

Statistical Matching Theory



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Abstract In this paper, we survey some recent developments on statistical properties of matchings of very large and infinite graphs. We discuss extremal graph theoretic results like Schrijver's theorem on the number of perfect matchings of regular bipartite graphs and its variants from the point of view of graph limit theory. We also study the number of matchings of finite and infinite vertex-transitive graphs.

Keywords Matchings · Permanent · Graph limits · Lattices · Bethe approximation

1 Introduction

In this paper, we survey some recent developments on statistical properties of matchings of very large and infinite graphs.

We will focus on two topics: *extremal graph theory* and *vertex-transitive bipartite graphs*. Both topics are intimately related to graph limit theory. In the first case, when we consider extremal graph theoretical problems, it turns out that in certain extremal problems concerning matchings of d -regular bipartite graphs, the extremal graph is not a finite graph, but the infinite d -regular tree. The proper understanding of this phenomenon leads not only to new proofs of classical theorems, but also to new results such as the Lower Matching Conjecture and other new theorems. In the second case, when we study matchings of finite vertex-transitive bipartite graphs, the direction is, in some sense, exactly the opposite: we would like to understand the matchings of infinite lattices through finite graphs. These finite graphs exhibit certain properties that can be utilized to study their matchings. Then these new observations transfer to the original lattices.

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Extremal graph theory. To give an example for the discussed problems we offer the following theorem of A. Schrijver. This theorem asserts that if G is a d -regular bipartite graph on $v(G)$ vertices, and $\text{pm}(G)$ denotes the number of perfect matchings, then

$$\text{pm}(G)^{1/v(G)} \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^{1/2}.$$

It is a natural question whether we can improve on the constant on the right hand side. The answer is no. Then it is natural to ask whether there is a finite graph G for which

$$\text{pm}(G)^{1/v(G)} = \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^{1/2}.$$

The answer is again no! The two negative answers together mean that

$$\inf_G \text{pm}(G)^{1/v(G)} = \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^{1/2},$$

where the infimum is taken over all d -regular bipartite graphs, but this infimum is never achieved by a finite graph. Can we still use classical extremal graph theoretic arguments to prove Schrijver's theorem? The answer is yes, and this is what Sect. 4 is about. On the other hand, there will be a little twist in the argument, and this is where graph limit theory comes into the picture. In extremal graph theory it is a natural idea to find a graph transformation φ such that for a given graph parameter $p(\cdot)$ we have

$$p(G) \leq p(\varphi(G)),$$

and the studied class of graphs is closed under φ . An example for this strategy is Zykov's symmetrization which does not decrease the number of edges and the size of the largest clique, and so it provides a powerful tool to prove Turán's theorem, and as it turns out, many other theorems where we expect the Turán-graph to be extremal. In general, we apply this transformation as long as we can, and then we solve an optimization problem on a much restricted class of graphs. In case of Turán's theorem, this restricted class is the class of complete multipartite graphs. This is the point where we will deviate from this strategy.

We will find a graph transformation φ such that for the graph parameter $p(G) = \text{pm}(G)^{1/v(G)}$ we have $p(G) \geq p(\varphi(G))$. In general, we will be able to apply φ in many different ways, so $\varphi(G)$ refers to one of the possible applications of the graph transformation φ to G . As a next step we prepare a graph sequence G_i such that $G_0 = G$, $G_{i+1} = \varphi(G_i)$, and consequently we have

$$p(G_0) \geq p(G_1) \geq p(G_2) \geq p(G_3) \geq \dots$$

The point is that the sequence (G_i) will not stabilize as in the proof of Turán’s theorem using Zykov’s symmetrization, but it will converge in the sense of Benjamini–Schramm. We will explain this convergence in Sect. 3. This convergence enables us to extend the universe of finite graphs with some new elements which we will call *random unimodular graphs*. We will define $p(\cdot)$ for these new elements as well. It turns out that in this extended topological space there will be a minimizer of the parameter $p(G)$, namely the infinite d –regular tree \mathbb{T}_d . Section 2 contains a brief survey on related results and in Sect. 4 we give an almost complete proof of Schrijver’s theorem along these lines.

Vertex-transitive bipartite graphs. To motivate the other main topic of this survey let us consider the following classical result of Kasteleyn [34] and independently Fisher and Temperley [50]. Let $Z_{m,n}$ be the number of perfect matchings of the $m \times n$ grid. Then

$$Z_{m,n} = \left(\prod_{j=1}^m \prod_{k=1}^n \left(4 \cos^2 \left(\frac{\pi j}{m+1} \right) + 4 \cos^2 \left(\frac{\pi k}{n+1} \right) \right) \right)^{1/4}.$$

This leads to the limit formula

$$\lim_{\substack{m,n \rightarrow \infty \\ 2 \mid mn}} \frac{1}{mn} \log Z_{m,n} = \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \log(4 \cos^2(x) + 4 \cos^2(y)) \, dx \, dy.$$

It is intuitively clear that the grids converge to the lattice \mathbb{Z}^2 . To make this statement precise once again we need the concept of Benjamini–Schramm convergence (see Sect. 3). Benjamini–Schramm convergence primarily grasps the local structure of a graph. It turns out that perfect matchings are especially fragile: a small change in the graph can lead to a dramatic change in the number of perfect matchings even if we restrict our attention to graphs with even number of vertices, or even if we only consider bipartite regular graphs. Fortunately, vertex-transitive bipartite graphs behave well in this respect, and so we can build a graph limit theory using them. More details can be found in Sect. 5.

This paper is organized as follows: in the next section we survey extremal graph theoretic results concerning matchings of finite (regular) graphs. In the third section we give the definition of Benjamini–Schramm convergence together with some examples. In the fourth section we give a sample proof of an extremal graph theoretic result on matchings that utilizes graph limit theory. In the fifth section we will study matchings of vertex-transitive bipartite graphs. In the final section we mention some results about matchings of dense graphs that are naturally connected to our discussion.

Basic notations: Throughout the paper, G denotes a graph, and $v(G)$ denotes the number of vertices of G . Recall that a matching of size k is set of k edges covering $2k$ vertices together. In other words, the edges e_1, \dots, e_k form a matching of size k , or shortly a k -matching if e_i and e_j have disjoint set of endpoints for any $1 \leq i, j \leq k$.

The number of matchings of size k will be denoted by $m_k(G)$. The size of the largest matching is denoted by $\nu(G)$. A matching is called perfect if it covers all vertices, that is, it has size $\nu(G)/2$. The number of perfect matchings will be denoted by $\text{pm}(G)$.

2 Extremal Graph Theory

In this section we will consider lower and upper bounds for the number of (perfect) matchings of bipartite graphs. Recall that $m_k(G)$ denotes the number of matchings of size k , and $\text{pm}(G)$ denotes the number of perfect matchings. When G is a bipartite graph with classes of size n , then the problem of counting the number of perfect matchings of G is equivalent to computing the permanent of a 0 – 1 matrix of size n by n . Recall that the permanent of a matrix A is defined as follows:

$$\text{per}(A) = \sum_{\pi \in S_n} a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}.$$

Let us suppose for a moment that all $a_{ij} \in \{0, 1\}$, and define a graph G on the vertex set $R \cup C$, where $R = \{r_1, r_2, \dots, r_n\}$ and $C = \{c_1, c_2, \dots, c_n\}$ correspond to the rows and columns of the matrix, respectively. If $a_{ij} = 1$, then put an edge between the vertices r_i and c_j . Now it is clear that $\text{per}(A) = \text{pm}(G)$, the number of perfect matchings of G .

