mid C \subseteq X\}$, is called a *principal filter*. In that case, the set C can be considered to be the set of *important*, or *relevant* worlds.[2] So in that case it may not be quite accurate to say that a set $X \in F$ is necessarily large. It is, however, *sufficiently* large in the sense that it contains all important worlds. Another way to think of this is that while C

[1] An epistemic modal use of that is the *Majority Logic* of [24].
[2] This is similar to the approach advocated in [21], see also Remark 3.

may have a small cardinality, the fact that they are important gives C a larger weight, so any set containing C has large weight.

The language of ELF is the same as that of standard single-agent modal logic.

Definition 2. *Let At be a countable set of propositional atoms. The language $\mathcal{L}$ is given by the following normal form, where $p \in At$:*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi,$$

As usual, we omit parentheses where this should not cause confusion, and use $\wedge, \rightarrow, \leftrightarrow, \Diamond$ and $\bigwedge$ as abbreviations.

The models of ELF are based on the usual Kripke models, but they are enriched with filter structures.

Definition 3. *A* model *is a tuple $\mathcal{M} = (W, R, \mathcal{F}, V)$, where W is a set of worlds, $R \subseteq W \times W$ is an accessibility relation, $\mathcal{F} : W \rightarrow \{F \mid F \text{ is a filter on } W\}$ assigns to each world a filter and $V : At \rightarrow 2^W$ is a valuation.*

We write $R(w)$ for $\{w' \mid (w, w') \in R\}$.

Now we can define the semantics.

Definition 4. *The* satisfaction relation $\models$ *is defined recursively by*

$$\begin{array}{lcl} \mathcal{M}, w \models p & \Leftrightarrow & w \in V(p) \\ \mathcal{M}, w \models \neg\varphi & \Leftrightarrow & \mathcal{M}, w \not\models \varphi \\ \mathcal{M}, w \models \varphi \vee \psi & \Leftrightarrow & \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi \\ \mathcal{M}, w \models \Box\varphi & \Leftrightarrow & [\![\varphi]\!]_{\mathcal{M}} \cap R(w) \in \mathcal{F}(w) \end{array}$$

We use $\models \varphi$ and $\Gamma \models \varphi$ in the usual way to denote φ being valid and φ being entailed by Γ, respectively.

Note that we have $\mathcal{M}, w \models \Box\varphi$ if there is a large set of accessible φ worlds. This is not exactly the same as there being a small set of accessible $\neg\varphi$ worlds: the accessible φ worlds being large always implies that the accessible $\neg\varphi$ worlds are small, but if $R(w) \notin \mathcal{F}(w)$ it is possible for $[\![\neg\varphi]\!]_{\mathcal{M}} \cap R(w)$ to be small without $[\![\varphi]\!]_{\mathcal{M}} \cap R(w)$ being large. The reason for this "largeness requirement" is that we consider knowledge to require some amount of intellectual effort and honesty.

It is generally held (e.g., [9,13,22]) that a necessary[3] condition for knowing φ is that φ is a justified true belief. So, in particular, for an agent to know φ it must be the case that there is a justification for the agent to believe φ. In the case of standard epistemic logic, this justification derives from the fact that all accessible worlds satisfy φ. Here, in ELF, the justification derives from the slightly weaker condition that the accessible $\neg\varphi$ worlds are negligible compared to the accessible φ worlds.

In order to obtain this justification it does not suffice that the set of accessible $\neg\varphi$ worlds is small in an absolute sense; if the agent considers three possible worlds and all three of them satisfy $\neg\varphi$, then it would be strange to say that

[3] But, unless one uses a very strong notion of justification, not sufficient [13].

they are justified in believing φ simply because there are few counterexample. Instead, the set of accessible counterexamples must be small compared to the set of accessible φ worlds. But even that is not quite enough; if there are no accessible $\neg\varphi$ worlds and at least one accessible φ world, then one could argue that the set of $\neg\varphi$ worlds is small compared to the set of φ worlds. In some cases, we would endorse the claim that this single accessible φ world, in the absence of accessible $\neg\varphi$ worlds, provides a justification for believing φ. But in other cases, the fact that there is only one world that the agent considers possible can betray a lack of effort and imagination by the agent.

We want the model $\mathcal{M}$ to describe what the agent knows, not what the agent *thinks* they know. This means that the model is drawn from the perspective of an outside observer who knows the agent's mental state, not from the perspective of the agent themselves. So the relation R describes objectively which worlds the agent considers possible. But the mental state is itself of course a subjective opinion of the agent: we are objectively describing a subjective mental state. If the agent has never thought of a world w_2, it would therefore be inaccurate to say that the agent considers w_2 possible, even if w_2 happens to be consistent with the agent's observations.

As a result, if the agent does not consider a world w_2 to be accessible, this could be either because the agent has thought of w_2 and determined it to be incompatible with their information, or because the agent never though of w_2. So if the agent considers only one world to be accessible, this could be because the agent is lazy, and didn't think of any other worlds. In that case, even if the only accessible world satisfies φ, this would not be a justification for believing φ. In order for the agent to be justified in believing φ, they should first consider sufficiently many worlds.

Note that the agent must consider *sufficiently* many worlds. This is not a cardinality requirement: in some situations, a finite number of worlds might be sufficient, while in another case even a continuum of worlds might not be enough. Instead, sufficiency is determined from the perspective of the objective, outside observer who designs the model. Specifically, the model designer determines sufficiency using the filter function $\mathcal{F}$.

If the agent considered sufficiently many worlds, so $R(w) \in \mathcal{F}(w)$, and all but a negligible amount of these worlds satisfy φ, so $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{F}(w)$, this yields the justification for the agent's belief that φ. These two conditions together are equivalent to $[\![\varphi]\!]_{\mathcal{M}} \cap R(w) \in \mathcal{F}(w)$, our condition for knowledge.

Remark 1. Note that we allow the filter $\mathcal{F}(w)$ to depend on the world w. This is because, otherwise, it would be impossible to have $\mathcal{M}, w_1 \models \Box\varphi$ and $\mathcal{M}, w_2 \models \Box\neg\varphi$. After all, $\mathcal{M}, w_1 \models \Box\varphi$ requires $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{F}(w_1)$ and $\mathcal{M}, w_2 \models \Box\neg\varphi$ requires $[\![\neg\varphi]\!]_{\mathcal{M}} \in \mathcal{F}(w_2)$. So if $\mathcal{F}(w_1) = \mathcal{F}(w_2)$, then we would have $[\![\varphi]\!]_{\mathcal{M}} \cap [\![\neg\varphi]\!]_{\mathcal{M}} \in \mathcal{F}(w_1)$, which is a contradiction since $\emptyset \notin \mathcal{F}(w_1)$.

Remark 2. The semantics presented above do not guarantee that knowledge in ELF satisfies certain properties that knowledge is often considered to have, such as truthfulness and introspection. ELF is, in this sense, similar to the basic modal logic K. And, like K, ELF can be extended with axioms and frame properties to guarantee truthfulness and introspection.

Remark 3. We recall example (*iii*) of a filter F given by $F = \{X \subseteq S \mid C \subseteq S\}$. This can be seen as an implementation of the contextualist view from [21] of knowledge as truth in all relevant accessible worlds. Among yet other precisions, Lewis writes:

> Then S knows that P iff S's evidence eliminates every possibility in which not-P—Psst!—except for those possibilities that conflict with our proper presuppositions. [21, page 554]

In other words, a proposition (called P) is known (by an agent S) if it is true in the intersection of the accessible worlds and the relevant worlds. On the assumption that $R(w) \in F(w)$, this corresponds exactly to the semantics of ELF when S is the set of relevant worlds.

ELF is more general however, because we do not require that $R(w) \in F(w)$ and not every filter is of the form $\{X \subseteq S \mid C \subseteq S\}$.

Now that we have defined the semantics of our logic, we can consider a few examples in some detail.

3 Examples

Example 1. Bob is a mathematics student. On an exam, he writes a proof by case distinction for a proposition p in some mathematical theory $\mathfrak{T}$. Because Bob is not very experienced in writing proofs, however, he is not certain that his case distinction is exhaustive. But even though Bob does not know this, his case distinction is in fact exhaustive and his proof is correct.

We will represent Bob's situation by a pointed model $\mathcal{M}, w$, where $M = (W, R, \mathcal{F}, V)$. The possible worlds of $\mathcal{M}$ are closely related to the models of $\mathfrak{T}$. Specifically, for every model $\mathcal{T}$ of $\mathfrak{T}$, there is a world where $\mathcal{T}$ is the "true" model. Having one such world per model is not quite enough, however, because there are other facts that may differ per world. In particular, if two worlds w_1 and w_2 have the same model $\mathcal{T}$ but Bob's beliefs differ between w_1 and w_2, then they must be different worlds. This can be represented by considering these worlds to be pairs $w = (\mathcal{T}, i)$, where i is simply some index used to differentiate between worlds with the same model of $\mathfrak{T}$.[4]

In Bob's proof, he considered the cases $q_1, \cdots, q_n$. Because the case distinction is in fact exhaustive, every model of $\mathfrak{T}$ satisfies at least one of these cases. So for every $(\mathcal{T}, i)$, there is at least one j such that $\mathcal{M}, (\mathcal{T}, i) \models q_j$. Furthermore, since the proof is correct, any world that satisfies one of these cases also satisfies p. So we have $\mathcal{M}, (\mathcal{T}, i) \models p$.

[4] Because we require W to be a set, as opposed to a class, we may have to restrict ourselves to the models of $\mathfrak{T}$ in some set-theoretic universe $\mathcal{U}$, where $W \notin \mathcal{U}$.

In order to represent Bob's uncertainty about whether his case distinction is exhaustive, we need some further worlds where none of the cases apply. While there are no models of $\mathfrak{T}$ that fall outside the case distinction, Bob thinks that there might be. So we need to add a number of worlds of the form $(\mathcal{N}, i)$, where $\mathcal{M}, (\mathcal{N}, i) \not\models q_j$ for every j. Here $\mathcal{N}$ is objectively not a model of $\mathfrak{T}$, but Bob is not certain that it is not a model. Because none of the cases apply, it is uncertain whether these worlds satisfy p. It is possible for p to be true there, but it is also possible for p to be false in these worlds.

The worlds that Bob considers possible are those that fall inside his case distinction. This is true for every world, so $R(w') = V(q_1) \cup \cdots \cup V(q_n)$ for every $w' \in W$. The filters represent the "relevant" worlds, in the sense that one is justified in believing a proposition after verifying that it holds in every world of the filter. Bob's belief in p is justified if p is true in every model of $\mathfrak{T}$, so $\mathcal{F}(w) = \{F \mid C \subseteq F\}$, where C is the set of worlds of the form $(\mathcal{T}, i)$. We have $R(w) \cap [\![p]\!]_{\mathcal{M}} \supseteq C$, and therefore $\mathcal{M}, w \models \Box p$. So Bob knows that p is true.

However, even though Bob's case distinction was exhaustive, he is uncertain about this. So in some of the accessible worlds $w' = (\mathcal{T}, i)$ his case distinction is not exhaustive. In such a world we have $\mathcal{F}(w') = \{F \mid C' \subseteq F\}$, where C' contains not only the worlds of the form $(\mathcal{T}, j)$, but also some worlds of the form $(\mathcal{N}, j)$. In these worlds, we have $R(w') \cap [\![p]\!]_{\mathcal{M}} \notin \mathcal{F}(w')$ and therefore $\mathcal{M}, w' \not\models \Box p$. Note that it does not matter whether p holds in the worlds $C' \setminus C$. Even if p happens to be true in all of C', the fact that his case distinction was non-exhaustive means that his belief in the truth of p would be unjustified.

Example 2. Suppose that we are about to draw a random real number uniformly from the interval $[0, 1]$. This situation can be modeled in the following way:

- For every $x \in [0, 1]$ there is a world w_x where x is the number that is drawn.
- Every world is accessible from every other world, i.e., $R = \{(w_x, w_y) \mid x, y \in [0, 1]\}$.
- The large sets are those that have full measure, i.e., for every $x \in [0, 1]$, we have $\mathcal{F}(w_x) = \{w_Y \mid \mu(Y) = 1\}$, where μ is the Lebesgue measure and $w_Y = \{w_y \mid y \in Y\}$.

Under these circumstances, we can say that we know that the drawn number x will be irrational, since the rationals have measure 0. Note that this knowledge is fallible: even in those worlds where we will draw a rational number, we know that the number will be irrational. Such failure is infinitely unlikely, however.

Example 3. Claire is a software engineer, who is demonstrating a program to a client. The program has been given its input, and is now running. Claire tells the client that she knows that the program will terminate and return the output "TRUE". In saying so, she ignores a number of possible worlds. In particular, if there is a power failure then the program will not terminate at all. Claire has thought of such possibilities, but she considers the conversation with the client to have a number of underlying unspoken assumptions, including the assumption that there will be no power failure. So while there are possible worlds where the program is interrupted by power failure or some other outside factor, the unspoken assumptions render such worlds irrelevant.

The set of worlds W of our model is given by $W = W_1 \cup W_2$, where W_1 is the set of worlds where the program will be allowed to run normally and W_2 is the set of worlds where the program will be interrupted by some outside event, such as a power cut or a meteorite strike. The accessibility relation is given by $R = W \times W$. Finally, for every world w the filter $\mathcal{F}(w)$ is the set of all sets containing the relevant worlds. In this case, as discussed above, we consider the relevant worlds to be those where the program is allowed to run uninterrupted, so $\mathcal{F}(w) = \{F \mid W_1 \subseteq F\}$. We let p stand for "the program terminates and returns TRUE", so $V(p) = W_1$.

For any world w of this model $\mathcal{M}$ we have $\mathcal{M}, w \models \Box p$. Note that, as in the previous example, this knowledge is fallible: Claire knows p in every world, including those where a power failure occurs. Unlike the previous example, however, such failure is not necessarily infinitely unlikely. The probability of power failures et cetera is low, but not infinitely so, after all. But this possibility of failure does not stop Claire from knowing p, under the conversational assumptions.

Example 4. As above, except now the possibility of a power failure or other outside event has not crossed Claire's mind. The accessibility relation is now given by $R = W \times W_1$. But because W_1 contains all relevant worlds, Claire still knows that the program will return TRUE.