A well-known result concerning permanents of 0 – 1 matrices is due to L. M. Brégman [8].

Theorem 2.1 (L. M. Brégman [8]) *Let A be a 0 – 1 matrix of size $n \times n$, and let r_i denote the number of 1's in the i -th row. Then*

$$\text{per}(A) \leq \prod_{i=1}^n (r_i!)^{1/r_i}.$$

Since $\text{pm}(K_{d,d}) = d!$, Theorem 2.1 immediately implies the following result about d -regular bipartite graphs.

Theorem 2.2 *Let $\text{pm}(G)$ denote the number of perfect matchings. Then for a d -regular (bipartite) graph we have*

$$\text{pm}(G)^{1/\nu(G)} \leq \text{pm}(K_{d,d})^{1/\nu(K_{d,d})}.$$

A priori Brégman's theorem implies the above result only for bipartite graphs but using the observation

$$\text{pm}(G)^2 \leq \text{pm}(G \times K_2) \tag{1}$$

one can deduce the general case from the bipartite case. Here $G \times K_2$ is the graph with vertex set $V(G) \times \{0, 1\}$, in which (u, i) and (v, j) are adjacent if $i \neq j$ and $(u, v) \in E(G)$. This is clearly a bipartite graph. Observation 1 was rediscovered several times, see, for instance, [4]. A generalization of this observation will be proved in Theorem 4.3.

Let us mention that one can prove an analogue of Brégman’s result for the number of all matchings, or even for weighted sums of matchings. Let

$$M(G, \lambda) = \sum_{k=0}^{\lfloor v(G)/2 \rfloor} m_k(G) \lambda^k.$$

It is the matching generating function of the graph G . In statistical physics it is known as the partition function of the monomer-dimer model.

Theorem 2.3 (E. Davies, M. Jenssen, W. Perkins, B. Roberts [16]) *For a d -regular graph G and $\lambda > 0$ we have*

$$M(G, \lambda)^{1/v(G)} \leq M(K_{d,d}, \lambda)^{1/v(K_{d,d})}.$$

For $\lambda = 1$ this result simplifies to the number of all matchings, and as $\lambda \rightarrow \infty$ it also implies Theorem 2.2.

Concerning lower bounds for perfect matchings of regular graphs, M. Voorhoeve ($d = 3$) and A. Schrijver (general d) proved the following result.

Theorem 2.4 (A. Schrijver [45], for $d = 3$ M. Voorhoeve [52]) *Let G be a d -regular bipartite graph on $v(G) = 2n$ vertices, and let $\text{pm}(G)$ denote the number of perfect matchings of G . Then*

$$\text{pm}(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^n.$$

In other words, for every d -regular bipartite graph G we have

$$\frac{\ln \text{pm}(G)}{v(G)} \geq \frac{1}{2} \ln \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right).$$

There are various different proofs for Schrijver’s theorem and its generalizations in the literature. Schrijver’s original proof is elementary but tricky. L. Gurvits [27] gave another proof using stable polynomials. For an account of this proof, see also [37]. This is a beautiful proof, probably one from The Book. D. Straszak and N. Vishnoi [49] found a generalization for certain graphical models also using stable polynomials. Another proof, revealing the extremal graph, was given in [13] and is sketched in this survey. M. Lelarge [38] gave a variant of this proof for another generalization of Schrijver’s theorem.

A result of Wilf [53] (see also [6, 46]) shows that the constant $\frac{1}{2} \ln \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)$ is the best possible. This can be shown by computing the expected value of $\text{pm}(G)$ for d -regular random bipartite graphs. There was no explicit construction for regular bipartite graphs with small number of perfect matchings for a long time. Very recently it turned out that if a d -regular bipartite graph has a small number of short cycles, then it has asymptotically the same number of perfect matchings as a random d -regular graph, the more precise formulation is the following.

Theorem 2.5 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1]) *Let (G_i) be a sequence of d -regular graphs such that $g(G_i) \rightarrow \infty$, where g denotes the girth, that is, the length of the shortest cycle.*

(a) *For the number of perfect matchings $\text{pm}(G_i)$, we have*

$$\limsup_{i \rightarrow \infty} \frac{\ln \text{pm}(G_i)}{v(G_i)} \leq \frac{1}{2} \ln \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right).$$

(b) *If, in addition, the graphs G_i are bipartite, then*

$$\lim_{i \rightarrow \infty} \frac{\ln \text{pm}(G_i)}{v(G_i)} = \frac{1}{2} \ln \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right).$$

We note that it is enough to assume that (G_i) converges to \mathbb{T}_d in Benjamini-Schramm sense. See Sect. 3 for more details.

In [28] L. Gurvits derived a version of Schrijver’s theorem for permanents using Schrijver’s theorem itself.

Theorem 2.6 (L. Gurvits [28]) *Let A be an n by n non-negative matrix. Then*

$$\text{per}(A) \geq \sup_{B \in \text{DS}_n} \exp \left(\sum_{i,j} B_{ij} \ln \frac{A_{ij}}{B_{ij}} + \sum_{i,j} (1 - B_{ij}) \ln(1 - B_{ij}) \right),$$

where DS_n is the set of doubly stochastic matrices of size n by n .

In the same paper [28] L. Gurvits also extended Schrijver’s theorem from perfect matchings to matchings of arbitrary size.

Theorem 2.7 (L. Gurvits [28]) *Let G be an arbitrary d -regular bipartite graph on $v(G) = 2n$ vertices. Let $m_k(G)$ denote the number of k -matchings. Set $p = \frac{k}{n}$. Then*

$$\frac{\ln m_k(G)}{v(G)} \geq \frac{1}{2} \left(p \ln \left(\frac{d}{p} \right) + (d - p) \ln \left(1 - \frac{p}{d} \right) - 2(1 - p) \ln(1 - p) \right) + o_{v(G)}(1).$$

Gurvits’s theorem was previously conjectured by Friedland, Krop and Markström [22] under the name Asymptotic Lower Matching Conjecture. They also had a more precise form of this conjecture known as Lower Matching Conjecture.

Conjecture 2.8 (Lower Matching Conjecture [22]) *Let G be a d -regular bipartite graph on $v(G) = 2n$ vertices, and let $m_k(G)$ denote the number of matchings of size k , then*

$$m_k(G) \geq \binom{n}{k}^2 \left(\frac{d-p}{d}\right)^{n(d-p)} (dp)^{np},$$

where $p = \frac{k}{n}$.

To see the connection between the Lower Matching Conjecture and its asymptotic version, it is worth introducing two notations. The first one is the function appearing in Gurvits's theorem:

$$\mathbb{G}_d(p) = \frac{1}{2} \left(p \ln \left(\frac{d}{p} \right) + (d-p) \ln \left(1 - \frac{p}{d} \right) - 2(1-p) \ln(1-p) \right).$$

Furthermore, let $p = \frac{k}{n}$, and let $p_\mu = \binom{n}{k} p^k (1-p)^{n-k}$. This is the probability that a random variable X with distribution Binomial(n, p) takes its mean value. It turns out that

$$\binom{n}{k}^2 \left(\frac{d-p}{d}\right)^{n(d-p)} (dp)^{np} = p_\mu^2 \cdot \exp(2n\mathbb{G}_d(p)).$$

Using these notations Gurvits's theorem says that $m_k(G) \geq \exp(2n(\mathbb{G}_d(p) + o_n(1)))$, while the Lower Matching Conjecture claims that $m_k(G) \geq p_\mu^2 \cdot \exp(2n\mathbb{G}_d(p))$. It turns out that the truth is even more beautiful.

Theorem 2.9 ([13]) *Let G be a d -regular bipartite graph on $v(G) = 2n$ vertices, and let $m_k(G)$ denote the number of matchings of size k , then*

$$m_k(G) \geq p_\mu \cdot \exp(2n\mathbb{G}_d(p)),$$

where $p = \frac{k}{n}$.

Furthermore, there exists a d -regular bipartite graph G on $2n$ vertices such that

$$m_k(G) \leq \sqrt{\frac{1-p/d}{1-p}} \cdot p_\mu \cdot \exp(2n\mathbb{G}_d(p)).$$

Next we introduce the function $\lambda_G(p)$, called the entropy function in statistical physics, which is closely related to counting matchings. The simplest way to define it is as follows: let rG be the disjoint union of r copies of G and let the sequence k_r be chosen in such a way that

$$\lim_{r \rightarrow \infty} \frac{2k_r}{v(rG)} = p;$$

then

$$\lambda_G(p) := \lim_{r \rightarrow \infty} \frac{\ln m_{k_r}(rG)}{v(rG)}.$$

If G contains a perfect matching, then the limit indeed exists whenever $p < 1$, and for $p = 1$ one can define it as

$$\lambda_G(1) := \frac{\ln \text{pm}(G)}{v(G)}.$$

The above definition for $\lambda_G(p)$ is not the original definition and is really hard to work with, but at least it is very easy to explain. Intuitively, it counts a normalized number of matchings covering p fraction of the vertices, but it has the advantage that it is meaningful even if p is irrational, thereby providing a continuous function. We also mention that it is really easy to extend $\lambda_G(p)$ to random rooted graphs \mathcal{G} . One can prove that for $p = \frac{k}{n}$ the following inequalities hold true:

$$p_\mu \cdot \exp(2n\lambda_G(p)) \leq m_k(G) \leq \exp(2n\lambda_G(p)).$$

This means that

$$\lambda_G(p) \approx \frac{\ln m_k(G)}{v(G)}.$$

Surprisingly, it turns out that Gurvits’s theorem is equivalent to $\lambda_G(p) \geq \mathbb{G}_d(p)$ for all $0 \leq p \leq 1$ provided that G is a d -regular bipartite graph, see [13] for details. So practically Gurvits’s theorem implies its more precise form.

At this moment, it may be mysterious why and how the function $\mathbb{G}_d(p)$ appears in these theorems. The mystery vanishes as soon as we realize that the function $\mathbb{G}_d(p)$ is nothing else but the entropy function of the infinite d -regular tree:

$$\lambda_{\mathbb{T}_d}(p) = \mathbb{G}_d(p).$$

We offer three more results in the spirit of Theorem 2.9. In Sect. 4 we will give a detailed sketch of the proof of Theorem 2.10. One can prove Theorems 2.11 and 2.12 with similar tools.

Theorem 2.10 ([13]) *Let G be a d -regular bipartite graph on $v(G) = 2n$ vertices, and let $m_k(G)$ denote the number of matchings of size k . For $\lambda \geq 0$ let $M(G, \lambda) = \sum_{k=0}^n m_k(G)\lambda^k$. Then*

$$\frac{1}{v(G)} \ln M(G, \lambda) \geq \frac{1}{2} \ln S_d(\lambda),$$

where

$$S_d(\lambda) = \frac{1}{\eta_\lambda^2} \left(\frac{d-1}{d-\eta_\lambda} \right)^{d-2} \quad \text{and} \quad \eta_\lambda = \frac{\sqrt{1+4(d-1)\lambda}-1}{2(d-1)\lambda}.$$

Alternatively, for $0 \leq p \leq 1$ we have

$$\sum_{k=0}^n m_k(G) \left(\frac{p}{d} \left(1 - \frac{p}{d} \right) \right)^k (1-p)^{2(n-k)} \geq \left(1 - \frac{p}{d} \right)^{nd}.$$

Note that this theorem directly reduces to Schrijver’s theorem if $p = 1$, and strongly suggests that there might be some probabilistic proofs for some results appearing in this survey.

The following theorem shows that one can extend a few theorems from d -regular bipartite graphs to arbitrary bipartite graphs or even to the permanent of a non-negative matrix. Before we state this result we need some notations.

The matching polytope $MP(G)$ of a graph G is defined as the convex hull of incidence vectors of matchings in G . We define the fractional matching polytope as

$$FMP(G) = \left\{ \underline{x} \in \mathbb{R}^{E(G)} \mid x_e \geq 0 \ \forall e \in E(G), \sum_{e:v \in e} x_e \leq 1 \ \forall v \in V(G) \right\}.$$

It is known that $MP(G) = FMP(G)$ if and only if G is bipartite. Similarly, we can define by $MP_k(G)$ the convex hull of incidence vectors of matchings in G of size k , and

$$FMP_k(G) = \left\{ \underline{x} \in FMP(G) \mid \sum_{e \in E(G)} x_e = k \right\}.$$

Again, if G is bipartite, then $MP_k(G) = FMP_k(G)$. Finally, let $\nu(G)$ be the size of the largest matching in G .

Theorem 2.11 (M. Lelarge [38]) *For a vector $\underline{x} \in [0, 1]^E$ let*

$$\mathbb{F}_G(\underline{x}) = \sum_{e \in E} (-x_e \ln x_e + (1 - x_e) \ln(1 - x_e)) - \sum_{v \in V(G)} \left(1 - \sum_{e:v \in e} x_e \right) \ln \left(1 - \sum_{e:v \in e} x_e \right).$$

Then for any bipartite graph G and $\lambda \geq 0$ we have

$$\ln M(G, \lambda) \geq \max_{\underline{x} \in MP(G)} \left\{ \left(\sum_{e \in E(G)} x_e \right) \ln \lambda + \mathbb{F}_G(\underline{x}) \right\}.$$

Furthermore, for all $k < \nu(G)$ we have

$$m_k(G) \geq b_{\nu(G),k} \left(\frac{k}{\nu(G)} \right) \exp \left(\max_{\underline{x} \in MP_k(G)} \mathbb{F}_G(\underline{x}) \right),$$

where $b_{n,k}(p) = \binom{n}{k} p^k (1 - p)^{n-k}$, that is, the probability that a binomial random variable $\text{Bin}(n, p)$ takes the value k .