Example 5. As above, except that Claire is now less careful in considering all possible executions of her program. Instead of considering all possible executions W_1, she makes some implicit assumptions and only thinks of $W_1' \subset W_1$. We have $R = W \times W_1'$. The set of relevant worlds remains the same, however: $\mathcal{F}(w) = \{F \mid W_1 \subseteq F\}$.

In this situation, Claire does not know that the program will return TRUE, because $R(w) \notin \mathcal{F}(w)$. Note that this is independent of whether the program returns TRUE in the relevant worlds that she failed to consider: Claire's belief that the program will return TRUE is not justified, so even if she happens to be right she doesn't *know* that the program will return TRUE.

4 Axiomatization

We introduce the proof system **WKL**. The **W** in **WKL** stands for "weak", since **WKL** is strictly weaker than **KL**, which is obtained by adding the axiom **L** to the standard proof system **K** for modal logic.[5]

Definition 5. *The proof system* **WKL** *is given by the following rules and axiom schemata.*

[5] The axiom **L** is, using the other axioms and rules, interderivable with the axiom **D**, given by $\Box\varphi \to \Diamond\varphi$. One could, therefore, think of **WKL** as "weak **KD**" instead of "weak **KL**". Our reason for preferring **L** over **D** in this context is that **L** more closely follows the semantical constraint that $\emptyset \notin \mathcal{F}(w)$.

P *all substitution instances of propositional tautologies*
K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
L $\neg\Box\bot$
RM *if* $\varphi \to \psi$ *is a theorem, infer* $\Box\varphi \to \Box\psi$
MP *from* $\varphi \to \psi$ *and* φ*, infer* ψ.

Definition 6. *A formula φ is a* theorem *of* **WKL**, *denoted* $\vdash \varphi$ *if it can be derived in a finite number of steps using the rules and axioms of* **WKL**. *A formula φ is* entailed by *a set Γ of formulas, denoted $\Gamma \vdash \varphi$ if φ can be derived in a finite number of steps using the rules and axioms of* **WKL** *and using Γ as premises.*

Note that **WKL** does not have a necessitation rule, i.e., we cannot infer from $\vdash \varphi$ that $\vdash \Box\varphi$. Instead, we use a strictly weaker *monotonicity rule* **RM**. In particular, $\Box\top$ is not provable in **WKL**.[6] ELF is therefore not a normal modal logic, although it is a *regular modal logic* [23]. In Sect. 5 we discuss ELF's position in the landscape of non-normal modal logics.

Soundness of **WKL** follows immediately from the semantics.

Lemma 1 (Soundness). *For all $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$, if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

Completeness of **WKL** is shown in the usual way, i.e., by constructing a canonical model and proving that every consistent formula is satisfied in that model (see for example [5]). Some of the following lemmas can be proven in the exact same way as the corresponding lemmas in other completeness proofs. We therefore omit the proofs of those lemmas.

We start with a lemma that allows us to switch between three different characterizations of entailment.

Lemma 2. *The following are equivalent.*

1. $\Gamma \vdash \varphi$
2. *there is a finite subset Γ' of Γ such that $\Gamma' \vdash \varphi$*
3. *there is a finite subset Γ' of Γ such that $\vdash \bigwedge \Gamma' \to \varphi$*

As usual, maximal consistent sets will serve as worlds for the canonical model.

Definition 7. *A set Γ of formulas is* consistent *if $\Gamma \nvdash \bot$,* maximal *if for every formula φ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ and* maximal consistent *if it is both maximal and consistent.*

Lemma 3 (Lindenbaum lemma). *Let Γ be a consistent set. Then there is a maximal consistent set Δ such that $\Gamma \subseteq \Delta$.*

Definition 8. *If Γ is a set of formulas, then $\Box^{-1}\Gamma = \{\varphi \mid \Box\varphi \in \Gamma\}$.*

The proof of the following lemma is slightly more complicated than usual, since we only have access to the monotonicity rule **RM** as opposed to the more powerful necessitation rule. We therefore provide a detailed proof.

[6] Note that $\not\models \Box\top$ in ELF, since $\mathcal{M}, w \not\models \Box\top$ when $R(w) \notin \mathcal{F}(w)$.

Lemma 4. *If Γ is consistent, then so is $\Box^{-1}\Gamma$.*

Proof. Suppose towards a contradiction that $\Box^{-1}\Gamma \vdash \bot$. Then there is a finite subset of $\Phi \subseteq \Box^{-1}\Gamma$ such that

$$\vdash \bigwedge \Phi \to \bot.$$

By **RM**, this yields

$$\vdash \Box \bigwedge \Phi \to \Box\bot. \tag{1}$$

Now, note that by repeatedly applying **K** and **MP**, we also have

$$\vdash \bigwedge \Box\Phi \to \Box \bigwedge \Phi. \tag{2}$$

Together, (1) and (2) imply that

$$\Box\Phi \vdash \Box\bot$$

and therefore by **L** and the fact that $\Box\Phi \subseteq \Gamma$

$$\Gamma \vdash \bot,$$

contradicting the consistency of Γ. □

Lemma 5. *If Γ is maximal consistent and $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.*

Proof. By maximality, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. Since $\Gamma \cup \{\neg\varphi\} \vdash \bot$ it follows from consistency that $\varphi \in \Gamma$. □

Now, let us define the canonical model.

Definition 9. *The* canonical model $\mathcal{M}^c = (W^c, R^c, V^c, \mathcal{F}^c)$ *is given by:*

- *W^c is the set of maximal consistent sets of formulas,*
- *$\mathcal{F}(w) = \{F \subseteq W^c \mid S(w) \subseteq F\}$, where $S(w) = \{w' \mid \Box^{-1}w \subseteq w'\}$,*
- *if $\Box\top \in w$, then $R^c(w) = S(w)$ otherwise $R^c(w) = \emptyset$,*
- *$V^c(p) = \{w \in W^c \mid p \in w\}$.*

Lemma 6 (Truth Lemma). *For every $w \in W^c$ and every formula φ, $\mathcal{M}^c, w \models \varphi$ if and only if $\varphi \in w$.*

Proof. By induction on the complexity of φ. If φ is atomic, then the lemma follows immediately from the definition of V^c. So assume as induction hypothesis that φ is not atomic and that the lemma holds for all strict subformulas of φ.

We continue by a case distinction on the main connective of φ. If it is a Boolean connective, then the lemma is once again trivial. So let us consider the interesting case, $\varphi = \Box\psi$.

Suppose $\Box\psi \in w$. By the definition of $S(w)$, we have $[\![\psi]\!]_{\mathcal{M}^c} \supseteq S(w)$. Furthermore, by **P** we have $\vdash \psi \to \top$, so by **RM** we have $\vdash \Box\psi \to \Box\top$. Since w

is maximal and consistent, this implies that $\Box\top \in w$. So $R^c(w) = S(w)$. We therefore have $[\![\psi]\!]_{\mathcal{M}^c} \cap R^c(w) = S(w) \in \mathcal{F}(w)$, so $\mathcal{M}^c, w \models \Box\psi$.

Suppose, on the other hand, that $\Box\psi \notin w$. We distinguish two sub-cases. First, suppose that $\Box\top \notin w$. Then $R^c(w) = \emptyset \notin \mathcal{F}(w)$, so $\mathcal{M}^c, w \not\models \Box\psi$.

The other case is if $\Box\top \in w$ but $\Box\psi \notin w$. In this case, suppose towards a contradiction that $\Box^{-1}w \cup \{\neg\psi\}$ is inconsistent. Then there is a finite subset Φ of $\Box^{-1}w$ such that $\Phi \cup \{\neg\psi\}$ is inconsistent. It follows that $\vdash \bigwedge\Phi \to \psi$ and therefore $\vdash \Box\bigwedge\Phi \to \Box\psi$. Since $\Box\varphi \in w$ for every $\varphi \in \Phi$, we have $\Box\bigwedge\Phi \in w$ and therefore $\Box\psi \in w$, contradicting our assumption.

So $\Box^{-1}w \cup \{\neg\psi\}$ is consistent, and can therefore be extended to a maximally consistent set w'. By the definition of $\mathcal{F}$, we have that $w' \in F$ for every $F \in \mathcal{F}(w)$. Since $w' \notin [\![\psi]\!]_{\mathcal{M}^c}$, it follows that $[\![\psi]\!]_{\mathcal{M}^c} \cap R(w) \notin \mathcal{F}(w)$, and therefore $\mathcal{M}^c, w \not\models \Box\psi$. $\square$

Completeness now follows immediately.

Lemma 7 (Completeness). *For all $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$, if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*

Proof. If $\Gamma \not\vdash \varphi$ then $\Gamma \cup \{\neg\varphi\}$ is consistent, so by Lemma 3 there is a maximal consistent set $w \supseteq \Gamma \cup \{\neg\varphi\}$. By Lemma 6 this implies that $\mathcal{M}^c, w \models \psi$ for every $\psi \in \Gamma$ and $\mathcal{M}^c, w \not\models \varphi$. Therefore, $\Gamma \not\models \varphi$. $\square$

We have now proven both soundness and completeness.

Theorem 1. *For all $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{K}$, we have $\Gamma \models \varphi$ if and only if $\Gamma \vdash \varphi$.*

Remark 4. In $\mathcal{M}^c$, every filter is of the form $\mathcal{F}(w) = \{F \subseteq W^c \mid S(w) \subseteq F\}$. So all filters in the canonical model are principal filters. It follows that the proof system is also sound and complete for the class of models

$$\mathfrak{M} := \{\mathcal{M} = (W, R, \mathcal{F}, V) \mid \forall w \in W : \mathcal{F}(w) \text{ is principal}\}.$$

5 Comparison to Other Non-normal Modal Logics

ELF is a so-called non-normal modal logic. In this section, we therefore compare the semantics of ELF to the commonly used neighborhood semantics, and the proof system **WKL** to other proof systems for non-normal modal logics.

In neighborhood semantics [8,23], a model is a tuple $\mathcal{M} = (W, \mathcal{N}, V)$, where W is a set of worlds, $\mathcal{N} : W \to 2^{2^W}$ is a neighborhood function that assigns to each world a set of sets of worlds and $V : At \to 2^W$ is a valuation. We then say that $\mathcal{M}, w \models \Box\varphi$ if and only if $[\![\varphi]\!]_{\mathcal{M}} \in \mathcal{N}(w)$.

The semantics of ELF can be reduced to neighborhood semantics. This is not very surprising, both because the formalism of neighborhood semantics is sufficiently versatile to encompass almost everything done in modal logic, and because the filter $\mathcal{F}(w)$ already looks a lot like a neighborhood function $\mathcal{N}$. Still, translating from ELF to neighborhood semantics is not entirely trivial; after all, whether $\mathcal{M}, w \models \Box\varphi$ depends not only of $\mathcal{F}(w)$ but also on $R(w)$, so $\mathcal{F}$ is not exactly the neighborhood function that we are looking for. Instead, given an ELF model $\mathcal{M} = (W, R, \mathcal{F}, V)$ we find a neighborhood model $\mathcal{M}' = (W, \mathcal{N}, V)$ by taking

$$\mathcal{N}(w) = \begin{cases} \mathcal{F}(w) \text{ if } R(w) \in \mathcal{F}(w) \\ \emptyset \quad\quad \text{otherwise} \end{cases}$$

Proposition 1. *$\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w \models \varphi$.*

Proof. By induction. As base case, suppose that φ is an atom p. Since $\mathcal{M}$ and $\mathcal{M}'$ have the same valuation, we have $\mathcal{M}, w \models p \Leftrightarrow \mathcal{M}', w \models p$. Suppose then as induction hypothesis that φ is not atomic, and that for every strict subformula ψ of φ we have $\mathcal{M}, w \models \psi \Leftrightarrow \mathcal{M}', w \models \psi$. We continue by case distinction on the main connective of φ.

If the main connective of φ is Boolean, then it follows immediately from the induction hypothesis that $\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}', w \models \varphi$. Suppose then that $\varphi = \Box\psi$. Then

$$\begin{aligned} \mathcal{M}, w \models \Box\psi &\Leftrightarrow [\![\psi]\!]_{\mathcal{M}} \cap R(w) \in \mathcal{F}(w) \Leftrightarrow [\![\psi]\!]_{\mathcal{M}} \in \mathcal{F}(w) \text{ and } R(w) \in \mathcal{F}(w) \\ &\Leftrightarrow [\![\psi]\!]_{\mathcal{M}} \in \mathcal{N}(w) \Leftrightarrow [\![\psi]\!]_{\mathcal{M}'} \in \mathcal{N}(w) \Leftrightarrow \mathcal{M}', w \models \Box\psi \end{aligned}$$

This completes the case distinction and thereby the induction step. □

Note that while $\mathcal{F}(w)$ is always a filter, $\mathcal{N}(w)$ need not be one. After all, a filter is by definition non-empty, whereas $\mathcal{N}(w)$ is empty whenever $R(w) \notin \mathcal{F}(w)$.

The neighborhood function $\mathcal{N}$ is, in the terminology of [23], consistent, closed under (binary) intersection and closed under supersets. From the fact that $\mathcal{N}$ is consistent, it immediately follows that $\models \neg\Box\bot$, from the fact that $\mathcal{N}$ is closed under intersection it follows that $\models (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ and from the fact that $\mathcal{N}$ is closed under supersets it follows that $\models \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

We continue by comparing the proof system **WKL**, where we recall Definition 5 on page 8, to other proof systems for non-normal modal logics. We write **WK** for the proof system containing **P**, **K**, **RM** and **MP**. So **WK** is **WKL** minus the axiom **L**.

A *regular modal logic* [8] contains the following axioms and rules.

P	all propositional tautologies
Dual	$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
M	$\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$
C	$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$
RM	from $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
MP	from $\varphi \rightarrow \psi$ and φ, infer ψ

We refer to the proof system containing exactly these six axioms and rules as the minimal regular modal logic **MRML**.

The proof system **WK** is an alternative presentation of a regular modal logic [8, Exercise 8.13a, page 241], i.e., a formula is provable in **MRML** if and only if it is provable in **WK**. It follows that **WKL** is a regular modal logic.

The axiom **L** is not provable in **MRML**. This can, for example, be seen by noting that **MRML** is sound and complete for relational models with impossible worlds (see, e.g., [23]), and that **L** is not valid on those models. So **WKL** is a strictly stronger proof system than **MRML**.

Regular modal logics have been studied quite extensively, see the aforementioned [23] for an overview. But the extension of a regular modal logic with the axiom **L** specifically has not, to the best of our knowledge, been studied before.