This theorem has a counterpart for permanents. For a non-negative matrix A of size n by n let

$$\text{allper}(A) = \sum_{|I|=|J|} \text{per}(A_{I,J}),$$

where $A_{I,J}$ is the submatrix of A induced by the rows I and J .

Theorem 2.12 (M. Lelarge [38]) *Let A be a non-negative matrix of size n by n . Let SDS_n be the set of non-negative matrices such that in each row and each column the sum of the elements is at most 1. For $B \in \text{SDS}_n$ let*

$$B_{i,0} = 1 - \sum_{j=1}^n B_{i,j} \quad \text{and} \quad B_{0,j} = 1 - \sum_{i=1}^n B_{i,j},$$

and

$$\begin{aligned} \mathbb{F}_A(B) &= \sum_{1 \leq i, j \leq n} B_{ij} \ln \frac{A_{ij}}{B_{ij}^2} + \sum_{i=1}^n B_{i0} \ln \frac{1}{B_{i0}} + \sum_{j=1}^n B_{0j} \ln \frac{1}{B_{0j}} \\ &+ \sum_{1 \leq i, j \leq n} (B_{ij} \ln B_{ij} + (1 - B_{ij}) \ln(1 - B_{ij})). \end{aligned}$$

Then for any non-negative matrix A we have

$$\text{allper}(A) \geq \sup_{B \in \text{SDS}_n} \exp(\mathbb{F}_A(B))$$

Remark 2.13 Theorems 2.6 and 2.12 are special cases of a more general phenomenon. Permanents, the number of matchings or the number of homomorphisms of graphs can be expressed as partition functions of so-called graphical models. To a graphical model one can associate two objects: the partition function $Z(G)$ and the Bethe partition function $Z_B(G)$. There is no general inequality between them, but in certain cases $Z(G) \geq Z_B(G)$. This happens for attractive graphical models [44], and for certain bipartite graphical models [49]. In Theorems 2.6 and 2.12 $Z(G)$ appears on the left hand side, and $Z_B(G)$ on the right hand side. For more details on this subject see [44, 49].

3 Graph Limits and Examples

In the previous section we have seen that $\mathbb{G}_d(p) = \lambda_{\mathbb{T}_d}(p)$ gives a lower bound on $\lambda_G(p)$ if G is a d -regular bipartite graph. This establishes a claim to handle infinite graphs and connect them to the theory of finite graphs. This is exactly the goal of this section. In what follows we introduce the concept of Benjamini–Schramm convergence with some examples. Before we define this concept one more remark is in order: in this paper we will always assume that there is some Δ such that the largest degree of any graph G_i in a given sequence of graphs is at most Δ . In such a case we say that the graph sequence (G_i) is a *bounded-degree* graph sequence. This assumption simplifies our task significantly.

Definition 3.1 Let L be a probability distribution on (infinite) connected rooted graphs; we will call L a *random rooted graph*. For a finite connected rooted graph α and a positive integer r , let $\mathbb{P}(L, \alpha, r)$ be the probability that the r -ball centered at the root vertex is isomorphic to α , where the root is chosen from the distribution L .

For a finite graph G , a finite connected rooted graph α and a positive integer r , let $\mathbb{P}(G, \alpha, r)$ be the probability that the r -ball centered at a uniform random vertex of G is isomorphic to α .

We say that a bounded-degree graph sequence (G_i) is *Benjamini–Schramm convergent* if for all finite rooted graphs α and $r > 0$, the probabilities $\mathbb{P}(G_i, \alpha, r)$ converge. Furthermore, we say that (G_i) *Benjamini–Schramm converges to L* , if for all positive integers r and finite rooted graphs α , $\mathbb{P}(G_i, \alpha, r) \rightarrow \mathbb{P}(L, \alpha, r)$.

The Benjamini–Schramm convergence is also called *local convergence* as it primarily grasps the local structure of the graphs (G_i) .

Note that if (G_i) is a sequence of d -regular graphs such that the girth $g(G_i)$ tends to infinity, then it is Benjamini–Schramm convergent and we can even see its limit object: the rooted infinite d -regular tree \mathbb{T}_d , so the corresponding random rooted graph L is simply the distribution which takes a rooted infinite d -regular tree with probability 1. When L is a certain rooted infinite graph with probability 1, then we simply say that this rooted infinite graph is the limit without any further reference on the distribution.

There are other very natural graph sequences that are Benjamini–Schramm convergent, for instance if we take larger and larger boxes in the d -dimensional grid \mathbb{Z}^d , then it will converge to the rooted \mathbb{Z}^d .

The following problem is one of the main problems in the area, and will be especially crucial for us.

Problem. For which graph parameters $p(G)$ is it true that the sequence $(p(G_i))_{i=1}^{\infty}$ converges whenever the graph sequence $(G_i)_{i=1}^{\infty}$ is Benjamini–Schramm convergent?

The problem in such a generality is intractable, but there are various tools to attack it in special cases. One of the most popular tools is the so-called belief propagation. For matchings we will use another way to attack this problem using certain empirical measures called matching measures.

Concerning the general problem the reader might wish to consult with the papers [7, 17–19, 47, 48] and the book [40] and the references therein.

4 A Proof Strategy

We have seen in Sect. 2 that many results suggest that the extremal graph for matchings is not finite but the infinite d -regular tree. This raises the question: how can we attack a problem if we conjecture that the d -regular tree is the extremal graph for a given graph parameter:

Problem: Given a graph parameter $p(G)$. We would like to prove that among d -regular graphs we have

$$p(G) \geq p(\mathbb{T}_d).$$

Proof (A possible two-step solution.)

- Find a graph transformation φ for which $p(G) \geq p(\varphi(G))$, and for every graph G there exists a sequence of graphs (G_i) such that $G = G_0$ and $G_i = \varphi(G_{i-1})$, and $G_i \rightarrow \mathbb{T}_d$.
- Show that if (G_i) is Benjamini–Schramm-convergent, then $p(G_i)$ is convergent, and compute $p(\mathbb{T}_d)$. (Or at least, show it in the case of $G_i \rightarrow \mathbb{T}_d$.) Then

$$p(G) = p(G_0) \geq p(G_1) \geq p(G_2) \geq \dots \geq p(\mathbb{T}_d).$$

Concerning the first step we will be more explicit: it seems that the 2-lift transformation can be used in a wide range of problems. Experience shows that the second step can be the most difficult, but the first step can also be tricky. Nevertheless, in the special case when we only consider a graph sequence converging to \mathbb{T}_d , there are many available tools: see for instance the paper of D. Gamarnik and D. Katz [24]. If $p(G) = \ln \tau(G)/v(G)$, where $\tau(G)$ denotes the number of spanning trees, then the second step is carried out in [41], and B. McKay proved [43] that $p(G)$ is maximized by the d -regular infinite tree among d -regular graphs. However, to prove McKay’s result with our approach, the first step requires some modification. If $p(G) = \ln I(G)/v(G)$, where $I(G)$ denotes the number of independent sets, then the first step is very easy for d -regular bipartite graphs, while the second step concerning the limit theorem was established by A. Sly and N. Sun [47].