6 Conclusion

We have introduced ELF, an epistemic logic that uses filters in order to represent situations where an agent knows (or has a justified belief) that a proposition φ is true even though there are some epistemically accessible worlds where φ is false. We have shown that the proof system **WKL** is sound and complete for ELF. This proof system is similar to **KD**, except that the necessitation rule of that proof system is replaced by a strictly weaker monotonicity rule.

In the basic version of ELF that we discussed in this paper, the properties of truthfulness, positive introspection and negative introspection are not valid. As with normal modal logics, we can enforce these properties by restricting to a smaller class of models. However, unlike normal modal logic, there is not something as elegant and general as correspondence, and there are also additional properties to consider, such as $\Box\top$ (**N**), the dual of our $\neg\Box\bot$ (**L**) axiom. We can then create a sound and complete proof system for ELF on such smaller classes of models by adding a number of axioms to **WKL**. Such technical explorations are relevant, as intuitive scenarios involving certainty and knowledge often satisfy, or fail to satisfy, such constraints. Due to space constraints we must leave the reporting of such frame conditions and axioms for future work.

Acknowledgements. We are grateful to Ramanujam, who suggested the idea of defining knowledge through filters to us. We also thank two anonymous reviewers for their helpful comments. Hans van Ditmarsch is also affiliated to IMSc, Chennai, as associate researcher.

References

1. Aucher, G.: How our beliefs contribute to interpret actions. In: Pěchouček, M., Petta, P., Varga, L.Z. (eds.) CEEMAS 2005. LNCS (LNAI), vol. 3690, pp. 276–285. Springer, Heidelberg (2005). https://doi.org/10.1007/11559221_28
2. Baltag, A., Bezhanishvili, N., Özgün, A., Smets, S.: Justified belief and the topology of evidence. In: Väänänen, J., Hirvonen, Å., de Queiroz, R. (eds.) WoLLIC 2016. LNCS, vol. 9803, pp. 83–103. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-52921-8_6
3. Baltag, A., Smets, S.: A qualitative theory of dynamic interactive belief revision. In: Proceedings of 7th LOFT. Texts in Logic and Games, vol. 3, pp. 13–60. Amsterdam University Press (2008)

4. Ben-David, S., Ben-Eliyahu, R.: A modal logic for subjective default reasoning. Artif. Intell. **116**, 217–236 (2000). https://doi.org/10.1016/S0004-3702(99)00081-8
5. Blackburn, P., van Benthem, J., Wolter, F. (eds.): Handbook of Modal Logic. Elsevier (2006)
6. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Tracts in Theoretical Computer Science, vol. 53. Cambridge University Press, Cambridge (2001)
7. Cartan, H.: Théorie des filtres. Comptes Rendus de l'Académie des Sciences de Paris **205**, 595–598 (1937)
8. Chellas, B.: Modal Logic: An Introduction. Cambridge University Press, Cambridge (1980)
9. Chisholm, R.: Perceiving: A Philosophical Study. Cornell University Press, Ithaca (1957)
10. Conitzer, V.: A puzzle about further facts. Erkenntnis (2018).https://doi.org/10.1007/s10670-018-9979-6
11. DeRose, K.: Contextualism and knowledge attributions. Philos. Phenomenol. Res. **52**, 913–929 (1992). https://doi.org/10.2307/2107917
12. van Ditmarsch, H., Halpern, J., van der Hoek, W., Kooi, B. (eds.): Handbook of Epistemic Logic. College Publications (2015)
13. Gettier, E.: Is justified true belief knowledge? Analysis **23**, 121–123 (1963). https://doi.org/10.2307/3326922
14. Harman, G.: Thought. Princeton University Press, Princeton (1973)
15. Hintikka, J.: Knowledge and Belief. Cornell University Press, Ithaca (1962)
16. van der Hoek, W.: Systems for knowledge and belief. J. Log. Comput. **3**(2), 173–195 (1993). https://doi.org/10.1093/logcom/3.2.173
17. Keisler, H.: Elementary Calculus: An Approach Using Infinitesimals. Prindle Weber & Schmidt (1986)
18. Kraus, S., Lehmann, D.: Knowledge, belief and time. Theor. Comput. Sci. **58**, 155–174 (1988). https://doi.org/10.1016/0304-3975(88)90024-2
19. Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics. Artif. Intell. **44**, 167–207 (1990). https://doi.org/10.1016/0004-3702(90)90101-5
20. Kyburg, H.: Probability and the Logic of Rational Belief. Wesleyan University Press, Middletown (1961)
21. Lewis, D.: Elusive knowledge. Australas. J. Philos. **74**, 549–567 (1996). https://doi.org/10.1080/00048409612347521
22. Nozick, R.: Philosophical Explanations. Harvard University Press, Cambridge (1981)
23. Pacuit, E.: Neighborhood Semantics for Modal Logic. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-319-67149-9
24. Pacuit, E., Salame, S.: Majority logic. In: Proceedings of Ninth KR, pp. 598–605 (2004)
25. Putnam, H.: Reason, Truth and History. Cambridge University Press, Cambridge (1982)
26. Segerberg, K.: Irrevocable belief revision in dynamic doxastic logic. Notre Dame J. Formal Log. **39**(3), 287–306 (1998). https://doi.org/10.1305/ndjfl/1039182247
27. Stalnaker, R.: On logics of knowledge and belief. Philos. Stud. **128**(1), 169–199 (2005). https://doi.org/10.1007/s11098-005-4062-y
28. Unger, P.: Philosophical Relativity. Oxford University Press, New York (1984)

A Framework for Distributional Formal Semantics

Noortje J. Venhuizen[1](✉), Petra Hendriks[2], Matthew W. Crocker[1], and Harm Brouwer[1]

[1] Saarland University, Saarbrücken, Germany
{noortjev,crocker,brouwer}@coli.uni-saarland.de
[2] University of Groningen, Groningen, The Netherlands
p.hendriks@rug.nl

Abstract. Formal semantics and distributional semantics offer complementary strengths in capturing the meaning of natural language. As such, a considerable amount of research has sought to unify them, either by augmenting formal semantic systems with a distributional component, or by defining a formal system on top of distributed representations. Arriving at such a unified framework has, however, proven extremely challenging. One reason for this is that formal and distributional semantics operate on a fundamentally different 'representational currency': formal semantics defines meaning in terms of models of the world, whereas distributional semantics defines meaning in terms of linguistic co-occurrence. Here, we pursue an alternative approach by deriving a vector space model that defines meaning in a distributed manner relative to formal models of the world. We will show that the resulting *Distributional Formal Semantics* offers probabilistic distributed representations that are also inherently compositional, and that naturally capture quantification and entailment. We moreover show that, when used as part of a neural network model, these representations allow for capturing incremental meaning construction and probabilistic inferencing. This framework thus lays the groundwork for an integrated distributional and formal approach to meaning.

Keywords: Distributionality · Compositionality · Probability · Inference · Incrementality

1 Introduction

In traditional formal semantics, the meaning of a logical expression is typically evaluated in terms of the truth conditions with respect to a formal model M, in which the basic meaning-carrying units (i.e., the basic expressions that are assigned a truth value) are propositions [14]. The meaning of a linguistic expression, then, is defined in terms of the truth conditions its logical translation poses upon a formal model. Critically, these truth conditions define meaning in a segregated manner; distinct propositions obtain separate sets of truth conditions.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 633–646, 2019.
https://doi.org/10.1007/978-3-662-59533-6_39

As a result, the relation between individual meanings is not inherently part of their truth-conditional interpretation, but rather follows indirectly from models satisfying these conditions. The core strength of the distributional semantics approach, by contrast, is that (word) meanings are defined in relation to each other, thus directly capturing semantic similarity [18]. It has, however, proven extremely difficult to incorporate well-known features from formal semantics (e.g., compositionality, entailment, etc.) into a distributional semantics framework [3] (but cf. [1,2,7,16,20]).

Here, we take the inverse approach: We introduce distributionality into a formal semantic system, resulting in a framework for Distributional Formal Semantics (DFS). This framework is based on the cognitively inspired meaning representations developed by Golden and Rumelhart [15] and adapted by Frank et al. [12]. In DFS, insights from formal and distributional semantics are combined by defining meaning distributionally over a set of logical models: individual models are treated as observations, or cues, for determining the truth conditions of logical expressions—analogous to how individual linguistic contexts are cues for determining the meaning of words in distributional semantics. Based on a set of logical models $\mathcal{M}$ that together reflect the state of the world both truth-conditionally and probabilistically (i.e., reflecting the probabilistic structure of the world), and a set of propositions $\mathcal{P}$, we can define a vector space for DFS: $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$. The meaning of a proposition is defined as a vector in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$, which reflects its truth or falsehood relative to each of the models in $\mathcal{M}$. The resulting meaning vector captures the *probabilistic* truth conditions of individual propositions indirectly by identifying the models that satisfy the proposition. Critically, the distributional meaning of individual propositions is defined in relation to all other propositions; propositions that have related meanings will be true in many of the same models, and hence have similar meaning vectors. In other words, the meaning of a proposition is defined in terms of the propositions that it co-occurs with—or, to paraphrase the distributional hypothesis formulated by Firth [10]: "You shall know a *proposition* by the company it keeps".

In what follows, we will show how a well-defined vector space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ can be derived from a high-level description of the structure of the world, how the resulting meaning space offers distributed representations that are probabilistic and inferential, and how it captures basic concepts from formal semantics, such as compositionality, quantification and entailment. As a proof-of-concept, we present a computational (Prolog) implementation of the DFS framework.[1] Finally, we will show how the DFS representations can be employed in a neural network model for incremental meaning construction. Crucially, we will show how this approach to incremental meaning construction allows for the representation of sub-propositional meaning by exploiting the continuous nature of the meaning space.

[1] DFS-TOOLS is publicly available at http://github.com/hbrouwer/dfs-tools under the Apache License, version 2.0.

2 A Framework for Distributional Formal Semantics

In DFS, the meaning of a proposition $p \in \mathcal{P}$ is defined as a vector $\boldsymbol{v}(p)$ in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$, such that each unit corresponds to a $M \in \mathcal{M}$, and is assigned a 1 iff M satisfies p, and a 0 otherwise. Consequently, for $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ to be well-defined, the set of models $\mathcal{M}$ that constitutes the meaning space must capture the relevant truth conditions for each proposition $p \in \mathcal{P}$, and conversely, the set of propositions $\mathcal{P}$ must contain all propositions that are captured by each model $M \in \mathcal{M}$. Beyond being well-defined, the meaning space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ should capture the structure of the world. First of all, the world can enforce hard world knowledge constraints on the co-occurrence of propositions; for instance, certain combinations of propositions may never co-occur, that is, never be simultaneously satisfied within the same model (e.g., a person cannot be at two different places). Secondly, there may be probabilistic constraints on the co-occurrence of propositions; a proposition p may co-occur more frequently with p' than with p'' (for some $p, p', p'' \in \mathcal{P}$), that is, there should be more models $M \in \mathcal{M}$ that satisfy $p \wedge p'$ than models $M' \in \mathcal{M}$ satisfying $p \wedge p''$ (e.g., one prefers reading in bed over reading on the sofa). For $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ to reflect our high-level knowledge about the structure of the world regarding the probabilistic truth-conditions of each proposition $p \in \mathcal{P}$, we thus need its constituent set of models $\mathcal{M}$ to approximate this knowledge. One way of arriving at a satisfactory $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ is to induce this set of models by sampling each model $M \in \mathcal{M}$ from a high-level specification of the structure of the world.

2.1 Sampling $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$

For a given set of propositions $\mathcal{P}$, there are theoretically $2^{\mathcal{P}}$ possible models. Hard constraints in the world rule out any model that satisfies illegal combinations of propositions, while probabilistic constraints require the set of models $\mathcal{M}$ that constitutes $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ to reflect that a proposition p may co-occur more frequently with p' than with p'' (for some $p, p', p'' \in \mathcal{P}$). Hence, the goal is to find a set of models $\mathcal{M}$ such that each $M \in \mathcal{M}$ satisfies all hard constraints, and $\mathcal{M}$ as a whole reflects the probabilistic structure of the world. To this end, we employ an inference-driven, non-deterministic sampling algorithm (inspired by [13]) that stochastically generates models from a set of hard and probabilistic co-occurrence constraints on the propositions $\mathcal{P}$.

As in traditional formal semantics, a model $M \in \mathcal{M}$ is defined as the tuple $\langle U_M, V_M \rangle$, where U_M defines the universe of M, and V_M is the interpretation function that assigns (sets of) entities to the individual constants and properties that constitute $\mathcal{P}$. Given the set of constants $c_1 \ldots c_n$ defined by $\mathcal{P}$, the universe of each $M \in \mathcal{M}$ is defined as $U_M = \{e_1 \ldots e_n\}$, and the interpretation function is initialized to map each constant onto a unique entity: $V_M(c_i) = e_i$. The next step is to stochastically define an interpretation for all propositions in $\mathcal{P}$, while taking into account hard and probabilistic constraints on world structure. To this end, we start out with the initialized interpretation function, which will be incrementally expanded with the interpretation of individual propositions.

We call this interpretation function the Light World[2] (LV_M), to which will be assigned all propositions that are satisfied in M. To facilitate the incremental, inference-driven construction of M, we will in parallel construct a Dark World interpretation function (DV_M), to which will be assigned all propositions that are *not* satisfied in M. Finally, we assume hard constraints to be represented by a set of well-formed formulas $\mathcal{C}$, while probabilistic constraints are represented by a function $Pr(\phi)$ that assigns a probability to a property ϕ. A model M is then sampled by iterating the following steps:

1. Given the constants $c_1 \dots c_n$ defined by $\mathcal{P}$, let $U_M = \{e_1 \dots e_n\}$, $LV_M(c_i) = e_i$ and $DV_M(c_i) = e_i$.
2. Randomly select a proposition $\phi = P(t_1, \dots, t_n)$ from $\mathcal{P}$ that is not yet assigned in LV_M or DV_M.
3. Let LV'_M be the function that extends LV_M with the interpretation of ϕ, such that $\langle t_1, \dots, t_n \rangle \in LV'_M(P)$.
4. Let Light World Consistency $LWC = \top$ iff each constraint in $\mathcal{C}$ is either satisfied by $\langle U_M, LV'_M \rangle$, or if its complement[3] is not satisfied by $\langle U_M, DV_M \rangle$.
5. Let DV'_M be the function that extends DV_M with the interpretation of ϕ , such that $\langle t_1, \dots, t_n \rangle \in DV'_M(P)$.
6. Let Dark World Consistency $DWC = \top$ iff each constraint in $\mathcal{C}$ is either satisfied by $\langle U_M, LV_M \rangle$, or if its complement is not satisfied by $\langle U_M, DV'_M \rangle$.
7. Provided the outcome of step 4 and step 6:
   - $LWC \wedge DWC$: ϕ can be true in both worlds, let $LV_M = LV'_M$ with probability $Pr(\phi)$ and $DV_M = DV'_M$ with probability $1 - Pr(\phi)$;
   - $LWC \wedge \neg DWC$: ϕ can be inferred to the Light World, let $LV_M = LV'_M$;
   - $\neg LWC \wedge DWC$: ϕ can be inferred to the Dark World, let $DV_M = DV'_M$;
   - $\neg LWC \wedge \neg DWC$: ϕ cannot be inferred to either world, meaning the model thus far is inconsistent, and sampling is restarted from step 1.[4]
8. Repeat from step 2 until each proposition in $\mathcal{P}$ is assigned in LV_M or DV_M.
9. If LV_M satisfies each constraint in $\mathcal{C}$, LV_M will be the interpretation function of the resultant model $M = \langle U_M, LV_M \rangle$.

Repeating this sampling procedure n times will yield a set of models $\mathcal{M}$ with cardinality $|\mathcal{M}| = n$. Crucially, while this procedure only samples one model at a time, the probabilistic assignment of non-inferrable propositions to the Light World in step 7 will assure that each probability $Pr(\phi)$ is approximated by the fraction of models in $\mathcal{M}$ that satisfy ϕ, provided that $\mathcal{M}$ is of sufficient size. An efficient implementation of this sampling algorithm is available as part of DFS-TOOLS (see Footnote 1).

[2] cf. The Legend of Zelda: A Link to the Past (Nintendo, 1992).

[3] While a constraint is a well-formed formula that specifies its truth-conditions relative to the Light World (LV_M), its complement specifies its falsehood-conditions relative to the Dark World (DV_M); e.g., the Light Word constraint $\forall x.sleep(x)$ can be proven to be violated if $\exists x.sleep(x)$ is satisfied in the Dark World. See the appendix for a full set of translation rules.

[4] The sampling of inconsistent models strongly depends on the interdependency of the constraints in $\mathcal{C}$ and can be prevented by defining $\mathcal{C}$ in such a way that all combinations of propositions are explicitly handled.

2.2 Formal Properties of $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$

Compositionality. A well-defined semantic space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ defines the meaning vectors for a set of individual propositions $\mathcal{P}$ relative to a set of logical models $\mathcal{M}$. Given that the meaning vector $\boldsymbol{v}(p)$ of a proposition $p \in \mathcal{P}$ defines its truth values relative to $\mathcal{M}$, we can define the negation $\neg p$ as the vector that assigns 1 to all $M \in \mathcal{M}$ such that p is not satisfied in M, and 0 otherwise:

$$\boldsymbol{v}_i(\neg p) = 1 \text{ iff } M_i \nvDash p \text{ for } 1 \leq i \leq |\mathcal{M}|$$

The meaning of the conjunction $p \wedge q$, given $p, q \in \mathcal{P}$, is defined as the vector $\boldsymbol{v}(p \wedge q)$ that assigns 1 to all $M \in \mathcal{M}$ such that M satisfies both p and q, and 0 otherwise:

$$\boldsymbol{v}_i(p \wedge q) = 1 \text{ iff } M_i \vDash p \text{ and } M_i \vDash q \text{ for } 1 \leq i \leq |\mathcal{M}|$$

Using the negation and conjunction operators, the meaning of any other logical combination of propositions in the semantic space can be defined, thus allowing for meaning vectors representing expressions of arbitrary logical complexity. Critically, these operations also allow for the definition of quantification. Since $\mathcal{P}$ fully describes the set of propositions expressed in $\mathcal{M}$, the (combined) universe of $\mathcal{M}$ ($U_{\mathcal{M}} = \{u_1, \ldots, u_n\}$) directly derives from $\mathcal{P}$. Universal quantification, then, can be formalized as the conjunction over all entities in $U_{\mathcal{M}}$:

$$\boldsymbol{v}_i(\forall x \phi) = 1 \text{ iff } M_i \vDash \phi[x \backslash u_1] \wedge \ldots \wedge \phi[x \backslash u_n] \text{ for } 1 \leq i \leq |\mathcal{M}|$$

Existential quantification, in turn, is formalized as the disjunction over all entities in $U_{\mathcal{M}}$:

$$\boldsymbol{v}_i(\exists x \phi) = 1 \text{ iff } M_i \vDash \phi[x \backslash u_1] \vee \ldots \vee \phi[x \backslash u_n] \text{ for } 1 \leq i \leq |\mathcal{M}|$$

The vectors from $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ are thus fully compositional at the propositional level. Furthermore, in Sect. 3, we will show how sub-propositional meaning can be constructed by incrementally mapping expressions onto vectors in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$.

Probability. The semantic space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ is inherently probabilistic, as the meaning vectors for individual propositions in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ inherently encode their probability. Given a set of models $\mathcal{M}$ that reflects the probabilistic nature of the world, the probability of p can be defined by the number of models that satisfy p, divided by the total number of models:

$$P(p) = |\{M \in \mathcal{M} \mid M \vDash p\}| / |\mathcal{M}|$$

Thus, propositions that are true in a large set of models will obtain a high probability. Given the notion of compositionality discussed above, the probability of $a \wedge b$ can be defined as the probability of the conjunctive vector $\boldsymbol{v}(a \wedge b)$, where a and b may be atomic propositions in $\mathcal{P}$ or any arbitrarily complex combination thereof. Finally, the conditional probability of b given a is defined as:

$$P(b|a) = P(a \wedge b)/P(a)$$

Inference. As described above, the meaning of individual propositions in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ is defined in terms of their co-occurrence with other propositions. As a result, the vector representations inherently encode how propositions, and logical combinations thereof, are logically related to each other. Entailment, for instance, is reflected in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ by means of vectors with overlapping truth values; (complex) proposition a entails b ($a \vDash b$) iff b is satisfied by all models that satisfy a. Based on the definition of conditional probability described above, we can moreover formalize probabilistic inference. Intuitively, a high conditional probability of b given a would indicate that b can be inferred from a, since b is satisfied by a large number of models that satisfy a. However, this conditional probability alone is insufficient, as inference requires quantifying the degree to which a increases (or decreases) the probability of b above and beyond its prior probability $P(b)$. We therefore adopt a score for logical inference that factors out this prior probability [11]:

$$inf(b, a) = \begin{cases} \frac{P(b|a)-P(b)}{1-P(b)} & \text{if } P(b|a) > P(b) \\ \frac{P(b|a)-P(b)}{P(b)} & \text{otherwise} \end{cases}$$

This score yields a value ranging from $+1$ to -1, where $+1$ indicates that (complex) proposition b is perfectly inferred from a (i.e., a entails b; $a \vDash b$), whereas a value of -1 indicates that the negation of b is perfectly inferred from a ($a \vDash \neg b$). Any inference score in between these extremes reflects probabilistic inference in either direction. In what follows, we will employ this notion of inference in a neural network model of incremental meaning construction.

3 Incremental Meaning Construction

The meaning space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ defines meaning vectors for all propositions in $\mathcal{P}$, and using the compositional operations described above, vectors can be derived for complex logical combinations of propositions. The meaning space also naturally captures sub-propositional meaning. That is, while vectors representing propositional meaning are binary—reflecting truth- and falsehood within models in $\mathcal{M}$—the meaning space itself is continuous, which means that it also captures meanings that are not directly expressible as (combinations of) propositions. We can exploit this continuous nature of $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ to model the word-by-word, context-dependent construction of (sentence-final) propositional meaning. That is, the meaning of a sub-propositional expression is a real-valued vector that defines a point in the vector space, which is positioned in between those points that instantiate the propositional meanings that the expression pertains to. In contrast to traditional semantic approaches, the DFS approach does not define an operation that simply combines the sub-propositional meanings of two subsequent expressions. Rather, sequences of words $w_1 \dots w_n$ define a trajectory $\langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_n \rangle$ through $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$, where each $\boldsymbol{v}_i$ represents the (sub-propositional) meaning induced by the sequence of words $w_1 \dots w_i$; that is, each word w_i induces a meaning in the context of the meaning assigned to its preceding words $w_1 \dots w_{i-1}$. Sub-propositional meaning thus critically derives from the

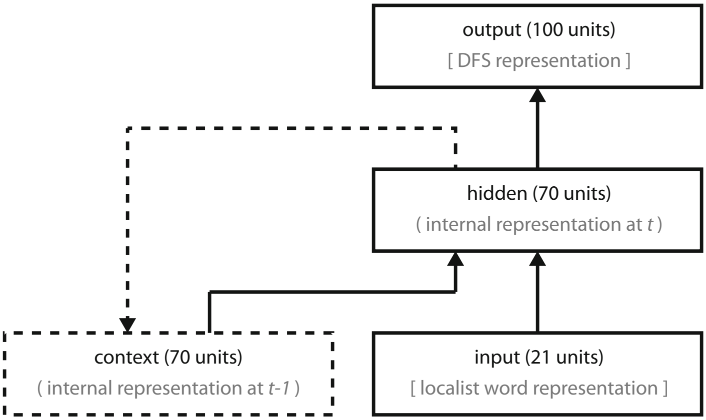


Fig. 1. Simple Recurrent neural Network. Boxes represent groups of artificial neurons, and solid arrows between boxes represent full projections between the neurons in a projecting and a receiving group. The dashed lines indicate that the CONTEXT layer receives a copy of the activation pattern at the HIDDEN layer at the previous time-step. See text for details.

incremental, context-dependent mapping from word sequences onto (complex) propositional meanings. One piece of machinery that is particularly good at approximating such an incremental, context-dependent mapping is the Simple Recurrent neural Network (SRN) [8]. Below, we describe an SRN for incremental meaning construction (cf. [22]) and show how it navigates the meaning space on a word-by-word basis, allowing for incremental (sub-propositional) meaning construction and inferencing.

3.1 Model Specification

We employ an SRN consisting of three groups of artificial logistic dot-product neurons: an INPUT layer (21 units), HIDDEN layer (70), and OUTPUT layer (100) (see Fig. 1). Time in the model is discrete, and at each processing time-step t, activation flows from the INPUT through the HIDDEN layer to the OUTPUT layer. In addition to the activation pattern at the INPUT layer, the HIDDEN layer also receives its own activation pattern at time-step $t-1$ as input (effectuated through an additional CONTEXT layer, which receives a copy of the activation pattern at the HIDDEN layer prior to feedforward propagation). The HIDDEN and the OUTPUT layers both receive input from a bias unit (omitted in Fig. 1). We trained the model using bounded gradient descent [19] to map sequences of localist word representations constituting the words of a sentence, onto a meaning vector from $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ representing the meaning of that sentence.

The sentences on which the model is trained describe situations in a confined world. This world is defined in terms of two persons ($p \in \{john, ellen\}$), two places ($x \in \{restaurant, bar\}$), and two types of food ($f \in \{pizza, fries\}$) and drinks ($d \in \{wine, beer\}$), which can be combined using the following 7 predicates: *enter(p,x)*, *ask_menu(p)*, *order(p,f/d)*, *eat(p,f)*, *drink(p,d)*, *pay(p)* and *leave(p)*. The resulting set of propositions $\mathcal{P}$ ($|\mathcal{P}| = 26$) fully describes the world. A meaning space was constructed from these atomic propositions by sampling a set of 10 K models $\mathcal{M}$ (using the sampling algorithm described in Sect. 2.1), while taking into account world knowledge in terms of hard and probabilistic constraints on proposition co-occurrence; for instance, a person can only enter a single place (hard), and *john* prefers to drink *beer* over *wine* (probabilistic). In order to employ meaning vectors derived from this meaning space in the SRN, we algorithmically selected a subset $\mathcal{M}'$ consisting of 100 models from $\mathcal{M}$, such that $\mathcal{M}'$ adequately reflected the structure of the world (using the algorithm described in [22]). Situations in the world were described using sentences from a language $\mathcal{L}$ consisting of 21 words. The grammar of $\mathcal{L}$ generates a total of 124 sentences, consisting of simple (NP VP) and coordinated (NP VP and VP) sentences. The sentence-initial NPs may be *john*, *ellen*, *someone*, or *everyone*, and the VPs map onto the aforementioned propositions. The corresponding meaning vectors for the sentences in $\mathcal{L}$ were derived using the compositional operations described in Sect. 2.2 (where *someone* and *everyone* correspond to existential and universal quantification, respectively). The model was trained on the full set of sentences generated by $\mathcal{L}$, without any frequency differences between sentences.[5]

Prior to training, the model's weights were randomly initialized using a range of $(-.5, +.5)$. Each training item consisted of a sentence (a sequence of words represented by localist representations) and a meaning vector representing the sentence-final meaning. For each training item, error was backpropagated after each word, using a zero error radius of 0.05, meaning that no error was backpropagated if the error on a unit fell within this radius. Training items were presented in permuted order, and weight deltas were accumulated over epochs consisting of all training items. At the end of each epoch, weights were updated using a learning rate coefficient of 0.1 and a momentum coefficient of 0.9. Training lasted for 5000 epochs, after which the mean squared error was 0.69. The overall performance of the model was assessed by calculating the cosine similarity between each sentence-final output vector and each target vector for all sentences in the training data. All output vectors had the highest cosine similarity to their own target (mean = .99; sd = .02), indicating that the model successfully learned to map sentences onto their corresponding semantics. We moreover computed how well the intended target could be inferred from the output of the model: $inf(\boldsymbol{v}_{target}, \boldsymbol{v}_{output})$.[6] The average inference score over the entire training set was 0.88, which means that after processing a sentence, the model almost perfectly infers the intended meaning of the sentence.

---

[5] The specification of the world described here, including the definition of the language $\mathcal{L}$, is available as part of DFS-TOOLS (see Footnote 1).

[6] For real-valued vectors, we can calculate the probability of vector $\boldsymbol{v}(a)$ as follows: $P(a) = \sum_i \boldsymbol{v}_i(a)/|\mathcal{M}|$.