In this section we demonstrate this approach by sketching the proof of Theorem 2.10. In the following sections we study each step separately.

4.1 First Step: Graph Transformation

In this section we introduce the concept of 2-lift (Fig. 1).

Definition 4.1 A k -cover (or k -lift) H of a graph G is defined as follows. The vertex set of H is $V(H) = V(G) \times \{0, 1, \dots, k-1\}$, and if $(u, v) \in E(G)$, then we choose a perfect matching between the vertices (u, i) and (v, j) for $0 \leq i, j \leq k-1$. If $(u, v) \notin E(G)$, then there are no edges between (u, i) and (v, j) for $0 \leq i, j \leq k-1$.

When $k = 2$ one can encode the 2-lift H by putting signs on the edges of the graph G : the $+$ sign means that we use the matching $((u, 0), (v, 0)), ((u, 1), (v, 1))$ at the edge (u, v) , the $-$ sign means that we use the matching $((u, 0), (v, 1)), ((u, 1), (v, 0))$ at the edge (u, v) . For instance, if we put $+$ signs to every edge, then we simply get $G \cup G$ as H , and if we put $-$ signs everywhere, then the obtained 2-cover H is simply $G \times K_2$.

The following result will be crucial for our argument.

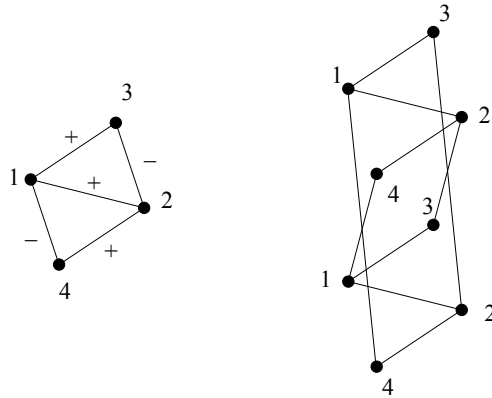


Fig. 1 A 2-lift

Lemma 4.2 (N. Linial [39]) *For any graph G , there exists a graph sequence $(G_i)_{i=0}^\infty$ such that $G_0 = G$, G_i is a 2-lift of G_{i-1} for $i \geq 1$, and $g(G_i) \rightarrow \infty$, where $g(H)$ is the girth of the graph H , that is, the length of the shortest cycle. In particular, if G_0 is d -regular, then $G_i \rightarrow \mathbb{T}_d$.*

Proof It is clear that if H' is a 2-lift of H , then $g(H') \geq g(H)$. Hence it is enough to show that for every H there exists an H'' obtained from H by a sequence of 2-lifts such that $g(H'') > g(H)$. We show that if the girth $g(H) = k$, then there exists a lift of H with fewer k -cycles than H . Let X be the random variable counting the number of k -cycles in a random 2-lift of H . Every k -cycle of H lifts to two k -cycles or a $2k$ -cycle with probability $1/2$ each, so $\mathbb{E}X$ is exactly the number of k -cycles of H . But $H \cup H$ has two times as many k -cycles than H , so there must be a lift with strictly fewer k -cycles than H has. Choose this 2-lift and iterate this step to obtain an H'' with girth at least $k + 1$. \square

Note that if G is a bipartite d -regular graph, and H is a 2-lift of G , then H is again a d -regular bipartite graph.

The following theorem shows that the first step of the plan works for matchings of bipartite graphs.

Theorem 4.3 *Let G be a graph, and let H be an arbitrary 2-lift of G . Then*

$$m_k(H) \leq m_k(G \times K_2),$$

where $m_k(\cdot)$ denotes the number of matchings of size k .

In particular, if $H = G \cup G$, then $m_k(G \cup G) \leq m_k(G \times K_2)$ for every k . It follows that $\text{pm}(G)^2 \leq \text{pm}(G \times K_2)$.

Furthermore, if G is a bipartite graph and H is a 2-lift of G , then

$$\frac{\ln M(G, \lambda)}{v(G)} = \frac{\ln M(G \cup G, t)}{v(G \cup G)} \geq \frac{\ln M(H, \lambda)}{v(H)},$$

where $M(G, \lambda) = \sum_k m_k(G)\lambda^k$. (Note that $M(G \cup G, \lambda) = M(G, \lambda)^2$.)

Proof Let M be any matching of a 2-lift of G . Let us consider the projection of M to G , then it will consist of cycles, paths and “double-edges” (i.e. when two edges project to the same edge). Let \mathcal{R} be the set of these configurations. Then

$$m_k(H) = \sum_{R \in \mathcal{R}} |\phi_H^{-1}(R)|$$

and

$$m_k(G \times K_2) = \sum_{R \in \mathcal{R}} |\phi_{G \times K_2}^{-1}(R)|,$$

where ϕ_H and $\phi_{G \times K_2}$ are the projections from H and $G \times K_2$ to G . Note that

$$|\phi_{G \times K_2}^{-1}(R)| = 2^{k(R)},$$

where $k(R)$ is the number of cycles and paths of R . Indeed, in each cycle or path we can lift the edges in two different ways. The projection of a double-edge is naturally unique. On the other hand,

$$|\phi_H^{-1}(R)| \leq 2^{k(R)},$$

since in each cycle or path if we know the inverse image of one edge, then we immediately know the inverse images of all other edges. Clearly, there is no equality in general for cycles. Hence

$$|\phi_H^{-1}(R)| \leq |\phi_{G \times K_2}^{-1}(R)|$$

and consequently,

$$m_k(H) \leq m_k(G \times K_2).$$

Note that if G is bipartite, then $G \times K_2 = G \cup G$, and so

$$\frac{1}{v(H)} \ln M(H, \lambda) \leq \frac{1}{v(G \cup G)} \ln M(G \cup G, \lambda) = \frac{1}{v(G)} \ln M(G, \lambda).$$

This finishes the proof. □

Remark 4.4 In certain cases it is also possible to prove that for a graph parameter $p(\cdot)$ one has $p(G) \geq p(H)$ for all k -cover H of G . Such a result was given by N. Ruozzi in [44] for attractive graphical models. The advantage of using k -covers is that one can spare the graph limit step in the above approach, and replace it with a much simpler averaging argument over all k -covers of G with k converging to infinity. For homomorphisms this averaging argument was given by P. Vontobel [51]. For matchings such a result was established by C. Greenhill, S. Janson and A. Ruciński [26].

4.2 Second Step: Graph Limit Theory

In this subsection we carry out the second step of our plan. First we develop the necessary terminology.

Recall that if $G = (V, E)$ is a finite graph, then $v(G)$ denotes the number of vertices, and $m_k(G)$ denotes the number of k -matchings ($m_0(G) = 1$). Let

$$\mu(G, x) = \sum_{k=0}^{\lfloor v(G)/2 \rfloor} (-1)^k m_k(G) x^{v(G)-2k}.$$

We call $\mu(G, x)$ the matching polynomial. Clearly, the matching generating function $M(G, \lambda)$ introduced in Sect. 2 and the matching polynomial encode the same information.

The following theorem is crucial in the development of the theory of matching measure.