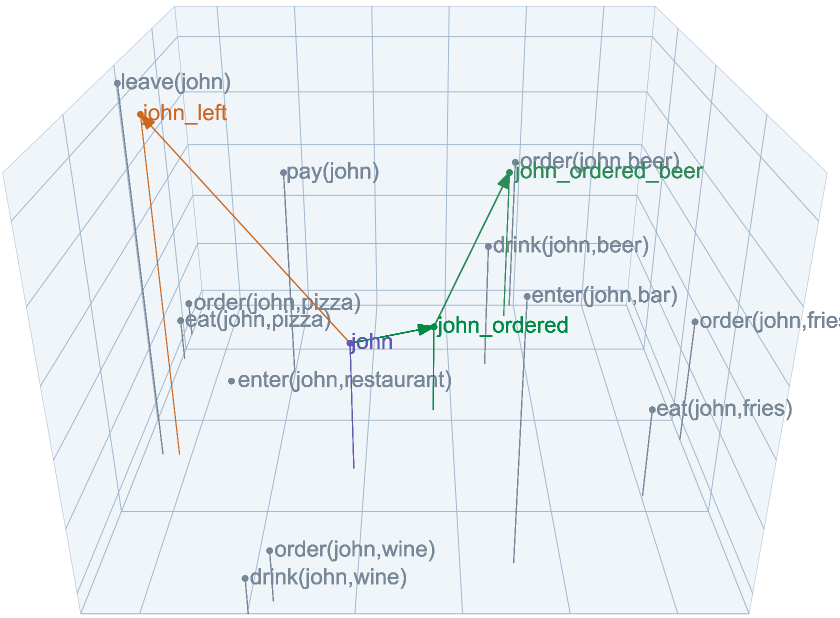


Fig. 2. Visualization of the meaning space into three dimensions (using multidimensional scaling; MDS) for a subset of the atomic propositions (those pertaining to *john*). Grey points represent propositional meaning vectors. Coloured points and arrows show the word-by-word navigational trajectory of the model for the sentences "*john ordered beer*" and "*john left*". See also Footnote 7. (Colour figure online)

3.2 Incremental Inferencing in DFS

On the basis of its linguistic input, the model incrementally constructs a meaning vector at its OUTPUT layer that captures sentence meaning; that is, the model effectively navigates the meaning space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ on a word-by-word basis. Figure 2 provides a visualization of this navigation process. This figure is a three-dimensional representation of the 100-dimensional meaning space (for a subset of the atomic propositions), derived using multidimensional scaling (MDS). The grey points in this space correspond to propositional meaning vectors. As this figure illustrates, meaning in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ is defined in terms of co-occurrence; propositions that co-occur frequently in $\mathcal{M}$ (e.g., *order(john,wine)* and *drink(john,wine)*) are positioned close to each other in space.[7] The coloured points show the model's word-by-word output for the sentences "*john ordered beer*" and "*john left*". The navigational trajectory (indicated by the arrows) illustrates how the model assigns intermediate points in meaning space to sub-propositional expressions, and instantiates propositional meanings at

[7] Multidimensional scaling from 100 into 3 dimensions necessarily results in a significant loss of information. Therefore, distances between points in the meaning space shown in Fig. 2 should be interpreted with care.

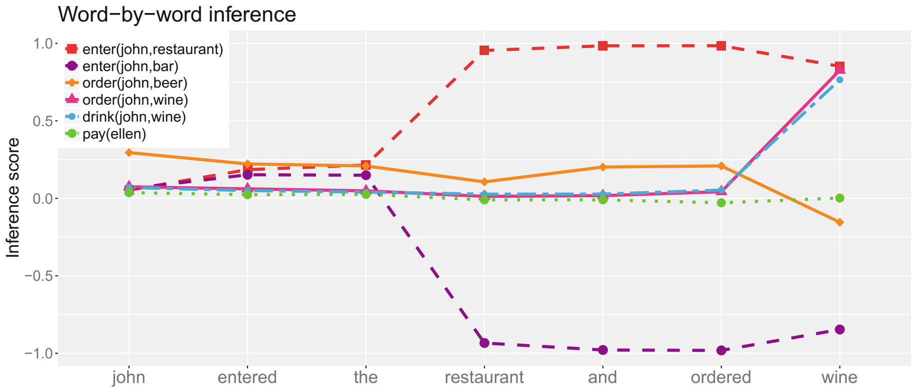


Fig. 3. Word-by-word inference scores of selected propositions for the sentence "*John entered the restaurant and ordered wine*" with the semantics: $enter(john, restaurant) \wedge order(john, wine)$. At a given word, a positive inference score for proposition p indicates that p is positively inferred to be the case; a negative inference score indicates that p is inferred not to be the case (see text for details). (Colour figure online)

sentence-final words. For instance, at the word "*john*", the model navigates to a point in meaning space that is in between the meanings of the propositions pertaining to *john*. The prior probability of propositions in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ ("world knowledge"), as well as the sentences on which the model was trained ("linguistic experience") together determine the model's trajectory through meaning space. For instance, while the model was exposed to the sentences "*john ordered beer*" and "john ordered wine" equally often, the vector for the expression "*john ordered*" is closer to *order(john,beer)* than *order(john,wine)*, because the former is more probable in the model's knowledge of the world (see [22] for an elaborate investigation of the influence of world knowledge and linguistic experience on meaning space navigation).

Using the inference score described in Sect. 2.2, we can moreover study what the model 'understands' at each word of a sentence (i.e., $inf(b, a)$, where b is the vector of a proposition of interest, and a the output vector of the SRN). Figure 3 shows the word-by-word inference scores for the sentence "*john entered the restaurant and ordered wine*" with respect to 6 propositions. First of all, this figure shows that by the end of the sentence, the model has understood its meaning: the inference scores of *enter(john,restaurant)* and *order(john,wine)* are both ≈ 1 at the sentence-final word. What is more, it does so on an incremental basis: at the word "restaurant", the model commits to the inference *enter(john,restaurant)*, which rules out *enter(john,bar)* since these do not co-occur in the world ($P($*enter(john,restaurant)* $\wedge$ *enter(john,bar)*$) = 0$). At the word "ordered", the model finds itself in state that is closer to the inference that *order(john,beer)* than *order(john,wine)*, as John prefers beer over wine ($P($*order(john,beer)*$) = 0.81 > P($*order(john,wine)*$) = 0.34$). However, at the word "*wine*" this inference is reversed, and the model understands that *order(john,wine)* is the case, and that *order(john,beer)* cannot be inferred. In addition, the word "*wine*" also leads the model to infer *drink(john,wine)*, even though this proposition is not explicitly part of the semantics of the sentence. This happens because the world stipulates that given that John ordered wine, it is likely that he also drank it ($P($*drink(john,wine)* $\mid$ *order(john,wine)*$) = 0.88$). Finally, no significant inferences are drawn about the unrelated proposition $pay(ellen)$.

4 Discussion

The DFS framework defines the meaning of a proposition p in terms of models that satisfy it and those that do not. Hence, the framework relies on finding a set of models $\mathcal{M}$ that truth-conditionally and probabilistically capture the structure of the world with respect to a set of propositions $\mathcal{P}$. Here, we focused on how this space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ can be induced from a high-level description of the structure of the world. We would like to emphasize, however, that none of the described formal properties of the meaning space hinges upon this sampling procedure. An alternative approach towards arriving at $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$, for instance, is to induce it empirically from a semantically annotated corpus (e.g., [4]) or from crowd-sourced human data on propositional co-occurrence (e.g., [23]). The only requirements are that the resultant space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ is well-defined, and that it accurately approximates the structure of the world in terms of hard and probabilistic constraints on propositional co-occurrence.

DFS representations are inherently compositional at the level of propositions in that atomic propositions can be compositionally combined into complex propositions. At the sub-propositional level, however, meaning is constructed by incrementally navigating $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$. Arriving at the meaning of "john ordered" does not simply involve combining the meaning of "john" with the meaning of "ordered", but rather entails the context-dependent integration of the word "ordered" into the meaning representation constructed after processing "john" (cf. [5]). Crucially, this is possible due to the continuous nature of $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$. Hence, in the DFS framework, compositionality at the propositional level and incrementality at the sub-propositional level interact in context-dependent meaning construction.

The relatively simple neural network model presented here served to illustrate the incremental meaning construction procedure. More sophisticated models, however, instantiating earlier formulations of the DFS framework (cf. [12]), have already highlighted various other interesting properties of the approach. For one, while the current model was trained and tested on the same sentence-semantics pairs, other models have shown generalization to unseen sentences and semantics, in both comprehension [11] and production [6]. Crucially, this semantic systematicity derives from the structure of the world as encoded by the meaning space. Moreover, since in a comprehension model—such as the one described here—each word serves as a contextualized cue for meaning space navigation, a relatively simple SRN architecture (as compared to more complex architectures such as Long Short-Term Memory, LSTM, [17] networks), suffices for this systematicity to manifest. Secondly, other models have explored the dynamics of meaning-space navigation using information-theoretic notions such as surprisal and entropy [13,22].

In DFS, there are two levels at which semantic phenomena can be modeled: the level of the meaning space $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$, and the mapping from words onto points within this meaning space. Starting with the meaning space itself, one could explore different schemes for encoding the atomic propositions, for instance to explicitly capture tense and aspect, or Davidsonian event semantics. Moreover, by varying temporally-dependent proposition co-occurrence within and across models, we obtain different encodings of time within $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ (see [22] for a within-model approach). At the level of the mapping between words and points in $\mathcal{S}_{\mathcal{M}\times\mathcal{P}}$ space, in turn, the DFS framework allows for different ways to capture discourse-level phenomena, such as modality, reference, information structure, and implicature. Crucially, the fact that inference directly follows from incremental semantic meaning construction circumvents the need for a separate pragmatic inference mechanism. This thus blurs the strict line between semantics and pragmatics, thereby directly implementing recent theorizing in formal semantics [21].

While the DFS framework combines formal and distributional approaches to meaning, we take the framework to be complementary to lexically-driven distributional semantics (e.g., LSA; [18]). In DFS, the 'representational currency' is *propositions*, whereas in distributional semantics it is *words*. As a result, DFS allows us to model similarity at the propositional level (e.g., $order(john, beer)$ is similar to $drink(john, beer)$ as they co-occur in $\mathcal{M}$), while distributional semantics models lexical similarity ("*beer*" is similar to "*wine*" as they occur in similar linguistic contexts; e.g., [9]). Crucially, the DFS approach and distributional semantics thus capture different notions of semantic similarity: while the latter offers representations that inherently encode feature-based lexical similarity between words, the former provides representations instantiating the truth-conditional similarity between propositions. The complementary nature of these meaning representations is underlined by recent advances in the neurocognition of language, where evidence suggests that lexical retrieval (the mapping of words onto lexical semantics) and semantic integration (the integration of word meaning into the unfolding representation of propositional meaning) are two distinct processes involved in word-by-word sentence processing [5]. Crucially, this perspective on language comprehension suggests that compositionality is only at play at the level of propositions, thus eschewing the need for compositionality at the lexical level.

5 Conclusion

The DFS framework offers a novel approach to distributional semantics, by defining the meaning of propositions distributionally over a set of formal models. As a consequence, the approach inherits the entire apparatus of (first-order) logic that powers formal semantics, while offering contextualized and probabilistic distributed meaning representations similar to distributional semantics. Crucially, the meaning representations differ from those from distributional semantics in that they offer probabilistic information that reflects the state of the world, rather than linguistic co-occurrence, thereby offering a complementary perspective on meaning representation. To illustrate the approach, we have shown how the DFS meaning space can be derived from a high-level specification of the world, and how it naturally captures well-known concepts from formal semantics, such as compositionality and entailment. Moreover, when employed in an incremental model of meaning construction, it naturally captures sub-propositional meaning and inferencing. As such, we believe that the DFS framework—implemented by DFS-TOOLS—offers a powerful synergy between formal and distributional approaches that paves the way towards novel investigations into formal meaning representation and construction.

Appendix

The complement of any well-formed formula is found by recursively applying the following translations, where ϕ' is the complement of ϕ:

$$\begin{array}{llllll}
\neg\phi & \mapsto \neg\phi' & \phi \veebar \psi & \mapsto (\phi' \wedge \psi') \vee (\neg\phi' \wedge \neg\psi') & \exists x.\phi & \mapsto \forall x.\phi' \\
\phi \wedge \psi & \mapsto \phi' \vee \psi' & \phi \rightarrow \psi & \mapsto \neg\phi' \vee \psi' & \forall x.\phi & \mapsto \exists x.\phi' \\
\phi \vee \psi & \mapsto \phi' \wedge \psi' & \phi \leftrightarrow \psi & \mapsto (\neg\phi' \wedge \psi') \vee (\phi' \wedge \neg\psi') & p & \mapsto p
\end{array}$$

References

1. Baroni, M., Bernardi, R., Zamparelli, R.: Frege in space: a program of compositional distributional semantics. Linguist. Issues Lang. Technol. (LiLT) **9**, 241–346 (2014)
2. Baroni, M., Zamparelli, R.: Nouns are vectors, adjectives are matrices: representing adjective-noun constructions in semantic space. In: Proceedings of the 2010 Conference on Empirical Methods in Natural Language Processing, pp. 1183–1193. Association for Computational Linguistics (2010)
3. Boleda, G., Herbelot, A.: Formal distributional semantics: introduction to the special issue. Comput. Linguist. **42**(4), 619–635 (2016)
4. Bos, J., Basile, V., Evang, K., Venhuizen, N.J., Bjerva, J.: The Groningen Meaning Bank. In: Ide, N., Pustejovsky, J. (eds.) Handbook of Linguistic Annotation, pp. 463–496. Springer, Dordrecht (2017). https://doi.org/10.1007/978-94-024-0881-2_18
5. Brouwer, H., Crocker, M.W., Venhuizen, N.J., Hoeks, J.C.J.: A neurocomputational model of the N400 and the P600 in language processing. Cogn. Sci. **41**, 1318–1352 (2017). https://doi.org/10.1111/cogs.12461
6. Calvillo, J., Brouwer, H., Crocker, M.W.: Connectionist semantic systematicity in language production. In: Papafragou, A., Grodner, D., Mirman, D., Trueswell, J.C. (eds.) Proceedings of the 38th Annual Conference of the Cognitive Science Society, Austin, TX, pp. 2555–3560 (2016)
7. Coecke, M.S.B., Clark, S.: Mathematical foundations for a compositional distributed model of meaning. In: Lambek Festschrift, Linguistic Analysis, vol. 36 (2010)

8. Elman, J.L.: Finding structure in time. Cogn. Sci. **14**(2), 179–211 (1990)
9. Erk, K.: What do you know about an alligator when you know the company it keeps? Semant. Pragmat. **9**(17), 1–63 (2016). https://doi.org/10.3765/sp.9.17
10. Firth, J.R.: A synopsis of linguistic theory, 1930–1955. In: Studies in linguistic analysis. Philological Society, Oxford (1957)
11. Frank, S.L., Haselager, W.F.G., van Rooij, I.: Connectionist semantic systematicity. Cognition **110**(3), 358–379 (2009)
12. Frank, S.L., Koppen, M., Noordman, L.G.M., Vonk, W.: Modeling knowledge-based inferences in story comprehension. Cogn. Sci. **27**(6), 875–910 (2003)
13. Frank, S.L., Vigliocco, G.: Sentence comprehension as mental simulation: an information-theoretic perspective. Information **2**(4), 672–696 (2011)
14. Frege, G.: Über Sinn und Bedeutung. Zeitschrift für Philosophie und philosophische Kritik **100**, 25–50 (1892)
15. Golden, R.M., Rumelhart, D.E.: A parallel distributed processing model of story comprehension and recall. Discourse Process. **16**(3), 203–237 (1993)
16. Grefenstette, E., Sadrzadeh, M.: Experimental support for a categorical compositional distributional model of meaning. In: Proceedings of the Conference on Empirical Methods in Natural Language Processing, pp. 1394–1404. Association for Computational Linguistics (2011)
17. Hochreiter, S., Schmidhuber, J.: Long short-term memory. Neural Comput. **9**(8), 1735–1780 (1997)
18. Landauer, T.K., Dumais, S.T.: A solution to Plato's problem: the latent semantic analysis theory of acquisition, induction, and representation of knowledge. Psychol. Rev. **104**(2), 211–240 (1997)
19. Rohde, D.L.T.: A connectionist model of sentence comprehension and production. Ph.D. thesis, Carnegie Mellon University (2002)
20. Socher, R., Huval, B., Manning, C.D., Ng, A.Y.: Semantic compositionality through recursive matrix-vector spaces. In: Proceedings of the 2012 Joint Conference on Empirical Methods in Natural Language Processing and Computational Natural Language Learning, pp. 1201–1211. Association for Computational Linguistics (2012)
21. Venhuizen, N.J., Bos, J., Hendriks, P., Brouwer, H.: Discourse semantics with information structure. J. Semant. **35**(1), 127–169 (2018). https://doi.org/10.1093/jos/ffx017
22. Venhuizen, N.J., Crocker, M.W., Brouwer, H.: Expectation-based comprehension: modeling the interaction of world knowledge and linguistic experience. Discourse Process. **56**(3), 229–255 (2019). https://doi.org/10.1080/0163853X.2018.1448677
23. Wanzare, L.D., Zarcone, A., Thater, S., Pinkal, M.: DeScript: a crowdsourced corpus for the acquisition of high-quality script knowledge. In: The International Conference on Language Resources and Evaluation (2016)

Weak Conservativity

Richard Zuber(✉)

CNRS, Laboratoire de Linguistique Formelle, Paris, France
Richard.Zuber@linguist.univ-paris-diderot.fr

Abstract. This paper focuses on formal properties of functions satisfying a weak conservativity, a generalisation of classical *conservativity*, a well known constraint on the denotations of unary determiners. Informally, classically conservative determiners are determiners which are conservative "on the right" whereas weakly conservative determiners can be conservative "on the right" or "on the left". These notions are made precise and it is shown in particular that the constraint of weak conservativity remains a very strong constraint excluding most type $\langle 1,1\rangle$ functions and that the Boolean closure of weakly conservative functions equals the set of all type $\langle 1,1\rangle$ functions.

1 Introduction

A by now well-known and often discussed semantic universal has to do with restrictions on the denotations of determiners: all determiners denote functions which are restricted by the property of conservativity or, less formally, all determiners are conservative. In terms of categorial grammar, determiners are functional expressions which take common nouns (CNs) as arguments and form expressions which can play the role of verbal arguments.

We will be interested mainly in unary determiners, that is determiners which take one CN as argument and verbal arguments we will take into account are subject noun phrases (NPs). Of course NPs can also occur in other grammatical positions, in particular in the direct object position but exact study of determiners in non-subject positions necessitates some additional definitions.

Determiners in the subject position denote binary relations between sets, sub-sets of a given universe, and conservativity is a specific constraint on such relations.

We will thus analyse formal properties of denotations of subject determiners that is determiners *Det* as they occur in sentences of the form given in (1), where *VP* is a verb phrase:

(1) *Det CN VP*

Given the syntactic status of *Det* in (1) and supposing that CNs and VPs denote sets, one can consider that unary determiners denote functions taking two sets as arguments and giving a truth value as result. Conservativity is a constraint on such functions. Classically this constraint is formulated as in D1:

Thanks to Ed Keenan for various suggestions.

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, pp. 647–660, 2019.
https://doi.org/10.1007/978-3-662-59533-6_40

Definition 1. *Let F be a function taking two sets as arguments and giving a truth value as result. Then F is conservative iff the following holds: $F(X)(Y) = F(X)(X \cap Y)$, for any set X and Y.*

We will call conservativity as formulated in D1 *classical conservativity.*

The *conservativity universal* is the claim that natural language determiners denote conservative functions (Barwise and Cooper 1981, Keenan and Faltz 1985, Keenan and Stavi 1986, Keenan and Westerståhl 1997, von Fintel and Keenan 2018).

Recall that the conservativity constraint is a very strong constraint. It can be shown (cf. Keenan and Stavi 1986, Thijsse 1984) that if the universe considered has n elements then the cardinality of all functions of the same type as F in D1 is 2^m for $m = 4^n$ whereas the cardinality of conservative functions is 2^k, for $k = 3^n$. So in the universe with just 2 elements there are $2^{16} = 65.536$ functions unrestricted by conservativity and only $2^9 = 512$ conservative functions. Thus conservativity excludes a great majority of functions as possible denotations for unary natural language determiners.

Recent research in the semantics of NLs shows that the classical conservativity is a too strong constraint since there are expressions which can play the role of determiners and which do not denote classically conservative functions (see Zuber and Keenan forthcoming, and various papers cited there). In particular there are empirical reasons to consider that items like *only, mostly, at most, at least, even, also* etc. can form NPs such as *only students, at most liberals, at least monks, mostly linguists, also students, even Japanese* and consequently these items can play the role of unary determiners. As one can check, and this will be done below for some of them, functions denoted by these items are not classically conservative.

It follows from the above that the conservativity constraint should be weakened to *weak conservativity* in order to account for various counter-examples to classical conservativity. Such a weakened version of the constraint is justified in Zuber and Keenan, forthcoming. In this paper we analyse various formal properties of functions constrained by weak conservativity and illustrate them occasionally by various examples.

2 Classical Conservativity

We will consider binary relations and functions over a universe E. The set $\{0, 1\}$ is the Boolean algebra of truth values. If A is a set then $\wp(A)$ is the power set of A that is the set of all subsets of A. Type $\langle 1 \rangle$ quantifiers are functions from $\wp(E)$ onto $\{0, 1\}$. They are denotation of NPs in the subject position. Type $\langle 1, 1 \rangle$ quantifiers are functions from $\wp(E)$ to type $\langle 1 \rangle$ quantifiers. They are denoted by unary determiners we are interested in. The set of all type $\langle 1, 1 \rangle$ functions that is the set of unrestricted type $\langle 1, 1 \rangle$ quantifiers will be denoted by $URDET$. This set forms an atomic Boolean algebra with Boolean operations defined pointwise.

Let F be an element of $URDET$. Then $\neg F$, the negation (or Boolean complement) of F and $F\neg$, the post-negation of F, are defined as follows:

Definition 2. *(i) Let F be an element of $URDET$. Then $\neg F$ is that element of $URDET$ for which $\neg F(X)(Y) = 1$ iff $F(X)(Y) = 0$.*
(ii) $F\neg$ is that element of $URDET$ for which $F\neg(X)(Y) = 1$ iff $F(X)(Y') = 1$, where Y' is the Boolean complement of Y.

Conservativity is a property of some functions belonging to $URDET$. In addition to definition D1 (classical) conservativity can be equivalently defined in two other ways. Keenan and Faltz (1985) indicate that conservativity can be equivalently defined as in Fact 1. Finally, using simple set-theoretical properties one can show that D1 and Fact 1 are equivalent to Fact 2 (Zuber 2005):

Fact 1. *F is conservative iff $F(X)(Y) = F(X)(Z)$ whenever $X \cap Y = X \cap Z$.*

Fact 2. *F is conservative iff $F(X)(Y) = F(X)(X' \cup Y)$, where X' is the Boolean complement of X.*

Facts 1 and 2 provide equivalent statements of classical conservativity often useful in establishing that one or another function is (classically) conservative.

As an illustration we show that the function $MOST$ given in (2), denoted by the determiner *most*, is conservative:

(2) $MOST(X)(Y) = 1$ iff $|X \cap Y| > |X \cap Y'|$.

We show that $MOST(X)(Y) = MOST(X)(X \cap Y)$. We have: $MOST(X)(X \cap Y) = 1$ iff $|X \cap X \cap Y| > |X \cap (X \cap Y)'|$ iff $|X \cap Y| > |X \cap (X' \cup Y')|$ iff $|X \cap Y| > |X \cap Y'|$.

The set of classically conservative functions forms a Boolean algebra, a subalgebra of $URDET$, which will be noted $CONS2$. For this reason we will also sometimes refer to members of $CONS2$, that is to classically conservative functions, as *cons2* functions.

The operation of Boolean complementation and of post-negation preserve cons2. That is we have:

Fact 3. *(i) $F \in CONS2$ iff $\neg F \in CONS2$, for any $F \in URDET$.*
(ii) $F \in CONS2$ iff $F\neg \in CONS2$, for any $F \in URDET$.

The algebra $CONS2$ has two important sub-algebras, the algebra INT of intersective functions, and the algebra $CO\text{-}INT$ of co-intersective functions. By definition (cf. Keenan 1993):

Definition 3. *$F \in INT$, iff for all properties X, Y, Z and W, if $X \cap Y = Z \cap W$ then $F(X)(Y) = F(Z)(W)$.*

Definition 4. *$F \in CO\text{-}INT$ iff for all properties X, Y, Z and W, if $X \cap Y' = Z \cap W'$ then $F(X)(Y) = F(Z)(W)$.*

One can check that numerals (considered as determiners) and the determiners *no* and *some* denote intersective functions. Similarly, the determiner *every* (but not *most*) and the determiner *every...but Dan*, as it occurs in (3), denote co-intersective, and thus cons2, functions. The function denoted by *every... but Dan* is given in (4):

(3) Every student but Dan is dancing.

(4) $F(X)(Y) = 1$ iff $X \cap Y' = \{d\}$, where d is the referent of *Dan*.

Both sets, INT and $CO\text{-}INT$ form atomic (and complete) Boolean algebras. Their atoms (and also their co-atoms, Boolean complements of atoms) are determined by a set (Keenan 1993): for any set P the function F_P such that $F_P(X)(Y) = 1$ iff $X \cap Y = P$ is an atom of INT and the function G_P such that $G_P(X)(Y) = 1$ iff $X \cap Y' = P$ is an atom of $CO\text{-}INT$. For instance exclusion determiners denote such atomic functions: *no...except Leo and Lea* denotes an atom of the algebra of intersective function determined by the set composed of two elements, Leo and Lea. More importantly, the quantifier ALL is an atom of the algebra $CO\text{-}INT$ determined by the empty set: $ALL(X)(Y) = 1$ iff $X \cap Y' = \emptyset$. Similarly, the quantifier NO is the atom of the algebra INT determined by the empty set: $NO(X)(Y) = 1$ iff $X \cap Y = \emptyset$. Consequently, the quantifier $SOME$ is the co-atom of INT determined by the empty set: $SOME(X)(Y) = 1$ iff $X \cap Y \neq \emptyset$.

It follows from D3 and D4 that elements of INT and of $CO\text{-}INT$ have in particular the following properties (Zuber 2005):

Fact 4. *The following three conditions are equivalent: (i)* $F \in INT$*, (ii)* $F(X)(Y) = F(X \cap Y)(X \cap Y)$*, (iii)* $F(X)(Y) = F(E)(X \cap Y)$

Fact 5. *The following four conditions are equivalent: (i)* $F \in CO\text{-}INT$*, (ii)* $F(X)(Y) = F(X - Y)(X' \cup Y)$*, (iii)* $F(X)(Y) = F(X \cap Y)(\emptyset)$*, (iv)* $F(X)(Y) = F(E)(X' \cup Y)$

Let us use Fact 5(iv) to show that the function denoted by the determiner *every... but Dan* and given in (4), is co-intersecttve: $F(E)(X' \cup Y) = 1$ iff $E \cap (X' \cup Y)' = \{d\}$ iff $E \cap (X \cap Y') = \{d\}$ iff $X \cap Y' = \{d\}$.

The operation of post-negation relates intersective and co-intersective functions in the following way:

Fact 6. $F \in INT$ *iff* $F\neg \in CO\text{-}INT$ *and* $F \in CO\text{-}INT$ *iff* $F\neg \in INT$*, for any* F *of type* $\langle 1, 1 \rangle$*.*

The Boolean algebras INT and $CO\text{-}INT$ are important because of the following theorem (Keenan 1993):

Theorem 1. *The Boolean closure of* $INT \cup CO\text{-}INT = CONS2$*.*

The theorem says that the full set of functions constructible from intersective or co-intersective by forming meets, joins and complements equals precisely the set of cons2 functions.

According to the Theorem 1, one can characterise classically conservative functions in terms of much smaller sets of functions. For indeed, given the definitions of atoms of INT and $CO\text{-}INT$ given above, if the domain $|E| = n$ then $|INT| = |CO\text{-}INT| = 2^m$, for $m = 2^n$. Hence, when $|E| = 2$ there are 16 intersective and 16 co-intersective functions whereas in this case, as we have seen, there are 512 conservative functions.