Theorem 4.5 (O. J. Heilmann and E. H. Lieb [32]) *The zeros of the matching polynomial $\mu(G, x)$ are real, and if the largest degree Δ is greater than 1, then all zeros lie in the interval $[-2\sqrt{\Delta - 1}, 2\sqrt{\Delta - 1}]$.*

Now we introduce a key concept of this theory, the matching measure.

Definition 4.6 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1]) The matching measure of a finite graph G is defined as

$$\rho_G = \frac{1}{v(G)} \sum_{z_i: \mu(G, z_i)=0} \delta(z_i),$$

where $\delta(s)$ is the Dirac-delta measure on s , and we take every z_i into account with its multiplicity. In other words, it is the uniform distribution on the zeros of $\mu(G, x)$.

Example: Let us consider the matching measure of the cycle on 6-vertices, C_6 .

$$\begin{aligned} \mu(C_6, x) &= x^6 - 6x^4 + 9x^2 - 2 = \\ &= (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{2 + \sqrt{3}})(x + \sqrt{2 + \sqrt{3}})(x - \sqrt{2 - \sqrt{3}})(x + \sqrt{2 - \sqrt{3}}). \end{aligned}$$

Hence

$$\begin{aligned} &\int f(z) d\rho_{C_6}(z) = \\ &\frac{1}{6} \left(f(\sqrt{2}) + f(-\sqrt{2}) + f(\sqrt{2 + \sqrt{3}}) + f(-\sqrt{2 + \sqrt{3}}) + f(\sqrt{2 - \sqrt{3}}) + f(-\sqrt{2 - \sqrt{3}}) \right). \end{aligned}$$

The following theorem enables us to consider the matching measure of a unimodular random graph which can be obtained as a Benjamini–Schramm limit of finite graphs. In particular, it provides an important tool to establish the second step in our plan.

Theorem 4.7 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1]) *Let (G_i) be a Benjamini–Schramm convergent bounded degree graph sequence. Let ρ_{G_i} be the matching measure of the graph G_i . Then the sequence (ρ_{G_i}) is weakly convergent, that is, there exists some measure ρ_G such that for every bounded continuous function f , we have*

$$\lim_{i \rightarrow \infty} \int f(z) d\rho_{G_i}(z) = \int f(z) d\rho_G(z).$$

Theorem 4.7 is originated in the work of M. Abért and T. Hubai [3]. They showed a similar result for measures arising from the chromatic polynomial. This result has been generalized to a wide class of graph polynomials including the chromatic polynomial and the matching polynomial by P. Csikvári and P. E. Frenkel [14]. It turns out that for the matching polynomial, one does not need to use this general theorem as it also follows from a result of C. Godsil [25], for details see [2]. Theorem 4.5 asserts that the matching measure is supported on a bounded interval. Then to show that ρ_{G_i} is weakly convergent, it is enough to show that for every fixed k the sequence $\int z^k d\rho_{G_i}(z)$ is convergent. For many graph polynomials one can show that knowing only the statistics of the k -balls already determines this integral. For instance, for the matching polynomial the integral is directly related to the enumeration of the so-called tree-like walks of length k , see [25]. A better-known example is that for the spectral measure, that is, the probability measure of uniform distribution on the eigenvalues of the adjacency matrix of the graph, this integral is determined by the number of closed walks of length k .

To illustrate the power of Theorem 4.7, let us consider an application that also provides us the second step of our plan.

Theorem 4.8 (M. Abért, P. Csikvári, T. Hubai [2]) *Let (G_i) be a Benjamini–Schramm convergent graph sequence of bounded degree graphs. Then the sequences of functions*

$$\frac{\ln M(G_i, \lambda)}{v(G_i)}$$

is pointwise convergent.

Proof If G is a graph on $v(G) = 2n$ vertices and

$$M(G, \lambda) = \prod_{i=1}^{v(G)/2} (1 + \gamma_i \lambda),$$

then

$$\mu(G, x) = \prod_{i=1}^{v(G)/2} (x - \sqrt{\gamma_i})(x + \sqrt{\gamma_i}).$$

Thus

$$\frac{\ln M(G, \lambda)}{v(G)} = \frac{1}{v(G)} \sum_{i=1}^{v(G)/2} \ln(1 + \gamma_i \lambda) = \int \frac{1}{2} \ln(1 + \lambda z^2) d\rho_G(z).$$

Since $\frac{1}{2} \ln(1 + \lambda z^2)$ is a continuous function for every fixed positive λ , the theorem immediately follows from Theorem 4.7. \square

It is worth introducing the notation

$$p_\lambda(G) = \frac{\ln M(G, \lambda)}{v(G)}.$$

We can even introduce $p_\lambda(\mathcal{G})$ if \mathcal{G} is a Benjamini–Schramm-limit of sequence of finite graphs (G_i) . (In fact, it is possible to define the function $p_\lambda(\mathcal{G})$ even if \mathcal{G} is not the Benjamini–Schramm-limit of finite graphs.) In particular, we can speak about $p_\lambda(\mathbb{T}_d)$.

If we know the matching measure of a random unimodular graph, then it is just a matter of computation to derive various results on matchings.

In the particular case when the sequence (G_i) converges to the infinite d -regular tree \mathbb{T}_d , the limit measure $\rho_{\mathbb{T}_d}$ turns out to be the so-called Kesten–McKay measure. It is true in general that for any finite tree or infinite random rooted tree the matching measure coincides with the so-called spectral measure, for details see [1]. In particular, this is true for the infinite d -regular tree \mathbb{T}_d . Its spectral measure is computed explicitly in the papers [36, 42]. The Kesten–McKay measure is given by the density function

$$f_d(x) = \frac{d\sqrt{4(d-1) - x^2}}{2\pi(d^2 - x^2)} \chi_{[-\omega, \omega]},$$

where $\omega = 2\sqrt{d-1}$. Hence for any continuous function $h(z)$ we have

$$\int h(z) d\rho_{\mathbb{T}_d}(z) = \int_{-\omega}^{\omega} h(z) f_d(z) dz.$$

In particular,

$$p_\lambda(\mathbb{T}_d) = \int \frac{1}{2} \ln(1 + \lambda z^2) d\rho_{\mathbb{T}_d}(z) = \frac{1}{2} \ln S_d(\lambda),$$

where

$$S_d(\lambda) = \frac{1}{\eta_\lambda^2} \left(\frac{d-1}{d-\eta_\lambda} \right)^{d-2} \quad \text{and} \quad \eta_\lambda = \frac{\sqrt{1+4(d-1)\lambda}-1}{2(d-1)\lambda}.$$

It is worth introducing the following substitution:

$$\lambda = \frac{\frac{p}{d} \left(1 - \frac{p}{d}\right)}{(1-p)^2}.$$

As p runs through the interval $[0, 1)$, λ runs through the interval $[0, \infty)$ and we have

$$\eta_\lambda = \frac{1-p}{1-\frac{p}{d}} \quad \text{and} \quad S_d(\lambda) = \frac{\left(1 - \frac{p}{d}\right)^d}{(1-p)^2}.$$

One can also prove that

$$\lambda_{\mathbb{T}_d}(p) = \frac{1}{2} \left(p \ln \left(\frac{d}{p} \right) + (d-p) \ln \left(1 - \frac{p}{d} \right) - 2(1-p) \ln(1-p) \right).$$

Remark 4.9 Instead of using measures one can use belief propagation to establish the convergence of certain graph parameters. For instance, M. Lelarge [38] used this method to prove Theorems 2.11 and 2.12. In general, one can choose the sequence $G = G_0, G_1, \dots$ of covering graphs such that the sequence (G_i) converges to the universal cover tree of G . The advantage of belief propagation over matching measures is that it is not always easy to compute the matching measure of such a tree. On the other hand, when it is possible to compute the limiting measure, integration yields a wide variety of results without any difficulty.