It follows from Theorem 1 in particular that there are conservative functions which are Boolean combinations of intersective and co-intersective functions but which are neither intersective nor co-intersective. As an example consider the denotation of the determiner *some but not all* given in (5):

(5) $F(X)(Y) = SOME(X)(Y) \wedge \neg ALL(X)(Y)$.

Clearly, F is a Boolean combination of an intersective and a co-intersective function. To show that F is neither intersective nor co-intersective consider the values of X and Y that satisfy the following conditions: (i) $X \neq \emptyset$, (ii) $X \neq Y$, and (ii) $Y \subseteq X$. When all these conditions hold then $F(X)(Y) = 1$. However, in this case $F(X \cap Y)(X \cap Y) = 0$ and thus, given Fact 4(ii), F is not intersective. Similarly, in this situation $F(X \cap Y')(\emptyset) = 0$ and thus, given Fact 5(iii), F is not co-intersective.

Consider now the determiner *all* interpreted with the existential import that is when it denotes the function ALL_{ei} given in (6a). This function can be equivalently represented as in (6b) and thus ALL_{ei} is a Boolean combination of the intersective function $SOME$ and the co-intersective function ALL:

(6) a. $ALL_{ei}(X)(Y) = 1$ iff $X \neq \emptyset$ and $X \subseteq Y$.
b. $ALL_{ei} = SOME \wedge ALL$.

The classically conservative function in (5) is a meet of two co-atoms (i.e. Boolean complements of atoms): $SOME$ is a Boolean complement of NO and $\neg ALL$ is a Boolean complement of ALL. In (6b) we have a meet of a co-atom of INT and of an atom of $CO\text{-}INT$.

An important class of classically conservative determiners which are neither intersective no co-intersective is given by *proportional* determiners:

Definition 5. *A type $\langle 1,1 \rangle$ function F is proportional iff $F(X)(Y) = F(W)(Z)$ whenever $|X \cap Y|/|X| = |W \cap Z|/|W|$, for $X \neq \emptyset$ and $W \neq \emptyset$.*

The quantifier $MOST$ is proportional and it is a Boolean combination of intersective and co-intersective functions as indicated in (7):

(7) $MOST = \bigcup_{B \in MAJ(A)} (F_B \wedge H_{A \cap B'})$, where $A, B \subseteq E$, $MAJ(X) = \{Y : Y \subseteq X \wedge 2 \times |Y| > |X|\}$, $F_B(X)(Y) = 1$ iff $X \cap Y = B$ and $H_{A \cap B'}(X)(Y) = 1$ iff $X \cap Y' = A \cap B'$.

Determiners such as *10% of...* and *at least half of...* also denote proportional type $\langle 1,1 \rangle$ functions.

3 Weakly Conservative Determiners

As indicated above recent research on the semantics of determiners shows that NLs have many determiners which are not classically conservative. There are various sub-classes of such determiners. Let me mention first two classes of possibly non -conservative determiners, which strictly speaking will not be discussed in this paper given their specificity.

The first class concerns *vague* determiners. Westerståhl (1985) observes that *many* (and *few*) in addition to generally accepted cardinal and proportional readings also have a *proportional reversed* reading which is not expressed by a classically conservative function. The cardinal reading of *many* $CMANY$ is defined as $CMANY(A)(B) = 1$ iff the set $A \cap B$ is, roughly speaking, large. The proportional reading of *many* corresponds to the function $PMANY(A)(B) = 1$ iff the set $A \cap B$ is large compared to the set A. Westerståhl (1985) considers that *many* taken in the reversed proportional reading corresponds to $RPMANY(A)(B)$ such that $RPMANY(A)(B) = 1$ iff the set $A \cap B$ is large compared to the set B. It is easy to see that $CMANY$ and $PMANY$ are classically conservative whereas $RPMANY$ is not classically conservative beacause $A \cap B$ can be large compared to B but $A \cap B$ is never large compared to itself. The function $RPMANY$ is cons1 because $A \cap A \cap B = A \cap B$.

The counter-example to classical conservativity based on *many* and *few* is special because it is not clear to what extent formal definition of cons2 should apply to them. The semantics for possible readings of *many* presented above has been challenged in Cohen 2001. In addition one cannot exclude that this vagueness may involve intensionality (cf. Bastiaanse 2014) and in principle conservativity is restricted to extensional determiners.

Another problematic case of classical conservativity is discussed in Yi (2016). He notices that (8a) and (8b) have different truth values in a situation in which among four boys surrounding Bo three are funny and these three are not surrounding Bo (these three are not enough to surround):

(8) a. Most of the boys surrounding Bo are funny.
    b. Most of the boys surrounding Bo are funny boys surrounding Bo.

Consequently, concludes Yi, the compound determiner *most of the* should be considered as denoting a function which is not classically conservative. We notice, without further discussion that this determiner has as arguments collective predicates which do not denote sets.

Let us see now some more "natural" counter-examples do conservativity. One of early such examples is the determiner *mostly* mentioned in Jensen 1987. The function denoted by this determiner is given in Zuber (2004): the function $MOSTLY$ defined in (9) as the inverse of $MOST$. This function is denoted by the determiner *mostly* as it occurs in (10) since (10) has plausibly the same truth conditions as (11):

(9) $MOSTLY(X)(Y) = MOST(Y)(X) = |X \cap Y| > |X' \cap Y|$

(10) Mostly monks are vegetarians.

(11) Most vegetarians are monks.

Examples in (12) and (13) show that *mostly* can be used as a determiner:

(12) Some teachers but mostly students danced.

(13) There are mostly freshmen in that course.

Conservativity defined in the previous section (cons2) can be said to be "conservativity with respect to the second argument" or "conservativity on the right" of a type $\langle 1, 1\rangle$ function. This is because conservativity thus defined permits a restriction of the second argument (the argument "on the right") of the function to its intersection with the first argument. For this reason we call the set of classically conservative functions $CONS2$.

In our discussion of weak conservativity, an essential role will be played by "conservativity with respect to the first argument" or "conservativity on the left" (cons1, for short). The set of functions conservative on the left will be noted $CONS1$. It is defined as follows:

Definition 6. *A type $\langle 1, 1\rangle$ function $F \in CONS1$ (or F is cons1) iff $F(X)(Y) = F(X \cap Y)(Y)$.*

For functions which are cons1 the following is true (cf. Zuber 2005):

Fact 7. *$F \in CONS1$ iff $F(X)(Z) = F(Y)(Z)$ whenever $X \cap Z = Y \cap Z$.*

Fact 8. *$F \in CONS1$ iff $F(X)(Z) = F(X \cup Z')(Z)$.*

There is an interesting relation between cons1 and cons2 functions: they are related by the relation of inversion of their arguments. Consider the bijection ^{i}, called *inversion*, from $URDET$ to $URDET$ defined as follows (Zuber 2005):

Definition 7. *Let $F \in URDET$. Then, by definition $F^i(X)(Y) = F(Y)(X)$.*

The following proposition indicates the relation between $CONS1$ and $CONS2$:

Proposition 1. *$(CONS1)^i = CONS2$ and $(CONS2)^i = CONS1$, where for any $K \subseteq URDET$, $K^i = \{F^i : F \in K\}$.*

Proof. We prove only the first equality. Suppose that $F \in CONS1$ and let $G = F^i$ and $X \cap Y = X \cap Z$ Then $G(X)(Y) = F(Y)(X)$ and $G(X)(Z) = F(Z)(X)$. This means, given Fact 7, that $F(Y)(X) = F(Z)(X)$ and thus $G(X)(Y) = G(X)(Z)$. Hence, given Fact 1, $G \in CONS2$ □

Given definition D3 of intersective functions and Proposition 1 one can see that if F is an intersective function then F^i is also intersective. Consequently, the algebra INT is also a sub-algebra of $CONS1$ and thus any intersective function is also cons1. In other words we have:

Proposition 2. *$CONS1 \cap CONS2 = INT$.*

Thus functions which are cons1 and cons2 are precisely intersective functions. This means that denotations of numerals (considered as determiners) and the quantifiers NO and $SOME$ are cons1 and also cons2.

Clearly $CONS1$ forms a Boolean algebra. Moreover we have:

Proposition 3. *$CONS1$ is an atomic Boolean algebra. Its atoms are functions $h_{A,B}$ for $A \subseteq B$ such that $h_{A,B}(X)(Y) = 1$ iff $X \cap Y = A$ and $Y = B$.*

The algebra $CONS1$ has also a sub-algebra of (classically) non-conservative functions called $CO\text{-}INT1$. By definition:

Definition 8. *$F \in CO\text{-}INT1$ iff if $X_1' \cap Y_1 = X_2' \cap Y_2$ then $F(X_1)(Y_1) = F(X_2)(Y_2)$*

Clearly $CO\text{-}INT1 = (CO\text{-}INT)^i$. Thus atoms of $CO\text{-}INT1$ are, for any $A \subseteq E$, functions F_A such that $F_A(X)(Y) = 1$ iff $X' \cap Y = A$. By analogy with Fact 5 we have for functions belonging to $CO\text{-}INT1$ the following property:

Fact 9. *$F \in CO\text{-}INT1$ iff $F(X)(Y) = F(\emptyset)(X' \cap Y)$.*

It is not surprising that algebra $CONS1$ has a similar characterisation to algebra $CONS2$:

Theorem 2. *The Boolean closure of $INT \cup CO\text{-}INT1$ equals $CONS1$.*

Proof. The proof of Theorem 2 uses the fact that any atom of $CONS1$ can be represented as a meet of an atom of INT and of an atom of $CO\text{-}INT1$. Indeed the following equality holds: $h_{A,B} = f_A \wedge g_{B-A}$, where $h_{A,B}$ is the atom of $CONS1$, for $A \subseteq B$, f_A is the atom of INT and G_{B-A} is the atom of $CO-INT1$, that is $g_{B-A}(X)(Y) = 1$ iff $B - A = Y \cap X'$. From this Theorem 2 follows since $CONS1$ is an atomic and complete Boolean algebra and any element of such an algebra is a join of some atoms of the algebra. □

A special class of cons1 functions (which are not cons2) is obtained from intersective functions by replacing their first argument by its complement. More precisely we have:

Proposition 4. *Let $G \in INT$. Then the function $F(X)(Y) = G(X')(Y)$ is cons1 but not cons2.*

Proof. Indeed, in this case, given Fact 3(iii), we have $F(X)(Y) = G(X')(Y) = G(E)(X' \cap Y)$ and $F(X \cap Y)(Y) = G(E)((X \cap Y)' \cap Y) = G(E)(X' \cap Y)$. Thus F is cons1. Similarly, $F(X)(X \cap Y) = G(X')(X \cap Y) = G(E)(\emptyset)$. Thus $F(X)(X \cap Y)$ is constant whereas $F(X)(Y)$ is not. Hence F is not cons2. □

In fact cons1 functions indicated in Proposition 4 are functions belonging to $CO\text{-}INT1$. Using Fact 4(iii) and Fact 9 we prove the following proposition:

Proposition 5. *Let $F, G \in URDET$ and $G(X)(Y) = F(X')(Y)$, for any $X, Y \subseteq E$. Then $F \in INT$ iff $G \in CO\text{-}INT1$.*

Proof. Suppose first that $F \in INT$. Then, by definition, $G(X)(Y) = F(X')(Y)$. Hence, given Fact 4(iii), $G(X)(Y) = F(E)(X' \cap Y)$ and thus $G(X)(Y) = G(\emptyset)(X' \cap Y)$ which means, given Fact 9, that $G \in CO\text{-}INT1$.

Suppose now that $G \in CO\text{-}INT1$. Then, by definition, $F(X)(Y) = G(X')(Y)$. Hence, given Fact 9, $F(X)(Y) = G(\emptyset)(X' \cap Y)$ and, thus, by definition $F(X)(Y) = F(E)(X \cap Y)$. It follows from this and from Fact 4(iii) that F is intersective. □

It follows from Proposition 5 that for any n the function $F(X)(Y) = n(X')(Y)$, where $n(X)(Y) = 1$ iff $|X \cap Y| \geq n$, is cons1 but not cons2. Similarly functions $NONNO(X)(Y) = NO(X')(Y)$ and $NONSOME(X)(Y) = SOME(X')(Y)$ are cons1 but not cons2.

Functions which are cons1 or cons2 will be called *weakly conservative*:

Definition 9. $WCONS = CONS1 \cup CONS2$.

The proposal defended in Zuber and Keenan, forthcoming, concerning NL determiners is the following *weak conservativity universal* **WCU**:

WCU: All unary subject determiners denote weakly conservative functions.

It does not follow from the **WCU** that all weakly conservative functions are expressible by some determiners in English, even over a finite domain. Very likely functions indicated in Proposition 5 are functions which are not expressible by NL determiners. This fact contrasts with the claim of Keenan and Stavi 1986 who show that over a finite domain E for each functions F which is cons2, one can construct a (sometimes very tedious) determiner that can be interpreted as F.

Weak conservativity remains a very strong constraint and most of type $\langle 1, 1\rangle$ functions, that is most of elements of $URDET$, are not weakly conservative. As shown in Zuber and Keenan, forthcoming, in a model with $|E| = 2$ there are, as noted in the introduction, 65.526 (unrestricted) type $\langle 1, 1\rangle$ functions of which $2 \times 512 - 2^4 = 1024 - 16$ are weakly conservative (and 512 of these functions are classically conservative).

Of course any classically conservative function is weakly conservative. Functions indicated in Proposition 4 are weakly conservative. Let us see now some examples of weakly conservative and denotable functions.

The function $MOSTLY$ is not conservative because $MOSTLY(X)(Y) = 1$ iff $|X \cap Y| > |X' \cap Y|$ and $MOSTLY(X)(X \cap Y) = 1$ iff $|X \cap Y| > 0$. It is, however, weakly conservative since $MOST$ is conservative.

The determiner *only* (understood with the existential import) is a well-known example of a determiner which denotes a classically non-conservative function $ONLY_{ei}$ given in (14):

(14) $ONLY_{ei}(X)(Y) = 1$ iff $Y \neq \emptyset \wedge (Y \subseteq X)$

Usually, one shows that this function is not classically conservative by pointing out that $F(X)(X \cap Y)$ is true for any $X, Y \subseteq E$, whereas $F(X)(Y)$ is not constant. Another way to see that $ONLY_{ei}$ is not cons2 is to notice that $ONLY_{ei}$ is the inverse of ALL_{ei} and $ALL_{ei} \notin INT$. This observation also indicates that $ONLY_{ei}$ is cons1 because ALL_{ei} is classically conservative.

As indicated above expressions *at most* and *at least* can also be used as determiners. In this case *at most* is interpreted by ATM given in (15) and *at least* (without the existential import) is interpreted by ATL given in (16):

(15) $ATM(X)(Y) = 1$ iff $(Y \subseteq X) \vee (X \cap Y = \emptyset)$

(16) $ATL(X)(Y) = 1$ iff $(Y \subseteq X) \vee (SOME(X)(Y) \wedge SOME(X')(Y))$

Notice that function ATM is an atom of $CO\text{-}INT1$: this is the function F_A, where $A = \emptyset$, such that $F(X)(Y) = 1$ iff $X' \cap Y = \emptyset$. Thus, given Proposition 3 and Fact 9, ATM is not classically conservative but is weakly conservative. Similarly, ATL is cons1.

The particle *also* used as a determiner and in this case it denoted the function $ALSO$ given in (17):

(17) $ALSO(X)(Y) = SOME(X)(Y) \wedge SOME(X')(Y)$

Function $ALSO$ is not classically conservative: when $X \cap Y \neq \emptyset$ and $X' \cap Y \neq \emptyset$ then $ALSO(X)(Y) = 1$ and $ALSO(X)(X \cap Y) = 0$. This function is, however, weakly conservative: $ALSO(X \cap Y)(Y) = 1$ iff $SOME(X)(Y) \wedge SOME((X \cap Y)')(Y) = 1$ iff $ALSO(X)(Y) = 1$.

The four classically non-conservative determiners discussed above, *mostly, at most, at least* and *also*, clearly also have non-determiner use. In particular they can be used in an adverbial position or as modifiers of numerals. What is important, however, is the fact that they also can be used as determiners, since the result of their application to common nouns is like that of an "ordinary" NP. In particular it can be Booleanly combined with other NPs in which classically conservative determiners occur.

Finally, as observed in Sauerland (2015) many proportional determiners expressing percentage and fractions have non-conservative interpretation, in particular when they occur in the direct object position and without the definite article. Consider (18a) and (18b) (cf. Zuber and Keenan, forthcoming):

(18) a. The company hired fifty percent of the woman (last year)
b. The company fired fifty percent woman (last year)

Though we need to define conservativity of determiners occurring in the object position, informally the determiner *fifty percent of the* behaves conservatively: it says that, roughly, half of the woman have the property of being hired by the company. Sentence (18b) says something different. It says that half of the people that the company hired were women and this interpretation is not classically conservative (see Zuber and Keenan, forthcoming for more details).

4 Formal Properties

We have already seen various properties of some weakly conservative functions. In this section we indicate some other, more general properties.

The function ATM discussed above can be used to show one of differences between the classical and the weak conservativity:

Proposition 6. *$WCONS$ is not closed with respect to post-negation.*

Proof. $ATM\neg$ is not cons2 because ATM is not cons2 and, as one can easily check, post-negation preserves conservativity. To show that $ATM\neg$ is not cons1 we have, given the definition of post-negation and Fact 9, $ATM\neg(X \cap Y)(Y) = ATM(X \cap Y)(Y') = ATM(\emptyset)(Y')$. This means, given (11), that $ATM\neg(X \cap Y)(Y)$ is always false. But $ATM\neg(X)(Y)$ is not constant. Hence $ATM\neg$ is not weakly conservative. $\square$

In fact it is the set $CONS1$ which is not closed under the post-negation. This set is closed under the right argument restriction in the following sense:

Definition 10. *Let h be a restrictive function on sets (that is $h(X) \leq X$, for any set X) and let $F \in URDET$. Then F^h, the right argument restriction of F by h, is that element of $URDET$ for which $F^h(X)(Y) = F(X)(h(Y))$.*

The following proposition is true:

Proposition 7. *$CONS1$ is closed under right argument restrictions.*

Proof. Let $F \in CONS1$ and h be a restrictive function. Then $F^h(X)(Y) = F(X)(h(Y)) = F(X \cap h(Y))(h(Y))$ and $F^h(X \cap Y)(Y) = F(X \cap Y)(h(Y)) = F(X \cap Y \cap h(Y))(h(Y))$. Since $Y \cap h(Y) = h(Y)$, we have that $F^h(X)(Y) = F^h(X \cap Y)(Y)$ which means that $F^h \in CONS1$. $\square$

Following Proposition 7 sentences (19) and (20) are logically equivalent:

(19) Only students were dancing in the park.

(20) Only students who were dancing in the park were dancing in the park.

Though weak conservativity is not preserved by post-negation and in spite of the fact that the set of weakly conservative functions does not form a Boolean algebra, the Boolean combination of weakly conservative functions equals the set of all type $\langle 1, 1\rangle$ functions since we have:

Theorem 3. *The Boolean closure of $CONS1 \cup CONS2 = URDET$*

Proof. Let $F_{A,B}$ be an atom of the algebra $URDET$. Thus $F_{A,B}(X)(Y) = 1$ iff $X = A$ and $Y = B$. Let $G_{P,Q}$ be an atom of the algebra of classically conservative (CONS2) functions. This means that $Q \subseteq P$ and $G_{P,Q}(X)(Y) = 1$ iff $X = P$ and $X \cap Y = Q$. Finally let H_P be an atom of non-classically co-intersective functions, of that is inverses of co-intersective functions. These non-classically co-intersective functions are functions H defined for instance as: $H \in CO{-}INT1$ iff if $X_1' \cap Y_1 = X_2' \cap Y_2$ then $H(X_1)(Y_1) = H(X_2)(Y_2)$. They are weakly conservative since they are CONS1. Since H_P is an atom of $CO - INT1$ this means that $H_P(X)(Y) = 1$ iff $X' \cap Y = P$. One shows now by set-theoretic calculation that $F_{A,B}(X)(Y) = G_{A,A\cap B}(X)(Y) \wedge H_{A'\cap B}(X)(Y)$. This means that any atom of a non-restricted type $\langle 1, 1\rangle$ function is a meet of atoms of weakly conservative functions. $\square$

A class of cons1 but not cons2 functions is obtained by taking the inverses of co-intersective functions. We have:

Proposition 8. *If F is non-trivial and $F \in CO\text{-}INT$ then $F^i \in (CONS1 \cap CONS2')$.*

Proof. Indeed, F^i is cons1 because it is obtained by the inversion of a cons2 function. F^i is not cons2 because, given Fact 4(iii), $F^i(X)(Y) = F(Y)(X) = F(Y \cap X')(\emptyset)$ and $F^i(X)(X \cap Y) = F(X \cap Y)(X) = F(X \cap Y \cap X')(\emptyset) = F(\emptyset)(\emptyset)$. □

A class of examples which illustrate Proposition 8 is discussed in Zuber (2004). To construct them, one uses the fact that there are exceptive determiners which denote atoms of $CO\text{-}INT$, corresponding systematically to some (complex) inverted determiners, which denote in the algebra $CO\text{-}INT1$ and thus in $CONS1$. Consider:

(21) a. Every dancer except Leo is a student.
 b. Apart from Leo only students are dancing.

These two sentences have very likely the same truth-conditions (though probably not the same presuppositions). The determiner *every..., except Leo* in (21a) and the determiner *apart from Leo only...* in (21b) denote type $\langle 1, 1\rangle$ functions which are inverses of each other. Since the determiner *every..., except Leo* denotes the atom $F_{\{l\}}$ of the algebra $CO\text{-}INT$ such that such that $F_{\{l\}}(X)(Y) = 1$ iff $X \cap Y' = \{l\}$, where l =Leo, given Proposition 8, the inverse function $F^i_{\{l\}}$ is not cons2 but is cons1.

Observe that if we admit that determiners of the form *apart from CN, only...* are acceptable in English that is if we accept that the NP *apart from two teachers only students* is obtained by the application of the determiner *apart from two, only...* to the CN *students* then there is an infinite number of determiners in English which denote weakly conservative functions which are not classically conservative. Indeed, the function in (22) interpreting such determiners is weakly conservative but not classically conservative for any $n \neq 0$:

(22) $AP\text{-}ONLY_{A,n}(X)(Y) = 1$ iff $|A \cap Y| = n \wedge |Y| \geq 2 \wedge Y \subseteq (A \cup X)$

It is easy to give examples of functions which are not weakly conservative and which are very likely not denotable by natural language determiners. One of the simplest examples of a function which is neither cons1 nor cons2 is the function EQ given in (23):

(23) $EQ(X)(Y) = 1$ iff $X = Y$

Observe that when $X \neq Y$ and $X \subseteq Y$ then $EQ(X)(Y) = 0$ and $EQ(X)(X \cap Y) = 1$. This means that EQ is not classically conservative. Similarly, when $X \neq Y$ and $Y \subseteq X$ then $EQ(X)(Y) = 0$ and $EQ(X \cap Y)(Y) = 1$ and thus EQ is not cons1.

Similarly, the three functions given in (24) are neither cons1 nor cons2:

(24) a. $U_A(X)(Y) = 1$ iff $X \cup Y = A$.
 b. $F_{m,n}(X)(Y) = 1$ iff $|X| = m$ and $|Y| = n$.
 c. $ORINCL(X)(Y) = 1$ iff $X \neq \emptyset \wedge Y \neq \emptyset \wedge (X \subseteq Y \vee Y \subseteq X)$.

The above examples show incidentally that the meet and the join of two weakly conservative functions do not need to be weakly conservative. This means that weakly conservative functions do not form a Boolean algebra as do classically conservative functions.

5 Conclusions

In this paper we have been interested in some formal properties of functions denoted by determiners found in NLs. As is well-known, such functions are mainly classically conservative and, occasionally, non-conservative. Consequently it is necessary to distinguish two sub-classes of non-consrvative functions: on the one hand those which are cons1 or conservative on the left or with respect to the first argument and, on the other hand, those which are neither conservative with respect to the first argument nor conservative with respect to the second argument. This distinctions leads to a weakening of the classical constraint of conservativity concerning the class of natural language determiners. Such a weakened constraint says that all (unary) natural language determiners, are, roughly speaking, weakly conservative in the sense that they are conservative either with respect to their first argument (they are cons1) or with respect to the second argument (they are cons2). There do not seem to exist in NLs attested unary determiners which denote functions which are neither conservative on the left nor conservative on the right. In addition it has been suggested that there exists an infinite number of determiners, related to exceptive determiners, which are not classically conservative but are weakly conservative. We have also shown that weak conservativity, in contradistinction to classical conservativity, is not preserved by post-negation. Furthermore, weakly conservative functions can generate by (infinite) Boolean operations the set of all type $\langle 1, 1\rangle$ functions, that is all members of $URDET$.

More importantly, weak conservativity is, on the one hand, a proper, small and natural extension of classical conservativity and, on the other hand, it still remains a very strong constraint on type $\langle 1, 1\rangle$ functions since, as it has been shown, most of type $\langle 1, 1\rangle$ functions are not weakly conservative.

References

Barwise, J., Cooper, R.: Generalised quantifiers and natural language. Linguist. Philos. **4**, 159–219 (1981)

Bastiaanse, H.A.: Conservativity reclaimed. J. Philos. Log. **43**(5), 883–901 (2014)

Cohen, A.: Relative readings of many, often, and generics. Nat. Lang. Semant. **9**, 41–67 (2001)

von Fintel, K., Keenan, E.L.: Determiners, conservativity and witnesses. J. Semant. **35**, 207–217 (2018)

Johnsen, L.G.: There-sentences and generalized quantifiers. In: Gärdenfors, P. (ed.) Generalized Quantifiers, pp. 93–109. D. Reidel (1987)

Keenan, E.L.: Natural language, sortal reducibility and generalised quantifiers. J. Symb. Log. **58**, 314–325 (1993)

Keenan, E.L., Faltz, L.M.: Boolean Semantics for Natural Language. D. Reidel Publishing Company, Dordrecht (1985)
Keenan, E.L., Stavi, J.: A semantic characterisation of natural language deterniners. Linguist. Philos. **9**, 253–326 (1986)
Keenan, E.L., Westerståhl, D.: Generalized quantifiers in linguistics and logic. In: van Benthem, J., ter Meulen, A. (eds.) Handbook of Logic and Language, Elsevier (1997)
Sauerland, U.: Surface Non-conservativity in German. In: Piñón, C. (ed.) Empirical Issues in Syntax and Semantics, vol. 10, pp. 125–142 (2015)
Thijsse, E.: Counting quantifiers. In: van Benthem, J., ter Meulen, A. (ed.) Generalised Quantifiers in Natural Language, pp. 127–146. Foris Publications (1984)
Westerståhl, D.: Logical constants in quantifier languages. Linguist. Philos. **8**, 387–413 (1985)
Yi, B.: Quantifiers, determiners and plural constructions. In: Carrara, M., et al. (eds.) Plurality, and Unity: Philosophy, Logic and Semantics, pp. 121–170. Oxford University Press (2016)
Zuber, R.: A class of non-conservative determiners in Polish. Linguisticae Investigationes **27**, 147–165 (2004)
Zuber, R.: More algebras for determiners. In: Blache, P., Stabler, E., Busquets, J., Moot, R. (eds.) LACL 2005. LNCS (LNAI), vol. 3492, pp. 347–362. Springer, Heidelberg (2005). https://doi.org/10.1007/11422532_23
Zuber, R., Keenan, E.L.: J. Semant. (forthcoming)

Correction to: Algebraic and Topological Semantics for Inquisitive Logic via Choice-Free Duality

Nick Bezhanishvili, Gianluca Grilletti, and Wesley H. Holliday

Correction to: Chapter "Algebraic and Topological Semantics for Inquisitive Logic via Choice-Free Duality" in: R. Iemhoff et al. (Eds.): *Logic, Language, Information, and Computation*, LNCS 11541, https://doi.org/10.1007/978-3-662-59533-6_3

In the original version of the chapter titled "Algebraic and Topological Semantics for Inquisitive Logic via Choice-Free Duality", the acknowledgement was missing. It has been added.

The updated version of this chapter can be found at
https://doi.org/10.1007/978-3-662-59533-6_3

R. Iemhoff et al. (Eds.): WoLLIC 2019, LNCS 11541, p. C1, 2019.
https://doi.org/10.1007/978-3-662-59533-6_41

Author Index