4.3 The End of the Proof of Theorem 2.10

For every sequence of 2-covers we know from Theorem 4.3 that

$$p_\lambda(G_0) \geq p_\lambda(G_1) \geq p_\lambda(G_2) \geq p_\lambda(G_3) \geq \dots$$

Furthermore, from Theorems 4.2 and 4.8 we know that we can choose the sequence of 2-covers such that the sequence $p_\lambda(G_i)$ converges to $p_\lambda(\mathbb{T}_d)$, hence $p_\lambda(G) \geq p_\lambda(\mathbb{T}_d)$ for any d -regular bipartite graph G . In other words,

$$\frac{1}{2n} \ln M(G, \lambda) \geq \frac{1}{2} \ln S_d(\lambda).$$

With the substitution $\lambda = \frac{\frac{p}{d}(1-\frac{p}{d})}{(1-p)^2}$ we obtain the inequality

$$M\left(G, \frac{\frac{p}{d}\left(1-\frac{p}{d}\right)}{(1-p)^2}\right) \geq \frac{1}{(1-p)^{2n}} \left(1-\frac{p}{d}\right)^n.$$

After multiplying by $(1-p)^{2n}$, we get that

$$\sum_{k=0}^n m_k(G) \left(\frac{p}{d}\left(1-\frac{p}{d}\right)\right)^k (1-p)^{2(n-k)} \geq \left(1-\frac{p}{d}\right)^{nd}.$$

This is true for all $p \in [0, 1)$ and so by continuity it is also true for $p = 1$, where it directly reduces to Schrijver’s theorem since all but the last term vanish on the left hand side.

5 Perfect Matchings of Vertex-Transitive Bipartite Graphs and Lattices

The starting point of this section is the following theorem of R. Kenyon, A. Okounkov, S. Sheffield [35]. We will not be able to fully understand this theorem as we did not define the characteristic function of a lattice. On the other hand, we can see that this theorem provides a sufficient condition ensuring the convergence of

$$\lim_{i \rightarrow \infty} \frac{\ln \text{pm}(G_i)}{v(G_i)}$$

for a given graph sequence (G_i) . Moreover, it also provides a way of computing this limit explicitly (as long as we accept an integral as an explicit expression).

Theorem 5.1 (R. Kenyon, A. Okounkov, S. Sheffield [35]) *Let \mathcal{G} be a \mathbb{Z}^2 -periodic bipartite planar graph, and let G_i be the quotient graph of \mathcal{G} by the action of $i\mathbb{Z}^2$. Let $P(z, w)$ be the characteristic function of \mathcal{G} . Assume that $P(z, w)$ has a finite number of zeros on the unit torus $\mathbb{T}^2 = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$. Then*

$$\lim_{i \rightarrow \infty} \frac{\ln \text{pm}(G_i)}{v(G_i)} = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \ln P(z, w) \frac{dz}{z} \frac{dw}{w}.$$

To help one understand the setting of this theorem we provide a figure of the square-octagon lattice, and its quotient graph by the action of $(4\mathbb{Z})^2$. The fundamental domain is given by the dotted lines (Fig. 2).

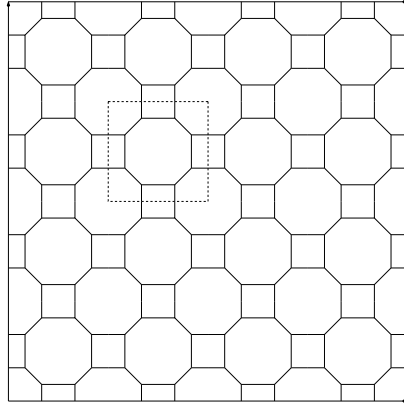


Fig. 2 The 4–8 lattice

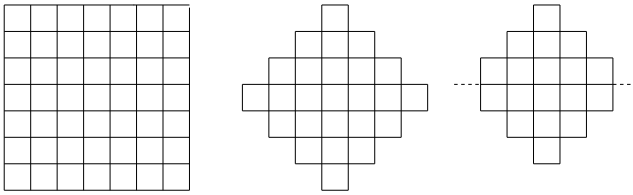


Fig. 3 Boxes, Aztec diamonds and modified Aztec diamonds

In particular cases, the characteristic function $P(z, w)$ can be computed (but not the integral!). For instance, for the square–octagon graph we have

$$P(z, w) = z + \frac{1}{z} + w + \frac{1}{w} + 5.$$

This theorem naturally raises the question why we needed such a special graph sequence, why did we simply not choose larger and larger subgraphs of the lattice? The problem is that in case of perfect matchings the boundary of the graph heavily affects the number of perfect matchings. Let us look at the following three sequences of graphs. All of them are Benjamini–Schramm convergent to \mathbb{Z}^2 (Fig. 3).

The first graph sequence is the sequence of boxes B_i (the graph B_8 is depicted in the figure). A classical result of Kasteleyn [34] and independently Fisher and Temperley [50] claims that

$$\lim_{i \rightarrow \infty} \frac{\ln \text{pm}(B_i)}{v(B_i)} = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \approx 0.291.$$

The second sequence of graphs are called Aztec diamonds, A_4 is depicted in the figure. A surprising fact due to N. Elkies, G. Kuperberg, M. Larsen and J. Propp [20] is that for all i we have

$$\text{pm}(A_i) = 2^{i(i+1)/2}.$$

Therefore,

$$\lim_{i \rightarrow \infty} \frac{\ln \text{pm}(A_i)}{v(A_i)} = \frac{\ln \text{pm}(A_i)}{v(A_i)} = \frac{\ln 2}{4} \approx 0.173.$$

In the third sequence, we slightly modify the Aztec diamonds. Now it turns out that $\text{pm}(D_i) = 1$ for all i , since one has to include the dotted edges in the perfect matching and this completely determines the whole perfect matching. Hence

$$\lim_{i \rightarrow \infty} \frac{\ln \text{pm}(D_i)}{v(D_i)} = \frac{\ln \text{pm}(D_i)}{v(D_i)} = 0.$$

This example shows that some nice boundary conditions are required for the graphs appearing in our convergent graph sequence. Unfortunately, it is unclear what would be such a boundary condition for nonplanar graph. One way to overcome this difficulty is to consider vertex-transitive bipartite graphs.

Theorem 5.2 ([12]) *Let (G_i) be a Benjamini–Schramm convergent sequence of vertex-transitive bipartite graphs. Then the sequence*

$$\frac{\ln \text{pm}(G_i)}{v(G_i)}$$

is convergent.

One might wonder whether vertex-transitivity is really necessary or it suffices to assume regularity. The next theorem shows that regularity is actually not sufficient.

Theorem 5.3 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1]) *Fix $d \geq 3$. Then there exists a sequence of d -regular bipartite graphs (G_i) such that (G_i) is Benjamini–Schramm convergent and*

$$\frac{\ln \text{pm}(G_i)}{v(G_i)}$$

is not convergent.

Let us see what goes wrong with the proof of Theorem 4.8 if we apply it to perfect matchings. If G is a graph on $v(G) = 2n$ vertices and

$$M(G, \lambda) = \prod_{i=1}^{v(G)/2} (1 + \gamma_i \lambda),$$

then

$$\mu(G, x) = \prod_{i=1}^{v(G)/2} (x - \sqrt{\gamma_i})(x + \sqrt{\gamma_i}),$$

and therefore

$$\frac{\ln \text{pm}(G)}{v(G)} = \frac{1}{v(G)} \sum_{i=1}^{v(G)/2} \ln(\gamma_i) = \int \ln |z| d\rho_G(z).$$

Now we see that $\ln |z|$ is not a bounded continuous function and this causes the problem.

On the other hand, we also see that the situation is not as bad as one might have previously thought. The function $\ln |z|$ is only discontinuous at 0. If we could prove that only a small measure is supported on the neighborhood of 0, then it would immediately resolve our problem. This is exactly the case when we consider vertex-transitive bipartite graphs.

Theorem 5.4 ([12]) *Let G be a d -regular vertex-transitive bipartite graph on $2n$ vertices, and*

$$M(G, t) = \prod_{i=1}^n (1 + \gamma_i t),$$

where $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n$. Then

$$\gamma_k(G) \geq \frac{d^2}{d-1} \frac{k^2}{4n^2}.$$

Consequently,

$$\rho_G([-s, s]) \leq \frac{2\sqrt{d-1}}{d} s$$

for all $s \in \mathbb{R}$.

For vertex-transitive graphs one can also extend certain extremal graph theoretic results. For instance, the following theorem is a strengthened form of the fact that the infinite d -regular tree plays the role of the extremal graph for regular bipartite graphs.

Theorem 5.5 ([12]) *Let G be a finite d -regular vertex-transitive bipartite graph, where $d \geq 2$. Furthermore, let the gap function $g(p)$ be defined as*

$$g(p) = \lambda_G(p) - \mathbb{G}_d(p).$$

Then $g(p)$ is a monotone increasing function with $g(0) = 0$, and hence $g(p)$ is non-negative. Furthermore, if G contains an ℓ -cycle, then

$$g(p) \geq \int_0^p f(x)^\ell dx,$$

where

$$f(x) = \frac{1}{4d} \min(x, (1 - x)^2).$$

5.1 Computational Results

In statistical physics matchings of large and infinite graphs are studied under the name monomer-dimer model. Let B_n be a box of size $n \times n \times \dots \times n$ in \mathbb{Z}^d , and let $M(B_n)$ be the number of all matchings in B_n . It has been known for a long time that the limit

$$\tilde{\lambda}(\mathbb{Z}^d) := \lim_{n \rightarrow \infty} \frac{\ln M(B_n)}{v(B_n)}$$

exists. When we count only perfect matchings, then the corresponding model is called the dimer model:

$$\lambda(\mathbb{Z}_d) := \lim_{2|n, n \rightarrow \infty} \frac{\ln \text{pm}(B_n)}{v(B_n)}.$$

The quantities $\tilde{\lambda}(\mathbb{Z}_d)$ and $\lambda(\mathbb{Z}_d)$ are called monomer-dimer and dimer free energies.

The computation of monomer-dimer and dimer free energies has a long history. The precise value is known only in very special cases. Such an exceptional case is the Fisher-Kasteleyn-Temperley formula [34, 50] for the dimer model on \mathbb{Z}^2 . There is no such exact result for monomer-dimer models if $d \geq 2$. The first approach for getting estimates was the use of the transfer matrix method. Hammersley [29, 30], Hammersley and Menon [31] and Baxter [5] obtained the first (non-rigorous) estimates for the free energy. Then Friedland and Peled [23] proved the rigorous estimates $0.6627989727 \pm 10^{-10}$ for $d = 2$ and the range $[0.7653, 0.7863]$ for $d = 3$. Here the upper bounds were obtained by the transfer matrix method, while the lower bounds relied on the Friedland-Tverberg inequality. The lower bound in the Friedland-Peled paper was subsequently improved by newer and newer results (see e.g. [21]) on Friedland’s asymptotic matching conjecture which was finally proved by L. Gurvits [28]. Meanwhile, a non-rigorous estimate $[0.7833, 0.7861]$ was obtained via matrix permanents [33]. Concerning rigorous results, the most significant improvement was obtained recently by D. Gamarnik and D. Katz [24] via their new method which they called sequential cavity method. They obtained the range $[0.78595, 0.78599]$. More precise but non-rigorous estimates can be found in [10]. This paper uses Mayer-series with many coefficients computed in the expansion. The related paper [9] may lead to further development through the so-called Positivity conjecture of the authors.

Here we give some computational results arising from estimating certain integrals along matching measures.

Theorem 5.6 ([2]) *We have*

$$\tilde{\lambda}(\mathbb{Z}^3) = 0.7859659243 \pm 9.88 \cdot 10^{-7},$$

$$\tilde{\lambda}(\mathbb{Z}^4) = 0.8807178880 \pm 5.92 \cdot 10^{-6}.$$

$$\tilde{\lambda}(\mathbb{Z}^5) = 0.9581235802 \pm 4.02 \cdot 10^{-5}.$$

The bounds on the error terms are rigorous.

6 Dense Graphs and Matchings

It is possible to extend several ideas of this paper to dense graph limits. Here we assume some partial familiarity with the theory of dense graph limits. We only mention some simple results.

Theorem 6.1 *Suppose that (G_n) is a sequence of graphs convergent in the dense model. Let π_n be the uniform probability measure on roots of the matching polynomial $\mu(G_n, x)$. Then the rescaled measures $\frac{1}{\sqrt{v(G_n)}} \cdot \pi_n$ converge weakly.*

In particular, this allows us to associate “matching measures” to graphons. For instance, with this method one can prove the following result for the constant p graphon, in other words, the limiting distribution of Erdős–Rényi random graphs. This result was independently and prior proved in [11].

Theorem 6.2 ([11, 15]) *Let $p \in (0, 1)$, and let $(G_n)_n$ be a sequence of Erdős–Rényi random graphs $G_n \sim \mathbb{G}_{n,p}$. Let π_n be the uniform probability distribution on the roots of the matching polynomial of G_n . Then almost surely, the measures $\lambda_n := \frac{1}{\sqrt{n}} \pi_n$ converge weakly to the semicircle distribution SC_p whose density function is*

$$\rho_p(x) := \frac{1}{2\pi} \sqrt{4 - \frac{x^2}{p}}, \quad -2p \leq x \leq 2p .$$

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